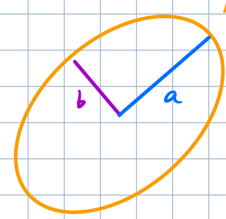
HW9 due 3/23 (wed)
6.2: 2, 12, 14, 38, 42
6.3: 2, 18(*)

Midterm2 3/25
3.4–6.3
Some practice on Canvas
Also look at old HW & Textbook T/F

(*) Area of ellipse $= \pi ab$



Diagonalization

Diagonal matrices are great!

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad \text{What is } A^3 = \begin{pmatrix} 2^3 & 0 & 0 & 0 \\ & 0^3 & 0 & 0 \\ & & 1^3 & 0 \\ & & & 3^3 \end{pmatrix}$$

$\text{rank}(A) = 3$

$\det(A) = 2 \cdot 0 \cdot 1 \cdot 3 = 0$

Basis for $\ker(A) = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ $= \vec{e}_2$

since

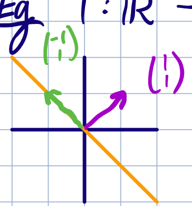
$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 0 \\ 1x_3 \\ 3x_4 \end{pmatrix}$$

Given a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$, we can find a matrix for T relative to $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$:

$$\begin{pmatrix} | & & | \\ [T(\vec{v}_1)]_{\mathcal{B}} & \cdots & [T(\vec{v}_n)]_{\mathcal{B}} \\ | & & | \end{pmatrix}$$

Can we find a basis so this $\mathcal{B}$-matrix is diagonal?

Eg $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by reflecting over line $y = -x$.



$$\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \rightsquigarrow \left(\left[T\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]_{\mathcal{B}} \left[T\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{\mathcal{B}} \right) = \left(\left[\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]_{\mathcal{B}} \left[\begin{pmatrix} -1 \\ -1 \end{pmatrix}\right]_{\mathcal{B}} \right)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\begin{pmatrix}-1\\-1\end{pmatrix} = 0\begin{pmatrix}-1\\1\end{pmatrix} + (-1)\begin{pmatrix}1\\1\end{pmatrix}$

Def Linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\vec{x}) = A\vec{x}$ is <u>diagonalizable</u> if the matrix B of T with respect to some basis $\mathcal{B}$ is diagonal. A is also called diagonalizable.

Prop A matrix A is diagonalizable if and only if it is similar to a diagonal matrix. I.e., there exists invertible matrix S and diagonal matrix B such that $S^{-1}AS = B$.

Eg $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by reflecting over $y=-x$ is given by $T(\vec{x}) = A\vec{x}$ for $A = \begin{pmatrix}0&-1\\-1&0\end{pmatrix}$. Take $S = \begin{pmatrix}1&1\\-1&1\end{pmatrix}$ $S^{-1} = \begin{pmatrix}1/2&-1/2\\1/2&1/2\end{pmatrix}$. Then

$$
\begin{aligned}
S^{-1}AS &= \frac{1}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}0&-1\\-1&0\end{pmatrix}\begin{pmatrix}1&1\\-1&1\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}1&-1\\-1&-1\end{pmatrix}\begin{pmatrix}1&1\\-1&1\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}2&0\\0&2\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix} = B
\end{aligned}
$$

So $\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = 1\cdot\begin{pmatrix}1\\-1\end{pmatrix}$ and $\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = -1\cdot\begin{pmatrix}1\\1\end{pmatrix}$.

Def For a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\vec{x}) = A\vec{x}$, $\vec{v} \in \mathbb{R}^n$ for $\vec{v} \neq \vec{0}$ is called an <u>eigenvector</u> of A or T if

$$A\vec{v} = \lambda\vec{v} \quad \text{for some scalar } \lambda.$$

λ is the <u>eigenvalue</u> associated to $\vec{v}$.

A basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ is called an <u>eigenbasis</u> of A or T if every $\vec{v}_i$ is an eigenvector of A or T.

Eg $A = \begin{pmatrix}0&-1\\-1&0\end{pmatrix}$ has eigenvectors $\vec{v}_1 = \begin{pmatrix}1\\-1\end{pmatrix}$ with eigenvalue $\lambda_1 = 1$

$\vec{v}_2 = \begin{pmatrix}1\\1\end{pmatrix}$ with eigenvalue $\lambda_2 = -1$

Together $\{\vec{v}_1, \vec{v}_2\}$ is an eigenbasis of A.

Thm a) A diagonalizable $\iff$ A has an eigenbasis.

b) Furthermore, if $\vec{v}_1, \dots, \vec{v}_n$ is an eigenbasis for A with

$A\vec{v}_i = \lambda_i \vec{v}_i$, then

$$S = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \lambda_1 & 0 \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix} \quad \text{Satisfy}$$

$S^{-1}AS = B$ ("S and B diagonalize A")

c) Conversely, if S and B diagonalize A, then

i) Column vectors of S give an eigenbasis for A

ii) diagonal entries of B are the associated eigenvalues.

Eg a) Above, we see this with reflection over $y = -x$

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$\vec{v}_1$ $\vec{v}_2$

$(S^{-1}AS = B)$

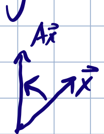
Also works with $S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}$

$\vec{v}_2$ $\vec{v}_1$

Note All reflections are diagonalizable.

b) Consider rotation CCW by $45°$ in $\mathbb{R}^2$. $A = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

Diagonalizable? No!



Because no vector is sent to a scalar multiple of itself

I.e., no vector is sent to a parallel vector.

c) For A an orthogonal matrix, $\|A\vec{x}\| = \|\vec{x}\|$

If $\vec{v}$ is an eigenvector $\|\vec{v}\| \underset{A \text{ is orthogonal}}{=} \|A\vec{v}\| = \|\lambda\vec{v}\| = |\lambda| \|\vec{v}\|$,

So the only possible (real) eigenvalues are ± 1.

2) If A satisfies $A\vec{v} = \vec{0}$, then $\vec{v}$ is an eigenvector with eigenvalue $\lambda = 0$.
$= 0 \cdot \vec{v}$. $\Uparrow$ $(\vec{v} \neq \vec{0})$

$\vec{v}$ is an eigenvector with eigenvalue $0 \iff \vec{v} \in \ker A$

$\iff A$ is not invertible.

Finding eigenvalues

$$
\begin{aligned}
A\vec{v} = \lambda\vec{v} &\iff A\vec{v} - \lambda\vec{v} = \vec{0} \\
&\iff (A - \lambda I_n)\vec{v} = \vec{0} \\
&\iff \vec{v} \in \ker(A - \lambda I_n)
\end{aligned}
$$

If $\vec{v} \neq \vec{0}$, this can only happen if $A - \lambda I_n$ is not invertible.

$$\Updownarrow$$

$$\underbrace{\det(A - \lambda I_n) = 0.}_{\text{"characteristic equation"}}$$