http://timbaumann.info/svd-image-compression-demo/

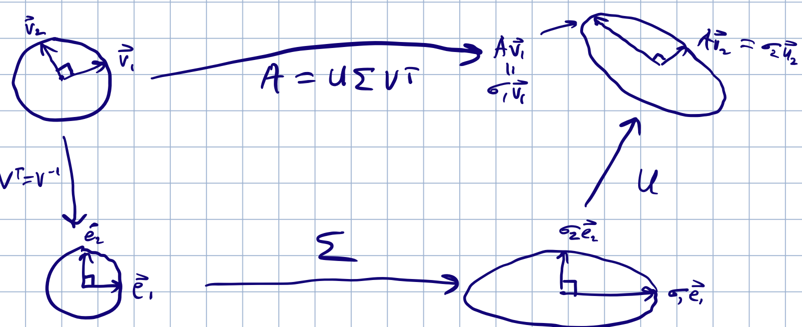
Written HW 11 due 4/15
7.5: 14, 20
7.1: 68, 70
7.4: 4, 34

Reflection 3 due 4/15
Canvas

Course Evaluations! (52%)
Close 4/20

Optional Suggested Problems
8.1: 4, 14, 16
8.3: 4, 6

Final 4/26 1:30p-3:30p
Here (EH 1068)



<u>Eg</u> $T(\vec{x}) = A\vec{x}$ $\quad A = \underset{2\times 3}{\begin{pmatrix} 4 & 11 & 14 \\ 4 & 7 & -2 \end{pmatrix}}$

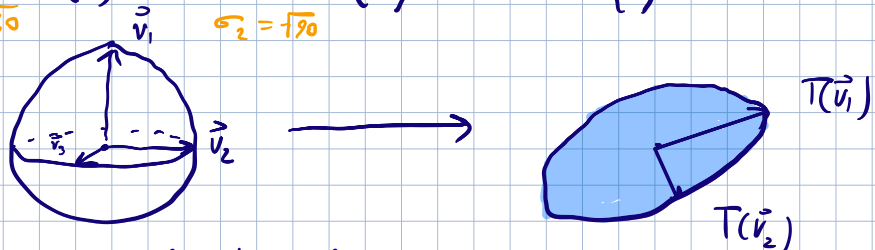
(Last time)

$$A^TA = \begin{pmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{pmatrix} \rightsquigarrow \text{Eigenvalues } \lambda_1 = 360,\ \lambda_2 = 90, \text{ and } \lambda_3 = 0.$$

$$E_{360} = \text{span}\left\{\underbrace{\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}}_{\vec{v}_1}\right\} \quad E_{90} = \text{span}\left\{\underbrace{\begin{pmatrix} -2/3 \\ -1/3 \\ 2/3 \end{pmatrix}}_{\vec{v}_2}\right\} \quad E_0 = \text{span}\left\{\underbrace{\begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}}_{\vec{v}_3}\right\}$$

$\sigma_1\vec{u}_1 = T(\vec{v}_1) = \begin{pmatrix} 18 \\ 6 \end{pmatrix}$, $\sigma_2\vec{u}_2 = T(\vec{v}_2) = \begin{pmatrix} 3 \\ -9 \end{pmatrix}$, $T(\vec{v}_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are orthogonal.

$\rightarrow \sigma_1 = \sqrt{360}$ $\quad \sigma_2 = \sqrt{90}$



Find SVD of A $(n = 2, m = 3)$:

$$\underset{2\times 3}{A} = \underset{2\times 2}{\begin{pmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{pmatrix}} \underset{2\times 3}{\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{pmatrix}} \underset{3\times 3}{\begin{pmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ - & \vec{v}_3^T & - \end{pmatrix}}$$

$$U = \begin{pmatrix} 18/\sqrt{360} & 3/\sqrt{90} \\ 6/\sqrt{360} & -9/\sqrt{90} \end{pmatrix} \quad V = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{pmatrix}$$

Why? For large matrices with high rank, many σ_i will be close to 0 relative to σ_1. Set those to 0 to approximate A:

$$\overset{n\times m}{A} = \underbrace{\sigma_1 \vec{u}_1 \vec{v}_1^T}_{n+m+1} + \dots + \sigma_r \vec{u}_r \vec{v}_r^T \quad \rightsquigarrow r(n+m+1)$$

$n \cdot m$ numbers

$$\approx \sigma_1 \vec{u}_1 \vec{v}_1^T + \dots + \sigma_s \vec{u}_s \vec{v}_s \quad \text{if } \sigma_{s+1}, \dots, \sigma_r \ll \sigma_1$$

$$\rightsquigarrow s(n+m+1)$$

Works well if $s(n+m+1) < n \cdot m$

8. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ -2 & 1 \end{pmatrix}$.

(a) (4 pts) Find the eigenvalues and corresponding eigenvectors of the matrix $A^T A$.

$$A^TA = \begin{pmatrix} 1 & 2 & -2 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -1 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$$

char poly of A^TA: $\lambda^2 - tr(A^TA)\lambda + det(A^TA)$

$$= \lambda^2 - 15\lambda + (54-4)$$
$$= \lambda^2 - 15\lambda + 50$$
$$= (\lambda-10)(\lambda-5)$$

Eigenvalues of A^TA are $\lambda = 5, 10$

$$E_{10} = ker(A^TA - 10 I_2)$$
$$= ker\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} = span\left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right\}$$

$$E_5 = ker(A^TA - 5I_2)$$
$$= ker\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} = span\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$$

Eigenvectors are $\vec{w}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\vec{w}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$(n\times m) = (n\times n)(n\times m)(m\times m)$

(b) (6 pts) Find the singular value decomposition $A = U\Sigma V^T$. Explicitly compute all three matrices U, Σ, and V.

orthonormal basis of domain: $\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\quad \vec{v}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \rightsquigarrow \quad V = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

$\sigma_1 = \sqrt{10}$ $\quad \sigma_2 = \sqrt{5}$ $\quad \rightsquigarrow \quad \Sigma = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{5} \\ 0 & 0 \end{pmatrix}$

$\downarrow A$

orthogonal vectors in range: $A\vec{v}_1 = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ -2 & 1 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix}$ $\quad A\vec{v}_2 = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ -2 & 1 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$

$A\vec{v}_1 = \sigma_1\vec{u}_1$ $\quad A\vec{v}_2 = \sigma_2\vec{u}_2$

(b/c $\|A\vec{v}_1\| = \sigma_1$, $\|A\vec{v}_2\| = \sigma_2$)

$$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{\sqrt{5}\sqrt{10}}\begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} = \frac{1}{5\sqrt{2}}\begin{pmatrix} 0 \\ 5 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2 = \frac{1}{\sqrt{5}\sqrt{5}}\begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$U = \begin{pmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{pmatrix}$$

orthogonal

$$\begin{cases} \vec{u}_3 \cdot \vec{u}_1 = 0 \\ \vec{u}_3 \cdot \vec{u}_2 = 0 \end{cases} \rightsquigarrow \begin{cases} 0\cdot x_1 + \frac{1}{\sqrt{2}}\cdot x_2 + \frac{1}{\sqrt{2}}\cdot x_3 = 0 \\ 1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 0 \end{cases} \Rightarrow \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\vec{u}_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = -x_3 \end{cases} \Rightarrow \vec{u}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

1. (20 pts) For the following statements, circle True if the statement is **always** true, and circle False otherwise. Make sure it is completely clear which is your final answer. No explanations are required for this question, and no partial credit. Read the questions very carefully!

   All matrices/eigenvalues/eigenvectors are assumed to have coefficients in $\mathbb{R}$ **unless otherwise specified**, and similarly for the property of diagonalizability.

   (a) If the reduced row echelon form of a square matrix A is the identity, then A is invertible.

   True False

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3 ( )

   (b) If $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation whose image is a plane, then the kernel of T is also a plane. dim = 2 dim = 2

rank(T) + nullity(T) = 4

2 + nullity(T) = 4 => nullity(T) = 2

   True False

   (c) If $\vec{x}$ is a vector in $\mathbb{R}^n$ and V is a subspace of $\mathbb{R}^n$, then $\|\text{proj}_V(\vec{x})\| \leq \|\vec{x}\|$.

   True False

   (d) If $\vec{v}$ and $\vec{w}$ are linearly independent vectors in $\mathbb{R}^4$, then $\vec{v} \cdot \vec{w} = 0$.

   True False

   (e) If A is a 100×100 matrix and $AA^T = \text{Id}_{100}$ (the identity matrix), then $\det A = 1$.

   True False

   (f) All symmetric $n \times n$ matrices are diagonalizable.

   True False

   (g) All diagonalizable $n \times n$ matrices are symmetric.

   True False

   (h) If an $n \times n$ matrix A is diagonalizable, then every vector $v \in \mathbb{R}^n$ may be written as a linear combination of the eigenvectors of A.

   True False

   (i) If $\vec{v}$ is an eigenvector of a matrix A, then $\vec{v}$ is an eigenvector of A^{1000}.

   True False

   (j) The following matrix has negative determinant: $\begin{pmatrix} 0 & 1 & 1 & 1 & 99999 \\ 99999 & 0 & 0 & 0 & 0 \\ 2 & 99999 & 1 & 1 & 2 \\ 1 & -2 & 99999 & 1 & 0 \\ 1 & 0 & 0 & 99999 & 2 \end{pmatrix}$

   True False