

MAT

1002



MAT1002 Calculus II

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I. Infinite Sequences and Series.

10.1 Sequences.

Representing Sequences

Def. A set of sample of a function $f(x)$, which $a_n = f(x=n)$.
A list of numbers $a_1, a_2, a_3, \dots, a_n, \dots$

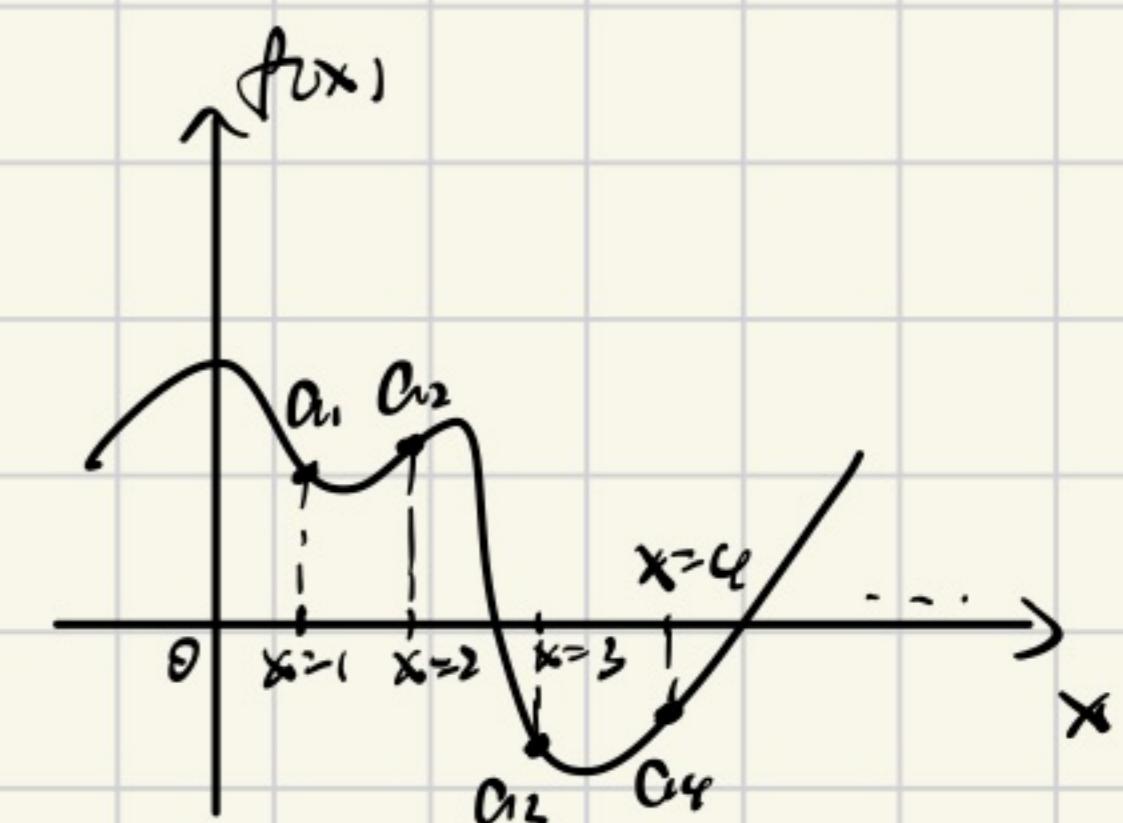
The integer n is the index of a_n .

An infinite sequence of numbers is a function whose domain is \mathbb{N}^* .

e.g. $a_n = \sqrt{n}$.

$$\Rightarrow \{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n}, \dots\}$$

$$= \{\sqrt{n}\}_{n=1}^{\infty}$$



P.S. we are only interested in Infinite sequences.

Convergence and Divergence

Def. The sequence $\{a_n\}$ converges to the number L if for every positive number ϵ there corresponds an integer N such that for all n ,

$$n > N \Rightarrow |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ diverges.

If $\{a_n\}$ converges to L , we write $\lim_{n \rightarrow \infty} a_n = L$ or $a_n \rightarrow L$, and call L the limit of $\{a_n\}$.

Def. The sequence $\{a_n\}$ diverges to infinity if for every number M there is an integer N s.t. for all $n > N$, $a_n > M$, we write:

$$\lim_{n \rightarrow \infty} a_n = \infty \quad \text{or} \quad a_n \rightarrow \infty. \quad (\text{negative infinity is similar}).$$

Calculating Limits of Sequences.

* Theorem 1: let $\{a_n\}, \{b_n\}$ be sequences of \mathbb{R} , one let $A, B \in \mathbb{R}$.

If $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$, we have:

$$\textcircled{1} \quad \lim_{n \rightarrow \infty} (a_n + b_n) = A + B$$

Sum & Difference Rule

$$\textcircled{2} \quad \lim_{n \rightarrow \infty} (k \cdot a_n) = k \cdot A \quad (\forall k \in \mathbb{R}) \quad \text{Constant Multiple Rule}$$

$$\textcircled{3} \quad \lim_{n \rightarrow \infty} (a_n \cdot b_n) = A \cdot B$$

Product Rule

$$\textcircled{4} \quad \lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = \frac{A}{B}.$$

Quotient Rule.

* Theorem 2: The Sandwich Theorem for Sequences.

let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences of \mathbb{R} . If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N , and if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$. We have $\lim_{n \rightarrow \infty} b_n = L$.

e.g. Since $\frac{1}{n} \rightarrow 0$, we have

1° $\frac{\cos n}{n} \rightarrow 0$ because $-\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$.

2° $\frac{1}{2^n} \rightarrow 0$ because $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$.

3° $(-1)^n \frac{1}{n} \rightarrow 0$ because $-\frac{1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$.

Theorem 3: The Continuous Function Theorem for Sequences

Let $\{a_n\}$ be a sequence of \mathbb{R} . If $a_n \rightarrow L$ and f is a function that is continuous at L and defined at all a_n , then $f(a_n) \rightarrow f(L)$.

$$\text{D} \lim_{n \rightarrow \infty} f(a_n) = f(\lim_{n \rightarrow \infty} a_n) = f(L)$$

Theorem 4: Suppose that $f(x)$ is a function defined for all $x \geq n_0$, and that $\{a_n\}$ is a sequence of \mathbb{R} s.t. $a_n = f(n)$ for $n \geq n_0$, then:

$$\text{D} \lim_{n \rightarrow \infty} f(x) = L \Rightarrow \lim_{n \rightarrow \infty} a_n = L. \quad \text{Treat } n \text{ as } x.$$

e.g. Show that $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$.

A: By theorem 4. We have $\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = 0$.

So that $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$.

L'Hôpital's Rule

Commonly Occurring Limits

① $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$

② $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$

③ $\lim_{n \rightarrow \infty} x^{\frac{1}{n}} = 1 \quad (x > 0)$.

④ $\lim_{n \rightarrow \infty} x^n = 0 \quad (|x| < 1)$.

⑤ $\lim_{n \rightarrow \infty} (1 + \frac{x}{n})^n = e^x \quad (\forall x \in \mathbb{R})$.

⑥ $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad (\forall x \in \mathbb{R})$.

Bounded Monotonic Sequences

Def. A sequence $\{a_n\}$ is bounded from above if $\exists M$ s.t. $a_n \leq M$ for all n .

We called M is an upper bound for $\{a_n\}$. And M is the least upper bound if it's the smallest (lower bound & greatest lower bound is similar).

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ indeed, unless it's unbounded.

Def. A sequence $\{a_n\}$ is nondecreasing if $a_n \leq a_{n+1}$ for all n , that is, $a_1 \leq a_2 \leq a_3 \leq \dots$ (nonincreasing is similar).

The sequence $\{a_n\}$ is monotonic if it is either nondecreasing or nonincreasing.

Theorem 6: The Monotonic Sequence Theorem

D If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

Recursive Definitions

Sequences are often defined recursively by giving

① The value(s) of the initial term or terms

② A rule, called recursion formula, for calculating the later terms.

e.g. Fibonacci numbers.

10.2 Infinite Series

Concepts

Def. Given a sequence of numbers $\{c_n\}$, an expression of the form

$$c_1 + c_2 + c_3 + \dots + c_n + \dots$$

is an infinite series. c_n is the n th term of the series.

And we have $S_n = c_1 + c_2 + \dots + c_n = \sum_{i=1}^n c_i$, which is the n th partial sum.

If the sequence of partial sums converges to a limit L , we say that the series converges. We write:

$$c_1 + c_2 + \dots + c_n + \dots = \sum_{n=1}^{\infty} c_n = L$$

otherwise, the series diverges.

Geometric Series

Def. Geometric series are series of the form:

$$a + ar + ar^2 + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n \quad (a \neq 0)$$

And $S_n = a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \quad (r \neq 1)$

If $|r| < 1$, the geometric series converges to $\frac{a}{1-r}$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1$$

If $|r| \geq 1$, the series diverges.

The n th-Term Test

Theorem: If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \rightarrow 0$.

Test: $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n \rightarrow \infty} a_n$ fails to exist or is different from 0.

e.g. $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \rightarrow \infty$.

$\sum_{n=1}^{\infty} (-1)^n$ diverges because $\lim_{n \rightarrow \infty} (-1)^n$ DNE.

Combining Series

Theorem:

If both $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

① $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n = A \pm B$ Sums and Difference Rule

② $\sum k a_n = k \cdot \sum a_n = kA$ ($k \in \mathbb{R}$) Constant Multiple Rule

If $\sum a_n$ diverges and $\sum b_n$ converges,

① $k \cdot \sum a_n$ diverges ($k \neq 0$).

② $\sum (a_n \pm b_n)$ diverges.

If both $\sum a_n$ and $\sum b_n$ diverges, $\sum (a_n \pm b_n)$ is not for sure.

e.g. $\sum a_n = 1+1+1+\dots$, $\sum b_n = (-1)+(-1)+(-1)+\dots$, $\sum (a_n + b_n) = 0$, which converges.

but $\sum (a_n + b_n)$ diverges.

10.3 The Integral Test

• Nondecreasing Partial Sums

Theorem: A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if and only if its partial sums are bounded from above.

$a_n \geq 0$ for all n . $\Rightarrow S_1 \leq S_2 \leq S_3 \leq \dots \leq S_n \leq S_{n+1} \leq \dots$

(Corollary of the Monotonic Sequence Theorem).

• The Integral Test

Theorem: Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous positive decreasing function of $x \geq N$.

Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x) dx$ both converge or both diverge.

e.g. Show that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \dots + \frac{1}{n^p} + \dots$ converges if $p > 1$, diverges if $p \leq 1$.

1° If $p > 1$, then $f(x) = \frac{1}{x^p}$ is a positive decreasing function of x .

Since $\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \int_1^b x^{-p} dx = \frac{1}{p-1}$, which converges.

we have the series also converges.

2° If $p \leq 0$, since $\lim_{n \rightarrow \infty} a_n \neq 0$, the series diverges.

3° If $0 < p < 1$, then $1-p > 0$.

Since $\int_1^{\infty} \frac{1}{x^p} dx = \frac{1}{1-p} \lim_{b \rightarrow \infty} (b^{1-p} - 1) = \infty$, which diverges.

we have the series diverges.

4° If $p = 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots$
which diverges. (we called it the harmonic series).

• Error Estimation

Suppose that a series $\sum a_n$ is shown to be convergent, and we want to estimate the size of the remainder R_n measuring the difference between S and S_n , that is:

$$R_n = S - S_n = a_{n+1} + a_{n+2} + \dots$$

According to the Integral Test, $a_n = f(n)$, where f is a continuous positive decreasing function for all $x \geq n$, and that $\sum a_n$ converges to S , we have:

$$\Delta \int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx.$$

If we add S_n to each side of the inequality, we get

$$S_n + \int_{n+1}^{\infty} f(x) dx \leq S \leq S_n + \int_n^{\infty} f(x) dx \Rightarrow \text{Estimate } S!$$

e.g. Estimate $\sum \frac{1}{n^2}$ and $n=10$.

We have that:

$$\int_n^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_n^b = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + \frac{1}{n} \right) = \frac{1}{n},$$

$$\text{Thus } S_{10} + \frac{1}{11} \leq S \leq S_{10} + \frac{1}{10}.$$

$$1.64068 \leq S \leq 1.64977.$$

If we approximate S by the midpoint, we find that $\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.6452$.

10.4 Comparison Test

• Definition

Theorem: The Comparison Test

Let $\sum a_n$, $\sum c_n$ and $\sum d_n$ be series with nonnegative terms.

Suppose that for some N : $d_n \leq a_n \leq c_n$ for all $n > N$.

① If $\sum c_n$ converges, then $\sum a_n$ also converges.

② If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

e.g. The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n-1}}$ diverges because

$$\frac{1}{\sqrt{n-1}} = \frac{1}{n - \frac{1}{4}} > \frac{1}{n}.$$

And $\sum \frac{1}{n}$ diverges due to harmonic series.

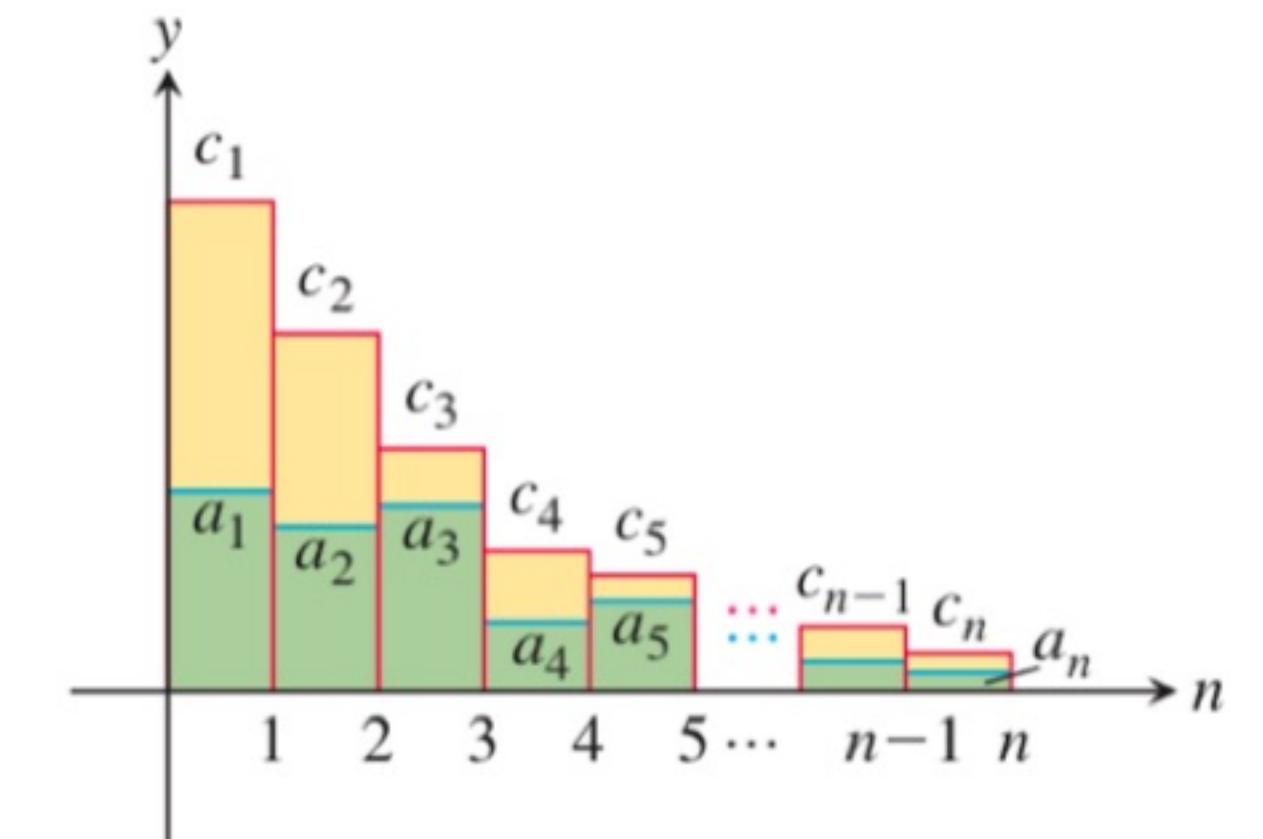


FIGURE 10.12 If the total area $\sum c_n$ of the taller c_n rectangles is finite, then so is the total area $\sum a_n$ of the shorter a_n rectangles.

• The Limit Comparison Test

Theorem: Limit Comparison Test

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$.

① If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

② If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

③ If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

e.g. Is $\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \dots = \sum_{n=1}^{\infty} \frac{1}{2^n-1}$ convergent or divergent?

Let $a_n = \frac{1}{2^n-1}$. For large n , we expect a_n to behave like $\frac{1}{2^n}$. So we let $b_n = \frac{1}{2^n}$.

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2^n}{2^n-1} = 1$.

$\sum a_n$ converges by Part 1 of the Limit Comparison Test.

10.5 Absolute Convergence

• Definition

A series $\sum a_n$ converges absolutely (is absolutely convergent) if the corresponding series of absolute values, $\sum |a_n|$, converges.

Theorem: The Absolute Convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

e.g. For $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \dots$, which contains both positive and negative terms,

the corresponding series of absolute value is:

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \dots \text{ which converges by comparison with } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

because $|\sin n| \leq 1$ for every n .

The original series converges absolutely, therefore it converges.

The Ratio Test

Theorem: The Ratio Test

let $\sum a_n$ be any series and suppose that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = p$$

Then ① the series converges absolutely if $p < 1$.

② the series diverges if $p > 1$ or p is infinite.

③ the test is inconclusive if $p = 1$.

e.g. Investigate the convergence of $\sum_{n=1}^{\infty} \frac{(2n)!}{n! \cdot n!}$

$$\text{If } a_n = \frac{(2n)!}{n! \cdot n!}, \quad a_{n+1} = \frac{(2n+2)!}{(n+1)! \cdot (n+1)!} \text{ and}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n! n! (2n+2)(2n+1)(2n)!}{(n+1)! (n+1)! (2n)!} = \lim_{n \rightarrow \infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = 4.$$

The series diverges since $p = 4 > 1$ by the Ratio Test.

The Root Test

Theorem: The Root Test

let $\sum a_n$ be any series and suppose that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = p$$

Then ① the series converges absolutely if $p < 1$.

② the series diverges if $p > 1$ or p is infinite.

③ the test is inconclusive if $p = 1$.

e.g. Investigate the convergence of $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$. $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

$$\text{Since } \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^2}{2^n}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n^2}}{2} = \lim_{n \rightarrow \infty} \frac{(\sqrt[n]{n})^2}{2} = \frac{1^2}{2} = \frac{1}{2} < 1.$$

The series converges due to $p = \frac{1}{2} < 1$ by the Root Test.

10.6 Alternating Series and Conditional Convergence

Alternating Series

Def. A series which the terms are alternately positive and negative.

e.g. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n+1}}{n} + \dots$ (alternating harmonic series).

$$1 - 2 + 3 - 4 + \dots + (-1)^{n+1} + \dots$$

Theorem: The Alternating Series Test

The series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots$ converges if all of the following satisfied:

1. The u_n 's are all positive

2. The u_n 's are eventually nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some N .

3. $u_n \rightarrow 0$.

Theorem: The Alternating Series Estimate Theorem

If $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges, then for all $n \geq N$,

$$S_n = u_1 - u_2 + \dots + (-1)^{n+1} u_n$$

approximate the sum S of the series with an error ϵ that $|S - S_n| < u_{n+1}$.

• Conditional Convergence

Def. A convergent series that is not absolutely convergent is conditionally convergent.

e.g. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p} = -\frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots$, $p > 0$. converges.

① If $p > 1$, the series converges absolutely as an ordinary p -series.

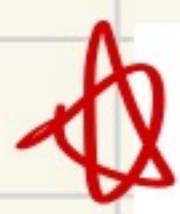
② If $0 < p \leq 1$, the series converges conditionally by the alternating series test.

• Rearranging Series

Theorem: The Rearrangement Theorem for Absolutely Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \dots, b_n, \dots$ is an arrangement of $\{a_n\}$, then

$$\sum b_n \text{ converges absolutely and } \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$



Summary of Tests

We have developed a variety of tests to determine convergence or divergence for an infinite series of constants. There are other tests we have not presented which are sometimes given in more advanced courses. Here is a summary of the tests we have considered.

1. **The n th-Term Test:** If it is not true that $a_n \rightarrow 0$, then the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. **p -series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test or try comparing to a known series with the Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember, absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

10.7 Power Series

• Power Series and Convergence

Def. A power series about $x=0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

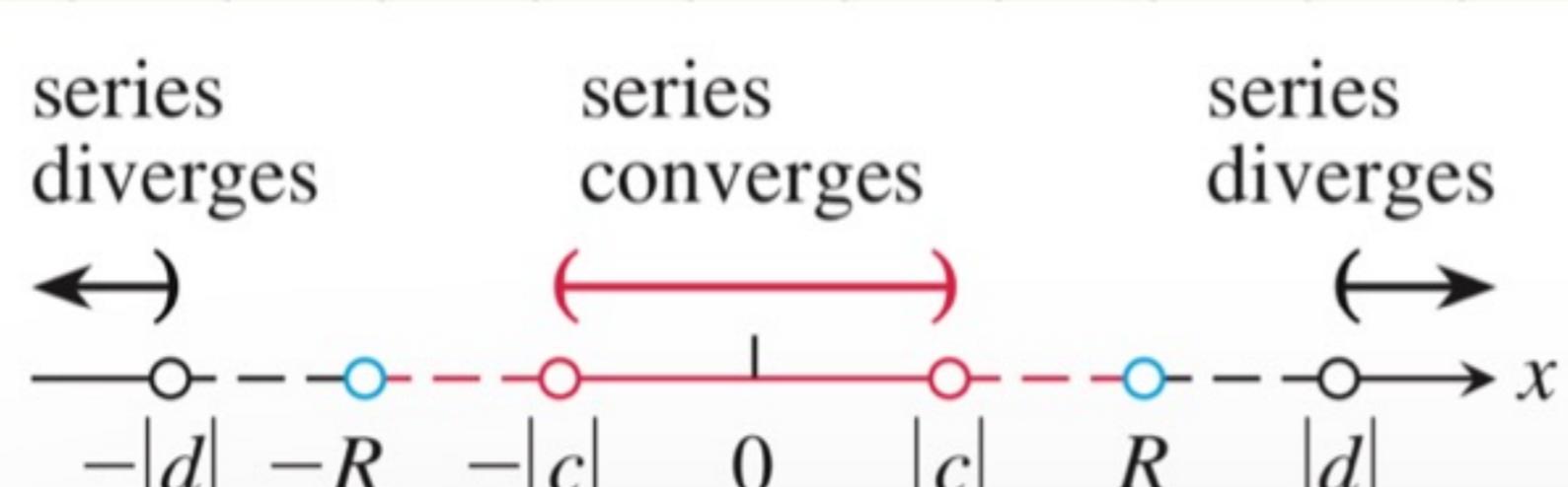
A power series about $x=c$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-c)^n = c_0 + c_1 (x-c) + c_2 (x-c)^2 + \dots + c_n (x-c)^n + \dots$$

in which the center c and the coefficients $c_0, c_1, c_2, \dots, c_n, \dots$ are constants.

► Theorem: The Convergence Theorem for Power Series

If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x=c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x=d$, then it diverges for all x ($|x| > |d|$).



The Radius of Convergence of a Power Series

Theorem: The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following cases:

1. There is a positive number R s.t. the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x=a-R$ and $x=a+R$.

2. The series converges absolutely for every x ($R = \infty$).

3. The series converges at $x=a$ and diverges elsewhere ($R=0$).

R is called the radius of convergence of the power series, and the interval of radius R centered at $x=a$ is called the interval of convergence.

Q: How to test a power series for convergence?

A: 1. Use the Ratio Test (or Root Test) to find the interval where the series converges absolutely.

Ordinarily, this is an open interval $|x-a| < R$

2. If the interval is finite, test for convergence or divergence at each endpoint.

3. If the interval is $|x-a| < R$, the series diverges for $|x-a| > R$.

Operations on Power Series

Theorem: The Series Multiplication Theorem for Power Series.

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and $c_n = \sum_{k=0}^n a_k b_{n-k}$,

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} c_n x^n.$$

Theorem: If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function f on $|f(x)| < R$.

Theorem: The Term-by-Term Differentiation Theorem

If $\sum c_n (x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ on the interval } a-R < x < a+R.$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}.$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2},$$

and so on. Each of these derived series converges at every point of the interval $(a-R, a+R)$.

Theorem: The Term-by-Term Integration Theorem

Suppose that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $a-R < x < a+R$ ($R > 0$). Then

$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ converges for $a-R < x < a+R$ and

$$\int f(x) dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C \text{ for } a-R < x < a+R.$$

10.8 Taylor and MacLaurin Series

Series Representations

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n + \dots \text{ with a positive radius of convergence.}$$

By repeated term-by-term differentiation with interval I, we obtain:

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \dots + na_n(x-a)^{n-1} + \dots,$$

$$f''(x) = 1 \cdot 2a_2 + 2 \cdot 3a_3(x-a) + 3 \cdot 4a_4(x-a)^2 + \dots,$$

⋮

With the n^{th} derivative, for all n , being

$$f^{(n)}(x) = n! a_n + \text{a sum of terms with } (x-a) \text{ as a factor.}$$

Since these eq's all hold at $x=a$, we have:

$$f'(a) = a_1, f''(a) = 1 \cdot 2a_2 \dots \text{ In general, } f^{(n)}(a) = n! a_n \Rightarrow a_n = \frac{f^{(n)}(a)}{n!}.$$

Thus, if f has a series representation, the the series is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

Taylor and MacLaurin Series

Def. Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the Taylor series generated by f at $x=a$ is:

$$\Delta \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

The MacLaurin Series of f is the Taylor series generated by f at $x=0$, or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

Taylor Polynomials

Def. let f be a function with derivatives of order k for $k=1, 2, \dots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N , the Taylor polynomial of order n generated by f at $x=a$ is the polynomial

$$\Delta P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

e.g. Let $f(x)=e^x$. then

Taylor series generated by f at $x=0$ is

$$\begin{aligned} & f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots \\ &= 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!}. \end{aligned}$$

The Taylor polynomial of order n at $x=0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}.$$

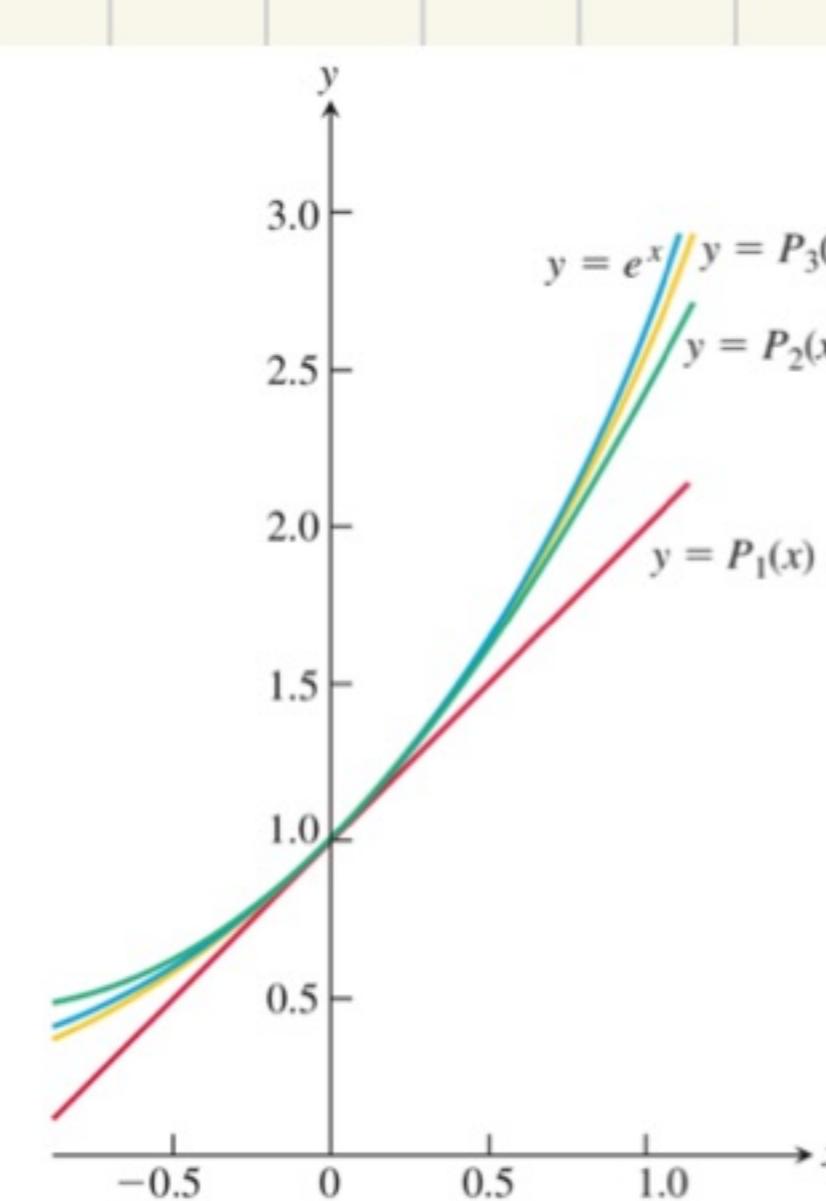


FIGURE 10.17 The graph of $f(x) = e^x$ and its Taylor polynomials

$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x + \frac{x^2}{2!}$$

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

⋮

It's getting more accurate.

e.g. Let $f(x)=\cos x$. then

The Taylor series generated by f at 0 is:

$$\begin{aligned} & f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots \\ &= 1 + 0 \cdot x - \frac{x^2}{2!} + 0 \cdot x^3 + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}. \end{aligned}$$

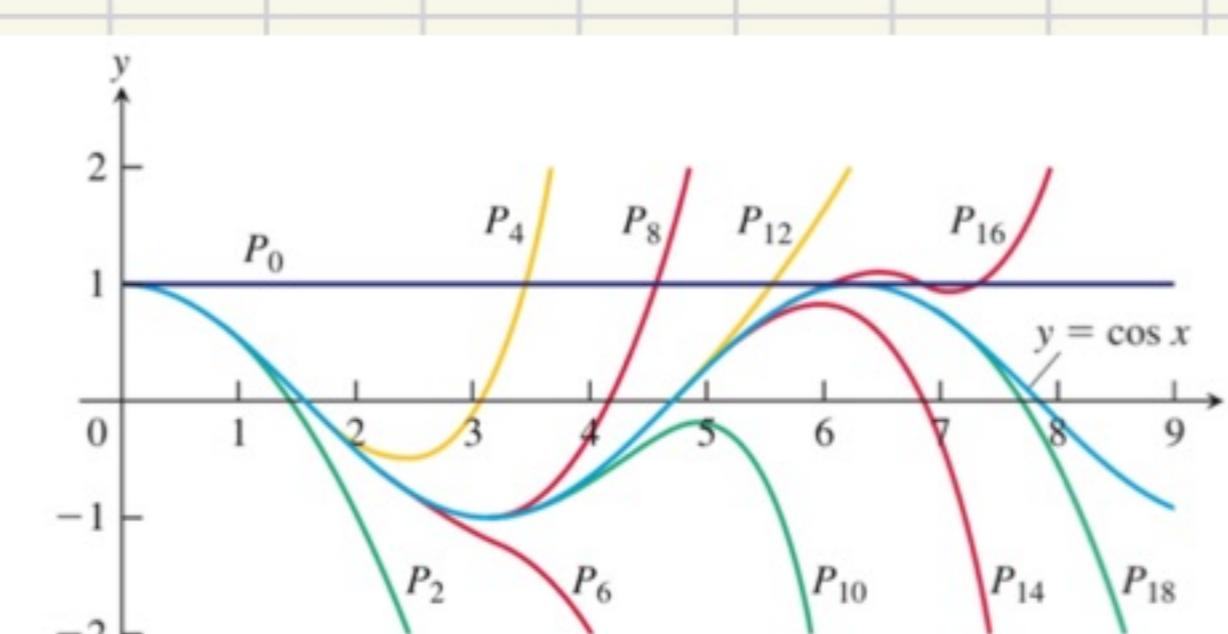


FIGURE 10.18 The polynomials

$$P_{2n}(x) = \sum_{k=0}^n \frac{(-1)^k x^{2k}}{(2k)!}$$

10.9 Convergence of Taylor Series

Motivation

Theorem: Taylor's Theorem

If f and its first n derivatives f' , f'' , ..., $f^{(n)}$ are continuous on the close interval between a and b , and $f^{(n)}$ is differentiable on the open interval between a and b , then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Taylor's Theorem is a generalization of the Mean Value Theorem.

Taylor's Formula

Def. If f has derivatives of all orders in an interval I containing a , then for each positive integer n and for each x in I ,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x).$$

where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x .

10.10 The Binomial Series and Applications of Taylor Series

The Binomial Series for Powers and Roots

Def. For $-1 < x < 1$, $(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k$, is called the binomial series.

where we define $\binom{m}{1} = m$, $\binom{m}{2} = \frac{m(m-1)}{2!}$, and $\binom{m}{k} = \frac{m(m-1)(m-2)\dots(m-k+1)}{k!}$ for $k \geq 3$.

The binomial series converges absolutely for $|x| < 1$.

Evaluating Nonelementary Integrals

Sometimes we can use a familiar Taylor series to find the sum of a given power series in terms of a known function. And it can be used to express non-elementary integrals.

e.g. $\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \dots$

$$\int \sin x^2 dx = C + \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \dots$$

It can be used for estimating.

e.g. For $\tan^{-1} x$, we have $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$, integrating to get:

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

We put $x=1$ into it and we get Leibniz's formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots + \frac{(-1)^n}{2n+1} + \dots$$

By multiplying by 4, we get π .

Evaluating Indeterminate Forms

We can sometimes evaluate indeterminate forms by expressing the functions as Taylor series.

e.g. Evaluate $\lim_{x \rightarrow 1} \frac{\ln x}{x-1}$.

We have $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \dots$

$$\text{So } \lim_{x \rightarrow 1} \frac{\ln x}{x-1} = \lim_{x \rightarrow 1} \left(1 - \frac{1}{2}(x-1) + \dots\right) = 1.$$

Euler's Identity

Def. For any real number θ , $e^{i\theta} = \cos \theta + i \sin \theta$. \star

It enables us to define e^{a+bi} to be $e^a \cdot e^{bi}$ for any complex number $a+bi$.

One consequence of the identity is $e^{i\pi} + 1 = 0$

Taylor Series

TABLE 10.1 Frequently used Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \dots + (-x)^n + \dots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \leq 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \leq 1$$

II. Parametric Equations and Polar Coordinates

11.1 Parametrizations of Plane Curves

• Parametric Equations

Def. If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

over an interval I of t -values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a parametric curve. The equations are parametric equations for the curve.

The variable t is a parameter for the curve, and its domain I is the parameter interval. If I is a closed interval, $a \leq t \leq b$, the point $(f(a), g(a))$ is the initial point of the curve and $(f(b), g(b))$ is the terminal point.

e.g. $x = t^2, \quad y = t + 1, \quad -\infty < t < \infty$.

TABLE 11.1 Values of $x = t^2$ and $y = t + 1$ for selected values of t .		
t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4

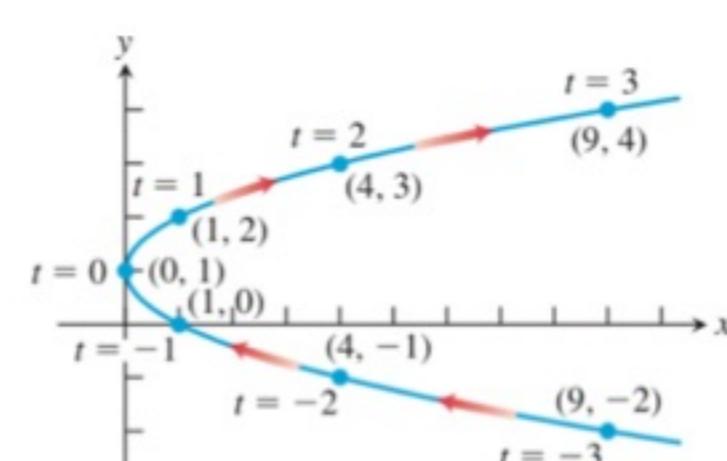


FIGURE 11.2 The curve given by the parametric equations $x = t^2$ and $y = t + 1$ (Example 1).

• Cycloids

Motivation: A wheel of radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference.

The path is called a cycloid.

We can get:

$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

• Brachistochrones and Tautochrones

?

11.2 Calculus with Parametric Curves

• Tangents and Areas

A parametrized curve $x = f(t)$ and $y = g(t)$ is differentiable at t if f and g are differentiable at t . At a point on a differentiable parametrized curve where y is also a differentiable function of x , the derivatives are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $\frac{dx}{dt} \neq 0$, we can have:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Which is the Parametric Formula for $\frac{dy}{dx}$.

If parametric equations define y as twice-differentiable function of x , we can get

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Length of a Parametrically Defined Curve

Def. If a curve C is defined parametrically by $x=f(t)$ and $y=g(t)$, $a \leq t \leq b$, where f' and g' are continuous and not simultaneously zero on $[a,b]$, and C is traversed exactly once as t increases from $t=a$ to $t=b$, then the length of C is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

e.g. Find the length of the circle of radius r defined parametrically by

$$x = r\cos t \text{ and } y = r\sin t, \quad 0 \leq t \leq 2\pi.$$

As t varies from 0 to 2π , the circle is traversed exactly once, we have

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2} dt = \int_0^{2\pi} r dt = r \cdot [t]_0^{2\pi} = 2\pi r.$$

Areas of Surfaces of Revolution for Parametrized Curves

Def. If a smooth curve $x=f(t)$, $y=g(t)$, $a \leq t \leq b$, is traversed exactly once as t increases from a to b , then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows:

1. Revolution about the x -axis ($y \geq 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

2. Revolution about the y -axis ($x \geq 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

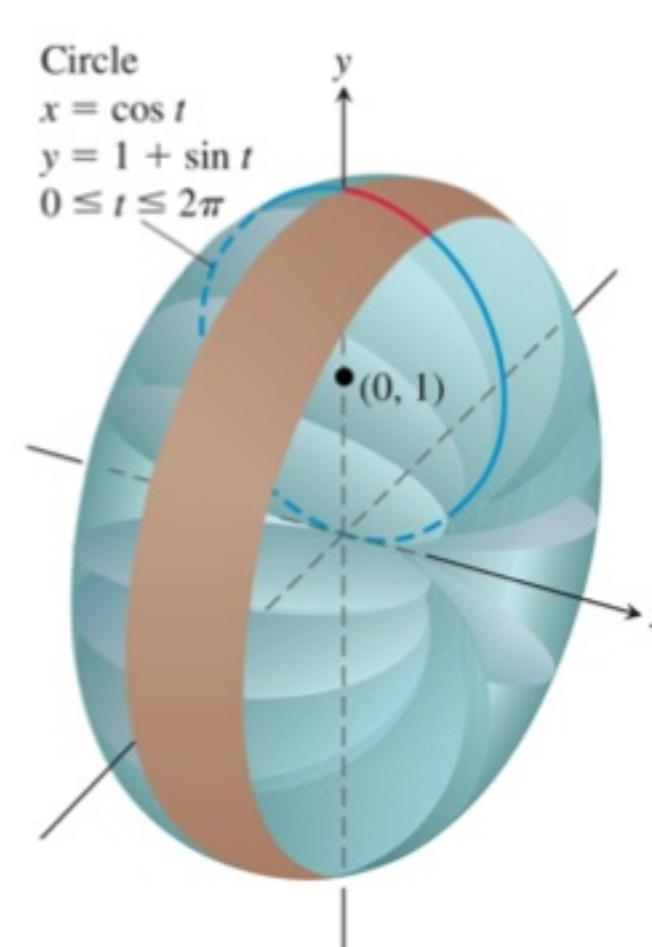
e.g. Find the area of the surface swept out by revolving the circle about the x -axis.

$$x = \cos t, \quad y = 1 + \sin t, \quad 0 \leq t \leq 2\pi.$$

$$\text{We have } S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \int_0^{2\pi} 2\pi(1 + \sin t) \sqrt{(-\sin t)^2 + (\cos t)^2} dt$$

$$= 2\pi \left[t - \cos t\right]_0^{2\pi} = 4\pi^2.$$



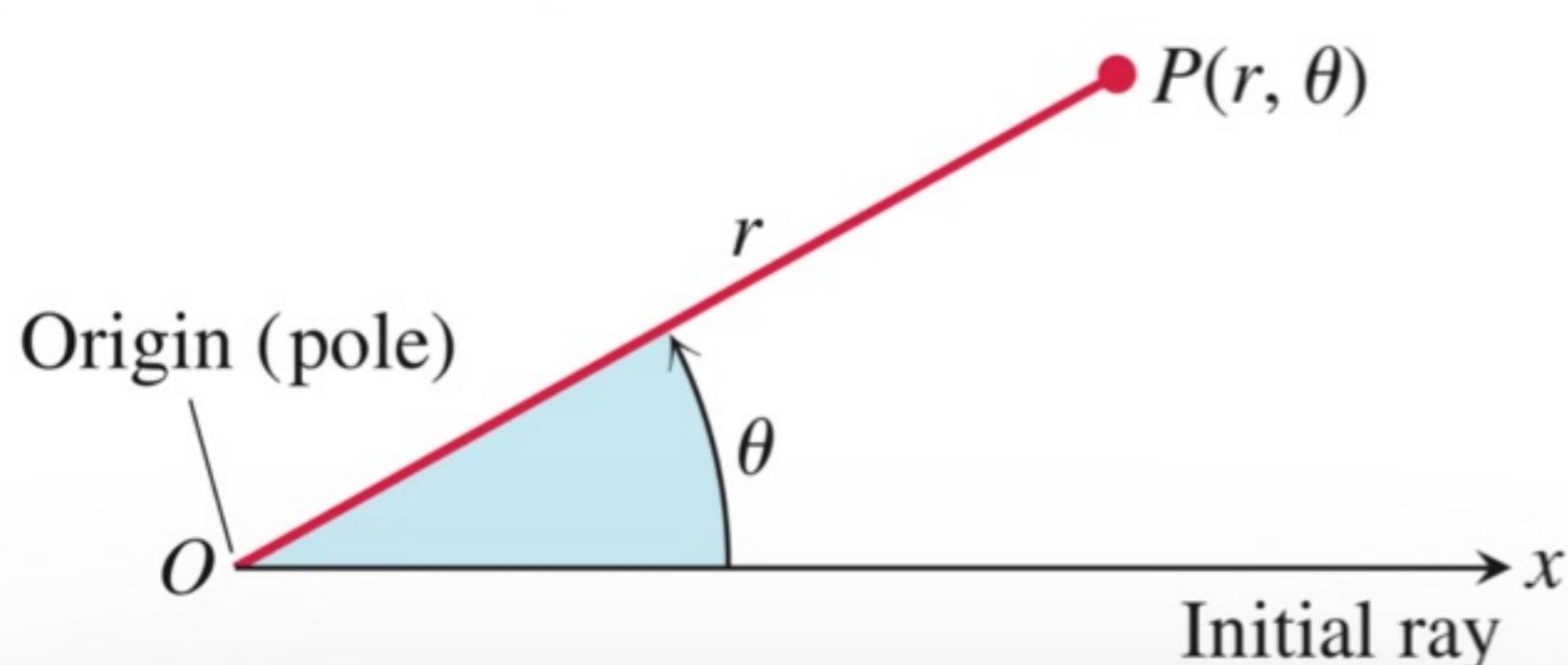
11.3 Polar Coordinates

Definition of Polar Coordinates

We first fix an origin O ("pole"), and an initial ray from O . Usually the positive x -axis is chosen as the initial ray. Then each point P can be located by assigning to it a polar coordinate pair (r, θ) , in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP . We label the point P as

$P(r, \theta)$.

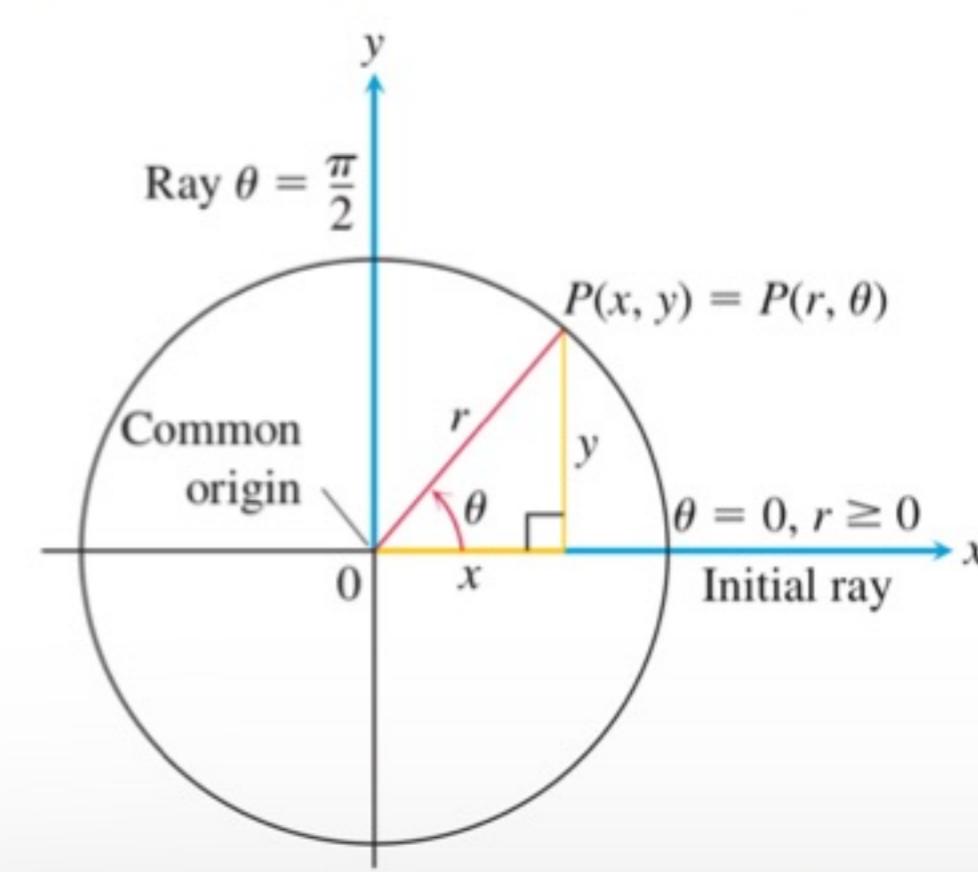
r for distance, θ for angle.



Relating Polar and Cartesian Coordinates

Equations of x, y, r, θ .

$$\begin{aligned} x &= r \cos \theta & r^2 &= x^2 + y^2 \\ y &= r \sin \theta & \tan \theta &= \frac{y}{x} \end{aligned}$$



11.4 Graphing Polar Coordinate Equations

Symmetry

Symmetry Test for Polar Graphs in the Cartesian xy -plane:

1. x -axis: $(r, \theta) \rightarrow (r, -\theta)$ or $(-r, \pi - \theta)$.
2. y -axis: $(r, \theta) \rightarrow (r, \pi - \theta)$ or $(-r, -\theta)$.
3. origin: $(r, \theta) \rightarrow (-r, \theta)$ or $(r, \theta + \pi)$.

Slope

Slope of the Curve $r = f(\theta)$ in the Cartesian xy -plane.

$$\left. \frac{dy}{dx} \right|_{(r, \theta)} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}. \quad \left(\frac{dx}{d\theta} \neq 0 \text{ at } (r, \theta) \right).$$

Graphing Polar Curves Parametrically

We can convert polar function $r = f(\theta)$ into parametric form using the equations

$$x = r \cos \theta = f(\theta) \cos \theta$$

$$y = r \sin \theta = f(\theta) \sin \theta.$$

Then we can draw a parametrized curve in the Cartesian xy -plane.

11.5 Areas and Lengths in Polar Coordinates

Area in the Plane

Def. Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$.

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta.$$

This is the integral of the area differential

$$dA = \frac{1}{2} r^2 d\theta = \frac{1}{2} (f(\theta))^2 d\theta.$$

Def. Area of the Region $0 \leq r(\theta) \leq r \leq r_2(\theta)$, $\alpha \leq \theta \leq \beta$.

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} (r_2^2 - r_1^2) d\theta.$$

Length of a Polar Curve

Def. If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β , then the length of the curve is:

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta.$$

III. Vectors and the Geometry Space

12.1 Three-Dimensional Coordinate Systems

Distance and Spheres in Space

Distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$.

$$|P_1 P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Standard Equation for the Sphere of Radius a and Center (x_0, y_0, z_0) .

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

12.2 Vectors

All learnt in High School

12.3 Dot Products

All learnt in High School

12.4 The Cross Product

The Cross Product of Two Vectors in Space

Def. The cross product $\vec{u} \times \vec{v}$ is the vector

$$\Delta \vec{u} \times \vec{v} = (\|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin \theta) \vec{n}$$

Where \vec{n} is a unit vector perpendicular to the plane of \vec{u} and \vec{v} .

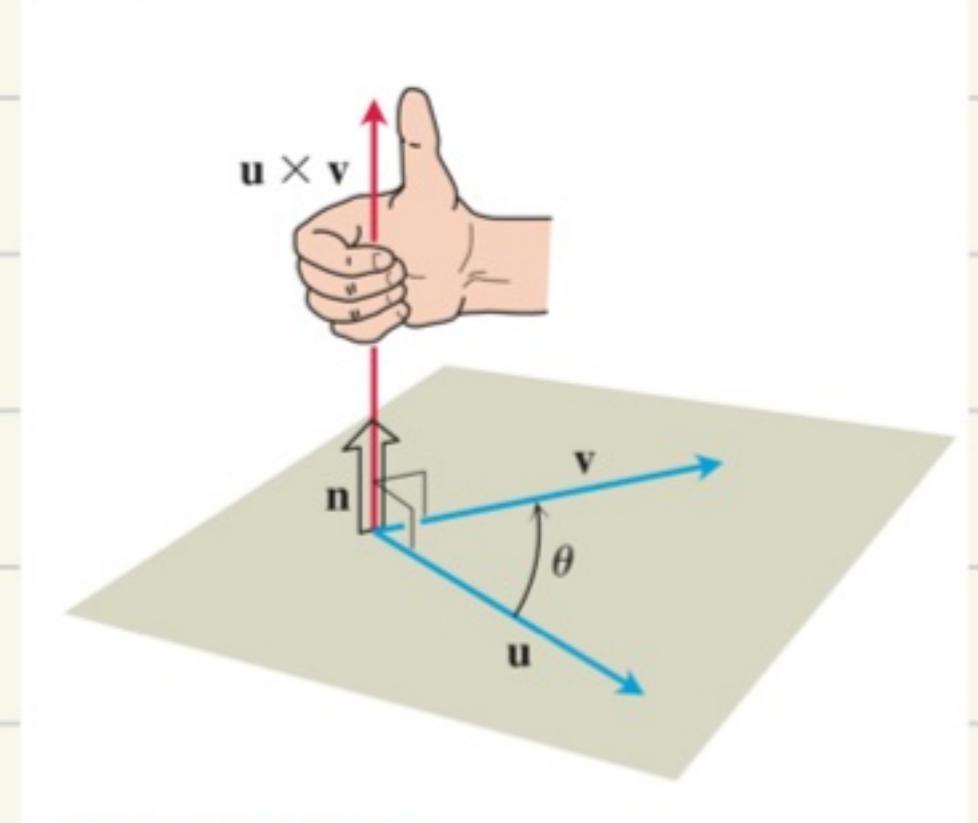


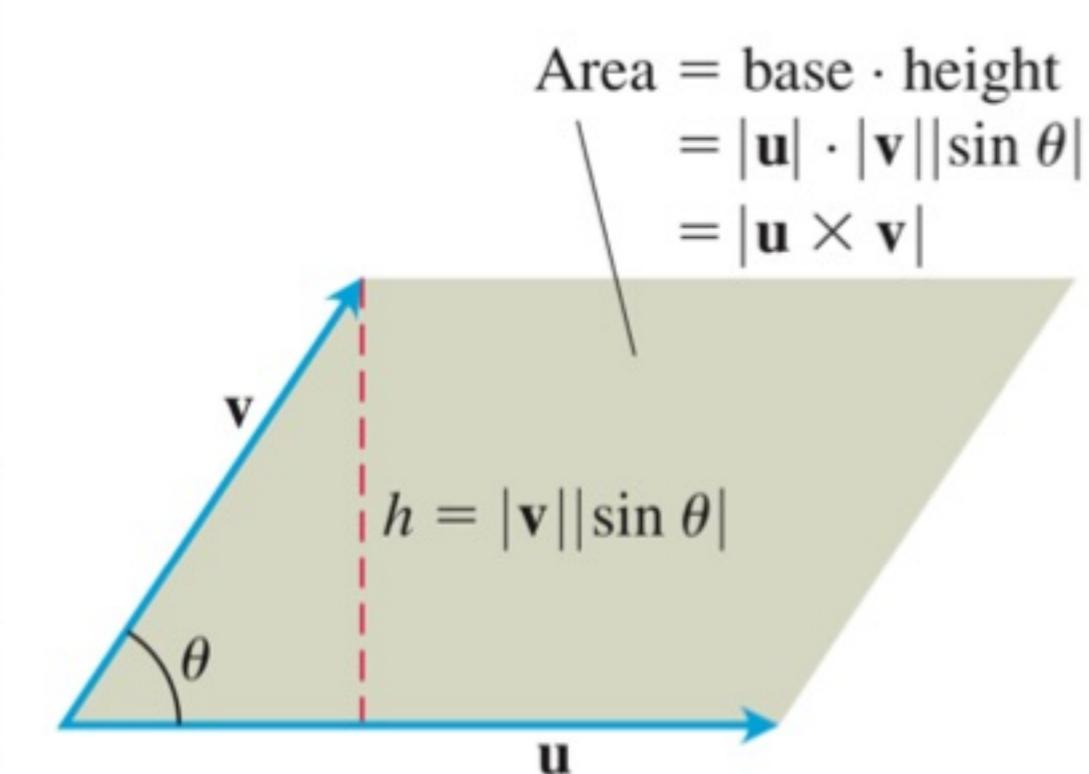
FIGURE 12.27 The construction of $\vec{u} \times \vec{v}$.

We have nonzero vectors \vec{u} and \vec{v} are parallel if and only if $\vec{u} \times \vec{v} = 0$.

The Area of a Parallelogram

Def. The area of a parallelogram determined by \vec{u} and \vec{v} is

$$|\vec{u} \times \vec{v}| = \|\vec{u}\| \cdot \|\vec{v}\| \cdot |\sin \theta| \cdot \|\vec{u}\| = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin \theta.$$



Determined Formula for $\vec{u} \times \vec{v}$

Def. Calculating the cross product as a determined:

If $\vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$ and $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$, then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Triple Scalar or Box Product

Def. The $(\vec{u} \times \vec{v}) \cdot \vec{w}$ is called the triple scalar product of \vec{u} , \vec{v} and \vec{w} (in that order).

$$\text{And we have } (\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

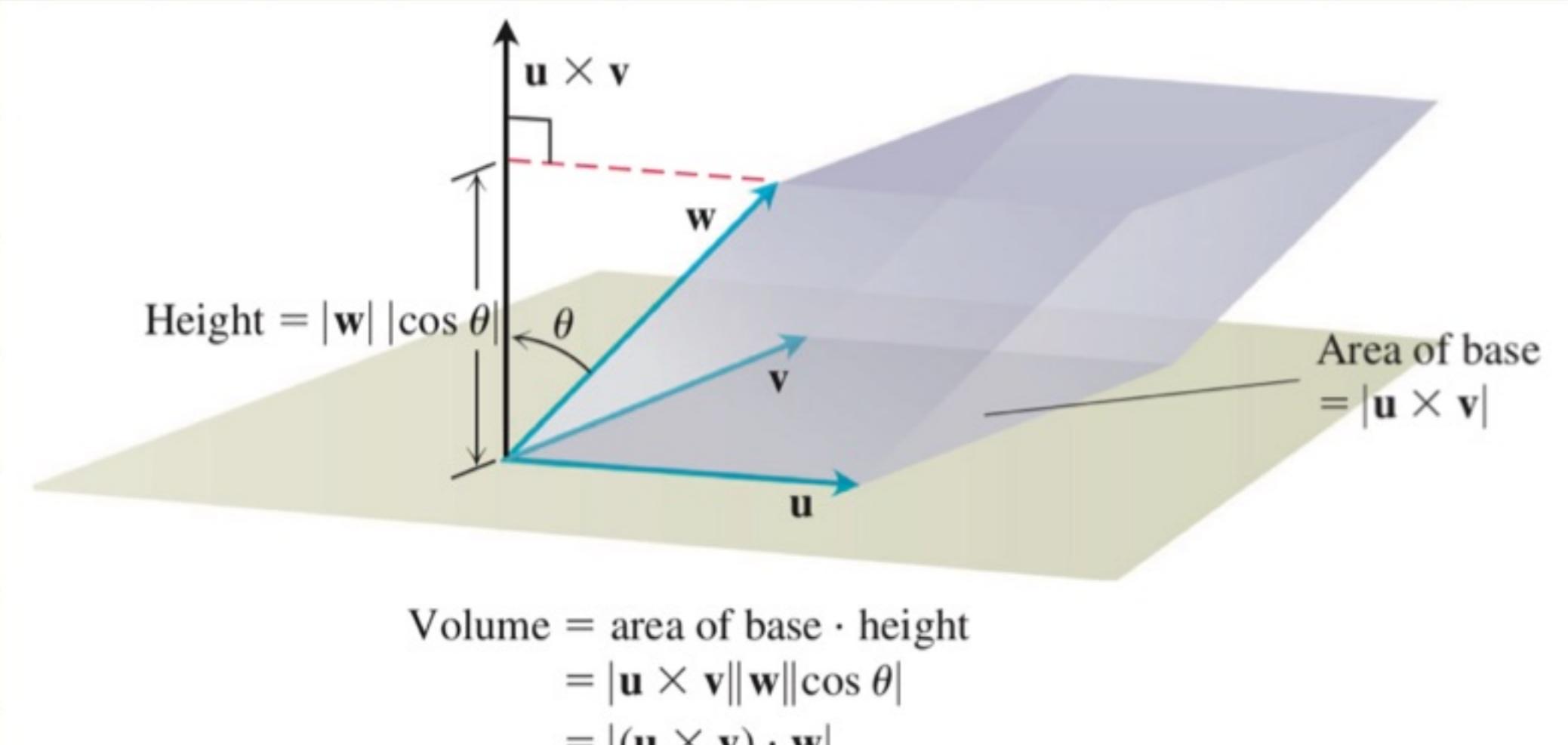


FIGURE 12.34 The number $|(u \times v) \cdot w|$ is the volume of a parallelepiped.

12.5 Lines and Planes in Space

• Lines and Line Segments in Space

Def. Vector Equation for a Line:

A vector equation for the line L through $P(x_0, y_0, z_0)$ parallel to \vec{v} is

$$\Delta \vec{r}(t) = \vec{r}_0 + t\vec{v}, \quad -\infty < t < \infty$$

where r is the position vector of a point $P(x, y, z)$ on L and \vec{r}_0 is the position vector of $P(x_0, y_0, z_0)$.

Def. Parametric Equation for a Line:

The standard parametrization of the line through $P(x_0, y_0, z_0)$ parallel to $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is

$$\Delta x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty.$$

• The Distance from a Point to a Line in Space

Def. Distance from a Point S to a Line Through P Parallel to \vec{v} :

$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$

• An equation for a Plane in Space

The plane through $P(x_0, y_0, z_0)$ normal to $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$ has

$$\text{Vector equation: } \vec{n} \cdot \vec{P} = 0$$

$$\text{Component equation: } A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$\text{Component equation simplified: } Ax + By + Cz = D, \text{ where } D = Ax_0 + By_0 + Cz_0$$

• The Distance from a Point to a Plane

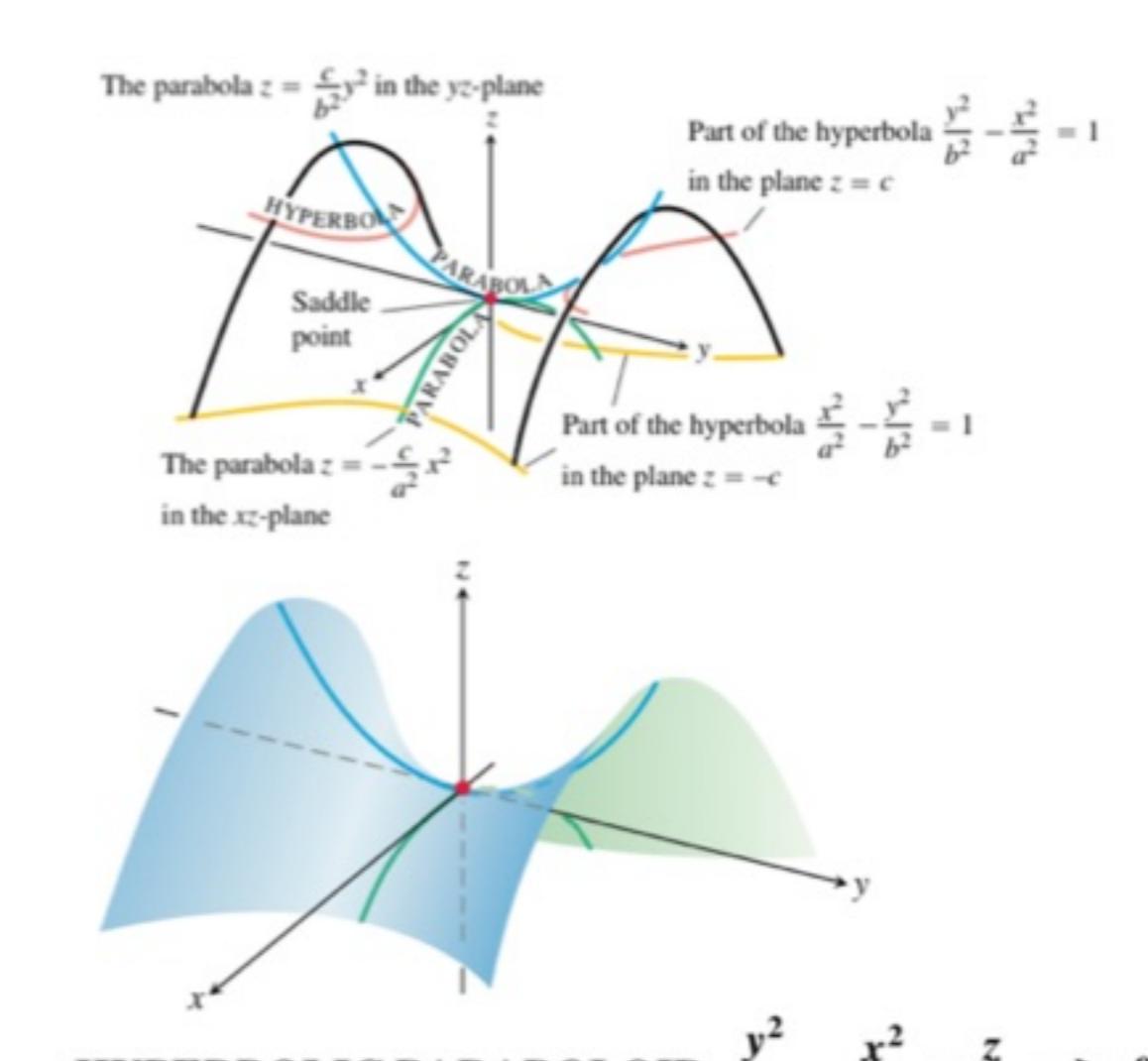
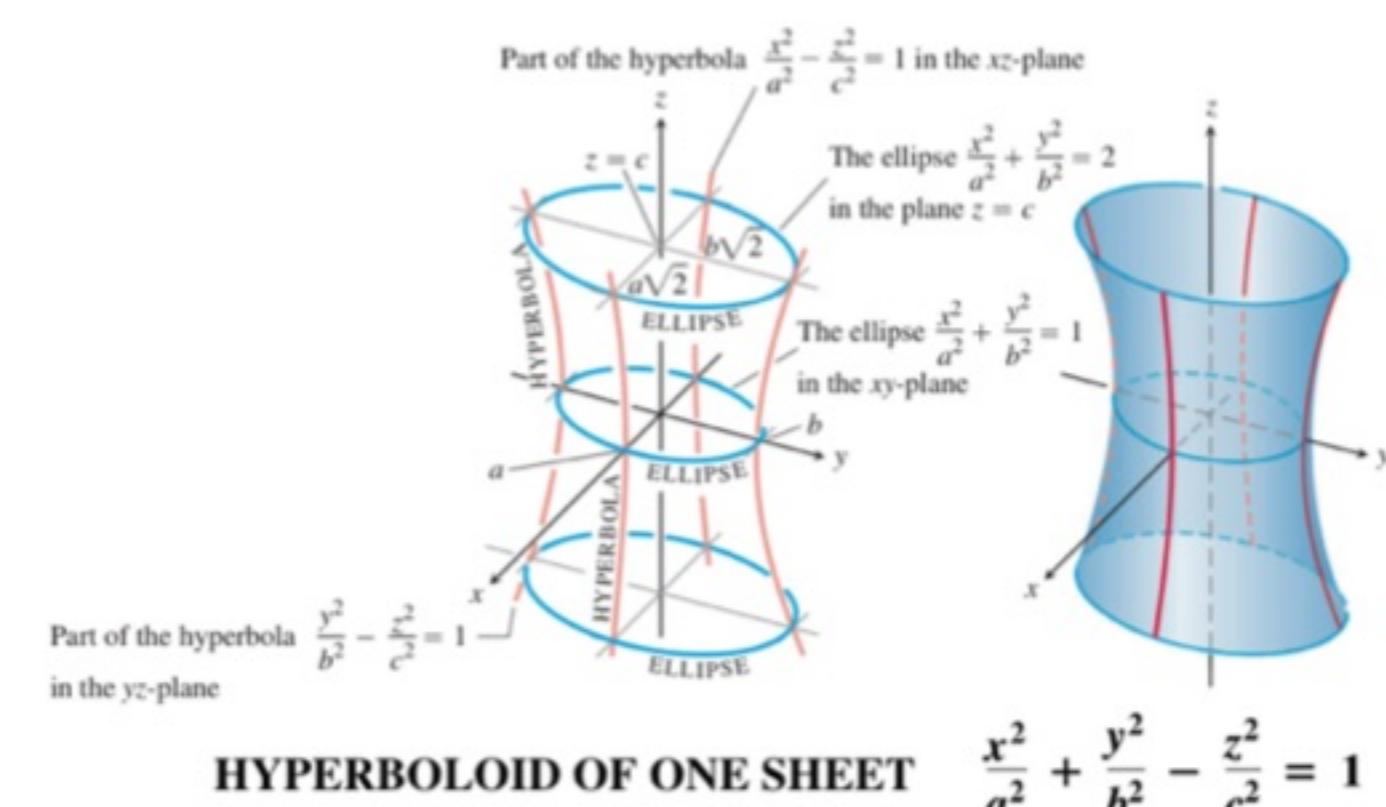
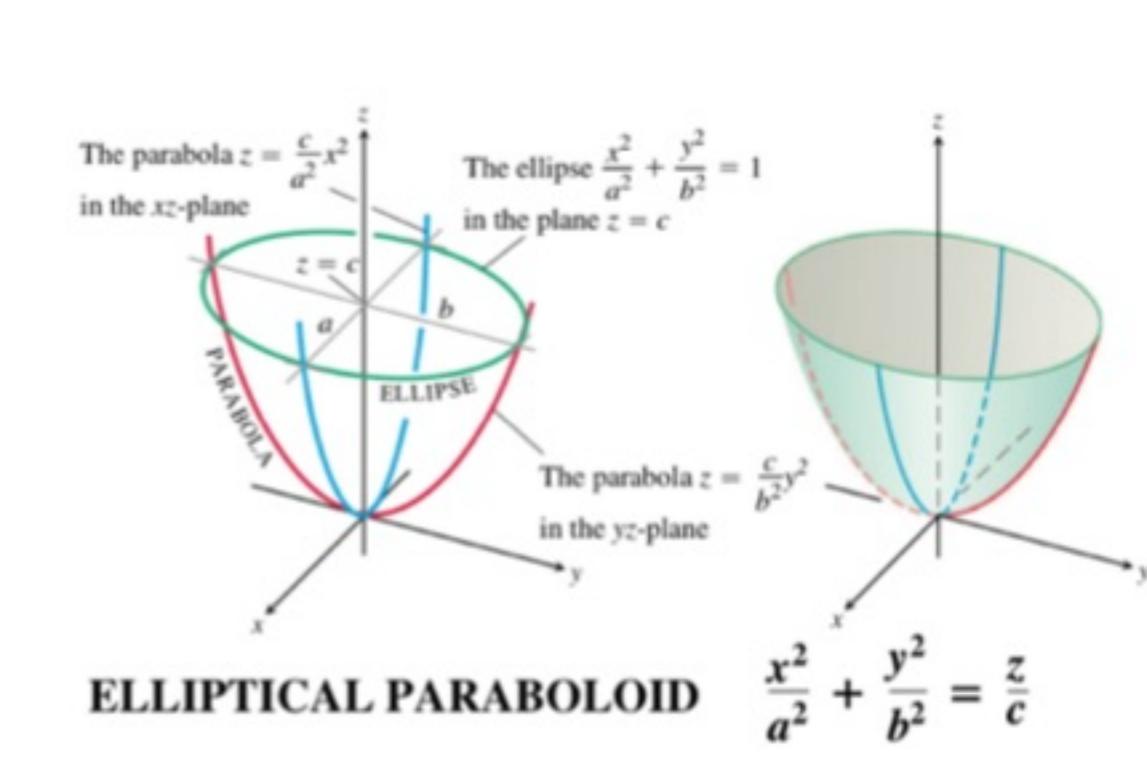
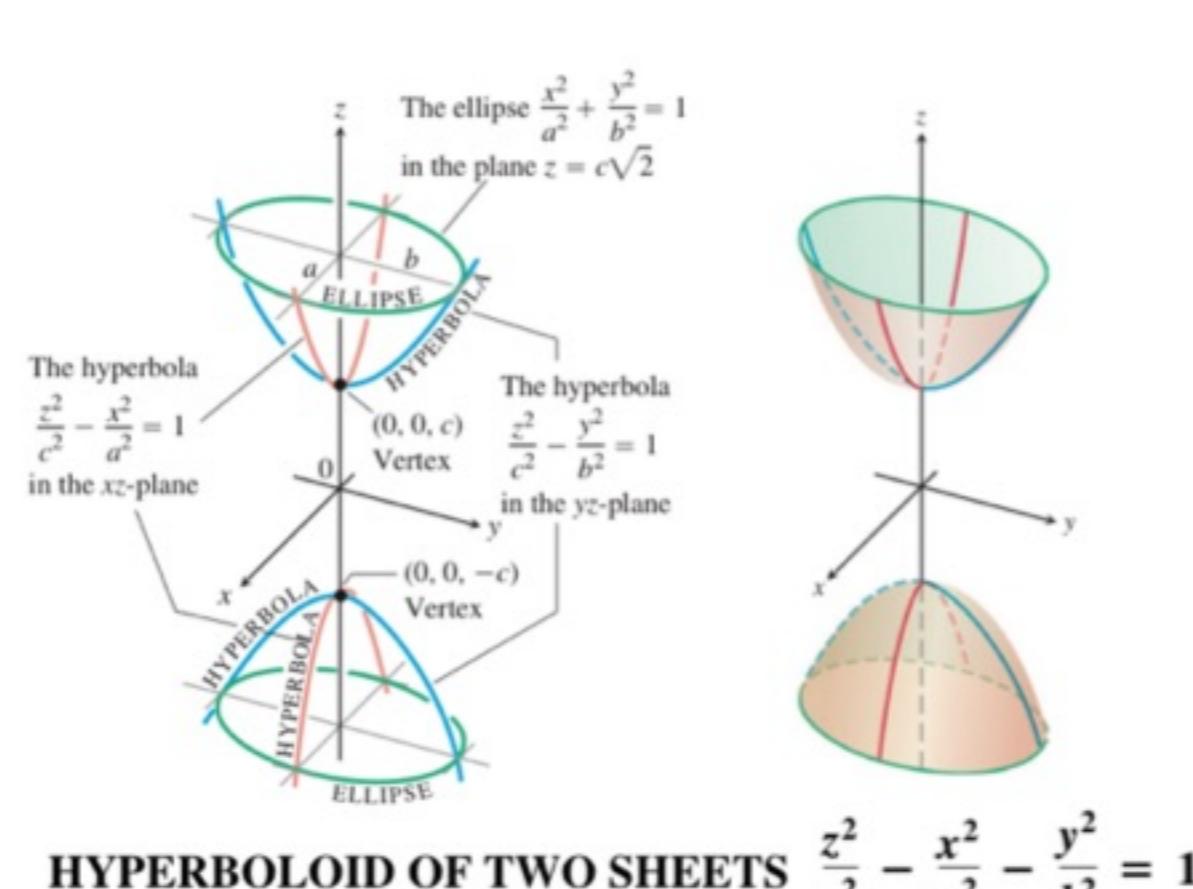
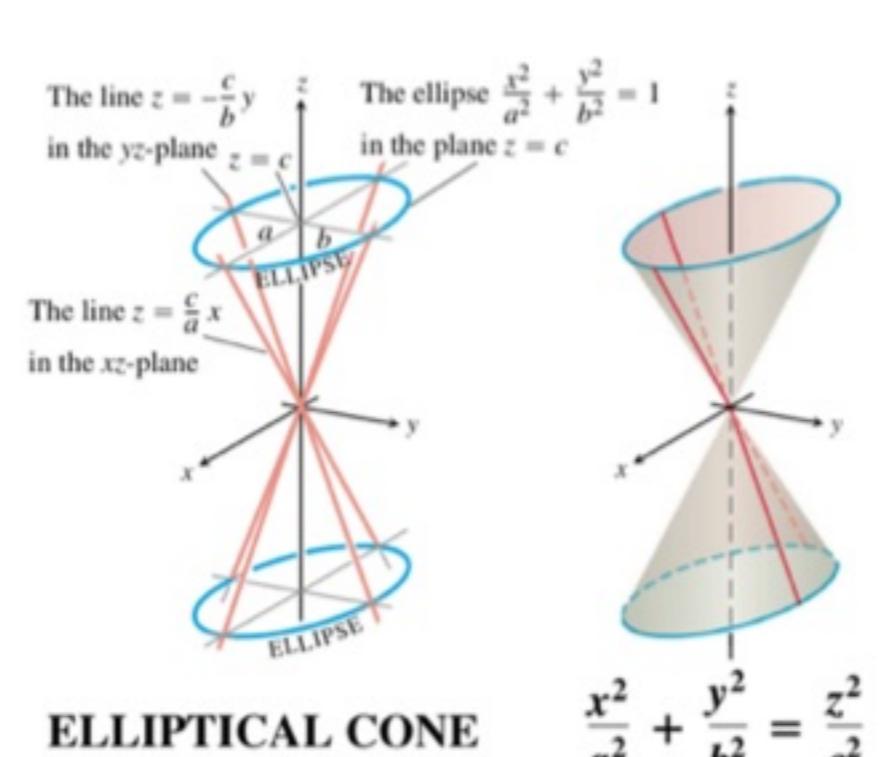
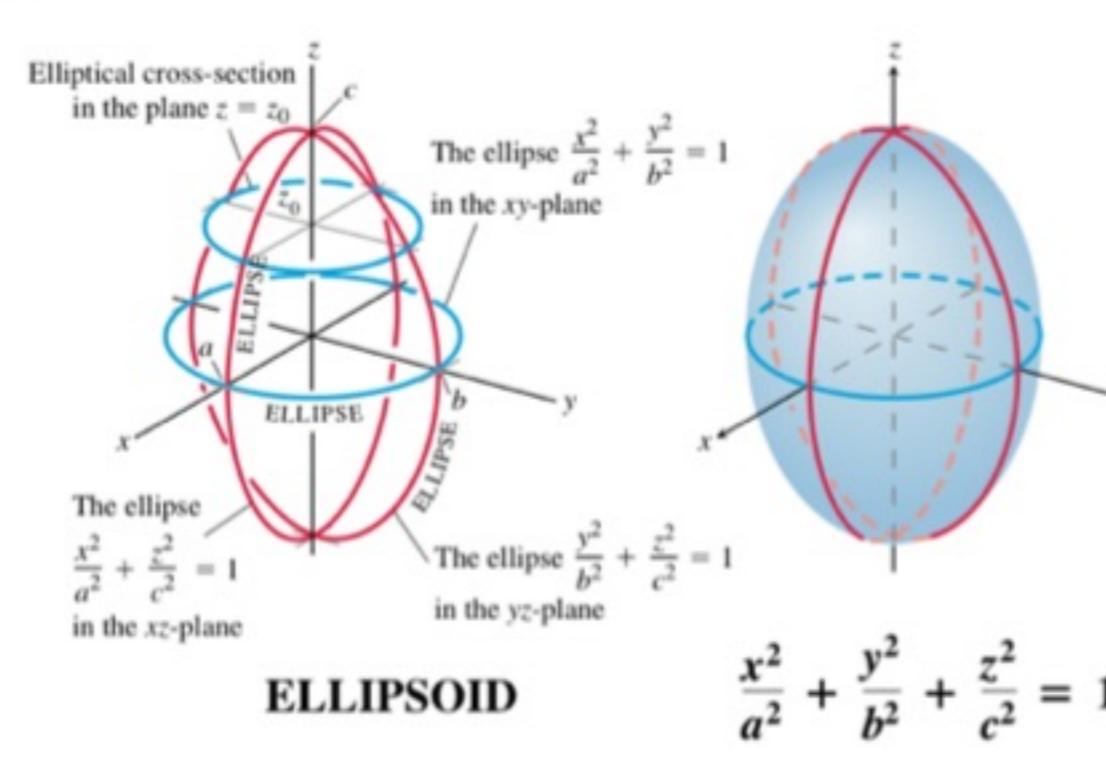
Def. If P is a point with normal \vec{n} , then the distance from any point S to the plane is

$$d = |\vec{PS} \cdot \vec{n}|$$

12.6 Cylinders and Quadratic Surfaces

As shown in the right image \rightarrow

TABLE 12.1 Graphs of Quadric Surfaces



IV. Vector-Valued Functions and Motion in Space

13.1 Curves in Space and Their Tangents

• Curves in Space

Def. When a particle moves through space during a time interval I , we think of the particle's coordinates as functions defined on I .

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I.$$

The points $(x, y, z) = (f(t), g(t), h(t))$, $t \in I$, make up the curve in space.

It can also be represented in vector form:

$$\vec{r}(t) = \vec{OP} = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

We say f, g, h are the component function.

• Limit and Continuity

Def. Let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ be a vector function with domain D , and L a vector. We say that \vec{r} has limit L as t approaches t_0 and write

$$\Delta \lim_{t \rightarrow t_0} \vec{r}(t) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ with that for all $t \in D$.

$$|\vec{r}(t) - L| < \epsilon \text{ whenever } 0 < |t - t_0| < \delta$$

Def. A vector function $r(t)$ is continuous at a point $t = t_0$ in its domain if $\lim_{t \rightarrow t_0} r(t) = r(t_0)$.

The function is continuous if it is continuous over its interval domain.

• Derivatives and Motion

Def. The vector function $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ has a derivative (is differentiable) at t if f, g, h have derivatives at t . The derivative is the vector function

$$\Delta \vec{r}'(t) = \frac{d\vec{r}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{df}{dt}\vec{i} + \frac{dg}{dt}\vec{j} + \frac{dh}{dt}\vec{k}.$$

The curved traced by \vec{r} is smooth if $\frac{d\vec{r}}{dt}$ is continuous and never $\vec{0}$.

Def. If \vec{r} is the position vector of a particle moving along a smooth curve in space, then

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

is the particle's velocity vector, tangent to the curve. At any time t , the direction of \vec{v} is the direction of motion, the magnitude of \vec{v} is the particle's speed, and the derivative $\vec{a} = \frac{d\vec{v}}{dt}$, when it exists, is the particle's acceleration vector.

• Vector Functions of Constant Length

Def. If r is a differentiable vector function of t of constant length, then

$$\vec{r} \cdot \frac{d\vec{r}}{dt} = 0.$$

13.2 Integrals of Vector Functions; Projectile Motion

Integrals of Vector Functions

Def. The indefinite integral of \vec{r} with respect to t is the set of all antiderivatives of \vec{r} , denoted by $\int \vec{r}(t) dt$. If \vec{R} is any antiderivative of \vec{r} , then

$$\int \vec{r}(t) dt = \vec{R}(t) + C.$$

Def. If the components of $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ are integrable over $[a, b]$, then so is \vec{r} , and the definite integral of \vec{r} from a to b is

$$\int_a^b \vec{r}(t) dt = \left(\int_a^b f(t) dt \right) \vec{i} + \left(\int_a^b g(t) dt \right) \vec{j} + \left(\int_a^b h(t) dt \right) \vec{k}$$

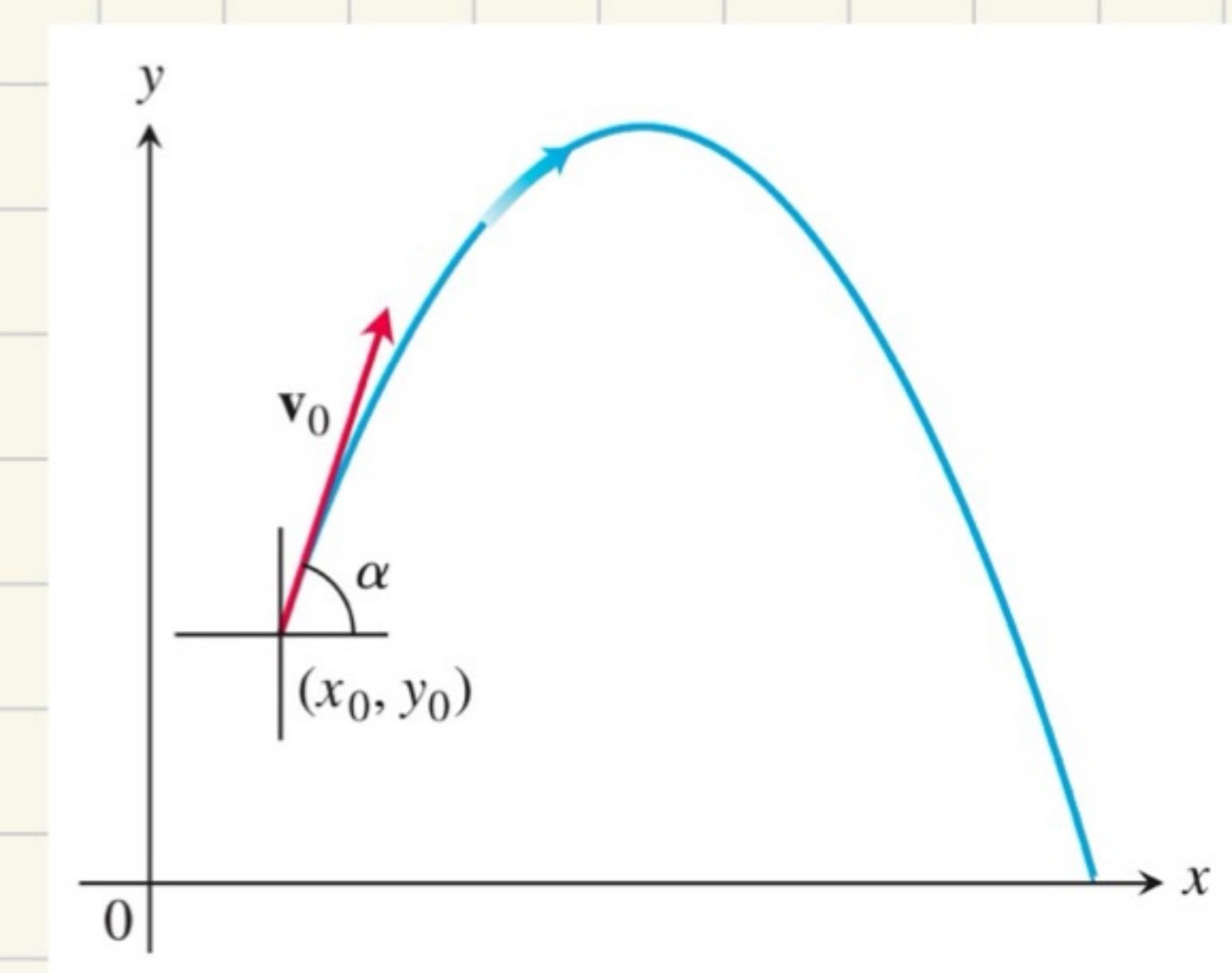
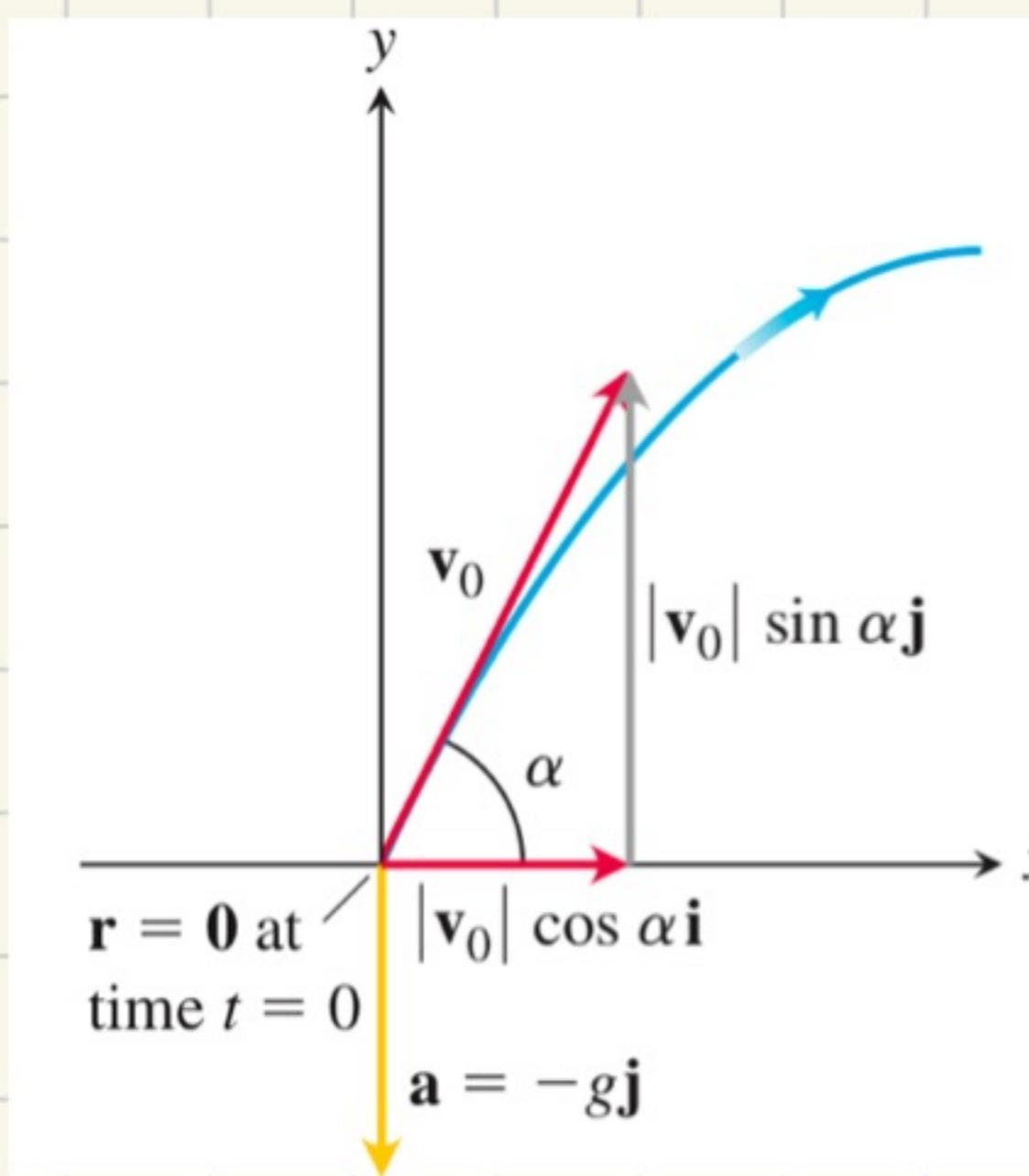
Local Projectile Motion Equations

$$\vec{r} = (v_0 \cos \alpha) \vec{i} + ((v_0 \sin \alpha)t - \frac{1}{2}gt^2) \vec{j}$$

$$y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g}$$

$$t = \frac{2v_0 \sin \alpha}{g}$$

$$R = \frac{v_0^2}{g} \sin 2\alpha$$



13.3 Arc Length in Space

Arc Length Along a Space Curve

Def. The length of a smooth curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $a \leq t \leq b$, that is traced exactly once as t increases from $t=a$ to $t=b$, is

$$\Delta L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_a^b |\vec{v}| dt$$

Arc Length Parameter with Base Point $P(t_0) = (x(t_0), y(t_0), z(t_0))$.

$$s(t) = \int_{t_0}^t \sqrt{(x'(T))^2 + (y'(T))^2 + (z'(T))^2} dT = \int_{t_0}^t |\vec{T}(T)| dT \text{ where } \vec{T} = \frac{d\vec{r}}{dt}$$

T : "tan".

13.4 Curvature and Normal Vectors of a Curve

Curvature of a Plane Curve

Def. If \vec{T} is the unit vector of a smooth curve, the curvature function of the curve is

$$\Delta K = \left| \frac{d\vec{T}}{ds} \right| \quad K: \text{"kappa"}$$

Formula for Calculating Curvature

If $\vec{r}(t)$ is a smooth curve, then the curvature is the scalar function

$$\Delta K = \frac{1}{|\vec{r}'|} \left| \frac{d\vec{T}}{dt} \right|.$$

where $\vec{T} = \frac{\vec{r}'}{|\vec{r}'|}$ is the unit tangent vector.

The Principal Unit Normal Vector

Def. At a point where $\kappa \neq 0$, the principle unit normal vector for a smooth curve in the plane is

$$\Delta \vec{N} = \frac{1}{\kappa} \frac{d\vec{T}}{ds}$$

Formula for Calculating N .

If $\vec{r}(t)$ is a smooth curve, then the principle unit normal is

$$\cancel{\Delta} \vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left| \frac{d\vec{T}}{dt} \right|}$$

Circle of Curvature for Plane Curves

The circle of curvature or osculating circle at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has)
2. has the same curvature the curve has at P .
3. has center that lies toward the concave or inner side of the curve

The radius of curvature of the curve at P is the radius of the circle of curvature, which is

$$\text{Radius of curvature} = \rho = \frac{1}{\kappa}$$

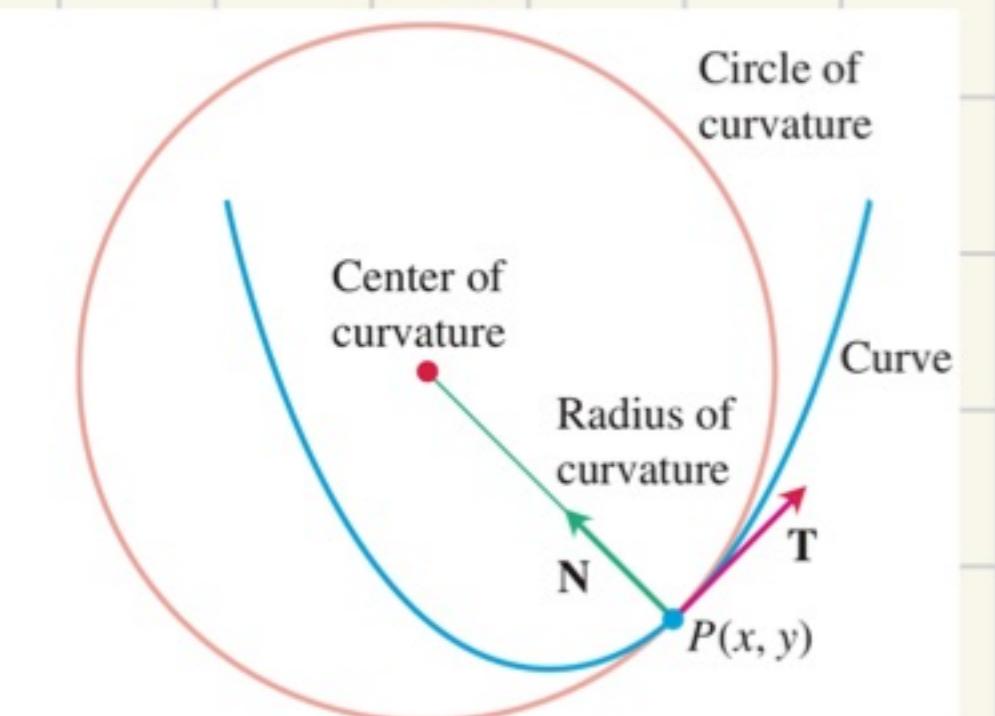


FIGURE 13.20 The center of the osculating circle at $P(x, y)$ lies toward the inner side of the curve.

Curvature and Normal Vectors for Space Curves

If a smooth curve in space is specified by the position vector $\vec{r}(t)$ as a function of some parameter t , and if s is the arc length parameter of the curve, then the unit tangent vector \vec{T} is $\frac{d\vec{r}}{ds} = \frac{\vec{v}}{\|\vec{v}\|}$. The curvature in space is then defined to be

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = \frac{1}{\|\vec{v}\|} \left| \frac{d\vec{T}}{dt} \right|$$

just as for plane curves. The vector $\frac{d\vec{T}}{ds}$ is orthogonal to \vec{T} , and we defined the principle unit normal to be

$$\vec{N} = \frac{1}{\kappa} \cdot \frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{\|d\vec{T}/dt\|}$$

V. Partial Derivatives

14.1 Functions of Several Variables

• Real-valued Function

Def. Suppose D is a set of n -tuples of real numbers (x_1, x_2, \dots, x_n) . A real-valued function f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \dots, x_n)$$

to each element in D . The set D is the function's domain. The set of w -values taken on by f is the function's range. The symbol w is the dependent variable of f , and f is said to be a function of the n independent variables x_1 to x_n . We also call the x_i 's the function's input variables and call w the function's output variable.

• Functions of Two Variables

Def. A point (x_0, y_0) in a region (set) R in the xy -plane is an interior point of R if it is the center of a disk of positive radius that lies entirely in R . A point (x_0, y_0) is a boundary point of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R .

The interior points of a region, as a set, make up the interior of the region. The region's boundary points make up its boundary. A region is open if it consists entirely of interior points. A region is closed if it contains all its boundary points.

Def. A region in the plane is bounded if it lies inside a disk of finite radius. A region is unbounded if it is not bounded.

Def. The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a level curve of f . The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f , is called the graph of f . The graph of f is also called the surface $z = f(x, y)$.

P.S. Three variables are similar.

14.2 Limits and Continuity in Higher Dimensions

• Limits for Function of Two Variables

Def. We say that a function $f(x, y)$ approaches the limit L as (x, y) approaches (x_0, y_0) , write

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L.$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f ,

$$|f(x, y) - L| < \epsilon \text{ whenever } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

Theorem : Properties of Limits of Functions of Two Variables

◇ Same as one variable! No need to record.

• Continuity

Def. A function $f(x, y)$ is continuous at the point (x_0, y_0) if

1. f is defined at (x_0, y_0)
2. $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y)$ exists and $= f(x_0, y_0)$.

A function is continuous if it is continuous at every point of its domain.

Two Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0) , then $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y)$ does not exist. □

Continuity of Composites

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0) .

14.3 Partial Derivatives

Partial Derivatives of a Function of Two Variables

Def. The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is

$$\frac{\partial f}{\partial x} \Big|_{(x_0, y_0)} = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

provided the limit exists. Also noted as $f_x(x_0, y_0)$.

And respect to y is similar

Second-Order Partial Derivatives

Def. When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right).$$

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$$

The Mixed Derivative Theorem

Def. If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy} , and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b) , then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Differentiability

Def. A function $z = f(x, y)$ is differentiable at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + e_1 \Delta x + e_2 \Delta y.$$

in which each of $e_1, e_2 \rightarrow 0$ as both $\Delta x, \Delta y \rightarrow 0$. We call f differentiable if it is differentiable at every point in its domain, and say that its graph is a smooth surface.

Theorem: The Increment Theorem for Functions of Two Variables

Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0) . Then

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

In the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y,$$

in which each of $\epsilon_1, \epsilon_2 \rightarrow 0$ as both $\Delta x, \Delta y \rightarrow 0$.

14.4 The Chain Rule

Functions of Two Variables and Three Variables

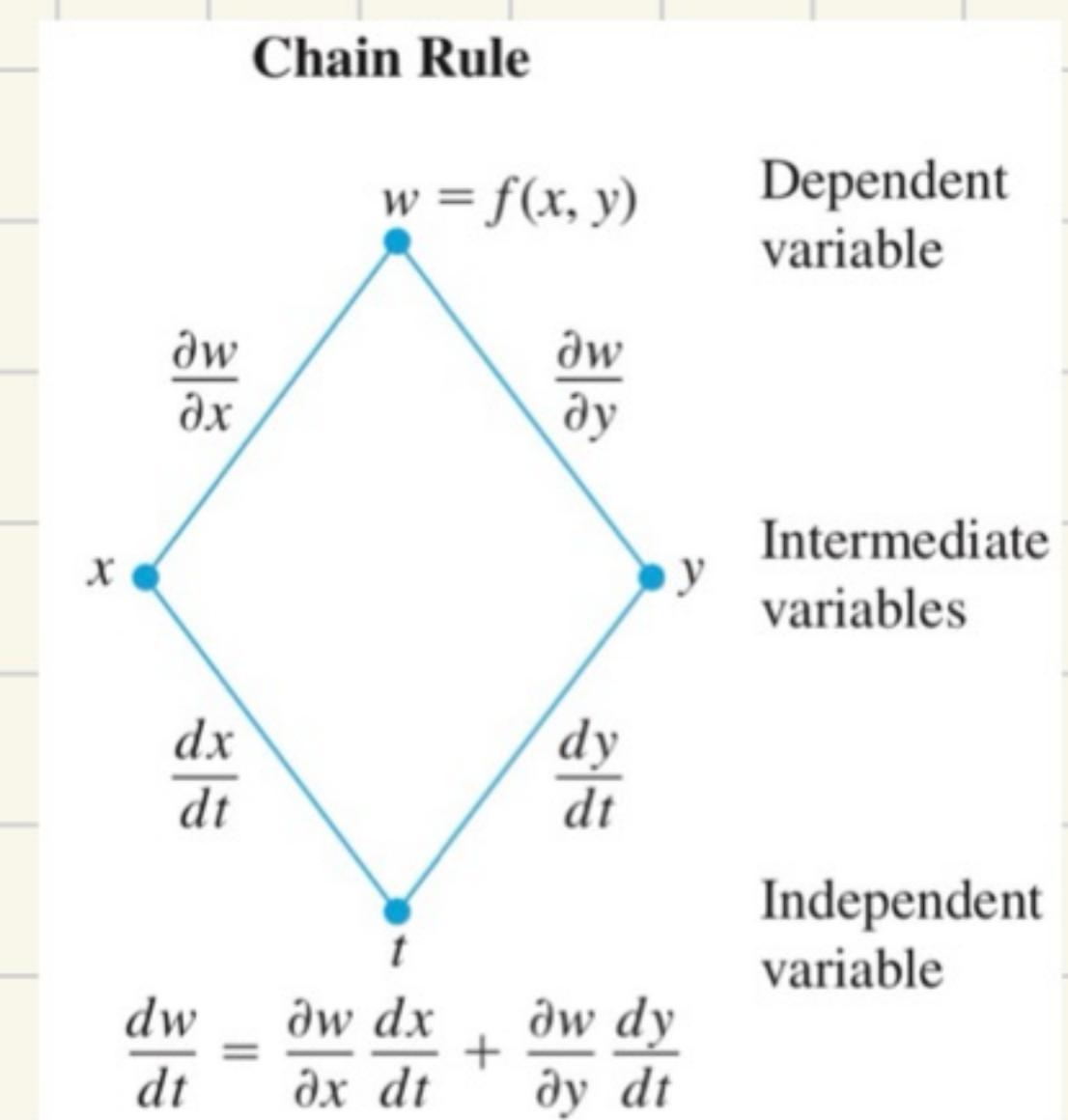
Theorem: Chain Rule For Functions of One Independent Variable and Two Intermediate Variables

If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t , then the composite $w = f(x(t), y(t))$ is a differential function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

If $w = f(x, y, z)$ and is differentiable, then

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$



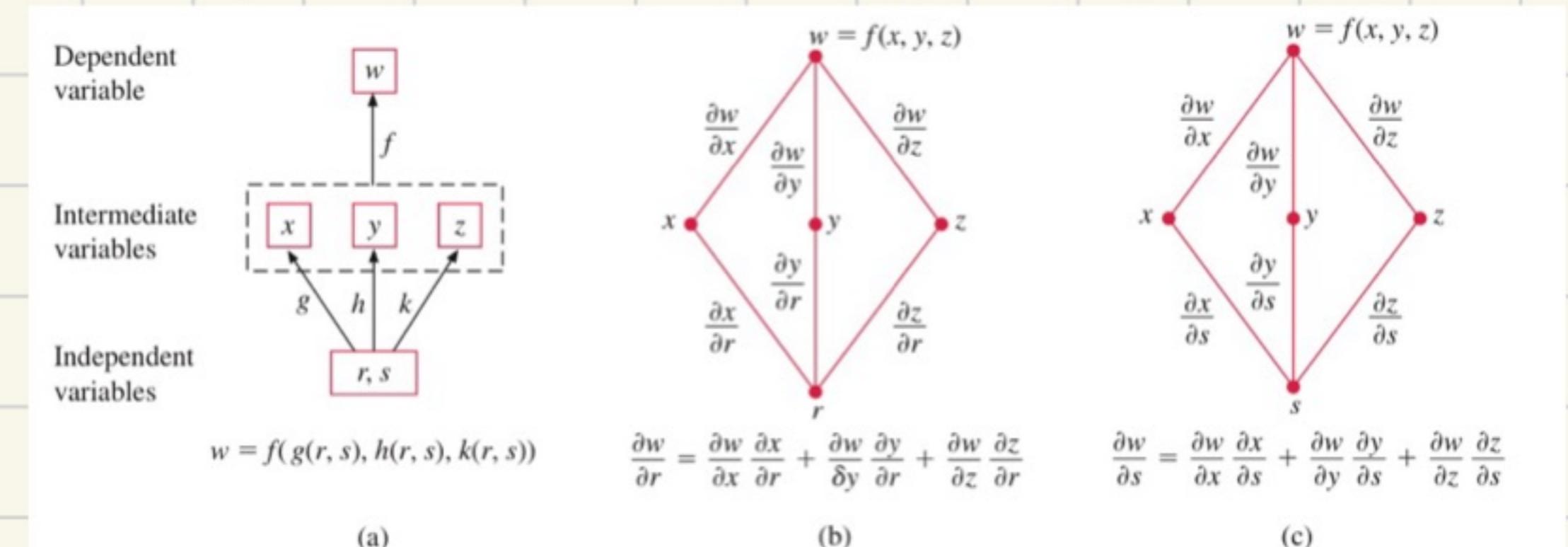
Functions Defined on Surfaces

Theorem: Chain Rule for Two Independent Variables and Three Intermediate Variables

Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s , given by

$$\begin{cases} \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r} \\ \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s} \end{cases}$$

P.S. Two Intermediate Variables is similar.



Implicit Differential Revisited

Theorem: A Formula for Implicit Differentiation

Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x . Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

14.5 Directional Derivatives and Gradient Vectors

Directional Derivatives in the Plane

Def. The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is

$$(\frac{df}{ds})_{\vec{u}, P_0} = \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided the limit exists. Also denoted by $(D_u f)_P$

