Written HW 10 (due 4/8)
7.1: 4, 6, 12, 16, 18
7.2: 8, 12, 18, 38
7.3: 8, 10, 24

Quiz on 7.1–7.3 4/11

Written HW 11 (due 4/15)
7.5: 14, 20
7.1: 68, 70
7.4: 4, 34

Eg (from last time)

$$A = \begin{pmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{pmatrix}$$

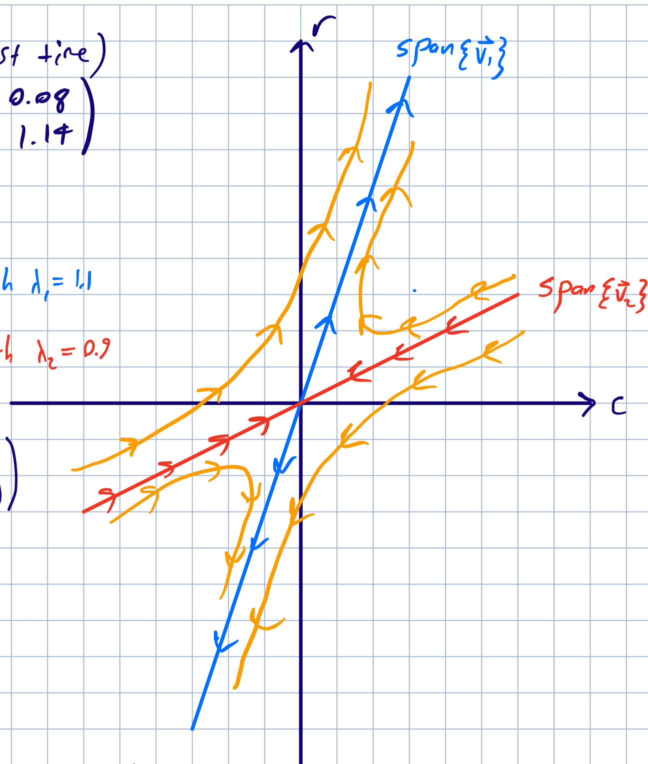
Eigenvectors

$\vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ with $\lambda_1 = 1.1$

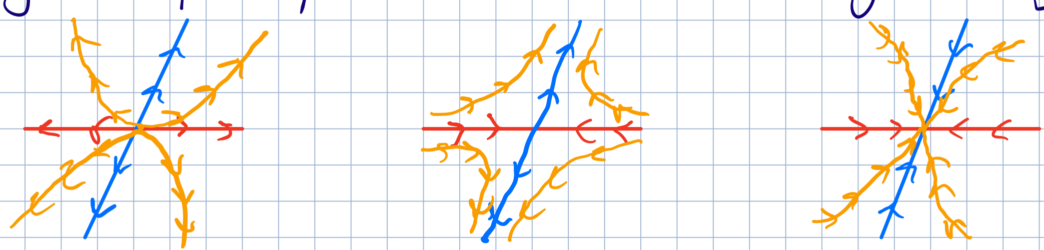
$\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with $\lambda_2 = 0.9$

$\vec{x}(t) = \begin{pmatrix} c(t) \\ r(t) \end{pmatrix}$

$\vec{x}(t+1) = A\vec{x}(t)$



In general, phase portraits for 2 dimensional dynamical systems



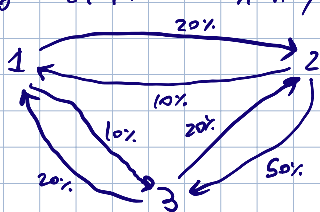
$\lambda_1 > \lambda_2 > 1$ $\qquad$ $\lambda_1 > 1 > \lambda_2 > 0$ $\qquad$ $1 > \lambda_1 > \lambda_2 > 0$

Thm Dynamical System $\vec{x}(t+1) = A\vec{x}(t)$ and $\vec{x}(0) = \vec{x}_0$

a) $\vec{x}(t) = A^t \vec{x}_0$

b) If A has an eigenbasis $\vec{v}_1, \dots, \vec{v}_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$, then for $\vec{x}_0 = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$, $\vec{x}(t) = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 + \dots + c_n\lambda_n^t\vec{v}_n$.

Eg Mini-web
(§7.4 ex 1)



$$A = \begin{pmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.6 \end{pmatrix} \qquad A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.7x_1 + 0.1x_2 + 0.2x_3 \\ 0.2x_1 + 0.4x_2 + 0.2x_3 \\ 0.1x_1 + 0.5x_2 + 0.6x_3 \end{pmatrix}$$

For instance, if $\vec{x}_0 = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$, $A\vec{x}_0 = \begin{pmatrix} 1/3 \\ 4/15 \\ 2/5 \end{pmatrix}$

Def a) $\vec{x} \in \mathbb{R}^n$ is a <u>distribution vector</u> if i) Components sum to 1 $(\Sigma x_i = 1)$
ii) all components ≥ 0 $(x_i \geq 0)$

b) A square matrix A is a <u>transition matrix</u> if every column is a distribution vector.

<u>Note</u> $\vec{x}$ a distribution vector, A a transition matrix
$\Rightarrow A\vec{x}$ is a distribution vector.

<u>Questions</u> 1) Formula for $A^t \vec{x}_0$?

2) For various choices of initial distribution vectors, what is the long term behavior of dynamical system modeled by a transition matrix?

<u>Eg</u> What happens for $\vec{x}_0 = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$?

$$A^5 \vec{x}_0 \approx \begin{pmatrix} 0.348 \\ 0.25 \\ 0.401 \end{pmatrix} \qquad A^{100} \vec{x}_0 \approx \begin{pmatrix} 0.35 \\ 0.25 \\ 0.4 \end{pmatrix}, \quad A^{101} \vec{x}_0 \approx \begin{pmatrix} 0.35 \\ 0.25 \\ 0.4 \end{pmatrix}$$

Def An <u>equilibrium distribution</u> is a distribution vector $\vec{x}$ such that
$$A\vec{x} = \vec{x}.$$

In other words, equilibrium distribution is a distribution vector that is also an eigenvector with eigenvalue 1.

Approach from last time on our mini-web:

1) Compute eigenvalues of A: $\longrightarrow \lambda_1 = 1, \ \lambda_2 = 0.5, \ \lambda_3 = 0.2$
$\Rightarrow A$ is diagonalizable.

2) Compute eigenbasis

$$E_1 = \ker \begin{pmatrix} -0.3 & 0.1 & 0.2 \\ 0.2 & -0.6 & 0.2 \\ 0.1 & 0.5 & -0.4 \end{pmatrix} \longrightarrow \operatorname{rref} \left(\begin{array}{ccc|c} -0.3 & 0.1 & 0.2 & 0 \\ 0.2 & -0.6 & 0.2 & 0 \\ 0.1 & 0.5 & -0.4 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & -7/8 & 0 \\ 0 & 1 & -5/8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$= \left\{ \begin{pmatrix} 7/8\, t \\ 5/8\, t \\ t \end{pmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 7/8 \\ 5/8 \\ 1 \end{pmatrix} \right\} = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} 7 \\ 5 \\ 8 \end{pmatrix}}_{=\vec{v}_1} \right\}$$

$A\vec{v}_1 = \vec{v}_1$

Similarly, $E_{0.5} = \text{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{= \vec{v}_2} \right\}$ $A\vec{v}_2 = 0.5\vec{v}_2$ $E_{0.2} = \text{Span}\left\{ \underbrace{\begin{pmatrix} -1 \\ -3 \\ 4 \end{pmatrix}}_{= \vec{v}_3} \right\}$ $A\vec{v}_3 = 0.2\vec{v}_3$

Thus, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ form an eigenbasis for A.

For any distribution vector $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

$$A^t\vec{x}_0 = c_1A^t\vec{v}_1 + c_2A^t\vec{v}_2 + c_3A^t\vec{v}_3$$

$$= c_1\vec{v}_1 + c_2(0.5)^t\vec{v}_2 + c_3(0.2)^t\vec{v}_3$$

as $t \to \infty$: $c_1\vec{v}_1 \to c_1\vec{v}_1$, $c_2(0.5)^t\vec{v}_2 \to \vec{0}$, $c_3(0.2)^t\vec{v}_3 \to \vec{0}$

However, A is a transition matrix and $\vec{x}_0$ is a distribution vector

$$\Longrightarrow \underbrace{c_1\vec{v}_1}_{= \begin{pmatrix} c_1 7 \\ c_1 5 \\ c_1 8 \end{pmatrix}} \text{ is a distribution vector}$$

$$\Rightarrow 7c_1 + 5c_1 + 8c_1 = 1 \Rightarrow 20c_1 = 1 = c_1 = 1/20$$

Thus $\lim\limits_{t\to\infty}(A^t\vec{x}_0) = \frac{1}{20}\begin{pmatrix} 7 \\ 5 \\ 8 \end{pmatrix}$ for any distribution vector $\vec{x}_0$ for this mini-web.

Def a) A transition matrix is <u>positive</u> if it has no zero entries (so every entry > 0)

b) A transition matrix A is <u>regular</u> if A^m is positive for some m.

Eg $A = \begin{pmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.6 \end{pmatrix}$ is positive. $A = \begin{pmatrix} 0.5 & 1 \\ 0.5 & 0 \end{pmatrix}$ is regular

because $A^2 = \begin{pmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{pmatrix}$ is positive.

Thm For diagonalizable regular transition matrix A of size $n \times n$

a) A has exactly one equilibrium distribution vector $\vec{x}_{equ}$.
Note, $\vec{x}_{equ}$ is an eigenvector w/ eigenvalue 1.

b) For any distribution vector $\vec{x}_0 \in \mathbb{R}^n$, $\lim_{t\to\infty} (A^t \vec{x}_0) = \vec{x}_{equ}$

c) $\lim_{t\to\infty} A^t = \begin{pmatrix} | & | & & | \\ \vec{x}_{equ} & \vec{x}_{equ} & \cdots & \vec{x}_{equ} \\ | & | & & | \end{pmatrix}$

Why? A a regular transition matrix $\Longrightarrow$ i) $\lambda_1 = 1$ is eigenvalue with geom mult 1.

ii) All other eigenvalues satisfy $|\lambda_j| < 1$ $(j \geq 2)$.