HW 5 due 2/18
3.4: 2, 4, 20, 26, 44

Quiz on 2/23
3.4, 4.1, 4.2

HW 6 due 2/25
4.1: 47, 48
4.2: 10, 14, 22, 54
4.3: 23, 24, 46

Reflection 2 due 2/25 (Canvas)

Matrix of a linear transformation between linear spaces

Eg. $D: P_2 \longrightarrow P_2$ given by $D(f) = f'$

$\mathcal{B} = \{x^2, x, 1\}$

Can you find a matrix B such that $B[f]_{\mathcal{B}} = [f']_{\mathcal{B}}$?

Picture

$$\begin{tikzcd} P_2 \arrow[r, "D"] \arrow[d, "L_{\mathcal{B}}"'] & P_2 \arrow[d, "L_{\mathcal{B}}"] \\ \mathbb{R}^3 \arrow[r, dashed, "B"'] & \mathbb{R}^3 \end{tikzcd}$$

$$D(ax^2 + bx + c) = 2ax + b$$

$\rightsquigarrow$ Matrix B such that $B\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 2a \\ b \end{pmatrix}$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Def If $T: V \longrightarrow V$ is a linear transformation and $\dim(V) = n$, let $\mathcal{B}$ be a basis of V. Then $L_{\mathcal{B}} \circ T \circ L_{\mathcal{B}}^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation represented by a matrix, say B. We say B is the <u>$\mathcal{B}$-matrix of transformation T</u>.

So, for all $f \in V$

$$[T(f)]_{\mathcal{B}} = B[f]_{\mathcal{B}}. \iff L_{\mathcal{B}}(T(f)) = B L_{\mathcal{B}}(f)$$

Diagram

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "L_{\mathcal{B}}"'] & V \arrow[d, "L_{\mathcal{B}}"] \\ \mathbb{R}^n \arrow[r, "B"'] & \mathbb{R}^n \end{tikzcd}$$

Thm For $\mathcal{B} = \{f_1, \dots, f_n\}$ a basis of V

$$B = \begin{pmatrix} | & & | \\ [T(f_1)]_{\mathcal{B}} & \cdots & [T(f_n)]_{\mathcal{B}} \\ | & & | \end{pmatrix}$$

Eg $V = \text{Span}\underbrace{\{\cos(x), \sin(x)\}}_{\mathcal{B}}$

$T: V \longrightarrow V$ given by $T(f) = 3f + 2f' - f''$

$$T(\cos(x)) = 3\cos(x) - 2\sin(x) + \cos(x)$$
$$= 4\cos(x) - 2\sin(x)$$

$$[T(\cos(x))]_{\mathcal{B}} = [4\cos(x) - 2\sin(x)]_{\mathcal{B}} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$T(\sin(x)) = 3\sin(x) + 2\cos(x) + \sin(x)$$
$$= 2\cos(x) + 4\sin(x)$$

$$[T(\sin(x))]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$B = \begin{pmatrix} | & | \\ [T(\cos(x))]_{\mathcal{B}} & [T(\sin(x))]_{\mathcal{B}} \\ | & | \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -2 & 4 \end{pmatrix}$$

Is T invertible? $\det\begin{pmatrix} 4 & 2 \\ -2 & 4 \end{pmatrix} = 16 - (-4) = 20 \neq 0$

Yes!

Question How are $[f]_{\mathcal{B}}$ and $[f]_{\mathcal{B}'}$ related for two different bases $\mathcal{B}, \mathcal{B}'$ of V?

$$\begin{tikzcd} & \overset{f_n}{V} \arrow[dr, "L_{\mathcal{B}'}"] & \\ \mathbb{R}^n \arrow[ur, "L_{\mathcal{B}}^{-1}"] \arrow[rr, dashed, "L_{\mathcal{B}'} \circ L_{\mathcal{B}}^{-1}"'] & & \mathbb{R}^n \end{tikzcd}$$

$[f]_{\mathcal{B}}$ $\qquad$ $[f]_{\mathcal{B}'}$

Linear transformation between vector spaces $\longrightarrow$ = $n \times n$ matrix

$$(L_{\mathcal{B}'} \circ L_{\mathcal{B}}^{-1})([f]_{\mathcal{B}})$$
$$:= L_{\mathcal{B}'}(L_{\mathcal{B}}^{-1}([f]_{\mathcal{B}}))$$

Def The matrix S representing $L_{\mathcal{B}'} \circ L_{\mathcal{B}}^{-1}$ is called the <u>change of basis matrix</u> from $\mathcal{B}$ to $\mathcal{B}'$ (Sometimes denoted $S_{\mathcal{B}\to\mathcal{B}'}$)

So, $[f]_{\mathcal{B}'} = S[f]_{\mathcal{B}}$

Furthermore, if $\mathcal{B} = (b_1, \dots, b_n)$, then

$$[b_i]_{\mathcal{B}'} = S[b_i]_{\mathcal{B}} = S\vec{e}_i = i^{th} \text{ column of } S$$

So, $$S = \begin{pmatrix} | & & | \\ [b_1]_{\mathcal{B}'} & \cdots & [b_n]_{\mathcal{B}'} \\ | & & | \end{pmatrix}$$

Picture

$$\begin{tikzcd}
 & {[f]_{\mathcal{B}'} \in \mathbb{R}^n} \\
f \in V \arrow[ur, "L_{\mathcal{B}'}"] \arrow[dr, "L_{\mathcal{B}}"'] & \\
 & {[f]_{\mathcal{B}} \in \mathbb{R}^n} \arrow[uu, "S"']
\end{tikzcd}$$

Rmk S is always invertible.

Eg a) P_2 deg ≤ 2 polynomials

$\mathcal{B}' = \{x^2, x, 1\}$

$\mathcal{B} = \{(x-1)^2, x+1, 1\}$

$[(x-1)^2]_{\mathcal{B}'} = [x^2 - 2x + 1]_{\mathcal{B}'} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$[x+1]_{\mathcal{B}'} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\quad [1]_{\mathcal{B}'} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad \Rightarrow S = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

$\rightsquigarrow$ Write $a(x-1)^2 + b(x+1) + c$ as $a'x^2 + b'x + c'$

$$\Rightarrow \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

b) Change of coordinates of $\mathbb{R}^n$ with $\mathcal{B}' = \{\vec{e}_1, \dots, \vec{e}_n\}$

$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$

Then, if $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $[\vec{x}]_{\mathcal{B}'} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ So $L_{\mathcal{B}'}$ is Id_n

and $S = \begin{pmatrix} | & & | \\ [\vec{v}_1]_{\mathcal{B}'} & \cdots & [\vec{v}_n]_{\mathcal{B}'} \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$ is the transformation

$\vec{x} = S[\vec{x}]_{\mathcal{B}}$ from §3.4.

Thm For subspace V of $\mathbb{R}^n$ with bases $\mathcal{B}' = \{\vec{a}_1, \dots, \vec{a}_m\}$
$\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_m\}$

we have $\begin{pmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_m \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_m \\ | & & | \end{pmatrix} S_{\mathcal{B}\to\mathcal{B}'}$.

Picture

$$\begin{tikzcd}
{[\vec{x}]_{\mathcal{B}'}} \arrow[dr, "\begin{pmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_m \\ | & & | \end{pmatrix}"] & \\
& \vec{x} \\
{[\vec{x}]_{\mathcal{B}}} \arrow[uu, "S_{\mathcal{B}\to\mathcal{B}'}"] \arrow[ur, "\begin{pmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_m \\ | & & | \end{pmatrix}"'] &
\end{tikzcd}$$

Thm If V a linear space with bases $\mathcal{B}$ and $\mathcal{B}'$

$T: V \longrightarrow V$ with $\mathcal{B}'$-matrix A $\quad (A[f]_{\mathcal{B}'} = [T(f)]_{\mathcal{B}'})$
$\mathcal{B}$-matrix B $\quad (B[f]_{\mathcal{B}} = [T(f)]_{\mathcal{B}})$

then, A is similar to B and, for $S = S_{\mathcal{B}\to\mathcal{B}'}$,
we have $AS = SB$.

Eg P_2 deg ≤ 2 polynomials
$\mathcal{B}' = \{x^2, x, 1\}$
$\mathcal{B} = \{(x-1)^2, x+1, 1\}$
$D: P_2 \longrightarrow P_2$ with $D(f) = f'$

Then, $A = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} | & | & | \\ [2(x-1)]_{\mathcal{B}} & [1]_{\mathcal{B}} & [0]_{\mathcal{B}} \\ | & | & | \end{pmatrix}$

$2x-2 = 2(x+1) - 4 \cdot 1$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -4 & 1 & 0 \end{pmatrix}$$

and $S = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ So check

$$\underbrace{\overset{A}{\overset{\shortparallel}{\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}} \overset{S}{\overset{\shortparallel}{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}}}_{\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -2 & 1 & 0 \end{pmatrix}} = \underbrace{\overset{S}{\overset{\shortparallel}{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}} \overset{B}{\overset{\shortparallel}{\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -4 & 1 & 0 \end{pmatrix}}}}_{\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -2 & 1 & 0 \end{pmatrix}}$$

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