# Mathematical modelling in biology

Giacomo Fantoni
telegram: @GiacomoFantoni
Github: https://github.com/giacThePhantom/mathematical-modelling-in-biology

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# Contents

# Chapter 1

# Introduction to ordinary differential equations

## 1.1 General concepts and methods

### 1.1.1 First order differential equations

A first order differential equation is written in normal form as:

$$y'(t) = f(t, y(t))$$

Where $f$ is a function of variables $t$ and $y$.

#### 1.1.1.1 Autonomous differential equations

In the case that $f$ does not depend on the variable $t$ the equation is said to be autonomous: the law that regulates how $y(t)$ changes does not depend on time.

#### 1.1.1.2 Solution of a differential equation

The solution of a differential equation if a function $y(t)$. To check whether $y(t)$ is a solution it is sufficient to compute its derivative and check that it is equal to the right hand side of the equation. So a solution if a function $y : I \to \mathbb{R}$ where $I$ is an interval of $\mathbb{R}$ sich that $y'(t) = f(t, y(y)) \forall t \in I$.

#### 1.1.1.3 Cauchy problems

A differential equation has infinite solutions as a function has infinite primitives. In order to predict the future evolution of a quantity it is necessary to know, besides the law regulating its dynamics or differential equation, an initial condition. An initial value problem or Cauchy problem is constituted by a differential equation and an initial condition:

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

A solution to an initial value problem is a function defined on an interval $I$ that contains the point $t_0$ such that for every $t \in I, y'(t) = f(t, y(t)))$ and also it is true that $y(t_0) = y_0$.

### 1.1.2 Direction fields

It is usually difficult or impossible to find exact solutions of differential equations. Because of this it is necessary to settle for something less complete. Direction fields allow to perform a graphical study of the qualitative behaviour of the solutions. The idea sit that according to the equation $y' = f(t, y)$ if a solution satisfies $y(t_0) = y_0$, the slope of the graph of $y(t)$ calculated at $t_0$, which is by definition $y'(t_0)$ must be equal to $f(t_0, y_0)$. Consequently, if in every $(t_0, y_0)$ a small segment of slop $f(t_0, y_0)$ is traced the slope of a solution that goes through that point is indicated.

#### 1.1.2.1 Draying a direction field

Many point $(\bar{t}, \bar{y})$ in a Cartesian plane are chosen an in each of them a small segment of slope $f(\bar{t}, \bar{y})$ are drawn. In this way the field of direction of $y' = f(t, y)$ can be built. Then a solution can be drawn making sure it is at all point tangent to the direction field.

#### 1.1.2.2 Autonomous equations

Autonomous equations are a class of differential equations in which the right hand side does not depend on $t$:

$$y' = g(y)$$

This means that the law that regulates the dynamics of the $y$ variable does not change over time. In particular if $y(t)$ is a solution, also $y(t - t_0)$ is a solution for all $t_0$. Graphically a solution can be moved horizontally and reach another one. Considering direction fields, each column in it will be equal to all the others, so it is sufficient to draw them at a given $t$ and repeat them for all the values of $t$.

### 1.1.3 Equilibria

Given the autonomous differential equation $y' = f(y(t))$ the points $\bar{y}$ such that $f(\bar{y}) = 0$ are said to be equilibria because if $y(0) = \bar{y}$ then $y(t) = \bar{y}$ for every $t$. Since solutions cannot cross each other if $f(y_0) > 0$ then $f(y(t)) > 0$ for every $t$ and then $y'(t) = f(y(t)) > 0$ or $y(t)$ is an increasing function. The same happens in the case of $f(y_0) < 0$.

## 1.2 Solution through separation of variables

The separation of variables is a technique that allows to find an explicit expression for the exact solutions of a class of equations.

### 1.2.1 Equations in which the right hand-side does not depend on $y$

In the case of an equation where the right-had side does not depend on $y$, it can be written as:

$$y' = f(t)$$

In this case the derivative of $y(t)$ is explicitly assigend. Solutions are all the primitives of $f$, so:

$$y(t) = \int f(t)dt$$

### 1.2.2 Separable equations

Separable equations are equations of the form:

$$y' = f(t)g(y)$$

They can be solved through the following steps:

1. The derivative $y'$ is written using the Leibniz notation:

$$\frac{dy}{dt} = f(t)g(y)$$

2. Then $dt$ gets multiplied as if it were a number:

$$\frac{dy}{g(y)} = f(t)dt$$

3. The integral is then applied to each of the members:

$$\int \frac{dy}{g(y)} = \int f(t)dt$$

4. Then the last equality is interrogated: a primitive $H(y)$ of $\frac{1}{g(y)}$ and $F(t)$ of $f(t)$ are chosen, such that for each $t$:

$$H(y(t)) = F(t) + C$$

Is true, where $C$ is a constant and $y(t)$ is the value of the solution at time $t$.

#### 1.2.2.1 Theorem

Consider the equation $y' = f(t)g(y)$. Let $F(t)$ be a primitive of $f$: $F'(t) = f(t)$ and $H(y)$ a primitive of $\frac{1}{g}$, $H'(y) = \frac{1}{g(y)}$. Then:

- If $y(t)$ is a solution of $y' = f(t)g(y)$ such that $g(y(t)) \neq 0$ then there exists a constant $C$ such that $H(y(t)) = F(t) + C$ for each $t$.
- If $y(t)$ satisfies $H(y(t)) = F(t) + C$ and $g(y(t)) \neq 0$ for each $t$ or, in other words, $H(y(t)) - F(t)$ is constant, then $y(t)$ is a solution of $y' = f(t)g(y)$.

### 1.2.3 Linear equations

A linear differential equation is an equation in which the second member is linearly dependent on $y$, so that it has form:

$$y' = a(t)y + b(t)$$

#### 1.2.3.1 Type of linear equation

- $b(t) = 0$: homogeneous.
- $b(t) \neq 0$: non-homogeneous.
- $a(t) \equiv a \wedge b(t) \equiv b$: autonomous.
- $a(t) \equiv a \wedge b(t) \equiv$ *any*: constant coefficients.

#### 1.2.3.2 Solution of linear equations

Autonomous and homogeneous linear equations can be solved by separating variables if and only if a primitive of $a(t)$ is found following the same procedure described above. Non-homogeneous linear equations can be solved creating a new variable equal to $y(t)$ over the solution of the corresponding homogeneous equation.

### 1.2.4 Separable equations that cannot be solved

To be able to explicitly write the solutions of a differential equation with the method of separating variables, after making sure that the equation can be written in $y' = f(t)g(y)$. First the primitives $\int \frac{dy}{g(y)}$ and $\int f(t)dt$ need to be computed and this is not always possible. Once this is done, the solution of the equation satisfy $H(y(t)) = F(t) + C$. So, in order to be able to explicitly write the solutions $y(t)$ it must be isolable in that expression: $H$ must be invertible.

## 1.3 Systems of differential equations

A system of two autonomous differential equation can be written as:

$$\begin{cases} x'(t) = f(x(t), y(t)) \\ y'(t) = g(x(t), y(t)) \end{cases}$$

### 1.3.1 Vector field and isoclines

An isocline is a set of point where one of the two derivatives is equal to zero. In general:

$$\{(x, y) : f(x, y) = 0\} \qquad \wedge \qquad \{(x, y) : g(x, y) = 0\}$$

After having found the isoclines the regions where the derivatives of $x$ are $y$ are positive or negative:

$$\{(x, y) : f(x, y) > 0\} \; \{(x, y) : f(x, y) < 0\} \; \{(x, y) : g(x, y) > 0\} \; \{(x, y) : g(x, y) < 0\}$$

It can be noted how these regions will be divided by the isoclines. At the intersections of the isoclines there are the equilibria.