Chapter 6

Discrete-Time Fourier Series

As with the CT Fourier series, there exists a discrete-time Fourier series (DTFS) representation for periodic DT signals, as part of a collection of discrete-time *Fourier analysis* techniques. Of course, while the DTFS is applicable for periodic DT signals, the DTFT is applicable for any DT signal regardless of periodicity or causality.

However, understanding how the DTFS works is a precursor to understanding the third DT Fourier analysis technique: the discrete Fourier transform (DFT). Before delving into the DFT, we briefly examine the DTFS.

6.1 Discrete-Time Fourier Series

Fourier's theorem can be applied to the discrete-time domain in the sense that any periodic DT signal $x[n]$ can be represented as a finite sum of DT sinusoids. Specifically, given that the period of $x[n]$ is N_0 and that sinusoids are closely related to complex exponentials, the signal can be expressed as a sum of N_0 complex exponentials.

The *discrete-time Fourier series* (DTFS) of a N_0-periodic DT signal $x[n]$ for $N_0 = 2\pi/\Omega_0$ is given by

$$x[n] = \sum_{k=0}^{N_0-1} x_k e^{jk\Omega_0 n}, \tag{6.1}$$

where the *Fourier coefficients* are calculated from the summation:

$$x_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-jk\Omega_0 n}. \tag{6.2}$$

Additionally, from the equation for calculating Fourier coefficients, since $x[n]e^{-jk\Omega_0 n}$ is N_0-periodic, the coefficients x_k themselves must also be N_0-periodic, despite only needing one period to compute such coefficients. That is,

$$x_k = x_{k+mN_0}, \text{ for } m \in \mathbb{Z}. \tag{6.3}$$

For $x_k = |x_k| \exp(j\phi_k)$, the magnitudes $|x_k|$ and phases ϕ_k can be collected and plotted against their corresponding harmonic frequencies on a *magnitude spectrum* and a *two-sided phase spectrum*, respectively. Often the two spectra together are collectively referred to as the *two-sided Fourier spectrum.*

As with the DTFT, since periodic DT signals are sequences, there is only one single necessary Dirichlet condition for the DTFS. It follows that if N_0-periodic signal $x[n]$ is absolutely summable over a period such that

$$\sum_{n=0}^{N_0-1} |x[n]| < \infty, \tag{6.4}$$

then $x[n]$ has a DTFS. Unless the sequence contains the value $\pm\infty$ at a certain value of n, the sequence will almost always converge absolutely. Note here that the condition is necessary, not sufficient. The implication is that the DTFS applies to strictly DT power signals.

Properties of the DTFS are outlined in Table 6.1.

Table 6.1: Properties of the DTFS.

Property	Periodic signal, $x[n]$	Fourier coefficients, x_k
Superposition	$A_1x_1[n] + A_2x_2[n]$	$A_1(x_1)_k + A_2(x_2)_k$
Time shift	$x[n-n_0]$	$e^{-jk\Omega_0 n_0}x_k$
Frequency shift	$e^{-jk_0\Omega_0 n}x[n]$	x_{k+k_0}
Time reversal	$x[-n]$	x_{-k}
Conjugate symmetry	$x[n]$ real	$\begin{cases} x_{-k} = x_k^* \\ \mathrm{Re}(x_k) = \mathrm{Re}(x_{-k}) \\ \mathrm{Im}(x_k) = -\mathrm{Im}(x_{-k}) \\ \lvert x_k\rvert = \lvert x_{-k}\rvert \\ \phi_k = -\phi_{-k} \end{cases}$
Real and even signals	$x[n]$ real and even	x_k purely real and even
Real and odd signals	$x[n]$ real and odd	x_k purely imaginary and odd
Even-odd decomposition of real signals	$\begin{cases} x_e[n] = \frac{1}{2}[x[n] + x[-n]] \\ x_o[n] = \frac{1}{2}[x[n] - x[-n]] \end{cases}$	$\begin{cases} \mathrm{Re}(x_k) \\ j\,\mathrm{Im}(x_k) \end{cases}$

Example 6.1.1. Compute the DTFS of periodic sequence

$$x[n] = \{..., \underline{24}, 8, 12, 16, 24, 8, 12, 16, ...\}.$$

SOLUTION

From visual inspection, we see that the period of $x[n]$ is $N_0 = 4$. Therefore, we need to find four Fourier coefficients.

$$x_0 = \frac{1}{4}\sum_{n=0}^{3} x[n] = \frac{1}{4}[24 + 8 + 12 + 16] = 15$$

$$x_1 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j2\pi n/4} = \frac{1}{4}[24 + 8e^{-j\pi/2} + 12e^{-j2\pi/2} + 16e^{-j3\pi/2}] = 3.6e^{j33.7^\circ}$$

$$x_2 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j4\pi n/4} = \frac{1}{4}[24 + 8e^{-j\pi} + 12e^{-j2\pi} + 16e^{-j3\pi}] = 3$$

$$x_3 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j6\pi n/4} = \frac{1}{4}[24 + 8e^{-j3\pi/2} + 12e^{-j6\pi/2} + 16e^{-j9\pi/2}] = 3.6e^{-j33.7^\circ}$$

Then for $\Omega_0 = 2\pi/N_0 = \pi/2$, the DTFS of $x[n]$ is

$$\begin{aligned} x[n] = \sum_{k=0}^{N_0-1} x_k e^{jk\Omega_0 n} &= 15 + 3.6e^{j33.7^\circ}e^{j\pi n/2} + 3e^{j2\pi n/2} + 3.6e^{-j33.7^\circ}e^{j3\pi n/2} \\ &= 15 + 3(-1)^n + 7.2\cos\left(\frac{\pi}{2}n + 33.7^\circ\right). \end{aligned}$$

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6.2 Parseval's Theorem for DTFS

Parseval's theorem for the DTFS is essentially a "conservation of (average) power" theorem. When periodic signals are mapped from the DTFS to the discrete-frequency Fourier spectrum, the total signal average power is conserved. It follows that the total average power of a DT periodic signal $x[n]$ can be evaluated as:

$$P_x = \frac{1}{N_0}\sum_{n=0}^{N_0-1} |x[n]|^2 = \sum_{k=0}^{N_0-1} |x_k|^2. \tag{6.5}$$

Here, the DC power is given by $|x_0|^2$ whereas the rest of the summation is the average AC power.

Additionally, we can define the *one-sided power spectral density* (1-sided PSD) to be

$$PSD_1 = \begin{cases} |x_0|^2, & k = 0 \\ 2|x_k|^2, & k > 0 \end{cases} \tag{6.6}$$

and the *two-sided power spectral density* (2-sided PSD) to be

$$PSD_2 = |x_k|^2. \tag{6.7}$$

Note that the 2-sided PSD is defined for all frequencies, whereas the 1-sided PSD is defined for only nonnegative frequencies. Because of this, the AC component of the 2-sided PSD are half the values of the 1-sided PSD. In signal processing, we tend to be more interested in the 1-sided PSD.

6.3 DT LTI Systems with DTFS

As with the CT case, the superposition principle can be applied to find the output of a DT LTI system, especially if the input signal is a periodic DT signal which can be represented as a finite sum of DT complex exponentials.

DTFS ANALYSIS.

1. Express the input signal $x[n]$ as a DTFS to obtain the Fourier coefficients x_k.
2. Use the DTFT to find the generic frequency response function $H(e^{j\Omega})$ of the system.
3. Compute the output Fourier coefficients $y_k = x_k H(e^{j\Omega})\big|_{\Omega=k\Omega_0}$.
4. Determine the output signal $y[n] = \displaystyle\sum_{k=0}^{N_0-1} y_k e^{jk\Omega_0 n}$.

Example 6.3.1. Suppose a periodic sequence

$$x[n] = \{..., \underline{4}, -2, 0, -2, 4, -2, 0, -2, ...\}$$

is inputted into a system characterized by impulse response

$$h[n] = \frac{\sin\left(\frac{\pi n}{3}\right)}{\pi n}.$$

Find the system response $y[n]$.

SOLUTION

From visual inspection, we see that the period of $x[n]$ is $N_0 = 4$. Therefore, we need to find four Fourier coefficients.

$$x_0 = \frac{1}{4}\sum_{n=0}^{3} x[n] = \frac{1}{4}[4-2+0-2] = 0$$
$$x_1 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j2\pi n/4} = \frac{1}{4}[4-2e^{-j\pi/2}+0e^{-j2\pi/2}-2e^{-j3\pi/2}] = 1$$
$$x_2 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j4\pi n/4} = \frac{1}{4}[4-2e^{-j\pi}+0e^{-j2\pi}-2e^{-j3\pi}] = 2$$
$$x_3 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j6\pi n/4} = \frac{1}{4}[4-2e^{-j3\pi/2}+0e^{-j6\pi/2}-2e^{-j9\pi/2}] = 1$$

Using the DTFT table, we find that

$$H(e^{j\Omega}) = \mathrm{DTFT}[h[n]] = \sum_{k=-\infty}^{+\infty} \mathrm{rect}\left(\frac{\Omega-2\pi k}{2(\pi/3)}\right).$$

Isolate one period such that the frequency response over one period is

$$H_{\langle 2\pi\rangle}(e^{j\Omega}) = \mathrm{rect}\left(\frac{\Omega}{2(\pi/3)}\right) = \begin{cases} 1, & 0 \le |\Omega| \le \pi/3 \\ 0, & \pi/3 < |\Omega| < \pi. \end{cases}$$

Then for $\Omega_0 = 2\pi/N_0 = \pi/2$,

$$H_{\langle 2\pi\rangle}(e^{j\Omega})\Big|_{\Omega=0} = 1$$
$$H_{\langle 2\pi\rangle}(e^{j\Omega})\Big|_{(\Omega=\Omega_0=\pi/2)} = 0$$
$$H_{\langle 2\pi\rangle}(e^{j\Omega})\Big|_{(\Omega=2\Omega_0=\pi)} = 0$$
$$H_{\langle 2\pi\rangle}(e^{j\Omega})\Big|_{(\Omega=3\Omega_0=3\pi/2=-\pi/2)} = 0.$$

The output Fourier coefficients are then

$$y_0 = x_0 H(e^{j\Omega})\big|_{\Omega=0} = 0$$
$$y_1 = x_1 H(e^{j\Omega})\big|_{\Omega=\Omega_0} = 0$$
$$y_2 = x_2 H(e^{j\Omega})\big|_{\Omega=2\Omega_0} = 0$$
$$y_3 = x_3 H(e^{j\Omega})\big|_{\Omega=3\Omega_0} = 0,$$

and the system response is thus

$$y[n] = \sum_{k=0}^{N_0-1} y_k e^{jk\Omega_0 n} = \sum_{k=0}^{3} 0\, e^{jk(2\pi/4)n} = 0.$$

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Example 6.3.2. Given the periodic sequences

$$x[n] = \{..., \underline{4}, 2, 1, 0, 4, 2, 1, 0, ...\}$$
$$y[n] = \{..., \underline{10}, 4, 10, 4, 10, 4, ...\}$$

find $h[n] = \{\underline{a}, b, c\}$ for $y[n] = x[n] * h[n]$.

SOLUTION

Because $h[n]$ is finite, we can avoid using the DTFS representation altogether by using the expanded tabular method for linear convolution. Since $x[n]$ has a period of 4 (which is greater than the period of $h[n]$ which is 2), only 4 rows for $h[\cdot]$ are needed.

$n \rightarrow$	-2	-1	0	1	2	3	4	$y[\cdot]$
$x[n]$	1	0	4	2	1	0	4	
$h[-n]$	c	b	a					$y[0] = c + 4a = 10$
$h[1-n]$		c	b	a				$y[1] = 4b + 2a = 4$
$h[2-n]$			c	b	a			$y[2] = 4c + 2b + a = 10$
$h[3-n]$				c	b	a		$y[3] = 2c + b = 4$

Then solving the system of equations from the right pane, we get

$$a = 2$$
$$b = 0$$
$$c = 2.$$

Therefore, $h[n] = \{\underline{2}, 0, 2\}$.