Mathematical modelling in biology

Methods to solve differential equations

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1 First order differential equations

1.1 Right hand term does not depend on $\mathbf{y(t)}$

First order ODEs whose right hand term does not depend on $y(t)$:

$$\frac{dy}{dt} = k \qquad \wedge \qquad \frac{dy}{dt} = f(t)$$

Where k is a constant and $f(t)$ is a function of t, can be solved by integration:

$$\begin{aligned}\frac{dy(t)}{dt} &= k\\ \int \frac{dy(t)}{dt}dt &= \int kdt\\ y(t) &= kt + c\\ y(t) &= \int_{t_0}^{t_1} kds = k(t_1 - t_0)\end{aligned}$$

$$\begin{aligned}\frac{dy(t)}{dt} &= f(t)\\ \int \frac{dy(t)}{dt}dt &= \int f(t)dt\\ y(t) &= \int f(t)dt\\ y(t) &= \int_{t_0}^{t_1} f(s)ds\end{aligned}$$

2 Separation of variables

Separable equations are in the form:

$$\frac{dy(t)}{dt} = f(t)g(y(t))$$

The method consists if three steps:

- The derivative is written using Leibnitz' notation.
- The equation is multiplied by dt and divided by $g(y(t))$.
- Both sides are integrated.

So, in practice:

$$\begin{aligned}\frac{dy(t)}{dt} &= f(t)g(y(t))\\ \frac{dy(t)}{g(y(t))} &= f(t)dt\\ \int \frac{dy(t)}{g(y(t))} &= \int f(t)dt\\ \int \frac{1}{g(y(t))}dy(t) &= \int f(t)dt\\ \int \frac{1}{g(y(t))}dy(t) &= \int_{t_0}^{t_1} f(s)ds\end{aligned}$$

3 Variation of constants

Variation of constants solves non-homogeneous ODEs:

$$\frac{dy(t)}{dt} = a(t)y(t) + b(t)$$

First the assciated homogeneous function is solved through separation of variables:

$$\begin{aligned}
\frac{du(t)}{dt} &= a(t)u(t) \\
\frac{du(t)}{u(t)} &= a(t)dt \\
\int \frac{du(t)}{u(t)} &= \int a(t)dt \\
\ln|u(t)| &= \int a(t)dt + c \\
u(t) &= e^{\int a(t)dt + c} \\
u(t) &= C_1 e^{\int a(t)dt} e
\end{aligned}$$

Then the constant is turned into a generic function of t:

$$y(t) = C(t)e^{\int a(t)dt}$$

Then the derivative is computed:

$$\frac{du(t)}{dt} = \frac{C(t)}{dt}e^{\int a(t)dt} + C(t)a(t)e^{\int a(t)dt}$$

Then the two are equated:

$$\begin{aligned}
\frac{du(t)}{dt} &= a(t)y(t) + b(t) \\
\frac{C(t)}{dt}e^{\int a(t)dt} + C(t)a(t)e^{\int a(t)dt} &= a(t)y(t) + b(t)
\end{aligned}$$

Considering that $y(t) = C(t)e^{\int a(t)dt}$:

$$\begin{aligned}
\frac{C(t)}{dt}e^{\int a(t)dt} + C(t)a(t)e^{\int a(t)dt} &= a(t)C(t)e^{\int a(t)dt} + b(t) \\
\frac{C(t)}{dt}e^{\int a(t)dt} &= b(t) \\
\frac{C(t)}{dt} &= b(t)e^{-\int a(t)dt} \\
C(t) &= \int b(t)e^{-\int a(t)dt}dt
\end{aligned}$$

So that, in the end:

$$y(t) = \left(\int b(t) e^{-\int a(t)dt} dt \right) e^{\int a(t)dt}$$

4 Multiplication by an integrating factor

An integrating factors is a function $u(t)$ that facilitates the solution of a differential equation. It applied to ODE in the form:

$$\frac{dy(t)}{dt} + a(t)y(t) = b(t)$$

The goal is to make the left hand side of the equation the derivative of a product:

$$u(t)\frac{dy(t)}{dt} + u(t)a(t)y(t) = u(t)b(t)$$

So that the left side can be expressed as:

$$\frac{d}{dt}\left[u(t)y(t)\right] = \frac{du(t)}{dt}b(t)$$

To make this happen:

$$\begin{aligned} \frac{du(t)}{dt}\cancel{y(t)} &= u(t)a(t)\cancel{y(t)} \\ \frac{du(t)}{dt} &= u(t)a(t) \end{aligned}$$

Which can be solved through separation of variables:

$$\begin{aligned} \frac{du(t)}{dt} &= u(t)a(t) \\ \frac{du(t)}{u(t)} &= a(t)dt \\ \int \frac{du(t)}{u(t)} &= \int a(t)dt \\ \ln|u(t)| &= \int a(t)dt + c \\ u(t) &= e^{\int a(t)dt + c} \\ u(t) &= C_1 e^{\int a(t)dt} \end{aligned}$$

The constant C_1 can be ignored because it will be cancelled out when the integrating factor is applied to the ODE. So in general:

$$u(t) = e^{\int a(t)dt}$$

Now both sides can be multiplied by $u(t)$:

$$\begin{aligned}
u(t)\frac{dy(t)}{dt} + u(t)a(t)y(t) &= u(t)b(t) \\
\underbrace{e^{\int a(t)dt}\frac{dy(t)}{dt} + e^{\int a(t)dt}a(t)y(t)}_{\text{derivative of a product}} &= e^{\int a(t)dt}b(t) \\
\int \frac{d}{dt}\left[e^{\int a(t)dt}y(t)\right] &= e^{\int a(t)dt}b(t) \\
e^{\int a(t)dt}y(t) + C = \int e^{\int a(t)dt}b(t)dt &
\end{aligned}$$

Now, solving for $y(t)$:

$$\begin{aligned}
y(t) &= \frac{\int e^{\int a(t)dt}b(t)dt + C}{e^{\int a(t)dt}} \\
&= \frac{\int u(t)db(t)dt + C}{u(t)}
\end{aligned}$$

5 Direction field

Many point in the Cartesian plane are choen and in each a vector $h, hf(t_i, y_i)$, with h small is drawn. Then a particular solution can be drawn by chosing an initial point and following the path tangent to the vectors.

6 Equilibrium points

They are identified in a direction field as the solutions always parallel to the t axis. They are computed by setting the ODE to 0 and solving for $y(t)$.

6.1 Classifying equilibrium points

An equilibrium is stable [unstable] if the direction of the derivative is negative [positive] above the equilibrium and positive [negative] below it.
In the case of autonomous DE, an alternative way to do it is to plot $y(t)$ against $\frac{dy(t)}{dt}$. Equilibria will be points with $x = 0$. The derivative allow to classify it: if $y(t) >$ [<]0 unstable [asymptotically stable].