A modified Graham's convex hull algorithm for finding the connected orthogonal convex hull of a finite planar point set

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A B S T R A C T

Graham's convex hull algorithm outperforms the others on those distributions where most of the points are on or near the boundary of the hull (Allison and Noga, 1984). To use this algorithm for finding an orthogonal convex hull of a finite planar point set, we introduce the concept of extreme points of a connected orthogonal convex hull of the set, and show that these points belong to the set. Then we prove that the connected orthogonal convex hull of a finite set of points is an orthogonal (x, y)-polygon where its convex vertices are its connected orthogonal convex hull's extreme points. As a result, an efficient algorithm, based on the idea of Graham's convex hull algorithm, for finding the connected orthogonal convex hull of a finite planar point set is presented. We also show that the lower bound of computational complexity of such algorithms is $O(n \log n)$. Some numerical results for finding the connected orthogonal convex hulls of random sets are given.

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1. Introduction

The computation of the convex hull of a finite planar point set has been studied extensively. It is natural to relate convex hulls to orthogonal convex hulls. Orthogonal convexity is also known as rectilinear, or (x, y) convexity. It has found applications in several research fields, including illumination [1], polyhedron reconstruction [8], geometric search [21], and VLSI circuit layout design [22], digital images processing [19]. The notation of orthogonal convexity has been widely studied since early eighties [23] and some of its optimization properties are given in [11]. However, unlike the classical convex hulls, finding the orthogonal convex hull of a finite planar point set is fraught with difficulties. An orthogonal convex hull of a finite planar point set may be disconnected. Unfortunately the connected orthogonal convex hulls of a finite planar point set might be not unique, even countless. There exist several algorithms to find a connected orthogonal convex hull of a finite planar point set [12,15–17]. In previous works, the definition of orthogonal convex set was used to find a connected orthogonal convex hull of a finite planar point set, and no numerical result has been shown. The first question is "what the

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explicit form of a connected orthogonal convex hull is?". To answer this question, first, we consider some assumption when the connected orthogonal convex hull of a finite planar point set is unique. Secondly, we introduce the concept of extreme points of the connected orthogonal convex hull of a finite planar point set, and show that this hull of a finite planar point set is totally determined by its extreme points and these points belong to the given finite planar point set. There arises the second question "How to detect these extreme points from the given finite planar point set?"

Our QuickhullDisk technique [13] does not work in this case. Let us consider Graham convex hull algorithm [10] that depends on an initial ordering of the given finite planar point set. The initial point with the other points in this order actually forms a star-convex set. Based on this shape, they constructed Graham's convex hull algorithm. Some advantages of Graham's convex hull algorithm can be seen in [7]. In case of connected orthogonal convex hulls, if we have a reasonably ordered points, we then can scan these ordered points to detect extreme points of the connected orthogonal convex hull from these points. As a result, an efficient algorithm to find the connected orthogonal convex hull of a finite planar point set which is based on the idea of Graham's convex hull algorithm [10] will be presented in this paper. As can been seen later in this paper, our new algorithm takes only $O(n \log n)$ time (Theorem 1).

In Sections 2 and 3, we detect in what circumstances, there exists the connected orthogonal convex hull of a planar points set (Propositions 2 - 3). Following the uniqueness of the connected orthogonal convex hull, we provide the construction of the hull which is an (x, y)-orthogonal polygon (Proposition 5 and Corollary 2), and its extreme vertices belong to the given points (Proposition 6). We present a procedure to determine if a given finite planar point set has the connected orthogonal convex hull. Section 4 contains the main algorithm, which is based on the idea of Graham's convex hull algorithm, for finding the connected orthogonal convex hull of a finite planar point set (Algorithm 1) and it states that the

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Algorithm 1 Finding the connected orthogonal convex hull.

Require: A set of finite distinct points P in the plane.
Ensure: List of extreme points of COCH(P) in order.
1: Find $a, b, c, d, e, f, g, h \in P$ satisfying (B) and P_{cb}, P_{ah}, P_{gf}, and P_{ed} satisfying (C).   ▷ *Now we reorder points of P in* $P_{cb} \cup P_{ah} \cup P_{gf} \cup P_{ed}$ *due to their y-coordinates.*
2: Sort all points of $P_{ah} \cup P_{gf}$ in decreasing their y-coordinates.If two points have the same y-coordinate, the one having smaller x-coordinateis chosen.
3: Sort all points of $P_{ed} \cup P_{cb}$ in ascending their y-coordinates.If two points have the same y-coordinate, the one having bigger x-coordinateis chosen.
4: Label these points to $p_0, p_1, \ldots, p_n$.   ▷ *Due to Proposition 6* p_0, p_1 *are extreme points of* COCH(P).
5: Stack $S = < p_0, p_1 > = < p_{t-1}, p_t >$; t indexes top.
6: **for** i = 2 to n **do**
7:     **if** p_i is left of the orthogonal linel(p_{t-1}, p_t) through p_t and p_{t-1} **then**   ▷ *where* $p(p_t, p_{t-1})$ *is defined by* (D).
8:        Push(p_i, S)
9:     **else**
10:        Pop(S).
11:     **end if**
12: **end for**
13: **return** S   ▷ *By Corollary 2, we obtain* COCH(P)*from S.*

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lower bound of computational complexity of such algorithm is $O(n \log n)$ (Proposition 8). Some numerical results show the connected orthogonal convex hulls of some sets of a finite number of points which is randomly positioned in the interior of a given square (Table 1).

2. Preliminaries

Let be given an Euclidean space $(\mathbb{R}^2, \|.\|)$. For $u, v \in \mathbb{R}^2$, denote $[u, v] := \{(1 - \lambda)u + \lambda v : 0 \le \lambda \le 1\}$. A *path* in $\mathbb{R}^2$ is a continuous mapping γ from an interval $[t_0, t_1] \subset \mathbb{R}$ to $\mathbb{R}^2$. We say that γ *joins* the point $\gamma(t_0)$ to the point $\gamma(t_1)$. The *length* of $\gamma : [t_0, t_1] \to \mathbb{R}^2$ is the quantity $length(\gamma) = \sup_{\sigma} \sum_{i=1}^{k} \|\gamma(\tau_{i-1}) - \gamma(\tau_i)\|$, where the supremum is taken over the set of partitions $t_0 = \tau_0 < \tau_1 < \cdots < \tau_k = t_1$ of $[t_0, t_1]$. We assume that all paths in this paper have finite length. We have $length(\gamma) \ge \|\gamma(t_0) - \gamma(t_1)\|$ and equality holds only if $\gamma([t_0, t_1]) = [\gamma(t_0), \gamma(t_1)]$.

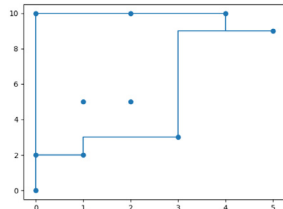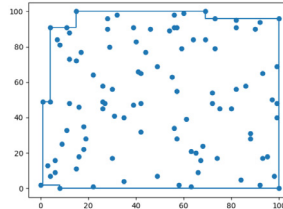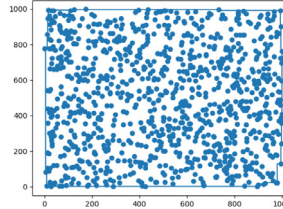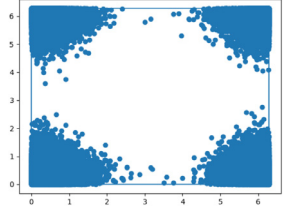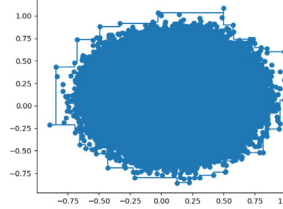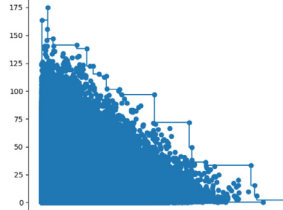Most of definitions and results above can be found in [18]. By abuse of notation, sometimes we also call the image $\gamma([t_0, t_1])$ the path $\gamma : [t_0, t_1] \to \mathbb{R}^2$. We denote by x_u and y_u, respectively the x-coordinate and y-coordinate, respectively of a point $u \in \mathbb{R}^2$.

3. Orthogonal convex sets and their properties

Definition 1 ([23])**.** A set $K \subset \mathbb{R}^2$ is said to be *orthogonal convex* if its intersection with any horizontal or vertical line is convex.

Table 1

Time (an average of 100 runs) required to compute the connected orthogonal convex hull of the set of *n* points with integer coordinates randomly positioned in the interior of a square of size $10,000,000$ having sides parallel to the coordinate lines.

Number of points	Time(s)	Illustrations of connected orthogonal convex hulls of some planar point sets
10	1.21E-03	
100	7.86E-03	
1000	7.75E-02	
10,000	9.53E-01	
100,000	38.91	
1,000,000	8919.93	

In some previous papers (see [12] and [23]), a slightly different definition of orthogonal convexity were given. Here, we use the term "convex" to cover line segments with or without its endpoints. Furthermore, our definition can be extended for $\mathbb{R}^n$. Observe that any convex set is orthogonal convex as seen in Fig. 1 (a), but the vice versa may be not true as seen in Fig. 1 (b).

K is said to be *connected orthogonal convex* if it is orthogonal convex and connected.
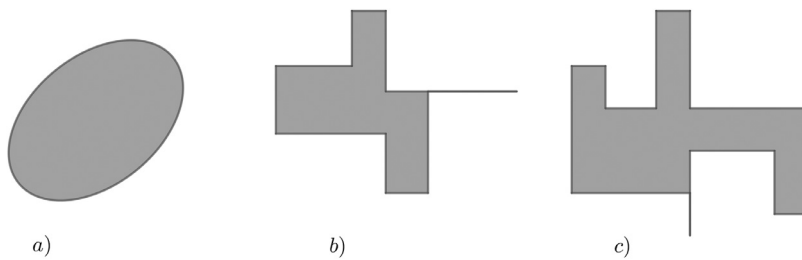
Fig. 1. (a) and (b) two orthogonal convex sets. (c) not an orthogonal convex set.
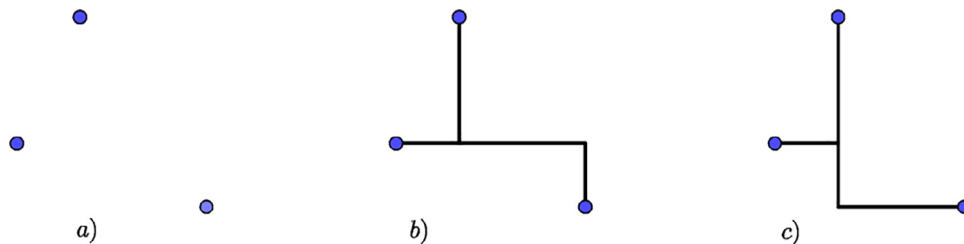

Fig. 2. (a) the set of three points in the plane and its orthogonal convex hull consists of these points. (b), (c) connected orthogonal convex hulls of these points.

Proposition 1. *Let $K \subset \mathbb{R}^2$. Then, K is connected orthogonal convex iff for all $a, b \in K$, there exists a shortest path $\mathrm{SP}(a, b) \subset K$ joining a and b with L_1 norm, and $length(\mathrm{SP}(a, b)) = \|a - b\|_1$, where L_1 norm is determined by $\|u - v\|_1 = |x_u - x_v| + |y_u - y_v|$, for $u = (x_u, y_u), v = (x_v, y_v)$. In addition, $\mathrm{SP}(a, b)$ is an increasingly monotone path (i.e., for $u, v \in \mathrm{SP}(a, b)$, $(x_u - x_v)(y_u - y_v) \geq 0$).*

The proof is given in the Appendix. It is obvious that the intersection of any family (finite or infinite) of orthogonal convex sets is orthogonal convex. An *orthogonal convex hull* [15] of a set $K \subset \mathbb{R}^2$ is the smallest orthogonal convex set which contains K. Thus, the orthogonal convex hull of K is the intersection of all orthogonal convex sets containing K and therefore, the orthogonal convex hull of a set is unique. But it may be not connected.

Definition 2 ([17])**.** A *connected orthogonal convex hull* of K, (*co-convex hull*, for brevity) is a smallest connected orthogonal convex set containing K.

In Fig. 2 we display a set of three distinct points in the plane. Observe that the orthogonal convex hull of the set is itself, and it is disconnected as in Fig. 2 (a). The connected orthogonal convex hulls of the set are not unique, as in Fig. 2 (b, c).

Note that a *co*-convex hull of a finite planar point set is compact. We define a line to be *rectilinear* if the line is parallel to either x-axe or y-axe. A half line or a line segment are *rectilinear* if the lines on which they lie are rectilinear.

Let $a \neq b$ be two given points in the plane. We define the *orthogonal line* $\mathrm{l}(a, b)$ through a, b as follow: if $x_a \neq x_b$ and $y_a \neq y_b$, $\mathrm{l}(a, b)$ is the union of two rectilinear half lines having the same starting point, and if $x_a = x_b$ or $y_a = y_b$, $\mathrm{l}(a, b)$ is the line through a and b.

Thus, an orthogonal line $\mathrm{l}(a, b), (x_a \neq x_b, y_a \neq y_b)$ separates the plane into two regions, as shown in Fig. 3. The quadrant region together with the orthogonal line $\mathrm{l}(a, b)$ will be called a *quadrant* determined by the orthogonal line. In Fig. 3, the quadrant regions are shaded.

Definition 3. Given a set $K \subset \mathbb{R}^2$. An $\mathrm{l}(a, b)$ is an orthogonal supporting line (*O-support*, for brevity) of a set K (a and b might not belong to K) if the intersection of $\mathrm{l}(a, b)$ with K is non-empty and either all points of $K \setminus (K \cap \mathrm{l}(a, b))$ are not on the quadrant of $\mathrm{l}(a, b)(x_a \neq x_b, y_a \neq y_b)$, or all points of $K \setminus (K \cap \mathrm{l}(a, b))$ are on one open half plane which is determined by the line $\mathrm{l}(a, b)(x_a = x_b, \text{or } y_a = y_b)$.

Two *O*-supports $\mathrm{l}(a, b)$ and $\mathrm{l}(c, d)$ of a set K are said to be *opposite* if their half lines meet in exactly two points. Such *O*-supports are indicated in Fig. 4.

We denote by $\mathcal{F}(K)$ the family of all connected orthogonal convex hulls of K. For $E \in \mathcal{F}(K)$, if there exist two opposite *O*-supports H and L of K intersecting in only two points, say p and q, with $x_p \neq x_q, y_p \neq y_q$, then there exists a staircase path[1] connecting p and q in E (for example, the boundary from p to q of the rectangle having rectilinear edges and diagonal $[p, q]$). We define all points on such path (not including p and q) to be *semi-isolated* points of E. Thus, if E has a

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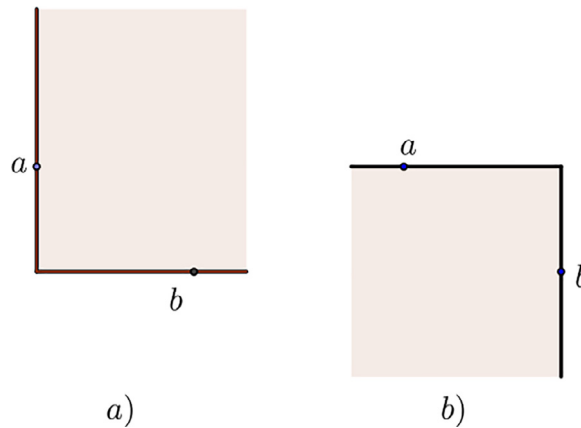[1] For definition of staircase path see [9]

Fig. 3. Orthogonal lines through the points a and b with $x_a \neq x_b$ and $y_a \neq y_b$ and their quadrants (shadow regions).
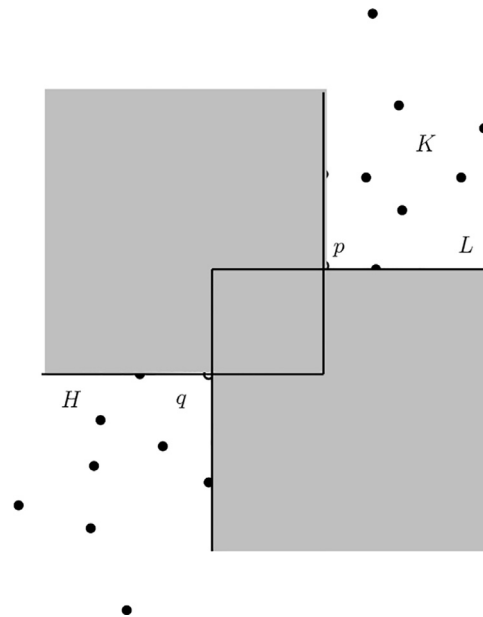

Fig. 4. Two opposite O-supports H and L of the set K and the set of semi-isolated points (the rectangle with the diagonal $[p, q]$ excepting $\{p, q\}$).

semi-isolated point, then E has infinity number of semi-isolated points. Therefore, the set of all semi-isolated points of elements of $\mathcal{F}(K)$ is the rectangle with the diagonal $[p, q]$ excepting $\{p, q\}$. Some semi-isolated points are illustrated in Fig. 4.

It follows immediately from the definition of semi-isolated points the following result.

Remark 1. If $E \in \mathcal{F}(K)$ has no semi-isolated point then every $F \in \mathcal{F}(K)$ also has no semi-isolated point. Therefore, the existence of semi-isolated points does not depend on each element $E \in \mathcal{F}(K)$.

Indeed, on the contrary, suppose that E has no semi-isolated point and there exists $F \in \mathcal{F}(K)$, F has semi-isolated points. Since E has no semi-isolated point, there do not exist two O-supports of K such that they meet each other in exactly two points. By the definition of semi-isolated points, F can not have semi-isolated point. Therefore, our contrary assumption is not true. By the above argument, the existence of semi-isolated points does not depend on each element $E \in \mathcal{F}(K)$.

Note that the intersection of all connected orthogonal convex hulls of a set may be a disconnected set, and it also might not be the orthogonal convex hull of the set. As shown in Fig. 2, connected orthogonal convex hulls of a finite planar point set may be countless. The following proposition provides some conditions such that the family of all connected orthogonal convex hulls of a given set has only one element.

Proposition 2. *Let P be a finite planar point set and $\mathcal{F}(P)$ be the family of all connected orthogonal convex hulls of P. If there exists an element of $\mathcal{F}(P)$ that has no semi-isolated point then $\bigcap_{E \in \mathcal{F}(P)} E$ is a connected orthogonal convex hull of P. Therefore, $\mathcal{F}(P)$ has only one element.*
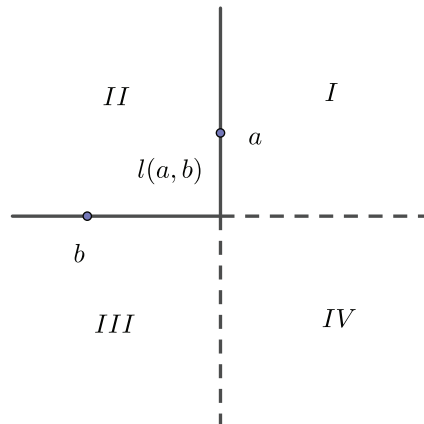
Fig. 5. Four corners I, II, III and IV of an orthogonal line l(a, b). Corner II belongs to the quadrant of l(a, b).

Proof. We will prove that $S = \bigcap_{E \in \mathcal{F}(P)} E$ is orthogonal convex and connected. i) We first prove that S is orthogonal convex. Suppose that the intersection of S and a vertical (horizontal, respectively) line contains two points e and f, where e is the highest point and f is the lowest point (e is the leftmost point and f is the rightmost point, respectively). Then $e, f \in E$, for all $E \in \mathcal{F}(P)$. Since E is orthogonal convex, we get $[e, f] \subset E$, for all $E \in \mathcal{F}(P)$. Therefore, $[e, f] \subset S$. This implies that S is orthogonal convex. ii) We now prove that S is connected. Suppose that S is disconnected. Then it has at least two connected closed components. As S is orthogonal convex, we can assume that two such connected closed components are T_1, T_2 and $x_p < x_q, y_p < y_q$ for all $p \in T_1$ and $q \in T_2$. Let a be the highest rightmost of T_1, b be the rightmost highest point of T_1; c be the lowest leftmost of T_2, d be the leftmost lowest point of T_2. Then there exist two O-supports of P: one connects a and d and the other connects b and c. These O-supports intersect in each other. This contradicts the fact that every element of $\mathcal{F}(P)$ has no semi-isolated point. Thus, S is connected.

Thus, S is the smallest connected orthogonal convex set containing P. Hence, $S \in \mathcal{F}(P)$ and therefore $\mathcal{F}(P)$ has only one element. The proof is complete. □

By Proposition 2, when there is no semi-isolated point, $\mathcal{F}(P)$ has only one element, we then denoted it by COCH(P). From now on, when the notation COCH(P) is used, we understand that elements of $\mathcal{F}(P)$ have no semi-isolated point.

Corollary 1. *Let P be a finite set of points in the plane. Then* COCH(P) *is the intersection of all connected orthogonal convex sets containing P.*

Proof. Let T be the intersection of all connected orthogonal convex sets containing P. We now prove that COCH(P) = T. Since COCH(P) is a connected orthogonal convex set, $T \subset$ COCH(P). For each F is a connected orthogonal convex set, we have COCH(P) $\subset F$. Thus, COCH(P) $\subset T$. □

The intersection of all connected orthogonal convex sets which contains P might not be the orthogonal convex hull of P, even when we assume that every connected orthogonal convex hull of P has no semi-isolated point.

Remark 2. Let P be a finite set of points in the plane. If the orthogonal convex hull of P is connected, then it is also the connected orthogonal convex hull of P.

Let P_1, P_2 be two finite point sets in the plane and $P_1 \subset P_2$. Then, the convex hull of P_1 is also a subset of the convex hull of P_2. Here, COCH(P_1) $\subset$ COCH(P_2) holds under the assumption of no semi-isolated point.

Remark 3. Let P_1, P_2 be two finite point sets in the plane and $P_1 \subset P_2$. Suppose that there exist $E \in \mathcal{F}(P_1), F \in \mathcal{F}(P_2)$ such that E and F have no semi-isolated point. Then COCH(P_1) $\subset$ COCH(P_2).

Note that $P_1 \subset P_2$ yields $P_1 \subset P_2 \subset$ COCH(P_2) and COCH(P_2) is a connected orthogonal convex set containing P_1. By Proposition 2, COCH(P_1) is the connected orthogonal convex hull of P_1. Therefore, COCH(P_1) $\subset$ COCH(P_2).

In the plane, two perpendicular rectilinear lines that define an orthogonal line l(a, b) divide the plane into four open *corners*: I, II, III and IV, and two rectilinear lines, as shown in Fig. 5. Two corners I and III (or II and IV) are opposite. Thus, II and two half lines through a, b form the quadrant of l(a, b).

Proposition 3. *Let P be a finite set of points in the plane and the following condition hold*

(A) *There exists an O-support* l(a, b) *of P ($a, b \in P$) such that all points of $P \setminus \{a, b\}$ lie only in both two opposite corners of* l(a, b).

Then there exists a connected orthogonal convex hull of P that has semi-isolated points and vice versa.

Proof. Suppose that all points of $P \setminus \{a, b\}$ lie only in two opposite corners I and III of $l(a, b)$. Let c be a smallest y-coordinate point among these points in corner I, and d be a greatest x-coordinate point among these points in corner III (because P is finte, such c and d exist). Then an orthogonal line $l(c, d)$ is an O-support and the intersection of $l(c, d)$ and $l(a, b)$ is exact two distinct points. Therefore, connected orthogonal convex hulls of P have semi-isolated points. The vice versa is obtained from the definition of semi-isolated points. Thus the proof is complete. $\square$

Let us describe COCH(P) when P is a finite planer point set. Based on the "throw-away" principle [2], we take the points a, b, c, d, e, f, g, and h belonging to P such that

(B) a (b, respectively) is the leftmost (rightmost, respectively) of highest points of P,
  2. e (f, respectively) is the rightmost (leftmost, respectively) of lowest points of P,
  3. c (d, respectively) is the highest (lowest, respectively) of rightmost points of P,
  4. g (h, respectively) is the lowest (highest, respectively) of leftmost points of P.

Then the rectangle $pquv$ formed by a, b, c, d, e, f, g, h is the minimum rectangle containing P and its edges are rectilinear (p is the intersection of the lines through ab and gh, q is the intersection of the lines through ab and cd, u is the intersection of the lines through cd and ef, v is the intersection of the lines through ef and gh). Assume without loss of generality that $p \neq q \neq u \neq v$. This rectangle $pquv$ is connected orthogonal convex and therefore contains COCH(P). In addition, the rectilinear line segments $[a, b], [c, d], [e, f], [g, h]$ belong the boundary of COCH(P) as (B) holds and COCH(P) is orthogonal convex. Assume without loss of generality that $p \neq q \neq u \neq v$ and $a \neq p$.

Suppose that $E \in \mathcal{F}(P)$ has semi-isolated points. According to Proposition 3, there are two opposite O-supports of P such that one lies in $l(a, h)$ and the other lies in $l(d, e)$, or one lies in $l(b, c)$ and the other lies in $l(f, g)$. Based on Proposition 3, procedure Semi-Isolated_Point($P, flag$) that is presented in the Appendix finds out if a connected orthogonal convex hull of a set of points has semi-isolated points or not.

From now on, we only consider the finite set of points P in the plane such that $|P| > 1$ and P does not satisfy the condition (A).

Proposition 4. *Let P be a finite planar point set. For two distinct points a and b in P there exists at most one O-support of P through a and b.*

Proof. Suppose $l_1(a, b)$ through a and b having starting point at (x_a, y_b) is an O-support of P and $l_2(a, b)$ is another orthogonal line through a and b having starting point (x_b, y_a). Then $l_1(a, b)$ and $l_2(a, b)$ are opposite O-supports of P. Since P does not satisfy the condition (A), no semi-isolated point can arise. Hence, there exists a point of P that is in the rectangle determined by $l_1(a, b)$ and $l_2(a, b)$. Therefore, $l_2(a, b)$, from the definition of O-support, can not be an O-support. $\square$

Under the assumptions of Proposition 4, two given points of a finite planar point set P determine at most one O-support of P.

4. Construction of the connected orthogonal convex hull of a finite planar point set

We need some definitions.

A *rectilinear polygon* (see [15]) is a simple polygon whose edges are rectilinear (i.e., they are parallel to either x or y axis). The polygon has therefore only 90 and 270 degree internal angles.

Definition 4 ([15])**.** An $(x, y)-polygon$ is one of the following: (a) a point; (b) connected rectilinear line segments; (c) a rectilinear polygon; and (d) a connected union of type (b) and or type (c) (x, y)-polygons.

Let us describe COCH(P) when P is a finite planar point set. Take the points a, b, c, d, e, f, g, and h belonging to P satisfying (B).

Consider the case $a \neq h, g \neq f, e \neq d$ and $c \neq b$. Take the orthogonal lines $l(a, h)$ through a and h, $l(g, f)$ through g and f, $l(e, d)$ through e and d, and $l(c, b)$ through c and b such that $p \notin l(a, h)$, $v \notin l(g, f)$, $u \notin l(e, d)$, and $q \notin l(c, b)$ (see Fig. 6). We define

(C) If $a \neq h$, P_{ah} is the set of points of P in the quadrant of $l(a, h)$. Otherwise, $P_{ah} := \{a\}$.
  2. If $b \neq c$, P_{cb} is the set of points of P in the quadrant of $l(c, b)$. Otherwise, $P_{cb} := \{b\}$.
  3. If $g \neq f$, P_{gf} is the set of points of P in the quadrant of $l(g, f)$. Otherwise, $P_{gf} := \{f\}$.
  4. If $e \neq d$, P_{ed} is the set of points of P in the quadrant of $l(e, d)$. Otherwise, $P_{ed} := \{e\}$.

We use the scanline technique given in [12] to find the successive layers of maximal elements of P_{ah}, where a point $w \in P_{ah}$ is maximal if there are no other points z of P_{ah} such that $x_w \geq x_z$ and $y_w \leq y_z$. Assume that $p_0 = a, p_1, \ldots, p_{k-1}, p_k = h$ are such maximal elements. They form a staircase path, say, $\mathcal{P}_{ah}$ (i.e., a union of parts of O-supports through p_i and p_{i+1}, $i = 0, \ldots, k - 1$) joining a and h such that the region formed by this path and $l(a, h)$ consists of P_{ah} and the area of this region is minimum.

Thus there is an orthogonal convex (x, y)-polygon, say $T(P)$, formed by the rectilinear line segments $[a, b], [c, d], [e, f], [g, h]$ and staircase paths $\mathcal{P}_{cb}$, $\mathcal{P}_{ah}$, $\mathcal{P}_{gf}$, $\mathcal{P}_{ed}$, respectively joining c and b, a and h, g and f, e and d, respectively.
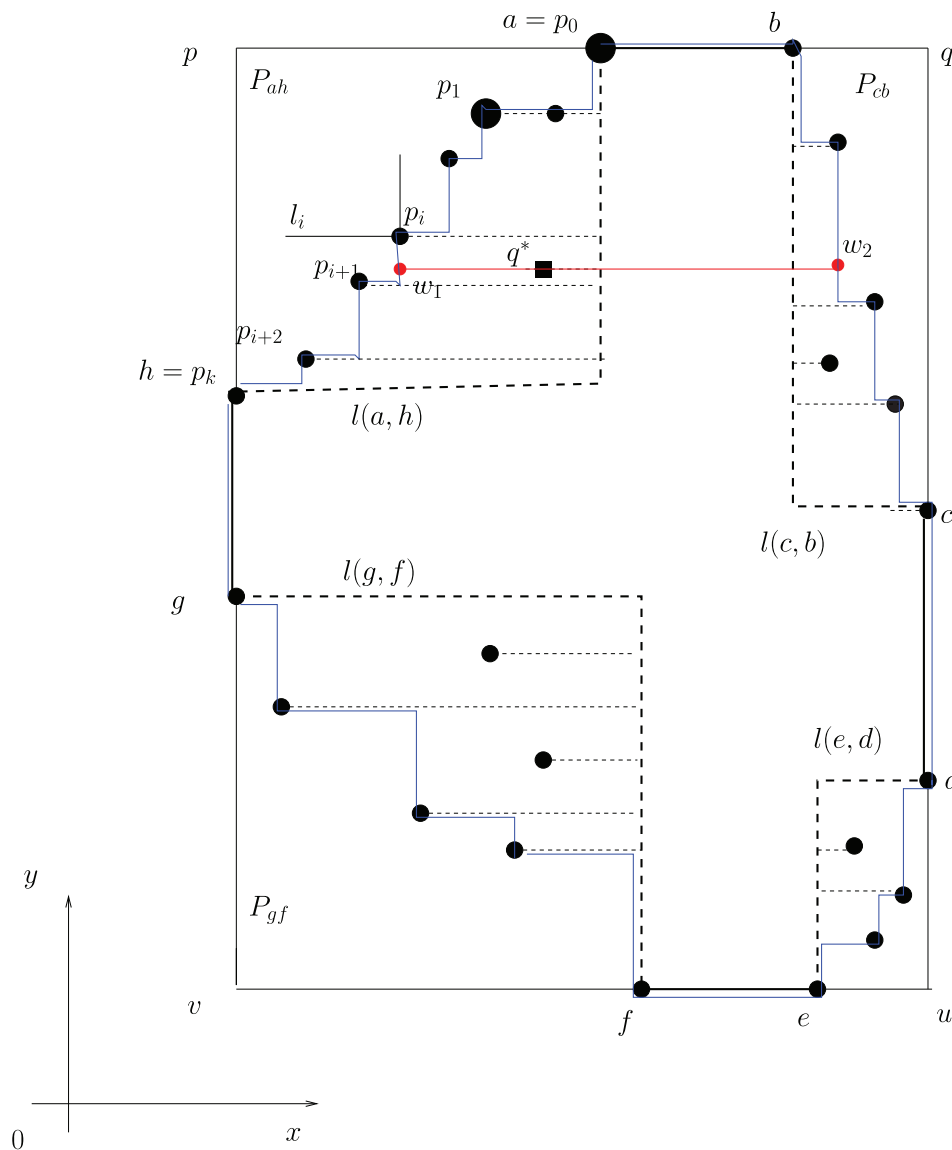
Fig. 6. The case $a \neq h$. A staircase path (coloured in blue) joining a and h (formed by maximal elements $p_0 = a, p_1, \ldots, p_{k-1}, p_k = h$ of the set P_{ah}). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Proposition 5. *Let P be a finite planar point set and $\mathcal{F}(P)$ be the family of all connected orthogonal convex hulls of P. Then the intersection COCH(P) of all connected orthogonal convex sets is an orthogonal convex (x, y)-polygon formed by the rectilinear line segments $[a, b], [c, d], [e, f], [g, h]$ and staircase paths $\mathcal{P}_{cb}, \mathcal{P}_{ah}, \mathcal{P}_{gf}, \mathcal{P}_{ed}$.*

Proof. Suppose that $T(P)$ is the region formed by $[a, b], [c, d], [e, f], [g, h], \mathcal{P}_{cb}, \mathcal{P}_{ah}, \mathcal{P}_{gf}, \mathcal{P}_{ed}$. By the construction of $T(P)$, $P \subset T(P)$ and $T(P)$ has no semi-isolated point. It follows from Remark 3 that COCH$(P) \subset T(P)$. We are in position to prove that $T(P) \subset$ COCH(P). Assume the contrary that there exists $q^* \in T(P)$. Because COCH(P) has no semi-isolated point, a horizonal line through q^* intersects two of staircase paths $\mathcal{P}_{cb}, \mathcal{P}_{ah}, \mathcal{P}_{gf}, \mathcal{P}_{ed}$, say at $w_1 \in \mathcal{P}_{ah}$ and $w_2 \in \mathcal{P}_{cb}$ (see Fig. 6). Assume that w_1 belongs to the O-support L of P through p_i, p_{i+1} and between these points. Because P does not satisfy (A), the part of L between p_i and p_{i+1} belongs to COCH(P). It implies that $w_1 \in$ COCH(P). Similarly, $w_2 \in$ COCH(P). Therefore, $q^* \in [w_1, w_2] \subset$ COCH(P). Hence, $T(P) \subset$ COCH(P). □

Definition 5. Let P be a finite planar point set. We define a point $u \in$ COCH(P) to be *extreme* of COCH(P) if there exists an orthogonal line L (u is the starting point of two half lines of L) whose intersection with COCH(P) is only u and there is no point of COCH$(P) \setminus \{u\}$ which lies in the quadrant determined by L. We denote all extreme points of COCH(P) briefly by o-ext(COCH(P)).

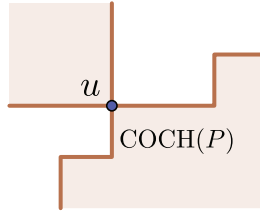**Fig. 7.** Illustration of an extreme point of COCH(P).

In Fig. 7 we display a point $u \in$ COCH(P) that is an extreme point of COCH(P). Consider the boundary of COCH(P) between a and h. It is a staircase path formed by the maximal elements $p_0 = a, p_1, \ldots, p_{k-1}, p_k = h$ (Fig. 6). We claim that p_i is an extreme point of COCH(P). Indeed, if it was not an extreme point of COCH(P), there was some point of P_{ah} in the quadrant of an orthogonal line, say l_i at p_i. It follows that p_i was not a maximal element, a contradiction. Thus, all points $p_0 = a, p_1, \ldots, p_{k-1}, p_k = h$ are extreme points of COCH(P). Therefore, we obtain the following

Proposition 6. *Let P be a finite planar point set. All extreme points of* COCH(P) *in P_{ah} are maximal elements of P_{ah}. They are formed by the staircase paths $\mathcal{P}_{ah}$. Consequently, all extreme points of* COCH(P) *are formed by the staircase paths $\mathcal{P}_{cb}$, $\mathcal{P}_{ah}$, $\mathcal{P}_{gf}$, $\mathcal{P}_{ed}$ and therefore they belong to P.*

Proposition 7. *Let P be a finite planar point set. Then,*

$$\text{COCH}(P) = \text{COCH}(\text{o} - \text{ext}(\text{COCH}(P))).$$

Proof. Let $T = \text{COCH}(\text{o-ext}(\text{COCH}(P)))$. By Proposition 6, o-ext(COCH(P)) $\subset P$. Then, by Remark 3, $T = \text{COCH}(\text{o-ext}(\text{COCH}(P))) \subset \text{COCH}(P)$.

On the other hand, by Proposition 5, the COCH(P) is formed by the rectilinear line segments determined by extreme points of COCH(P). It implies that COCH(P) $\subset T$. The proof is complete. $\square$

Obviously, the staircase path $\mathcal{P}_{cb}$ ($\mathcal{P}_{ah}$, $\mathcal{P}_{gf}$, $\mathcal{P}_{ed}$, respectively) can be seen as an union of finite set of O-supports, and each O-support goes through two extreme points of P_{cb} P_{ah}, P_{gf}, P_{ed}, respectively). It follows directly from Prop. 6 the following

Corollary 2. *The connected orthogonal convex hull of a finite planar point set P is an orthogonal convex (x, y)-polygon whose boundary is union of finite set of O-supports, and each O-support goes through two extreme points of P.*

5. Algorithm

5.1. New algorithm based on Graham's scan

Let P be a finite planar point set and (A) do not hold. In case of connected orthogonal convex hulls, if we have a reasonably ordered points, we then can scan these ordered points to get candidates for extreme points of COCH(P).

Assume without loss of generality that $p \neq q \neq u \neq v$ and $a \neq p$. First of all, for each set P_{cb} P_{ah}, P_{gf}, P_{ed}, we reorder points due to their y-coordinates only. Then we use Graham's scan. More details can be seen in Algorithm 1.

Before starting the algorithm we need to determine an orthogonal line $l(p_t, p_{t-1})$ through two points p_t, p_{t-1} in P_{ha}, P_{gf}, P_{ed}, P_{cb} as follows

(D) In P_{ha}: If $x_{p_t} \leq x_{p_{t-1}}$, $l(p_{t-1}, p_t)$ is parallel to $[q, u] \cup [u, v]$. Otherwise, $l(p_t, p_{t-1})$ is parallel to $[p, q] \cup [q, u]$.

  2. In P_{gf}: If $x_{p_t} \leq x_{p_{t-1}}$, $l(p_t, p_{t-1})$ is parallel to $[q, u] \cup [u, v]$. Otherwise, $l(p_t, p_{t-1})$ is parallel to $[p, q] \cup [q, u]$.

  3. In P_{ed}: If $x_{p_t} \leq x_{p_{t-1}}$, $l(p_t, p_{t-1})$ is parallel to $[u, v] \cup [v, p]$. Otherwise, $l(p_t, p_{t-1})$ is parallel to $[v, p] \cup [p, q]$.

  4. In P_{cb}: If $x_{p_t} \leq x_{p_{t-1}}$, $l(p_t, p_{t-1})$ is parallel to $[u, v] \cup [v, p]$. Otherwise, $l(p_t, p_{t-1})$ is parallel to $[v, p] \cup [p, q]$.

Let $l(q_1, q_2)$ be an orthogonal line through two points q_1, q_2 and its rectilinear half lines have starting point q_3. If the triple (q_1, q_3, q_2) forms a clockwise circuit, and a point q_4 is not in the quadrant determined by l, then q_4 is to the *left* of l. In other words, a point q_4 is to the left of l if q_4 is to the left of the directed line $q_1 q_3$ or q_4 is to the left of the directed line $q_3 q_2$.[2]

Theorem 1. *Algorithm 1 determines* COCH(P). *The time complexity is $O(n \log n)$, where n is the number of points of P.*

Proof. As we have seen that a, b, c, d, e, f, g, h are extreme point of COCH(P), we can assume without loss of generality, that $P = P_{ah}$ ($b = c, g = f, e = d$). Firstly, we claim that each point which is poped from the stack S is impossible to be an extreme point of COCH(P). Indeed, suppose that point p_i is poped from the stack, i.e., there is some p_j such that p_j is not on the left of $l(p_{t-1}, p_t)$ and $p_t = p_i$. Assume that $p_l = p_{t-1}$. Then $l < i < j$ as we order points of P_{cb} in decreasing their y-coordinates.
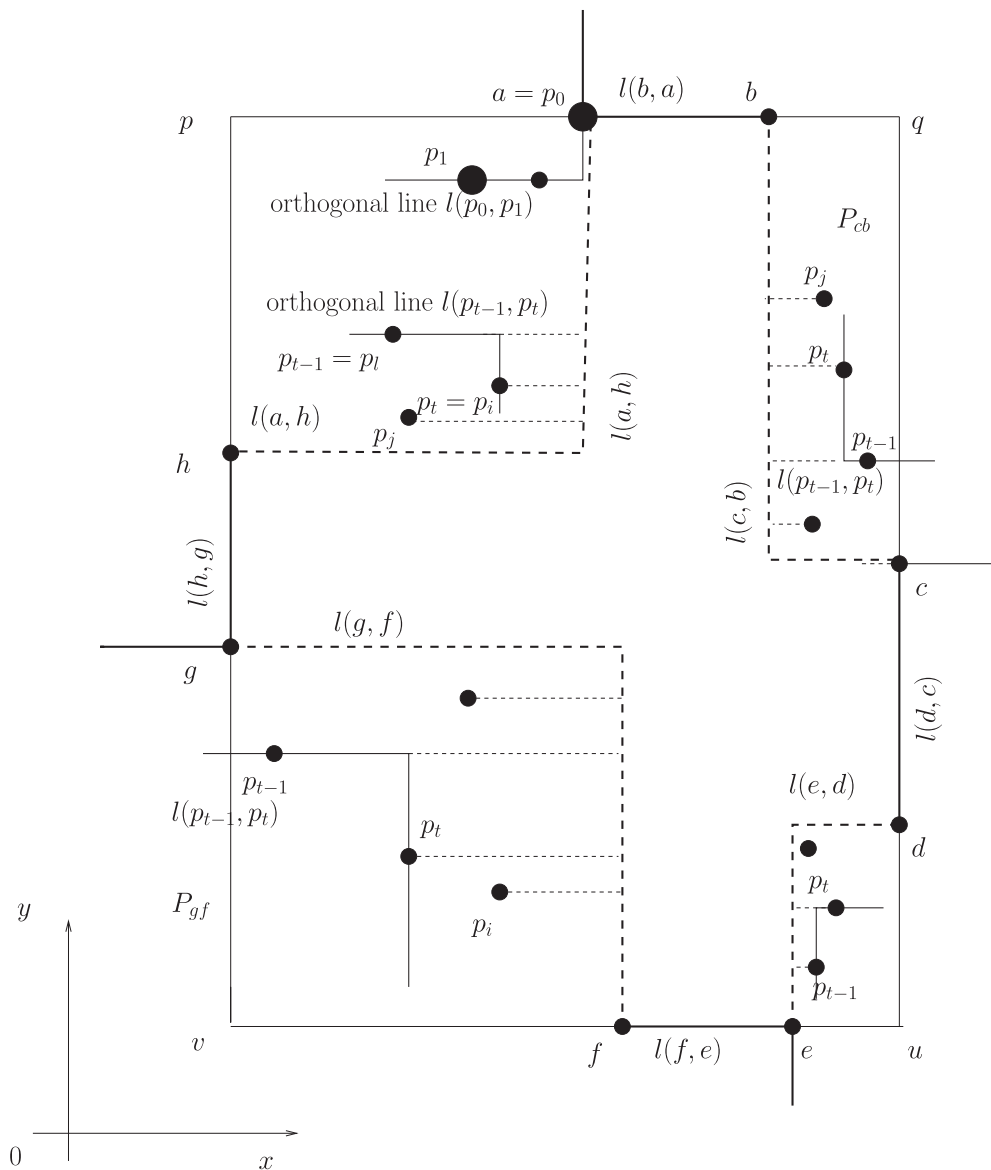
Fig. 8. Illustration of the orthogonal line l(p_t, p_{t-1}) (defined by the relation between x_{p_t} and $x_{p_{t-1}}$ and by the location of p_t, p_{t-1} in P_{ha}, P_{gf}, P_{ed} and P_{cb}).

Hence, $p_i \in P_{ah}$, $x_{p_i} > x_{p_j}$ and $y_{p_i} > y_{p_j}$ (see Fig. 8). Thus, p_i is not a maximal element of P_{ah}. According to Proposition 6, p_j is not an extreme point of COCH(P).

Next, we claim that when the algorithm stops, the points on stack always are extreme points of COCH(P). Indeed, by Proposition 6, $a = p_0$, p_1 are extreme points of COCH(P). We now prove by induction. Assume that $p_{t-1} \in S$ with $t \geq 2$ is an extreme point of COCH(P), we prove that p_t is an extreme point of COCH(P), too. Indeed, we have p_j is left of l(p_{t-1}, p_t) for all $j > i$, where $p_t = p_i$, $p_{t-1} = i'$, $i' < i$. As $j > i$ and the sort of points of P_{ah} is in decreasing their y-coordinates, we have $y_{p_j} < y_{p_i}$. On the other hand, as p_{t-1}, p_t are consecutive points in S, we get that all points $p_m \in P_{ah}$ ($i' < m < i$) are left of l(p_{t-1}, p_t). It follows from Proposition 6 and the fact that p_{t-1} is a maximal element of P_{ah} that $x_{p_t} > x_{p_{t-1}}$. Thus, p_t is a maximal element of P_{ah}. It follows from Prop. 6 that p_t is an extreme point of COCH(P).

We now turn to analysis the complexity of the algorithm. Step 1 needs $O(n)$ time. Steps 2 and 3 need $O(n \log n)$ time. Steps 6–13 take $O(n)$ time. Therefore, Algorithm 1 takes $O(n \log n)$ time. □

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[2] c is left of a directed line ab iff $(x_b - x_a)(y_c - y_a) - (x_c - x_a)(y_b - y_a) > 0$.
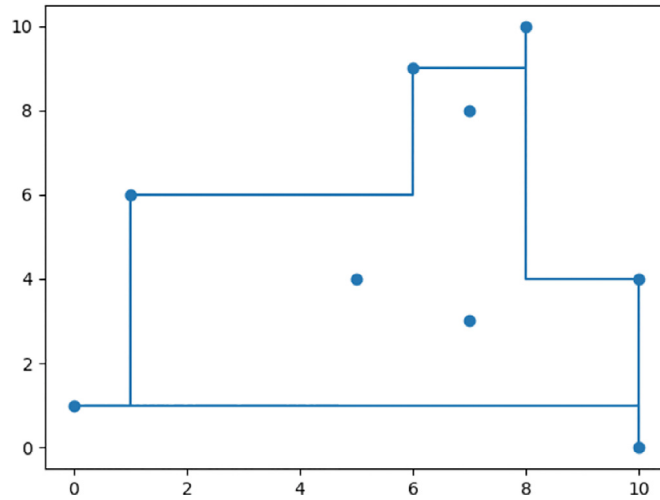
Fig. 9. Illustration of the Example 1.

Example 1. A demonstration of the algorithm is shown in Fig. 9. The input is $P = \{(0, 1), (10, 0), (7, 3), (5, 4), (1, 6), (10, 4), (7, 8), (6, 9), (8, 10)\}$ that does not satisfy (A). A highest point is $a = b = (8, 10)$, a leftmost point is $g = h = (0, 1)$, a lowest point is $e = f = (10, 0)$, the rightmost point is $c = d = (10, 4)$.

After sorting via y-coordinates, we have the list of points

$$P = \{(8, 10), (6, 9), (7, 8), (1, 6), (5, 4), (7, 3), (0, 1), (10, 0), (10, 4)\}.$$

Below is shown the stack S and the value of i at the for loop:

$i = 2 : (8, 10), (6, 9)$

$i = 3 : (8, 10), (6, 9), (7, 8)$

$i = 4 : (8, 10), (6, 9), (1, 6)$

$i = 5 : (8, 10), (6, 9), (1, 6), (5, 4)$

$i = 6 : (8, 10), (6, 9), (1, 6), (5, 4), (7, 3)$

$i = 7 : (8, 10), (6, 9), (1, 6), (0, 1)$

$i = 8 : (8, 10), (6, 9), (1, 6), (0, 1), (10, 0)$

$i = 9 : (8, 10), (6, 9), (1, 6), (0, 1), (10, 0), (10, 4).$

Hence, COCH(P) is determined by the extreme points $(8, 10)$, $(6, 9)$, $(1, 6)$, $(0,1)$, $(10, 0)$, $(10, 4)$ and their order.

5.2. Lower bound of computational complexity

The lower bound of computational complexity of all algorithms for finding the connected orthogonal convex hull can be proved similarly to lower bound of computational complexity of finding convex hulls (see [20]).

Proposition 8. *Lower bound on computational complexity of algorithms for finding the connected orthogonal convex hull of a finite planar point set is the same as for sorting, it means $O(n \log n)$.*

Proof. We have presented Algorithm 1 that runs in $O(n \log n)$ time to find the connected convex hull of a finite set of points. We will prove that any algorithm for finding the connected convex hull of a finite set of points cannot run faster than sorted algorithms (Hence, since the lower bound of computational complexity of sorted algorithms is $O(n \log n)$, this implies the required proof).

Suppose that problem A is an unsorted list P_1 of numbers to be sorted, $x_1, x_2, \ldots, x_n$ and we have some algorithm B that constructs the connected orthogonal convex hull as a (x, y)-polygon of n points in $T(n)$ time. Now we will use B to solve A in time $T(n) + O(n)$, where the additional $O(n)$ represents the time to convert the solution of B to a solution of A.

Let $P_1 \subset [w_1, w_2]$. Take $w = (w_1 + w_2)/2$. Now we have the set

$$P := \{(x_i \in P_1, |x_i - w|), i = 1, \ldots, n\} \cup \{(w_1, w - w_1), (w_2, w_2 - w)\}$$

in the plane, as shown in Fig. 10, where $\{(w_1, w - w_1), (w_2, w_2 - w)\}$ are artificial points. P lies on the graph of the function $y = |x - w|$ and does not satisfy (A). We use algorithm B to construct the connected orthogonal convex hull COCH(P) of these points. It follows from Proposition 6 that every point of P is extreme point of COCH(P). The order in which the points
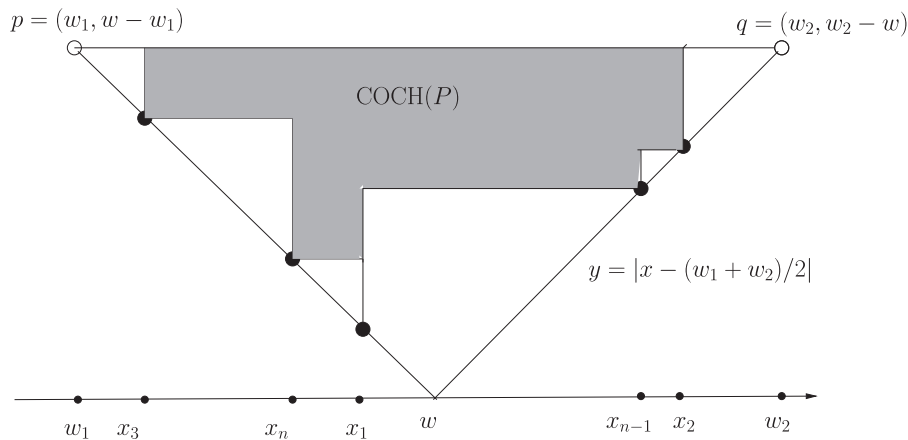
Fig. 10. Illustrating the proof of Proposition 8.

of P occur on the hull in counterclockwise from p is the sorted order for P_1. Thus we can use any algorithm for finding the connected orthogonal convex hull to sort the list P_1, but it cannot run faster than sorted algorithms. $\quad\square$

6. Implementation

Our algorithm was implemented in Python. Tests were run on a PC 3.20GHz with an intel Core i5 and 8 GB of memory. The actual run times of our algorithm on the set of a finite number of points which is randomly positioned in the interior of a square of size 10000000 having sides parallel to the coordinate lines are given in Table 1. In our experiments, the more points we add the less cases of semi-isolated points happen. In the code we consider the case $a = h = p$ as follows: we take the last point in the ordered points in Step 4 of Algorithm 1 to be the starting point p_0 and $a = h$ to be the second point p_1. Following the way of sorting all points of P we obtain p_0 as an extreme point of COCH(P).

7. Concluding remarks

We have provided a similar result as in convex analysis, that is the connected orthogonal convex hull of a planar point set is the intersection of all connected orthogonal convex sets of the given set under the assumption of no semi-isolated point. We then have introduced the concept of extreme vertices of the connected orthogonal convex hull of a finite planar point set under the same condition. The construction of the connected orthognal convex hull via extreme points is also presented. Then an efficient algorithm and the lower bound of computational complexity of all algorithms for finding the connected orthognal convex hull of a finite planar point set are presented.

The method of orienting curves [3–5], can be used for finding the connected orthogonal convex hull of a finite planar point set. A similar definition of extreme points can be used for $\mathcal{O}_\beta$-convex sets, where $\mathcal{O}_\beta$-convex was introduced in [6]. The connected orthogonal convex hull can be applied in aircraft type recognition [14]. Instead of QuickhullDisk [13], we can use the idea of Graham's convex hull algorithm for finding the convex hull of disks. They will be the subject of another paper.

Acknowledgments

Appendix

The proof of Proposition 1:

Proof. (i) Suppose that K is connected orthogonal convex, $a \neq b \in K$. If $x_a = x_b$ or $y_a = y_b$ then SP(a, b) $= [a, b] \subset K$ and $length(\text{SP}(a, b)) = \|a - b\|_1$. Otewise, we assume, without loss of generality, that b is on the north east of a ($x_a < x_b$ and $y_a < y_b$). Take a path γ joining a and b in K (since K is connected, such a path exists).

Consider the rectangle $acbd$ determined by a and b such that the lines go through its egdes $[a, c], [b, d]$, respectively are vertical lines, respectively and the lines go through its egdes $[a, d], [b, c]$, respectively are horizonal lines, respectively.
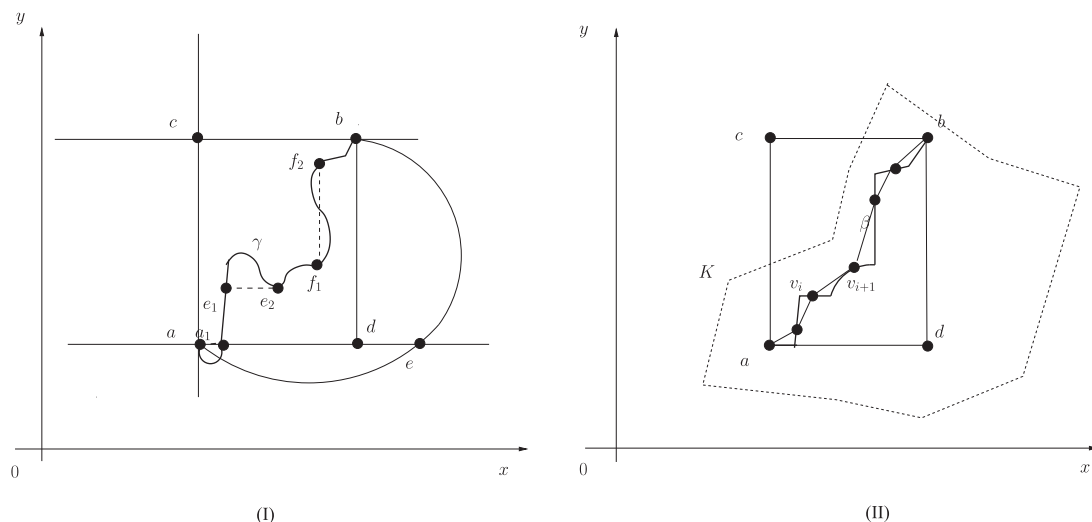
Fig. 11. (I) γ is an arbitrary path joining a and b in K. We replace its parts between a and a_1, e_1 and e_2, f_1 and f_2 by $[a, a_1]$, $[e_1, e_2]$, $[f_1, f_2]$ to get β. (II) β is an increasingly monotone path joining a and b in $K \cap acbd$.

We first prove that there exists an insreasingly monotone path β joining a and b (geometrically speaking, this path only goes to right or goes up from a to b) and in $K \cap adbc$ and $length(\gamma) \geq length(\beta)$. Indeed, if $\gamma \setminus \{a, b\} \cap abcd = \emptyset$ then $\gamma \setminus \{a, b\}$ intersects with at least one of vertical and horizonal lines through a. Suppose that $\gamma \setminus \{a, b\}$ intersects with the horizonal line through a at e (see Fig. 11 (I)). By the orthogonalness of K, $[a, e] \subset K$. If $d \in [a, e]$ then $[b, d] \subset K$. It follows that $[a, d] \cup [b, d]$ is an insreasingly monotone path joining a and b and lies in K. If there are two points f_1, f_2 of γ in $K \cap adbc$ having the same x-coordinates (or having the same y-coordinates) then by the orthogonalness of K, $[f_1, f_2] \subset K$ and we replace the part of γ between f_1 and f_2, say γ_f, by $[f_1, f_2]$. Clearly,

$$length(\gamma_f) \geq l([f_1, f_2]) = \|f_1 - f_2\|_1.$$

As a result, we get an insreasingly monotone path β in $K \cap adbc$ joining a and b (see Fig. 11 (I)) and $length(\gamma) \geq length(\beta)$.

Next, we prove that $length(\beta) = \|a - b\|_1$. Take a partition of $[a, b]$ as follows: $v_0 = a, v_1, \ldots, v_i, \ldots, v_n = b \in \beta$ such that $x_{v_i} < x_{v_{i+1}}, i = 0, \ldots, n - 1$. Since β is increasingly staircase, we have $y_{v_{i+1}} \geq y_{v_i}, i = 0, \ldots, n - 1$. It follows that

$$\begin{aligned}
length(\beta) &= \sup_\sigma \sum_{i=0}^{n-1} \|v_i - v_{i+1}\|_1 \\
&= \sup_\sigma \sum_{i=0}^{n-1} (|x_{v_{i+1}} - x_{v_i}| + |y_{v_{i+1}} - y_{v_i}|) \\
&= \sup_\sigma \sum_{i=0}^{n-1} (x_{v_{i+1}} - x_{v_i} + y_{v_{i+1}} - y_{v_i}) \\
&= \sup_\sigma (x_{v_n} - x_{v_0} + y_{v_n} - y_{v_0}) = x_b - x_a + y_b - y_a \\
&= \|b - a\|_1
\end{aligned}$$

where the supremum is taken over the set of such partitions. Therefore, $length(\gamma) \geq length(\beta) = \|b - a\|_1$ for an arbitrary path γ joining a and b in K. We conclude that β is a shortest path joining a and b in K with norm l_1 and $length(\beta) = \|b - a\|_1$. ii) It remains to prove that for every $a, b \in K$, if there exists a shortest path $SP(a, b)$ joining a and b with l_1 norm and lying in K and $length(SP(a, b)) = \|a - b\|_1$, then K is connected orthogonal convex. Suppose p, q belong to the intersection of some horizontal line l to K and $x_p < x_q$ (the case of a vertical line l is similar). Then $y_p = y_q$ and therefore $\|p - q\|_1 = x_q - x_p$. Take a point $v \in SP(p, q)$ such that $x_p < x_v < x_q$. If $y_v \neq y_p$ then by the triangle inequality, we have $\|q - v\|_1 + \|v - p\|_1 \leq length(SP(p, q))$. It implies that

$$\begin{aligned}
x_q - x_p &= x_q - x_v + x_v - x_p \\
&< x_q - x_v + |y_q - y_v| + |y_q - y_v| + x_v - x_p \\
&= \|q - v\|_1 + \|v - p\|_1 \leq length(SP(p, q)).
\end{aligned}$$

Hence, $\|p - q\|_1 < length(SP(p, q)$, a contradiction. Thus, $y_v = y_p$. It follows that $[p, q] \subset SP(p, q)$. As $SP(p, q) \subset K$, we get $[p, q] \subset K$. By the defintion of orthogonal convex sets, K is orthogonal convex. It is easy to see that K is connected. $\square$
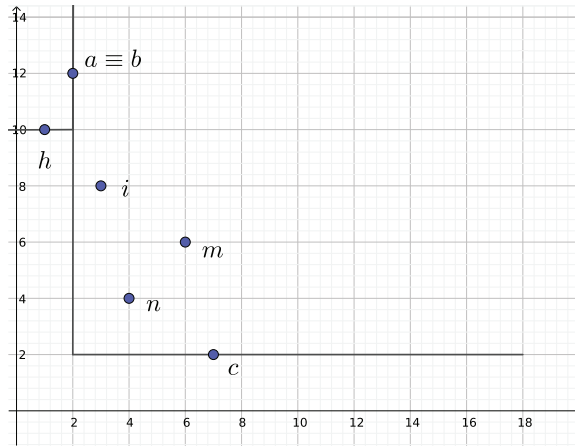
Fig. 12. $P = \{a, b, c, h, i, m, n\}$ and a connected orthogonal convex hull of P has semi-isolated points.

Based on Proposition 3, procedure Semi-Isolated_Point(P, $flag$) below finds out if a connected orthogonal convex hull of a set of points has semi-isolated points or not. Note that we need only to check points of P in the $l(a, h)$'s quadrant and in the $l(b, c)$'s quadrant. If all points of P in $l(a, h)$'s quadrant do not sastify the condition (A), then we continue to check the points of P in $l(b, c)$'s quadrant. Here our pseudocode is only for the $l(a, h)$'s quadrant. The case of the $l(b, c)$'s quadrant is similar.

1: **procedure** Semi-Isolated_Point(P, flag)
Require: A set of finite points P in the plane.
Ensure: Determine whether a *co*-convex hull of P has semi-isolated points.
2:     Take a and h satisfying (B).
3:     Consider the orthogonal line $l(a, h)$ through a and h parallel to the line segment $[q, v]$ and $[u, v]$.
4:     Find $A := \{$all points of P lie in the quadrant of $l(a, h)\}$.
5:     Find the set S of all O-supports $l(r, s)$ of $P(r, s \in P)$ in the corner II of $l(a, h)$
6:     **if** $S \neq \emptyset$ **then**
7:         flag = 0
8:         **for** each $l(r, s) \in S$ **do**
9:             $k = 0, Q = l(r, s) \cap P$
10:            **for** $w \in P \backslash Q$ **do**
11:                **if** w is in corner I or corner III of $l(r, s)$ **then**
12:                    k:=k+1
13:                **end if**
14:            **end for**
15:            **if** $k = |P| - |Q|$ **then**
16:                flag:=1
17:                return flag   ▷ If flag=1, a *co*-convex hull of P has semi-isolated points. Otherwise, it has no ones.
18:            **end if**
19:        **end for**
20:    **end if**
21: **end procedure**

Example 2. A demonstration of the procedure is shown in Fig. 12. The input is $P = \{(1, 10), (2, 12), (3, 8), (4, 4), (6, 6), (7, 2)\}$. The highest points is $a = (2, 12)$, the leftmost point is $h = (1, 10)$. $S := \{l(a, b)\}$. Consider $(3, 8), (4, 4), (6, 6), (7, 2) \in P$. Step gives $k = 0$. Thus, flag $= 0$. Therefore, we continue to check points of P in $l(b, c)$'s quadrant. $b \equiv a \equiv (2, 12)$, $c = (7, 2)$. $T = \{l(b, i), l(i, m), l(m, c)\}$. Similarly, the procedure gives $k = 4 = |P| - 2$. Thus, flag $= 1$. Therefore, all elements of $\mathcal{F}(P)$ have semi-isolated points.

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.amc.2020.125889

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