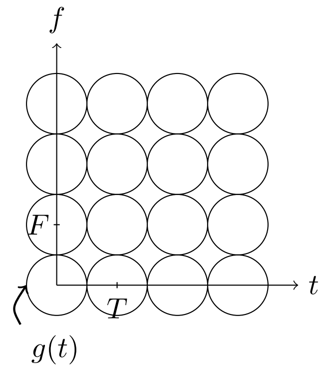
Harmonic Analysis

Weyl-Heisenberg frames

There are two important classes of highly structured frames that have been studied extensively, namely Weyl-Heisenberg (Gabor) frames and wavelets. We will start with Weyl-Heisenberg frames.

1 Motivation

There are many applications where a localized time-frequency representation of a signal $x(t) \in \mathscr{L}^2(\mathbb{R})$ is needed. In 1946, Gabor formulated an approach to decompose signals in terms of elementary signals. He partitioned the time-frequency plane into non-overlapping rectangles and intended to find a collection of elementary signals with small corresponding areas in the time-frequency plane:



We consider modulated and translated versions of a so called window function $g(t)$, i.e.,

$$g_{m,n}(t) = e^{i2\pi nFt} g(t - mT)$$

where $m, n \in \mathbb{Z}$ and $T, F > 0$ are fixed time and frequency parameters. $\{g_{m,n}(t)\}_{m,n\in\mathbb{Z}}$ is called a Weyl-Heisenberg system.

The following discussion is primarily concerned with the question for what choices of $g(t)$ and T and F a Weyl-Heisenberg system is a frame.

For convenience, we define the *Weyl operator* $W_{m,n}^{(T,F)}$ as

$$\left(W_{m,n}^{(T,F)} g\right)(t) = g_{m,n}(t) = e^{i2\pi nFt} g(t - mT) \,.$$

Example 1.1: Consider the Fourier series. The Fourier series is a signal expansion in the functions $e^{i2\pi n\frac{t}{T}}$. It is convenient to normalize these functions, thus we define

$$\phi_n(t) = \begin{cases} \sqrt{1/T} e^{i2\pi n\frac{t}{T}} & \text{for } 0 \le t \le T \\ 0 & \text{otherwise.} \end{cases}$$

Setting

$$g(t) = \begin{cases} \sqrt{1/T} & \text{for } 0 \le t \le T \\ 0 & \text{otherwise,} \end{cases}$$

we can write $\phi_n(t) = W_{0,n}^{(T,\frac{1}{T})} g(t)$. The set

$$\left\{ \left(W_{0,n}^{(T,\frac{1}{T})} g \right)(t) = \phi_n(t) \right\}_{n\in\mathbb{Z}}$$

is an ONB for $\mathscr{L}^2([0,T))$. Thus $\mathscr{L}^2([Tm, T(m+1)))$ has the ONB

$$\left\{ \left(W_{m,n}^{(T,\frac{1}{T})} g \right)(t) = \phi_n(t-mT) \right\}_{n\in\mathbb{Z}}.$$

Putting these bases together, we obtain that $\mathscr{L}^2(\mathbb{R})$ has the ONB

$$\left\{ \left(W_{m,n}^{(T,\frac{1}{T})} g \right)(t) \right\}_{m,n\in\mathbb{Z}}. \tag{1.1}$$

Thus (1.1) is a frame for $\mathscr{L}^2(\mathbb{R})$. Note that $g(t)$ has finite support, thus its support in the frequency domain is infinite.

Definition 1.1: When the set of functions $\left\{ g_{m,n}(t) = \left(W_{m,n}^{(T,F)} g \right)(t) \right\}$ with $m, n \in \mathbb{Z}$ and $T, F > 0$ is a frame for $\mathscr{L}^2$, it is called a *Weyl-Heisenberg* frame.

Proposition 1.1. *The Weyl-operator $W_{m,n}^{(T,F)}$ satisfies the following properties*

i. $W_{m,n}W_{k,l} = e^{-i2\pi(mT)(lF)}W_{m+k,n+l}$ *(Composition)*

ii. $W_{m,n}^{*} = e^{-i2\pi mnTF}W_{-m,-n}$ *(Adjoint)* linearity!

iii. $W_{m,n}W_{m,n}^{*} = W_{m,n}^{*}W_{m,n} = \mathbb{I}$ *(Unitarity)*

Proof. We prove the three properties separately.

i. Note that

$$\begin{aligned}(W_{m,n}W_{k,l}x)(t) &= W_{m,n}\left[e^{i2\pi lFt}x(t-kT)\right] \\ &= e^{i2\pi nFt}e^{i2\pi lF(t-mT)}x(t-(m+k)T) \\ &= e^{-i2\pi lmFT}\left(W_{m+k,n+l}x\right)(t).\end{aligned}$$

ii. By straightforward calculation

$$\begin{aligned}\langle W_{m,n}x, y\rangle &= \int_{-\infty}^{\infty} e^{i2\pi nFt}x(\underbrace{t-mT}_{t'})\overline{y(t)}dt \\ &= \int_{-\infty}^{\infty} e^{i2\pi nF(t'+mT)}x(t')\,\overline{y(t'+mT)}dt' \\ &= \int_{-\infty}^{\infty} x(t')\,\overline{e^{-i2\pi nF(t'+mT)}y(t'+mT)}dt' \\ &= \left\langle \mathrm{x}, e^{-i2\pi nmFT}W_{-m,-n}\mathrm{y}\right\rangle,\end{aligned}$$

where the second equality follows from the substitution $t = t' + mT$. Hence, the adjoint operator of $W_{m,n}$ is given by $W_{m,n}^{*} = e^{-i2\pi mnTF}W_{-m,-n}$.

iii. Verify that

$$W_{m,n}W_{m,n}^{*} \overset{\text{ii.}}{=} W_{m,n}W_{-m,-n}e^{-i2\pi mnTF} \overset{\text{i.}}{=} \mathbb{I}.$$

Analogously, we find that

$$W_{m,n}^{*}W_{m,n} = \mathbb{I}.$$

$\square$

Definition 1.2: The *frame operator* is defined as

$$(\mathbb{S}x)(t) = \sum_m \sum_n \langle \mathrm{x}, \mathrm{g}_{m,n} \rangle \, g_{m,n}(t) \, .$$

Lemma 1.1. *The frame operator* $\mathbb{S}$ *and its inverse* $\mathbb{S}^{-1}$ *commute with the Weyl operator* $W_{m,n}$*, i.e.,*

$$\begin{aligned} W_{m,n}\mathbb{S} &= \mathbb{S}W_{m,n} \\ W_{m,n}\mathbb{S}^{-1} &= \mathbb{S}^{-1}W_{m,n}. \end{aligned}$$

Proof. Notice that the second property follows from the first by left- and right-multiplication with $\mathbb{S}^{-1}$. Hence, it suffices to prove the first identity. We find that

$$\begin{aligned} (W_{m,n}\mathbb{S}x)(t) &= W_{m,n} \sum_{k,l} \langle \mathrm{x}, W_{k,l}\mathrm{g} \rangle \, (W_{k,l}g)(t) \\ &= \sum_{k,l} \langle \mathrm{x}, W_{k,l}\mathrm{g} \rangle \, (W_{m,n}W_{k,l}g)(t) \\ &= \sum_{k,l} \langle \mathrm{x}, W_{k,l}\mathrm{g} \rangle \, e^{-i2\pi mTlF} \, (W_{m+k,n+l}g)(t) = (*) \, , \end{aligned}$$

where used Proposition 1.1, i. to come from the second to the third equality. On the other hand, we have

$$\begin{aligned} (\mathbb{S}W_{m,n}x)(t) &= \sum_{k,l} \langle W_{m,n}\mathrm{x}, W_{k,l}\mathrm{g} \rangle \, (W_{k,l}g)(t) \\ &= \sum_{k,l} \left\langle \mathrm{x}, W_{m,n}^{*}W_{k,l}\mathrm{g} \right\rangle \, (W_{k,l}g)(t) \\ &= \sum_{k,l} e^{i2\pi(n-l)mTF} \left\langle \mathrm{x}, W_{\underbrace{k-m}_{k'},\underbrace{l-n}_{l'}}\mathrm{g} \right\rangle (W_{k,l}g)(t) \\ &= \sum_{k',l'} \left\langle \mathrm{x}, W_{k',l'}\mathrm{g} \right\rangle e^{-i2\pi mTl'F} \left(W_{m+k',n+l'}g\right)(t) = (*) \end{aligned}$$

implying that $W_{m,n}\mathbb{S} = \mathbb{S}W_{m,n}$. $\square$

Proposition 1.2. *Let* $\{g_{m,n}(t) = (W_{m,n}g)(t)\}$ *be a Weyl-Heisenberg frame. Then, the dual frame is again a Weyl-Heisenberg frame, i.e.,*

$$\tilde{g}_{m,n}(t) = (W_{m,n}\tilde{g})(t)$$

with $\tilde{g}(t) = \left(\mathbb{S}^{-1}g\right)(t)$.

Proof.

$$\tilde{g}_{m,n}(t) = \left(\mathbb{S}^{-1}g_{m,n}\right)(t) = \left(W_{m,n}\mathbb{S}^{-1}g\right)(t) = (W_{m,n}\tilde{g})(t) \, .$$

$\square$

2 Zak Transform

To answer for which T and F the family $\{g_{m,n}(t)\}$ is a frame, we need to study a new tool.
Definition 2.1: The Zak transform (ZT) of a signal $x(t)$ is defined as

$$\left(\mathbb{Z}^{T,F}x\right)(t,f) = \sum_{k=-\infty}^{\infty} x(t+kT)\, e^{-i2\pi k \frac{f}{F}}.$$

We sometimes use the shorthand notations $(\mathbb{Z}\mathrm{x})(t,f)$ or $\mathbb{Z}_x(t,f)$.
Proposition 2.1. *The Zak transform exhibits the following properties (see [**?**, Sec. 8.3] for the assumptions on the signals)*

i Frequency-Domain Expression

$$\mathbb{Z}_x(t,f) = \frac{1}{T} e^{i2\pi \frac{f}{R} t} \sum_{k=-\infty}^{\infty} \widehat{x}\left(\frac{f+kF}{R}\right) e^{i2\pi k \frac{t}{T}}$$

where $\widehat{x}(f) = \int_{\mathbb{R}} e^{-i2\pi f t} dt$ is the Fourier transform of the signal $x(t)$.

ii Periodicity
$\mathbb{Z}_x(t,f)$ is periodic in the frequency variable and quasi-periodic in the time variable, i.e.,

$$\begin{aligned}\mathbb{Z}_x(t+T,f) &= e^{i2\pi \frac{f}{F}} \mathbb{Z}_x(t,f)\\ \mathbb{Z}_x(t,f+F) &= \mathbb{Z}_x(t,f)\,.\end{aligned}$$

iii Symmetry Properties

$$\begin{aligned}\mathbb{Z}_{x^-}(t,f) &= \mathbb{Z}_x(-t,-f)\\ \mathbb{Z}_{\overline{x}}(t,f) &= \overline{\mathbb{Z}_x(t,-f)},\end{aligned}$$

where x^- denotes $x(-t)$.

iv Scaling

$$\begin{aligned}&\tilde{x}(t) \triangleq x(at)\\ &\mathbb{Z}_{\tilde{x}}^{(T,F)}(t,f) = \mathbb{Z}_x^{(aT,F)}(at,f) = \mathbb{Z}_x^{\left(aT,\frac{F}{a}\right)}\left(at,\frac{f}{a}\right).\end{aligned}$$

v Shift Properties

$$\begin{aligned}\tilde{x}(t) \triangleq x(t-\tau) \qquad &\Rightarrow \mathbb{Z}_{\tilde{x}}(t,f) = \mathbb{Z}_x(t-\tau,f)\\ \tilde{x}(t) \triangleq x(t)\,e^{i2\pi\nu t} \qquad &\Rightarrow \mathbb{Z}_{\tilde{x}}(t,f) = e^{i2\pi\nu t}\mathbb{Z}_x(t,f-\nu R)\,.\end{aligned}$$

vi Marginal Properties (inverse transforms)

a. $x(t) = \frac{1}{F}\int_0^F \mathbb{Z}_x(t,f)\,df$

b. $\widehat{x}(f) = \int_0^T e^{-i2\pi f t}\mathbb{Z}_x(t,fR)\,dt$

vii Inner Product

$$\frac{1}{F}\langle \mathbb{Z}\mathrm{x}, \mathbb{Z}\mathrm{y}\rangle \triangleq \frac{1}{F}\int_0^T \int_0^F \mathbb{Z}_x(t,f)\,\overline{\mathbb{Z}_y(t,f)} dt df = \langle \mathrm{x}, \mathrm{y}\rangle$$

.

viii Norm

$$\frac{1}{F}\|\mathbb{Z}_{\mathrm{x}}\|^2 = \|\mathrm{x}\|^2.$$

ix Multiplication Property

$$\begin{aligned}
&\tilde{x}(t) = x(t)\, y(t) \\
&\mathbb{Z}_{\tilde{x}}(t,f) = \frac{1}{F}\int_0^F \mathbb{Z}_x(t, f-\nu)\, \mathbb{Z}_y(t,\nu)\, d\nu.
\end{aligned}$$

x Convolution Property

$$\begin{aligned}
&\tilde{x}(t) = x(t) \star y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, y(\tau)\, d\tau \\
&\mathbb{Z}_{\tilde{x}}(t,f) = \int_0^T \mathbb{Z}_x(t-\tau, f)\, \mathbb{Z}_y(\tau, f)\, d\tau.
\end{aligned}$$

xi Product of Two Zak Transforms

$$\mathbb{Z}_x(t,f)\,\overline{\mathbb{Z}_y(t,f)} = F \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \left\langle \mathrm{x}, \tilde{\mathrm{y}}^{\left(mT,\frac{n}{T}\right)}\right\rangle e^{-i2\pi m\frac{f}{F}} e^{i2\pi n\frac{t}{T}}$$

where $\tilde{y}^{\left(mT,\frac{n}{T}\right)}(t) = y(t-mT)\, e^{i2\pi n\frac{t}{T}}$.

xii Cascading the Zak Transform Operator with the Weyl Operator: For integer $N = TF$*, we have*

$$(\mathbb{Z}W_{m,n}g)(t,f) = e^{i2\pi(nFt - m\frac{f}{F})}\mathbb{Z}_g(t,f)$$

Proof. In the following, we prove the above properties separately.

i. Recall the Poisson summation formula

$$\sum_{k=-\infty}^{\infty} y(t+kT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \widehat{y}\left(\frac{k}{T}\right) e^{i2\pi k\frac{t}{T}}.$$

Substituting $y(t) = x(t)\, e^{-i2\pi\tilde{f}t} \circ\!\!-\!\!\bullet\ \widehat{y}\,(f) = \widehat{x}\left(f+\tilde{f}\right)$ into the Poisson summation formula yields

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} x(t+kT)\, e^{-i2\pi\tilde{f}(t+kT)} = \frac{1}{T}\sum_{k=-\infty}^{\infty} \widehat{x}\left(\tilde{f}+\frac{k}{T}\right) e^{i2\pi k\frac{t}{T}} \\
\Rightarrow &\sum_{k=-\infty}^{\infty} x(t+kT)\, e^{-i2\pi\tilde{f}kT} = \frac{1}{T}e^{i2\pi\tilde{f}t}\sum_{k=-\infty}^{\infty} \widehat{x}\left(\tilde{f}+\frac{k}{T}\right) e^{i2\pi k\frac{t}{T}}.
\end{aligned}$$

Setting $f = \tilde{f}R = \tilde{f}TF$ gives the result.

ii. - v. Follows directly from the definition.

vi. The proofs of both properties are based on the same idea.

a. Using the definition and swapping integration and summation yields

$$\frac{1}{F}\int_0^F \mathbb{Z}_x(t,f)\,df = \frac{1}{F}\int_0^F \sum_{k=-\infty}^{\infty} x(t+kT)\,e^{-i2\pi k\frac{f}{F}}df$$
$$= \frac{1}{F}\sum_{k=-\infty}^{\infty} x(t+kT)\underbrace{\left[\int_0^F e^{-i2\pi k\frac{f}{F}}df\right]}_{F\delta[k]} = x(t)\,.$$

b. Use the same idea but exploit property i. and express $\mathbb{Z}_x$ through $\widehat{x}(f)$.

vii. Direct calculation yields

$$\begin{aligned}
&\int_0^T\int_0^F \mathbb{Z}_x(t,f)\,\overline{\mathbb{Z}_y(t,f)}dtdf \\
&= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\int_0^T\int_0^F x(t+kT)\,\overline{y(t+lT)}e^{i2\pi(l-k)\frac{f}{F}}dtdf \\
&= F\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta[l-k]\int_0^T x(t+kT)\,\overline{y(t+lT)}dt \\
&= F\sum_{k=-\infty}^{\infty}\int_{kT}^{(k+1)T} x(t)\,\overline{y(t)}dt \\
&= F\int_{-\infty}^{\infty} x(t)\,\overline{y(t)}dt.
\end{aligned}$$

viii. Substitute $y(t)=x(t)$ into the expression for the inner product given in vii.

ix. Note that $\tilde{x}(t) = x(t)\,y(t) \circ\!\!-\!\!\bullet\ \widehat{\tilde{x}} = \widehat{x}(f) \star \widehat{y}(f) = \int_{-\infty}^{\infty} \widehat{x}(f-\nu)\,\widehat{y}(\nu)\,d\nu$. It follows that

$$
\begin{aligned}
&\mathbb{Z}_{\tilde{x}}(t,f)\\
&= \frac{1}{T} e^{i2\pi \frac{f}{R} t} \sum_{k=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \widehat{x}\Big(\frac{f+kF}{R} - \nu\Big)\,\widehat{y}(\nu)\,d\nu \right] e^{i2\pi k \frac{t}{T}}\\
&= \frac{1}{T} e^{i2\pi \frac{f}{R} t} \sum_{k=-\infty}^{\infty} e^{i2\pi k \frac{t}{T}} \sum_{l=-\infty}^{\infty} \int_{l\frac{F}{R}}^{(l+1)\frac{F}{R}} \widehat{x}\Big(\frac{f+kF}{R} - \nu\Big)\,\widehat{y}(\nu)\,d\nu\\
&= \frac{1}{T} e^{i2\pi \frac{f}{R} t} \sum_{k=-\infty}^{\infty} e^{i2\pi k \frac{t}{T}} \sum_{l=-\infty}^{\infty} \int_{0}^{\frac{F}{R}} \widehat{x}\Big(\frac{f+kF-lF}{R} - \nu\Big)\,\widehat{y}\Big(\nu + l\frac{F}{R}\Big)\,d\nu\\
&\overset{k'=k-l}{=} \frac{1}{T} e^{i2\pi \frac{f}{R} t} \sum_{k'=-\infty}^{\infty} e^{i2\pi k' \frac{t}{T}} \sum_{l=-\infty}^{\infty} e^{i2\pi l \frac{t}{T}} \int_{0}^{\frac{F}{R}} \widehat{x}\Big(\frac{f+k'F}{R} - \nu\Big)\,\widehat{y}\Big(\nu + l\frac{F}{R}\Big)\,d\nu\\
&= T \int_0^{\frac{F}{R}} \underbrace{\frac{1}{T} e^{i2\pi (\frac{f}{R}-\nu) t} \sum_{k=-\infty}^{\infty} \widehat{x}\Big(\frac{f+kF}{R} - \nu\Big)\, e^{i2\pi k \frac{t}{T}}}_{\mathbb{Z}_x(t,f-\nu R)} \times\\
&\quad \underbrace{\frac{1}{T} e^{i2\pi \nu t} \sum_{l=-\infty}^{\infty} \widehat{y}\Big(\nu + l\frac{F}{R}\Big)\, e^{i2\pi l \frac{t}{T}}\,d\nu}_{\mathbb{Z}_y(t,\nu R)}\\
&= T \int_0^{\frac{F}{R}} \mathbb{Z}_x(t, f-\nu R)\,\mathbb{Z}_y(t,\nu R)\,d\nu\\
&= \frac{1}{F} \int_0^{F} \mathbb{Z}_x(t, f-\nu)\,\mathbb{Z}_y(t,\nu)\,d\nu.
\end{aligned}
$$

x. We obtain by direct calculation

$$
\begin{aligned}
\mathbb{Z}_{\tilde{x}}(t,f) &= \sum_{k=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t+kT-\tau)\,y(\tau)\,d\tau \right] e^{-i2\pi k \frac{f}{F}}\\
&= \sum_{k=-\infty}^{\infty} e^{-i2\pi k \frac{f}{F}} \sum_{l=-\infty}^{\infty} \int_{lT}^{(l+1)T} x(t+kT-\tau)\,y(\tau)\,d\tau\\
&= \sum_{k=-\infty}^{\infty} e^{-i2\pi k \frac{f}{F}} \sum_{l=-\infty}^{\infty} \int_{0}^{T} x(t+kT-lT-\tau)\,y(\tau+lT)\,d\tau\\
&= \int_0^T \underbrace{\sum_{k'=-\infty}^{\infty} x\big(t+k'T-\tau\big)\, e^{-i2\pi k' \frac{f}{F}}}_{\mathbb{Z}_x(t-\tau,f)} \underbrace{\sum_{l=-\infty}^{\infty} y(\tau+lT)\, e^{-i2\pi l \frac{f}{F}}\,d\tau}_{\mathbb{Z}_y(\tau,f)}\,.
\end{aligned}
$$

xi. It follows from ii that $\mathbb{Z}_x(t,f)\,\overline{\mathbb{Z}_y(t,f)}$ is periodic in both variables t and f. Since $\mathbb{Z}_x(t,f)\,\overline{\mathbb{Z}_y(t,f)}$ is periodic, it can be represented by its Fourier series

$$
\mathbb{Z}_x(t,f)\,\overline{\mathbb{Z}_y(t,f)} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} e^{-i2\pi m \frac{f}{F}} e^{i2\pi n \frac{t}{T}}\,.
$$

The Fourier coefficients $a_{m,n}$ are

$$\begin{aligned}
a_{m,n} &= \int_0^T \int_0^F \mathbb{Z}_x(t,f)\,\overline{\mathbb{Z}_y(t,f)}e^{-i2\pi n\frac{t}{T}}e^{i2\pi m\frac{f}{F}}dtdf \\
&= \int_0^T \int_0^F \sum_{k=-\infty}^{\infty} x(t+kT)\,e^{-i2\pi k\frac{f}{F}} \sum_{l=-\infty}^{\infty} \overline{y(t+lT)}e^{i2\pi l\frac{f}{F}}e^{-i2\pi n\frac{t}{T}}e^{i2\pi m\frac{f}{F}}dtdf \\
&= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\int_0^T x(t+kT)\,\overline{y(t+lT)}e^{-i2\pi n\frac{t}{T}}dt \underbrace{\int_0^F e^{i2\pi(m-k+l)}df}_{\delta(m-k+l)} \\
&= F\sum_{l=-\infty}^{\infty}\int_0^T x(t+(l+m)\,T)\,\overline{y(t+lT)}e^{-i2\pi n\frac{t}{T}}dt \\
&= F\sum_{l=-\infty}^{\infty}\int_{(l+m)T}^{(l+m+1)T} x(t)\,\overline{y(t-mT)}e^{-i2\pi n\frac{t}{T}}dt \\
&= F\int_{-\infty}^{\infty} x(t)\,\overline{y(t-mT)}e^{-i2\pi n\frac{t}{T}}dt \\
&= F\left\langle \mathrm{x}, \tilde{\mathrm{y}}^{(mT,\frac{n}{T})}\right\rangle
\end{aligned}$$

xii. Observe that

$$\begin{aligned}
(\mathbb{Z}W_{m,n}g)(t,f) &= \mathbb{Z}_{g_{m,n}}(t,f) \\
&= \sum_{k=-\infty}^{\infty} g\left(t+\overbrace{(k-m)}^{k'}T\right)e^{i2\pi nF(t+kT)}e^{-i2\pi k\frac{f}{F}} \\
&= \sum_{k'=-\infty}^{\infty} g\big(t+k'T\big)\,e^{i2\pi nF(t+(k'+m)T)}e^{-i2\pi(k'+m)\frac{f}{F}} \\
&= e^{i2\pi(nF(t+mT)-m\frac{f}{F}))}\sum_{k'=-\infty}^{\infty} g\big(t+k'T\big)\,e^{-i2\pi k'(\frac{f}{F}-nFT)}
\end{aligned}$$

which becomes, for integer $N=TF$

$$\begin{aligned}
(\mathbb{Z}W_{m,n}g)(t,f) &= e^{i2\pi(nFt-m\frac{f}{F})}\sum_{k'=-\infty}^{\infty} g\big(t+k'T\big)\,e^{-i2\pi k'\frac{f}{F}} \\
&= e^{i2\pi(nFt-m\frac{f}{F})}\mathbb{Z}_g(t,f).
\end{aligned}$$

□

Proposition 2.2. *$\sqrt{\frac{1}{F}}\mathbb{Z}^{T,F}$ is a unitary map from $\mathscr{L}^2(\mathbb{R})$ onto $\mathscr{L}^2([0,T]\times[0,F])$.*

Proof. Consider again the ONB for $\mathscr{L}^2(\mathbb{R})$

$$\left\{\left(W_{m,n}^{(T,\frac{1}{T})}\sqrt{\frac{1}{T}}\,\mathrm{rect}_{[0,T]}\right)(t)\right\}_{m,n\in\mathbb{Z}}$$

and apply the Zak transform

$$
\begin{aligned}
&\sqrt{\frac{1}{F}}\mathbb{Z}^{T,F}W_{m,n}^{(T,\frac{1}{T})}\sqrt{\frac{1}{T}}\,\mathrm{rect}_{[0,T]}(t)\\
&=\sqrt{\frac{1}{F}}\mathbb{Z}^{T,F}\left(e^{i2\pi n\frac{t}{T}}\sqrt{\frac{1}{T}}\,\mathrm{rect}_{[0,T]}(t-mT)\right)\\
&=\sqrt{\frac{1}{TF}}\sum_{k\in\mathbb{Z}}e^{i2\pi n\frac{t+kT}{T}}\,\mathrm{rect}_{[0,T]}(t+kT-mT)\,e^{-i2\pi k\frac{f}{F}}\\
&\overset{k'=k-m}{=}\sqrt{\frac{1}{TF}}e^{i2\pi n\frac{t}{T}}\sum_{k'\in\mathbb{Z}}\underbrace{\mathrm{rect}_{[0,T]}\big(t+k'T\big)}_{\neq 0\Leftrightarrow k'=0}\,e^{-i2\pi k'\frac{f}{F}}e^{-i2\pi m\frac{f}{F}}\\
&=\sqrt{\frac{1}{TF}}e^{i2\pi n\frac{t}{T}}e^{-i2\pi m\frac{f}{F}}.
\end{aligned}
$$

This set of functions is an ONB for $\mathscr{L}^2([0,T]\times[0,F])$. Hence, $\sqrt{\frac{1}{F}}\mathbb{Z}^{T,F}$ maps an ONB for $\mathscr{L}^2(\mathbb{R})$ into an ONB for $\mathscr{L}^2([0,T]\times[0,F])$ and is consequently a surjective, unitary map. □

3 When is a Weyl-Heisenberg System a Frame?

We now answer the question for which T and F the family $\{g_{m,n}(t)\}$ is a frame. In doing so, we discern three different cases, namely

i. $TF=1$, the critical case

ii. $TF<1$, the oversampled case

iii. $TF>1$, the undersampled case.

3.1 Undersampled Case

In the undersampled case $TF>1$, the TF-grid defined by T and F is 'too loose', i.e., the $g_{m,n}(t)$ cannot span $\mathscr{L}^2(\mathbb{R})$.

Theorem 3.1. *If $TF>1$ and $g\in\mathscr{L}^2(\mathbb{R})$, then $\{g_{m,n}(t)\}$ is incomplete in $\mathscr{L}^2(\mathbb{R})$.*

Proof. We omit the proof. □

3.2 Critical Sampling

It is natural to ask under what conditions a Weyl-Heisenberg frame can be a ONB. It can be shown that this is only possible in the critical case, i.e., when $TF=1$.

In this section, we first show that in the critical case, a Weyl-Heisenberg frame and its dual must be biorthogonal.

Next we next check whether there exits "useful" windows such that the corresponding Weyl-Heisenberg frame is an ONB.

Theorem 3.2. $(\mathbb{Z}\mathbb{S}x)(t,f)=T\mathbb{Z}_x(t,f)\,|\mathbb{Z}_g(t,f)|^2$.

Proof. First note that using $TF=1$, we obtain by application of Proposition 2.1 xii.

$$(\mathbb{Z}W_{m,n}g)(t,f)=e^{i2\pi\left(n\frac{t}{T}-m\frac{f}{F}\right)}\mathbb{Z}_g(t,f)\,.$$

It follows from direct calculation that

$$
\begin{aligned}
(\mathbb{Z}\mathbb{S}x)(t,f) &= \sum_{m,n} \langle \mathrm{x}, \mathrm{g}_{m,n} \rangle \, \mathbb{Z}_{g_{m,n}}(t,f) \\
&= \sum_{m,n} \langle \mathrm{x}, \mathrm{g}_{m,n} \rangle \, e^{i2\pi\left(n\frac{t}{T}-m\frac{f}{F}\right)} \mathbb{Z}_g(t,f) \\
&= \frac{1}{F} \mathbb{Z}_x(t,f)\, \overline{\mathbb{Z}_g(t,f)} \mathbb{Z}_g(t,f) \\
&= T\mathbb{Z}_x(t,f)\, |\mathbb{Z}_g(t,f)|^2
\end{aligned}
$$

where we used Proposition 2.1 xi. to come from the second to the third equality. □

Theorem 3.3. *In the critical case, the set $\{g_{m,n}(t)\}$ is a frame for $\mathscr{L}^2(\mathbb{R})$ with bounds A, B if and only if*

$$
A \leq T|\mathbb{Z}_g(t,f)|^2 \leq B \tag{3.2}
$$

with $A > 0, B < \infty$.

Proof. We shall first show that (3.2) implies that $\{g_{m,n}(t)\}$ is a frame with frame bounds A and B. Starting from (3.2)

$$
\begin{aligned}
&A|\mathbb{Z}_x(t,f)|^2 \leq T|\mathbb{Z}_x(t,f)|^2|\mathbb{Z}_g(t,f)|^2 \leq B|\mathbb{Z}_x(t,f)|^2 \\
&\underbrace{A\int_0^T\int_0^{\frac{1}{T}} |\mathbb{Z}_x(t,f)|^2 dtdf}_{F\|\mathrm{x}\|^2} \leq \underbrace{T\int_0^T\int_0^{\frac{1}{T}} |\mathbb{Z}_x(t,f)|^2|\mathbb{Z}_g(t,f)|^2 dtdf}_{F\langle \mathbb{S}\mathrm{x},\mathrm{x}\rangle} \\
&\quad \leq \underbrace{B\int_0^T\int_0^{\frac{1}{T}} |\mathbb{Z}_x(t,f)|^2 dtdf}_{F\|\mathrm{x}\|^2}
\end{aligned}
$$

where the expressions under the braces follow from Theorem 3.2 and Proposition 2.1 vii. and viii.

To show the other direction, we assume that $\{g_{m,n}(t)\}$ is a frame. Hence, it holds that

$$
A\|\mathrm{x}\|^2 = AT\int_0^T\int_0^{\frac{1}{T}} |\mathbb{Z}_x(t,f)|^2 dtdf \leq T^2\int_0^T\int_0^{\frac{1}{T}} |\mathbb{Z}_x(t,f)|^2|\mathbb{Z}_g(t,f)|^2 dtdf.
$$

Reformulating the above expression yields

$$
T\int_0^T\int_0^{\frac{1}{T}} |\mathbb{Z}_x(t,f)|^2 \left(T|\mathbb{Z}_g(t,f)|^2 - A\right) dtdf \geq 0
$$

and since $\mathbb{Z}_x(t,f)$ is arbitrary, this implies that $T|\mathbb{Z}_g(t,f)|^2 - A \geq 0$. Hence, it follows that

$$
\begin{aligned}
T|\mathbb{Z}_x(t,f)|^2 &\geq A \\
T|\mathbb{Z}_x(t,f)|^2 &\leq B
\end{aligned}
$$

g, nicht x

where the lower bound for B follows analogously. □

Corollary 3.1. *In the critical case, $\{g_{m,n}(t)\}$ is a tight frame with $A = B$ if and only if $T|\mathbb{Z}_g(t,f)|^2 = A$.*

Theorem 3.4. *Let $\{g_{m,n}(t)\}$ be a Weyl-Heisenberg frame and $TF = 1$. Then the Zak transform of the canonical dual function $\tilde{g(t)}$ is given by*

$$\mathbb{Z}_{\tilde{g}}(t,f) = \frac{1}{T\overline{\mathbb{Z}_g(t,f)}}.$$

Proof. Application of Theorem 3.2 yields

$$\begin{aligned}\mathbb{S}\tilde{\mathrm{g}} = \mathrm{g} &\Rightarrow \mathbb{Z}\mathbb{S}\tilde{g} = \mathbb{Z}g \\ &\Rightarrow T\mathbb{Z}_{\tilde{g}}(t,f)\,|\mathbb{Z}_g(t,f)|^2 = \mathbb{Z}_g(t,f).\end{aligned}$$

Since $\{g_{m,n}(t)\}$ is a frame, $|\mathbb{Z}_g(t,f)|^2$ is bounded by the frame bounds and hence we can divide both sides by $\mathbb{Z}_g(t,f)$ to obtain

$$\mathbb{Z}_{\tilde{g}}(t,f) = \frac{1}{T\overline{\mathbb{Z}_g(t,f)}}.$$

□

Theorem 3.5. *Let $\{g_{m,n}(t)\}$ be a Weyl-Heisenberg frame in the critical case $TF = 1$. Then $\{g_{m,n}(t)\}$ is exact, i.e., $\{g_{m,n}(t)\}$ and $\{\tilde{g}_{m,n}(t)\}$ are biorthonormal*

$$\left\langle \mathrm{g}_{m,n}, \tilde{\mathrm{g}}_{m',n'} \right\rangle = \delta_{m-m'}\delta_{n-n'}.$$

Proof. We have

$$\begin{aligned}\langle \tilde{\mathrm{g}}, g_{m,n} \rangle &= T\,\langle \mathbb{Z}\tilde{\mathrm{g}}, \mathbb{Z}\mathrm{g}_{m,n} \rangle \\ &= T\int_0^T\int_0^F \mathbb{Z}_{\tilde{g}}(t,f)\,e^{-i2\pi n\frac{t}{T}}e^{i2\pi m\frac{f}{F}}\overline{\mathbb{Z}_g(t,f)}dtdf \\ &= \int_0^T\int_0^F e^{-i2\pi n\frac{t}{T}}e^{i2\pi m\frac{f}{F}}dtdf \\ &= \delta_m\delta_n,\end{aligned}$$

where we applied Theorem 3.4 to obtain the last line. The claim now follows from

$$\left\langle \tilde{\mathrm{g}}_{m',n'}, g_{m,n} \right\rangle = \left\langle W_{m',n'}\tilde{\mathrm{g}}, W_{m,n}g \right\rangle \tag{3.3}$$

$$= \left\langle \tilde{\mathrm{g}}, W^*_{m',n'}W_{m,n}g \right\rangle \tag{3.4}$$

$$= \left\langle \tilde{\mathrm{g}}, g_{m-m',n-n'} \right\rangle \tag{3.5}$$

$$= \delta_{m-m'}\delta_{n-n'}. \tag{3.6}$$

□

Theorem 3.6 (Balian-Low)**.** *If $\{g_{m,n}(t)\}$ is a Weyl-Heisenberg frame for $\mathscr{L}^2(\mathbb{R})$ in the critical case $TF = 1$, then either $tg(t) \notin \mathscr{L}^2(\mathbb{R})$ or $f\widehat{g}(f) \notin \mathscr{L}^2(\mathbb{R})$, i.e.,*

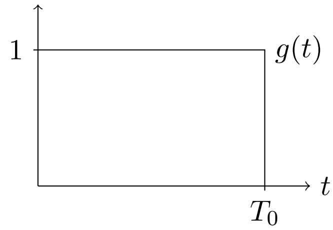
$$\int_{-\infty}^{\infty} t^2 g(t)^2 dt = \infty \text{ or } \int_{-\infty}^{\infty} f^2\widehat{g}(f)^2 df = \infty.$$

Proof. We omit the proof. □

Remember that we motivated Weyl-Heisenberg frames saying that they are a useful tool to represent functions that are concentrated in time and frequency provided that the prototype function g is concentrated in time and frequency. The above theorem basically says that this is impossible in the critical case, i.e., to achieve our aim, we have to choose $TF < 1$. Thus we have to resort to frames.

3.3 Oversampled Case

Even for $TF < 1$, a given window $g(t)$ gives rise to a frame only if T and F are properly chosen.
Example 3.1: Consider the prototype function $g(t)$ depicted in the figure below. If $T > T_0$, $g(t)$ does not give rise to a frame independently of how small F is.



Under certain restrictions on $g(t)$ and T, one can always find a frequency shift parameter F_0, i.e.,

$$\begin{array}{ll} \forall F,\ 0 < F < F_0: & (\mathrm{g}, T, F) \text{ is a frame.} \\ \forall F,\ F > F_0: & (\mathrm{g}, T, F) \text{ is not a frame.} \end{array}$$

Theorem 3.7. *Let*

$$m(g(t)\,;T) = \min_{t\in[0,T)} \sum_m |g(t-mT)|^2 > 0$$
$$M(g(t)\,;T) = \max_{t\in[0,T)} \sum_m |g(t-mT)|^2 < \infty$$

be the minimum and the maximum of the T-periodic function $\sum_m |g(t-mT)|^2$, respectively, and assume $m(g(t)\,;T) > 0$ and $M(g(t)\,;T) < \infty$. Furthermore, assume that for some $\epsilon > 0$

$$\max_{\tau\in\mathbb{R}} \left(1+\tau^2\right)^{1+\frac{\epsilon}{2}} \beta(\tau) = C_\epsilon < \infty$$

with

$$\beta(\tau) = \max_{t\in[0,T)} \sum_m |g(t-mT)|\,|g(t+\tau-mT)|\,.$$

Then, there exists an $F_0 > 0$ s.t.

i. $\forall F \in (0, F_0)$ $\{g_{m,n}(t)\}$ is a frame with frame bounds

$$A \geq \frac{1}{F}\left[m(g(t)\,;T) - \sum_{n\neq 0}^{\infty} \sqrt{\beta\Big(\frac{n}{F}\Big)\,\beta\Big(-\frac{n}{F}\Big)}\right]$$
$$B \leq \frac{1}{F}\left[M(g(t)\,;T) + \sum_{n\neq 0}^{\infty} \sqrt{\beta\Big(\frac{n}{F}\Big)\,\beta\Big(-\frac{n}{F}\Big)}\right].$$

ii. $\forall \delta > 0$, there exists an $F \in [F_0, F_0+\delta]$ s.t. $\{g_{m,n}(t)\}$ associated with (g, T, F) is not a frame.

Proof. See 'The wavelet transform, T-F localization and signal analysis' by I. Daubechies. □

Remark:

i. The condition $m(g(t)\,;T) > 0$ means that there may not be gaps between translates $g(t-mT)$ of $g(t)$.

ii. The condition $M(g(t)\,;T) < \infty$ is satisfies if $g(t)$ decays sufficiently at $t \to \infty$.
For $|g(t)| \leq C\left(t+t^2\right)^{-3/2}$, the condition $M(g(t)\,;T) < \infty$ is always satisfied.

Theorem 3.8. *Let g be a Gaussian function, i.e., $g(t) = 2^{1/4}e^{-\pi t^2}$ for all $t \in \mathbb{R}$. Then $\{g_{m,n}\}_{m,n\in\mathbb{Z}}$ is a frame for $\mathscr{L}^2(\mathbb{R})$ if and only if $TF < 1$.*

3.3.1 Zak Transform Methods

In the following, we assume $TF = \frac{p}{q}$. First, we need an alternative expression for the frame operator.
Theorem 3.9. *The frame operator for a Weyl-Heisenberg frame $\{g_{m,n}(t)\}$ can be written as*

$$(\mathbb{S}x)(t) = \frac{1}{F}\sum_{n=-\infty}^{\infty} x\Big(t-\frac{n}{F}\Big)\sum_{m=-\infty}^{\infty} g(t-mT)\,\overline{g\Big(t-mT-\frac{n}{F}\Big)}.$$

Proof. Left as an exercise. □

We next restrict ourself to the case $TF = \frac{1}{N}$, slightly more involved results also hold for rational TF. The analogy of Theorem 3.2 is

$$(\mathbb{Z}\mathbb{S}x)(t,f) = \frac{1}{\underbrace{NF}_{=T}}\,\mathbb{Z}_x(t,f)\sum_{k=0}^{N-1}\left|\mathbb{Z}_g\Big(t, f-\frac{k}{NT}\Big)\right|^2.$$

In the case of oversampling by integer factors, the application of the frame operator to an arbitrary signal $x(t)$ corresponds to multiplication of $\mathbb{Z}_x(t,f)$ by $\sum_{k=0}^{N-1}|\mathbb{Z}_g(t,f-kF)|^2$ in the Zak transform domain.

Now, we can generalize Theorem 3.3.
Theorem 3.10. *The set $\{\mathrm{g}_{m,n}\}$ is a frame for $\mathscr{L}^2(\mathbb{R})$ with bounds A, B in the oversampled case $TF = \frac{1}{N}$ if and only if*

$$A \leq T\sum_{i=0}^{N-1}|\mathbb{Z}_g(t,f-iF)|^2 \leq B.$$

Proof. The proof is analogous to the proof of Theorem 3.3. □

Corollary 3.2. *$\{\mathrm{g}_{m,n}\}$ is a tight frame if and only if*

$$T\sum_{i=0}^{N-1}|\mathbb{Z}_g(t,f-iF)|^2 = A$$

Theorem 3.11. *Assume that $TF = \frac{1}{N}$. Then, the Zak transform of the canonical dual function $\tilde{g}(t)$ is given by*

$$\mathbb{Z}_{\tilde{g}}(t,f) = \frac{\mathbb{Z}_g(t,f)}{T\sum_{i=0}^{N-1}|\mathbb{Z}_g(t,f-iF)|^2}.$$

Proof. Start from

$$(\mathbb{Z}\mathbb{S}\tilde{g})(t,f) = T\mathbb{Z}_{\tilde{g}}(t,f)\sum_{i=0}^{N-1}|\mathbb{Z}_g(t,f-iF)|^2$$

and use $\mathbb{Z}\mathbb{S}\tilde{\mathrm{g}} = \mathbb{Z}\mathrm{g}$. □