Mathematical modelling in biology

Definitions and theorems

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Contents

1 Differential equations

A differential equation that relates a function to its derivative. They are characterized by the order of the derivative and other criteria, which are useful in determining the approach to a solution.

1.1 Ordinary differential equation

An ordinary differential equation is a differential equation whose unknown consists of a function:

$$y(t) : \mathbb{R} \to \mathbb{R}^n$$

Of one variable t and involves the derivative in dt of that function. ODEs have the form:

$$\frac{dy(t)}{dt} = f(t, y(t))$$

To check whether a candidate solution is valid it is enough to compute its derivative and check that it is equal to $f(t, y(t))$.

1.2 Cauchy problem

In general differential equations have infinite solutions, but if we impose an initial condition we can find a unique solution. This is the initial value or Cauchy problem, which is in the form:

$$\begin{cases} \frac{dy(t)}{dt} = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

1.3 Autonomous equations

A first order ODE iss said to be autonomous if its right hand side does not explicitly depend on t. It will be in the form:

$$\frac{dy(t)}{dt} = f(y(t))$$

Givena a particular solution $y_\alpha(t)$ for a Cauchy problem with $y(0) = y_0$ and another $t_\beta(t)$ for which $t(t_0) = t_0$, then:
$y_\beta(t) = y_\alpha(t - t_0)$

1.4 Separable equations

An equation is separable if it can be written in the form:

$$\frac{dy(t)}{dt} = f(t)g(y(t))$$

All autonomous equations are separable, but not all separable equations are autonomous. Moreover all seaparble ODE with $f(t) = k$ are called constant coefficient problems.

1.4.1 Separability

Consider a differential equation in the form:

$$\frac{dy(t)}{dt} = f(t)g(y(t))$$

Let $F(t)$ be the primitive of $f(t)$ and $H(y(t))$ the primitive of $\frac{1}{g(y(t))}$. Then:

- If $y(t)$ is a solution of $\frac{dy(t)}{dt} = f(t)g(y(t))$ such that $g(y(t)) \neq 0$, there exists a constant c such that $H(y(t)) = F(t) + c \ \forall t$.
- If $y(t)$ satisfies $H(y(t)) = F(t) + c \ \forall t$ such that $g(y(t)) \neq 0$, then $y(t)$ is a solution of the equation.

1.5 Linear ODE

A first order linear ODE is in the form:

$$\frac{dy(t)}{dt} = a(t)y(t) + b(t)$$

- If $b(t) = 0$ the equation is homogeneous and can be solved by the separation of variables.
- If $b(t) \neq 0$ it is non-homogeneous, for which in general the separatio of variables is not effective.
- If $a(t) = a \wedge b(t) = b$ it is autonomous.
- If $a(t) = a$ and $b(t)$ any, this becomes a constant coefficient problem.

1.6 Direction field

The direction field allows to graphically fid some properties of a solution of a DE, without explicitly solving it. The DE tells that if a solution satisfies an initial condition then the slope of the graph of $y(t)$ computed at t_0, which is $y'(t_0)$, must be equal to $f(t_0, y_0)$. Consequently, if in every point (t_0, y_0) a small segment of slope $f(t_0, y_0)$ is drawn, then the solution must be tangent to all of them.

1.7 Autonomous equations

Autonomous equations will show the same pattern for each t. So all columns in the cartesian plane will looke the same.

1.8 Equilibrium points

Given a first order ODE, equilibrium points are particular solutions such that:

$$\frac{dy(\bar{t})}{dt} = 0$$

Their derivative is zero for any value of t. They are constant solutions.

1.8.1 Stability

The stability of an equilibrium solution is classified according to the behavior of the solutions generated by initial conditions close to the point. In particular:

- An equilibrium $y_e(t)$ is stable if $\forall \epsilon > 0 \exists U$ neighbourhood of (t_e, y_e) such that $(t_i, y_i) \in U \rightarrow y_i(t) - y_e(t) \leq \epsilon \ \forall t$. An equilibrium is stalbe if solution arising from initial point close to the initial point remain close to the equilibrium solution.
- An equilibrium $y_e(t)$ is asymptotically stable or attractive if, in addition to being stalbe, it is true that:

  $$\lim_{t \to \infty} y_i(t) = y_e(t)$$

  If solution arising close to the equilibrium converge to it.
- An equilibrium $y_e(t)$ is unstable or repulsive if $\exists \eta : \forall \epsilon > 0 \exists (t_i, y_i) \Rightarrow |(t_e, y_e) - (t_i, y_i)| < \epsilon \wedge |y_e(t) - y_i(t)| \geq \eta$. If there are solutions that diverge from the equilibrium.