AMath 574
Homework 1 by Xin Yang, yangxin@uw.edu
Due by 11:00pm on January 15, 2015

For submission instructions, see:

`http://faculty.washington.edu/rjl/classes/am574w2015/homework1.html`

Problem #2.1 in the book

Solution:

Equations (2.47) are

$$\begin{aligned}\tilde{\rho}_t + (\widetilde{\rho u})_x &= 0\\ (\widetilde{\rho u})_t + (-u_0^2 + P'(\rho_0))\tilde{\rho}_x + 2u_0(\widetilde{\rho u})_x &= 0\end{aligned}$$

As shown in the textbook, the perturbation in the pressure is related to the perturbation in the density:

$$p_0 + \tilde{p} = P(\rho_0 + \tilde{\rho}) \approx P(\rho_0) + P'(\rho_0)\tilde{\rho} \Longrightarrow \tilde{\rho} \approx \frac{\tilde{p}}{P'(\rho_0)}$$

and the perturbation in the velocity also has relation

$$(\rho u)_0 + \widetilde{\rho u} = (\rho_0 + \tilde{\rho})(u_0 + \tilde{u}) \Longrightarrow \widetilde{\rho u} = \rho_0\tilde{u} + u_0\tilde{\rho}$$

As a result, plug in the relations into (2.47), we get equations (2.48)

$$\begin{aligned}\tilde{p}_t + \rho_0 P'(\rho_0)\tilde{u}_x + u_0\tilde{p}_x &= 0\\ \rho_0\tilde{u}_t + \tilde{p}_x + u_0\rho_0\tilde{u}_x &= 0\end{aligned}$$

Problem #2.3 in the book

Solution: Via direct calculation,

$$\tilde{A} = \begin{pmatrix} 0 & c_0^2 \\ 1 & 0 \end{pmatrix}$$

has eigenvalues $\lambda_1 = -c_0$ and $\lambda_2 = c_0$ with corresponding eigenvectors

$$\mu_1 = (-c_0, 1), \quad \mu_2 = (c_0, 1).$$

Since $c_0 = \sqrt{K_0/\rho_0}$, not hard to observe

$$A = \begin{pmatrix} 0 & K_0 \\ 1/\rho_0 & 0 \end{pmatrix} = \begin{pmatrix} 1/\rho_0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & c_0^2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \rho_0 & 0 \\ 0 & 1 \end{pmatrix}$$

which gives the similarity transform.

Problem #2.4 in the book

Solution: Via direct calculation,

$$A = \begin{pmatrix} 0 & 1 \\ -u_0^2 + P'(\rho_0) & 2u_0 \end{pmatrix}$$

has eigenvalues $\lambda_1 = u_0 - \sqrt{P'(\rho_0)}$ and $\lambda_2 = u_0 + \sqrt{P'(\rho_0)}$ with corresponding eigenvectors

$$\mu_1 = \left(\frac{1}{\sqrt{u_0 - P'(\rho_0)}}, 1 \right), \quad \mu_2 = \left(\frac{1}{u_0 + \sqrt{P'(\rho_0)}}, 1 \right).$$

As we have $c_0 = \sqrt{P'(\rho_0)}$, these results are the same as (2.57). To figure out the similarity transform, notice that

$$\begin{pmatrix} \rho \\ \rho u \end{pmatrix} = \begin{pmatrix} 1/P'(\rho_0) & 0 \\ u_0/P'(\rho_0) & \rho_0 \end{pmatrix} \begin{pmatrix} p \\ u \end{pmatrix} := R \begin{pmatrix} p \\ u \end{pmatrix}$$

Then $R^{-1}AR$ gives the matrix in (2.51) provided $u_0 = 0$.

Problem #2.5 in the book

Solution: Rewrite (2.91)

$$\begin{aligned} \epsilon_t - u_x &= 0 \\ u_t - \frac{\sigma'(\epsilon)}{\rho}\epsilon_x &= 0 \end{aligned}$$

we need the coefficient matrix

$$A = \begin{pmatrix} 0 & -1 \\ -\frac{\sigma'(\epsilon)}{\rho} & 0 \end{pmatrix}$$

to be diagonalizable with real eigenvalues. Via direct calculation, we get

$$\lambda_1 = -\lambda_2 = \sqrt{\frac{\sigma'(\epsilon)}{\rho}}$$

therefore the system is hyperbolic iff $\frac{\sigma'(\epsilon)}{\rho} > 0$ (physically, $\rho > 0$). In the case, $\frac{\sigma'(\epsilon)}{\rho} = 0$, we do not have a complete eigenvector set.

Problem #2.7 in the book

Solution: Here the coefficient matrix is

$$A = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}$$

with eigenvalues

$$\lambda_1 = -\lambda_2 = \sqrt{-p'(v)}$$

Using the same calculation as the previous problem, we get a complete set of eigenvectors iff $p'(v) < 0$. (Imaginary eigs when negative, incomplete eigenspace when zero)

Problem #3.3 in the book

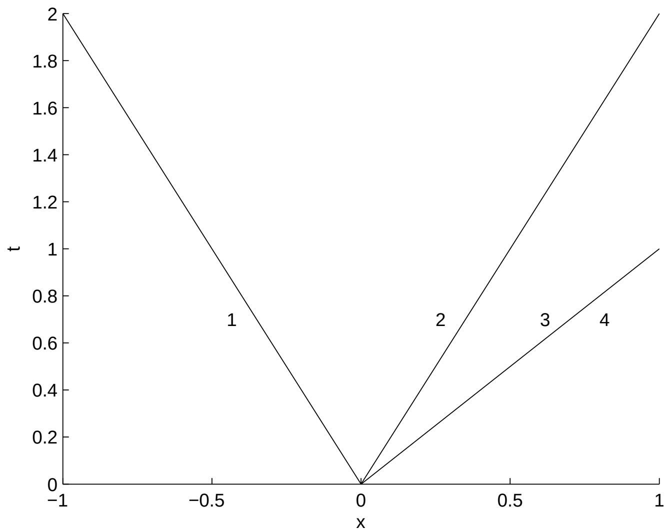
Following the sort of thing done in script `problem_3_5.py` might be useful if you want to insert a figure in your solution, or you can draw with another programming language, or sketch the solution by hand and scan.

Solution:

Given a Riemann problem with A, q_l, q_r, following the procedure in the textbook, what we need to do is finding the characteristic lines along which the discontinuity propagates. Let λ be the vector of eigenvalues, and R be the matrix made of the normalized the eigenvectors. Then the lines are given by $x = \lambda t$. These lines partition the whole $x - t$ domain into different regions. The solution in each region is determined by solving $R\alpha = q_r - q_l$. Here, R is the orthogonal matrix made of eigenvectors and α gives the jump across each corresponding line of discontinuities.

1.

$$A = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad q_l = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad q_r = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$$

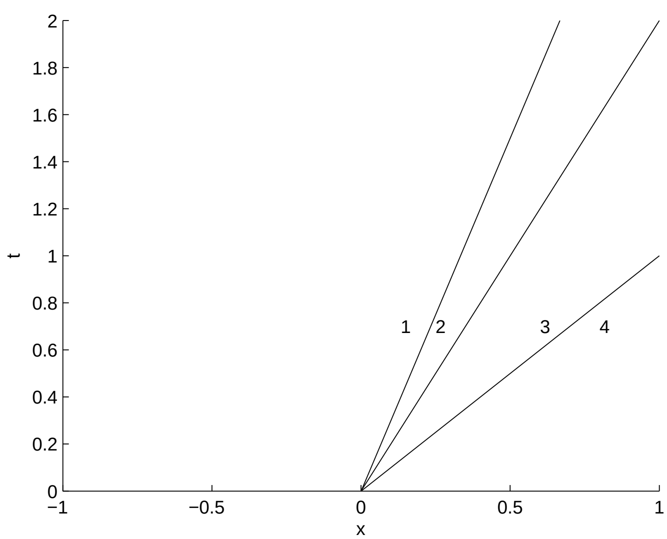


$\lambda = (-2, 2, 1)$ corresponds to the lines of discontinuities from the left to the right. The solution in regions from 1 to 4 is

$$q_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 0 \\ 2 \\ 0.5 \end{pmatrix}, \quad q_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad q_4 = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$$

2.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad q_l = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad q_r = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

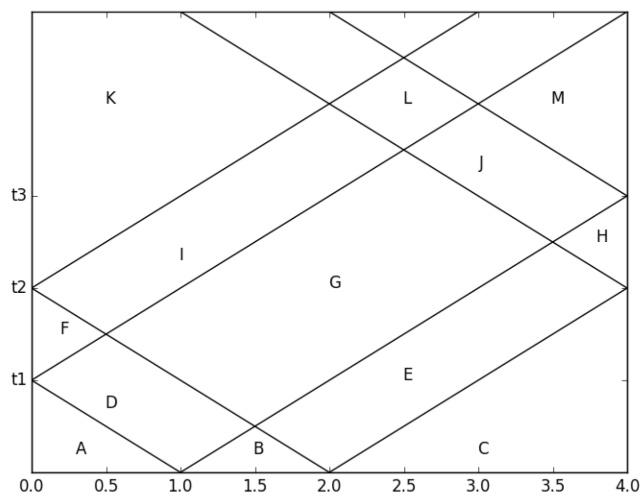


$\lambda = (3, 2, 1)$ corresponds to the lines of discontinuities from the left to the right. The solution in regions from 1 to 4 is

$$q_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}, \quad q_3 = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}, \quad q_4 = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

Problem #3.5 in the book

The script `problem_3_5.py` was used to generate this figure:



To solve this problem, determine the states $A,\ B,\ \ldots,\ M$ and also the times $t_1,\ t_2,\ t_3$. The times can be written in terms of the parameters ρ_0 and K_0, which were not stated in the problem.

For example,

$$A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad ,B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad ,C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \ldots$$

Solution: The figure illustrates the propagation and reflection of the sound waves. Based on the illustration, at time t_1, the left wave reaches the left boundary for the first time. Since the wave speed is $c = \pm\sqrt{K_0/\rho_0}$,

$$t_1 = 1/|c|.$$

Therefore,

$$A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since the eigenvectors corresponding to $(-\sqrt{K_0/\rho_0}, \sqrt{K_0/\rho_0})$ are

$$v_1 = \begin{bmatrix} -\sqrt{K_0\rho_0} \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} \sqrt{K_0\rho_0} \\ 1 \end{bmatrix},$$

let $Z = \sqrt{K_0\rho_0}$ and left going and right going components are

$$D = v_l = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2\sqrt{K_0\rho_0}} \end{bmatrix} = -\frac{1}{2Z}v_1, \quad E = v_r = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{K_0\rho_0}} \end{bmatrix} = \frac{1}{2Z}v_2.$$

Then we can find

$$G = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Boundary conditions $u = 0$ in F and H show that

$$F = H = B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

And $t_2 = 2/|c|$ is the first time that the right going wave hits the right boundary. $t_3 = 3/|c|$ is the time the right going wave leaving the right boundary.
Mimicking the same procedure, we can find the pattern, and as a result,

$$A = C = G = K = M = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

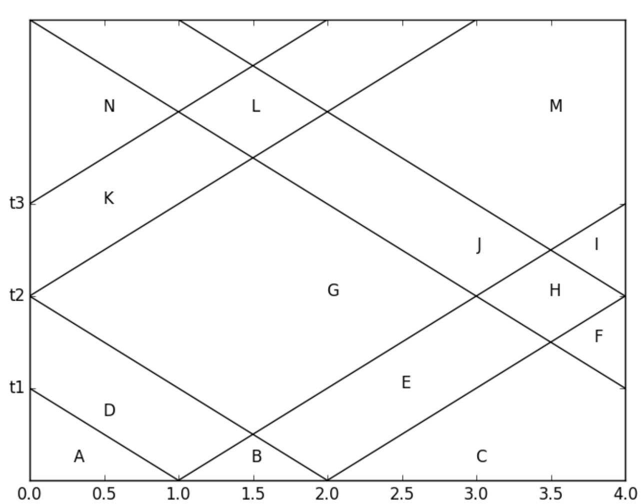
$$B = F = H = L = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$D = J = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2\sqrt{K_0\rho_0}} \end{bmatrix}.$$

$$E = I = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{K_0\rho_0}} \end{bmatrix}.$$

Problem #3.5A Solve #3.5 with *periodic* boundary conditions instead of reflecting walls. Sketch the solution in the x–t plane up to at least time t_3 (as in #3.5 the time the right-going wave from $x_0 = 1$ hits the right boundary) and indicate the state in each section. You might want to modify the script `problem_3_5.py` to make the plot.

Solution:



As we have the same equation as the previous problem, the same procedure applies here. However, instead of using reflection on the boundary, we need to shift the solution periodically from the other side. Based on the illustration in the figure, time $t_1 = 1/|c|$, $t_2 = 2/|c|$, $t_3 = 3/|c|$ are the same. Mimicking the same procedure of superposition of the left and right going wave, we can find the result,

$$A = C = G = N = M = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$B = H = L = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$D = F = J = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2\sqrt{K_0\rho_0}} \end{bmatrix}.$$

$$E = I = K = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{K_0\rho_0}} \end{bmatrix}.$$

Optional

For more practice with simple Riemann problems you might want to work through Problem 3.1 and perhaps tackle 3.2. If you want to try writing a program in Python, the module `numpy.linalg` contains an `eig` function similar to Matlab.

Solution: Since uploading files other than pdf are not allowed in the canvas, as the code is short, I put it here.
Matlab code:

```
function [ V,D,alpha ] = myRS( A,ql,qr )

[V,D]=eig(A);
dim=size(A);
[DD,ind]=sort(atan2(1./diag(D),0*diag(D)+1));
D=diag(D);
D=D(ind);
V=V(:,ind);
alpha=V\(qr-ql);

figure(2), clf, hold on
x=[-1,1];
t=0.7;
for j=1:dim(1)
    y=D(j)*x;
    plot(x,y,'k');
    text(t/D(j)-0.1,t,num2str(j))
```

```
end
xlim([-1,1]);
ylim([0,2]);
text(t/D(j)+0.1,t,num2str(j+1))
xlabel('x')
ylabel('t')
end
```

The inputs are A, q_l, q_r and the program outputs the eigenvalue, eigenvector and solution to the linear system α with a plot of the region partition. For this "toy" program, no efforts were spent on documentation or extension.