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STAT 2132
March 18, 2021

Homework 5

a) $Y_{ij} = \mu_i + \epsilon_{ij}$ for $i = 1, \dots, r$ (r treatments)
$j = 1, \dots, n_i$ (each treatment i has n_i levels)

$Y = X\beta + \epsilon$; $Y = \begin{bmatrix} Y_{11} \\ \vdots \\ Y_{1n_1} \\ \vdots \\ Y_{r1} \\ \vdots \\ Y_{rn_r} \end{bmatrix}$, $\beta = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_r \end{bmatrix}$; $\epsilon = \begin{bmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{1n_1} \\ \vdots \\ \epsilon_{r1} \\ \vdots \\ \epsilon_{rn_r} \end{bmatrix}$

$\Rightarrow$ X must be

$$\underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ & \dots & & \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & \dots & & \\ 0 & 1 & \dots & 0 \\ & \dots & & \\ 0 & 0 & \dots & 1 \\ & \dots & & \\ 0 & 0 & \dots & 1 \end{bmatrix}}_{r \text{ columns}} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} n_1 \text{ rows} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n_2 \text{ rows} \\ \dots \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n_r \text{ rows} \end{matrix}$$

This is a $n \times r$ matrix, where $n = \sum_{i=1}^{r} n_i$

b) With X given above, X^TX must be a diagonal matrix

$X^TX = \begin{bmatrix} n_1 & 0 & \dots & 0 \\ 0 & n_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & n_r \end{bmatrix}$, which is a $r \times r$ matrix

$(X^TX)_{ii} = n_i$

$\Rightarrow$ $(X^TX)^{-1}$ is also a diagonal matrix with dimension $r \times r$

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$$(X^TX)^{-1} = \begin{bmatrix} 1/n_1 & 0 & \cdots & 0 \\ 0 & 1/n_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/n_r \end{bmatrix} \quad \text{where } (X^TX)^{-1}_{ii} = 1/n_i$$

$$\Rightarrow \underset{n\times r}{X}\,\underset{r \times r}{(X^TX)^{-1}} = \begin{array}{c} \underbrace{\begin{bmatrix} n_1^{-1} & 0 & \cdots & 0 \\ n_1^{-1} & 0 & \cdots & 0 \\ \cdots \\ n_1^{-1} & 0 & \cdots & 0 \\ \cdots \\ 0 & 0 & \cdots & n_r^{-1} \\ \cdots \\ 0 & 0 & \cdots & n_r^{-1} \end{bmatrix}}_{r \text{ columns}} \end{array} \begin{array}{l} \left.\right\} n_1 \text{ rows} \\ \vdots \\ \left.\right\} n_r \text{ rows} \end{array}$$

$$\Rightarrow H = X(X^TX)^{-1}X^T$$

$$= \begin{bmatrix} n_1^{-1} & \cdots & n_1^{-1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots \\ n_1^{-1} & \cdots & n_1^{-1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots \\ 0 & \cdots & & \cdots & & & \cdots & n_r^{-1} & \cdots & n_r^{-1} \\ \vdots \\ 0 & \cdots & & \cdots & & & \cdots & n_r^{-1} & \cdots & n_r^{-1} \end{bmatrix} \begin{array}{l} \left.\right\} n_1 \text{ rows} \\ \vdots \\ \left.\right\} n_r \text{ rows} \end{array}$$

$$\underbrace{\qquad\qquad}_{n_1 \text{ cols}} \quad \underbrace{\qquad}_{n_2 \text{ cols}} \quad \cdots \quad \underbrace{\qquad}_{n_r \text{ cols}}$$

H is a $n\times n$ matrix

c) $$\underset{n\times n}{H}\,\underset{n \times 1}{Y} = \begin{bmatrix} n_1^{-1} & \cdots & n_1^{-1} & \cdots & 0 & \cdots & 0 \\ \vdots \\ n_1^{-1} & \cdots & n_1^{-1} & \cdots & 0 & \cdots & 0 \\ \vdots \\ 0 & \cdots & 0 & \cdots & n_r^{-1} & \cdots & n_r^{-1} \\ 0 & \cdots & 0 & \cdots & n_r^{-1} & \cdots & n_r^{-1} \end{bmatrix} \begin{bmatrix} Y_{11} \\ \vdots \\ Y_{1n_1} \\ \vdots \\ Y_{r1} \\ \vdots \\ Y_{rn_r} \end{bmatrix}$$

$$= \left[\frac{1}{n_1}(Y_{11}+\dots+Y_{1n_1}) \;\cdots\; \frac{1}{n_1}(Y_{11}+\dots+Y_{1n_1}) \;\cdots\; \frac{1}{n_r}(Y_{r1}+\dots+Y_{rn_r}) \cdots \frac{1}{n_r}(Y_{r1}+\dots+Y_{rn_r})\right]^T$$

$$= [\underbrace{\bar{Y}_{1\cdot} \cdots \bar{Y}_{1\cdot}}_{n_1 \text{ elements}} \;\; \bar{Y}_{2\cdot} \cdots \bar{Y}_{2\cdot} \;\cdots\; \underbrace{\bar{Y}_{r\cdot} \cdots \bar{Y}_{r\cdot}}_{n_r \text{ elements}}]^T = [\bar{Y}_{1\cdot}\mathbf{1}_{n_1} \;\cdots\; \bar{Y}_{r\cdot}\mathbf{1}_{n_r}]^T$$

d) $H = \begin{bmatrix} n_1^{-1} J_{n_1} & \cdots & \\ \vdots & \ddots & \\ & & n_r^{-1} J_{n_r} \end{bmatrix}_{n \times r}$ where $J_{n_i} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \\ 1 & \cdots & 1 \end{bmatrix}_{n_i \times n_i} = 1_{n_i} 1_{n_i}^T$

We already defined $n = \sum_{i=1}^{r} n_i$ previously, hence $n = n_T$

i) $Y^T(H - L)Y = Y^T H Y - Y^T L Y = Y^T(HY) - Y^T L Y$

$HY = (\bar{Y}_{1.} 1_{n_1}, \dots, \bar{Y}_{r.} 1_{n_r})^T$

$\Rightarrow \underset{1\times n}{Y^T} H \underset{n \times 1}{Y} = Y_{11}\bar{Y}_{1.} + \dots + Y_{1n_1}\bar{Y}_{1.} + \dots + Y_{r1}\bar{Y}_{r.} + \dots + Y_{rn_r}\bar{Y}_{r.}$

$= \sum_{i=1}^{r} \bar{Y}_{i.}(Y_{i1} + \dots + Y_{in_i}) = \sum_{i=1}^{r} n_i \bar{Y}_{i.}^2$

$LY = \underset{n\times n}{(n^{-1} 1_n 1_n^T)} \underset{n \times 1}{Y} = \begin{bmatrix} n^{-1} Y_{11} + \dots + n^{-1} Y_{rn_r} \\ \vdots \\ n^{-1} Y_{11} + \dots + n^{-1} Y_{rn_r} \end{bmatrix}$

$= \begin{bmatrix} n^{-1} \sum_{i=1}^{r}\sum_{j=1}^{n_i} Y_{ij} \\ \vdots \\ n^{-1} \sum_{i=1}^{r}\sum_{j=1}^{n_i} Y_{ij} \end{bmatrix} = \begin{bmatrix} \bar{Y}_{..} \\ \vdots \\ \bar{Y}_{..} \end{bmatrix} = \bar{Y}_{..} 1_n$

$\Rightarrow \underset{1\times n}{Y^T} L \underset{n \times 1}{Y} = Y_{11}\bar{Y}_{..} + \dots + Y_{1n_1}\bar{Y}_{..} + Y_{r1}\bar{Y}_{..} + \dots + Y_{rn_r}\bar{Y}_{..}$

$= \bar{Y}_{..}(Y_{i1} + \dots + Y_{in_i}) = \bar{Y}_{..}^2 n$

$\Rightarrow Y^T(H-L)Y = Y^T H Y - Y^T L Y = \sum_{i=1}^{r} n_i \bar{Y}_{i.}^2 - \bar{Y}_{..}^2 n$ (1)

On the other hand, we have

$SSTR = \sum_i n_i (\bar{Y}_{i.} - \bar{Y}_{..})^2 = \sum_i n_i (\bar{Y}_{i.}^2 + \bar{Y}_{..}^2 - 2\bar{Y}_{i.}\bar{Y}_{..})$

$= \sum_i n_i \bar{Y}_{i.}^2 + \bar{Y}_{..}^2 \sum_i n_i - 2\bar{Y}_{..} \sum_i n_i \bar{Y}_{i.}$

$= \sum_i n_i \bar{Y}_{i.}^2 + \bar{Y}_{..}^2 n - 2\bar{Y}_{..} \sum_i \sum_j Y_{ij}$

$$\Rightarrow SSTR = \sum_i n_i \bar{Y}_{i.}^2 + \bar{Y}_{..}^2 n - 2n\bar{Y}_{..}\left(\frac{\sum_i\sum_j Y_{ij}}{n}\right)$$

$$= \sum n_i \bar{Y}_{i.}^2 + \bar{Y}_{..}^2 n - 2n\bar{Y}_{..}^2$$

$$\Rightarrow SSTR = \sum n_i \bar{Y}_{i.}^2 - \bar{Y}_{..}^2 n \qquad (2)$$

From (1) and (2)

$$\Rightarrow SSTR = Y^T(H-L)Y$$

From last semester, we learned $H = X(X^TX)^{-1}X^T$ is symmetric and idempotent $\Rightarrow H^2 = H$, $H^T = H$

We have $L = n^{-1} 1_n 1_n^T$

$$\Rightarrow L^T = n^{-1}(1_n^T)^T 1_n^T = n^{-1} 1_n 1_n^T = L$$

$$L^2 = L \cdot L = n^{-1} 1_n 1_n^T n^{-1} 1_n 1_n^T$$

$$= n^{-2} J_n J_n = n^{-2}(J_n^2) = n^{-2} n^1 J_n$$

$$= n^{-1} J_n = n^{-1} 1_n 1_n^T = L$$

(J_n is $n\times n$ matrix of ones, it has property $J_n^k = n^{k-1}J_n$ for $k = 1, 2, \ldots$)

$\Rightarrow L$ is also symmetric and idempotent

Now look at $H - L$

$(H-L)^T = H^T - L^T = H - L \Rightarrow H - L$ is symmetric

$(H-L)^2 = H^2 - HL - LH + L^2 = H - HL - LH + L$

$$HL = \begin{bmatrix} \frac{1}{n_1} \cdots \frac{1}{n_1} & & \\ \vdots & & \\ \frac{1}{n_1} \cdots \frac{1}{n_1} & \ddots & \\ & & 1/n_r \cdots 1/n_r \\ & & \vdots \\ & & 1/n_r \cdots 1/n_r \end{bmatrix} \begin{bmatrix} 1/n & \cdots & 1/n \\ \vdots & & \\ 1/n & \cdots & 1/n \end{bmatrix} = \begin{bmatrix} n_1 \frac{1}{n_1} n^{-1} & \cdots & n_1 \frac{1}{n_1} n^{-1} \\ \vdots & & \\ n_r \frac{1}{n_r} n^{-1} & \cdots & n_r \frac{1}{n_r} n^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} n^{-1} & \cdots & n^{-1} \\ \vdots & & \\ n^{-1} & \cdots & n^{-1} \end{bmatrix} = L \quad ; \quad \text{Similarly } LH = L$$

$\Rightarrow (H-L)^2 = H - L - L + L = H - L \Rightarrow H - L$ is idempotent

1) d)

(ii) $\text{rank}(H) = r$

$\text{rank}(L) = 1$

$\Rightarrow \text{rank}(H - L) = r - 1$

(iii) Let $\vec{a} \in Im(H - L)$

$$\Rightarrow \underset{n_T \times 1}{\vec{a}} = \underbrace{(H - L)}_{n_T \times n_T} \underset{n_T \times 1}{\vec{v}_1}$$

Let $\vec{b} \in Im(I_n - H)$

$$\Rightarrow \underset{n_T \times 1}{\vec{b}} = \underset{n_T \times n_T}{(I_n - H)} \underset{n_T \times 1}{\vec{v}_2}$$

$$\vec{a}^T \cdot \vec{b} = \vec{v}_1^T (H - L)^T (I_n - H) \vec{v}_2$$

$$= \vec{v}_1^T (H - L)(I_n - H) \vec{v}_2 \quad (H-L \text{ is symmetric})$$

$$= \vec{v}_1^T (H - H - L + LH) \vec{v}_2$$

$$= \vec{v}_1^T (H - H - L + L) \vec{v}_2$$

$$= 0$$

$\Rightarrow$ $Im(H - L)$ is orthogonal to $Im(I_n - H)$

(iv) From last semester, $Y \sim N$, A & B are symmetric and idempotent, A & B orthogonal

$\Rightarrow$ Quadratic forms $Y^T A Y$ and $Y^T B Y$ are independent

In this case $Y \sim N$

$$\left.\begin{array}{l} SSTR = Y^T (H - L) Y \\ SSE = Y^T (I_n - H) Y \end{array}\right\} \text{quadratic forms}$$

And we proved $(H - L)$ and $(I_n - H)$ to be orthogonal

$(H-L)$ and $(I_n - H)$ are symmetric and idempotent

$\Rightarrow$ SSTR and SSE are independent

2) $Z_{n\times 1} \in \mathbb{R}^n$, $E(Z) = \mu$, $Var(Z) = V$

$A_{n\times n}$

a) $$E(Z^TAZ) = E(Z^TAZ) + \mu^TA\mu - \mu^TA\mu$$
$$= E(Z^TAZ) - \mu^TA\mu + \mu^TA\mu$$
$$= E\left[(Z^T - \mu^T)A(Z-\mu)\right] + \mu^TA\mu$$
$$= E\left[\underbrace{(Z-\mu)^T}_{1\times n}\underbrace{A}_{n\times n}\underbrace{(Z-\mu)}_{n\times 1}\right] + \mu^TA\mu$$

this whole expectation is a scalar

$$= E\left[Tr\left(A(Z-\mu)(Z-\mu)^T\right)\right] + \mu^TA\mu$$
$$= Tr\left[E\left(A(Z-\mu)(Z-\mu)^T\right)\right] + \mu^TA\mu$$
$$= Tr\left(A\,E\left((Z-\mu)(Z-\mu)^T\right)\right) + \mu^TA\mu$$
$$= Tr(A\,Var(Z)) + \mu^TA\mu$$
$$\Rightarrow E(Z^TAZ) = Tr(AV) + \mu^TA\mu$$

(because $E[(Z-\mu)(Z-\mu)^T] = Cov(Z) = Var(Z)$)

b) $$MSE = \frac{1}{n-r}SSE = \frac{1}{n-r}Y^T(I_n - H)Y$$
$$\Rightarrow E(MSE) = \frac{1}{n-r}E(Y^T(I_n-H)Y)$$
$$= \frac{1}{n-r}\left[Tr(I_n - H)Var(y) + (X\beta)^T(I_n-H)X\beta\right]$$
$$= \frac{1}{n-r}\left[(n-r)\sigma^2 + 0\right] \quad \left(\text{because } \begin{array}{l} HX = X \\ I_nX = X\end{array}\right)$$
$$= \sigma^2$$

$$MSTR = \frac{SSTR}{r-1} = \frac{1}{r-1}Y^T(H-L)Y$$

$$\Rightarrow E(MSTR) = \frac{1}{r-1} E(Y^T(H-L)Y)$$

$$= \frac{1}{r-1}\left(Tr(H-L)Var(Y) + (X\beta)^T(H-L)X\beta\right)$$

$$= \frac{1}{r-1}\left((r-1)\sigma^2 + \beta^T X^T H X\beta - \beta^T X^T L X\beta\right)$$

$$= \sigma^2 + \frac{1}{r-1}\left(\beta^T X^T X\beta - \beta^T X^T L X\beta\right) \quad (1)$$

$$\sum_i n_i(\mu_i - \mu_.)^2 = \sum_i n_i(\mu_i^2 + \mu_.^2 - 2\mu_i\mu_.)$$

$$= \sum_i n_i\mu_i^2 + \sum_i n_i\mu_.^2 - 2\sum_i n_i\mu_i\mu_.$$

$$= \sum_i n_i\mu_i^2 + \frac{(\sum n_i\mu_i)^2}{\sum_i n_i} - 2\mu_. \sum_i n_i\mu_i$$

$$= \sum_i n_i\mu_i^2 + \frac{(\sum_i n_i\mu_i)^2}{\sum_i n_i} - 2\frac{(\sum n_i\mu_i)^2}{\sum n_i}$$

$$= \sum_i n_i\mu_i^2 - n^{-1}\left(\sum n_i\mu_i\right)^2$$

$$= \mu^T\mu - \mu^T L\mu = \beta^T X^T X\beta - \beta^T X^T L X\beta \quad (2)$$

From (1) and (2)

$$\Rightarrow E(MSTR) = \sigma^2 + \frac{1}{r-1} \cdot \sum_{i=1}^{r} n_i(\mu_i - \mu_.)^2$$