

HW 5 due 2/18
3.4: 2, 4, 20, 26, 44

Quiz on 2/23
3.4, 4.1, 4.2

HW 6 due 2/25
4.1: 47, 48
4.2: 10, 14, 22, 54
4.3: 23, 24, 46

Reflection 2 due 2/25 (Canvas)

Matrix of a linear transformation between linear spaces

Ex $D: P_2 \rightarrow P_2$ given by $D(f) = f'$

$$B = \{x^2, x, 1\}$$

Can you find a matrix B such that $B[f]_B = [f']_B$?

Picture

$$\begin{array}{ccc} P_2 & \xrightarrow{D} & P_2 \\ \downarrow L_B & & \downarrow L_B \\ \mathbb{R}^3 & \xrightarrow{B} & \mathbb{R}^3 \end{array}$$

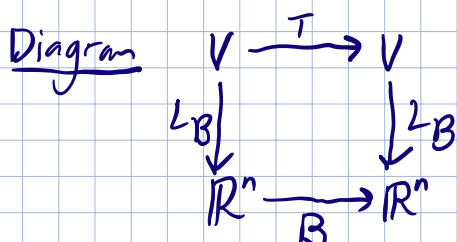
$$D(ax^2 + bx + c) = 2ax + b$$

\leadsto Matrix B such that $B \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 2a \\ b \end{pmatrix}$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Def If $T: V \rightarrow V$ is a linear transformation and $\dim(V) = n$,
let B be a basis of V . Then $L_B \circ T \circ L_B^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$
is a linear transformation represented by a matrix,
say B . We say B is the B -matrix of transformation T .
So, for all $f \in V$

$$[T(f)]_B = B[f]_B \iff L_B(T(f)) = B L_B(f)$$



Thm For $B = \{f_1, \dots, f_n\}$ a basis of V

$$B = \begin{pmatrix} [T(f_1)]_B & \dots & [T(f_n)]_B \end{pmatrix}$$

Ex $V = \text{Span} \{ \underbrace{\cos(x), \sin(x)}_B \}$

$T: V \rightarrow V$ given by $T(f) = 3f + 2f' - f''$

$$\begin{aligned}
 T(\cos(x)) &= 3\cos(x) - 2\sin(x) + \cos(x) \\
 &= 4\cos(x) - 2\sin(x)
 \end{aligned}$$

$$[T(\cos(x))]_B = [4\cos(x) - 2\sin(x)]_B = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

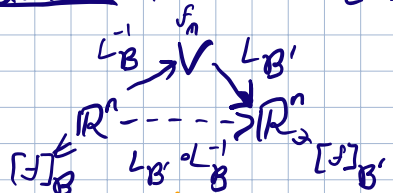
$$\begin{aligned}
 T(\sin(x)) &= 3\sin(x) + 2\cos(x) + \sin(x) \\
 &= 2\cos(x) + 4\sin(x)
 \end{aligned}$$

$$[T(\sin(x))]_B = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$B = \begin{pmatrix} [T(\cos(x))]_B & [T(\sin(x))]_B \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -2 & 4 \end{pmatrix}$$

Is T invertible? $\det \begin{pmatrix} 4 & 2 \\ -2 & 4 \end{pmatrix} = 16 - (-4) = 20 \neq 0$
Yes!

Question How are $[f]_B$ and $[f]_{B'}$ related for two different bases B, B' of V ?



Linear transformation
between vector spaces
 $\rightarrow = n \times n$ matrix

$$\begin{aligned}
 (L_{B'} \circ L_B^{-1})([f]_B) \\
 := L_{B'}(L_B^{-1}([f]_B))
 \end{aligned}$$

Def The matrix S representing $L_{B'} \circ L_B^{-1}$ is called the change of basis matrix from B to B'

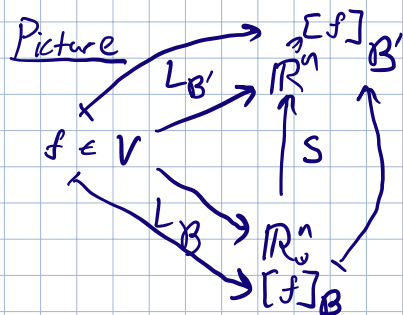
(sometimes denoted $S_{B \rightarrow B'}$)

$$\text{so, } [f]_{B'} = S[f]_B$$

Furthermore, if $B = (b_1, \dots, b_n)$, then

$$[b_i]_{B'} = S[b_i]_B = S\vec{e}_i = i^{\text{th}} \text{ column of } S$$

$$\text{so, } S = \begin{pmatrix} [b_1]_{B'} & \dots & [b_n]_{B'} \end{pmatrix}$$



Remark S is always invertible.

Ex a) P_2 deg ≤ 2 polynomials

$$B' = \{x^2, x, 1\}$$

$$B = \{(x-1)^2, x+1, 1\}$$

$$[(x-1)^2]_{B'} = [x^2 - 2x + 1]_{B'} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$[x+1]_{B'} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad [1]_{B'} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow S = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

\leadsto Write $a(x-1)^2 + b(x+1) + c$
as $a'x^2 + b'x + c'$

$$\Rightarrow \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

b) Change of coordinates of \mathbb{R}^n with $B' = \{\vec{e}_1, \dots, \vec{e}_n\}$

$$B = \{\vec{v}_1, \dots, \vec{v}_n\}$$

Then, if $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $[\vec{x}]_{B'} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ so $L_{B'}$ is I_n

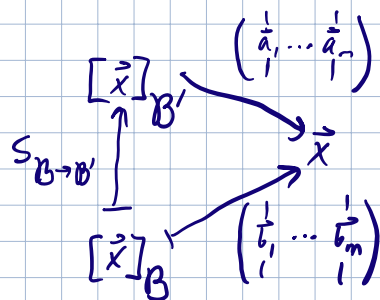
and $S = \left([\vec{v}_1]_{B'} \dots [\vec{v}_n]_{B'} \right) = \begin{pmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{pmatrix}$ is the transformation

$$\vec{x} = S[\vec{x}]_{B'} \text{ from } \S 3.4.$$

Thm For subspace V of \mathbb{R}^n with bases $B' = \{\vec{a}_1, \dots, \vec{a}_m\}$
 $B = \{\vec{b}_1, \dots, \vec{b}_m\}$

$$\text{we have } \begin{pmatrix} | & \dots & | \\ \vec{b}_1 & \dots & \vec{b}_m \\ | & & | \end{pmatrix} = \begin{pmatrix} | & \dots & | \\ \vec{a}_1 & \dots & \vec{a}_m \\ | & & | \end{pmatrix} S_{B \rightarrow B'}.$$

Picture



Thm If V a linear space with bases B and B'

$T: V \rightarrow V$ with B' -matrix A ($A[f]_{B'} = [T(f)]_{B'}$)

B -matrix B ($B[f]_B = [T(f)]_B$)

then, A is similar to B and, for $S = S_{B \rightarrow B'}$,

$$\text{we have } AS = SB.$$

Eg P_2 deg ≤ 2 polynomials

$$B' = \{x^2, x, 1\}$$

$$B = \{(x-1)^2, x+1, 1\}$$

$D: P_2 \rightarrow P_2$ with $D(f) = f'$

$$\text{Then, } A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} | & & | \\ [2(x-1)]_B & [1]_B & [0]_B \\ | & & | \end{pmatrix}$$

$$2x-2 = 2(x+1) - 4 \cdot 1$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -4 & 1 & 0 \end{pmatrix}$$

ans $S = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ So Check

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{I''} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -4 & 1 & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -2 & 1 & 0 \end{pmatrix}}$$

