Chapter 3

Discrete-Time Convolution

In the previous chapter, we were introduced to the concept of *linear convolution*, an operation $(*)$ defined by the *convolution sum*

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k].$$

We have seen that the operation is commutative since by change of variables,

$$\begin{aligned} x[n] * h[n] &= \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{+\infty} x[n-k]h[k] = h[n] * x[n]. \end{aligned}$$

We have also already seen another linear convolution property in effect with the sampling property of the unit impulse sequence.

$$\sum_{k=-\infty}^{+\infty} x[k]\delta[k-N] = x[N]$$

$$\implies \sum_{k=-\infty}^{+\infty} x[k]\delta[k-n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] = x[n]$$

$$x[n] * \delta[n] = x[n].$$

We also introduced the concept that any DT LTI system characterized by impulse response $h[n]$ will always have predictable outputs since the output is simply the linear convolution of the input signal and the impulse response, provided that all initial conditions of the system are zero. That is,

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]. \tag{3.1}$$

In this chapter, different techniques to computing the linear convolution of two sequences are explored.

3.1 Analytical Convolution in Discrete-Time

One way to compute the linear convolution is to directly solve the convolution sum itself. Refer to Table 3.1 for some linear convolution properties.

Table 3.1: Linear convolution properties.

Property	Description
Commutativity	$x[n] * h[n] = h[n] * x[n]$
Associativity	$x[n] * [h[n] * g[n]] = [x[n] * h[n]] * g[n]$
Distributivity	$x[n] * [h_1[n] + \cdots + h_N[n]] = (x[n] * h_1[n]) + \cdots + [x[n] * h_N[n]]$
Time-shift	$x[n - N_1] * h[n - N_2] = y[n - N_1 - N_2]$
Convolution with impulse	$x[n] * \delta[n] = x[n]$
Convolution with step	$x[n] * u[n] = \sum_{k=-\infty}^{n} x[k]$ (ideal accumulator)
Causal signals and systems	$y[n] = u[n] \sum_{k=0}^{n} h[k] x[n - k]$
Width (duration)	$\text{Width}[y[n]] = \text{Width}[x[n]] + \text{Width}[h[n]] - 1$
	$\begin{cases} (\text{domain } x[n]) = [a, b], \\ (\text{domain } h[n]) = [c, d] \end{cases} \implies (\text{domain } y[n]) = [a + c, b + d]$
"Area" (infinite sum)	$\sum_{k=-\infty}^{+\infty} y[n] = \left(\sum_{k=-\infty}^{+\infty} x[n] \right) \left(\sum_{k=-\infty}^{+\infty} h[n] \right)$

When convolving two causal sequences, one of the sequences will become time-reversed in the convolution sum. We can use this to our advantage in redefining the limits of summation. That is, for $f[n]$ representing the non-step factors,

$$\sum_{n=-\infty}^{+\infty} f[n] \cdot \underbrace{u[n - a]}_{n - a \, \geq \, 0} \cdot \underbrace{u[b - n]}_{b - n \, \geq \, 0} = u[b - a] \sum_{n=a}^{b} f[n]. \tag{3.2}$$

The inequalities above are defined for when the step sequences are "switched on", and combining the two inequalities gives us an interval of validity $a \leq n \leq b$, which is what is used to update the convolution sum.

When solving convolution sums, the need to compute the geometric series come up often. As a reminder, the (length-N) finite sum of a geometric series is defined as

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r} \quad \text{for } |r| < 1, \tag{3.3}$$

whereas the infinite sum of a geometric series is defined as

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \ \text{ for } |r| < 1,$$

(3.4)

or more generally,

$$\sum_{n=\alpha}^{\infty} r^n = \frac{r^\alpha}{1-r} \ \text{ for } |r| < 1.$$

(3.5)

While we are mostly interested in intervals of validity that follow causal signals and systems, there are other possible intervals of validity for different types of systems that may or may not be causal. Refer to Table 3.2 for these summation redefinitions.

Table 3.2: Summation redefinitions based on unit step sequences.

$$\sum_{n=-\infty}^{+\infty} f[n]u[n-a] = \sum_{n=a}^{+\infty} f[n]$$

(3.6)

$$\sum_{n=-\infty}^{+\infty} f[n]u[b-n] = \sum_{n=-\infty}^{b} f[n]$$

(3.7)

$$\sum_{n=-\infty}^{+\infty} f[n]u[n-a]u[b-n] = u[b-a]\sum_{n=a}^{b} f[n]$$

(3.8)

$$\sum_{n=-\infty}^{+\infty} f[n]u[n-a]u[n-b] = \sum_{n=\max(a,b)}^{+\infty} f[n]$$

$$= u[a-b]\sum_{n=a}^{+\infty} f[n] + u[b-a]\sum_{n=b}^{+\infty} f[n]$$

(3.9)

$$\sum_{n=-\infty}^{+\infty} f[n]u[a-n]u[b-n] = \sum_{n=-\infty}^{\min(a,b)} f[n]$$

$$= u[b-a]\sum_{n=-\infty}^{a} f[n] + u[a-b]\sum_{n=-\infty}^{b} f[n]$$

(3.10)

Example 3.1.1. Find $y[n] = x[n] * h[n]$ for

$$x[n] = 2^{-n}u[n],$$
$$h[n] = u[n-2].$$

SOLUTION

Using the convolution sum, it follows that

$$y[n] = [2^{-n}u[n]] * u[n-2] = \sum_{k=-\infty}^{+\infty} 2^{-k} \underbrace{u[k]}_{k \geq 0} \underbrace{u[n-k-2]}_{n-2 \geq k}$$

$$= u[n-2] \sum_{k=0}^{n-2} 2^{-k} = u[n-2] \sum_{k=0}^{n-2} \left(\frac{1}{2}\right)^k$$

$$= u[n-2] \sum_{k=0}^{n-2} 2^{-k} = u[n-2] \left(\frac{1 - \left(\frac{1}{2}\right)^{n-1}}{1 - \frac{1}{2}}\right)$$

$$= \left(2 - \left(\frac{1}{2}\right)^{n-2}\right) u[n-2].$$

$\blacksquare$

Example 3.1.2.   Find $y[n] = x[n] * h[n]$ for

$$x[n] = 2^{-|n|},$$
$$h[n] = u[n-2].$$

SOLUTION

First, rewrite $x[n]$ such that

$$x[n] = 2^{-|n|}$$
$$= 2^{-n}u[n] + 2^{n}u[-n-1].$$

Using the convolution sum, it follows that

$$y[n] = [2^{-n}u[n] + 2^{n}u[-n-1]] * u[n-2]$$

$$= \sum_{k=-\infty}^{+\infty} [2^{-k}u[k] + 2^{k}u[-k-1]]u[n-k-2]$$

$$= \sum_{k=-\infty}^{+\infty} 2^{-k} \underbrace{u[k]}_{k \geq 0} \underbrace{u[n-k-2]}_{n-2 \geq k} + \sum_{k=-\infty}^{+\infty} 2^{k} \underbrace{u[-k-1]}_{k \leq -1} \underbrace{u[n-k-2]}_{n-2 \geq k}$$

$$= u[n-2] \sum_{k=0}^{n-2} 2^{-k} + u[n-1] \sum_{k=-\infty}^{-1} 2^{k} + u[1-n] \sum_{k=-\infty}^{n-2} 2^{k}$$

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Via change of variables, the output is given by

$$y[n] = \left(2 - \left(\frac{1}{2}\right)^{n-2}\right) u[n-2] + u[n-1] \sum_{\ell=1}^{+\infty} \left(\frac{1}{2}\right)^{\ell} + u[1-n] \sum_{\ell=2-n}^{+\infty} \left(\frac{1}{2}\right)^{\ell}$$

$$= \left(2 - \left(\frac{1}{2}\right)^{n-2}\right) u[n-2] + \left(\frac{\frac{1}{2}}{1-\frac{1}{2}}\right) u[n-1] + \left(\frac{\left(\frac{1}{2}\right)^{2-n}}{1-\frac{1}{2}}\right) u[1-n]$$

$$= \left(2 - \left(\frac{1}{2}\right)^{n-2}\right) u[n-2] + u[n-1] + \left(\frac{1}{2}\right)^{1-n} u[1-n]$$

$$= \begin{cases} 2^{n-1}, & n < 1 \\ 2, & n = 1 \\ 3 - \left(\frac{1}{2}\right)^{n-2}, & n \geq 2 \end{cases}.$$

■

3.2   Pointwise Graphical Convolution in Discrete-Time

Another way to compute the convolution is to interpret the convolution graphically. We will first introduce the pointwise method, best used when only a few output samples are of interest. Since

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k],$$

the output at a certain single time n is given by the sum of the points formed by multiplying $x[k]$ and $h[n-k]$, which is a time-reversed and translated version of $h[k]$.

POINTWISE GRAPHICAL CONVOLUTION.

Step 1: Determine the set of time values of interest $n_0, ..., n_N$.

Step 2: Plot $x[k]$ and $h[-k]$ along the k-axis.

Step 3: Let $n = n_0$ and shift $h[-k]$ to the right by n units to represent $h[n-k]$.

Step 4: Using the region of overlap between $x[k]$ and $h[n-k]$, take the sum of the point-by-point product.

Step 5: Repeat Steps 2–4 for the remaining time values of interest.

Example 3.2.1.   Find $y[n] = x[n] * h[n]$ for

$$x[n] = \{\underline{3}, 1, 2\},$$
$$h[n] = \{\underline{3}, 2, 1\}.$$

SOLUTION

From the width property of linear convolution, we know that the duration of the output is five samples, which is the sum of the two widths less one. Additionally, both $x[n]$ and $h[n]$ are finite signals that start at $n = 0$ and end at $n = 2$. Therefore, we are interested when $n = 0, 1, 2, 3, 4$.
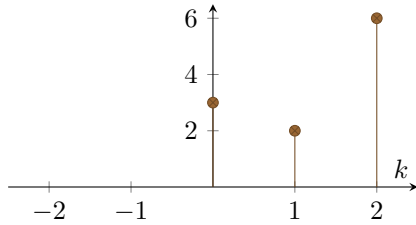
Suppose we are interested in only finding $y[2]$. First, we plot $x[k]$ and $h[-k]$ on the k-axis.
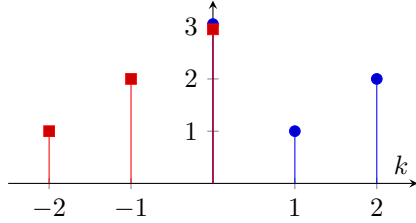
Since we want to observe at time $n = 2$, we shift the red plot to the right by 2 units:
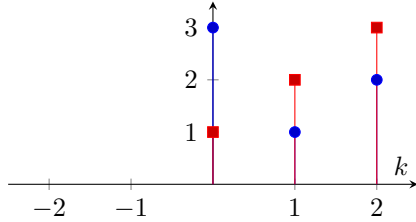
Performing point-by-point multiplication, we get:

Taking the sum of the points, it follows that

$$y[2] = 3 + 2 + 6 = 11.$$

We can repeat the process for other values of $n \in [0, 4]$ until we get $y[n] = \{\underline{9}, 9, 11, 5, 2\}$. ∎

3.3   Region-Based Graphical Convolution in Discrete-Time

The pointwise method is impractical if we are interested in the output response over a larger, if not entire, time domain. Here, we introduce the region-based graphical convolution, which utilizes a signal's *leading and trailing edges*, if any.

REGION-BASED GRAPHICAL CONVOLUTION.

Step 1: Plot $x[k]$ and $h[-k]$ along the k-axis, with $h[-k]$ being the simpler sequence to flip.

Step 2: Label the leading and trailing edges of $h[-k]$.

Step 3: Identify a region of overlap and use the edges to determine the interval of n-values. Then evaluate the convolution sum during that region of overlap.

Step 4: Repeat Steps 2–3 for any other regions of overlap when shifting $h[-k]$ from left-to-right along the k-axis. Keep track of the width of $h[n-k]$.
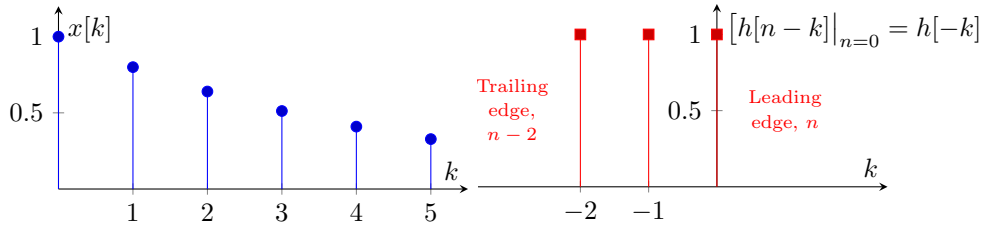

Example 3.3.1.   Find $y[n] = x[n] * h[n]$ for

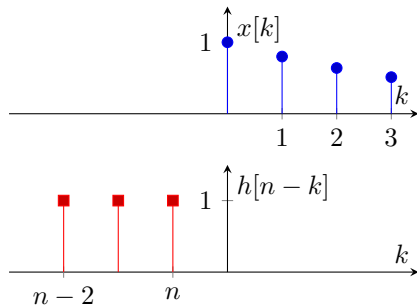$$x[n] = (0.8)^n u[n],$$
$$h[n] = u[n] - u[n-3].$$

SOLUTION

First, plot $x[k]$ and $h[-k]$. Then identify the leading and trailing edges of $h[-k]$.

From $h[-k]$ above, we see that the leading edge is $k = n$ and the trailing edge is $k = n - 2$. Now we place $h[n-k]$ to the furthest left and gradually shift to the right.
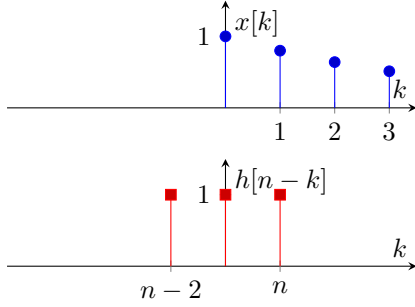
Region 1: No overlap

Leading edge: $n \leq -1$

No overlap, so $y[n] = 0$ for $n \leq -1$.
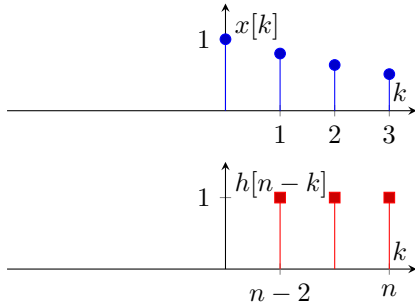
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Region 2: Partial overlap

$$\text{Leading edge: } n \geq 0$$
$$\text{Trailing edge: } n - 2 \leq -1 \Rightarrow n \leq 1$$

$$\Longrightarrow 0 \leq n \leq 1$$

$$\Longrightarrow y[n] = \sum_{k=0}^{n} [(0.8)^k][1] = \frac{1 - (0.8)^{n+1}}{1 - 0.8}$$

Region 3: Complete overlap

$$\text{Trailing edge: } n - 2 \geq 0 \Rightarrow n \geq 2$$

$$\Longrightarrow y[n] = \sum_{k=n-2}^{n} [(0.8)^k][1] = \sum_{\ell=0}^{2} (0.8)^{\ell+n-2}$$
$$\Longrightarrow y[n] = (0.8)^{n-2} \left(\frac{1 - (0.8)^3}{1 - 0.8} \right) = 2.44(0.8)^{n-2}$$

Aggregating the outputs from the five regions, we can write a piecewise function for the overall output:

$$y[n] = \begin{cases} 0, & n \leq -1 \\ \frac{1 - (0.8)^{n+1}}{0.2}, & 0 \leq n \leq 1 \\ 2.44(0.8)^{n-2}, & n \geq 2 \end{cases} .$$

3.4   Tabular Methods for Finite-Length Sequences

Since offline processing (or even real-time processing with limited memory) involves analyzing data with finite duration (due to the sampled nature of existing collected data), linear convolutions in practice mostly involves finite-length sequences – meaning the output sequence will also be of finite-length. There are various shortcut methods to solving these specific types of linear convolutions by hand.

3.4.1   Expanded Tabular Method

The expanded tabular method essentially is a tabular version of the pointwise graphical convolution, albeit saving time on sketching multiple plots.

EXPANDED TABULAR METHOD. For (domain $x[n]$) = $[a, b]$ and (domain $h[n]$) = $[c, d]$:

Step 1: Create a header row for the values of n.

Step 2: Populate the first row with values of $x[n]$ under their respective n.

Step 3: Populate the second row with the time-reversed $h[-n]$ under the respective n.

Step 4: For every necessary time shift N, populate another row corresponding to $h[N - n]$.

Step 5: Vertically multiply for each $x[n]h[N - n]$ and take the sum of the pointwise products to get $y[N]$.

Example 3.4.1.   Find $y[n] = x[n] * h[n]$ for

$$x[n] = \{\underline{3}, 4, 5\},$$
$$h[n] = \{\underline{6}, 7, 8\}.$$

SOLUTION

From the width property, the output $y[n]$ should be of length 5 and defined for $n = 0, 1, 2, 3, 4$. Setting up the table:

$n \rightarrow$	-2	-1	0	1	2	3	4	$y[\cdot]$
$x[n]$			3	4	5			
$h[-n]$	8	7	6					$y[0] = 3(6) = 18$
$h[1-n]$		8	7	6				$y[1] = 3(7) + 4(6) = 45$
$h[2-n]$			8	7	6			$y[2] = 3(8) + 4(7) + 5(6) = 82$
$h[3-n]$				8	7	6		$y[3] = 4(8) + 5(7) = 67$
$h[4-n]$					8	7	6	$y[4] = 5(8) = 40$

Therefore, the output is $y[n] = \{\underline{18}, 45, 82, 67, 40\}$. ■

3.4.2   Condensed Tabular Method

The condensed tabular method is visually more compact and the quickest to set up.

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CONDENSED TABULAR METHOD. For (domain $x[n]$) = $[a, b]$ and (domain $h[n]$) = $[c, d]$:

Step 1: Create a header column for the values of $x[n]$. Underline $x[0]$.

Step 2: Create a header row for the values of $h[n]$. Underline $h[0]$.

Step 3: Populate the table by taking the product of the corresponding $x[\cdot]$ and $h[\cdot]$ values.

Step 4: Add entries diagonally from right to left and list the sums in the order of their diagonals.

Step 5: Find where $x[0]$ and $h[0]$ intersect. The corresponding right-to-left diagonal represents $y[0]$.

Example 3.4.2.   Find $y[n] = x[n] * h[n]$ for

$$x[n] = \{\underline{3}, 4, 5\},$$
$$h[n] = u[n] - u[n - 4].$$

SOLUTION

First, note that $h[n] = u[n] - u[n - 4] = \{\underline{1}, 1, 1, 1\}$ is of length 4. Setting up the table:

$x[n]$ \ $h[n]$	$\underline{1}$	1	1	1
$\underline{3}$	3	3	3	3
4	4	4	4	4
5	5	5	5	5

Each right-to-left diagonal is highlighted a different color to show which entries should be summed. The output is then

$$y[n] = \{\underline{3},\ 3 + 4,\ 3 + 4 + 5,\ 3 + 4 + 5,\ 4 + 5,\ 5\} = \{\underline{3}, 7, 12, 12, 9, 5\}$$

◼

Example 3.4.3.   For $y[n] = x[n] * h[n]$, find $h[n]$ given that

$$x[n] = \{\underline{1}, 2, 3\},$$
$$y[n] = \{\underline{1}, 4, 7, 6\}.$$

SOLUTION

Since (domain $x[n]$) $= [0, 2]$ and (domain $y[n]$) $= [0, 3]$, it must be true by the width property for linear convolution that

$$(\text{domain } h[n]) = [0 - 0, 3 - 2] = [0, 1]$$

$$\implies h[n] = \left\{ \underline{h[0]}, h[1] \right\}$$

Using the condensed tabular method, we get:

$x[n]$ \\ $h[n]$	$h[0]$	$h[1]$
$\underline{1}$	$h[0]$	$h[1]$
2	$2h[0]$	$2h[1]$
3	$3h[0]$	$3h[1]$

Lining up each right-to-left diagonal with the corresponding $y[n]$ entry, we get the following system of equations:

$$h[0] = 1$$
$$2h[0] + h[1] = 4$$
$$3h[0] + 2h[1] = 7$$
$$3h[1] = 6$$

Solving the system of equations, we get:

$$h[n] = \left\{ \underline{h[0]}, h[1] \right\} = \{\underline{1}, 2\}$$

∎