Mathematical modelling in biology

Definitions and theorems

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June 7, 2023

Contents

1 Differential equations

A differential equation that relates a function to its derivative. They are characterized by the order of the derivative and other criteria, which are useful in determining the approach to a solution.

1.1 Ordinary differential equation

An ordinary differential equation is a differential equation whose unknown consists of a function:

$$y(t) : \mathbb{R} \to \mathbb{R}^n$$

Of one variable t and involves the derivative in dt of that function. ODEs have the form:

$$\frac{dy(t)}{dt} = f(t, y(t))$$

To check whether a candidate solution is valid it is enough to compute its derivative and check that it is equal to $f(t, y(t))$.

1.2 Cauchy problem

In general differential equations have infinite solutions, but if we impose an initial condition we can find a unique solution. This is the initial value or Cauchy problem, which is in the form:

$$\begin{cases} \frac{dy(t)}{dt} = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

1.3 Autonomous equations

A first order ODE iss said to be autonomous if its right hand side does not explicitly depend on t. It will be in the form:

$$\frac{dy(t)}{dt} = f(y(t))$$

Givena a particular solution $y_\alpha(t)$ for a Cauchy problem with $y(0) = y_0$ and another $t_\beta(t)$ for which $t(t_0) = t_0$, then:
$y_\beta(t) = y_\alpha(t - t_0)$

1.4 Separable equations

An equation is separable if it can be written in the form:

$$\frac{dy(t)}{dt} = f(t)g(y(t))$$

All autonomous equations are separable, but not all separable equations are autonomous. Moreover all seaparble ODE with $f(t) = k$ are called constant coefficient problems.

1.4.1 Separability

Consider a differential equation in the form:

$$\frac{dy(t)}{dt} = f(t)g(y(t))$$

Let $F(t)$ be the primitive of $f(t)$ and $H(y(t))$ the primitive of $\frac{1}{g(y(t))}$. Then:

- If $y(t)$ is a solution of $\frac{dy(t)}{dt} = f(t)g(y(t))$ such that $g(y(t)) \neq 0$, there exists a constant c such that $H(y(t)) = F(t) + c \ \forall t$.
- If $y(t)$ satisfies $H(y(t)) = F(t) + c \ \forall t$ such that $g(y(t)) \neq 0$, then $y(t)$ is a solution of the equation.

1.5 Linear ODE

A first order linear ODE is in the form:

$$\frac{dy(t)}{dt} = a(t)y(t) + b(t)$$

- If $b(t) = 0$ the equation is homogeneous and can be solved by the separation of variables.
- If $b(t) \neq 0$ it is non-homogeneous, for which in general the separatio of variables is not effective.
- If $a(t) = a \wedge b(t) = b$ it is autonomous.
- If $a(t) = a$ and $b(t)$ any, this becomes a constant coefficient problem.

1.6 Direction field

The direction field allows to graphically fid some properties of a solution of a DE, without explicitly solving it. The DE tells that if a solution satisfies an initial condition then the slope of the graph of $y(t)$ computed at t_0, which is $y'(t_0)$, must be equal to $f(t_0, y_0)$. Consequently, if in every point (t_0, y_0) a small segment of slope $f(t_0, y_0)$ is drawn, then the solution must be tangent to all of them.

1.7 Autonomous equations

Autonomous equations will show the same pattern for each t. So all columns in the cartesian plane will looke the same.

1.8 Equilibrium points

Given a first order ODE, equilibrium points are particular solutions such that:

$$\frac{dy(\bar{t})}{dt} = 0$$

Their derivative is zero for any value of t. They are constant solutions.

1.8.1 Stability

The stability of an equilibrium solution is classified according to the behavior of the solutions generated by initial conditions close to the point. In particular:

- An equilibrium $y_e(t)$ is stable if $\forall \epsilon > 0 \exists U$ neighbourhood of (t_e, y_e) such that $(t_i, y_i) \in U \rightarrow y_i(t) - y_e(t) \leq \epsilon \ \forall t$. An equilibrium is stalbe if solution arising from initial point close to the initial point remain close to the equilibrium solution.
- An equilibrium $y_e(t)$ is asymptotically stable or attractive if, in addition to being stalbe, it is true that:

$$\lim_{t \to \infty} y_i(t) = y_e(t)$$

If solution arising close to the equilibrium converge to it.

- An equilibrium $y_e(t)$ is unstable or repulsive if $\exists\eta : \forall\epsilon > 0 \exists(t_i, y_i) \Rightarrow |(t_e, y_e) - (t_i, y_i)| < \epsilon \wedge |y_e(t) - y_i(t)| \geq \eta$. If there are solutions that diverge from the equilibrium.

2 Systems of ODEs

2.1 Homogeneous linear systems

A homogenous linear system with constant coefficients is a system of ODES in the form:

$$\begin{cases} \frac{dy_1(t)}{dt} &= a_{11}y_1(t) + a_{12}y_2(t) + \cdots + a_{1n}y_n(t) \\ \frac{dy_2(t)}{dt} &= a_{21}y_1(t) + a_{22}y_2(t) + \cdots + a_{2n}y_n(t) \\ \vdots \\ \frac{dy_n(t)}{dt} &= a_{n1}y_1(t) + a_{n2}y_2(t) + \cdots + a_{nn}y_n(t) \end{cases}$$

Since the coefficient are constants, this is an autonomous system and can be rewritten using the vector form:

$$\frac{d\vec{Y}(t)}{dt} = A\vec{Y(t)}$$

Where:

$$\vec{Y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Which resembles the problem of $\frac{dy(t)}{dt} = ay(t)$, so it is tempting to use a solution in the form $\vec{Y}(t) = \vec{C}e^{At}$. To check this the derivative of $\vec{Y}(t)$ needs to be computed. The matrix exponential needs to be solved. To do so consider the Taylor expansion of the exponential:

$$e^{At} = \sum_{k=0}^{\infty} \frac{At^k}{k!} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \cdots + \frac{A^nt^n}{n!}$$

Which can be derived:

$$\begin{aligned} \frac{d\vec{Y}(t)}{dt} &= \vec{C}\frac{de^{At}}{dt} \\ &= \vec{C}\left[0 + A + A^2t + \frac{A^3t^2}{2} + \cdots + \frac{A^nt^{n-1}}{(n-1)!}\right] \\ &= \vec{C}A\left[I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \cdots + \frac{A^nt^n}{n!}]\right] \end{aligned}$$

The last factor is the Taylor expansion of e^{At}. So, in conclusion $\frac{d\vec{Y}(t)}{dt} = \vec{C}Ae^{At}$. The general solution for a system of homogeneous linear ODS is indeed:

$$\vec{Y}(t) = \vec{C}e^{At}$$

Consideing the Cauhy problem: $\vec{Y}(0) = \vec{Y}_0$, then $\vec{C} = \vec{Y}_0$ and the solution is:

$$\vec{Y}(t) = \vec{Y}_0 e^{At}$$

This method means evaluating the exponential A^k of a matrix as k grows, which is computationally expensive. Assume that $A \in \mathbb{R}^{n\times n}$, is diagonalizable: there exists an invertible matrix $P \in \mathbb{R}^{n\times n}$ and a diagonal one $D \in \mathbb{R}^{n\times n}$ such that:

$$\begin{aligned} P^{-1}AP &= D \\ AP &= PD \\ A &= PDP^{-1} \end{aligned}$$

Where D has as diagonal coefficients the eigenvalues of A. Consider additionally the set B, the union of the basis-vectors of each eigenspace:

$$B = B_{E_\alpha} \cup E_{E_\beta} \cup \cdots \cup E_{E_\omega}$$

Which is a basis of $\mathbb{R}^n$. There exists a basis B of $\mathbb{R}^n$ formed by eigenvectors of A. So every vector $\in \mathbb{R}^n$ can be written as a unique linear combination of the vectors in B. Once it has been built, the vectors can be used as column vectors to build P. Let $\vec{v}$ be an eigenvector, then:

$$e^{At}\vec{v} = \left(I + At + \frac{A^2t^2}{2} = \frac{A^3t^3}{3!} + \cdots + \frac{A^nt^n}{n!}\right)\vec{v}$$

Since $\vec{v}$ is an eigenvector, $A\vec{v} = \lambda\vec{v}$, where λ is the eigenvalue correlated with $\vec{v}$. If $A\vec{v} = \lambda\vec{v}$, then $A^k\vec{v} = \lambda^k\vec{v}$:

$$\begin{aligned} e^{At}\vec{v} &= \left(\vec{v} + \lambda\vec{v}t + \lambda^2\vec{v}\frac{t^2}{2} + \lambda^3\vec{v}\frac{t^3}{3!} + \cdots + \lambda^n\vec{v}\frac{n}{n!}\right) \\ &= \left(1 + \lambda t + \frac{\lambda^2t^2}{2} + \frac{\lambda^3t^3}{3!} + \cdots + \frac{\lambda^nt^n}{n!}\right)\vec{v} \end{aligned}$$

Noting that the right-hand ise is the power series for $e^{\lambda t}$, then, if $\vec{v}$ is an eigenvector of A with eigenvalue λ:

$$e^{At}\vec{v} = e^{\lambda t}\vec{v}$$

If $\vec{v}$ is a generic vector, there exists a basis B of $\mathbb{R}^n$ formed by eigenvectors of A, so it can be written as:

$$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 + \cdots + c_n\vec{b}_n$$

So, for linearity:

$$e^{At}\vec{v} = c_1e^{\lambda_1 t}\vec{b}_1 + c_2e^{\lambda_2 t}\vec{b}_2 + \cdots + c_ne^{\lambda_n t}\vec{b}_n = \sum_{j=1}^{n} c_je^{\lambda_j t}\vec{b}_j$$

Now, modifying the notation to bring back the context of the initial value problem $\vec{v} = \vec{C}$:

$$\vec{Y}(T) = \vec{C}e^{At} = \sum_{j=1}^{n} c_j e^{\lambda_j t}\vec{b}_j \Rightarrow \vec{Y}(t) = \vec{Y}_0 e^{At} = \sum_{j=1}^{n} c_j e^{\lambda_j t}\vec{b}_j$$

Where $c_1, c_2, \ldots, c_n$ are the coordinates of $\vec{Y}_0$ with respect to the basis formed by the eigenvectors of A:

$$\vec{Y}_0 = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$$

Complex eigenvalues always come in pair of complex conjugates: $\lambda_{1,2} = r \pm i\omega$. Also their eigenvectors are complex and complex conjugates. Also c_j will be complex, so, having as eigenvalues only the complex couple $\lambda_{1,2}$:

$$\vec{Y}_0 e^{At} = c_1 e^{\lambda_1 t}\vec{b}_1 + c_2 e^{\lambda_2 t}\vec{b}_2$$

Uing Euler's formula:

$$\begin{aligned} e^{\lambda_1 t} &= e^{(r+i\omega)t} = e^{rt}e & i\omega t &= e^{rt}\left[\cos(\omega t) + i\sin(\omega t)\right] \\ e^{\lambda_2 t} &= e^{(r-i\omega)t} = e^{rt}e & -i\omega t &= e^{rt}\left[\cos(\omega t) - i\sin(\omega t)\right] \end{aligned}$$

So that:

$$\begin{aligned} e^{At}\vec{v} &= c_1 e^{rt}\left[\cos(\omega t) + i\sin(\omega t)\right]\vec{b}_1 + c_2 e^{rt}\left[\cos(\omega t) - i\sin(\omega t)\right]\vec{b}_2 \\ &= \left(c_1 e^{rt}\vec{b}_1 + c_2 e^{rt}\vec{b}_2\right)\cos(\omega t) + e^{rt}\left(c_1\vec{b}_1 i - c_2\vec{b}_2 i)\sin(\omega t)\right) \\ &= e^{rt}\cos(\omega t)\vec{u} + e^{rt}\sin(\omega t)\vec{w} \end{aligned}$$

Where:

- $\vec{u} = c_1\vec{b}_1 + c_2\vec{b}_2 = \vec{Y}_0$
- $\vec{w} = c_1\vec{b}_1 i - c_2\vec{b}_2 i$

So that, for homogeneous linear systems with constant coefficients described by a diagonalizable matrix, the solution is:

$$\begin{aligned} \vec{Y}(t) &= \vec{Y}_0 e^{At} = \\ &= \sum_{j=1}^{k} c_j^{\lambda_j t}\vec{b}_j + \sum_{j=k+1}^{n} e^{r_j t}\cos(\omega_j t)\vec{u}_j + e^{r_j t}\sin(\omega_j t)\vec{w}_j \end{aligned}$$

Where the first part accounts for non complex eigenvalues, while the second for each couple of complex conjugates one.

2.1.1 Equilibrium points

Equilibrum solution are particular solution such that $\frac{d\vec{Y}(t)}{dt} = 0$, so that:

$$\frac{d\vec{Y}(t)}{dt} = A\vec{Y}(t) = 0$$

If the matrix has $Rank(A) = n$ the equation has a single solution, the null vector O, which is always present. The system has a single equilibrium in the origin of the plane having $y_i(t)$ in the axis. If $Rank(a) \neq n$, the matrix has infinite equilibrium solutions. The focus will be on $Rank(A) = n$. Considering the $2D$ case, $A \in \mathbb{R}^{n\times n}$ and there are two eigenvalues with corresopnding eigenvectors, the basis for the phase plane. Assuming non-complex eigenvalues the two eigenvector and their eigenspaces can be plotted into the phase plane.

2.1.2 Stability

2.1.2.1 Saddle

$$\lambda_2 < 0 < \lambda_1$$

If the initial poin is in $E_1<$ the eigenspace of $\vec{b}_1$ ($c_2 = 0$), the solution at time t falls in E_1, $c_1 e^{\lambda_1 t}\vec{b}_1$. If $c_2 = 0$ and $c_1 > 0$, increasing t measn that the solution travels along E_1 towards higher values. If $c_2 = 0$ and $c_1 < 0$, the solutions travel along E_1 towards higher absolute values. This is because λ_1 is positive.
Considering $\vec{b}_2$ and E_2, the opposite behavior is seen, whigh the solution approaching 0 despite the sign of c_2: λ_2 is negative.
Staring on a generic pont as t advances the influence of both is seen: projecting the starting point on the eigenspaces, its $\vec{b}_2$ component will shrink and the $\vec{b}_1$ will grow in absolute value.
With $t \to \infty$, the solutions will asymptically approach the origin.

2.1.2.2 Stable node

$$\lambda_1, \lambda_2 < 0$$

With $t \to \infty$ the solution will converge to the oriding no matter the starting point. Before falling into the oriding they will approach the eigenspace correlated with the less negative eigenvalue faster.

2.1.2.3 Unstable node

$$0 < \lambda_1, \lambda_2$$

With $t \to \infty$ the solution will diverge from the origin no matter the starting point.

2.1.3 Complex conjugate eigenvalues

Complex eigenvalue appear in conjugates $\lambda_{1,2} = a \pm i\omega$:

$$\vec{Y}_0 = c_1\vec{b}_1 + c_2\vec{b}_2$$

$$\vec{Y}(t) = e^{rt}\cos(\omega t)\vec{Y}_0 + e^{rt}\sin(\omega t)\vec{w}$$

Trying to compute the value of the solution after an interval $\frac{2\pi}{\omega}$:

$$\begin{aligned}
\vec{Y}\left(+\frac{2\pi}{\omega}\right) &= e^{rt}e^{\frac{2\pi}{\omega}r}\cos\left(\omega t+\omega\frac{2\pi}{\omega}\right)\vec{Y}_0 + e^{rt}e^{\frac{2\pi}{\omega}r}\sin\left(\omega t+\omega\frac{2\pi}{\omega}\right)\vec{w} \\
&= e^{rt}e^{\frac{2\pi}{\omega}r}\cos(\omega t+2\pi)\vec{Y}_0 + e^{rt}e^{\frac{2\pi}{\omega}}\sin(\omega t+2\pi)\vec{w} \\
&= e^{rt}e^{\frac{2\pi}{\omega}r}\cos(\omega t)\vec{Y}_0 + e^{rt}e^{\frac{2\pi}{\omega}}\sin(\omega t)\vec{w} \\
&= e^{\frac{2\pi}{\omega}r}\left[e^{rt}\cos(\omega t)\vec{Y}_0 + e^{rt}e\sin(\omega t)\vec{w}\right] \\
&= e^{\frac{2\pi}{\omega}r}\vec{Y}(t)
\end{aligned}$$

So the solution is the same but multiplied by $e^{\frac{2\pi}{\omega}r}$. Focussing on the real part r. if $r < 0$, then $e^{\frac{2\pi}{\omega}r} < 1$ and the solution becomes smaller. If $r > 0$, then $e^{\frac{2\pi}{\omega}r} > 1$ and the solution becomes bigger. Since this is true fora ll starting points, then if $r < 0$ the solution spirals towards the oriding (stable focus, damped oscillation), while if $r > 0$ the solution spirals away from the origin (unstable focus, amplified oscillation).

2.1.4 Fast criteria for stability

Suppose a homogeneous linear system with constant coefficients composed of n ODEs. In vector form the coefficient are in $A \in \mathbb{R}^{n\times n}$. The stability of the O equilibrium was not ifnluenced by the nature of the eigenvecture, but by the sign of the eigenvalues. So, when looking only for stability:

- If forall λ_j, $Re(\lambda_j) < 0$, then all solutions will converge to O, which is an asypmtotically stable equilibrium.
- If there exist at leas one λ_j such that $Re(\lambda_i) > 0$, for $t \to \infty$, almost all solutions will diverge from O, which is an unstable equilibrium. There are exception for some situations and for somes tarting point: in a saddle if as starting point any point on the eigenspace of the eigenvalue with negative value is chosen, the solution will converge to O.

Let's introduce the spectral bound $S(A)$ of a matrix, which is the maximum real part $Re(\lambda_j)$ of any eigenvalue of A. It is evident that:

- If $S(A) < 0$, then for $t \to \infty$ all solutions will converge to O.
- If $S(A) > 0$, then for $t \to \infty$, almost all solutions will diverge from O.

TO apply it there is still need to compute the eigenvalues, but this can be simplified:

- If A is diagonal the eigenvalues are the element of the diagonal.
- If A is upper or lower triangular, the eigenvalues are the elements on the main diagonal.
- If A is block-triangular the eigenvalues are the union of the eigenvalues of the blocks adjacent to the zero-block of the matrix.

2.1.4.1 Routh-Hurwitz criteria A set of rules that allow to understand if $S(A) < 0$. The criteria depend on the dimension of A.

If $A \in \mathbb{R}^{2\times 2}$, $S(A) < 0$ if and only if:

$$\begin{cases} det(A) > 0 \\ tr(A) < 0 \end{cases}$$

Where:

- $tr(A)$ is the trace of A, the sum of the elements of its main diagonal.

If $A \int \mathbb{R}^{3\times 3}$, $S(A) < 0$ if and only if:

$$\begin{cases} a_1 = -tr(A) > 0 \\ a_2 = \text{ sum of principal minors } > 0 \\ a_3 = -det(A) > 0 \\ a_1 a_2 - a_3 > 0 \end{cases}$$

- The principal minors of A are the determinants of all the $n-1$ matrices generated by removing the j and j row from A.

These criteria are necessary and sufficient. If they are not verified the equilibrium is unstable.

2.2 Non-homogeneous linear systems

They are in the form:

$$\frac{d\vec{Y}(t)}{dt} = A\vec{Y}(t) + \vec{F}(t)$$

The variation of constants and multiplication for an integrating factor can be used to solve them.

2.2.1 Solution

SUing the variation of of constants. First solve the associated homogeneous system:

$$\frac{d\vec{Y}(t)}{dt} = A\vec{Y}(t)$$

The solution, assuming no complex eigenvalues:

$$\vec{Y}(t) = \sum_{j=1}^{k} c_j e^{\lambda_j t} \vec{b}_j$$

Which, in matrix form:

$$\vec{Y}(t) + \begin{bmatrix} e^{\lambda_1} b_{11} & \dots & e^{\lambda_k} b_{1k} \\ \vdots & \ddots & \vdots \\ e^{\lambda_1} b_{n1} & \dots & e^{\lambda_k} b_{nk} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = M\vec{C}$$

At this point the constant $\vec{C}$ is made a function of time:

$$\vec{Y}(t) = M\vec{C}(t) = M \begin{bmatrix} c_1(t) \\ \vdots \\ c_k(t) \end{bmatrix}$$

Taking the derivative of $\vec{Y}(t)$ and considering that $\frac{d\vec{Y}(t)}{dt} = A\vec{Y}(t) + \vec{F}(t)$ and $\vec{Y}(t) = m\vec{C}(t)$:

$$\frac{d\vec{Y}(t)}{dt} = \frac{dM}{dt}\vec{C}(t) + M\frac{d\vec{C}(t)}{dt}$$
$$A\vec{Y}(t) + \vec{F}(t) = \frac{dM}{dt}\vec{C}(t) + M\frac{d\vec{C}(t)}{dt}$$

It is always true that $AM\vec{C}(t) = \frac{dM}{dt}\vec{C}(t)$, so:

$$\vec{F}(t) = M\frac{d\vec{C}(t)}{dt}$$
$$\frac{d\vec{C}(t)}{dt} = M^{-1}\vec{F}(t)$$
$$\vec{C}(t) = \int M^{-1}\vec{F}(t)dt$$

So the solution is:

$$\vec{Y}(t) = M\vec{C}(t)$$
$$\vec{Y}(t) = M\int M^{-1}\vec{F}(t)dt$$

Wich can be written in the form:

$$\vec{Y}(t) = \sum_{j=1}^{k} c_j e^{\lambda_j t}\vec{b}_j + \vec{G}(t)$$

2.2.2 Equilibria

Finding an equilibrium means to solve the problem:

$$\frac{d\vec{Y}(t) = A\vec{Y}(t) + \vec{F}(t)}{dt} = 0$$

Even if $Rank(A) = n$ the zero vector is not guaranteed due to $\vec{F}(t)$. In order to find equilibria there is a need to solve the corresponding system. In the case where the system is non-autonomous ($F(t)$ depends explicitly on time), the zeros of each component will depend on time, meaning that there will be no equilibrium points.

2.2.3 Stability

Stability is the same as in the homogeneous case.

2.3 Non-linear systems and linearization

Solving non linear system is not always possible, but local properties such as stability of an equilibrium can be studied exploiting linearization. This is a process that allows to derive a linear approximation of the system in the neighbourhood of a point of interest. Consider the non linear system:

$$\frac{d\vec{Y}(t)}{dt} = f(t, \vec{Y}(t))$$

And assume that the system will be linearized around the equilibrium solution $\bar{\vec{Y}}(t)$. That means that $f(t, ve\bar{c}Y(t)) = 0 \forall t$. Consider a small displacement from equilibrium: $\vec{V}(t) = \vec{Y}(t) - \bar{\vec{Y}}(t)$, where $\vec{Y}(t) \approx \bar{\vec{Y}}(t)$:

$$\frac{d\vec{V}(t)}{dt} = \frac{d\vec{Y}(t)}{dt}$$

Since $\bar{\vec{Y}}(t)$ is constant. Then:

$$\frac{d\vec{V}(t)}{dt} = \frac{d\vec{Y}(t)}{dt} = f(t, \bar{\vec{Y}}(t)) = f(t, \bar{\vec{Y}}(t) + \vec{V}(t))$$

Now, taking a Taylor expansion of $f(t, \bar{\vec{Y}}(t) + \vec{V}(t))$, with $\vec{X} = \bar{\vec{Y}}(t) + \vec{V}(t)$ and $a = \bar{\vec{Y}}(t)$ and stopping at the first order:

$$\begin{aligned} f(t, \bar{\vec{Y}}(t) + \vec{V}(t)) &= f(t, \bar{Y}) + \frac{d}{dt}\left[f(\bar{\vec{Y}})\right]\left[\bar{\vec{Y}}(t) + \vec{V}(t) - \bar{\vec{Y}}(t)\right] + o(\vec{V}(t)) \\ &= f(t, \bar{\vec{Y}}) + \frac{d}{dt}\left[f(\bar{\vec{Y}})\right]\vec{V}(t) + o(\vec{V}(t)) \end{aligned}$$

So that:

$$\frac{d\vec{V}(t)}{dt} = f(\bar{\vec{Y}}(t)) + \frac{d}{dt}\left[f(\bar{\vec{Y}})\right]\vec{V}(t) + o(\vec{V}(t))$$

Since $\bar{\vec{Y}}(t)$ is a constant function, $f(\bar{\vec{Y}}(t)) = 0$, $o(\vec{V}(t))$ can be ignored:

$$\frac{d\vec{V}(t)}{dt} = f(\bar{\vec{Y}}(t)) + \frac{d}{dt}\left[f(\bar{\vec{Y}})\right]\vec{V}(t)$$

$\frac{d}{dt}\left[f(\bar{\vec{Y}})\right]$ is a Jacobian matrix J, with form:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_n} \end{bmatrix} \qquad \frac{d\vec{V}(t)}{dt} = J\vec{V}(t)$$

In order to study the properties of the equilibrium solution, the Jacobian of the system at the equilibrium point, and then that is treated as the matrix A of the linear systems.