# Chapter 4

# Laplace Transform

Previously, we have seen that LTI systems can be characterized by their impulse responses $h(t)$, and the outputs of those systems can be computed by taking the convolution of the input signal and the impulse response.

In fact, all LTI systems can be characterized by some impulse response. Going back to the affine transformation on the independent variable, its impulse response is given by

$$h(t) = \delta(at - b) = \frac{1}{|a|}\delta\left(t - \frac{b}{a}\right). \tag{4.1}$$

As seen in the first two chapters, some LTI systems can be represented by LCCDEs. While one could find the system response by solving the differential equation, it is more difficult to solve it that way. Here, we introduce a concept called the Laplace transform which offers a simpler way to solve differential equations by mapping them to a domain that uses algebraic equations.

As signals and systems in the physical world are causal, we are particularly interested in the unilateral (one-sided) Laplace transform.
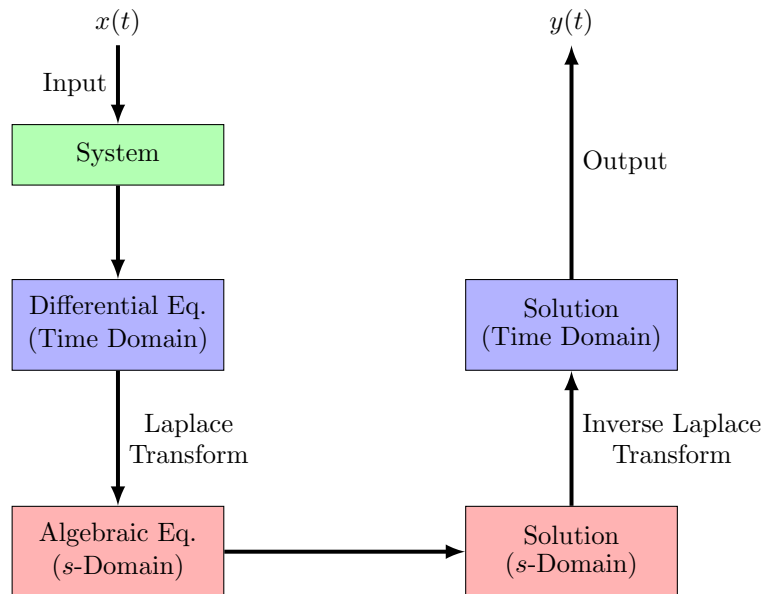
## 4.1   Unilateral Laplace Transform

The *unilateral Laplace transform* of a signal $x(t)$ is defined as

$$X(s) = \mathcal{L}[x(t)] = \int_{0^-}^{\infty} x(t)e^{-st}\,dt, \tag{4.2}$$

where $s = \sigma + j\omega$ is the *complex frequency*.

The Laplace transform essentially is an operator that maps a signal defined in the *time domain* to another signal defined in the *s-domain*. $x(t)$ and $X(s)$ constitute a *Laplace transform pair*. This relationship can be written as

$$x(t) \iff X(s). \tag{4.3}$$

Because of this relationship, there exists an *inverse Laplace transform* defined by

$$x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{j2\pi} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{+st}\,ds, \tag{4.4}$$

though direct computation of the inverse integral is often more cumbersome than referencing a table of Laplace transform pairs.

Additionally, for signal $x(t)$ to be transformable, the unilateral Laplace integral must be absolutely integrable such that

$$\int_{0^-}^{\infty} |x(t)e^{-st}|\,dt = \int_{0^-}^{\infty} |x(t)||e^{-(\sigma+j\omega)t}|\,dt$$

$$= \int_{0^-}^{\infty} |x(t)||e^{-\sigma t}||e^{-j\omega t}|\,dt < \infty. \tag{4.5}$$

$$\tag{4.6}$$

From above, $|e^{-j\omega t}| < 1$ and $\sigma$ is real. Therefore, the integrable condition simplifies to

$$\int_{0^-}^{\infty} |x(t)|e^{-\sigma t}\,dt < \infty. \tag{4.7}$$

The *region of convergence* (ROC) is then defined as the values for $s$ for which the unilateral Laplace integral is absolutely integrable for a given signal $x(t)$ for all $t$. Here, the ROC is defined by a set of values for $\sigma = \text{Re}(s)$.

For the unilateral Laplace transform, the ROC informs if there exists a unilateral Laplace transform for some signal $x(t)$ that can be mapped back to the time domain. Since we are more interested in causal signals and systems, signals that are physically realizable always have a unilateral Laplace transform.

Since all physically realizable signals and systems have a Laplace transform, the ROC is not necessary when modeling those entities. In fact, all causal signals have a unique unilateral Laplace transform.

Table 4.1 lists the properties of the unilateral Laplace transform.

**Table 4.1: Properties of the Unilateral Laplace Transform**

| Property | $x(t)$ | $X(s) = \mathcal{L}[x(t)]$ |
|---|---|---|
| Superposition | $K_1 x_1(t) + K_2 x_2(t)$ | $K_1 X_1(s) + K_2 X_2(s)$ |
| Time scaling | $x(at), \ a > 0$ | $\dfrac{1}{a} X\left(\dfrac{s}{a}\right)$ |
| Time shift | $x(t-T)u(t-T), \ T > 0$ | $e^{-Ts} X(s)$ |
| Frequency shift | $e^{-at} x(t)$ | $X(s+a)$ |
| Time 1st derivative | $x'(t) = \dfrac{dx(t)}{dt}$ | $sX(s) - x(0^-)$ |
| Time 2nd derivative | $x''(t) = \dfrac{d^2 x(t)}{dt^2}$ | $s^2 X(s) - sx(0^-) - x'(0^-)$ |
| Time $n$th derivative | $x^{(n)}(t) = \dfrac{d^n x(t)}{dt^n}$ | $s^n X(s) - \displaystyle\sum_{k=1}^{n} s^{n-k} x^{(k-1)}(0^-)$ |
| Time integral | $\displaystyle\int_{0^-}^{t} x(\tau)\,d\tau$ | $\dfrac{1}{s} X(s)$ |
| Frequency derivative | $t \cdot x(t)$ | $-\dfrac{dX(s)}{ds} = -X'(s)$ |
| Frequency integral | $\dfrac{x(t)}{t}$ | $\displaystyle\int_{s}^{\infty} X(\gamma)\,d\gamma$ |
| Convolution | $x_1(t) * x_2(t)$ | $X_1(s) X_2(s)$ |
| Multiplication | $x_1(t) x_2(t)$ | $X_1(s) * X_2(s)$ |
| Initial value theorem | $x(0^+)$ | $\displaystyle\lim_{s \to \infty} X(s)$ |
| Final value theorem | $x(\infty)$ | $\displaystyle\lim_{s \to 0} X(s)$ |

**Example 4.1.1.** Find the unilateral Laplace transform of $x(t) = e^{-at} u(t)$, for $a > 0$.

**SOLUTION**

Using the unilateral Laplace integral,

$$X(s) = \mathcal{L}[x(t)] = \int_{0^-}^{\infty} x(t) e^{-st}\,dt,$$

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we get

$$X(s) = \int_{0^-}^{\infty} [e^{-at}u(t)]e^{-st}\,dt = \int_{0^-}^{\infty} e^{-(s+a)t}\,dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]\Big|_{0^-}^{\infty} = \frac{1}{s+a}.$$

■

Table 4.2 lists some common unilateral Laplace transform pairs. While one could find the inverse Laplace transform using the inverse Laplace integral, it is much simpler to reference a table of transform pairs.

**Example 4.1.2.**    Suppose a signal $x(t) = u(t-2) - u(t-4)$ is inputted into an LTI system characterized by $h(t) = t \cdot u(t)$. Find the unilateral Laplace transform of the output signal $y(t)$.

**SOLUTION**

First, let $x_0(t) = u(t)$. From the convolution property, we can find the unilateral Laplace transforms of both $x_0(t)$ and $h(t)$, then take the product $X_0(s)H(s)$.

$$X_0(s) = \int_{0^-}^{\infty} u(t)e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]\Big|_{0^-}^{\infty} = \frac{1}{s}$$

$$H(s) = \int_{0^-}^{\infty} t \cdot u(t)e^{-st}\,dt = \left[\frac{e^{-st}}{s^2}(-st-1)\right]\Big|_{0^-}^{\infty} = \frac{1}{s^2}$$

$$\implies Y_0(s) = X_0(s)H(s) = \frac{1}{s^3}$$

Using the table of unilateral transform pairs, we inverse-transform the product back to the time domain to get

$$y_0(t) = \mathcal{L}^{-1}[Y_0(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^3}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{2} \cdot \frac{2}{s^3}\right]$$

$$= \frac{1}{2} \cdot \mathcal{L}^{-1}\left[\frac{2}{s^3}\right] = \frac{1}{2} \cdot t^2 u(t).$$

By the superposition principle for LTI systems, it follows that

$$x(t) = x_0(t-2) - x_0(t-4) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = y_0(t-2) - y_0(t-4)$$
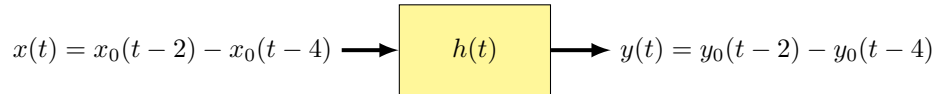
Therefore,

$$y(t) = y_0(t-2) - y_0(t-4)$$

$$= \frac{1}{2} \cdot (t-2)^2 u(t-2) - \frac{1}{2} \cdot (t-4)^2 u(t-4).$$

■

**Table 4.2: Unilateral Laplace Transform Pairs**

| $x(t)$ | $X(s) = \mathcal{L}[x(t)]$ |
|--------|---------------------------|
| $\delta(t)$ | $1$ |
| $u(t)$ | $\dfrac{1}{s}$ |
| $e^{-at}u(t)$ | $\dfrac{1}{s+a}$ |
| $t \cdot u(t)$ | $\dfrac{1}{s^2}$ |
| $t^2 \cdot u(t)$ | $\dfrac{2}{s^3}$ |
| $te^{-at}u(t)$ | $\dfrac{1}{(s+a)^2}$ |
| $t^2 e^{-at}u(t)$ | $\dfrac{2}{(s+a)^3}$ |
| $t^{n-1}e^{-at}u(t)$ | $\dfrac{(n-1)!}{(s+a)^n}$ |
| $\sin(\omega_0 t)u(t)$ | $\dfrac{\omega_0}{s^2 + \omega_0^2}$ |
| $\sin(\omega_0 t + \theta)u(t)$ | $\dfrac{s\sin\theta + \omega_0\cos\theta}{s^2 + \omega_0^2}$ |
| $\cos(\omega_0 t)u(t)$ | $\dfrac{s}{s^2 + \omega_0^2}$ |
| $\cos(\omega_0 t + \theta)u(t)$ | $\dfrac{s\cos\theta - \omega_0\sin\theta}{s^2 + \omega_0^2}$ |
| $e^{-at}\sin(\omega_0 t)u(t)$ | $\dfrac{\omega_0}{(s+a)^2 + \omega_0^2}$ |
| $e^{-at}\cos(\omega_0 t)u(t)$ | $\dfrac{s+a}{(s+a)^2 + \omega_0^2}$ |
| $2e^{-at}\cos(bt-\theta)u(t)$ | $\dfrac{e^{+j\theta}}{s+a+jb} + \dfrac{e^{-j\theta}}{s+a-jb}$ |
| $e^{-at}\cos(bt-\theta)u(t)$ | $\dfrac{(s+a)\cos\theta + b\sin\theta}{(s+a)^2 + b^2}$ |
| $\dfrac{2t^{n-1}}{(n-1)!}e^{-at}\cos(bt-\theta)u(t)$ | $\dfrac{e^{+j\theta}}{(s+a+jb)^n} + \dfrac{e^{-j\theta}}{(s+a-jb)^n}$ |

## 4.2 Bilateral Laplace Transform

The *bilateral Laplace transform* of a signal $x(t)$ is defined as

$$X(s) = \mathcal{L}[x(t)] = \int_{-\infty}^{+\infty} x(t)e^{-st}\, dt. \tag{4.8}$$

Similar to the unilateral Laplace transform, for a signal $x(t)$ to be transformable, the bilateral Laplace integral must be absolutely integrable such that the simplified integrable condition given by

$$\int_{-\infty}^{+\infty} |x(t)|e^{-\sigma t}\, dt < \infty \tag{4.9}$$

holds true. However, the region of convergence (ROC) plays a bigger role for the bilateral Laplace transform.

While the unilateral Laplace transform refers to the ROC to validate the existence of a unilateral transform pair, the bilateral Laplace transform needs the ROC to correctly map the $s$-domain solution to the time domain. That is, the ROC informs the mapping for the inverse bilateral transform. Two different signals may have the same Laplace transform and only differ by their ROCs.

For instance, consider the causal signal $x(t) = e^{-at}u(t)$ with the integrable condition

$$\int_{-\infty}^{+\infty} |e^{-at}u(t)|e^{-\sigma t}\, dt = \int_{0^-}^{\infty} e^{-(a+\sigma)t}\, dt < \infty.$$

If $\text{Re}(s) > -a$, then it follows that

$$\int_{0^-}^{\infty} e^{-(a+\sigma)t}\, dt = \left[ \frac{e^{-(a+\sigma)t}}{-(a+\sigma)} \right]_{0^-}^{\infty} < \infty.$$

If $\text{Re}(s) \leq -a$, then for $\epsilon = -(a+\sigma) \geq 0$, it follows that

$$\int_{0^-}^{\infty} e^{-(a+\sigma)t}\, dt = \int_{0^-}^{\infty} e^{+\epsilon t}\, dt = \left[ \frac{e^{\epsilon t}}{\epsilon} \right]_{0^-}^{\infty} = \infty.$$

However, the anticausal signal $x_A(t) = -e^{-at}u(-t)$ has an ROC of $\text{Re}(s) \leq -a$ such that

$$\int_{-\infty}^{+\infty} |-e^{-at}u(-t)|e^{-(a+\sigma)t}\, dt = \int_{-\infty}^{0^+} e^{-(a+\sigma)t}\, dt = \left[ \frac{e^{-(a+\sigma)t}}{-(a+\sigma)} \right]_{-\infty}^{0^+} < \infty.$$

Here, we see two signals – one of which is causal and the other anticausal – that would have the same bilateral Laplace transform $X(s) = \frac{1}{s+a}$ but with different ROCs.

Generally, while the unilateral Laplace transform is convenient for solving differential equations with initial conditions, the bilateral Laplace transform provides insight into system characteristics such as stability, causality, and frequency response.

However, we can avoid using the bilateral Laplace transform (and avoid ROC calculations) when analyzing system characteristics by setting zero initial conditions and/or using other tools such as the Fourier transform, which will be covered in a later chapter.

For the rest of the text, when referring to the Laplace transform, we will be using the unilateral form.

## 4.3 Partial Fraction Expansion

When converting a solution from $s$-domain back to time domain, we will often have a $s$-domain solution in the form of a rational function that may need to be rewritten as a sum of parts, with each part having a Laplace transform pair.

Consider an $s$-domain function with the form

$$X(s) = \frac{N(s)}{D(s)}, \tag{4.10}$$

written as a ratio of a polynomial numerator $N(s)$ to a polynomial denominator $D(s)$.

The roots of the numerator are called the *zeros* of $X(s)$ and are defined by

$$N(s) = 0, \tag{4.11}$$

whereas the roots of the denominator are called the *poles* of $X(s)$ and are defined by

$$D(s) = 0. \tag{4.12}$$

Before even performing partial fraction expansion, first we examine the orders of the numerator and denominator. Let

$$m = \deg[N(s)], \tag{4.13}$$
$$n = \deg[D(s)]. \tag{4.14}$$

There are three types of rational functions, each with a different task at hand.

- $m < n$: $X(s)$ is *strictly proper*. It is ready for partial fraction expansion, if needed.

- $m = n$: $X(s)$ is *proper*. Perform long division and express $X(s)$ in terms of the quotient and remainder, then determine if the remainder term needs partial fraction expansion.

- $m > n$: $X(s)$ is *improper*. Perform long division and express $X(s)$ in terms of the quotient and remainder, then determine if the remainder term needs partial fraction expansion.

A term would need partial fraction expansion if there is not an easily matching Laplace transform pair provided by the table. Usually this occurs when $\deg[D(s)] > 2$.

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**Example 4.3.1.**  Determine if

$$X(s) = \frac{2s^2 + 8s + 6}{s^2 + 2s + 1}$$

needs to undergo partial fraction expansion.

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**SOLUTION**

Since both the numerator and denominator are of the same polynomic order, first we need to long divide.

$$\begin{array}{r} 2 \phantom{aaaaaaaaaa} \\ s^2 + 2s + 1{\overline{\smash{\big)}\,}} 2s^2 + 8s + 6 \phantom{} \\ \underline{-\ 2s^2 - 4s - 2} \phantom{} \\ 4s + 4 \end{array}$$

From here, we can rewrite $X(s)$ as

$$X(s) = 2 + \frac{4s + 4}{s^2 + 2s + 1} = 2 + \frac{4(s+1)}{(s+1)^2} = 2 + \frac{4}{s+1}.$$

Since both addends are similar to the ones in the Laplace transform pairs table, partial fraction expansion is not needed, and

$$x(t) = \mathcal{L}^{-1}\left[2 + \frac{4}{s+1}\right] = 2\delta(t) + 4e^{-t}u(t).$$

■

**Example 4.3.2.** Determine if

$$X(s) = \frac{s^4 + 4}{s^3 - 3s - 1}$$

needs to undergo partial fraction expansion.

**SOLUTION**

Since both the numerator and denominator are of the same polynomic order, first we need to long divide.

$$\begin{array}{r} s \phantom{aaaaaaaaaa} \\ s^3 - 3s - 1{\overline{\smash{\big)}\,}} s^4 \phantom{aaaaaaa} + 4 \\ \underline{-\ s^4 + 3s^2 + s} \phantom{} \\ 3s^2 + s + 4 \end{array}$$

From here, we can rewrite $X(s)$ as

$$X(s) = s + \frac{3s^2 + s + 4}{s^3 - 3s - 1}.$$

Since the remainder term does not have a matching Laplace transform pair, partial fraction expansion is needed.

■

Given that a remainder term requires partial fraction expansion, all poles $p_n$ of $X(s)$ must be solved first so that $D(s)$ is expressed in factored form. That is,

$$X(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_{\ell=1}^{n}(s - p_\ell)}. \tag{4.15}$$

There are four different cases for partial fraction expansion. Depending on the poles, one or more cases may apply. Each case requires using a technique called the *residue method*, which evaluates a modified expression at a pole $s = p$ so that factors cancel and expansion coefficients can be calculated.

**Case 1: Distinct real poles.**

Suppose there are $n$ distinct real poles in $X(s)$. Then $X(s)$ can be expanded as

$$X(s) = \frac{N(s)}{(s - p_1)(s - p_2) \cdots (s - p_n)} = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2} + \cdots + \frac{A_n}{s - p_n}. \tag{4.16}$$

The expansion coefficients for Case 1 is then given by

$$A_n = (s - p_n)X(s)\big|_{s=p_n}. \tag{4.17}$$

**Example 4.3.3.** Use the residue method to expand $X(s) = \dfrac{s^2 - 4s + 3}{s(s + 1)(s + 3)}$.

**SOLUTION**

First, set up the expanded form

$$X(s) = \frac{A_1}{s} + \frac{A_2}{s + 1} + \frac{A_3}{s + 3}.$$

The expansion coefficients are given by

$$A_1 = sX(s)\big|_{s=0} = s\frac{s^2 - 4s + 3}{s(s + 1)(s + 3)}\bigg|_{s=0} = \frac{s^2 - 4s + 3}{(s + 1)(s + 3)}\bigg|_{s=0} = 1$$

$$A_2 = (s + 1)X(s)\big|_{s=-1} = (s + 1)\frac{s^2 - 4s + 3}{s(s + 1)(s + 3)}\bigg|_{s=-1} = \frac{s^2 - 4s + 3}{s(s + 3)}\bigg|_{s=0} = -4$$

$$A_3 = (s + 3)X(s)\big|_{s=-3} = (s + 3)\frac{s^2 - 4s + 3}{s(s + 1)(s + 3)}\bigg|_{s=-1} = \frac{s^2 - 4s + 3}{s(s + 1)}\bigg|_{s=0} = 4$$

The full expansion is then

$$X(s) = \frac{1}{s} - \frac{4}{s + 1} + \frac{4}{s + 3}.$$

Here, we see that inverse Laplace transform is

$$x(t) = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{4}{s + 1} + \frac{4}{s + 3}\right]$$
$$= u(t) - 4e^{-t}u(t) + 4e^{-3t}u(t).$$

**Case 2: Repeated real poles.**

Suppose there is a repeated pole at $s = p$. Then $X(s)$ can be expanded as

$$X(s) = \frac{N(s)}{(s-p)^m} = \frac{B_1}{s-p} + \frac{B_2}{(s-p)^2} + \cdots + \frac{B_m}{(s-p)^m}, \tag{4.18}$$

with expansion coefficients given by

$$B_m = (s-p)^m X(s)\big|_{s=p} \tag{4.19}$$

$$B_{m-1} = \left\{ \frac{d}{ds} \left[ (s-p)^m X(s) \right] \right\} \Big|_{s=p} \tag{4.20}$$

$$B_{m-2} = \left\{ \frac{1}{2!} \cdot \frac{d^2}{ds^2} \left[ (s-p)^m X(s) \right] \right\} \Big|_{s=p}, \tag{4.21}$$

or more generally,

$$B_n = \left\{ \frac{1}{(m-n)!} \cdot \frac{d^{m-n}}{ds^{m-n}} \left[ (s-p)^m X(s) \right] \right\} \Big|_{s=p}, \quad \text{for } n = 1, 2, ..., m. \tag{4.22}$$

**Example 4.3.4.** Use the residue method to expand $X(s) = \dfrac{s^2 + 3s + 3}{(s+2)(s+3)^3}$.

**SOLUTION**

Setting up the expanded form,

$$X(s) = \frac{A_1}{s+2} + \frac{B_1}{s+3} + \frac{B_2}{(s+3)^2} + \frac{B_3}{(s+3)^3}.$$

The expansion coefficients are given by

$$A_1 = sX(s)\big|_{s=-2} = s\frac{s^2+3s+3}{(s+2)(s+3)^3}\bigg|_{s=-2} = \frac{s^2+3s+3}{(s+3)^3}\bigg|_{s=-2} = 1$$

$$B_3 = (s+3)^3 X(s)\big|_{s=-3} = (s+3)^3 \frac{s^2+3s+3}{(s+2)(s+3)^3}\bigg|_{s=-3} = \frac{s^2+3s+3}{s+2}\bigg|_{s=-3} = -3$$

$$B_2 = \left\{ \frac{d}{ds}[(s+3)^3 X(s)] \right\}\bigg|_{s=-3} = \frac{d}{ds}\left[\frac{s^2+3s+3}{s+2}\right]\bigg|_{s=-3} = \frac{s^2+4s+3}{(s+2)^2}\bigg|_{s=-3} = 0$$

$$B_1 = \left\{ \frac{1}{2!} \cdot \frac{d^2}{ds^2}[(s+3)^3 X(s)] \right\}\bigg|_{s=-3} = \frac{1}{2} \cdot \frac{d^2}{ds^2}\left[\frac{s^2+3s+3}{s+2}\right]\bigg|_{s=-3} = \frac{1}{2} \cdot \frac{2}{(s+2)^3}\bigg|_{s=-3} = -1$$

The full expanded form is then

$$X(s) = \frac{1}{s+2} - \frac{1}{s+3} - \frac{3}{(s+3)^3},$$

with inverse Laplace transform

$$x(t) = e^{-2t}u(t) - e^{-3t}u(t) - \frac{3}{2}t^2 e^{-3t}u(t)$$

$$= \left[ e^{-2t} - e^{-3t} - \frac{3}{2}t^2 e^{-3t} \right] u(t).$$

■

**Case 3: Distinct complex poles.**

If $X(s)$ has a complex pole at $s = p$, it also follows that its complex conjugate $s = p^*$ is also a pole. Interestingly, the expansion coefficients associated with both complex poles are also complex conjugates of each other. That is,

$$X(s) = \frac{N(s)}{s^2 + as + b} = \frac{C}{s - p} + \frac{C^*}{s - p^*}, \tag{4.23}$$

with expansion coefficient

$$C = (s - p)X(s)\big|_{s=p}. \tag{4.24}$$

**Example 4.3.5.**    Use the residue method to expand $X(s) = \dfrac{s(s - 8)}{(s + 2)(s^2 + 16)}$.

**SOLUTION**

It follows that for $s^2 + 16 = 0$, there are poles at $s = \pm j4$, along with $s = -2$. The expanded form then is given by

$$X(s) = \frac{A_1}{s + 2} + \frac{C_1}{s + j4} + \frac{C_1^*}{s - j4},$$

with expansion coefficients

$$A_1 = (s + 2)X(s)\big|_{s=-2} = (s + 2)\frac{s(s - 8)}{(s + 2)(s^2 + 16)}\bigg|_{s=-2} = \frac{s(s - 8)}{(s^2 + 16)}\bigg|_{s=-2} = 1$$

$$C_1 = (s + j4)X(s)\big|_{s=-j4} = \frac{s(s - 8)}{(s + 2)(s - j4)}\bigg|_{s=-j4} = -j = e^{-j\pi/2}$$

$$C_1^* = C_1\big|_{j \leftarrow (-j)} = e^{+j\pi/2}$$

The full expanded form is then

$$X(s) = \frac{1}{s + 2} + \frac{e^{-j\pi/2}}{s + j4} + \frac{e^{+j\pi/2}}{s - j4},$$

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with inverse Laplace transform

$$x(t) = e^{-2t}u(t) + 2\cos(4t + \pi/2)u(t).$$

■

**Case 4: Repeated complex poles.**

Lastly, for repeated complex poles, the expansion is given by

$$X(s) = \frac{N(s)}{(s^2 + as + b)^m} = \frac{N(s)}{(s-p)^m(s-p^*)^m}$$
$$= \left(\frac{D_1}{s-p} + \frac{D_2}{(s-p)^2} + \cdots + \frac{D_m}{(s-p)^m}\right) + \left(\frac{D_1^*}{s-p^*} + \frac{D_2^*}{(s-p^*)^2} + \cdots + \frac{D_m^*}{(s-p^*)^m}\right), \quad (4.25)$$

with expansion coefficients

$$D_n = \left\{\frac{1}{(m-n)!} \cdot \frac{d^{m-n}}{ds^{m-n}}[(s-p)^m X(s)]\right\}\bigg|_{s=p}, \quad \text{for } n = 1, 2, ..., m. \quad (4.26)$$

Table 4.3 lists the corresponding Laplace transform pairs for the four partial fraction expansion cases.

**Table 4.3: Unilateral Laplace Transform Pairs Based on Poles**

| Pole | $X(s)$ | $x(t) = \mathcal{L}^{-1}[X(s)]$ |
|---|---|---|
| Distinct real | $\dfrac{A}{s+a}$ | $Ae^{-at}u(t)$ |
| Repeated real | $\dfrac{A}{(s+a)^n}$ | $\dfrac{At^{n-1}}{(n-1)!}e^{-at}u(t)$ |
| Distinct complex | $\dfrac{Ae^{+j\theta}}{s+a+jb} + \dfrac{Ae^{-j\theta}}{s+a-jb}$ | $2Ae^{-at}\cos(bt - \theta)u(t)$ |
| Repeated complex | $\dfrac{Ae^{+j\theta}}{(s+a+jb)^n} + \dfrac{Ae^{-j\theta}}{(s+a-jb)^n}$ | $\dfrac{2At^{n-1}}{(n-1)!}e^{-at}\cos(bt - \theta)u(t)$ |

## 4.4 Transfer Function

Given that all initial conditions of the input and output signals of an LTI system are zero, the *transfer function* is the Laplace transform of the impulse response and is defined as

$$H(s) = \frac{Y(s)}{X(s)} = \mathcal{L}[h(t)]. \quad (4.27)$$

In fact, by the convolution property, it follows that

$$y(t) = x(t) * h(t) \iff Y(s) = X(s)H(s). \quad (4.28)$$

Symbolically,
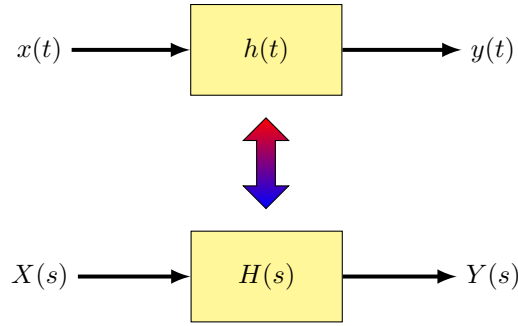
As the impulse response is the system response to input $x(t) = \delta(t)$, consider the Laplace transform of the impulse signal $\mathcal{L}[\delta(t)] = 1$. Then it also follows that
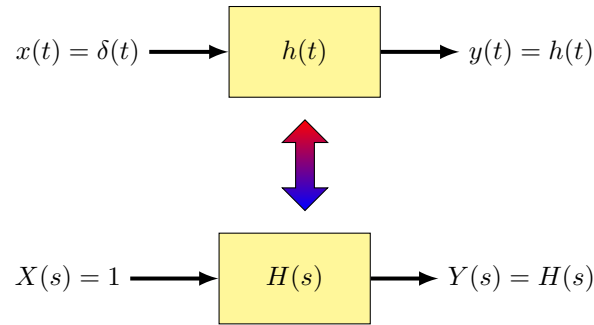


To find the transfer function of an LTI system, there are two approaches:

1. Find $X(s), Y(s)$. Then calculate $H(s) = Y(s)/X(s)$.

2. Find $X(s), Y(s)$. Then substitute $X(s) = 1$ and $Y(s) = H(s)$.

Typically, the transfer function $H(s)$ is used to analyze the characteristics of the LTI system. Sometimes, when one is interested in representing systems solely for the purpose of finding an output response, the *conditional transfer function* $H_C(s)$ is calculated, where initial conditions are not necessarily zero; this is when the substitution approach works best. However, when including nonzero initial conditions, the system can no longer be called LTI.

**Example 4.4.1.**    Find the transfer function for an LTI system characterized by LCCDE

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t),$$
$$x(0^-) = 0,$$
$$y(0^-) = y'(0^-) = 1.$$

Then find the conditional transfer function of the system.

**SOLUTION**

To find the transfer function $H(s)$, set all initial conditions to zero and take the Laplace trans-

form of the LCCDE.

$$\mathcal{L}\left[\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t)\right] = \mathcal{L}\left[\frac{dx(t)}{dt} + 3x(t)\right]$$

$$s^2Y(s) + 3sY(s) + 2Y(s) = sX(s) + 3X(s)$$

$$Y(s)[s^2 + 3s + 2] = X(s)[s + 3]$$

Rearranging the equation, we get the transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{s^2 + 3s + 2}.$$

Now we consider the initial conditions to solve the conditional transfer function.

$$[s^2Y(s) - s - 1] + 3[sY(s) - 1] + 2Y(s) = [sX(s) - 0] + 3X(s)$$

$$Y(s)[s^2 + 3s + 2] - (s + 4) = X(s)[s + 3].$$

Now let $X(s) = 1$ and $Y(s) = H_C(s)$. The conditional transfer function is given by

$$H_C(s)[s^2 + 3s + 2] - (s + 4) = [1][s + 3]$$

$$\implies H_C(s) = \frac{2s+7}{s^2 + 3s + 2}$$

$\blacksquare$

For the rest of this text, unless explicitly specified, only transfer functions with zero initial conditions will be considered.

### 4.4.1 System Stability

As previously introduced, zeros are the roots of the numerator of a transfer function ($N(s) = 0$), and poles are the roots of the denominator of a transfer function ($D(s) = 0$). We can visualize both zeros and poles on the *s-plane*, which plots $\mathrm{Im}(s)$ versus $\mathrm{Re}(s)$. As seen in Figure 4.1, zeros are plotted with circles, and poles are plotted with crosses.

The *open left half-plane* (OLHP) refers to the region of the *s*-plane which describes $\mathrm{Re}(s) < 0$.

- If an LTI system with transfer function $H(s)$ is strictly proper or proper, then the system is BIBO stable if and only if its poles reside in the OLHP.

- If $H(s)$ is improper, then the system is unstable.

Note that the imaginary axis is not included in the OLHP; a system with poles on the imaginary axis is not stable.

**Example 4.4.2.** Determine if an LTI system with the following impulse response is BIBO stable:

$$h(t) = (4 + j5)e^{(2+j3)t}u(t) + (4 - j5)e^{(2-j3)t}u(t)$$

**Figure 4.1:** $s$-plane plot of a system with zeros and poles.
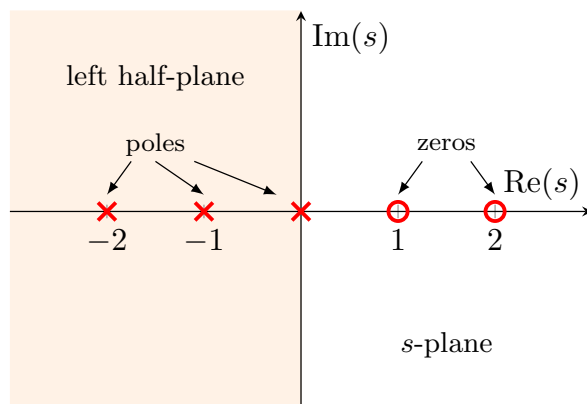
**SOLUTION**

First, find the transfer function by taking the Laplace transform of the impulse response.

$$H(s) = \mathcal{L}[h(t)] = (4 + j5)\mathcal{L}[e^{(2+j3)t}u(t)] + (4 - j5)\mathcal{L}[e^{(2-j3)t}u(t)]$$

$$= \frac{4 + j5}{s - 2 - j3} + \frac{4 - j5}{s - 2 + j3}$$

$$= \frac{8s - 46}{(s - 2 - j3)(s - 2 + j3)}$$

We can see that the poles are given by $p_1 = 2 + j3$ and $p_2 = 2 - j3$, and the zero is given by $z_1 = 46/8$. Since $\text{Re}(p_k) > 0$, the poles are not in the OHLP, and thus the system is unstable.

■

### 4.4.2 Invertible Systems

As previously introduced, a system is invertible if there exists an inverse system that maps an output signal $y(t)$ back to its input $x(t)$. In the LTI case, an inverse system characterized by $g(t)$ exists such that
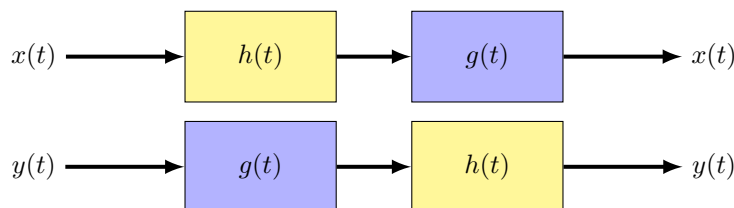


While an inverse system $g(t)$ may exist, it does not necessarily mean it is stable. In fact, for

$$G(s) = \frac{1}{H(s)}, \tag{4.29}$$

in order for a BIBO stable and causal LTI system $H(s)$ to have a stable and causal inverse LTI system $G(s)$, the transfer function $H(s)$ must be a proper rational function (i.e., $\deg[N(s)] = \deg[D(s)]$) with both its poles and zeros all residing in the OLHP of the $s$-plane. Such a system is called a *minimum phase system*.

**Example 4.4.3.** An LTI system is characterized by its impulse response

$$h(t) = \delta(t) - 4e^{-3t}u(t).$$

Determine if it is a minimum phase system.

**SOLUTION**

$$H(s) = \mathcal{L}[h(t)] = \mathcal{L}[\delta(t)] - 4\mathcal{L}[e^{-3t}u(t)]$$
$$= 1 - \frac{4}{s+3} = \frac{s-1}{s+3}.$$

From above, there is a pole $p = -3$ and a zero $z = 1$. While the forward system may be stable for $\mathrm{Re}(p) < 0$, the inverse is not since $\mathrm{Re}(z) > 0$. Therefore, the LTI system is not a minimum phase system. ∎

## 4.5 System Response of Second-Order LCCDEs

Consider second-order LCCDEs of the form

$$\frac{d^2y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_2 y(t) = b_1 \frac{dx(t)}{dt} + b_2 x(t). \tag{4.30}$$

Setting initial conditions to zero such that the system is LTI, the transfer function is given by

$$s^2 Y(s) + a_1 s Y(s) + a_2 Y(s) = b_1 s X(s) + b_2 X(s)$$
$$\implies H(s) = \frac{Y(s)}{X(s)} = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}. \tag{4.31}$$

Define the following system attributes

- Attenuation coefficient [Np/s]: $\alpha = \dfrac{a_1}{2}$

- Undamped natural frequency [rad/s]: $\omega_0 = \sqrt{a_2}$

- Damping coefficient [unitless]: $\xi = \dfrac{\alpha}{\omega_0} = \dfrac{a_1}{2\sqrt{a_2}}$

such that the solutions to the *characteristic equation*

$$s^2 + a_1 s + a_2 = 0 \tag{4.32}$$

can be written as

$$s = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = \omega_0 \left[ -\xi \pm \sqrt{\xi^2 - 1} \right]. \tag{4.33}$$

In fact, the solutions to the characteristic equation are the poles $p_1, p_2$ of the system. Since $\mathrm{Re}(p_k) < 0$ must be true for a system to be BIBO stable, it also holds from above that $a_1 > 0$ and $a_2 > 0$ must be true for the system to be BIBO stable.

Depending on the damping coefficient $\xi$, there are five possible system responses for an input impulse or step signal:

- $\xi > 1$: overdamped response

- $\xi = 1$: critically damped response

- $0 < \xi < 1$: underdamped response

- $\xi = 0$: undamped response

- $\xi < 0$: unstable response

**Figure 4.2: Second-order system response on the $s$-plane.**
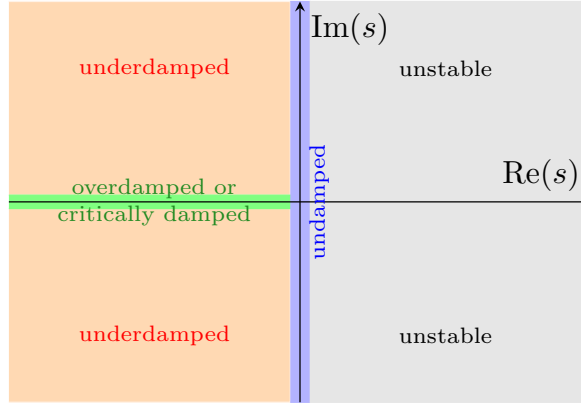


We are particularly interested in the first three damping responses as they describe how a system response reaches *steady-state*, or *equilibrium*. That is, it describes how fast and in what manner a system approaches the final value of the system response.

As seen in Figure 4.3, overdamped responses slowly approach steady-state, whereas underdamped responses quickly but oscillatorily approach steady-state at a *damped natural frequency* of

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \omega\sqrt{1 - \xi^2}. \tag{4.34}$$

Critically damped responses represent the quickest non-oscillatory path to steady-state.

**Figure 4.3: Damping responses.**
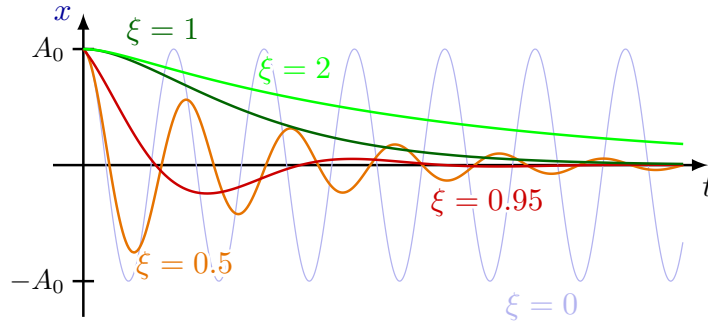
**Example 4.5.1.**    An LTI system is described by the LCCDE

$$\frac{d^2y(t)}{dt^2} + B\frac{dy(t)}{dt} + 25y(t) = \frac{dx(t)}{dt} + 23x(t).$$

Compute the range of values for constant $B$ such that the system impulse response is:

(a) overdamped

(b) critically damped

(c) underdamped

Then let $B = 26$ and compute the impulse and step responses of the system.

**SOLUTION**

The characteristic equation of the LCCDE is

$$s^2 + Bs + 25 = 0.$$

The damping coefficient is then given by

$$\xi = \frac{a_1}{2\sqrt{a_2}} = \frac{B}{2\sqrt{25}} = \frac{B}{10}.$$

It then follows that

- overdamped: $\xi > 1 \Longrightarrow B > 10$

- critically damped: $\xi = 1 \Longrightarrow B = 10$

- underdamped: $0 < \xi < 1 \Longrightarrow 0 < B < 10$

Letting $B = 26$ and setting initial conditions to zero, the Laplace transform of the LCCDE is

$$Y(s)[s^2 + 26s + 25] = X(s)[s + 23].$$

The transfer function is given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+23}{s^2 + 26s + 25} = \frac{s+23}{(s+1)(s+25)} = \frac{A_1}{s+1} + \frac{A_2}{s+25},$$

$$A_1 = (s+1)H(s)\big|_{s=-1} = 22/24 = 11/12,$$

$$A_2 = (s+25)H(s)\big|_{s=-25} = (-2)/(-24) = 1/12.$$

Therefore the impulse response is

$$h(t) = \mathcal{L}^{-1}[H(s)] = \frac{1}{12}\left[11e^{-t}u(t) + e^{-25t}u(t)\right].$$

The step response can be calculated from

$$y_{step}(t) = \int_{-\infty}^{t} h(\tau)\,d\tau \iff Y_{step}(s) = \frac{1}{s}H(s).$$

$$Y_{step}(s) = \frac{s+23}{s(s+1)(s+25)} = \frac{A_0}{s} + \frac{A_1}{s+1} + \frac{A_2}{s+25},$$
$$A_0 = sY_{step}(s)\big|_{s=0} = 23/25$$
$$A_1 = (s+1)Y_{step}(s)\big|_{s=-1} = 22/(-24) = -11/12,$$
$$A_2 = (s+25)Y_{step}(s)\big|_{s=-25} = (-2)/600 = -1/300.$$

$$y_{step}(t) = \mathcal{L}^{-1}[Y_{step}(s)] = \frac{23}{25}u(t) - \frac{11}{12}e^{-t}u(t) - \frac{1}{300}e^{-25t}u(t).$$
∎

## 4.6   System Response Partitions

Given some system response $y(t)$, there are three different ways to partition the response, with each partition describing a particular characteristic of the system response:

1. Zero-state / zero-input partition

2. Natural / forced partition

3. Transient / steady-state partition

**Type 1: Zero-state / zero-input partition.**

As the names imply,

- the *zero-state response* (ZSR) is the system response when there are no initial conditions, and

- the *zero-input response* (ZIR) is the system response when there is no input signal (but there are initial conditions).

It follows that

$$y(t) = y_{ZSR}(t) + y_{ZIR}(t). \tag{4.35}$$

Any exponentials $e^{-\lambda_k t}$ in the zero-input response $y_{ZIR}(t)$ are referred to as the *characteristic modes* (also called *natural modes*, or simply *modes*) of the system. The zero-input response is comprised of only natural modes, whereas the zero-state response has a mix of natural modes and "forced" modes from the input excitation function $x(t)$.

**Type 2: Natural / forced partition.**

On the discussion of natural modes and forced modes,

- the *natural response* is the part of the system response with only natural modes which describe the natural identity of the system, and

- the *forced response* is the part of the system response with only forced modes which mimic the character of the excitation function $x(t)$.

It follows that

$$y(t) = y_{nat}(t) + y_{forc}(t). \tag{4.36}$$

The natural response is closely related to the zero-input response; since the zero-input response is composed exclusively of modes, we can use it to identify the modes, then collect all terms with those modes to find the natural response. Naturally, the characteristic modes can also be determined by finding the solutions $s$ of the characteristic equation and calculating $e^{-st}$.

### Type 3: Transient / steady-state partition.

The end behavior of the terms of the system response can be analyzed such that

- the *transient response* is the part of the system response that decays to zero as $t \to \infty$, and

- the *steady-state response* is the part of the system response that remains after the transient response goes to zero.

It follows that

$$y(t) = y_{tr}(t) + y_{ss}(t). \tag{4.37}$$

We have seen this behavior before with the damping response of a stable LTI system described by a second-order LCCDE. The curves eventually approach a final value $K$, with the approach rate and shape dependent on the damping coefficient $\xi$. The damping response actually can be written as a sum of transient terms and the steady-state response $K \cdot u(t)$.

Again, notice here that we reserve the term "LTI system" for systems that have zero initial conditions, in addition to the superposition principle condition. Partitions can be applied to any LTI or "LTI-like" system response, whether or not the initial conditions are zero.

**Example 4.6.1.** Suppose a system can be characterized by the LCCDE

$$\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + 3x(t),$$
$$y(0) = y'(0) = 1,$$
$$x(t) = u(t).$$

Identify all six partitioned responses of the system response.

**SOLUTION**

Taking the Laplace transform of the LCCDE, it follows that

$$\mathcal{L}\left[\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t)\right] = \mathcal{L}\left[\frac{dx(t)}{dt} + 3x(t)\right]$$
$$[s^2 Y(s) - s - 1] + 3[sY(s) - 1] + 2Y(s) = [sX(s) - 0] + 3X(s)$$
$$Y(s)[s^2 + 3s + 2] - (s + 4) = X(s)[s + 3].$$

Since $X(s) = \mathcal{L}[u(t)] = \dfrac{1}{s}$, it follows that

$$Y(s)[s^2 + 3s + 2] - (s + 4) = \frac{s + 3}{s}$$

$$Y(s)[(s + 1)(s + 2)] = \frac{s^2 + 5s + 3}{s}$$

$$\implies Y(s) = \frac{s^2 + 5s + 3}{s(s + 1)(s + 2)} = \frac{A_1}{s} + \frac{A_2}{s + 1} + \frac{A_3}{s + 2}.$$

After partial fraction decomposition, the inverse Laplace transform gives

$$y(t) = 1.5u(t) + e^{-t}u(t) - 1.5e^{-2t}u(t).$$

To find the modes, first solve for $y_{ZIR}(t)$. We can do this by taking the Laplace transform of the LCCDE and setting $X(s) = 0$.

$$Y(s)[s^2 + 3s + 2] - (s + 4) = X(s)[s + 3] = 0$$

$$\implies Y_{ZIR}(s) = \frac{s + 4}{s^2 + 3s + 2} = \frac{s + 4}{(s + 1)(s + 2)} = \frac{3}{s + 1} - \frac{2}{s + 2}.$$

Then it follows that

$$y_{ZIR}(t) = 3e^{-t}u(t) - 2e^{-2t}u(t),$$
$$y_{ZSR}(t) = y(t) - y_{ZIR}(t) = 1.5u(t) - 2e^{-t}u(t) + 0.5e^{-2t}u(t).$$

From $y_{ZIR}(t)$, the characteristic modes are given by exponentials $\{e^{-t}, e^{-2t}\}$. Therefore, by grouping natural modes together and forced modes together,

$$y_{nat}(t) = e^{-t}u(t) - 1.5e^{-2t}u(t),$$
$$y_{forc}(t) = 1.5u(t).$$

Lastly, we can group decaying terms together and the remaining terms together such that

$$y_{tr} = e^{-t}u(t) - 1.5e^{-2t}u(t),$$
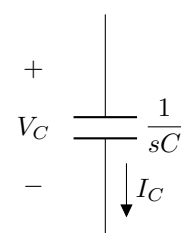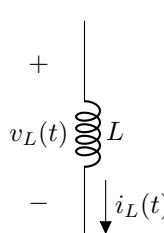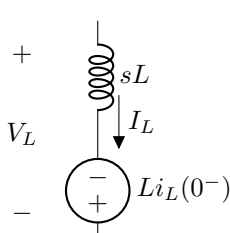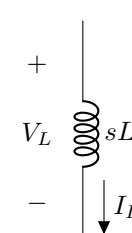$$y_{ss} = 1.5u(t).$$

$\blacksquare$
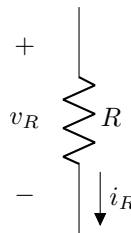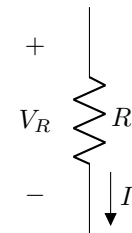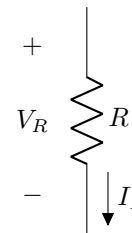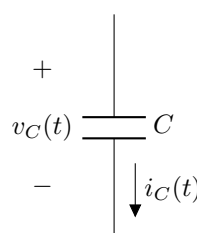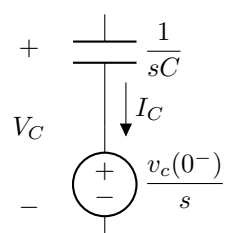
## 4.7   $s$–Domain Circuit Analysis

Just as differential equations can be solved in the $s$-domain, the systems represented by differential equations can be solved in the $s$-domain. For instance, an electric circuit can be transformed to an $s$-domain equivalent and solved using algebraic equations, with the $s$-domain electrical components referred to as impedances. The equivalent $s$-domain models can be seen in Table 4.4.

While only Thévenin equivalent models with voltage sources are shown in Table 4.4, Norton equivalent models with current sources can be drawn instead, though the relation between the Thévenin equivalents and the steady-state equivalents are easier to visualize.

**Table 4.4: Circuit models for electrical components in the $s$-domain.**

| Electrical Component | Time Domain | $s$-Domain with Initial Conditions | $s$-Domain at Steady State |
|---|---|---|---|
| Resistor | $+$ $v_R$ $R$ $-$ $i_R$ | $+$ $V_R$ $R$ $-$ $I_R$ | $+$ $V_R$ $R$ $-$ $I_R$ |
| Capacitor | $+$ $v_C(t)$ $C$ $-$ $i_C(t)$ | $+$ $\frac{1}{sC}$ $V_C$ $I_C$ $\frac{v_c(0^-)}{s}$ $-$ | $+$ $V_C$ $\frac{1}{sC}$ $-$ $I_C$ |
| Inductor | $+$ $v_L(t)$ $L$ $-$ $i_L(t)$ | $+$ $sL$ $V_L$ $I_L$ $Li_L(0^-)$ $-$ | $+$ $V_L$ $sL$ $-$ $I_L$ |

Recall that the unit step function $u(t)$ can be treated as an "on switch", where the signal being multiplied gets turned on at time $t = 0$. Similarly here, circuit switches can be represented using signals multiplied by $u(t)$, as seen in Figure 4.4. Because of how switches operate, here we see that the initial condition is zero.

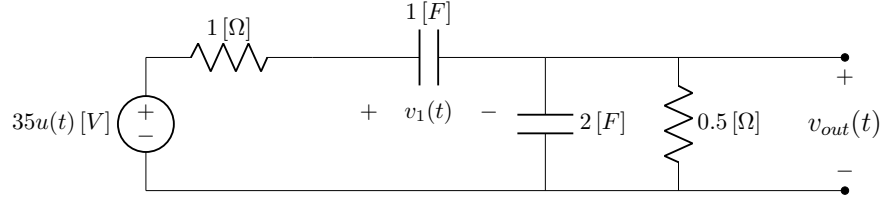**Figure 4.4: Equivalent signal representation of switches.**
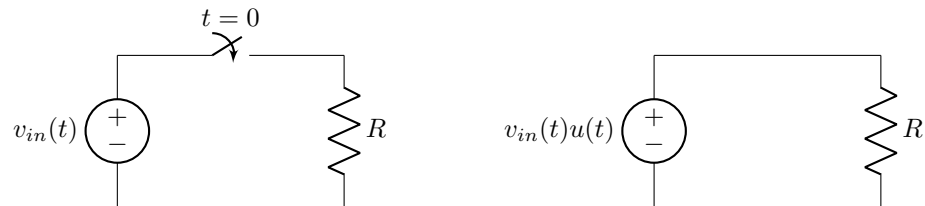
**Example 4.7.1.** Find the output response $v_{out}(t)$ of the following circuit, given that the switch closes at time $t = 0$ and $v_1(0^-) = 20\,[V]$.
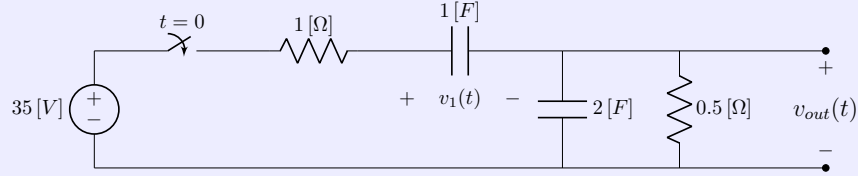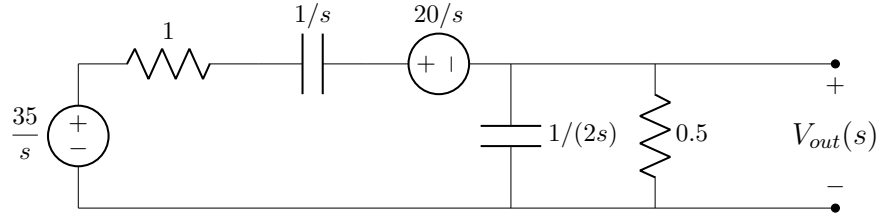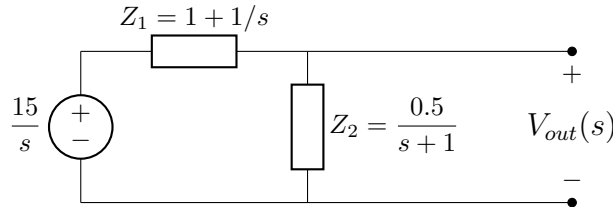


## SOLUTION

Using the equivalent model for switches, first we redraw the circuit as



Now we can transform the circuit into an $s$-domain equivalent model:



We can algebraically combine the two voltage sources together, the two series impedances together, and the two parallel impedances together:



From here, the $s$-domain output response can be calculated using the voltage divider:

$$V_{out}(s) = \frac{15}{s} \cdot \left[\frac{Z_2}{Z_1 + Z_2}\right] = \frac{7.5}{s^2 + 2.5s + 1} = \frac{5}{s + 0.5} - \frac{5}{s + 2}.$$
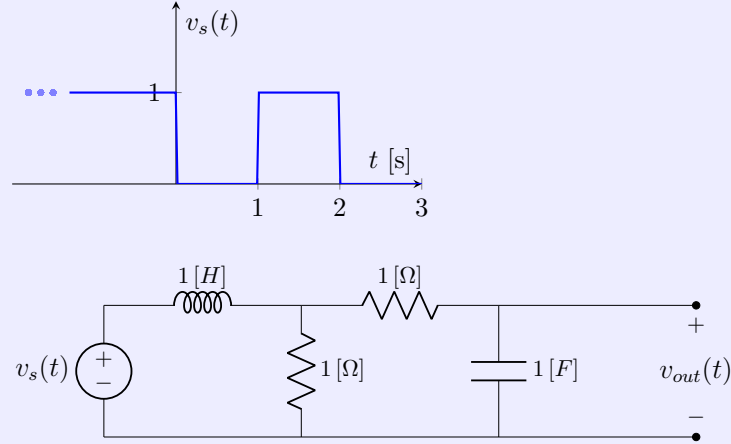
Lastly, taking the solution to the time domain, we get

$$v_{out}(t) = 5e^{-0.5t}u(t) - 5e^{-2t}u(t)\,[V].$$

■

**Example 4.7.2.** Given the plot of input excitation $v_s(t)$, find the conditional transfer function $H_C(s) = V_{out}(s)/V_s(s)$ of the circuit below for time $t \geq 0$. Then find $v_{out}(t)$ for $t \geq 0$. Note that an ellipsis indicates a continuing pattern on the corresponding end.





### SOLUTION

First, find the initial conditions at time $t = 0^-$. Since $v_s(t) = 1$ for $t < 0$, we can assume the circuit is at steady-state at $t = 0^-$. The steady-state circuit for $t < 0$ is given by
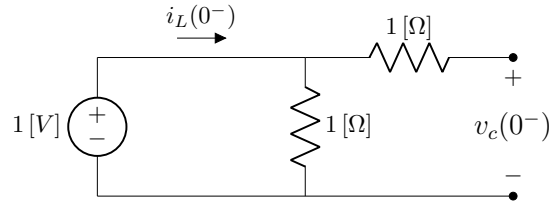


Here we see that $i_L(0^-) = 1\,[V]/1\,[\Omega] = 1\,[A]$ and $v_c(0^-) = 1\,[V]$. Using these initial conditions, we can set up an $s$-domain equivalent of the circuit for $t \geq 0$.
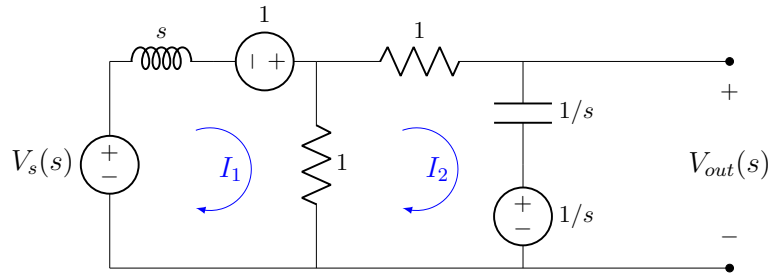


We can use the mesh current method to solve the circuit.

$$-V_s(s) + sI_1 - 1 + 1 \cdot (I_1 - I_2) = 0 \implies (s+1)I_1 - I_2 = V_s(s) + 1$$
$$1 \cdot (I_2 - I_1) + I_2(1 + 1/s) + 1/s = 0 \implies -sI_1 + (2s+1)I_2 = -1$$

In matrix form, the mesh current equations can be written as

$$\begin{bmatrix} s+1 & -1 \\ -s & 2s+1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_s(s)+1 \\ -1 \end{bmatrix}.$$

From here, we can use Cramer's rule to solve for $I_2$ which can then be used to solve for $V_{out}(s)$.

$$\Delta = \begin{vmatrix} s+1 & -1 \\ -s & 2s+1 \end{vmatrix} = 2s^2 + 2s + 1$$

$$\Delta_2 = \begin{vmatrix} s+1 & V_s(s)+1 \\ -s & -1 \end{vmatrix} = sV_s(s) - 1$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{sV_s(s) - 1}{2s^2 + 2s + 1}$$

The conditional transfer function can now be solved.

$$V_{out}(s) = I_2 \cdot \frac{1}{s} + \frac{1}{s} = \frac{sV_s(s) - 1}{s(2s^2 + 2s + 1)} + \frac{1}{s} = \frac{sV_s(s) + 2s^2 + 2s}{s(2s^2 + 2s + 1)}$$

$$\implies H_C(s) = [V_{out}(s)]\big|_{V_s(s)=1} = \frac{2s + 3}{2s^2 + 2s + 1} = \frac{s + 1.5}{s^2 + s + 0.5}$$

With the Laplace transform of input $v_s(t)$ for $t \geq 0$ given by

$$V_s(s) = \mathcal{L}[v_s(t)] = \mathcal{L}[u(t-1) - u(t-2)] = \frac{1}{s}(e^{-s} - e^{-2s}),$$

the equation for $V_{out}(s)$ is given by

$$V_{out}(s) = H_C(s)V_s(s) = (e^{-s} - e^{-2s}) \left[ \frac{s + 1.5}{s(s^2 + s + 0.5)} \right]$$

$$= (e^{-s} - e^{-2s}) \left[ \frac{3}{s} + \frac{-1.5 - j0.5}{s + 0.5 + j0.5} + \frac{-1.5 + j0.5}{s + 0.5 - j0.5} \right]$$

$$= (e^{-s} - e^{-2s}) \underbrace{\left[ \frac{3}{s} + \frac{1.581e^{-j2.82}}{s + 0.5 + j0.5} + \frac{1.581e^{+j2.82}}{s + 0.5 - j0.5} \right]}_{G(s)}.$$

The inverse Laplace transform of $G(s)$ is

$$g(t) = 3u(t) + 3.162e^{-0.5t}\cos(0.5t + 2.82 \text{ [rad]})u(t)$$

The output voltage $v_{out}(t)$ for $t \geq 0$ (with phase expressed in radians) is then

$$v_{out}(t) = \mathcal{L}^{-1}[(e^{-s} - e^{-2s})G(s)] = g(t-1) - g(t-2)$$

$$= 3u(t-1) + 3.162e^{-0.5(t-1)}\cos(0.5(t-1) + 2.82)u(t-1)$$

$$- 3u(t-2) - 3.162e^{-0.5(t-2)}\cos(0.5(t-2) + 2.82)u(t-2). \qquad \blacksquare$$
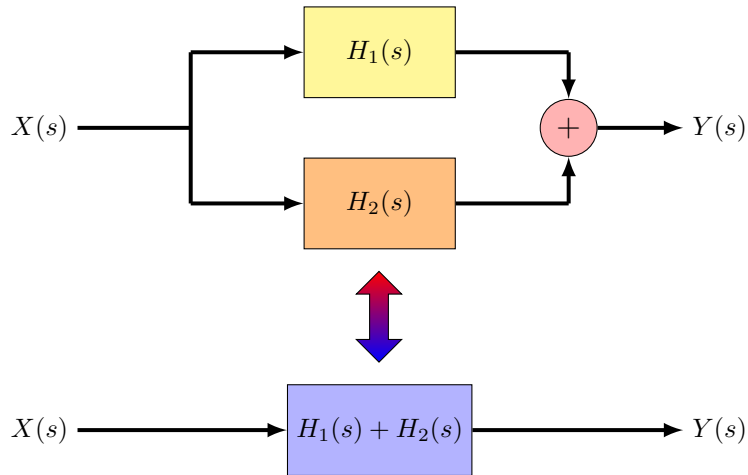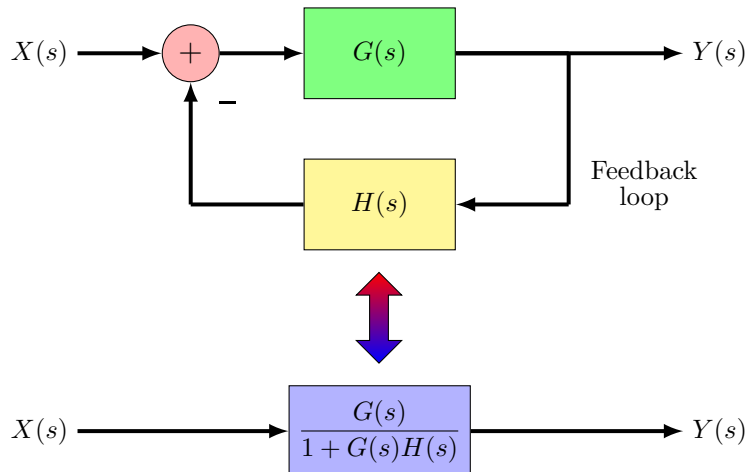
## 4.8  $s$–Domain Block Diagrams

Similar to the interconnection of LTI systems in the time domain, LTI systems can be combined in the $s$-domain. When LTI systems are connected in series, the interconnection has the following equivalence:

$$X(s) \longrightarrow \boxed{H_1(s)} \longrightarrow \boxed{H_2(s)} \longrightarrow Y(s)$$

$$X(s) \longrightarrow \boxed{H_1(s)H_2(s)} \longrightarrow Y(s)$$
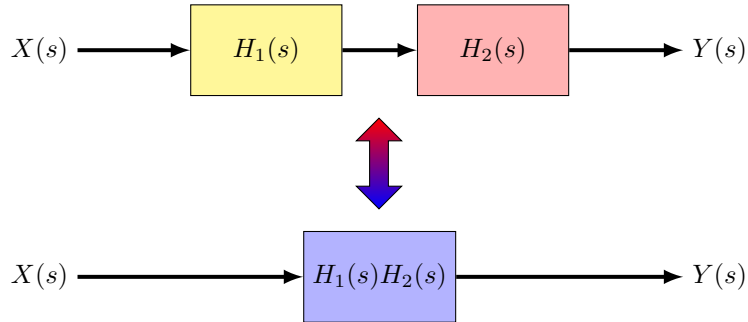
When LTI systems are connected in parallel, the interconnection has the following equivalence:

$$X(s) \longrightarrow \boxed{H_1(s)} \searrow \oplus \longrightarrow Y(s)$$
$$\boxed{H_2(s)} \nearrow$$

$$X(s) \longrightarrow \boxed{H_1(s) + H_2(s)} \longrightarrow Y(s)$$

A minus sign adjacent to an arrowhead at a summation node indicates that the corresponding addend signal is multiplied by $-1$ before being summed. When an interconnection of LTI systems implement *negative feedback*, it has the following equivalence:

$$X(s) \longrightarrow \oplus \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$
$$\boxed{H(s)} \quad \text{Feedback loop}$$

$$X(s) \longrightarrow \boxed{\dfrac{G(s)}{1 + G(s)H(s)}} \longrightarrow Y(s)$$

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*Feedback* takes a sample of the output signal and feeds it back to the input signal. A system without feedback is called an *open-loop system*, whereas a system that utilizes feedback to form a *feedback loop* is called a *closed-loop system*.

A closed-loop system has *positive feedback* if the strength of the input signal is increased as a result; otherwise, a closed-loop system is said to have *negative feedback* if the strength of the input signal is decreased.

An interconnection of LTI systems with feedback loops can be simplified to an equivalent single-transfer-function model.

1. Condense any systems in parallel or series first.

2. Count the number of summation nodes left. The number of equations needed is the number of summation nodes plus one, with the one representing the output.

3. Define an intermediary signal $E_n(s)$ after each summation node $n$.

4. Starting from the output signal $Y(s)$, write an equation for the output. Then working backwards from $Y(s)$, write equations for each summation node.

5. Begin substituting and rearranging equations until a single equation in terms of only $X(s)$ and $Y(s)$ is left.

**Example 4.8.1.** Find an equivalent transfer function for the following interconnection of LTI systems.
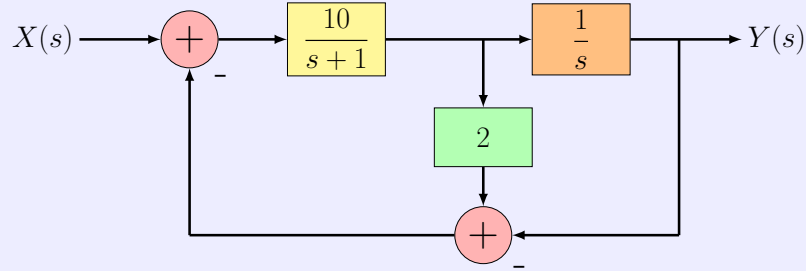


**SOLUTION**

There are two summation nodes. We will need to define $E_1(s), E_2(s)$ and write three equations.
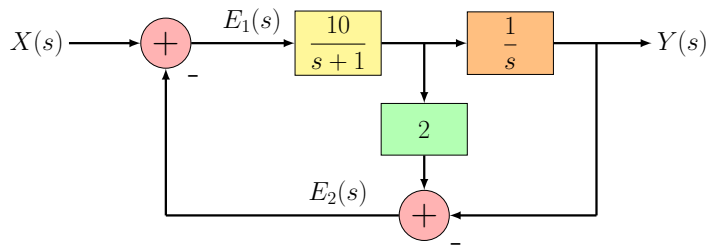
We can first write an equation for $Y(s)$, following the feedforward path at the top.

$$Y(s) = E_1(s) \cdot \frac{10}{s+1} \cdot \frac{1}{s}$$
$$= E_1(s) \left[ \frac{10}{s(s+1)} \right]$$

Now we write an equation for $E_2(s)$ using the rightmost branch.

$$E_2(s) = E_1(s) \cdot \frac{10}{s+1} \cdot 2 - Y(s)$$
$$= E_1(s) \cdot \frac{10}{s+1} \cdot 2 - E_1(s) \left[ \frac{10}{s(s+1)} \right]$$
$$= E_1(s) \cdot \frac{10}{s+1} \left[ 2 - \frac{1}{s} \right]$$
$$= E_1(s) \left[ \frac{10(2s-1)}{s(s+1)} \right]$$

Lastly, we write an equation for $E_1(s)$ using the leftmost branch.

$$E_1(s) = X(s) - E_2(s) = X(s) - E_1(s) \left[ \frac{10(2s-1)}{s(s+1)} \right]$$
$$\implies X(s) = E_1(s) \left[ 1 + \frac{10(2s-1)}{s(s+1)} \right]$$

We can rearrange the equation for $Y(s)$ and substitute into $X(s)$.

$$E_1(s) = Y(s) \left[ \frac{s(s+1)}{10} \right]$$
$$\implies X(s) = E_1(s) \left[ 1 + \frac{10(2s-1)}{s(s+1)} \right]$$
$$= Y(s) \left[ \frac{s(s+1)}{10} \right] \left[ 1 + \frac{10(2s-1)}{s(s+1)} \right]$$
$$= Y(s) \left[ \frac{s(s+1)}{10} + (2s-1) \right]$$
$$= Y(s) \left[ \frac{s^2 + 21s - 10}{10} \right]$$

Then the equivalent transfer function $H(s)$ is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{10}{s^2 + 21s - 10}.$$

■