Quiz on 7.1–7.3 4/11

Written HW 11 due 4/15
7.5: 14, 20
7.1: 68, 70
7.4: 4, 34

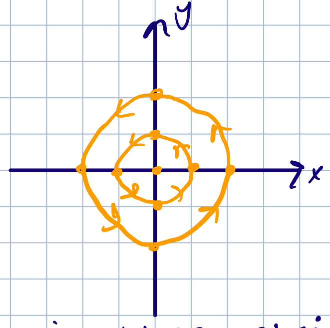
Reflection 3 due 4/15
Canvas

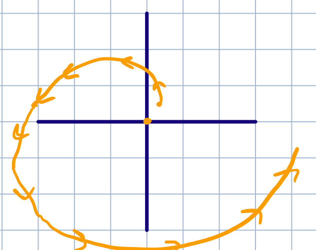
Optional Suggested Problems
8.1: 4, 14, 16

## Concluding Remark on Dynamical Systems

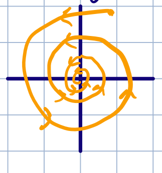
Note General dynamical systems (not modeled by transition matrix) can have other behaviors.

Eg a) $A =$ rotation matrix (complex eigenvalues) b) $A = \begin{pmatrix} 1.1 & -0.1 \\ 0.1 & 1.1 \end{pmatrix}$ $\lambda = \frac{11}{10} \pm \frac{i}{1}$

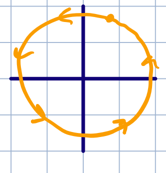


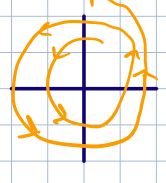
For eigenvalues $p \pm qi$, set $r = p^2 + q^2$, the following can happen



$r < 1$



$r = 1$



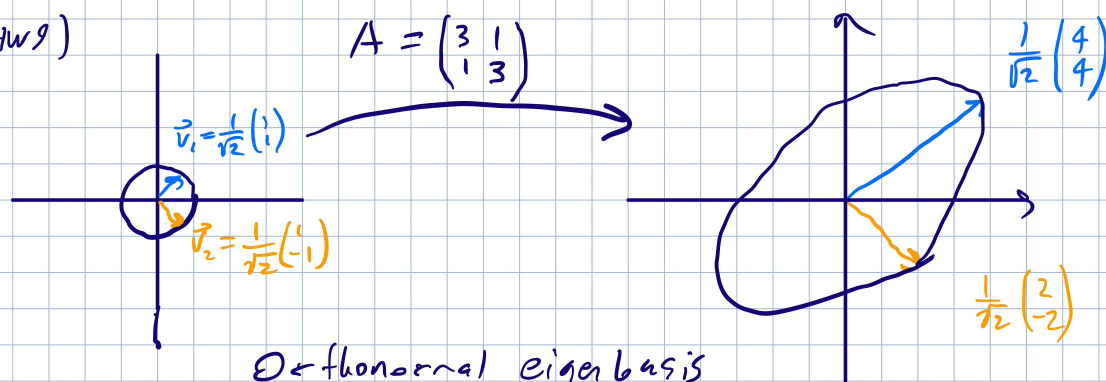
$r > 1$

## Symmetric Matrices

Recall $A$ diagonalizable $\Longleftrightarrow$ $A$ has an eigenbasis.

When does $A$ have an orthonormal eigenbasis?

Eg
(From HW9)




Orthonormal eigenbasis
$\{\vec{v}_1, \vec{v}_2\}$ because $\vec{v}_1 \cdot \vec{v}_2 = 0$.

$\Longrightarrow$ $A$ is diagonalized by $S = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}}_{\text{orthogonal}}$ and $B = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$

$(B = S^{-1}AS)$

<u>Def</u> $n \times n$ matrix $A$ is <u>orthogonally diagonalizable</u> if there exists orthogonal matrix $S$ such that $S^{-1}AS$ is diagonal.

<u>Note</u> $\underset{\text{diagonal}}{D} = \underset{\text{orthogonal}}{S^{-1}}AS \iff A = SDS^{-1} = SDS^T$

$\iff A^T = (SDS^T)^T = (S^T)^T D^T S^T$

$= SDS^T$

$= A$

Thus, $A$ orthogonally diagonalizable $\Longrightarrow$ $A$ is symmetric $(A^T = A)$.

<u>Spectral Thm</u> $A$ orthogonally diagonalizable $\iff$ $A$ is symmetric.

<u>Special Case</u> 1) For symmetric matrix $A$, if $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors with distinct eigenvalues $\lambda_1 \neq \lambda_2$, then $\vec{v}_1 \cdot \vec{v}_2 = 0$.

2) A symmetric $n \times n$ matrix $A$, has $n$ <u>real</u> eigenvalues when counted with algebraic multiplicity.

Eg For $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ find orthogonal matrix $S$ such that $S^{-1}AS$ is diagonal.

$\rightsquigarrow$ Eigenvalues $0, 3$

$$E_0 = \ker \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{\text{''}\vec{v}_1}, \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{\text{''}\vec{v}_2} \right\}$$

$$E_3 = \ker \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \rightsquigarrow \vec{v}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\rightsquigarrow$ $\vec{v}_3$ is orthogonal to $\vec{v}_1$ and $\vec{v}_2$, but $\vec{v}_1, \vec{v}_2$ are not orthogonal to each other ($\vec{v}_1 \cdot \vec{v}_2 = 1$).

Perform Gram-Schmidt on $E_0$:

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}}(1)\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$$\vec{u}_2 = \frac{1}{\sqrt{1/4 + 1/4 + 1}} \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

So, $\{\vec{u}_1, \vec{u}_2, \vec{v}_3\}$ forms an orthonormal eigenbasis

$$\Rightarrow S = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \text{ gives } S^{-1}AS = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 3 \end{pmatrix}.$$

Summary of orthogonal diagonalization for symmetric matrix $A$

1) Find eigenvalues and basis for each eigenspace.
2) Use Gram-Schmidt on each eigenspace to find an orthonormal basis of each

3) Form an eigenbasis $\{\vec{v}_1, \dots, \vec{v}_n\}$ by concatenating orthonormal bases from (2) and set $S = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$ for $S$ to be

i) orthonormal &

ii) $S^{-1}AS$ diagonal.