Valuation of a Financial Claim Contingent on the Outcome of a Quantum Measurement

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We consider a rational agent who at time 0 enters into a financial contract for which the payout is determined by a quantum measurement at some time $T > 0$. The state of the quantum system is given by a known density matrix $\hat{p}$. How much will the agent be willing to pay at time 0 to enter into such a contract? In the case of a finite dimensional Hilbert space, each such claim is represented by an observable $\hat{X}_T$ where the eigenvalues of $\hat{X}_T$ determine the amount paid if the corresponding outcome is obtained in the measurement. We prove, under reasonable axioms, that there exists a pricing state $\hat{q}$ which is equivalent to the physical state $\hat{p}$ on null spaces such that the pricing function Π_{0T} takes the form $\Pi_{0T}(\hat{X}_T) = P_{0T}\,\mathrm{tr}(\hat{q}\hat{X}_T)$ for any claim $\hat{X}_T$, where P_{0T} is the one-period discount factor. By "equivalent" we mean that $\hat{p}$ and $\hat{q}$ share the same null space: thus, for any $|\xi\rangle \in \mathcal{H}$ one has $\langle\bar{\xi}|\hat{p}|\xi\rangle = 0$ if and only if $\langle\bar{\xi}|\hat{q}|\xi\rangle = 0$. We introduce a class of optimization problems and solve for the optimal contract payout structure for a claim based on a given measurement. Then we consider the implications of the Kochen-Specker theorem in such a setting and we look at the problem of forming portfolios of such contracts.

Key words: Quantum mechanics, quantum measurement, contingent claims, discount bonds, absence of arbitrage, rate of return, density matrices, Gleason's theorem, Kochen-Specker theorem.

I. INTRODUCTION

By "quantum finance" we mean the valuation, optimization and risk management of financial contracts for which the outcomes (in the form of one or more payments made between the various parties involved) are contingent on the results of one or more quantum measurements. The financial contracts that we consider can be easily implemented, at least in principle, in a suitable laboratory. Our investigations are entirely within the scope of standard quantum mechanics and we are not concerned here with modifications of the standard framework or with interpretive issues. The resulting theory of quantum finance is distinctly non-Kolmogorovian, inheriting as it does the full generality of quantum probability.

Let time 0 be the present and T a fixed time in the future. We consider the situation where an agent A enters into a contract with another agent B in accordance with which A pays B an amount H_0 ("the price") at time 0 and then B pays A an amount H_T (the "payout") at time T, where H_T is contingent in some specified way on the outcome of a quantum measurement. More elaborate setups can be considered, with multiple payments as time progresses and multiple agents; but for simplicity we look at a one-period market involving two agents.

As an example, suppose the payout is determined by a measurement of the spin of a spin one-half particle along the z-axis. The outcome of such a measurement either gives $+\frac{1}{2}\hbar$, corresponding to spin up along that axis, or $-\frac{1}{2}\hbar$, corresponding to spin down. Henceforth, we work with physical units such that $\hbar = 1$. For the basics of quantum theory, see, for instance, reference [28]. We fix a two-dimensional Hilbert space $\mathcal{H}^2$ and on it we introduce the usual observable for the spin along the z-axis, given by the Hermitian operator

$$\hat{S}_z = \tfrac{1}{2}|z_1\rangle\langle\bar{z}_1| - \tfrac{1}{2}|z_2\rangle\langle\bar{z}_2|, \tag{1}$$

where $|z_1\rangle$ is a unit Hilbert space vector corresponding to the upward direction along the z-axis and $|z_2\rangle$ denotes an orthogonal unit Hilbert space vector corresponding to the downward direction along the z-axis. Thus, $\langle\bar{z}_1|z_1\rangle = 1$, $\langle\bar{z}_2|z_2\rangle = 1$, $\langle\bar{z}_1|z_2\rangle = 0$, $\langle\bar{z}_2|z_1\rangle = 0$, and the possible outcomes of the measurement are the eigenvalues of $\hat{S}_z$, which are $+\frac{1}{2}$ and $-\frac{1}{2}$.

The probabilities of these outcomes are determined by the *state* of the system, which is represented by a density matrix $\hat{p}$. The density matrix in quantum theory has a status that is analogous in certain respects to that of the probability measure in classical probability theory. The density matrix is assumed to be a positive Hermitian operator with trace unity, which in the case of a two-dimensional Hilbert space takes the form

$$\hat{p} = p_1|\psi_1\rangle\langle\bar{\psi}_1| + p_2|\psi_2\rangle\langle\bar{\psi}_2|, \tag{2}$$

for some orthonormal basis $\{|\psi_1\rangle, |\psi_2\rangle\}$ in $\mathcal{H}^2$, where $p_1 \geq 0$, $p_2 \geq 0$, and $p_1 + p_2 = 1$. In general, such a matrix will have rank two, but if $p_1 = 0$ or $p_2 = 0$ then it will have rank one. A state with rank one is called a "pure" state. The probability for a given outcome is the trace of the product of the state and the projection operator onto the Hilbert subspace associated to the eigenvalue corresponding to that outcome (the "Born rule"). Thus we have

$$\text{Prob}\,\left(S_z = \tfrac{1}{2}\right) = \langle\bar{z}_1|\hat{p}|z_1\rangle, \quad \text{Prob}\,\left(S_z = -\tfrac{1}{2}\right) = \langle\bar{z}_2|\hat{p}|z_2\rangle. \tag{3}$$

In the case of a contingent claim where the payout is determined by the result of such a spin measurement, it should be clear that the claim itself can also be represented by a Hermitian operator on $\mathcal{H}^2$, in this case, an operator of the form

$$\hat{Z}_T = z_1|z_1\rangle\langle\bar{z}_1| + z_2|z_2\rangle\langle\bar{z}_2|, \tag{4}$$

where z_1 denotes the payment made to agent A in the case the measurement outcome is spin $+\frac{1}{2}$ and z_2 is the payment made to A when the measurement outcome is spin $-\frac{1}{2}$. One can think of such a contract as being an example of a so-called real option [15, 27, 44]. Payments are understood to be made in some fixed numeraire or unit of account. Thus, we conclude that *a contingent claim for which the payouts are determined by the result of a quantum measurement can be represented by an observable, in the usual quantum mechanical sense, whose eigenvalues correspond to the possible cash flows at time T.*

Among the various observables that can be represented in the form (4) there is a special observable that takes the form

$$\hat{P}_{0T} = 1|z_1\rangle\langle\bar{z}_1| + 1|z_2\rangle\langle\bar{z}_2|, \tag{5}$$

which pays one unit of account at time T, regardless of the outcome of the spin measurement. This is evidently a "risk-free" asset, since the payout is fixed and guaranteed, and we write

$$\hat{P}_{0T} = \hat{\mathbf{1}}. \tag{6}$$

Here $\hat{\mathbf{1}}$ denotes the identity operator on $\mathcal{H}^2$. The risk-free asset $\hat{P}_{0T}$ can be thought of as a discount bond that pays one unit of account (e.g., one "dollar") at maturity T. It has the interesting property that it does not depend on the specific choice of axis along which the spin measurement is taken.

In addition to contracts of the form (4), we can more generally consider contracts of the same type, but where the measurement of the spin is taken along some other axis. Each such contract is characterized by (a) the choice of a basis in Hilbert space along which the spin measurement is made, together with (b) the payouts that take place as a consequence of the results of the measurement. Indeed, it is a theorem that any positive Hermitian operator $\hat{Z}_T$ on $\mathcal{H}^2$ other than multiples of the identity can expressed uniquely in the form (4) for some choice of the orthonormal basis $\{|z_1\rangle, |z_2\rangle\}$ in $\mathcal{H}^2$, modulo multiplicative phase factors.

To complete the discussion we need to determine the *price* paid by agent A to agent B in exchange for the payout corresponding to $\hat{X}_T$. In short, we need a *pricing function* that maps each observable $\hat{X}_T$ to a corresponding price X_0.

II. EXISTENCE OF PRICING OPERATOR

It will be useful going forward to generalize our considerations to the case of a Hilbert space $\mathcal{H}$ of arbitrary finite dimension n. As usual, we can write $|\xi\rangle$ for a typical element of $\mathcal{H}$ and $\langle\bar{\xi}|$ for its complex conjugate. The observable that determines the payout will in the generic situation be a non-degenerate Hermitian operator $\hat{X}_T$ on this space and hence admit n distinct real eigenvalues, each corresponding to a distinct cash flow.

For example, if the quantum system admits n different energy levels, and the underlying physical observable being measured is the energy of the system, then the contract will in general result in a different cash flow $\{x_j\}_{j=1,2,\ldots,n}$ for each of the possible energy outcomes. For the financial observable representing such a contract we can write

$$\hat{X}_T = \sum_{j=1}^{n} x_j |x_j\rangle\langle\bar{x}_j|, \quad \langle\bar{x}_j|x_k\rangle = \delta_{jk} \tag{7}$$

for some orthonormal basis $\{|x_j\rangle\}_{j=1,2,\ldots,n}$ in Hilbert space. More generally, the set of all financial observables associated with a given Hilbert space will include some that are degenerate in the sense that the same payout will result for two or more distinct values of the outcome j. Such a degeneracy can result either because there is a degeneracy in the spectrum of the underlying physical observable, or because two or more distinct eigenvalues of the physical observable are assigned the same cash flow. An example of the latter is a unit discount bond, for which $x_j = 1$ for all $j = 1, 2, \ldots, n$ even though the underlying energy levels may be distinct. Then the identity operator on $\mathcal{H}$ represents such a bond and again we write (6) in that case.

Another example of a degenerate observable is the analogue of a so-called Arrow-Debrue (A-D) security [1], which for each value of j has the payout $x_j = \mathbb{1}\{j = k\}$ for some fixed value of k. Here $\mathbb{1}\{E\}$ denotes the indicator function for the event E. Thus $x_j = 1$ if $j = k$, and $x_j = 0$ if $j \neq k$. The A-D securities are represented by pure projection operators, each with payout unity or zero, depending on the result of the underlying quantum measurement, whose outcome is also unity or zero. Thus the set of all Arrow-Debrue type contracts is precisely the set of all pure projection operators on $\mathcal{H}$.

The state of a quantum system in n dimensions is represented by a positive semidefinite Hermitian matrix with trace unity. Such a matrix can be put in the form

$$\hat{p} = \sum_{j=1}^{n} p_j |\psi_j\rangle\langle\bar{\psi}_j|, \tag{8}$$

for some orthonormal basis $\{|\psi_j\rangle\}_{j=1,2,\ldots,n}$, with $p_j \geq 0$ for $j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} p_j = 1$.

In the case of a density matrix of maximal rank with distinct eigenvalues, this basis is uniquely determined up to phase factors. If the density matrix is of maximal rank but with a degenerate spectrum, the basis is determined modulo unitary transformations on the degenerate subspaces. In particular, in the case of a density matrix of lower rank, the basis is determined at best only up to an arbitrary unitary transformation of the basis vectors that span the null space of the density matrix.

Given two density matrices $\hat{p}$ and $\hat{q}$, we say that $\hat{q}$ is *absolutely continuous* with respect to $\hat{p}$ if the null space of $\hat{p}$ is a subspace of the null space of $\hat{q}$. Thus, $\hat{q}$ is absolutely continuous with respect to $\hat{p}$ if and only if for all $|\psi\rangle \in \mathcal{H}$ such that $\hat{p}|\psi\rangle = 0$ it holds that $\hat{q}|\psi\rangle = 0$. We say that $\hat{p}$ and $\hat{q}$ are *equivalent* if each is absolutely continuous with respect to the other, that is to say, if they share the same null space. For example, in four dimensions the operators

$$\hat{p} = p_1|\psi_1\rangle\langle\bar{\psi}_1| + p_2|\psi_2\rangle\langle\bar{\psi}_2| + p_3|\psi_3\rangle\langle\bar{\psi}_3| + p_4|\psi_4\rangle\langle\bar{\psi}_4| \tag{9}$$

and

$$\hat{q} = q_1|\psi_1'\rangle\langle\bar{\psi}_1'| + q_2|\psi_2'\rangle\langle\bar{\psi}_2'| + q_3|\psi_3\rangle\langle\bar{\psi}_3| + q_4|\psi_4\rangle\langle\bar{\psi}_4| \tag{10}$$

are equivalent when $p_1 > 0, p_2 > 0, p_3 = 0, p_4 = 0,\ q_1 > 0, q_2 > 0, q_3 = 0, q_4 = 0$, if $\{|\psi_1\rangle, |\psi_2\rangle\}$ and $\{|\psi_1'\rangle, |\psi_2'\rangle\}$ are related by a unitary transformation in the Hilbert subspace orthogonal to the subspace spanned by $|\psi_3\rangle$ and $|\psi_4\rangle$. This is because the two density matrices share a common null space, spanned by the orthogonal vectors $|\psi_3\rangle$ and $|\psi_4\rangle$. It is easy to see that "equivalence" in this sense is an equivalence relation in the usual mathematical sense, and it follows that all density matrices of maximal rank are equivalent.

We shall say that a claim $\hat{X}_T$ is *positive* if $\langle\bar{\psi}|\hat{X}_T|\psi\rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}$ and *strictly positive* if $\langle\bar{\psi}|\hat{X}_T|\psi\rangle > 0$ for all $|\psi\rangle \in \mathcal{H}$. By (7) one sees that $\hat{X}_T$ is positive if and only if $x_j \geq 0$ for all j and strictly positive if and only if $x_j > 0$ for all j. It is legitimate to consider financial contracts resulting in negative cash flows as well, but it will suffice for our purpose to look at financial contracts with positive cash flows.

Let us now consider a one-period market represented by the set of all positive claims on an n-dimensional Hilbert space. If $\hat{X}_T$ and $\hat{Y}_T$ are two claims, then so is the linear combination

$$\hat{Z}_T = a\hat{X}_T + b\hat{Y}_T \tag{11}$$

for $a, b \geq 0$. Hence the space of positive claims has a natural convex structure. It should be evident that the experiments underlying the claims $\hat{X}_T$ and $\hat{Y}_T$ are in general different and that the experiment underlying $\hat{Z}_T$ is different yet again. If we write these claims in their diagonalized forms

$$\hat{X}_T = \sum_{j=1}^{n} x_j |x_j\rangle\langle\bar{x}_j|, \quad \hat{Y}_T = \sum_{j=1}^{n} y_j |y_j\rangle\langle\bar{y}_j|, \tag{12}$$

with respect to the relevant basis vectors, one sees that the payouts and basis vectors associated with these claims are uniquely determined, up to the usual ambiguities associated with degeneracies and null spaces, and at the same time the payouts and basis vectors of (11) are represented by the decomposition

$$\hat{Z}_T = \sum_{j=1}^{n} z_j |z_j\rangle\langle\bar{z}_j|. \tag{13}$$

Here we see a curious feature arising in the analysis of such financial instruments: if we have two contracts, each with positive payouts, depending on separate measurements, then any linear combination of the operators corresponding to the two contracts, with positive coefficients, will give rise to the operator corresponding to yet another contracts, with a different set of payouts, depending on still another measurement. Thus, a linear combination (11) is *not*, generally, to be understood as representing a "portfolio" of its constituents (we consider how to model portfolios in Section VI). One is tempted, nonetheless, to conjecture that the price of the contract represented by a linear combination of two contracts should equal the corresponding linear combination of the prices of the constituents. But it is not obvious that this will be the case, since the new contract involves a different payout structure and a different experiment – so we do not simply wish to *assume* linearity in general.

We can, however, quite reasonably assume that such a linear relationship holds in certain special situations. In particular, if two contracts $\hat{U}_T$ and $\hat{V}_T$ depend on the outcome of *the same experiment*, and differ from one another only in the amounts paid for the various outcomes of the experiment, then the price of the contract

$$\hat{W}_T = a\hat{U}_T + b\hat{V}_T \tag{14}$$

should indeed be equal to the corresponding linear combination of the prices of $\hat{U}_T$ and $\hat{V}_T$. More precisely, if the operators $\hat{U}_T$ and $\hat{V}_T$ *commute*, then the prices should be additive. The reasoning is as follows. If $\hat{U}_T$ and $\hat{V}_T$ commute, we can find a common orthogonal basis $\{|w_j\rangle\}_{j=1,2,\dots,n}$ in which both are diagonalized:

$$\hat{U}_T = \sum_{j=1}^{n} u_j |w_j\rangle\langle\bar{w}_j|, \quad \hat{V}_T = \sum_{j=1}^{n} v_j |w_j\rangle\langle\bar{w}_j|. \tag{15}$$

Then if we form the linear combination (14) we obtain

$$\hat{W}_T = \sum_{j=1}^{n} (au_j + bv_j) |w_j\rangle\langle\bar{w}_j|, \tag{16}$$

showing that the payouts for $\hat{W}_T$ are given by linear combinations of the payouts of the constituents. Thus, for commuting observables it is obvious that the price of a linear combination of contracts should be the corresponding linear combination of the prices of the individual contracts; but it is not obvious that linearity extends to non-commuting contracts.

At this point, it may be helpful if we codify our assumptions somewhat more explicitly. As usual, we write $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. We fix a quantum system with state $\hat{p}$ on an n-dimensional Hilbert space $\mathcal{H}$ and write $\mathcal{V}^+$ for the cone for positive contracts on $\mathcal{H}$. Thus our market is characterized by the triple $\{\mathcal{H}, \hat{p}, \mathcal{V}^+\}$. Let us write P_{0T} for the price of a unit discount bond. Our goal is to assign a price to each contract $\hat{X}_T \in \mathcal{V}^+$.

By a *pricing function* on the market $\{\mathcal{H}, \hat{p}, \mathcal{V}^+\}$ in a one-period setting we mean a mapping $\Pi_{0T} : \mathcal{V}^+ \to \mathbb{R}^+$ satisfying the following:

Axiom (1). For all $\hat{X}_T \in \mathcal{V}^+$ it holds that $\Pi_{0T}[\hat{X}_T] = 0$ if and only if $\mathrm{tr}(\hat{p}\hat{X}_T) = 0$.

Axiom (2). If the m contracts represented by the Hermitian matrices $\{\hat{X}_T^k\}_{k=1,2,\ldots,m}$ commute, then for all $\{a_k \geq 0\}_{k=1,2,\ldots,m}$ one has

$$\Pi_{0T}\left[\sum_{k=1}^{m} a_k\, \hat{X}_T^k\right] = \sum_{k=1}^{m} a_k\, \Pi_{0T}\left[\hat{X}_T^k\right]. \tag{17}$$

Axiom (3). $\Pi_{0T}[\,\hat{\mathbf{1}}\,] = P_{0T}$.

The axioms can be interpreted as follows. Axiom (1) ensures the absence of arbitrage: the price of a positive contract vanishes if and only if the expected payout vanishes. Axiom (2) ensures that the pricing function is linear when it acts on a collection of contracts represented by commuting observables. Axiom (3) fixes the price of the risk-free asset. Then we obtain the following general characterization of the price of a contract:

Proposition 1. *If $n \geq 3$ then there exists a state $\hat{q}$ on $\{\mathcal{H}, \hat{p}, \mathcal{V}^+\}$ that is equivalent to $\hat{p}$ such that for any contract $\hat{X}_T \in \mathcal{V}^+$ the price of $\hat{X}_T$ is given by*

$$\Pi_{0T}[\hat{X}_T] = P_{0T}\, \mathrm{tr}(\hat{q}\hat{X}_T). \tag{18}$$

Proof. It suffices to consider the pricing of A-D securities. For each such contract, the underlying measurement takes the form of a projection operator $\hat{\Lambda} = |\lambda\rangle\langle\bar{\lambda}|$ for some normalized vector $|\lambda\rangle \in \mathcal{H}$. The pricing function gives a map from the space of pure projection operators on $\mathcal{H}$ to $\mathbb{R}^+$. Now, it is well known that the space of pure projection operators on a Hilbert space of dimension n is isomorphic to the complex projective space $\mathbb{CP}^{n-1}$. Thus we obtain a real function $\Pi_{0T} : \mathbb{CP}^{n-1} \to \mathbb{R}^+$ with the property that for any set of n points $\{\lambda_j \in \mathbb{CP}^{n-1}\}_{j=1,2,\ldots,n}$ corresponding to an orthogonal basis in $\mathcal{H}$ one has

$$\sum_{j=1}^{n} \Pi_{0T}(\lambda_j) = P_{0T}. \tag{19}$$

This is because the projection operators associated with an orthonormal basis commute and hence by Axiom (2) the sum of the prices of the projection operators must equal the price of the sum of the projection operators. But the latter sum gives the identity operator, which offers a risk-free payout of unity. Thus we obtain a unit discount bond, for which the price is P_{0T} by Axiom (3). Gleason's theorem [21] can now be applied to the problem and it follows that there exists a state $\hat{q}$ such that the price of any claim of the form $\hat{\Lambda}$ is given by

$$\Pi_{0T}[\hat{\Lambda}] = P_{0T}\, \mathrm{tr}(\hat{q}\hat{\Lambda}). \tag{20}$$

Now, any contract $\hat{X}_T$ can be constructed as a linear combination of orthogonal pure projection operators with positive coefficients. Since these operators commute, Axiom (2) implies that the price of such a contract will be given by the sum of the prices of its elements, and this gives us (18). The fact that the "pricing" operator $\hat{q}$ must be equivalent to the "physical" state $\hat{p}$ follows as a consequence of Axiom (1), which ensures that the price of a positive contract $\hat{X}_T$ vanishes if and only if $\hat{X}_T$ is concentrated on the null space of $\hat{p}$. $\square$

The key point here is that we do not assume *a priori* the existence of a pricing state. The idea rather is to prove the existence of such a state under the *prima facie* much weaker assumptions implicit in our axioms. The requirement that the pricing function is linear when it is applied to any commuting family of A-D securities coupled with the assumption that the price of a one-period discount bond is known allows us to deduce that the pricing function takes the form (18). In the case of a finite-dimension Hilbert space, the associated projective Hilbert space takes the form of a complex projective space $\mathbb{CP}^{n-1}$ equipped with the Fubini-Study metric [8]. Gleason's theorem shows that any map $f : \mathbb{CP}^{n-1} \to [0,1]$ with the property that $\sum_{j=1}^{n} f(\lambda_j) = 1$ for any set of n points $\{\lambda_j\}_{j=1,2,\ldots n} \in \mathbb{CP}^{n-1}$ that are maximally distant from each other under the Fubini-Study metric necessarily takes the form $f(\lambda) = \langle\bar{\lambda}|\hat{q}|\lambda\rangle / \langle\bar{\lambda}|\lambda\rangle$ for some positive operator $\hat{q}$ with trace unity. The principle of no arbitrage ("no free lunch") then implies that $\hat{q}$ is equivalent to $\hat{p}$.

It should be noted that the physical state $\hat{p}$ refers to the state of the quantum system upon which measurement of a given physical observable determines the payment made under the terms of the financial contract. Thus $\hat{p}$ can be used to calculate the probability distribution of the payout, but gives no information about the price, except that minimal statement which is mandated by the absence of arbitrage – namely, that the price should be zero if and only if the probability of a payout greater than zero is zero.

The operator $P_{0T}\,\hat{q}$ plays the role of a pricing kernel in our theory. In the case of an n-dimensional Hilbert space the prices of any $n^2 - 1$ linearly independent financial contracts, alongside the price of the unit discount bond, will be sufficient to completely calibrate the pricing kernel, which can then be used to price other contracts. It may seem surprising that the knowledge of such a system of prices gives no information about the physical state $\hat{p}$, except to determine its null space, but the analogue of this phenomenon is well known in the classical theory of finance [7, 16–19]. At first glance, one might think that Proposition 1 is rather weak, since the pricing operator $\hat{q}$ is completely arbitrary apart from its having the same null space as $\hat{p}$; but such a conclusion would be incorrect – the point is that the existence of a pricing operator is not assumed but rather is *deduced* from the minimal axioms we have chosen to characterize a pricing function. Thus, beginning with only the assumed existence of a pricing function, which might in principle be nonlinear, one can whittle the candidates for such a map down to a linear function of the specific form (18).

III. OPTIMAL INVESTMENT

A typical problem in classical finance theory is to determine, given a budget X_0, the investment that maximizes the expectation of the utility gained by the investor at time T when the proceeds of the investment are liquidated. It is reasonable then to pose a similar problem in quantum finance. Let us assume (a) that agent A's attitudes towards risk and return can be expressed by a standard utility function $\{U(x)\}_{x>0}$, (b) that the physical state $\hat{p}$ of the quantum system is known, (c) that the basis under which the physical measurement is being made is known, and (d) that the pricing operator is known. Thus the investment will be characterized by an observable of the form (7), where the basis $\{|x_j\rangle\}_{j=1,\ldots,n}$ is fixed, and the cash flows $\{x_j\}_{j=1,\ldots,n}$ must be determined in such a way that the budget is saturated and the expected utility is maximized. What makes the problem interesting in the present setting is that the expected utility of the payout is calculated by use of the physical state $\hat{p}$ whereas the budget constraint involves the pricing state $\hat{q}$, and that neither $\hat{p}$ nor $\hat{q}$ necessarily has any special relation to the measurement basis.

Definition 1. *By a standard utility function we mean a map* $U : \mathbb{R}^+\backslash\{0\} \to \mathbb{R}$ *that satisfies the following conditions:* (i) $U \in \mathrm{C}^2(\mathbb{R}^+\backslash\{0\})$, (ii) $U'(x) > 0$ *for all* $x > 0$, (iii) $U''(x) < 0$ *for all* $x > 0$, (iv) $\lim_{x\to\infty} U'(x) = 0$, *and* (v) $\lim_{x\to 0} U'(x) = \infty$.

These requirements can be relaxed somewhat, but the "standard" conditions typically lead to well-posed problems for which solutions can be shown to exist and hence prove to be natural as a basis for modelling. Note that a standard utility function is a strictly convex, strictly increasing function defined for all strictly positive values of its argument. We refer to the function $U' : \mathbb{R}^+\backslash\{0\} \to \mathbb{R}^+\backslash\{0\}$ as the *marginal utility*. The final two conditions of the definition ensure that a standard utility function has the property that there exists an inverse marginal utility function $\{I(y)\}_{y>0}$ such that $I(U'(x)) = x$ for all $x > 0$. Examples of standard utility functions are (a) logarithmic utility, for which $U(x) = \log(x)$ for $x > 0$, and (b) power utility with index $p \in (-\infty, 1)\backslash\{0\}$, for which $U(x) = p^{-1}x^p$ for $x > 0$. For logarithmic utility one finds that $I(y) = 1/y$ and for power utility $I(y) = y^{1/(p-1)}$.

The goal of agent A's optimization problem is to determine the cash flows $\{x_j\}_{j=1,2,\ldots,n}$ that maximize the expected value of the utility, providing that these cash flows can be realized with the specified budget. Thus, given a standard utility function $\{U(x)\}_{x>0}$ we set

$$\{x_j^*\}_{j=1,2,\ldots,n} = \underset{\{x_j\}}{\mathrm{argmax}}\ \mathrm{tr}\left[\hat{p}\,\hat{U}(\{x_j\})\right] \tag{21}$$

where $\hat{U}(\{x_j\}) = \sum_{j=1}^n U(x_j)|x_j\rangle\langle\bar{x}_j|$ and the argmax is subject to the budget constraint

$$X_0 = P_{0T}\,\mathrm{tr}(\hat{q}\,\hat{X}_T), \quad \hat{X}_T = \sum_{j=1}^n x_j|x_j\rangle\langle\bar{x}_j|. \tag{22}$$

Proposition 2. *Let the physical state of a quantum system on an* n*-dimensional Hilbert space be* $\hat{p}$. *Let the pricing state for a financial market based on measurements of the system be* $\hat{q}$, *with one-period discount factor* P_{0T}. *Let the risk preferences of the investor be represented by a standard utility function* $U : \mathbb{R}^+\backslash\{0\} \to \mathbb{R}$ *and write* I *for the associated inverse marginal utility function. Then the optimal cash flow structure* $\{x_j^*\}$ *for an investment with budget* X_0 *paying out according to the measurement of a financial observable of the form*

$$\hat{X} = \sum_{j=1}^n x_j|x_j\rangle\langle\bar{x}_j|, \tag{23}$$

for some fixed orthonormal basis $\{|x_j\rangle\}_{j=1,\ldots,n}$, *is given by*

$$x_j^* = I\left[\lambda\frac{\langle\bar{x}_j|\hat{q}|x_j\rangle}{\langle\bar{x}_j|\hat{p}|x_j\rangle}\right], \tag{24}$$

where for any choice of $X_0 > 0$ *the parameter* λ *is uniquely determined by the relation*

$$P_{0T}\sum_{j=1}^n I\left[\lambda\frac{\langle\bar{x}_j|\hat{q}|x_j\rangle}{\langle\bar{x}_j|\hat{p}|x_j\rangle}\right]\langle x_j|\hat{q}|\bar{x}_j\rangle = X_0. \tag{25}$$

Proof. The method of Lagrange multipliers can be used to obtain a candidate for the argmax. We introduce a Lagrange multiplier λ and seek a solution to the unconstrained problem

$$\{x_j^*\} = \underset{\{x_j\}}{\mathrm{argmax}}\left(\mathrm{tr}\left[\hat{p}\,\hat{U}(\{x_j\})\right] - \lambda P_{0T}\,\mathrm{tr}(\hat{q}\,\hat{X}_T)\right), \tag{26}$$

or equivalently

$$\{x_j^*\} = \underset{\{x_j\}}{\operatorname{argmax}} \left(\sum_{j=1}^{n} U(x_j) \langle \bar{x}_j | \hat{p} | x_j \rangle - \lambda P_{0T} \sum_{j=1}^{n} x_j \langle \bar{x}_j | \hat{q} | x_j \rangle \right). \tag{27}$$

Differentiating with respect to x_j and setting the results to zero, we find that

$$U'(x_j) = \lambda \frac{\langle \bar{x}_j | \hat{q} | x_j \rangle}{\langle \bar{x}_j | \hat{p} | x_j \rangle} \tag{28}$$

for each value of j. Applying the inverse marginal utility function to each side of this equation, we are then led to (24) and the budget constraint (22) gives (25). That (25) admits a unique solution for λ for any $X_0 > 0$ follows from the fact that the monotonic decreasing map $I : \mathbb{R}^+ \backslash \{0\} \to \mathbb{R}^+ \backslash \{0\}$ is surjective, which is a consequence of the conditions (iv) and (v) satisfied by a standard utility function. That the candidate solution is indeed a true solution can be checked by use of the fundamental inequality $U(I(y)) - I(y)y \geq U(x) - xy$, which holds for all $x > 0$ and $y > 0$ in the case of a standard utility function. It follows then from (24) that for any *alternative* choice of payout structure $\{x_j\}$ we have

$$U(x_j^*) - x_j^* \lambda \frac{\langle \bar{x}_j | \hat{q} | x_j \rangle}{\langle \bar{x}_j | \hat{p} | x_j \rangle} \geq U(x_j) - x_j \lambda \frac{\langle \bar{x}_j | \hat{q} | x_j \rangle}{\langle \bar{x}_j | \hat{p} | x_j \rangle}, \tag{29}$$

for each $j = 1, 2, \ldots, n$. Multiplying by p_j and summing we obtain

$$\sum_{j=1}^{n} p_j U(x_j^*) - \sum_{j=1}^{n} p_j U(x_j) \geq \sum_{j=1}^{n} p_j x_j^* \lambda \frac{\langle \bar{x}_j | \hat{q} | x_j \rangle}{\langle \bar{x}_j | \hat{p} | x_j \rangle} - \sum_{j=1}^{n} p_j x_j \lambda \frac{\langle \bar{x}_j | \hat{q} | x_j \rangle}{\langle \bar{x}_j | \hat{p} | x_j \rangle}. \tag{30}$$

Then since $p_j = \langle x_j | \hat{p} | \bar{x}_j \rangle$ we have

$$\sum_{j=1}^{n} p_j U(x_j^*) - \sum_{j=1}^{n} p_j U(x_j) \geq \lambda \left[\sum_{j=1}^{n} x_j^* \langle \bar{x}_j | \hat{q} | x_j \rangle - \sum_{j=1}^{n} x_j \langle \bar{x}_j | \hat{q} | x_j \rangle \right]. \tag{31}$$

Now, we know by (25) that λ has been chosen to ensure that the candidate solution $\{x_j^*\}$ satisfies the budget constraint

$$P_{0T} \sum_{j=1}^{n} x_j^* \langle \bar{x}_j | \hat{q} | x_j \rangle = X_0. \tag{32}$$

If we require that the alternative choice of payout structure should also satisfy the budget constraint, or perhaps operate under budget, so

$$P_{0T} \sum_{j=1}^{n} x_j \langle \bar{x}_j | \hat{q} | x_j \rangle \leq X_0, \tag{33}$$

then the two terms on the right-hand side of (31) cancel, or else leave a difference that is positive (if the alternative choice is under budget), which gives

$$\sum_{j=1}^{n} p_j U(x_j^*) \geq \sum_{j=1}^{n} p_j U(x_j), \tag{34}$$

showing that the candidate solution for the optimal payout gives an expected utility that is no less than that of any alternative choice of payout structure with a budget no greater than that of the candidate solution. $\square$

IV. RATE OF RETURN

As an example, we can look in detail at the case of logarithmic utility. Suppose we set $U(x) = \log x$ for $x > 0$. Then the inverse marginal utility function is given by $I(y) = 1/y$ for $y > 0$. It follows that for log utility the optimal payout structure takes the form

$$x_j^* = \lambda^{-1}\frac{\langle \bar{x}_j|\hat{p}|x_j\rangle}{\langle \bar{x}_j|\hat{q}|x_j\rangle}. \tag{35}$$

Inserting this expression into the budget constraint (32) we obtain

$$P_{0T}\lambda^{-1}\sum_{j=1}^{n}\langle \bar{x}_j|\hat{p}|x_j\rangle = X_0. \tag{36}$$

But the sum appearing in the expression above is unity since $\sum_{j=1}^{n}|x_j\rangle\langle\bar{x}_j| = \hat{\mathbf{1}}$ and the trace of $\hat{p}$ is one. Thus for log utility we deduce that $P_{0T}\lambda^{-1} = X_0$ and hence

$$x_j^* = (P_{0T})^{-1}X_0\frac{\langle \bar{x}_j|\hat{p}|x_j\rangle}{\langle \bar{x}_j|\hat{q}|x_j\rangle}. \tag{37}$$

We observe that when the physical state and the pricing state are one and the same, the payouts of the optimal investment are identical for each outcome of chance, each giving $(P_{0T})^{-1}X_0$, the usual "future value" of the initial investment. In that case, the optimal investment is to put the initial endowment into unit discount bonds, totalling X_0 in value. Then we have $\hat{X}_T = (P_{0T})^{-1}X_0\,\hat{\mathbf{1}}$ and one sees that if the pricing state is the physical state, the market assigns no premium to the return on a risky investment, ensuring that the optimal investment is in a discount bond and the rate of return is the interest rate.

The same conclusion applies, more generally, for any choice of the utility. This follows from (24) and (25), from which one concludes that if $\hat{p} = \hat{q}$ then $x_j^* = (P_{0T})^{-1}X_0$ for all j. It is interesting therefore to enquire what happens when the pricing state is different from the physical state. The return R_{0T} on an investment $\hat{X}_T$ is given by the ratio of the expectation of $\hat{X}_T$ under $\hat{p}$ to the amount initially invested, namely X_0. Thus, quite generally, we have

$$R_{0T} = (X_0)^{-1}\mathrm{tr}(\hat{p}\hat{X}_T). \tag{38}$$

But $X_0 = P_{0T}\,\mathrm{tr}(\hat{q}\hat{X}_T)$ by (18), so we deduce that

$$R_{0T} = (P_{0T})^{-1}\frac{\mathrm{tr}(\hat{p}\hat{X}_T)}{\mathrm{tr}(\hat{q}\hat{X}_T)}, \tag{39}$$

and it should be clear that if $\hat{p} = \hat{q}$, except possibly on the null space of $\hat{X}_T$, then the rate of return on the investment is the one-period interest rate.

Specializing now to the case of an optimal investment for an agent with logarithmic utility, let us calculate the rate of return. We have

$$\hat{X}_T = \sum_{j=1}^{n} x_j^*\,|x_j\rangle\langle\bar{x}_j|, \tag{40}$$

where the optimal payout structure $\{x_j^*\}$ is given by (37). It follows then that

$$\begin{aligned} R_{0T} &= (X_0)^{-1}\mathrm{tr}(\hat{p}\hat{X}_T) \\ &= (X_0)^{-1}\mathrm{tr}\left(\hat{p}\sum_{j=1}^{n} x_j^*\,|x_j\rangle\langle\bar{x}_j|\right) \\ &= (X_0)^{-1}\sum_{j=1}^{n} x_j^*\,\langle\bar{x}_j|\hat{p}|x_j\rangle \\ &= (P_{0T})^{-1}\sum_{j=1}^{n}\frac{\langle\bar{x}_j|\hat{p}|x_j\rangle^2}{\langle\bar{x}_j|\hat{q}|x_j\rangle}. \end{aligned} \tag{41}$$

If we set $R_{0T} = \mathrm{e}^{\mu T}$ then the rate of return μ can be split into two parts, namely a risk-free one-period interest rate and a so-called excess rate of return, which is the part of the rate of return that exceeds the interest rate. We can represent this in a standard way by writing $R_{0T} = \mathrm{e}^{(r+\beta)T}$ where r is the interest rate and β denotes the excess rate of return. The interest rate is determined by the relation $\mathrm{e}^{rT} = (P_{0T})^{-1}$ and the excess rate of return is determined by the relation

$$\mathrm{e}^{\beta T} = \sum_{j=1}^{n}\frac{\langle\bar{x}_j|\hat{p}|x_j\rangle^2}{\langle\bar{x}_j|\hat{q}|x_j\rangle}. \tag{42}$$

Proposition 3. *The optimal investment in the case of an investor with logarithmic utility has a positive excess rate of return. The utility gained from such an investment in a market where the physical state and pricing state differ is greater than or equal to the utility gained from an investment in a risk-free bond.*

Proof. For general utility, the expected utility gained from the payout of an optimal investment is given by

$$\mathrm{tr}\left[\hat{p}\,\hat{U}\right] = \sum_{j=1}^{n} U(x_j^*)\langle\bar{x}_j|\hat{p}|x_j\rangle. \tag{43}$$

Let us set $U(x_j^*) = \log x_j^*$ for logarithmic utility and insert (37). The result is

$$\sum_{j=1}^{n} U(x_j^*)\langle\bar{x}_j|\hat{p}|x_j\rangle = \log\left[(P_{0T})^{-1}X_0\right] + \sum_{j=1}^{n}\left[\langle\bar{x}_j|\hat{p}|x_j\rangle\log\frac{\langle\bar{x}_j|\hat{p}|x_j\rangle}{\langle\bar{x}_j|\hat{q}|x_j\rangle}\right]. \tag{44}$$

The first term on the right-hand side of this equation isolates the part of the utility gain due to the interest rate. The second term can be interpreted as a *relative entropy*. In particular, if we set $p_j = \langle\bar{x}_j|\hat{p}|x_j\rangle$ and $q_j = \langle\bar{x}_j|\hat{q}|x_j\rangle$ then it is evident that $\{p_j\}_{j=1,2,\ldots,n}$ and $\{q_j\}_{j=1,2,\ldots,n}$ constitute a pair of absolutely continuous probability distributions. The second term on the right then takes the form of a Kullback-Liebler divergence [35]:

$$D_{KL}(p,q) = \sum_{j=1}^{n} p_j\log\left(\frac{p_j}{q_j}\right). \tag{45}$$

Thus, the utility thus gained gives a measure of the divergence between the physical density matrix and the pricing density matrix. Now, it is well known that the Kullback-Liebler divergence is *non-negative*. It follows, then, that the utility gained from an optimal risky investment in a market where $\hat{p}$ and $\hat{q}$ are distinct will be greater than or equal to the utility gained from a risk-free bond investment, as claimed. Moreover, we have the following. The standard logarithmic inequality $\log z \leq z - 1$, which holds for $z > 0$, implies that

$$\log\left(\frac{p_j}{q_j}\right) \leq \left(\frac{p_j}{q_j}\right) - 1 \tag{46}$$

for each j. Hence, multiplying by p_j and summing we obtain

$$\sum_{j=1}^{n} p_j \log\left(\frac{p_j}{q_j}\right) \leq \sum_{j=1}^{n} \left(\frac{p_j^2}{q_j}\right) - 1. \tag{47}$$

Thus, we have

$$\mathrm{e}^{\beta T} = \sum_{j=1}^{n} \left(\frac{p_j^2}{q_j}\right) \geq 1 + D_{KL}(p, q), \tag{48}$$

and by the positivity of the Kullback-Liebler divergence we deduce that the excess rate of return β is positive for an optimal investment under logarithmic utility, as claimed. □

V. KOLMOGOROV VS BELL-KOCHEN-SPECKER

It is often maintained by physicists that quantum probability is more general than Kolmogorov's well-established "classical" theory of probability [33] and that the latter is contained as a special case of the former. There is no doubt that quantum probability, when laid out as a mathematical theory, does have a different look and feel when it is compared to Kolmogorov's theory; but despite the fact that numerous well-argued accounts of quantum probability can be found in the literature [13, 14, 22, 25, 34, 36, 42, 43] (see also [3, 14, 45]), some even taking an axiomatic approach, it is not that easy to pinpoint the exact sense in which quantum theory is *essentially* non-Kolmogorovian – rather than, say, merely an intricate reworking of Kolmogorov's theory in a radically different form. This issue is compounded by the fact that, except in the most loose terms, it is difficult to say what one means by "probability" without embedding the concept in a rigorous framework; and it seems that there are at least two such frameworks available to us at the moment – namely, classical probability and quantum probability, each with a host of applications.

It is fortunate then that we have the results of Gleason [21], Bell [4–6], Kochen & Specker [24, 32], and others following in their footsteps, which add clarity to the matter. The point is that one has to work rather hard to come up with examples of situations in quantum probability that cannot be reduced to a classical probability model. But a number of such examples have been worked out involving finite-dimensional Hilbert spaces, so this creates the prospect of constructing financial models for claims based on the results of quantum measurements, in settings for which quantum probability is required in their analysis.

Since most of what we know of modern finance theory is based quite explicitly on Kolmogorov's framework, both for the derivation of rigorous theorems concerning the mathematics of financial markets as well as for the implementation of the large-scale computer models used for trading and risk management at financial institutions, it may be worthwhile to take note of a few examples of situations where quantum probability comes into play.

Among the numerous attempts that have been made to generalize or extend the Kochen-Specker construction [11, 12, 29, 30, 37–39]. Perhaps the simplest of these yet put forward is that of Cabello et al, which entails the specification of nine different non-commuting observables on a four-dimensional Hilbert space, each admitting four distinct eigenvalues. In a financial context, one can think of the setup as involving a single quantum system being prepared in a state $\hat{p}$ with nine different "draft financial contracts" drawn up, each requiring measurement of one of the nine observables. The contracts specify the payments that will be made in the event that one of the four possible results occurs for the measurement associated with a specific contract. It is of the nature of quantum probability that only one of the nine contracts can be implemented, so we can envisage a rational investor being presented with the alternatives and choosing one optimally in accordance with their needs.

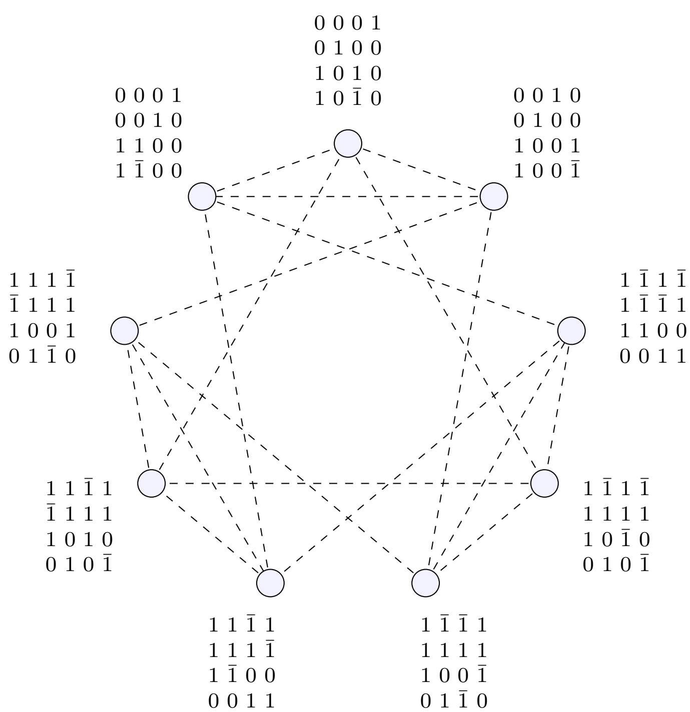

Figure 1: Diagram illustrating a result of the Kochen-Specker type in a 4-dimensional Hilbert space. Each of the 9 vertices are met by 4 lines and each of the 18 lines join 2 vertices. The 18 lines represent a set of normalized projection operators with the property that the 4 projection operators meeting a given vertex are mutually orthogonal and sum to the identity operator. It is easy to see that it is impossible to "colour" the lines so that one blue line meets each vertex and 3 red lines meet each vertex. This illustrates the fact that in the standard Kolmogorov setup one cannot find a set of 18 random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, each taking values in the set $\{0, 1\}$, such that when the 18 random variables are assigned to the 18 lines, the sum of the 4 random variables meeting any given vertex will be one for all $\omega \in \Omega$.

The setup is an elaborate although feasible one, and we can use the methods discussed to calculate the probabilities of the results for the nine different measurements and hence the expected utility gained from each choice. Each observable has four possible outcomes, thus determining an orthonormal tetrad in Hilbert space. These are the four eigenvectors of the Hermitian matrix corresponding to a given observable. The result of the measurement is to select one of these eigenvectors. Equivalently, each measurement measures four commuting projection operators, namely the projection operators associated with the four legs of the tetrad. The outcome of one of these four measurements will be unity and the rest nil.

The clever idea behind results of the Kochen-Specker type is to choose the observables so that some of the tetrads legs overlap when one moves from one observable to another. In the present situation, involving nine observables, the overlap structure is shown in Figure 1. Alongside each vertex of the enneagon one sees the corresponding tetrad, where to ease the typography we write $\bar{1}$ for -1. When two vertices are connected by a dotted line, this means that the associated tetrads share a vector in common. The analysis is simplified somewhat by the fact that the tetrads in this example can all be taken to be real.

If we label the nine observables $\{\hat{X}_r\}_{r=1,2,\ldots,9}$ and if for each value of r the four projection operators associated with $\hat{X}_r$ are denoted $\{\hat{\pi}_{rj}\}_{j=1,2,3,4}$, then the probability that outcome j will result, if contract r is chosen, is given by $\mathrm{tr}(\hat{\pi}_{rj}\,\hat{p})$. The construction of an analogous setup within Kolmogorov's system turns out to be impossible. Since this is a rather sweeping statement, let us be a little more precise about what is being claimed. The point is that in Kolmogorov's theory, one would have to model the setup with 36 random variables on a single probability space. The 36 random variables are grouped into nine sets of four. Let's call these hypothetical random variables $\{X_{rj}\}_{r=1,2,\ldots,9,\,j=1,2,3,4}$ (with no hats). Each random variable can take the value zero or one. Thus we have a total of 36 maps of the form

$$X_{rj} : \Omega \to \{0, 1\}. \tag{49}$$

There are two requirements that have to be satisfied to match the layout of the quantum setup. First, the sum of the four random variables for a given value of r must be unity. This means that one of them must be equal to one and the other three must be equal to zero for any given outcome of chance $\omega \in \Omega$. Secondly (this is where the rabbit goes into the hat) the 36 random variables have to be equal in pairs, in conformation with the structure of the diagram in Figure 1. Thus, the 36 random variables are cut down in effect to 18 by the requirement that they must match in pairs.

Can one find such a set of 18 random variables? The answer, perhaps surprisingly, is no. This can be checked by a colouring argument. Given Figure 1, can one colour each line red or blue in such a way that exactly one blue line meets each vertex? Suppose one finds a way of colouring four of the lines blue, no vertex being hit by more than one blue line. That would leave one vertex unmet by a blue line. Suppose then one tried to colour five lines blue. Well, that would mean at least one vertex was hit by more than one blue line. This shows that it is impossible to construct a set of 18 random variables on a probability space in such a way that the required properties are satisfied.

In financial terms, this means that we cannot model the payouts of the nine contracts as random variables on a probability space in such a way that the outcome of chance determines the payouts of all nine. A sceptic might ask, "Isn't it unlikely in practice that one will come up against such a configuration of contracts?" Well, that may be so, but the point is that quantum finance can handle such configurations whereas classical finance can't.

VI. PORTFOLIOS AS MULTI-PARTICLE SYSTEMS

Let's return now to the matter of portfolios. There are two rather distinct notions of portfolio that arise in quantum finance. The first notion involves a portfolio of contracts all depending in their payouts on the same experiment. In that case, we can fix the n axes of the n-dimensional Hilbert space determining the frame of the measurement and write $\{\hat{\pi}_j\}_{j=1,2,...n}$ for the associated projection operators. Then for a given outcome of the experiment one of these projection operators will give the result unity and the rest zero. The projection operators can be regarded as the A-D securities for that experiment and it should be evident that any contingent claim based on the outcome of the given experiment can be written as a portfolio of n such A-D securities. Thus, for such claims we can write

$$\hat{X} = \sum_{j=1}^{n} \theta_j \, \hat{\pi}_j, \tag{50}$$

where the $\{\theta_j\}_{j=1,2,...n}$ represent the holdings in the various A-D securities. More generally, if we allow short positions in the A-D securities, then the resulting overall position can be expressed uniquely as the difference between two positive claims, with the understanding that we net claims involving long and short positions in the same A-D security.

Clearly, a linear combination of two portfolios in this setting gives another portfolio. Furthermore, it should be evident that the operator corresponding to the portfolio can be represented as the sum of a trace part, proportional to the identity operator, and a trace-free part. The trace part represents a position (long or short) in the risk-free asset, and the remainder consists of investments in risky assets. For example, in two dimensions, a portfolio of the form $2|z_1\rangle\langle\bar{z}_1|+|z_2\rangle\langle\bar{z}_2|$ consists of a long position of three-halves of one unit of the risk free asset, a long position of one-half of a unit in the A-D security $\hat{\pi}_1$ and a short position of one-half of a unit in the A-D security $\hat{\pi}_2$, since we have $2\hat{\pi}_1 + \hat{\pi}_2 = \frac{3}{2}(\hat{\pi}_1 + \hat{\pi}_2) + \frac{1}{2}(\hat{\pi}_1 - \hat{\pi}_2)$. In this way we can isolate the risk-free part of a portfolio. This first notion of a portfolio corresponds rather closely to the notion of a portfolio in a one-period market that arises in classical finance theory [1, 7, 16–19] and can be pursued further in that spirit. The point is that once the measurement basis for the underlying experiment has been fixed, the various associated operators arising for positions with different portfolio weightings commute.

As we pointed out in Section II, it does not make sense to form a portfolio of several contracts each based on the same quantum system but with different measurement frames, since such measurements will in general be incompatible and cannot be simultaneously realized. In this respect, our approach differs completely from that of reference [2] who attempt to set up a portfolio theory on exactly that basis, where all of the assets are associated with the same Hilbert space. This leads us to conclude their main result (an analogue of the fundamental theorem of asset pricing) is in some respects ill-posed. The authors of [2] give no clear statement (let alone a definition) of what is meant in real terms by a "quantum asset" and merely assume the existence of a pricing state rather than deducing its existence. Nonetheless, that said, there is some interesting overlap with the present work.

In our approach to the problem we consider portfolios of assets for which the payouts are based on separate measurements being made on two or more distinct quantum systems. Imagine, for example, a financial institution where in one room an experiment is carried out on Quantum System I, with certain results obtained, and another experiment is carried out in another room on Quantum System II, with certain results obtained. In each case, there are contracts leading to payouts depending on the results obtained.

Since the measurements do not interfere with one another (after all, they are carried out in different rooms) they can be carried out simultaneously, each delivering a certain number of units of account, so it makes sense to speak of holding a portfolio in the two assets, for which the payout is simply the totality of the payouts of the constituents of the portfolio, with appropriate weightings.

Let's see how we model such a situation. To simplify the discussion, we stick with the case where there are just two quantum systems involved, with measurements made on each of them. The setup can be easily generalized to the case where there are N such systems. The key idea is that to model a portfolio of two such quantum assets, we need to consider the tensor product of the Hilbert spaces of the individual systems. In fact, the two Hilbert spaces might even be of different dimensions.

The usual Dirac notation does not hold up so well in such a setting, so we use an *index notation* instead, which works quite smoothly [8, 20]. Thus, let $\mathcal{H}_1$ be a Hilbert space of dimension n and let $\mathcal{H}_2$ be a Hilbert space of dimension n', where n and n' are not necessarily the same. We write ξ^a and $\xi^{a'}$ for typical elements of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, where $a = 1, 2, \dots n$ and $a' = 1, 2, \dots n'$. Thus indices without dashes refer to the first Hilbert space and indices with dashes refer to the second Hilbert space. We write η_a and $\eta_{a'}$ for typical elements of the corresponding dual spaces $\mathcal{H}_1^*$ and $\mathcal{H}_2^*$. The complex conjugates of ξ^a and $\xi^{a'}$ are denoted $\bar{\xi}_a$ and $\bar{\xi}_{a'}$ respectively. Then for the inner product between ξ^a and η_a we write $\xi^a \eta_a$ and for the inner product between $\xi^{a'}$ and $\eta_{a'}$ we write $\xi^{a'} \eta_{a'}$, with the usual summation convention.

We are interested in the product Hilbert space $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and we write $\xi^{aa'} \in \mathcal{H}_{12}$ for a typical element of this space. Then we write $\eta_{aa'}$ for a typical element of $\mathcal{H}_{12}^*$ and $\bar{\xi}_{aa'}$ for the complex conjugate of $\xi^{aa'}$, and for the inner product of $\xi^{aa'}$ and $\eta_{aa'}$ we write $\xi^{aa'} \eta_{aa'}$. The state of a two-particle system takes the form of a density matrix $p^{aa'}_{bb'}$. Thus we require that it should be Hermitian, of unit trace, and positive, so

$$p^{aa'}_{bb'} = \bar{p}^{aa'}_{bb'}, \quad p^{cc'}_{cc'} = 1, \quad p^{aa'}_{bb'} \alpha^b \bar{\alpha}_a \beta^{b'} \bar{\beta}_{a'} \geq 0 \tag{51}$$

for all $\alpha^a, \beta^{a'}$. A two-particle density matrix is *pure* if $p^{aa'}_{bb'} = \xi^{aa'} \bar{\xi}_{bb'}$ for some state vector $\xi^{aa'}$. We say that the particles are *independent* if

$$p^{aa'}_{bb'} = p^a_b \, p^{a'}_{b'} \tag{52}$$

for some pair of one-particle states p^a_b and $p^{a'}_{b'}$. The state is said to be *separable* if it can be written in the form

$$p^{aa'}_{bb'} = \sum_{r=1}^{k} p^a_b(r) \, p^{a'}_{b'}(r), \tag{53}$$

for some collection of $2k$ one-particle states $\{p^a_b(r)\}_{r=1,2,\dots k}$ and $\{p^{a'}_{b'}(r)\}_{r=1,2,\dots k}$. But if the two-particle state is not separable then we say that the particles are *entangled*.

Now we are in a position to discuss the idea of measurements on a two-particle system and the contracts one can associate with such measurements. A generic contract based on the outcome of a measurement made on a two-particle system is described by a Hermitian operator $X^{aa'}_{bb'}$. We are interested in the case when the measurement splits into a measurement on System I and a measurement on System II and one adds the results. Such a contract takes the form

$$X^{aa'}_{bb'} = U^a_b \delta^{a'}_{b'} + \delta^a_b V^{a'}_{b'}, \tag{54}$$

where δ^a_b and $\delta^{a'}_{b'}$ denote the identity operators on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. The eigenstates of such an operator are of the form

$$p^{aa'}_{bb'} = \alpha^a \bar{\alpha}_b \, \beta^{a'} \bar{\beta}_{b'}, \tag{55}$$

where α^a is an eigenvector of U^a_b and $\beta^{a'}$ is an eigenvector of $V^{a'}_{b'}$. Thus $U^a_b \alpha^b = u\alpha^a$ and $V^{a'}_{b'} \beta^{b'} = v\beta^{a'}$ for some $u, v \in \mathbb{R}^+ \backslash \{0\}$ and the sum $u + v$ gives the overall outcome of the measurement. Such a contract can be thought of as representing a portfolio consisting of one unit of a contract based on System I and one unit of a contract based on System II. More generally, for a portfolio consisting of θ_1 units of the first contract and θ_2 units of the second contract we have

$$X^{aa'}_{bb'}(\theta_1, \theta_2) = \theta_1 \, U^a_b \delta^{a'}_{b'} + \theta_2 \, \delta^a_b V^{a'}_{b'} \tag{56}$$

and the payout will be of the form $\theta_1 u + \theta_2 v$. The setup for a portfolio of arbitrary size can be constructed analogously. In particular, one can check that the expected payout of a portfolio is equal to the sum of the expectations of the constituents. This is because whenever the density matrix of the two-particle state hits one of the identity operators in the portfolio operator, all but one of the systems gets traced out and one is left with the trace of the product of a single particle density operator and the observable associated with that system. For example, in the case of a two-particle system one finds that

$$\begin{aligned} p^{bb'}_{aa'} X^{aa'}_{bb'}(\theta_1, \theta_2) &= p^{bb'}_{aa'} (\theta_1 \, U^a_b \delta^{a'}_{b'} + \theta_2 \, \delta^a_b V^{a'}_{b'}) \\ &= \theta_1 \, p^{bb'}_{aa'} \, U^a_b \delta^{a'}_{b'} + \theta_2 \, p^{bb'}_{aa'} \delta^a_b V^{a'}_{b'} \\ &= \theta_1 \, p^b_a \, U^a_b + \theta_2 \, p^{b'}_{a'} V^{a'}_{b'}, \end{aligned} \tag{57}$$

where $p^b_a = p^{bc'}_{ac'}$ and $p^{b'}_{a'} = p^{cb'}_{ca'}$. Likewise one can check that the price of a portfolio is equal to the weighted sum of the prices of its constituents. This may not be obvious, but the point is that the two-particle system is itself a quantum system with a financial observable based on it, of the form (54), so by Proposition 1 there exists a pricing operator $q^{bb'}_{aa'}$ such that

$$q^{bb'}_{aa'} X^{aa'}_{bb'}(\theta_1, \theta_2) = \theta_1 \, q^b_a \, U^a_b + \theta_2 \, q^{b'}_{a'} V^{a'}_{b'}, \tag{58}$$

where the traced-out operators $q^b_a = q^{bc'}_{ac'}$ and $q^{b'}_{a'} = q^{cb'}_{ca'}$ are the pricing operators associated with the respective individual systems.

There is one further aspect of the portfolio problem that can be analyzed and this concerns the matter of correlations. If the state of the two-particle system is of the form (52), so the two particles are independent, then the outcomes of the experiments on the two systems will be uncorrelated. But if the systems are entangled, then the correlation will in general be non-vanishing, leading to relations such as

$$p^{bb'}_{aa'} \left[U^a_b - \delta^a_b \, p^d_c \, U^c_d \right] \left[V^{a'}_{b'} - \delta^{a'}_{b'} \, p^{d'}_{c'} \, V^{c'}_{d'} \right] \neq 0. \tag{59}$$

The point about entanglement is that even if the two systems are in separate rooms (or even different cities) the outcomes may be correlated, owing to the original construction of the state of the two-particle system to which they belong. The same is true of the prices: if $q^{aa'}_{bb'}$ is entangled, then there will be correlations in the prices, as shown in relations such as

$$q^{bb'}_{aa'} \left[U^a_b - \delta^a_b \, q^d_c \, U^c_d \right] \left[V^{a'}_{b'} - \delta^{a'}_{b'} \, q^{d'}_{c'} \, V^{c'}_{d'} \right] \neq 0. \tag{60}$$

Thus, in the general situation we see that when there is a market based on contracts associated with measurements being made on a number of different quantum systems, there will be correlations between outcomes of measurements and correlations between prices, where the former are determined by the structure of physical density operator for the market as a whole and the latter by the structure of the pricing operator for the market as a whole.

The physical density operator is objective in nature, the only limitations in its determination being in the usual practicalities of the laboratory settings where the states are manufactured. The pricing operator, on the other hand, if classical finance theory is any guide in the matter [1, 7, 16–19], will be determined by the collective appetite for risk and reward among market participants and also perhaps by supply and demand related considerations. Hence, as in all markets, prices will be subject to fluctuation and change over time and may even be amenable to a Bayesian treatment. In the one-period setting that we have investigated here, the pricing operator simply is what it is, and all we can say of a definite nature about it is that it exists and that the pricing operator and the physical density operator are equivalent, as we have seen in Proposition 1.

In the one-period version of the theory we have presented, one can be agnostic on the matter of dynamics. This is because $\hat{p}$, $\hat{q}$ and P_{0T} are specified at time 0 and no further data are required. By the time one reaches T, however, the physical state will have evolved from the initial state $\hat{p}$ to a new state $\hat{p}_T$ given by

$$\hat{p}_T = \mathrm{e}^{\mathrm{i}\hat{H}T}\,\hat{p}\,\mathrm{e}^{-\mathrm{i}\hat{H}T}, \tag{61}$$

where $\hat{H}$ denotes the Hamiltonian, assuming that there is no physical interaction with the environment. This new state can then be used for working out the probabilities of outcomes for the next period in a multi-period setting. One might conjecture that $\hat{q}$ will undergo a similar unitary transformation, in the absence of any Bayesian updating. This will ensure that the physical state and the pricing state will continue to share the same null space. But one can also arrive at this conclusion if one works in the Heisenberg representation from the outset, in which case the sole function of the unitary operators is to rotate the measurement frames for each observable, leaving $\hat{p}$ and $\hat{q}$ fixed, in the absence of external interventions or state changes arising as a consequence of measurements having been made.

That the non-Kolmogorovian character of quantum probability may have implications for the development of quantum technologies is widely appreciated. See, e.g., [26] and references cited therein. And indeed, if quantum computers eventually replace the classical computers currently used for algorithmic trading by financial institutions, as they no doubt will, then the role of valuations of the type we have considered here may be important. There is also a widely held view that quantum probability may play a part in cognitive science and hence behavioural finance as well. See, e.g., [10, 23, 31, 40, 41, 46] and references cited therein.

It would be out of the scope of the present discussion to look at either of these proposals in any detail here, but on the latter point one could well imagine that if judgements and decisions are made on the basis of quantum probability then in some situations these assessments would involve *valuations*, rather than probability estimates, and it would be the pricing operator, rather than the physical density operator, that would come into play in making these valuations. In such cases, external intervention in the form of Bayesian updating could be modelled, for example, as in reference [9]. This is consistent with the point we made earlier about the pricing operator being specific to the risk and reward profiles of market operatives and in a state of flux as new information arrives. These and other further developments of the theory we hope to explore elsewhere.

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