Linear Algebra

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Chapter 1

Vector Spaces

Linear algebra is the study of linear maps on finite-dimensional vector spaces A vector space is a set wit operations of addition and scalar multiplications that satisfy natural algebraic properties.

1.1 $\mathcal{R}^n$ and $\mathcal{C}^n$

1.1.1 Complex number

Assume that there exists a square root of -1 called i, then:

- A complex number is an ordered pair (a, b) where $a, b \in \mathcal{R}$ written as $a + bi$.
- The set of all complex number is $\mathcal{C} = \{a + bi | a, b \in \mathcal{R}\}$.
- Addition and multiplications are defined on $\mathcal{C}$ as:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$$

Complex addition and multiplication have the expected properties:

- Commutativity: $\alpha + \beta = \beta + \alpha \wedge \alpha \cdot \beta = \beta \cdot \alpha \forall \alpha, \beta \in \mathcal{C}$.
- Associativity: $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda) \wedge (\alpha \cdot \beta) \cdot \lambda = \alpha \cdot (\beta \cdot \lambda) \forall \alpha, \beta, \lambda \in \mathcal{C}$.
- Identities: $\lambda + 0 = \lambda \wedge 1 \cdot \lambda = \lambda \forall \lambda \in \mathcal{C}$.
- Additive inverse: $\forall \alpha \in \mathcal{C}, \exists \beta \in \mathcal{C} \Rightarrow \alpha + \beta = 0$.
- Multiplicative inverse: $\forall \alpha \in \mathcal{C}, \exists \beta \in \mathcal{C} \Rightarrow \alpha \cdot \beta = 1$.
- Distributive property: $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta \forall \alpha, \beta, \lambda \in \mathcal{C}$.

Which are proved by using the same properties on the real numbers. Furthermore suppose $\alpha, \beta \in \mathcal{C}$, then:

- $-\alpha$ the additive inverse of α: $-\alpha$ is the unique complex number such that:

$$\alpha + (-\alpha) = 0$$

- Subtraction is then defined as:

$$\beta - \alpha = \beta + (-\alpha)$$

- For $\alpha \neq 0$, let $\frac{1}{\alpha}$ denote the multiplicative inverse of α, a number such that:

$$\alpha \cdot \left(\frac{1}{\alpha}\right) = 1$$

- For $\alpha \neq 0$ division by α is defined as:

$$\frac{\beta}{\alpha} = \beta \cdot \left(\frac{1}{\alpha}\right)$$

From now on definition and theorem can be proved on both real and complex numbers, adopting the notation where $\mathcal{F}$ stands for either $\mathcal{R}$ or $\mathcal{C}$.
Elements of $\mathcal{F}$ are called *scalars*. Furthermore $\forall \alpha \in \mathcal{F} \land m \in \mathcal{N}$, α^m is defined as:

$$\alpha^m = \underbrace{\alpha \cdots \alpha}_{m \text{ times}}$$

Which implies that:

$$(\alpha^m)^n = a^{m \cdot n} \qquad \land \qquad (\alpha\beta)^m = \alpha^m \beta^m$$

1.1.2 Lists

Suppose $n \in \mathcal{N}$, then a list of length n is an ordered collection of n elements. Two lists are equal if and only if they have the same length and the same elements in the same order. They are often noted $(z_1, \ldots, z_n)$. By definition a list has a finite length that is a non negative integer. A list of length 0 is ( ). They differ from set in the fact that order matters and repetitions are allowed.
Fixing n for the rest of the chapter, $\mathcal{F}^n$ is the set of all lists of length n of elements of $\mathcal{F}$:

$$\mathcal{F}^n = \{(x_1, \ldots x_n) | x_k \in \mathcal{F} \forall k \in [1, n]\}$$

Where x_k is the k^{th} coordinate of $(x_1, \ldots x_n)$.
Addition in $\mathcal{F}^n$ is defined by adding the corresponding coordinates:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

All of the same properties of addition hold.
Elements of $\mathcal{F}^n$, with $n > 1$ can be denoted as vectors with $\vec{x}$ for example. Furthermore each of this set has a zero: $\vec{0}$.
For all $\vec{x} \in \mathcal{F}^n$, the additive inverse of $\vec{x}$, $-\vec{x}$, is the vector $-\vec{x} \in \mathcal{F}^n$ such that:

$$\vec{x} + (-\vec{x}) = \vec{0}$$

Thus if $\vec{x} = (x_1, \ldots, x_n)$, then $-\vec{x} = (-x_1, \ldots, -x_n)$.
Scalar multiplication in $\mathcal{F}^n$ is the product of a number λ and a vector in $\mathcal{F}^n$. it is computed by multiplying each coordinate of the vector by λ:

$$\lambda\vec{x} = \lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

Where $\lambda \in \mathcal{F}$ and $\vec{x} \in \mathcal{F}^n$. From a geometric standpoint scalar multiplication shrinks or stretches the vector.

1.1.3 Digression on fields

A field is a set containing at least two distinct elements called $\vec{0}$ and $\vec{1}$, along with operations of addition and multiplication satisfying all the previously listed properties. The only fields in this work are $\mathcal{R}$ and $\mathcal{C}$, but many of the definition and theorems that work for them work without change in arbitrary fields. Except in the inner product chapters, results that have as a hypothesis that $\mathcal{F}$ is $\mathcal{C}$, the hypothesis can be replaced with the fact that $\mathcal{F}$ is an algebraically closed field: every nonconstant polynomial with coefficients in $\mathcal{F}$ has a zero.