Mathematical modelling in biology exercises

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June 4, 2023

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The thorium-uranium method for dating rocks is based on the fact that Uranium234 decays into Thorium-230 which in turn decays into other elements. Set $t = 0$ the rock formation time and denoting $U(t)$ $[T(t)]$ the amount of Uranium-234 [Thorium-230] in the rock at time t (measured in years), the following differential equation system is written:

$$\begin{cases} \frac{dU(t)}{dt} = -aU(t) \\ \frac{dT(t)}{dt} = aU(t) - bT(t) \\ U(0) = U_0 \\ T(0) = 0 \end{cases}$$

Where:

- $a \approx 5.9 \cdot 10^{-6} \frac{1}{years}$
- $b \approx 1.9 \cdot 10^{-5} \frac{1}{years}$
- U_0 is the initial amount of Uranium-234

Note that, based on geological principles, it is believed that there was no thorium at the time of rock formation.

1.3.1 Solve the equation for U(t)

$$\begin{aligned}
\frac{dU(t)}{dt} &= -aU(t) \\
\frac{dU(t)}{dU(t)} &= -adt \\
\int \frac{q}{dU(t)}dt &\int -adt \\
\log(U(t)) &= -at + C \\
U(t) &= e^{-at+C} \\
U(t) &= C_1 e^{-at}
\end{aligned}$$

Considering $U(0) = U_0$:

$$\begin{aligned}
C_1 e^{-a0} &= U_0 \\
C_1 &= U_0
\end{aligned}$$

So, in conclusion:

$$U(t) = U_0 e^{-at}$$

1.3.2 How can the quantities of a and b can be interpreted? From the data provided can we infer the half-live of Uranium-234 and Thorium-230?

- a is the rate at which Uranium-234 decays into Thorium-230.
- b is the rate at which Thorium-230 decays into other elements.

To compute the half-life of Uranium-234:

$$\begin{aligned}
U(t_1) &= \frac{1}{2}U(t_2) \\
\cancel{U_0} e^{-at_1} &= \frac{1}{2}\cancel{U_0} e^{-at_2} \\
2 &= \frac{e^{-at_2}}{e^{-at_1}} \\
e^{-at_2+at_1} &= 2 \\
e^{a(t_1-t_2)} &= 2 \\
a(t_1 - t_2) &= \log(2) \\
t_1 - t_2 &= \frac{\log(2)}{a} \\
t_1 - t_2 &= \frac{\log(2)}{5.9 \cdot 10^{-6}} = 117482.6 \text{ years}
\end{aligned}$$

1.3.3 Calculate T(t), solution of the second differential equation

$$\frac{dT(t)}{dt} = aU_0e^{-at} - bT(t)$$

Let:

$$w(t) = \frac{T(t)}{e^{-bt}}$$
$$w(t) = T(t)e^{bt}$$

Taking its derivative:

$$\frac{dw(t)}{dt} = bT(t)e^{bt} + [aU_0e^{-at} - bT(t)]e^{bt}$$
$$\frac{dw(t)}{dt} = bT(t)e^{bt} + aU_0e^{-at+bt} - bT(t)e^{bt}$$
$$\frac{dw(t)}{dt} = \cancel{bT(t)e^{bt}} + aU_0e^{-at+bt}\cancel{-bT(t)e^{bt}}$$
$$\frac{dw(t)}{dt} = aU_0e^{-at+bt}$$

Now taking the integral:

$$\int \frac{dw(t)}{dt}dt = \int aU_0e^{-at+bt}dt$$
$$w(t) = \int aU_0 \int e^{(b-a)t}dt$$
$$w(t) = \frac{aU_0}{b-a}e^{b-at} + C_2$$

Now, considering that:

$$T(t) = w(t)e^{-bt}$$

$$T(t) = \frac{aU_0}{b-a}e^{(b-a)t}e^{-bt} + C_2e^{-bt}$$
$$T(t) = \frac{aU_0}{b-a}\cancel{e^{bt}}e^{-at}\cancel{e^{-bt}} + C_2e^{-bt}$$
$$= \frac{aU_0}{b-a}e^{-at} + C_2e^{-bt}$$

Considering $T(0) = 0$:

$$\begin{aligned}
\frac{aU_0}{b-a}e^{-at} + C_2e^{-bt} &= 0 \\
\frac{aU_0}{b-a} + C_2 &= 0 \\
C_2 &= \frac{-aU_0}{b-a}
\end{aligned}$$

So, in the end:

$$T(t) = \frac{aU_0}{b-a}e^{-at} - \frac{aU_0}{b-a}e^{-bt}$$

1.3.4 Compute $\lim\limits_{\mathbf{t}\to\infty} \frac{\mathbf{T(t)}}{\mathbf{U(t)}}$

Computing the ratio:

$$\begin{aligned}
\frac{T(t)}{U(t)} &= \frac{\frac{aU_0}{b-a}e^{-at} - \frac{aU_0}{b-a}e^{-bt}}{U_0e^{-at}} \\
&= \frac{aU_0}{b-a}e^{-at}\frac{1}{U_0e^{a\,t}} - \frac{aU_0}{b-a}e^{-bt}\frac{1}{U_0e^{a\,t}} \\
&= \frac{a\cancel{U_0}}{b-a}\cancel{e^{-at}}\frac{1}{\cancel{U_0}\cancel{e^{a\,t}}} - \frac{a\cancel{U_0}}{b-a}e^{-bt}\frac{1}{\cancel{U_0}e^{-at}} \\
&= \frac{a}{b-a} - \frac{a}{b-a}e^{(a-b)t}
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to\infty}\frac{T(t)}{U(t)} = \lim_{t\to\infty}\frac{a}{b-a} - \frac{a}{b-a}e^{(a-b)t} &= \frac{a}{b-a} - \frac{a}{b-a}e^{(a-b)\infty} \\
&= \frac{a}{b-a} - \frac{a}{b-a}0 \\
&= \frac{a}{b-a}
\end{aligned}$$

1.3.5 Explain why it is possible to estimate the rock age from the knowledge of $\frac{\mathbf{T(t)}}{\mathbf{U(t)}}$ at current time

Hint: write down the function $\frac{T(t)}{U(t)}$ and evaluate its behaviour.

Consider:

$$\frac{T(t)}{U(t)} = \frac{a}{b-a} - \frac{a}{b-a}e^{(a-b)}$$

Now, this function does not depend on the initial concentration of Uranium and can be inverted for t:

$$
\begin{aligned}
\frac{T(t)}{U(t)} &= \frac{a}{b-a} - \frac{a}{b-a}e^{(a-b)t} \\
\frac{a}{b-a}e^{(a-b)t} &= \frac{a}{b-a} - \frac{T(t)}{U(t)} \\
e^{(a-b)t} &= \frac{a}{\cancel{b-a}}\frac{\cancel{b-a}}{a} - \frac{b-a}{a}\frac{T(t)}{U(t)} \\
e^{(a-b)t} &= 1 - \frac{b-a}{a}\frac{T(t)}{U(t)} \\
(a-b)t &= \log\left(1 - \frac{b-a}{a}\frac{T(t)}{U(t)}\right) \\
t\left(\frac{T(t)}{U(t)}\right) &= \frac{\log\left(1 - \frac{b-a}{a}\frac{T(t)}{U(t)}\right)}{a-b}
\end{aligned}
$$

So now from the ratio $\frac{T(t)}{U(t)}$ the rock age can be estimated.