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STAT 2131
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Homework 1

1) $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$ $\Rightarrow \begin{cases} AB \in \mathbb{R}^{m\times m} \\ BA \in \mathbb{R}^{n\times n} \end{cases}$

$$tr(AB) = \sum_{i=1}^{m} (AB)_{ii} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} b_{ji}$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m} b_{ji} a_{ij} = \sum_{j=1}^{n} (BA)_{jj} = tr(BA)$$

$\Rightarrow$ $tr(AB) = tr(BA)$

2) a) • Let $y \in Im(AG)$ $\Rightarrow y = AGx$ for some $x \in \mathbb{R}^n$

$\Rightarrow y = A(Gx) \Rightarrow y = Az$ for some $z = Gx$

$\Rightarrow y \in Im(A) \Rightarrow Im(AG) \subseteq Im(A)$

• Let $x \in ker(G) \Rightarrow Gx = 0$ $(x \in \mathbb{R}^n)$

$\Rightarrow A(Gx) = 0 \Rightarrow (AG)x = 0$

$\Rightarrow x \in ker(AG) \Rightarrow ker(G) \subseteq ker(AG)$

b) To prove that $Im(AG) = Im(A)$, with the conclusion from part A that $Im(AG) \subseteq Im(A)$, we need to prove that $Im(A) \subseteq Im(AG)$

let $y \in Im(A) \Rightarrow y = Ax$ for some x in $\mathbb{R}^n$

Since $G \in \mathbb{R}^{n\times q}$ $(G: \mathbb{R}^q \to \mathbb{R}^n)$ is surjective

$\Rightarrow$ there exists some z in $\mathbb{R}^q$ such that

$x = Gz$ (x in $\mathbb{R}^n$)

$\Rightarrow y = Ax = A(Gz) = AG(z)$

$\Rightarrow y \in Im(AG)$

$\Rightarrow Im(A) \subseteq Im(AG)$

$\Rightarrow Im(A) = Im(AG)$ if G is surjective

c) Let $y \in Im(A+B)$

$\Rightarrow$ There exists a vector x such that

$$y = (A+B)x = Ax + Bx$$

$$\Rightarrow \quad y \in Im(A) + Im(B)$$

$$\Rightarrow \quad Im(A+B) \subseteq Im(A) + Im(B)$$

We have

$$r(A+B) = \dim(Im(A+B)) \leq \dim(Im(A) + Im(B))$$

$$\leq \dim(Im(A)) + \dim(Im(B)) = rank(A) + rank(B)$$

$$\Rightarrow \quad r(A+B) \leq r(A) + r(B)$$

3) a) • Let $W \cap W^\perp = x \Rightarrow x \in W$ and $x \in W^\perp$

$$\Rightarrow x^T x = 0 \Rightarrow \|x\|_2^2 = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i^2 = 0 \Rightarrow \text{the } x_i\text{'s are all zero}$$

$\Rightarrow x$ has to be zero

$$\Rightarrow \quad W \cap W^\perp = \{0\}$$

•

Since $W^\perp$ is the orthogonal complement of W and W is a vector subspace of $\mathbb{R}^n$

$$\Rightarrow \dim(W) + \dim(W^\perp) = n$$

Since $W^{\perp\perp}$ is the orthogonal complement of $W^\perp$ and $W^\perp \subset \mathbb{R}^n$

$$\Rightarrow \dim(W^\perp) + \dim(W^{\perp\perp}) = n$$

$$\Rightarrow \dim(W) + \dim(W^\perp) = \dim(W^\perp) + \dim(W^{\perp\perp})$$

$$\Rightarrow \quad \dim(W) = \dim(W^{\perp\perp}) \quad (*)$$

Also, let $x \in W \Rightarrow y^T x = 0$ for all $y \in W^\perp$

This property, also holds for $z \in W^{\perp\perp}$ where

$$y^T z = 0 \text{ for all } y \in W^\perp$$

$$\Rightarrow x \in W^{\perp\perp} \Rightarrow W \subseteq W^{\perp\perp} \quad (**)$$

From $(*)$ and $(**)$ we can conclude

$$W = W^{\perp\perp}$$

b) • Prove $\ker(A^T) = \mathrm{Im}(A)^\perp$

Let $y \in \mathrm{Im}(A) \Rightarrow y = Ax$ for $x \in \mathbb{R}^n$

$\Rightarrow y^T = (Ax)^T = x^T A^T$

Let $z \in \ker(A^T) \Rightarrow A^T z = 0$

$\Rightarrow y^T z = (x^T A^T) z = x^T (A^T z) = x^T . 0 = 0$

$\Rightarrow \ker(A^T) = \mathrm{Im}(A)^\perp$

• Prove $\mathrm{Im}(A^T) = \ker(A)^\perp$

Let $y \in \mathrm{Im}(A^T) \Rightarrow y = A^T x$ for $x \in \mathbb{R}^m$

Let $z \in \ker(A) \Rightarrow Az = 0$

$\Rightarrow y^T z = (A^T x)^T z = x^T (A^T)^T z$

$= x^T . A . z = x^T (Az) = x^T . 0 = 0$

$\Rightarrow \mathrm{Im}(A^T) = \ker(A)^\perp$

c) From conclusion $W \cap W^\perp = \{0\}$ in part a)

and the two conclusions in part b)

$\Rightarrow \begin{cases} \mathrm{Im}(A^T) \cap \ker(A) = 0 \\ \mathrm{Im}(A) \cap \ker(A^T) = 0 \end{cases}$

d) As $\dim(\mathrm{Im}(A)) = \mathrm{rank}(A) = r$

$\Rightarrow$ a set of vectors with r members $\{u_1, \dots, u_r\}$ is a basis for $\mathrm{Im}(A)$ (u_i's are linearly independent)

Also $\dim(\mathrm{Im}(A)) + \dim(\ker(A^T)) = m$

(as $\mathrm{Im}(A) = \ker(A^T)^\perp$ and they are subspaces of $\mathbb{R}^m$)

$\Rightarrow \dim(\ker(A^T)) = m - r$

$\Rightarrow$ a set of $(m-r)$ linearly independent vectors $(u_{r+1}, \dots, u_m)$ forms a basis for $\ker(A^T)$

These two sets combined form $(m-r)+r = m$ total vectors that form a basis for $\mathbb{R}^m$

• Verify that $u_1, \dots u_r, u_{r+1}, \dots, u_m$ are linearly independent

Let $a_1 u_1 + a_2 u_2 + \dots + a_r u_r + b_1 u_{r+1} + \dots + b_m u_m = 0$

$$\Rightarrow \underbrace{\sum_{i=1}^{r} a_i u_i}_{\in \mathrm{Im}(A) \cap \ker(A^T)} = \underbrace{-\sum_{i=r+1}^{m} b_i u_i}_{\in \mathrm{Im}(A) \cap \ker(A^T)}$$

But $Im(A) \cap ker(A^T) = \{0\}$

$\Rightarrow \begin{cases} \sum_{t=1}^{r} a_i u_i = 0 \\ -\sum_{i=r+1}^{m} b_i u_i = 0 \end{cases}$

But u_i's $(u_1, \dots, u_r)$ are alrealy linearly independent $\Rightarrow a_i = 0$

But $u_{r+1}, \dots u_m$ are alrealy linearly independent $\Rightarrow b_i = 0$

$\Rightarrow$ $u_1, \dots, u_r, u_{r+1}, u_{r+2}, \dots, u_m$ are linearly independent

$\Rightarrow$ These m vectors form a basis for $\mathbb{R}^m$

$\Rightarrow$ Any vector x in $\mathbb{R}^m$ can be written as

$$x = \sum_{i=1}^{r} a_i u_i + \sum_{i=r+1}^{m} b_i u_i$$

(any x can be written as a linear combination of these m vectors that span $\mathbb{R}^m$)

$$\Rightarrow \quad x = x_0 + x_1$$

with $x_0 = \sum_{i=1}^{r} a_i u_i \in Im(A)$

$x_1 = \sum_{i=r+1}^{m} b_i u_i \in ker(A^T)$

And $Im(A) = ker(A^T)^{\perp}$

$$\Rightarrow \quad x_0^T x_1 = 0$$

We have $\|x\|_2^2 = \|x_0 + x_1\|_2^2$

$$= \|x_0\|_2^2 + \|x_1\|_2^2 + 2\|x_0\|\|x_1\|$$

$$= \|x_0\|_2^2 + \|x_1\|_2^2 + \underbrace{2x_0^T x_1}_{=0}$$

$$\Rightarrow \|x\|_2^2 = \|x_0\|_2^2 + \|x_1\|_2^2$$

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4) a) $H_0 : p_0 = 0.5$

$H_A : p_0 \neq 0.5$

(with p_0 = probability a child is born female

$\hat{p}$ = 49.6% – from our data of female population

n = 7.594 bn – from our data of Earth population)

In order to do hypothesis testing for proportion, we use a z-score calculated as follows

$$z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.496 - 0.5}{\sqrt{\frac{0.5(1-0.5)}{7.594bn}}} \simeq -697.1485$$

Using $\alpha = 0.05$ with a two-tailed test, $z_c = \pm 1.96$

We have $z_0 < -1.96 \Rightarrow$ we reject H_0

$\Rightarrow$ We cannot claim that the probability of a child being born female is exactly 50%

b) A 95% confidence interval for f will be

$$\hat{p} \pm z_c \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Lower bound $= 0.496 - 1.96 \cdot \sqrt{\frac{0.496(1-0.496)}{7.594bn}}$

$\simeq 0.49599$

Upper bound $= 0.496 + 1.96 \cdot \sqrt{\frac{0.496(1-0.496)}{7.594bn}}$

$\simeq 0.49601$

$\Rightarrow$ The 95% confidence interval is between 0.49599 and 0.49601

c) Our proposed $p_0 = 0.5$ in H_0 doesn't fall into the 95% interval

If repeated samples were taken and the 95% CI was computed each time, 95% of the intervals would contain the population mean, or 95% of the time the population mean would fall into the CI $\Rightarrow$ We reject the null because it is unlikely that the $p_0 = 0.5$ is close to the population mean

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5) a) $P(T_\vartheta \leq -t \text{ or } T_\vartheta \geq t)$

$= P(T_\vartheta^2 \geq t^2)$

$= P\left(\frac{N(0,1)^2}{\chi_\vartheta^2/\vartheta} \geq t^2\right)$

$= P\left(\frac{\chi_1^2/1}{\chi_\vartheta^2/\vartheta} \geq t^2\right)$ (as $N(0,1)^2 = \chi_1^2$)

$= P(F_{1,\vartheta} \geq t^2)$ (as $\frac{\chi_1^2/1}{\chi_\vartheta^2/\vartheta} = F_{1,\vartheta}$)

b) • $n^{-1}\chi_n^2 = \frac{1}{n}\sum_{i=1}^{n} Z_i^2$

$= \frac{1}{n}(Z_1^2 + Z_2^2 + \dots + Z_n^2)$

Also, $Z_i \overset{iid}{\sim} N(0,1) \Rightarrow Z_i^2 \overset{iid}{\sim} N(1,2)$

According to strong law of large numbers, as $n \to \infty$

$n^{-1}\chi_n^2 = \frac{1}{n}(Z_1^2 + Z_2^2 + \dots + Z_n^2)$ will get to $\frac{1}{n}\underbrace{(1+1+\dots+1)}_{n \text{ numbers}}$ which is 1 (as Z_i^2 will converge to the population mean $= 1$)

$\Rightarrow n^{-1}\chi_n^2 \xrightarrow{a.s.} 1$ as $n \to \infty$

• $P(T_n \leq t) = P\left(\frac{N(0,1)}{\sqrt{\chi_n^2/n}} \leq t\right)$

With the conclusion above that $n^{-1}\chi_n^2 \xrightarrow{a.s.} 1$ as $n \to \infty$

$\to \lim_{n\to\infty} P(T_n \leq t) = P\left(\frac{N(0,1)}{1} \leq t\right)$

$= P(N(0,1) \leq t)$

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