Written HW 10 (due 4/8)
7.1: 4, 6, 12, 16, 18
7.2: 8, 12, 18, 38
7.3: 8, 10, 24

$A\vec{v} = \lambda \vec{v}$ — $\vec{v}$ an eigenvector, λ its corresponding eigenvalue

Eigenvectors are rescaled but stay on same line under A.

$E_\lambda = \{\text{all eigenvectors of } A \text{ with eigenvalue } \lambda\} \cup \{\vec{0}\}$
$= \ker(A - \lambda I_n)$

$\dim E_\lambda$ = geometric multiplicity of λ

Last time $n \times n$ matrix A is diagonalizable
$\iff$ A has n eigenvalues when counted with geometric multiplicity

Eg $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ has $\lambda = 2$ geometric mult 1
$\lambda = -1$ geometric mult 2

so $\dim E_2 + \dim E_{-1} = 1 + 2 = 3$

So A is diagonalizable.

Thm If $n \times n$ matrix A has n distinct eigenvalues, then
A is diagonalizable because $\dim E_{\lambda_1} + \dim E_{\lambda_2} + \cdots + \dim E_{\lambda_n} = 1 + 1 + \cdots + 1$
$= n.$

Eigenvalues and Similarity

Thm For similar $n \times n$ matrices A and B

a) A and B have the same characteristic polynomial.

so, $\det A = \det B$ and $\operatorname{tr} A = \operatorname{tr} B$.

Why? Because det distributes multiplicatively

$$\det(B - \lambda I_n) = \det(S^{-1}AS - \lambda I_n)$$
$$= \det(S^{-1}(A - \lambda I_n)S)$$
$$= \det(S^{-1})\det(A - \lambda I_n)\det(S) = \det(A - \lambda I_n).$$

b) $\operatorname{rank} A = \operatorname{rank} B$ and $\operatorname{nullity} A = \operatorname{nullity} B$

Why? $B = S^{-1}AS$

$\{\vec{v}_1, \dots, \vec{v}_r\}$ basis for $\ker B \Rightarrow \{S\vec{v}_1, \dots, S\vec{v}_r\}$ is in $\ker A$ and linearly indep

$$B\vec{v}_i = \vec{0} \iff S^{-1}AS\vec{v}_i = \vec{0} \iff AS\vec{v}_i = \vec{0}$$

$\{\vec{w}_1, \dots, \vec{w}_s\}$ basis for $\ker A \Rightarrow \{S^{-1}\vec{w}_1, \dots, S^{-1}\vec{w}_s\}$ is in $\ker B$ and linearly indep.

$\Rightarrow \dim(\ker A) = \dim(\ker B)$ and rank-nullity.

c) A and B have same eigenvalues with same alg mult and same geom mult.

(⚠ But may not have the same eigenvectors!)

Why? Same Alg mult $\Leftarrow$ A and B have same characteristic poly

(b) $\Rightarrow \operatorname{nullity}(A - \lambda I_n) = \operatorname{nullity}(B - \lambda I_n)$

$\Rightarrow$ same geometric multiplicity.

Eg Is $\begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix}$ similar to $\begin{pmatrix} 4 & 6 \\ 2 & 1 \end{pmatrix}$?

No! trace $= 4$ trace $= 5$

Thm For any eigenvalue λ_0 of $n \times n$ matrix A,

geometric multiplicity of $\lambda_0 \leq$ algebraic multiplicity of λ_0.

Idea If λ_0 has geometric multiplicity m, then

$$A \text{ is similar to } \begin{array}{c} \\ m\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\ \\ \end{array} \begin{pmatrix} \overbrace{\begin{matrix} \lambda_0 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_0 \end{matrix}}^{m} & * \\ \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} & * \end{pmatrix} \leftarrow \text{has characteristic poly } (\lambda_0 - \lambda)^m g(\lambda)$$

$\dim(\ker(A - \lambda_0 I_n)) = m \Rightarrow E_{\lambda_0}$ has basis $\{\vec{v}_1, \dots, \vec{v}_m\}$

Pick S such that $S\vec{e}_i = \vec{v}_i$ $1 \leq i \leq m \iff \vec{e}_i = S^{-1}\vec{v}_i$ and S invertible.

(Eg $S = \begin{pmatrix} | & & | & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m & \vec{w}_1 & \cdots & \vec{w}_{n-m} \\ | & & | & | & & | \end{pmatrix}$ for $\{\vec{w}_1, \dots, \vec{w}_{n-m}\}$ basis of $E_{\lambda_0}^{\perp}$)

Thus, $(1 \le i \le m)$ $S^{-1}AS\vec{e}_i = S^{-1}A\vec{v}_i = S^{-1}\lambda_0\vec{v}_i = \lambda_0(S^{-1}\vec{v}_i) = \lambda_0\vec{e}_i$

$$\Rightarrow S^{-1}AS = \begin{array}{c} \\ m\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\ \\ \\ \end{array}\left(\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{\qquad\qquad}^{m}} & \\ \lambda_0 & 0 \cdots & 0 & \\ 0 & \ddots & 0 & * \\ 0 & \cdots 0 & \lambda_0 & \\ \hline 0 & \cdots & 0 & \\ \vdots & & \vdots & * \\ 0 & \cdots & 0 & \end{array}\right)$$

has characteristic poly $(\lambda_0-\lambda)^m g(\lambda)$

$\Rightarrow$ A has characteristic poly $(\lambda_0-\lambda)^m g(\lambda)$

$\Rightarrow$ λ_0 has alg mult at least m. $\square$

Summary How to determine if matrix is diagonalizable.

1) Find eigenvalues by solving
$\det(A - \lambda I_n) = 0$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\det(A-\lambda I_3) = \det\begin{pmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{pmatrix} = (2-\lambda)^2(1-\lambda)$$

$\rightsquigarrow$ Eigenvalues $\lambda = 1$ w/ alg mult 1
$\lambda = 2$ w/ alg mult 2.

2) For each eigenvalue λ, find a basis of eigenspace $E_\lambda = \ker(A - \lambda I_n)$

$\underline{\lambda = 1}$

$$\ker\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$$

$\underline{\lambda = 2}$

$$\ker\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}$$

See $1+2=3$ ✓

3) A diagonalizable $\iff$ dims of eigenspaces add up to n
$\iff$ (n eigenvalues when counted with geom mult)

In this case, eigenbasis $\{\vec{v}_1, \dots, \vec{v}_n\}$ is given by concatenating the bases for eigenspaces in step 2 and A is diagonalized by

$$S = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix} \qquad B = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Why diagonalization?

A is diagonalizable with $B = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$

$\det A = \lambda_1 \cdots \lambda_n \quad \Longleftarrow \quad \det B = \lambda_1 \cdots \lambda_n$

$$\begin{matrix} \text{rank } A \\ \text{nullity } A \end{matrix} \quad \equiv \quad \begin{matrix} \text{rank } B \\ \text{nullity } B \end{matrix}$$

$B = S^{-1}AS$

$\Rightarrow \quad SBS^{-1} = A$

$$B^{100} = \begin{pmatrix} \lambda_1^{100} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^{100} \end{pmatrix}$$

$$\begin{aligned} A^{100} &= (SBS^{-1})^{100} \\ &= (SB\cancel{S^{-1})(S}B\cancel{S^{-1})} \cdots \cancel{(S}BS^{-1}) \\ &= SB^{100}S^{-1} \end{aligned}$$