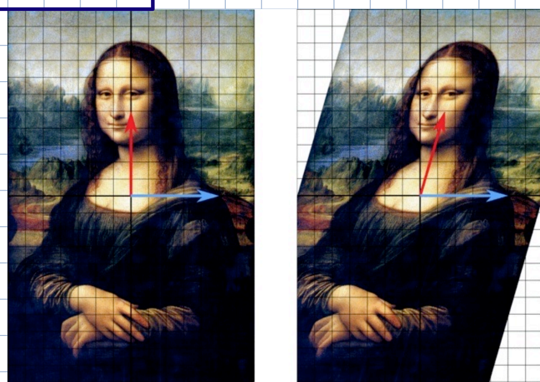
Written HW 10 (due 4/8)
7.1: 4, 6, 12, 16, 18
7.2: 8, 12, 18, 38

Picture:
(Wikipedia)



Recall $A\vec{v} = \lambda\vec{v}$ $\vec{v}$ is an eigenvector
λ is its assoc eigenvalue

Thm If $n\times n$ matrix has eigenvalues $\lambda_1, \dots, \lambda_n$ (with alg mult)
then $\det(A) = \lambda_1\lambda_2\cdots\lambda_n$ and $tr(A) = \lambda_1+\lambda_2+\cdots+\lambda_n$
because $f_A(\lambda) = \underbrace{(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)}_{\text{characteristic poly of } A} = \lambda^n - \underbrace{(\lambda_1+\cdots+\lambda_n)}_{tr(A)}\lambda^{n-1} + \cdots + \underbrace{\lambda_1\lambda_2\cdots\lambda_n}_{\det(A)}$

Eg (Last time) $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ has eigenvalues -1 w/ alg mult 2
& 2 w/ alg mult 1
$\rightsquigarrow \lambda_1 = -1,\ \lambda_2 = -1,\ \lambda_3 = 2$

$\det A = 2 = (-1)(-1)2$
$tr\, A = 0 = -1 + -1 + 2$

Computing Eigenvectors and Eigenspaces

For $n\times n$ matrix A, we can find eigenvalues, now what about eigenvectors?

$A\vec{v} = \lambda\vec{v} \iff (A - \lambda I_n)\vec{v} = \vec{0}$ So must compute basis of $\ker(A - \lambda I_n)$.

Eg $A = \begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix} \rightsquigarrow \lambda^2 + \lambda - 6 = 0$
$tr\, A = -1$ $\implies (\lambda+3)(\lambda-2) = 0$
$\det A = -6$ $\implies \lambda = -3, 2$ are eigenvalues.

$\underline{\lambda = 2}$ $\ker(A - 2I_2) = \ker\begin{pmatrix} -7 & 2 \\ -7 & 2 \end{pmatrix}$

$\iff$ solve $x_1\begin{pmatrix} -7 \\ -7 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Has a solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$

So, $\ker(A-2I_2) = \text{span}\left\{\begin{pmatrix}2\\7\end{pmatrix}\right\} \Longrightarrow \vec{v} = \begin{pmatrix}2\\7\end{pmatrix}$ is an eigenvector of A with eigenvalue 2.

$\underline{\lambda=-3}$ $\ker(A+3I_2) = \ker\begin{pmatrix}-2&2\\-7&7\end{pmatrix}$

$\Longleftrightarrow$ Solve $x_1\begin{pmatrix}-2\\-7\end{pmatrix} + x_2\begin{pmatrix}2\\7\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$. Has a solution $\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$

So $\ker(A+3I_2) = \text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$ so $\begin{pmatrix}1\\1\end{pmatrix}$ is an eigenvector of A with eigenvalue -3.

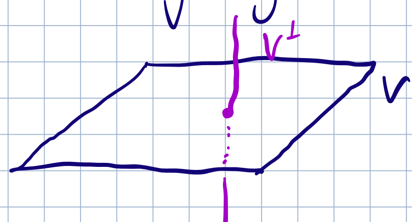
Thus, $\left\{\begin{pmatrix}2\\7\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$ forms an eigenbasis of A, so A is diagonalizable.

(diagonalized by $S = \begin{pmatrix}2&1\\7&1\end{pmatrix}$ and $B = \begin{pmatrix}2&0\\0&-3\end{pmatrix}$)

$\underline{\text{Def}}$ For eigenvalue λ of $n\times n$ matrix A, $\ker(A-\lambda I_n)$ is called the $\underline{\text{eigenspace}}$ associated to λ, denoted E_λ.

$\underline{\text{Rmk}}$ $E_0 = \ker(A)$.

$\underline{\text{Eg}}$ $\text{proj}_V : \mathbb{R}^3 \to \mathbb{R}^3$ given by projection onto $V = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : x+y+z=0\right\}$



$= \text{span}\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}\right\}$

$\uparrow$ Eigenvectors with eigenvalue 1

$\begin{pmatrix}x\\y\\z\end{pmatrix} \in V$ then $\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = x+y+z = 0$

$\Rightarrow \begin{pmatrix}1\\1\\1\end{pmatrix} \in V^\perp$, $\text{proj}_V \begin{pmatrix}1\\1\\1\end{pmatrix} = \vec{0} = 0\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}$

Since $\text{proj}_V \vec{v} = 1\cdot\vec{v}$ for $\vec{v} \in V$

$\Longrightarrow E_1 = V$

So $\begin{pmatrix}1\\1\\1\end{pmatrix}$ is an eigenvector of proj_V with eigenvalue 0, so $E_0 = V^\perp$

(Since $V^\perp = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$)

So if $\text{proj}_V(\vec{x}) = A\vec{x}$, then A is similar to $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ via

$$S = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Note, this is another way to compute A.

Eg $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad \det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2$ $\quad$ so $\lambda = 1$ is eigenvalue with alg mult 2.

$\ker(A - 1I_2) = \ker\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

so $E_1 = \ker\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

$\Rightarrow \dim E_1 = 1.$

Def The <u>geometric multiplicity</u> of eigenvalue λ of $n \times n$ matrix A is the dimension of $E_\lambda = \ker(A - \lambda I_n)$.

Note Algebraic multiplicity is not necessarily equal to geometric multiplicity! See e.g. above with $\lambda = 1$ having alg mult $= 2$, geo mult $= 1$.

Recall A diagonalizable $\Leftrightarrow$ A has an eigenbasis

Notice If A has eigenvalues $\lambda_1, \dots, \lambda_k$ and $\overbrace{(\dim E_{\lambda_1}) + (\dim E_{\lambda_2}) + \cdots + (\dim E_{\lambda_k})}^{s} = n$ then concatenating bases of $E_{\lambda_1}, \dots, E_{\lambda_k}$ gives basis of $\mathbb{R}^n$.

If $(\dim E_{\lambda_1}) + (\dim E_{\lambda_2}) + \cdots + (\dim E_{\lambda_k}) < n$, then there are not enough linearly independent eigenvectors to form a basis of $\mathbb{R}^n$.

Thm For $n \times n$ matrix A

a) if we concat the bases of each eigenspace of A, the resulting eigenvectors $\vec{v}_1, \dots, \vec{v}_s$ are linearly independent.

b) A is diagonalizable $\Leftrightarrow$ $s = n$.

<u>Eg</u> a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is <u>not</u> diagonalizable since $\dim E_1 = 1 < 2$.

$(s < n)$

b) $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ has eigenvalues $\lambda = 2$ w/ alg mult 1
$\lambda = -1$ w/ alg mult 2

$$E_2 = \ker(A - 2I_2) = \ker \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \qquad \operatorname{rref} \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} t \\ t \\ t \end{pmatrix} : t \text{ any real number} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$E_{-1} = \ker \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \operatorname{rref} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} -t-s \\ s \\ t \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$\dim E_2 + \dim E_{-1} = 2 + 1 = 3 = n$ so A is diagonalizable

and $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$ forms an eigenbasis.