Chapter 1

Discrete-Time Signals

Digital signal processing (DSP) usually refers to the analysis and modification of digital signals (which are generated by digitizing real-world analog signals). While digital signals generally refer to discrete-time, discrete-valued signals, technological advances in computing allows us to store data with increasingly greater precision. The most common floating-point formats are single-precision (32-bit) and double-precision (64-bit).

Because of the large precision of current computer number formats, the term digital signal processing effectively can be applied to *discrete-time signals* (which can have infinite precision). In fact, the modern usage of the term *digital signal processing* tends to be interchangeable with the outdated term *discrete-time signal processing*.

In the context of DSP, the distinction between discrete-time signals and digital signals only becomes prevalent when the rounding errors become non-neglibile.

For the remainder of this text, we will only be looking into signals with the following characteristics:

- single-channel, one-dimensional

- discrete-time (DT)

- deterministic

1.1 Discrete-Time Signal Representation

Since a sampler samples a continuous-time (CT) signal every T_s seconds, the relationship between discrete-time (DT) signals (also called discrete signals) and CT signals is
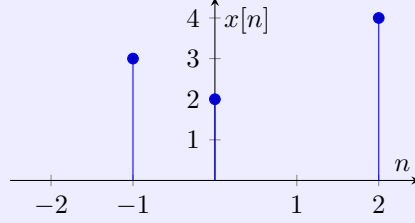
$$x[n] = x(t = nT_s). \tag{1.1}$$

Instead of dealing with the points of time t of a CT signal, the indices n are considered for a DT signal. $x[n]$ then is essentially a sequence, which is a special form of a function with an integer domain, where $x[N]$ represents the Nth term of the sequence. In discrete time, n can be referred to as "time," with units of number of samples.

Other than piecewise notation, DT signals can be written in bracket notation, where the value

corresponding to $x[0]$ is underlined, and any values with indices outside of the brackets are implicitly zero.

$$x[n] = \Big\{x[-N], ..., x[-1], \underline{x[0]}, x[1], ..., x[M]\Big\} \tag{1.2}$$

Example 1.1.1. Given the plot of $x[n]$ below, write an expression for $x[n]$.

SOLUTION

The expressions for $x[n]$ in both bracket notation and piecewise notation are provided:

$$x[n] = \{3, \underline{2}, 0, 4\} = \begin{cases} 3, & n = -1 \\ 2, & n = 0 \\ 4, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

1.2 Signal Transformations

Similar to the CT case, a DT signal can be modified by apply an *affine transformation on the independent variable* such that $x[n] \mapsto x[an + b]$.

1.2.1 Time Shifting

A DT signal that is time-shifted by N units can be expressed as

$$y[n] = x[n - N]. \tag{1.3}$$

If $N > 0$, then $y[n]$ is said to be delayed by N units relative to $x[n]$. If $N < 0$, then $y[n]$ is advanced by N units.

1.2.2 Time Scaling

A CT signal that is time-scaled by some integer factor a can be expressed as

$$y[n] = x[an]. \tag{1.4}$$

If $|a| > 1$, then $y[n]$ is a temporally compressed (or *downsampled*) version of $x[n]$, which keeps every a^{th} sample of $x[n]$. If $|a| < 1$, then $y(t)$ is a temporally expanded (or *upsampled*) version of $x[n]$, which stuffs $a - 1$ zeros between each sample of $x[n]$.

1.2.3 Time Reversal

A time reversal is a reflection of some signal over the vertical axis and can be expressed as

$$y[n] = x[-n]. \tag{1.5}$$

1.2.4 Combined Transformation

Let

$$y[n] = x[an - b] = x\left[a\left(n - \frac{b}{a}\right)\right]. \tag{1.6}$$

Since b/a is not guaranteed to be an integer, there is only one surefire approach to transforming the signal $x[n]$ to $y[n]$.

1. Shift by b units.
2. Time scale by $|a|$. Then reflect if $a < 0$.

1.2.5 Amplitude Transformation

Some applications may call for vertically scaling the amplitude of a DT signal or vertically shifting the signal. The amplitude transformation

$$y[n] = Ax[n] + B \tag{1.7}$$

scales the signal $x[n]$ by factor A and then vertically shifts the signal by B units.

1.3 Discrete-Time Sequences

1.3.1 Discrete-Time Exponential and Sinusoidal Signals

Complex exponential sequences take on the form

$$x[n] = Ae^{j\Omega n} = A\exp(j\Omega n), \tag{1.8}$$

with some complex scalar $A = |A|e^{j\theta}$ and *discrete-time angular frequency* Ω. In fact, by sampling the CT complex exponential function every T_s seconds, the DT complex exponential sequence can also be written as

$$x[n] = x(t = nT_s) = A\exp(j\omega_0(nT_s)) = A\exp(j\Omega n), \tag{1.9}$$

where $\Omega = \omega_0 T_s$ [rad/samp]. Unlike the CT form, the DT complex exponential sequence may or may not be periodic; in order for the DT sequence to be periodic, $2\pi/\Omega$ must be a rational number. In fact, if the DT signal is periodic, then

$$\frac{2\pi}{\Omega} = \frac{N_0}{k}, \tag{1.10}$$

where N_0 is the *discrete-time fundamental period*, and k is the smallest integer such that N_0 is an integer. If given the CT complex exponential signal with period T_0 and the sampling period T_s, the test for periodicity can be done using the following relation:

$$\frac{2\pi}{\Omega} = \frac{T_0}{T_s}. \tag{1.11}$$

If the DT complex exponential sequence is periodic, then $\Omega \triangleq \Omega_0 \in [0, \pi]$ is the *discrete-time funda-mental angular frequency*.

The real and imaginary parts of the complex exponential sequence can be isolated to obtain *discrete-time sinusoidal sequences*:

$$\text{Re}(Ae^{j\Omega n}) = A\cos(\Omega n) = |A|\cos(\Omega n + \theta) \tag{1.12}$$

$$\text{Im}(Ae^{j\Omega n}) = A\sin(\Omega n) = |A|\sin(\Omega n + \theta) \tag{1.13}$$

Similar to the DT complex exponential sequence, the DT sinusoidal sequences are not necessarily periodic and can only be periodic if and only if

$$\frac{2\pi}{\Omega} = \frac{T_0}{T_s} = \frac{N_0}{k} \tag{1.14}$$

yields a rational number. If periodic, the reciprocal of this ratio is called the *fundamental normalized frequency*, defined as $r_0 = f_0/f_s$.

1.3.2 Discrete-Time Counterparts of Singularity Signals

Since the concept of continuity does not apply in discrete time, sequences are limited to finite am-plitudes, and singularity points do not exist. For this reason, we will collectively refer to the DT counterparts of CT singularity signals as *discretized singularity signals*, which emphasizes the fact that the CT singularity signals are being sampled and quantized to a finite amplitude.

The *unit impulse sequence*, also called the *Kronecker delta sequence* or the *unit sample sequence*, is defined as

$$\delta[n - N] = \delta_{n,N} = \begin{cases} 1, & n = N \\ 0, & \text{otherwise} \end{cases}, \tag{1.15}$$

$$\text{with} \sum_{n=-\infty}^{+\infty} \delta[n - N] = 1. \tag{1.16}$$

Interestingly, when time-scaled by a factor a,

$$\delta[an] = \delta[n]. \tag{1.17}$$

The impulse sequence also has a *sampling property*, also called *sifting property*:

$$\sum_{n=-\infty}^{+\infty} x[n]\delta[n - N] = x[N], \tag{1.18}$$

$$x[n]\delta[n - N] = x[N]\delta[n - N]. \tag{1.19}$$

The cumulative summation of the unit impulse sequence is the *unit step sequence*, also called the *Heaviside step sequence*, and is defined as

$$u[n - N] = \begin{cases} 0, & n < N \\ 1, & n \geq N \end{cases}. \tag{1.20}$$

Similar to the CT case, the unit step sequence can be treated as an "on switch", where at time $n = N$, the signal being multiplied gets turned on.

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The cumulative summation of the unit step sequence is the *unit ramp sequence*, which is defined as

$$r[n - N] = \text{ramp}[n - N] = (n - N)u[n - N] \tag{1.21}$$

$$= \begin{cases} 0, & n < N \\ n - N, & n \geq N \end{cases}. \tag{1.22}$$

Notice that the ramp sequence can be written as a product of the linear sequence and the unit step sequence. Using the "on switch" concept, the linear sequence is suppressed for all times before N and is activated at $n = N$.

The *rectangular sequence* is defined as

$$u[n - N_1] - u[n - N_2] = \begin{cases} 1, & N_1 \leq n < N_2 \\ 0, & \text{otherwise} \end{cases}. \tag{1.23}$$

Unlike the CT case, here the DT rectangular sequence includes $n = N_1$ but excludes $n = N_2$.

The *signum sequence* is defined as

$$\text{sgn}[n] = u[n] - u[-n] \tag{1.24}$$

$$= \{..., -1, -1, \underline{0}, 1, 1, ...\} \tag{1.25}$$

$$= \begin{cases} -1, & n < 0 \\ 0, & n = 0 \\ 1, & n > 0 \end{cases}. \tag{1.26}$$

The *geometric sequence* is defined as

$$a^n u[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}. \tag{1.27}$$

Here, the value a can be real or complex. Note that when $a = e^{j\Omega}$, the geometric sequence essentially becomes the one-sided complex exponential sequence.

The stem plots of all DT signals described in this section can be found in Table 1.1. As with the CT case, these signals can also be linearly combined.
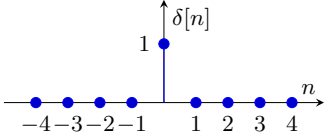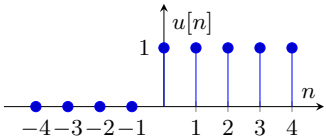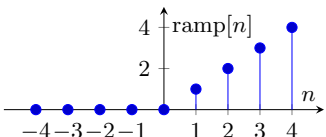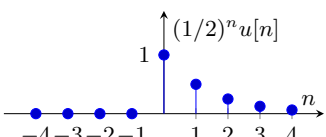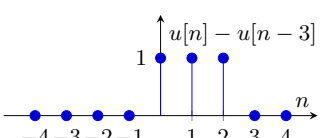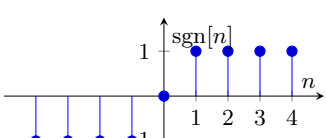
1.4 Properties of Discrete-Time Signals

1.4.1 Causality

Unlike the CT case, DT signals can either be causal, noncausal, or anticausal, depending on the application.

- Causal signals: $x[n] = 0$ for $n < 0$

- Noncausal signals: $x[n] \neq 0$ for $n < 0$

- Anticausal signals: $x[n] = 0$ for $n > 0$

Table 1.1: Common DT sequences.

Sequence	Expression	Stem plot
Unit impulse	$\delta[n-N] = \mathbb{1}_{\{n=N\}}[n]$	
Unit step	$u[n-N] = \mathbb{1}_{\{n \geq N\}}[n]$	
Unit ramp	$r[n-N] = \mathrm{ramp}[n-N] = (n-N)u[n-N]$	
Geometric	$a^n u[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$	
Rectangular	$u[n-N_1] - u[n-N_2] = \mathbb{1}_{[N_1,N_2)}[n]$	
Signum	$\mathrm{sgn}[n] = u[n] - u[-n]$	

DT signals must be causal if *real-time processing* is performed; however, the option to do *offline processing* gives a pathway for noncausal signals (and anticausal signals by extension).

1.4.2 Symmetry

A DT signal could have even symmetry, odd symmetry, or no symmetry.

- Even signals: $x[n] = x[-n]$
- Odd signals: $x[n] = -x[-n]$, or alternatively $x[-n] = -x[n]$

While a DT signal might not have any symmetry, any signal can be expressed as a sum of two component signals: one with even symmetry and the other with odd symmetry. Sometimes it is easier to analyze each individual component rather than analyze the whole signal on its own. For

$$x[n] = x_e[n] + x_o[n], \tag{1.28}$$

the even and odd components are given by

$$x_e[n] = \frac{1}{2}\left(x[n] + x[-n]\right) \tag{1.29}$$

$$x_o[n] = \frac{1}{2}\left(x[n] - x[-n]\right) \tag{1.30}$$

1.4.3 Periodicity

A DT signal $x[n]$ is said to be *periodic* if $x[n] = x[n + mN_0]$ for all integers m and time n with fundamental period N_0. Note that $\Omega_0 = 2\pi/N_0$ is the fundamental angular frequency.

Unlike the CT case, the sum of M periodic DT signals is always periodic with

- $\Omega_0 = \mathbf{GCD}(\Omega_1, \Omega_2, ..., \Omega_M)$, where Ω_k is the fundamental angular frequency of the kth signal, and $N_0 = 2\pi/\Omega_0 = \mathbf{LCM}(N_1, N_2, ..., N_M)$

- $\forall m \in \left\{ \frac{\Omega_1}{\Omega_0}, \frac{\Omega_2}{\Omega_0}, ..., \frac{\Omega_M}{\Omega_0} \right\}$ are integers, with each m called the mth harmonic (or mode) of fundamental angular frequency Ω_0

1.4.4 Signal Power and Energy

The total energy of a DT signal $x[n]$ is given by

$$E = \sum_{n=-\infty}^{+\infty} |x[n]|^2, \tag{1.31}$$

whereas the average power of a DT signal is given by

$$P_{av} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2. \tag{1.32}$$

Interestingly, all finite-duration DT signals are energy signals.

Furthermore, if a DT signal $x[n]$ is periodic with fundamental period N_0, then

$$P_{av} = \frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2. \tag{1.33}$$

It can then be said that all periodic signals are power signals.

Example 1.4.1. Fully describe what type of signal $x[n] = \exp(j\frac{5\pi}{31}n) + \cos(\frac{2\pi}{7}n + \frac{1}{6})$ is.

SOLUTION

First let $x_1[n] = \exp(j\frac{5\pi}{31}n)$. Then

$$\frac{2\pi}{\Omega_1} = \frac{2\pi}{5\pi/31} = \frac{62}{5} = \frac{N_1}{k}$$

is a rational number, and $x_1[n]$ thus is periodic with $N_1 = 62$.

Now let $x_2[n] = \cos(\frac{2\pi}{7}n + \frac{1}{6})$. Then

$$\frac{2\pi}{\Omega_2} = \frac{2\pi}{2\pi/7} = 7 = \frac{N_2}{k}$$

is a rational number, and $x_2[n]$ thus is periodic with $N_2 = 7$.

Therefore, $x[n] = x_1[n] + x_2[n]$ is periodic with $N_0 = \mathbf{LCM}(62, 7) = 434$. Since $x[n]$ is periodic, it is also a power signal.

Checking for symmetry, we find there is none since

$$x[-n] = \exp(-j5\pi n/31) + \cos(2\pi n/7 - 1/6) \neq x[n]$$
$$-x[-n] = -\exp(-j5\pi n/31) - \cos(2\pi n/7 - 1/6) \neq x[n]$$

Therefore, $x[n]$ is:

- a discrete-time signal,
- deterministic,
- noncausal,
- asymmetric,
- periodic,
- and a power signal.

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