Infinite-time blow-up for the nonlinear heat equations in low dimension

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Declaration of any previous Submission of the Work

The material presented here for examination for the award of a higher degree by research has not been incorporated into a submission for another degree.

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Declaration of Authorship

I am the author of this thesis, and the work described therein was carried out by myself personally in collaboration with my supervisor Manuel Del Pino.

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Summary

In this thesis, we investigate the asymptotic behaviour of positive global unbounded solutions to the critical semilinear heat equation. We construct the first example of non-radial infinite-time blow up solution in a 3-dimensional bounded domain. This generalizes the non-radial case proved by Cortázar, Del Pino and Musso in [7] for dimension $n \geq 5$ and the radial result for $n = 3$ by Galaktionov and King [14].

Our analysis starts by selecting a good ansatz, which encloses all the main asymptotic properties of the exact solution. We show that, after necessary improvements of the natural approximation, we get a sufficiently small error to start the second part of the proof. Then, we produce a correct perturbation of the approximate solution by adapting the parabolic inner-outer gluing method developed in [7, 8]. This consists in solving a weakly coupled system after suitably decomposing the problem near and far from the blow-up point. This approach is constructive and allows an accurate analysis of the asymptotic dynamics.

A fundamental feature and difficulty in the inner regime is the presence of a nonlocal operator that controls the second order asymptotic of the blow-up parameter. We show that such operator, similar to a half-fractional derivative, can be inverted but a loss of regularity in the parameter appears. We prove the invertibility of such operator using a Laplace transform type method combined with heat kernel estimates and we regain regularity of the parameters using smoothness of the solution in the outer regime.

For the unit ball, our construction works for any blow-up point sufficiently close to the center. In particular, we give a new proof of the infinite-time blow-up at the center of a ball, firstly proved in [14] using matched expansion techniques. Our construction applies to any domain under a natural analytic condition, given in terms of the Robin function of the domain.

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Contents

Chapter 1

Introduction

The aim of this thesis is to investigate the long term behaviour of solutions to the critical heat equation, a model in the class of nonlinear parabolic problems.

In this chapter we introduce the reader to the main topic by beginning with an overview on the Fujita problem. Then we present the motivation behind this work. In the main chapter 2 we prove the existence of a positive global unbounded solution in dimension 3 for the critical heat equation in a nonradial setting. In chapter 3 we present few steps towards the solution to the conjecture in dimension 4 and outlook.

1.1. The Fujita problem

Superlinear parabolic equations of the form

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

are often employed to describe reaction-diffusion systems in stellar dynamics and combustion. The term superlinear indicates that the function $f(u)$ grows more than linearly as the solution u tends to infinity. A typical model problem in the theory of blow-up analysis is the Dirichlet problem for the Fujita equation

$$\begin{cases} u_t = \Delta u + u^p & \text{in} \quad \Omega \times (0,\infty), \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{F_p}$$

where u_0 is a continuous datum, $p > 1$ and $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain in dimension $n \geq 1$. The dynamics of the solution to (F_p) depends on the domain's geometry Ω, nonlinear exponent p, and initial datum u_0.

Fujita started to investigate equation (F_p) in [13] as a first approach to more general superlinear problems. The recently updated monograph [21] by Quittner and Souplet contains methods and results about the qualitative study of this problem until 2019. Now, we describe how the admissible behaviour of global solutions drastically changes as a function of $p \in (1,\infty)$. A crucial role is played by the critical Sobolev exponent defined by

$$p_S := \begin{cases} \dfrac{n+2}{n-2} & \text{if} \quad n \geq 3, \\ \infty & \text{otherwise.} \end{cases}$$

In the subcritical case $p < p_S$, positive global solutions must be bounded. The first important result in this direction is the work [6] by Cazenave and Lions. Then, after many partial results, Quittner [20] proved that all positive global solutions to the Dirichlet problem in bounded domain possess the a priori bound

$$\sup_{t\geq 0}\|u(\cdot,t)\|_{L^\infty(\Omega)}\leq C, \quad \text{where} \quad C=C\Big(\|u_0\|_{L^\infty(\Omega)}\Big), \tag{1.1}$$

where C is bounded in any bounded sets. In other words, for any initial datum u_0 such that $\|u_0\|_\infty < K$, the evolved global solution satisfies the uniform bound $\|u(t)\|_\infty \leq C$, where $C = C(K)$. Instead, in the supercritical regime $p > p_S$, (1.1) is false when Ω is star-shaped (see Theorem 28.7 in [21]). Nevertheless, Blatt and Struwe [2] proved that, if Ω is convex and bounded, then we have

$$\sup_{t\geq 0}\|u(\cdot,t)\|_{L^\infty(\Omega)}<\infty. \tag{1.2}$$

This a priori L^∞-bound, is still an open question for bounded non-convex domains.
If $p = p_S$, even the a priori bound (1.2) is false. In other words, the critical case could admit unbounded global positive solutions. In fact, we shall see below that examples of such solutions have been detected.

1.1.1 Threshold solutions

Given an exponent $p \in (1,\infty)$, the behaviour of a solution to (F_p) is heavily dependent on the L^∞-size of the datum. Given any smooth function $\varphi(x) \geq 0, \varphi \not\equiv 0$, consider $\alpha > 0$ and $u_\alpha(x,0) := \alpha\varphi(x)$ as initial datum. On one hand, we show that if α is sufficiently small, then u_α tends uniformly to zero as $t \to \infty$. Indeed, consider a function of the form

$$v(x,t)=\varepsilon e^{-at}\phi_1(x),$$

where $a \in (0,\lambda_1)$ and λ_1, ϕ_1 are respectively the first Dirichlet eigenvalue and the positive eigenfunction of the Laplacian in Ω with $\|\phi_1\|_{L^1(\Omega)} = 1$. We prove that v is a supersolution for (F_p). Indeed,

$$\partial_t v-\Delta v-v^p=e^{-at}\phi_1(x)\varepsilon\Big\{(\lambda_1-a)-\varepsilon^{p-1}\phi_1^{p-1}e^{-a(p-1)t}\Big\}>0,$$

if we fix $\varepsilon < (\lambda_1 - a)^{\frac{1}{p-1}}\|\phi_1\|^{-1}_{L^\infty(\Omega)}$. Clearly, $v(x,t) = 0$ on $\partial\Omega$. Now, $u_\alpha(x,0) = \alpha\phi(x) < v(x,0) = \varepsilon\phi_1(x)$ if α is sufficiently small. Thus, it follows by the semilinear comparison principle that

$$u_\alpha(x,t)\leq\varepsilon\phi_1(x)e^{-at}.$$

In particular, $u_\alpha(x,t)$ decays uniformly as $t \to \infty$ when α is fixed sufficiently small. On the other hand, using the eigenfunction method of Kaplan [16], we prove that, for $\alpha > 0$ sufficiently large, $u_\alpha(x,t)$ is not globally defined in time. We multiply the equation by

ϕ_1 and integrate by parts to get

$$\partial_t \int_\Omega u_\alpha(x,t)\phi_1(x)\,dx = -\lambda_1 \int_\Omega u_\alpha(x,t)\phi_1(x)\,dx + \int_\Omega \phi_1(x) u_\alpha(x,t)^p\,dx.$$

Letting $\hat{u} = \int_\Omega \phi_1(x)u_\alpha(x,t)\,dx$, the Jensen's inequality implies

$$\begin{aligned}\hat{u}'(t) &= -\lambda_1 \hat{u}(t) + \int_\Omega \phi_1(x)u_\alpha(x,t)^p\,dx \\ &\geq -\lambda_1\hat{u}(t) + \hat{u}(t)^p.\end{aligned}$$

We observe that

$$\hat{u}(0) = \int_\Omega \phi_1(x)u_\alpha(x,0)\,dx = \alpha\int_\Omega \phi_1(x)\varphi(x)\,dx,$$

is positive. Also, for $\alpha > 0$ sufficiently large we have $-\lambda_1\hat{u}(0) + \hat{u}(0)^p > 0$. Thus, by comparison we have $\hat{u}(t) \geq \hat{u}(0)$ and hence $\hat{u}'(t) > 0$. Hence, supposing that $\hat{u}$ (and hence u) is well-defined in $[0,T]$, we can perform a change of variable to integrate the inequality above and we find

$$\begin{aligned}T = \int_0^T ds &\leq \int_{\hat{u}(0)}^{\hat{u}(T)} \frac{1}{r^p - \lambda_1 r}\,dr \\ &\leq \int_{\hat{u}(0)}^{\infty} \frac{1}{r^p - \lambda_1 r}\,dr < \infty.\end{aligned}$$

This shows that T cannot be arbitrarily large. We conclude that $\|u_\alpha(\cdot,t)\|_{L^\infty(\Omega)} \geq \hat{u}(t)$ blows-up in finite time.
Thus, we observe a dramatic transition of the dynamics from global existence with decay into blow-up in finite time. It was proved by Lions in [18] that these are the typical behaviors, in the sense that the set of non-negative initial values u_0 for which one of these scenarios occurs is dense in $C_0^1(\bar{\Omega})$. It follows from standard parabolic theory (see [17]) that the set

$$A = \{\alpha > 0 : u_\alpha(x,t) \text{ is uniformly bounded and } u_\alpha(x,t) \to 0 \quad \textbf{as} \quad t \to \infty\}$$

is open. We have just shown that A is nonempty and bounded above. Then, the threshold number $\alpha^* = \alpha^*(\varphi) \in (0,\infty)$ defined by

$$\alpha^* := \sup\left\{\alpha > 0 \,:\, \lim_{t\to\infty} \|u_\alpha(\cdot,t)\|_\infty = 0\right\},$$

satisfies the following properties:

- if $\alpha < \alpha^*$ the solution $u_\alpha(x,t)$ tends uniformly to zero as $t \to +\infty$;
- if $\alpha > \alpha^*$ the solution $u_\alpha(x,t)$ blows-up in finite time.

The first existence result of threshold solutions is due to Ni, Sacks and Tavantzis [19], who proved that u_{α^*} is well-defined as L^1-weak solution. It is known (see Theorem 28.7 in [21]) that

- for $p \in (1, p_S)$, $u_{\alpha^*}(x,t)$ is global, smooth and, up to subsequences, tends to a positive steady states of (F_p);
- if $p > p_S$, and Ω is convex then u_{α^*} blow-up in finite time;
- if $p = p_S$, $\Omega = B_1(0)$ and φ radial non-increasing, then Galaktionov and Vazquez [15] proved that u_{α^*} is smooth, global and

$$\lim_{t\to\infty} \|u_{\alpha^*}(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{1.3}$$

The main open question that motivates our work concerns the limit (1.3) and goes as follows:

What is the asymptotic behaviour of $\|u_{\alpha^*}(\cdot,t)\|_{L^\infty(\Omega)}$ in the non-radial setting?

1.1.2 The critical exponent

The critical exponent p_S is related to the Sobolev inequalities: for bounded domains Ω the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is compact when $p \in (1, p_S)$ and the best Sobolev constant

$$S_p(\Omega) := \inf_{0\neq u\in H_0^1(\Omega)} \frac{\|u\|^2_{H_0^1(\Omega)}}{\|u\|^2_{L^{p+1}(\Omega)}}$$

is attained. Due to the loss of compactness when $p = p_S$, the constant $S_{p_S}(\Omega)$ is attained only if $\Omega = \mathbb{R}^n$. $S_{p_S}(\mathbb{R}^n)$ is achieved by the Talenti bubbles (see [24])

$$U_{\mu,\xi}(x) := \mu^{-\frac{n-2}{2}} U\left(\frac{x-\xi}{\mu}\right), \tag{1.4}$$

defined for every $\mu \in (0,\infty)$ and $\xi \in \mathbb{R}^n$, where

$$U(x) := \alpha_n \left(\frac{1}{1+|x|^2}\right)^{\frac{n-2}{2}}, \quad \alpha_n = [n(n-2)]^{\frac{n-2}{4}}. \tag{1.5}$$

The work of Caffarelli, Gidas and Spruck [5] implies that these are all the positive solutions of the equation

$$\Delta U + U^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n.$$

These are the positive critical points of the energy

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x)|^2\, dx - \frac{n-2}{2n}\int_{\mathbb{R}^n} |u(x)|^{\frac{2n}{n-2}}\, dx.$$

The family (1.4) is energy invariant, meaning that

$$E(U_{\mu,\xi}) = E(U) = S_{p_S}(\mathbb{R}^n) \quad \forall\, (\mu, \xi) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

If we consider the limit $\mu \to 0$ we see that the function $U_{\mu,\xi}$ becomes unbounded at $x = \xi$. In other words, the family (1.4) becomes asymptotically singular when the parameter μ decays and the energy is concentrating around $x = \xi$.

When Ω is a bounded domain, $S_{p_S}(\Omega) = S_{p_S}(\mathbb{R}^n)$ is not attained. Struwe proved in [22] that every Palais-Smale sequence $\{u_j\}_{j=1}^{\infty} \in H_0^1(\Omega)$ associated to the energy functional E, namely satisfying $\sup_j |E(u_j)| < \infty$ and $\nabla E(u_j) \to 0$, has the decomposition

$$u_j(x) = u_\infty + \sum_{i=0}^{k} U_{\mu_j^i, \xi_j^i} + o(1) \quad \text{when} \quad j \to \infty, \tag{1.6}$$

up to subsequences, for some $k \in \mathbb{N}$, where $u_\infty \in H_0^1(\Omega)$ is a critical point of E and $\mu_j^i \to 0$, $\xi_j^i \in \Omega$. When the domain is star-shaped, the Pohozaev identity constrains u_∞ to vanish. It is worth noting that, in general, $k \geq 0$. However, if we restrict to non-negative Palais-Smale sequences $\{u_j\}_{j=1}^{\infty}$ with $E(u_j) \geq S_{p_S}$ it follows that k must be positive. In this case, we say that the compactness of E is lost by "bubbling" because the Talenti bubbles in (1.6) are preventing the existence of a converging subsequence. Furthermore Du [12] and Suzuki [23] have proved, that for every sequence of times $\{t_j\}_{j=1}^{\infty}$ with $t_j \to \infty$ as $j \to \infty$, the threshold solution $u_{\alpha^*}(x, t_j)$ of the energy-critical heat equation (F_{p_S}) has the asymptotic decomposition (1.6) up to subsequences. Thus, when constructing example of threshold solutions in the critical case, it is natural to look for solutions with the asymptotic shape (1.4).

1.2. Examples of threshold solutions for $p = p_S$

In this section we present an overview about the known examples of threshold solutions, in both radial and nonradial case, and we give a first introduction to the next chapters.

1.2.1 The radial case

Most of the results in the literature concerning the dynamics of the threshold solution pertain to the radial case. This setting allows the construction of specific solutions by means of matched asymptotic expansions. This technique has been used by Galaktionov and King [14] to prove that, as $t \to \infty$, the radial threshold solution has the asymptotic bubbling profile

$$u_{\alpha^*}(x,t) \sim [n(n-2)]^{\frac{n-2}{4}} \left(\frac{\mu(t)}{\mu(t)^2 + |x|^2} \right)^{\frac{n-2}{2}},$$

with

$$\log \|u_{\alpha^*}(\cdot, t)\|_\infty = \begin{cases} \dfrac{\pi^2}{4} t(1 + o(1)) & \text{if} \quad n = 3, \\ 2\sqrt{t}(1 + o(1)) & \text{if} \quad n = 4, \end{cases} \tag{1.7}$$

and

$$\|u_{\alpha^*}(\cdot,t)\|_\infty = (\gamma_n t)^{\frac{n-2}{2(n-4)}}, \quad \text{if} \quad n \geq 5, \tag{1.8}$$

with constants

$$\gamma_n = [n(n-2)]^{\frac{n-2}{2}} \frac{(n-4)^2 \Gamma_0(N)}{n(n+2)\Gamma_0(N/2)^2},$$

where Γ_0 denotes the Gamma function. As a by-product of our analysis in the following chapter, we prove a generalization of (1.7) for $n = 3$.

1.2.2 The nonradial case in higher dimension

The first example of global unbounded solution without radial symmetry has been constructed by Cortázar, Del Pino and Musso in [7] in dimension $n \geq 5$. Let $\Omega \subset \mathbb{R}^n$ with $n \geq 5$. For any $q \in \Omega$ there exists an initial datum $u_0(x)$ such that the positive solution to (F_{p_S}) has the form

$$u(x,t) = \mu^{-\frac{n-2}{2}} U\left(\frac{x-\xi(t)}{\mu(t)}\right) - \mu^{\frac{n-2}{2}}(H_0(x,q) + \theta(x,t)),$$

where

- $\theta(x,t)$ is bounded and decays uniformly away from the point q;
- $\|u(\cdot,t)\|_\infty$ satisfies (1.8) as $t \to \infty$;
- $\xi(t) = q + O\big(\mu(t)^2\big)$ as $t \to \infty$.

In particular, we observe that the time asymptotics at $x = q$, given by $\mu^{-\frac{n-2}{2}}$, does not depend on the position of q in Ω. This is in contrast with what we shall see in dimension 3 and 4.

In fact, they prove a more general result, giving the first example of a multi-spike threshold solution. Let $G(x,y)$ be the solution to

$$\begin{aligned} -\Delta_x G(x,y) &= c_n \delta(x-y) \quad \text{in } \Omega, \\ G(x,y) &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\delta(x)$ is the Dirac mass at the origin and $c_n = \alpha_n \omega_n$. Also, consider the regular part of the Green function

$$H(x,y) := \Gamma(x-y) - G(x,y),$$

where $\Gamma(z) = \alpha_n/|z|^{n-2}$ is a multiple of the fundamental solution. It is proved in [7] that, given a set of points $q_1, \ldots, q_k \in \Omega \subset \mathbb{R}^n$ with $n \geq 5$ and $k \in \mathbb{N}^+$, if the matrix $\mathcal{G}$ with entries

$$(\mathcal{G})_{ij} = \begin{cases} H(q_j,q_j) & \text{if} \quad i = j, \\ -G(q_i,q_j) & \text{otherwise}, \end{cases}$$

is positive definite, then a solution to (F_{ps}) exists and asymptotically looks like a sum of bubbles centered at q_j for $j = 1, \dots, k$. The condition on $\mathcal{G}$ guarantees that the interaction terms between the bubbles are sufficiently weak to treat them as lower order terms.

Such solutions enjoy a k codimension stability. In other words, there exists a codimension k manifold M in $C^1(\bar{\Omega})$, which contains the initial datum $u_q(x,0)$ that blows up in infinite time at points $\{q_i\}_{i=1}^k$ such that if $v(x) \in M$ and it is sufficiently close to $u_q(x,0)$ in C^1-sense, then the solution with initial datum $v(x)$ has exactly k blow up points $\{\tilde{q}_i\}_{i=1}^k$ with $\tilde{q}_i$ near the original q_i for $i = 1, \dots, k$.

1.2.3 The nonradial case in dimension 3

The existence of positive global and unbounded solutions of (F_{ps}) in nonradial case is an open problem when $n \in \{3, 4\}$. Chapter 2 of this thesis deals with the conjecture in dimension 3. We consider the Dirichlet problem

$$\begin{cases} u_t = \Delta u + u^5 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.9}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$. We observe that the equation is translation-invariant in time. It is convenient to construct $u(x,t)$ in $\Omega \times [t_0, \infty)$ for a sufficiently large initial time $t_0 > 0$; then, the function $u_0(x,t) := u(x, t - t_0)$ is a solution to (1.9) in $\Omega \times [0, \infty)$.

We discover that an important role in the analysis of this problem is played by the Green function G_γ of the elliptic operator $-\Delta - \gamma$, for a special number $\gamma \in [0, \lambda_1)$, in Ω under Dirichlet boundary conditions. The Green function G_γ satisfies

$$\begin{cases} -\Delta_x G_\gamma(x,y) - \gamma G_\gamma(x,y) = c_3 \delta_y(x) \quad \text{in } \Omega, \\ G_\gamma(x,y) = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Here $\delta_y(x)$ is the Dirac delta distribution centered at y. Also, $c_n := \omega_n \alpha_n$ where ω_n is the area of the unit sphere in dimension n and α_n is given in (1.5). In order to separate the singular part, we decompose

$$G_\gamma(x,y) = \Gamma(x-y) - H_\gamma(x,y)$$

where

$$\Gamma(x) := \frac{c_3}{|x|},$$

denotes (a multiple of) the fundamental solution of the Laplacian, and $H_\gamma(x,y)$ is the regular part of G_γ. The diagonal function $R_\gamma(x) = H_\gamma(x,x)$ is called Robin function, and it turns out that, given $q \in \Omega$, there exists a unique number $\gamma = \gamma(q)$ such that

$$R_{\gamma(q)}(q) = 0, \quad \text{with} \quad \gamma(q) \in (0, \lambda_1).$$

Equivalently, this number is defined by

$$\gamma(q) := \sup\{\gamma > 0 : R_\gamma(q) > 0\}.$$

We are now in position to state the main theorem proved in chapter 2.

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ a bounded smooth domain. Let q a point in Ω such that*

$$\gamma(q) < \frac{\lambda_1}{3}. \tag{1.10}$$

Then, there exist an initial datum $u_0(x) \in C^1(\bar{\Omega})$, smooth functions $\xi(t), \mu(t)$ and $\theta(x,t)$ such that the solution $u(x,t)$ to the problem (1.9) is a positive unbounded global solution with the asymptotic form

$$u(x,t) = \mu^{-1/2} U\left(\frac{x - \xi(t)}{\mu(t)}\right) - \mu^{1/2}(H_\gamma(x,\xi) + \theta(x,t)),$$

where θ is a bounded function. Also, θ decays uniformly away from the point q, that is: for all compact set $K \subset \bar{\Omega}$ with $q \notin K$, we have $\|\theta(\cdot,t)\|_{L^\infty(\Omega\setminus K)} \to 0$ as $t \to \infty$. Moreover, the parameters $\mu(t), \xi(t)$ are smooth functions of time and satisfy

$$\ln\left(\frac{1}{\mu(t)}\right) = 2\gamma(q)t(1 + o(1)), \quad \xi(t) - q = O(\mu(t)) \quad \text{as} \quad t \to \infty.$$

The assumption (1.10) The condition above seems necessary when we look for a stable solution in the sense of [7]. In our construction we need to consider the Dirichlet problem of the type

$$\begin{aligned} &u_t = \Delta u + \gamma u + e^{-2\gamma t} \quad \text{in } \Omega \times \mathbb{R}^+, \\ &u(x,t) = 0 \quad \text{on } \partial\Omega \times [0,\infty), \\ &u_0(x) = 0 \quad \text{in } \Omega. \end{aligned} \tag{1.11}$$

For $t > 1$, we have

$$|u(x,t)| \lesssim e^{-\min\{\lambda_1, 3\gamma\}t}.$$

This is a consequence of the long-term behaviour of the Dirichlet heat kernel

$$p_t^\Omega(x,t) \sim \phi_1(x)\phi_1(y)e^{-\lambda_1 t},$$

in bounded domains. Since we need to solve fixed point theorems in weighted-L^∞ spaces to find the exact remainder of the blow-up parameter $\mu(t)$, the long time behaviour of the solution to (1.11) has to be $e^{-2\gamma t}$.

In any domain, the number $\gamma(q)$, as a function of q, is smooth and tends to λ_1 as q approaches $\partial\Omega$. Hence, (1.10) necessarily requires q to be sufficiently far from the boundary. We have examples of domains where (1.10) is satisfied somewhere. We do not know if this is true for all domains. A study of Wang [25] suggests that (1.10) may be false everywhere in 'very thin cylinders'.

Relation between γ and $\mu_{\mathbf{BN}}$ The number $\gamma(q)$ is related to the Brezis-Nirenberg problem. Define

$$\mathcal{S}_a(\Omega) := \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u(x)|^2 - a\int_\Omega |u(x)|^2\,dx}{\left(\int_\Omega |u|^6\,dx\right)^{\frac{1}{3}}}.$$

In the celebrated work [3] Brezis and Nirenberg proved that the existence of a constant $\mu_{\mathrm{BN}} \in (0, \lambda_1)$ such that

$$\mu_{\mathrm{BN}} := \inf\{a > 0 : \mathcal{S}_a(\Omega) < \mathcal{S}_0\}.$$

Then, Druet [11] proved

$$\min_{q\in\Omega} \gamma(q) = \mu_{\mathrm{BN}}(\Omega).$$

Thus, when $3\mu_{BN}(\Omega) < \lambda_1(\Omega)$ is true, condition (1.10) is satisfied in some open set $\mathcal{O} \subset \Omega$, and Theorem 1 gives the desired solution with blow-up at any fixed point $q \in \mathcal{O}$.

The unit ball B_1 When we consider the radial case $\Omega = B_1(0)$ and $q = 0$, an explicit computation gives $\gamma(0) = \pi^2/4$, that is consistent with (1.7). In fact, this is the minimum value for $\gamma(q)$ since Brezis and Nirenberg computed $\mu_{BN}(B_1) = \pi^2/4$. Applying the formula in Remark 2.1 of Chapter 2 to the radial case we deduce that $\gamma(r)$ is a decreasing function of $r \in [0, 1]$. Thus, the condition (1.10) is satisfied in the ball B_{d^*}, where $d^* = |q|$ and q is a point such that $\gamma(q) = \lambda_1/3$.

The unit cube $\mathcal{C}_1$ For the unit cube $\mathcal{C}_1$ it is known (see Remark 4.3 in [25]) that $3\mu_{\mathrm{BN}}(\mathcal{C}_1) < \lambda_1(\mathcal{C}_1)$. Indeed, from $B_{1/2}(0) \subset \mathcal{C}_1$ and the strict monotonicityof $\mu_{BN}(\Omega)$ with respect to Ω we deduce $\mu_{\mathrm{BN}}(\mathcal{C}_1) < \mu_{\mathrm{BN}}\big(B_{1/2}\big) = \pi^2$. By separation of variables we easily compute $\lambda_1(\mathcal{C}_1) = 3\pi^2$, thus

$$3\mu_{\mathrm{BN}}(\mathcal{C}_1) < 3\mu_{\mathrm{BN}}\big(B_{1/2}\big) = 3\pi^2 = \lambda_1(\mathcal{C}_1).$$

In general, in Theorem 1 we need the smoothness of the domain Ω to get a smooth solution up to the boundary. In case of the cube, a slight modification of Theorem 1 applies: since C_1 is a Lipschitz domain, by the parabolic regularity theory we get a smooth solution $u(x,t)$ in $\Omega \times \mathbb{R}^+$ which is Lipschitz continuous in $\Omega \times [0, \infty)$.

Estimates for other domains Let Ω^* the ball with the same volume as Ω. The following estimate holds true:

$$\frac{\lambda_1(\Omega^*)}{4} \le \mu_{\mathrm{BN}}(\Omega) \le \frac{\lambda_1(\Omega^*)}{4} \min_{x\in\Omega} R_0(x)^2.$$

Thus, assuming without loss of generality Ω with volume $|\Omega| = |B_1|$, if it happens that we know $\min_{x\in\Omega} R_0(x)^2 < 4/3$ we can apply Theorem 1 to Ω. The first inequality was proved by Brezis and Nirenberg [3] by means of a symmetrization argument. Using harmonic transplantation Bandle and Flucher [1] proved the upper bound. Wang [25] conjectured that $\mu_{\mathrm{BN}}/\lambda_1 \in [1/4, 4/9)$. This range is supported by numerical computations made by Budd and Humphries in [4].

1.2.4 The nonradial case in dimension 4

The critical heat equation in dimension $n = 4$ is

$$u_t = \Delta u + u^3 \quad \text{in } \Omega \times \mathbb{R}^+.$$

If a nonradial positive global unbounded solution for the Dirichlet problem exists is still an open question. Solving this conjecture is a work in progress in collaboration with Manuel Del Pino and Juan Dávila.

In Chapter 3 we prove some steps towards an expected full solution. We have computed the natural generalization of (1.7) by beginning the bubbling construction. A nonlocal operator, less singular than in dimension 3, governs the dynamics of lower order terms of the blow-up rate. The nonradial extension of (1.7) is as follows:

$$\ln\Big(\|u(\cdot,t)\|_{L^\infty(\Omega)}\Big) = k\sqrt{t}(1+o(1)), \quad k = \Big(\sqrt{2}R_0(q)\Big)^{1/2}. \tag{1.12}$$

Here $R_0(x) = H_0(x,x)$ is the Robin function. When $\Omega = B_1(0)$ and $q = 0$ we explicitly compute $R_0(q) = \alpha_4 = 2\sqrt{2}$, hence the radial case (1.7) is recovered. Thus, as in dimension 3, we expect that the asymptotic behaviour depends on the position of q in the domain. More precisely, in chapter 3 we discuss the following steps regarding the bubbling construction:

- we show that the ansatz

  $$\mu^{-1}U\left(\frac{x-\xi}{\mu}\right) - \mu^{1/2}H_0(x,\xi)$$

  requires a nonlocal improvement to remove a slow-decay term of type

  $$\dot\lambda \frac{\alpha_4}{\mu^2 + |x-\xi|^2},$$

  where $\lambda(t) = -\ln(\mu(t))$;

- we invert such nonlinear operator at the main order;

- we obtain the behaviour (1.12) and we show that it matches the radial case;
- we prove a modification of [7, Lemma 7.2] in dimension $n = 4$. This is a necessary ingredient to get and extension of the linear theory developed in higher dimension and finding a decaying perturbation close to the blow-up point.

Chapter 2

Infinite blow-up solution for the critical heat equation in dimension 3

In this chapter we present the first example of global unbounded solution to the 3 dimensional critical heat equation without radial symmetry. The radial case was constructed by means of matched expansion techniques by Galaktionov and King [14]. We generalize this result to the nonradial case with a different method.

Our work is based on the new parabolic gluing method introduced in [7] by Cortázar, Del Pino and Musso. We extend their result to $n = 3$, thus leaving open only the 4-dimensional nonradial case. In [7] the assumption $n \geq 5$ allows to solve the problem avoiding nonlocal operators. To solve this main difficulty, we develop an invertibility theory for a half-fractional derivative type operator, combining asymptotic properties of the heat kernel of the domain with a Laplace transform argument.

We construct the solution assuming an analytical property of the domain, that can be stated in terms of the Brezis-Nirenberg number. This condition is verified for balls and cubes, where we can select the blow-up point in a suitable open set (far from the boundary). It is not known if there are domains that do not possess this property. As a consequence of the strategy we also obtain a 1-codimensional stability for the solution. This is a joint work with Manuel Del Pino.

2.1. Outline of the Article

The paper consists of three main parts. Firstly, we choose a natural ansatz u_1, and we compute the associated error. It turns out that u_1 is not sufficiently good to start our perturbation scheme. Thus, an improved approximation u_3 is constructed by adding global and local terms. In the second part, by means of the inner-outer gluing procedure, we prove the existence of a perturbation $\tilde{\phi}$ such that $u = u_3 + \tilde{\phi}$ is an exact solution to the problem. Basically, this strategy consists in decomposing $\tilde{\phi}$ to separate the regime close to the blow-up point and far away from it. We firstly solve the outer problem for given parameters and then, by fixed point arguments, we solve the inner problem. We need the validity of some orthogonality conditions with respect to the kernel of the main linear operator in the inner problem. These are satisfied by carefully selecting the free parameters of our ansatz. In particular, solving one of this orthogonality conditions requires a crucial invertibility theory for a nonlocal operator $\mathcal{J}$, that is proved in the third part of the paper.

Appendix B: Statement of Authorship

This declaration concerns the article entitled:									
Infinite time blow-up for the three dimensional energy critical heat equation in bounded domain									
Publication status (tick one)									
draft manuscript	X	**Submitted**		**In review**		**Accepted**		**Published**	
Publication details (reference)	Authors: Giacomo Ageno, Manuel Del Pino								
Candidate's contribution to the paper (detailed, and also given as a percentage).	The bulk of the calculations have been performed by the author of the thesis (80%). The presentation of the content has been fully performed by the author of the thesis (100%).								
Statement from Candidate	This paper reports on original research I conducted during the period of my Higher Degree by Research candidature.								
Signed						**Date**	15.6.2023		

INFINITE TIME BLOW-UP FOR THE THREE DIMENSIONAL ENERGY CRITICAL HEAT EQUATION IN BOUNDED DOMAIN

GIACOMO AGENO AND MANUEL DEL PINO

Abstract. We consider the Dirichlet problem for the energy-critical heat equation

$$\begin{cases} u_t = \Delta u + u^5 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^3$. Let $H_\gamma(x,y)$ be the regular part of the Green function of $-\Delta - \gamma$ in Ω, where $\gamma \in (0, \lambda_1)$ and λ_1 is the first Dirichlet eigenvalue of $-\Delta$. Then, given a point $q \in \Omega$ such that $3\gamma(q) < \lambda_1$, where

$$\gamma(q) := \sup\{\gamma > 0 : H_\gamma(q,q) > 0\},$$

we prove the existence of a non-radial global positive and smooth solution $u(x,t)$ which blows up in infinite time with spike in q. The solution has the asymptotic profile

$$u(x,t) \sim 3^{\frac{1}{4}} \left(\frac{\mu(t)}{\mu(t)^2 + |x - \xi(t)|^2} \right)^{\frac{1}{2}} \quad \text{as} \quad t \to \infty,$$

where

$$-\ln(\mu(t)) = 2\gamma(q)t(1 + o(1)), \quad \xi(t) = q + O(\mu(t)) \quad \text{as} \quad t \to \infty.$$

1. Introduction and statement of the main result

We investigate the asymptotic structure of global in time solutions $u(x,t)$ of the energy-critical semilinear heat equation

$$\begin{cases} u_t = \Delta u + u^5 & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ u(\cdot,0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain and u_0 is a smooth initial datum. The energy associated to the solution $u(x,t)$ is

$$E(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{6} \int |u|^6 \, dx.$$

Since classical solutions of (1.1) satisfy

$$\frac{d}{dt} E(u(\cdot,t)) = -\int_\Omega |u_t|^2 \, dx \leq 0,$$

the energy is a Lyapunov functional for (1.1). The stationary equation on the whole space is the Yamabe problem

$$\Delta U + U^5 = 0 \quad \text{in } \mathbb{R}^3.$$

All positive solutions of this equation are given by the Talenti bubbles (see [4])

$$U_{\mu,\xi}(x) = \mu^{-\frac{1}{2}} U\Big(\frac{x-\xi}{\mu}\Big), \tag{1.2}$$

where $\mu > 0, \xi \in \mathbb{R}^3$ and

$$U(x) = \alpha_3 \frac{1}{\Big(1 + |x|^2\Big)^{1/2}}, \quad \text{where} \quad \alpha_3 := 3^{1/4}.$$

Consider the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, which is compact for $p \in (1, p_S)$, where $p_S = \frac{n+2}{n-2}$, and the associated constant

$$S_p(\Omega) := \inf_{0 \neq u \in H_0^1(\Omega)} \frac{\|u\|^2_{H_0^1(\Omega)}}{\|u\|^2_{L^{p+1}(\Omega)}}.$$

The Talenti bubbles achieve the constant $S_{p_S}(\mathbb{R}^n)$. Thus, the energy $E(U_{\mu,\xi}) = S_{p_S}(\mathbb{R}^n)$ is invariant with respect to μ, ξ. When $\mu \to 0$ the Talenti bubble becomes singular. This is the reason for the loss of compactness in the Sobolev embedding for $p = p_S$. Struwe proved in [31] that every Palais-Smale sequence $\{u_j\}_{j=1}^{\infty} \in H_0^1(\Omega)$ associated to the energy functional E, namely satisfying $\sup_j |E(u_j)| < \infty$ and $\nabla E(u_j) \to 0$, has the decomposition

$$u_j(x) = u_\infty + \sum_{i=0}^{k} U_{\mu_j^i, \xi_j^i} + o(1) \quad \text{when} \quad j \to \infty, \tag{1.3}$$

up to subsequences, for some $k \in \mathbb{N}$, where $u_\infty \in H_0^1(\Omega)$ is a critical point of E and $\mu_j^i \to 0$, $\xi_j^i \in \Omega$. When the domain is star-shaped, the Pohozaev identity constrains u_∞ to vanish. It is worth noting that, in general, $k \geq 0$. However, if we restrict to non-negative Palais-Smale sequences $\{u_j\}_{j=1}^{\infty}$ with $E(u_j) \geq S_{p_S}$ it follows that k must be positive. In this case, we say that the compactness is lost by 'bubbling'. When the domain is star-shaped, the Pohozaev identity constrains u_∞ to vanish.

For classical finite-energy solutions $u(x,t)$ the problem (1.1) is well-posed in short time intervals. We refer to the monograph [29] by Quittner and Souplet for an extended review on this problem and more general semilinear parabolic problems.

The aim of this paper is exhibiting classical positive finite-energy solutions $u(x,t)$ of (1.1) which are globally defined in time and satisfy

$$\lim_{t \to \infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{1.4}$$

These global unbounded solutions are difficult to detect since the typical behaviour (in the sense of Lions [26]) of the solutions to (1.1) is blow-up in finite time or decay at infinity. On one hand, if the initial datum is sufficiently large than the solutions are defined until a maximum time $T < \infty$; on the other hand, if $\|u_0\|_\infty$ is small enough then the solution eventually decay. For this reason solutions with the property (1.4) are called 'threshold solutions'. In 1984 the first rigorous proof of the existence in L^1-weak sense of unbounded global solutions was found by Ni, Sacks and Tavantzis [28]. Du [15] and Suzuki [32] have proved, that for a global unbounded solution u of the energy-critical heat equation (1.1) and every sequence of times $\{t_n\}_{n=1}^{\infty}$ with $t_n \to \infty$ as $n \to \infty$, $u(x,t_n)$ have the asymptotic decomposition (1.3) up to subsequences. Thus, when constructing examples of threshold solutions in the critical case, it is natural to look for solutions with the asymptotic shape (1.2).

Most of the results about the dynamics of the threshold solution in the literature concern the radial case. This particular setting allows the construction of specific solutions by means of matched expansions. In [17], Galaktionov and King studied the problem for $\Omega = B_1(0)$ and radial initial datum. They found that the blow-up rate of the global unbounded solution is

$$\log \|u(\cdot,t)\|_\infty = \begin{cases} \frac{\pi^2}{4} t(1+o(1)) & \text{if} \quad n=3, \\ 2\sqrt{t}(1+o(1)) & \text{if} \quad n=4, \end{cases} \tag{1.5}$$

and

$$\|u(\cdot,t)\|_\infty = (\gamma_n t)^{\frac{n-2}{2(n-4)}}, \quad \text{if} \quad n \geq 5,$$

with some explicit constants γ_n. Our main theorem is the extension of this result in dimension $n=3$ to the nonradial case. The case of higher dimension $n \geq 5$ has been already extended to the nonradial case by Cortázar, Del Pino and Musso in [5], where they built positive multispike threshold solutions which blow-up by bubbling in infinite time. The term multispike refers to the fact that the constructed solution has k blow-up points as $t \to \infty$ for every choice of $k \in \mathbb{N}^+$.

Our solutions involve the Green function G_γ associated to the elliptic operator

$$L_\gamma = -\Delta - \gamma \quad \text{on} \quad \Omega,$$

where $\gamma \in [0, \lambda_1)$ and λ_1 is the principal Dirichlet eigenvalue. Namely, for all $y \in \Omega$, G_γ satisfies

$$\begin{aligned} &-\Delta_x G_\gamma(x,y) - \gamma G_\gamma(x,y) = c_3 \delta(x-y) \quad \text{in} \quad \Omega, \\ &G_\gamma(x,y) = 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\delta(x)$ is the Dirac delta, $c_3 := \alpha_3 \omega_3$, the constant ω_n indicates the area of the unit sphere and $\alpha_n = [n(n-2)]^{\frac{n-2}{4}}$. The Green function can be decomposed as

$$G(x,y) = \Gamma(x-y) - H_\gamma(x,y),$$

where $\Gamma(x) = \alpha_3 |x|^{-1}$ and the regular part $H_\gamma(x,y)$ is defined as the solution, for all $y \in \Omega$, to

$$\begin{aligned} &\Delta_x H(x,y) + \gamma H_\gamma(x,y) = \gamma \frac{\alpha_3}{|x-y|} \quad \text{in} \quad \Omega, \\ &H_\gamma(x,y) = \Gamma(x-y) \quad \text{in} \quad \partial\Omega. \end{aligned}$$

The diagonal $R_\gamma(x) := H_\gamma(x,x)$ is called Robin function associated to the operator $-\Delta - \gamma$ in Ω. It turns out that for any fixed $q \in \Omega$ there exists a unique number $\gamma(q) \in (0, \lambda_1)$ defined as

$$\gamma(q) := \{\gamma > 0 : R_\gamma(q) > 0\}.$$

Our main theorem shows that, for any $q \in \Omega$ such that $3\gamma(q) < \lambda_1$ holds, there exists a global solution to the problem (1.1) which blow-up at the point q.

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ a bounded smooth domain. Let q a point in Ω such that*

$$\gamma(q) < \frac{\lambda_1}{3}. \tag{1.6}$$

Then, there exist an initial datum $u_0(x) \in C^1(\bar{\Omega})$, smooth functions $\xi(t), \mu(t)$ and $\theta(x,t)$ such that the solution $u(x,t)$ to the problem (1.1) is a positive unbounded global solution with the asymptotic form

$$u(x,t) = \mu^{-1/2} U\left(\frac{x-\xi(t)}{\mu(t)}\right) - \mu^{1/2}(H_\gamma(x,\xi) + \theta(x,t)) \quad as \quad t \to \infty,$$

where θ is a bounded function. Also, θ decays uniformly away from the point q, that is: for all compact set $K \subset \bar{\Omega}$ with $q \notin K$, we have $\|\theta(\cdot,t)\|_{L^\infty(\Omega\setminus K)} \to 0$ as $t \to \infty$. Moreover, the parameters $\mu(t), \xi(t)$ are smooth functions of time and satisfy

$$\ln\left(\frac{1}{\mu(t)}\right) = 2\gamma(q)t(1+o(1)), \quad \xi(t) - q = O(\mu(t)) \quad as \quad t \to \infty. \tag{1.7}$$

Furthermore, thanks to the inner-outer gluing approach, which is based only on elliptic and parabolic estimates, as in [5] and [8] we get a codimension-1 stability of the solution stated by Theorem 1. In fact, under condition (1.6), the proof is identical to that one of Corollary 1.1 in [5] (see the remark in section 7).

Corollary 1.1. *Let u be the solution stated in Theorem 1 which blow us at q. Then, there exist a codimension 1 manifold $\mathcal{M}$ in $C^1(\bar{\Omega})$ with $u_0 \in \mathcal{M}$ and constants $C, \varepsilon_0 > 0$ such that if $\bar{u}_0 \in \mathcal{M}$ and $\|u_0 - \bar{u}_0\|_{C^1(\Omega)} < \varepsilon$ then the solution $\tilde{u}$ given by Theorem 1 with initial datum $\bar{u}_0$ is global with bubbling spike in some point $\bar{q}$ with $|q - \bar{q}| < C\varepsilon$.*

The condition (1.6) tells us that the point q cannot be very close to boundary, since $\gamma(q) \to \lambda_1^-$ as $q \to \partial\Omega$ (see Lemma A.2 in Appendix A). To prove the result, we will need to consider Dirichlet problems of the type

$$\begin{aligned} &u_t = \Delta u + \gamma u + e^{-2\gamma t} f(x) \quad \text{in } \Omega \times \mathbb{R}^+, \\ &u(x,t) = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ &u(x,0) = 0 \quad \text{in } \Omega, \end{aligned}$$

for some $f(x) \in L^p$ with $p > 2$. In order to get the natural estimate

$$\|u(\cdot,t)\|_\infty \le Ce^{-2\gamma t}$$

for $t > 1$, the condition (1.6) is necessary. This is due to the long-term behaviour of the Dirichlet heat kernel associated to Ω

$$p_t^\Omega(x,t) \sim \phi_1(x)\phi_1(y)e^{-\lambda_1 t} \quad \text{as} \quad t \to \infty.$$

More specifically, we use assumption (1.6) in the following steps of the proof:

- in Lemma 2.2 and Lemma 2.3 for improving the ansatz;
- in Lemma 4.1 for solving the outer problem;
- in Proposition 6.1 for the invertibility theory of the nonlocal operator $\mathcal{J}$.

The number $\gamma(q)$ is related to the Brezis-Nirenberg problem. Define

$$\mathcal{S}_a(\Omega) := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u(x)|^2 - a\int_\Omega |u(x)|^2\, dx}{\left(\int_\Omega |u|^6\, dx\right)^{\frac{1}{3}}}.$$

In the celebrated work [2] Brezis and Nirenberg proved that the existence of a constant $\mu_{\rm BN} \in (0, \lambda_1)$ such that

$$\mu_{\rm BN} := \inf\{a > 0 : \mathcal{S}_a(\Omega) < \mathcal{S}_0\}.$$

Then, Druet [14] proved

$$\min_{q\in\Omega}\gamma(q)=\mu_{\mathrm{BN}}(\Omega).$$

Thus, when $3\mu_{BN}(\Omega)<\lambda_1(\Omega)$ is true, condition (1.6) is satisfied in some open set $\mathcal{O}\subset\Omega$, and Theorem 1 gives the desired solution with blow-up at any fixed point $q\in\mathcal{O}$.

When we consider the radial case $\Omega=B_1(0)$ and $q=0$, an explicit computation gives $\gamma(0)=\pi^2/4$, that is consistent with (1.5). In fact, this is the minimum value for $\gamma(q)$ since Brezis and Nirenberg computed $\mu_{BN}(B_1)=\pi^2/4$. By radial symmetry we deduce that condition (1.6) is satisfied in the ball B_{d^*}, where $d^*=|q|$ and q is a point such that $\gamma(q)=\lambda_1/3$.

For the unit cube $\mathcal{C}$ it is known (see Remark 4.3 in [34]) that $3\mu_{\mathrm{BN}}(\mathcal{C})<\lambda_1(\mathcal{C})$. Indeed, from $B_{\frac{1}{2}}(0)\subset\mathcal{C}$ and the strict monotonicity of $\mu_{BN}(\Omega)$ with respect to Ω we deduce $\mu_{\mathrm{BN}}(\mathcal{C})<\mu_{\mathrm{BN}}\left(B_{1/2}\right)=\pi^2$. By separation of variables we easily compute $\lambda_1(\mathcal{C})=3\pi^2$, thus

$$3\mu_{\mathrm{BN}}(\mathcal{C})<3\mu_{\mathrm{BN}}\left(B_{1/2}\right)=3\pi^2=\lambda_1(\mathcal{C}).$$

Let Ω^* the ball with the same volume as Ω. The following estimate holds true:

$$\frac{\lambda_1(\Omega^*)}{4}\le\mu_{\mathrm{BN}}(\Omega)\le\frac{\lambda_1(\Omega^*)}{4}\min_{x\in\Omega}R_0(x)^2.$$

The first inequality was proved by Brezis and Nirenberg [2] by means of a symmetrization argument. Using harmonic transplantation Bandle and Flucher [1] proved the upper bound. Thus, if it happens that we know $\min_{x\in\Omega}R_0(x)^2<4/3$ we can apply Theorem 1 to Ω. Wang [34] conjectured that $\mu_{\mathrm{BN}}/\lambda_1\in[1/4,4/9)$. This range is supported by numerical computations made by Budd and Humphries in [3].

The main differences with respect to the result [5] in dimension $n\ge5$ are the following:

- in our result the blow-up rate is dependent on the position of the point $q\in\Omega$. This is a completely new phenomenon.
- The condition (1.6) does not allow us to construct multi-spike solutions, since, roughly speaking, the spikes need to be relatively far from each other and sufficiently close to the boundary in order to bound the interaction between the bubbles (see [5] for a rigorous condition in terms of the Green function G_0). However, it could still be possible to detect multi-spike.
- A nonlocal operator controls the dynamics of $\mu(t)$. The presence of a nonlocal operator has been treated also in [8], where the domain $\Omega=\mathbb{R}^3$ allows an explicit inversion of the Laplace transform.

The approach developed in this work is inspired by [5], [8] and [6]. It is constructive and allows an accurate analysis of the asymptotic dynamics and stability. Let describe the general strategy. The first step consists in choosing a good approximated solution u_3. Here the word 'good' means that the associated error function

$$S[u](x,t):=-\partial_t u+\Delta u+u^5$$

is sufficiently small in Ω. Part of the problem consists in understanding what smallness on $S[u]$ is sufficient to find a small perturbation $\tilde{\phi}$ such that

$$u=u_3+\tilde{\phi}$$

is an exact solution to (1.1). Our building block is the scaled Talenti bubble which we modify to match the boundary at the first order. Then we realize that we need two improvements. The first one is a global correction useful to get solvability conditions for the elliptic linearized operator around the standard bubble

$$L[\phi] := \Delta\phi + 5U^4(r)\phi.$$

Such improvement produces a nonlocal term which will govern the second order term in the expansion of the scaling parameters $\mu(t)$. This is a low-dimensional effect, which is ultimately due to the fact that

$$Z_{n+1}(r) := \frac{n-2}{2}U(r) + U'(r)r \notin L^2(\mathbb{R}^n) \quad \text{when} \quad n \in \{3,4\},$$

where Z_{n+1} is the unique (up to multiples) bounded radial function belonging to the kernel of $L[\phi]$. Actually, the dimensional restriction in [5] was especially designed to avoid this effect and the presence of the corresponding nonlocal term. Then, by choosing $\gamma(q)$ as in (1.6) we reduce the error close to $x = q$; this gives the asymptotic behaviour (1.7) of $\mu(t)$ at the first order. A second correction, local in nature, removes nonradial slow-decay terms and gives the asymptotic behaviour of ξ written in (1.7). At this point we have a sufficiently good ansatz to start the so called inner-outer gluing procedure: we decompose the problem in a system of nonlinear problems, namely an inner and an outer problem which are weakly coupled thanks to the smallness of $S[\tilde{u}]$. We solve the outer problem, that is a perturbation of the standard heat equation, for suitable decaying solutions of the inner equation. We can find the inner solution, by fixed point argument, using the adaptation to $n = 3$ of the linear theory for the inner problem developed in [5]. This requires the solvability of orthogonality conditions which are equivalent to a system in the parameters ξ, μ. To solve this system, we need the invertibility of a nonlocal equation, which we achieve by means of a Laplace transform argument using asymptotic properties of the heat kernel $p_t^\Omega(x, y)$.

Of course, the full problem consists in finding the exact initial datum that evolves in an infinite time solution. We find the positive initial condition

$$\begin{aligned} u(x, t_0) =& \mu(t_0)^{-1/2} U\Big(\frac{x - \xi(t_0)}{\mu(t_0)}\Big) - \mu(t_0)^{1/2} H_\gamma(x, \xi(t_0)) + \mu_0(t_0)^{1/2} J_1(x, t_0) \\ &+ \mu(t_0)^{-1/2}\phi_3\Big(\frac{x - \xi(t_0)}{\mu(t_0)}, t_0\Big)\eta_{l(t_0)}\Big(\Big|\frac{x - \xi(t_0)}{\mu(t_0)}\Big|\Big) \\ &+ \mu_0(t_0)^{1/2}\psi(x, t_0) + \eta_{R(t_0)}\Big(\Big|\frac{x - \xi(t_0)}{\mu(t_0)}\Big|\Big)\mu(t_0)^{-1/2} e_0 Z_0\Big(\frac{x - \xi(t_0)}{\mu(t_0)}\Big), \end{aligned}$$

for t_0 fixed sufficiently large, where the existence of μ, ξ, ϕ, ψ and the constant e_0 is a consequence of fixed point arguments, η_l, l, η_R, R are defined in (2.5), (2.17), (2.17) and the functions ϕ_3, J_1, J_2 solve the problems (2.21), (2.14) and (2.15) respectively.

To conclude the proof, it is necessary to establish the Lipschitz dependence of $\phi[\psi_0]$ and $e_0[\psi_0]$, where ϕ represents the solution to the inner problem with the initial datum $\phi(y, t_0) = e_0 Z_0(y)$ and ψ_0 denotes the initial outer condition. This property is crucial for obtaining contraction maps, requiring the initial datum in $C^1(\bar{\Omega})$ class to apply the Implicit Function Theorem. It is worth noting that, as a consequence of the smoothing property of the heat equation, a smooth solution $u(x, t)$ is guaranteed for $t > t_0$.

We conclude this introduction giving a short bibliographic overview on relate problems and recent developments. Concerning the Cauchy problem for (1.1) in the critical case

$p = p_S$, infinite blow-up solutions have been found in dimension $n = 3$ in [8] by Del Pino, Musso and Wei. Recently, Wei, Zhang and Zhou [35] detected analogue solutions in dimension $n = 4$. King and Galaktionov [17] used matched asymptotic methods to formally analyze the behaviour of infinite blow-up solutions in the radial case, conjecturing non-existence of positive infinite time blow-up solutions in dimension $n \geq 5$. In [9], sign-changing solutions in the form of "tower of bubbles", that is a superposition of a negative and a positive bubble concentrating at the same point, have been constructed in dimension $n \geq 7$.

Many articles in the literature have been dedicated to construct finite time blow-up solutions. A smooth solution of

$$\begin{aligned} &u_t = \Delta u + u^p \quad \text{in } \Omega \times (0,T), \\ &u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ &u(\cdot,0) = u_0 \quad \text{in } \Omega, \end{aligned}$$

blows-up at finite time if $\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty$ as $t \to T$ for some $T < \infty$. Finite time blow-up can be classified into two types:

- Type I if

$$\limsup_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_\infty < \infty,$$

- Type II if

$$\limsup_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_\infty = \infty.$$

Type I blow-up exhibits behavior similar to the corresponding ODE $u_t = u^p$, while Type II blow-up is considerably more difficult to identify. We know after [16] and [27] that if $p = p_S$ and $n \geq 3$ Type II blow-up is not admitted, but it is still admissible for sign-changing solutions, and in fact examples have been found in [9–11, 19, 20, 25, 30].

2. Approximate solution and estimate of the associated error

In this section we construct an approximation for a solution to the problem

$$\begin{cases} u_t = \Delta u + u^5 & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{2.1}$$

and we compute the associated error. The first approximation u_1 is chosen by selecting a time-scaled version of the stationary solution to

$$\Delta U + U^5 = 0 \quad \text{in } \mathbb{R}^3,$$

properly adjusted to be small at the boundary $\partial\Omega$. This is constructed in section 2.1. In order to make the error small at the blow-up point, we need to select a precise first order for the dilatation parameter $\mu(t)$, which matches the blow-up rate in the radial case found in [17]. However, we observe in section 2.2 that, for our rigorous proof, u_1 is not close enough to an exact solution to make our scheme rigorous. In section 2.3 we make a global improvement u_2. Such correction involves a nonlocal operator, similar to a $\frac{1}{2}$-fractional Caputo derivative, in the lower order term of $\mu(t)$. The last improvement u_3 is only local, and it removes slow-decaying terms in non-radial modes by selecting the first order asymptotic of the translation parameter $\xi(t)$.

2.1. **First global approximation.** Our building blocks are the scaled Talenti bubble (1.2) which satisfy

$$\Delta U_{\mu,\xi} + U_{\mu,\xi}^5 = 0 \quad \text{in } \mathbb{R}^3. \tag{2.2}$$

We look for a solution of the form $u_1(x,t) \approx U_{\mu(t),\xi(t)}(x)$. We make an ansatz for the parameters $\mu(t), \xi(t)$. Assuming that $\mu(t) \to 0$ as $t \to \infty$ and $\xi \to 0$ we notice that $U_{\mu,\xi}(x)$ is concentrating around $x = 0$ and uniformly small away from it. For this reason, we should have

$$\begin{aligned}\partial_t u_1 - \Delta u_1 &= u_1(x,t)^5 \\ &\approx \delta_0(x-\xi)\int_{\mathbb{R}^3}\left(\mu^{-1/2}U\left(\frac{x-\xi}{\mu}\right)\right)^5 dx \\ &= \delta_0(x-\xi)\mu^{1/2}\int_{\mathbb{R}^3} U(y)^5\, dy \\ &= \delta_0(x-\xi)\omega_3\alpha_3\mu^{1/2},\end{aligned} \tag{2.3}$$

where $\omega_3 = 4\pi$ is the surface area of S^2. Let $\mu_0(t)$ the first order of $\mu(t)$, that is

$$\mu(t) = \mu_0(t)(1 + o(1)) \quad \text{as} \quad y \to \infty.$$

From (2.3) we define the scaled function

$$v(x,t) := \mu^{-1/2}u_1(x,t),$$

should satisfy

$$\begin{aligned}&v_t \approx \Delta v + \left(-\frac{\dot{\mu}}{2\mu}\right)v + \omega_3\alpha_3\delta_0(x-\xi) \quad \text{in } \Omega\times\mathbb{R}^+, \\ &v = 0 \quad \text{on } \partial\Omega\times\mathbb{R}^+.\end{aligned} \tag{2.4}$$

We choose the parameter $\mu_0(t)$ such that

$$-\frac{\dot{\mu}_0(t)}{2\mu_0(t)} = \gamma,$$

for some $\gamma \in \mathbb{R}^+$ that will be fixed later. This is equivalent to choosing

$$\mu_0(t) = be^{-2\gamma t}, \tag{2.5}$$

for some $b \in \mathbb{R}^+$. We can fix $b = 1$. Indeed, the equation is translation-invariant in time: we construct, for a sufficiently large initial time t_0, a solution $u(x,t)$ in $\Omega \times [t_0, \infty)$ and we conclude that $u_0(x,t) := u(x, t - t_0)$ is a solution to (2.1) in $\Omega \times [0,\infty)$. We observe that after shifting the initial time, the main dilatation parameter μ_0 becomes $\mu_0(t - t_0) = e^{2\gamma t_0}e^{-2\gamma t}$. With this choice (2.4) reads

$$\begin{aligned}&v_t \approx \Delta v + \gamma v + \omega_3\alpha_3\delta_0(x-\xi) \quad \text{in } \Omega\times\mathbb{R}^+, \\ &v = 0 \quad \text{on } \partial\Omega\times\mathbb{R}^+.\end{aligned}$$

Hence, for large time we should have

$$v(x,t) \approx G_\gamma(x,\xi), \tag{2.6}$$

where $G_\gamma(x,y)$ is the Green function for the boundary value problem

$$\begin{aligned}&-\Delta_x G_\gamma(x,y) - \gamma G_\gamma(x,y) = \omega_3\alpha_3\delta(x-y) \quad \text{in } \Omega, \\ &G(\cdot,y) = 0 \quad \text{on } \partial\Omega.\end{aligned} \tag{2.7}$$

We write

$$G_\gamma(x,y) = \Gamma(x-y) - H_\gamma(x,y), \tag{2.8}$$

where

$$-\Delta_x \Gamma(x) = \omega_3 \alpha_3 \delta_0(x), \quad \Gamma(x) = \frac{\alpha_3}{|x|}$$

is (a multiple of) the fundamental solution of the Laplacian in $\mathbb{R}^3$ and the regular part H_γ satisfies

$$\begin{aligned} &-\Delta_x H_\gamma(x,y) - \gamma H_\gamma(x,y) = -\gamma \Gamma(x-y) \quad \text{in } \Omega, \\ &H_\gamma(\cdot, y) = \Gamma(\cdot - y) \quad \text{on } \partial\Omega. \end{aligned} \tag{2.9}$$

The function $H_\gamma(x,y) \in C^{0,1}(\Omega)$ when $\gamma \in (0, \lambda_1)$. Also, we decompose

$$H_\gamma(x,y) = \theta_\gamma(x-y) - h_\gamma(x,y), \tag{2.10}$$

where

$$\theta_\gamma(x) := \alpha_3 \frac{1 - \cos\big(\sqrt{\gamma}|x|\big)}{|x|} \tag{2.11}$$

and $h_\gamma(\cdot, y) \in C^\infty(\Omega)$ solves

$$\begin{aligned} &\Delta_x h_\gamma(x,y) + \gamma h_\gamma(x,y) = 0 \quad \text{in } \Omega, \\ &h_\gamma(x,y) = -\alpha_3 \frac{\cos\big(\sqrt{\gamma}|x-y|\big)}{|x-y|} \quad \text{on } \partial\Omega. \end{aligned} \tag{2.12}$$

We also define the Robin function

$$R_\gamma(x) := H_\gamma(x,x) = h_\gamma(x,x).$$

In terms of the original function u_1 the equation (2.6) reads as

$$u_1(x,t) \approx \mu^{1/2} \frac{\alpha_3}{|x-\xi|} - \mu^{1/2} H_\gamma(x,\xi).$$

We also notice that far away from the origin we have

$$U_{\mu,\xi}(x) \approx \mu^{1/2} \frac{\alpha_3}{|x-\xi|}.$$

This formal analysis suggests the ansatz

$$u_1(x,t) := U_{\mu,\xi}(x) - \mu^{1/2} H_\gamma(x,\xi).$$

2.1.1. *Dilatation parameter* $\mu(t)$. The full dilatation parameter is given by

$$\mu = \mu_0(t) e^{2\Lambda(t)},$$

where

$$\mu_0(t) = e^{-2\gamma t}, \quad \text{and} \quad \Lambda(t) = o(1) \quad \text{as} \quad t \to \infty.$$

In this notation we have

$$\begin{aligned} \frac{\dot\mu(t)}{2\mu(t)} &= \frac{\dot\mu_0 e^{2\Lambda}}{2\mu_0 e^{2\Lambda}} + \frac{2\dot\Lambda \mu_0 e^{2\Lambda}}{2\mu_0 e^{2\Lambda}} \\ &= -\gamma + \dot\Lambda(t), \end{aligned}$$

and

$$\Lambda(t) = -\int_t^\infty \dot\Lambda(s)\, ds,$$

where $\dot\Lambda(s)$ is an integrable function in any $[t_0, \infty)$.

2.2. **Error associated to u_1.** The next step consists in computing the error associated to the first ansatz u_1. We define the error operator

$$S[u] := -\partial_t u + \Delta u + u^5.$$

Of course, finding u such that $S[u] = 0$ is equivalent to solving the equation in (2.1). It is well-known that all bounded solutions to the linearized operator

$$\Delta_y \phi + 5U^4\phi = 0 \quad \text{in } \mathbb{R}^n,$$

are linear combinations of the functions

$$Z_i(y) := \partial_{y_i} U(y), \quad i = 1,2,3, \quad Z_4(y) := \frac{1}{2}U(y) + y \cdot \nabla U(y) = \frac{\alpha_3}{2}\frac{1 - |y|^2}{\left(1 + |y|^2\right)^{3/2}}.$$

We define the scaled variable

$$y = y(x,t) := \frac{x - \xi(t)}{\mu(t)}.$$

Now, we compute $S[u_1](x,t)$ for $x \neq \xi(t)$. We have

$$\begin{aligned}
\Delta u_1 &= \mu^{-1/2}\Delta_x U\left(\frac{x-\xi}{\mu}\right) - \mu^{1/2}\Delta_x H_\gamma(x,\xi) \\
&= -\mu^{-5/2}U(y)^5 + \mu^{1/2}\left(\gamma H_\gamma(x,\xi) - \frac{\gamma\alpha_3}{|x-\xi|}\right) \\
&= -\mu^{-5/2}U(y)^5 + \mu^{1/2}\gamma H_\gamma(x,\xi) - \mu^{1/2}\frac{\gamma\alpha_3}{|x-\xi|},
\end{aligned}$$

where we used equations (2.2) and (2.9) for U and H_γ. The time-derivative gives

$$\begin{aligned}
\partial_t u_1 =& -\frac{1}{2}\frac{\dot\mu}{\mu}\mu^{-1/2}U(y) + \mu^{-1/2}\nabla_y U(y) \cdot \left[-\frac{\dot\xi}{\mu} - \frac{\dot\mu}{\mu}y\right] \\
&- \frac{1}{2}\frac{\dot\mu}{\mu}\mu^{1/2}H_\gamma(x,\xi) - \mu^{1/2}\dot\xi \cdot \nabla_{x_2}H_\gamma(x,\xi) \\
=& -\left(\frac{\dot\mu}{2\mu}\right)\left[\mu^{-1/2}U + 2\mu^{-1/2}\nabla_y U \cdot y + \mu^{1/2}H_\gamma(x,\xi)\right] \\
&- \mu^{-3/2}\dot\xi \cdot \nabla_y U - \mu^{1/2}\dot\xi \cdot \nabla_{x_2}H_\gamma(x,\xi)
\end{aligned}$$

Hence, the error associated to u_1 is

$$\begin{aligned}
S[u_1] =& \dot\Lambda\left(\mu^{-1/2}2Z_4(y) + \mu^{1/2}H_\gamma(x,\xi)\right) - \gamma\mu^{-1/2}\left(2Z_4(y) + \frac{\alpha_3}{|y|}\right) \\
&+ \mu^{-3/2}\dot\xi \cdot \nabla_y U(y) + \mu^{1/2}\dot\xi \cdot \nabla_{x_2}H_\gamma(x,\xi) \\
&- \mu^{-3/2}5U(y)^4 H_\gamma(x,\xi) \\
&+ \mu^{-5/2}\left[(U(y) - \mu H_\gamma(x,\xi))^5 - U(y)^5 + \mu 5U(y)^4 H_\gamma(x,\xi)\right].
\end{aligned} \tag{2.13}$$

2.3. **Global improvement.** The remaining part of this section concerns the improvement of the natural ansatz u_1. Later in the argument we will divide the error in outer and inner part. We realize that solving the inner-outer system requires a global and local improvements. Based on Proposition 3.1 with $a = 2$,we say that a term is slow-decay (in space) if it is not controlled by

$$\frac{1}{1 + |y|^4}.$$

We can find an exact perturbation with our scheme if we remove such terms. Looking at (2.13) we observe that all the terms in the first two lines are slow-decay. For the moment we can assume $\dot\Lambda, \Lambda, \dot\xi, \xi$ bounded by some power of $\mu(t)$. Later in the argument we shall specify precise norms for these parameters. Firstly, we decompose

$$\begin{aligned}-\mu^{-3/2}5U(y)^4 H_\gamma(x,\xi) = &-\mu^{-3/2}5U(y)^4\theta_\gamma(x-\xi)\\ &+\mu^{-3/2}5U(y)^4 h_\gamma(x,\xi).\end{aligned}$$

Now, we select the solution $J_1[\dot\Lambda](x,t)$ to the problem

$$\partial_t J_1 = \Delta_x J_1 + \gamma J_1 + \left(\frac{\mu}{\mu_0}\right)^{\frac12}\dot\Lambda\left(\mu^{-1}2Z_4\left(\frac{x-\xi}{\mu}\right) + H_\gamma(x,\xi)\right) \quad \text{in } \Omega\times[t_0-1,\infty), \tag{2.14}$$

$$J_1(x,t) = 0 \quad \text{in } \partial\Omega\times[t_0-1,\infty),$$
$$J_1(x,t_0-1) = 0 \quad \text{on } \Omega,$$

and the solution $J_2(x,t)$ to

$$\partial_t J_2 = \Delta_x J_2 + \gamma J_2 - \left(\frac{\mu}{\mu_0}\right)^{\frac12}\left[\gamma\left(\mu^{-1}2Z_4(y) + \frac{\alpha_3}{|x-\xi|}\right) + \mu^{-2}5U(y)^4\theta_\gamma(\mu y)\right] \quad \text{in } \Omega\times[t_0,\infty), \tag{2.15}$$

$$J_2(x,t) = 0 \quad \text{on } \partial\Omega\times[t_0,\infty),$$
$$J_2(x,t_0) = 0 \quad \text{in } \Omega.$$

The choice of defining J_1 from the time t_0-1 as well as $\dot\Lambda(t)$ will become clear in section 8. We extend $\xi(t)=\xi(t_0)$ for $t\in[t_0-1,t_0)$. We define

$$u_2 := u_1 + \mu_0^{1/2} J[\dot\Lambda](x,t),$$

where

$$J[\dot\Lambda] := J_1[\dot\Lambda] + J_2.$$

The new error reads as

$$\begin{aligned}S[u_2] &= S[u_1] + (-\partial_t+\Delta_x)(\mu_0^{1/2}J(x,t)) + u_2^5 - u_1^5\\ &= S[u_1] + \mu_0^{1/2}\{-\partial_t J + \Delta_x J + \gamma J\} + u_2^5 - u_1^5.\end{aligned}$$

Inserting $S[u_1]$ given in (2.13) we get

$$\begin{aligned}S[u_2] =&\mu^{-3/2}\dot\xi\cdot\nabla_y U(y) + \mu^{1/2}\dot\xi\cdot\nabla_{x_2}H_\gamma(x,\xi(t)) + \mu^{-3/2}5U(y)h_\gamma(x,y)\\ &+\mu_0^{1/2}\left\{-\partial_t J_1 + \Delta_x J_1 + \gamma J_1 + \left(\frac{\mu}{\mu_0}\right)^{\frac12}\dot\Lambda\left(\mu^{-1}2Z_4\left(\frac{x-\xi}{\mu}\right) + H_\gamma(x,\xi)\right)\right\}\\ &+\mu_0^{1/2}\left\{-\partial_t J_2 + \Delta_x J_2 - \gamma J_2 - \left(\frac{\mu}{\mu_0}\right)^{\frac12}\left[\gamma\left(\mu^{-1}2Z_4(y) + \frac{\alpha_3}{|x-\xi|}\right) + \mu^{-2}5U(y)^4\theta_\gamma(\mu y)\right]\right\}\\ &+\mu^{-5/2}\left[\left(U(y) - \mu H_\gamma(x,\xi) + \mu\left(\frac{\mu_0}{\mu}\right)^{1/2} J[\dot\Lambda](x,t)\right)^5 - U(y)^5 + \mu 5U(y)^4 H_\gamma(x,\xi)\right].\end{aligned}$$

Using equations (2.14) and (2.15), the error associated to u_2 becomes

$$S[u_2] = \mu^{-3/2}\dot{\xi}\cdot\nabla_y U(y) + \mu^{1/2}\dot{\xi}\cdot\nabla_{x_2} H_\gamma(x,\xi) + \mu^{-3/2} 5U(y)^4 h_\gamma(x,\xi) \tag{2.16}$$
$$+ \mu^{-5/2}\left[\left(U(y) - \mu H_\gamma(x,\xi) + \mu\left(\frac{\mu_0}{\mu}\right)^{1/2} J[\dot{\Lambda}](x,t)\right)^5 - U(y)^5 + \mu 5U(y)^4 H_\gamma(x,\xi)\right].$$

2.3.1. *Choice of* γ. We observe that with the choice of J_2 we removed the singular term $|x-\xi|^{-1}$ from (2.13). At this point, the main error at $x = \xi(t)$ is given by the first order of the nonlinear term

$$\mu^{-3/2} 5U(0)^4 R_\gamma(\xi),$$

which, in general as size $\mu(t)^{-3/2}$. We realize that we can reduce this error by selecting γ such that $R_\gamma(0) = 0$. The existence of such number is given by the following lemma.

Lemma 2.1. *There exists a unique* $\gamma = \gamma^*(0) \in (0, \lambda_1)$ *such that* $R_{\gamma^*}(0) = 0$.

Proof. We consider the function $R_\gamma(0)$ as a function of γ. Lemma A.2 in [7] shows that

$$R_\gamma(0) : (0,\lambda_1) \to (-\infty, R_0(0))$$

is smooth in $(0,\lambda_1)$ and $\partial_\gamma R_\gamma(0) < 0$. Lemma A.1 shows that $R_\gamma(0) \to -\infty$ as $\gamma \to \lambda_1^-$. By the maximum principle $H_0(x,y) > 0$ for all $x, y \in \Omega$, hence we have $R_0(0) > 0$ and the intermediate value theorem gives the existence of a root

$$\gamma^*(0) := \max\{\gamma > 0 : \ R_\gamma(0) > 0\}.$$

Finally the monotonicity of $R_\gamma(0)$ implies the uniqueness of $\gamma^*(0)$. $\square$

Remark 2.1 (Regularity of $\gamma^*(x)$)**.** *Let* $R_\gamma(x) =: R(\gamma, x)$*. Since* $R(\gamma^*(x), x) = 0$ *and* $\partial_\gamma R(\gamma, x) < 0$ *for all* $x \in \Omega$*, the implicit function theorem implies that* $\gamma^*(x) \in C^1(\Omega)$ *with*

$$\nabla_x \gamma^*(x) = -\frac{\nabla_x R(\gamma, x)}{\partial_\gamma R(\gamma, x)}.$$

Remark 2.2 (radial case)**.** *We compute* $\gamma(0)$ *in case* $\Omega = B_1(0)$*. We look for a radial solution to*

$$\Delta H_\gamma + \gamma H_\gamma = \frac{\alpha_3}{|x|} \quad \text{in } B_1,$$
$$H_\gamma(x,0) = \frac{\alpha_3}{|x|} \quad \text{on } \partial B_1.$$

We define $l_0(|x|) := H_\gamma(x,0)$ *for a function* $l_0 : [0,1] \to \mathbb{R}$*. Then* l_0 *solves*

$$\partial_{rr} l_0 + \frac{2}{r}\partial_r l_0 + \gamma l_0 = \gamma\frac{\alpha_3}{r} \quad \text{in } [0,1],$$
$$l_0(1) = \alpha_3, \quad l_0(r) \quad \text{bounded at } r = 0.$$

We write $l_0(r) = \alpha_3 \frac{l(r)}{r}$*, where* $l(r)$ *solves*

$$\partial_{rr} l + \gamma l = \gamma \quad \text{in } [0,1],$$
$$l(1) = 1, \quad l(r) = O(r) \quad \text{for } r \to 0.$$

The solution to this problem is given by

$$l(r) = 1 - \cos(\sqrt{\gamma} r) + \cot(\sqrt{\gamma})\sin(\sqrt{\gamma} r),$$

and we conclude with

$$H_\gamma(r,0) = \alpha_3\left[\frac{1-\cos(\sqrt{\gamma}r)}{r} + \frac{\sin(\sqrt{\gamma}r)}{r\tan(\sqrt{\gamma})}\right].$$

In particular, for $r=0$ we find

$$R_\gamma(0) = H_\gamma(0,0) = \alpha_3\sqrt{\gamma}\cot(\sqrt{\gamma}).$$

Asking for $R_\gamma(0)=0$

$$\gamma = \left(\frac{\pi}{2}+k\pi\right)^2 \quad \textit{for} \quad k\in\mathbb{N},$$

and, recall that $\lambda_1(B_1)=\pi^2$, the unique value in $(0,\lambda_1)$ is

$$\gamma^* = \frac{\pi^2}{4},$$

as predicted in the analysis of Galaktionov and King [17].

For sake of simplicity we continue to use $\gamma=\gamma(0)$ to denote the selected number $\gamma^*(0)$. Since $R_\gamma(x)\in C^\infty(\Omega)$ we expand

$$R_\gamma(\xi) = R_\gamma(0) + \xi\cdot\nabla_x R_\gamma(0) + \frac{1}{2}\xi\cdot D^2_{xx}R_\gamma(\xi^*)\cdot\xi,$$

for some $\xi^*\in\overline{[0,\xi]}$. Assuming $|\xi(t)| = O(\mu(t))$ we conclude

$$\mu^{-3/2}5U(0)^4R_\gamma(\xi) = O\left(\mu^{-1/2}\right).$$

2.4. **Local improvement and final error computations.** In this section we make a further improvement and we obtain the final ansatz. We still need to remove from (2.13) the main order of the terms

$$\mu^{-3/2}\dot{\xi}\cdot\nabla_y U + \mu^{-3/2}5U(y)^4h_\gamma(x,\xi).$$

We define the final ansatz

$$u_3(x,t) := u_2(x,t) + \mu(t)^{-1/2}\phi_3\left(\frac{x-\xi(t)}{\mu(t)},t\right)\eta_{l(t)}\left(\left|\frac{x-\xi(t)}{\mu(t)}\right|\right).$$

where $\eta:[0,\infty)\to[0,1]$ denotes a smooth cut-off function such that $\eta(s)\equiv 1$ for $s<1$ and $\operatorname{supp}\eta\subset[0,2]$, and we define

$$\eta_{l(t)}(|y|) := \eta\left(\frac{|y|}{l(t)}\right), \qquad l(t) := \frac{1}{k\mu}, \tag{2.17}$$

where k is a constant such that $B_{\frac{2}{k}}(0)\subset\Omega$, to ensure that $\operatorname{supp}\eta_l(|\cdot|)\Subset\Omega$. Also we define the variable

$$z_3(x,t) := \frac{y(x,t)}{l(t)} = \frac{x-\xi(t)}{\mu(t)l(t)}.$$

We compute

$$\partial_t\left(\mu^{-1/2}\phi_3\eta_{l(t)}\right) = -\frac{\dot{\mu}}{2\mu}\mu^{-1/2}\phi_3\eta_{l(t)} + \mu^{-1/2}\eta_{l(t)}\left[\partial_t\phi_3 + \nabla_y\phi_3\cdot\left(-\frac{\dot{\mu}}{\mu}y - \frac{\dot{\xi}}{\mu}\right)\right] + \mu^{-1/2}\phi_3\partial_t\eta$$

$$\Delta_x\left(\mu^{-1/2}\phi_3\eta_l\right) = \mu^{-5/2}\Delta_y\phi_3 + 2\mu^{-3/2}\nabla_y\phi_3\cdot\frac{y}{|y|}\left(\frac{\eta'(|z_3|)}{\mu l}\right) + \mu^{-1/2}\phi_3\left(\frac{2}{|z_3|}\frac{\eta'(|z_3|)}{\mu^2l^2} + \frac{\eta''(|z_3|)}{\mu^2l^2}\right)$$

and define

$$\mathcal{N}_3(y,t) := \left(U(y)^5 - \mu H_\gamma(\mu y+\xi,\xi) + \mu\left(\frac{\mu_0}{\mu}\right)^{1/2} J(\mu y+\xi,t) + \phi_3(y,t)\eta_l\right)^5 - U(y)^5$$
$$- 5U(y)^4\left(-\mu H_\gamma(\mu y+\xi,\xi) + \mu\left(\frac{\mu_0}{\mu}\right)^{1/2} J + \phi_3\eta\right)$$

Thus, using (2.16),

$$\begin{aligned}
S[u_3] =& -\partial_t\left(\mu^{-1/2}\phi_3\eta_l\right) + \Delta_x\left(\mu^{-1/2}\phi_3\eta_l\right) + u_3^5 - u_2^5 + S[u_2]\\
=&\mu^{-3/2}\dot{\xi}\cdot\nabla_y U + \mu^{1/2}\dot{\xi}\cdot\nabla_x H_\gamma(x,\xi) + \mu^{-3/2}5U(y)^4 h_\gamma(x,\xi)\\
&+ \mu^{-5/2}\mathcal{N}_3(y,t) + 5U(y)^4\mu^{-3/2}\left(\frac{\mu_0}{\mu}\right)^{1/2} J(x,t) + \mu^{-5/2}\phi_3\eta_l 5U(y)^4\\
&- \left\{\left(-\frac{\dot{\mu}}{2\mu}\right)\mu^{-1/2}\phi_3\eta_{l(t)} + \mu^{-1/2}\eta_{l(t)}\left[\partial_t\phi_3 + \nabla_y\phi_3\cdot\left(-\frac{\dot{\mu}}{\mu}y - \frac{\dot{\xi}}{\mu}\right)\right] + \mu^{-1/2}\phi_3\partial_t\eta\right\}\\
&+ \mu^{-5/2}\eta_{l(t)}\Delta_y\phi_3 + 2\mu^{-3/2}\nabla_y\phi_3\cdot\frac{y}{|y|}\left(\frac{\eta'(|z_3|)}{\mu l}\right) + \mu^{-1/2}\phi_3\left(\frac{2}{|z_3|}\frac{\eta'(|z_3|)}{\mu^2 l^2} + \frac{\eta''(|z_3|)}{\mu^2 l^2}\right).
\end{aligned}$$

By Taylor expansion of $h_\gamma(x,\xi)$ centered at $x=\xi$ we have

$$h_\gamma(x,\xi) = R_\gamma(\xi) + \mu y\cdot\nabla_{x_1}h_\gamma(\xi,\xi) + \frac{1}{2}\mu^2 y^2 : D_{xx}h_\gamma(\bar{x},\xi) \tag{2.18}$$

for some $\bar{x}\in\overline{[\xi,x]}$. Now, we expand the first terms at $(\xi,\xi)=(0,0)$. By the Chain Rule we have $\nabla_{x_1}h_\gamma(x,x) = 2\nabla_x R_\gamma(x)$. Hence, we have

$$\nabla_{x_1}h_\gamma(\xi,\xi) = \frac{1}{2}\nabla_x R_\gamma(\xi) = \frac{1}{2}\nabla_x R_\gamma(0) + \frac{1}{2}\xi\cdot D_{xx}R_\gamma(\xi^{**}),$$

for some $\xi^{**}\in\overline{[0,\xi]}$. Furthermore, since $R_\gamma(0)=0$, we have

$$R_\gamma(\xi) = \xi\cdot\nabla_x R_\gamma(0) + \frac{1}{2}\xi^2 : D_{xx}R_\gamma(\xi^*)$$

for some $\xi^*\in\overline{[0,\xi]}$. Plugging these identities in (2.18) we obtain

$$\begin{aligned}
h_\gamma(x,\xi) =&\xi\cdot\nabla_x R_\gamma(0) + \frac{1}{2}\mu y\cdot\nabla_x R_\gamma(0)\\
&+ \frac{1}{2}\xi^2 : D_{xx}R_\gamma(\xi^*) + \frac{1}{2}\mu y\cdot D_{xx}R_\gamma(\xi^{**})\cdot\xi\\
&+ \frac{1}{2}\mu^2 y^2 : D_{xx}h_\gamma(\bar{x},\xi).
\end{aligned}\tag{2.19}$$

Let

$$\xi = \xi_0 + \xi_1.$$

Now, we assume the following decay for the parameters $\xi_1, \dot{\xi}_1, \Lambda, \dot{\Lambda}$:

$$\begin{aligned}
|\xi_1(t)| + |\dot{\xi}_1(t)| &\le C\mu(t)^{1+k},\\
|\Lambda(t)| &\le C\mu(t)^{l_0},\\
|\dot{\Lambda}(t)| &\le C\mu(t)^{l_1},
\end{aligned}$$

for some positive constants k, l_0, l_1 to be chosen. We write the full error

$$
\begin{aligned}
S[u_3] =& \mu^{-3/2}\nabla_y U(y)\cdot\Big[\dot\xi-\mu^{-1}\mu_0\dot\xi_0\Big]\eta_l\\
&+5U(y)^4\Big[\mu^{-3/2}h_\gamma(x,\xi)-\mu^{-5/2}\mu_0\Big(\frac12\mu_0 y\cdot\nabla_x R_\gamma(0)\Big)\Big]\eta_l\\
&+\Big[\mu^{-3/2}\nabla_y U(y)\cdot\dot\xi+5U(y)^4\mu^{-3/2}h_\gamma(x,\xi)\Big](1-\eta_l)\\
&+\mu^{1/2}\dot\xi\cdot\nabla_x H_\gamma(x,\xi)\\
&+\mu^{-5/2}\mathcal N_3(y,t)+5U(y)^4\mu^{-3/2}\Big(\frac{\mu_0}{\mu}\Big)^{1/2}J(x,t)\\
&-\Big\{\Big(-\frac{\dot\mu}{2\mu}\Big)\mu^{-1/2}\phi_3\eta_{l(t)}+\mu^{-1/2}\eta_{l(t)}\Big[\partial_t\phi_3+\nabla_y\phi_3\cdot\Big(-\frac{\dot\mu}{\mu}y-\frac{\dot\xi}{\mu}\Big)\Big]+\mu^{-1/2}\phi_3\partial_t\eta\Big\}\\
&+\mu^{-5/2}\eta_{l(t)}\Big[\Delta_y\phi_3+5U(y)^4\phi_3+\mathcal M[\mu_0,\xi_0]\Big]\\
&+2\mu^{-3/2}\nabla_y\phi_3\cdot\frac{y}{|y|}\Big(\frac{\eta'(|z_3|)}{\mu l}\Big)+\mu^{-1/2}\phi_3\Big(\frac{2}{|z_3|}\frac{\eta'(|z_3|)}{\mu^2l^2}+\frac{\eta''(|z_3|)}{\mu^2l^2}\Big).
\end{aligned}
$$

where

$$
\mathcal M[\mu_0,\xi_0]:=\mu_0\dot\xi_0\cdot\nabla_y U(y)-\frac52 U(y)^4\mu_0(\mu_0 y\cdot\nabla_x R_\gamma(0)) \tag{2.20}
$$

For any fixed $t>t_0$, we select $\phi_4(\cdot,t)$ as the bounded solution to the elliptic problem

$$
\Delta_y\phi_3(y,t)+5U(y)^4\phi_3(y,t)=-\mathcal M[\mu_0,\xi_0](y,t)\quad\text{in } \mathbb R^3, \tag{2.21}
$$

with the following orthogonality conditions on the right-hand side:

$$
\int_{\mathbb R^3}\mathcal M[\mu_0,\xi_0](y,t)Z_i(y)\,dy=0\quad\text{for}\quad t>t_0,\quad\text{and}\quad i=1,2,3,4. \tag{2.22}
$$

As we shall see, conditions (2.22) are essential to have ϕ_3 bounded in space (see (2.4)) and equivalent to choose $\xi_0(t)$. The condition corresponding to the index $i=4$ is satisfied by symmetry. When $i=1,2,3$ the orthogonality condition (2.22) is equivalent to

$$
\mu_0\dot\xi_{0,i}\Big(\int_{\mathbb R^3}|\partial_{y_i}U(y)|^2\,dy\Big)-\mu_0^2\Big(\int_{\mathbb R^3}5U(y)^4y_i\partial_{y_i}U(y)\,dy\Big)\frac12\partial_{x,i}R_\gamma(0)=0.
$$

Hence, we select $\xi_{0,i}$ such that

$$
\dot\xi_{0,i}(t)=\frac{\partial_{x,i}R_\gamma(0)\big(\int_{\mathbb R^3}5U(y)^4y_i\partial_{y_i}U(y)\,dy\big)}{2\Big(\int_{\mathbb R^3}|\partial_{y_i}U(y)|^2\,dy\Big)}\mu_0(t).
$$

With the condition $\lim_{t\to\infty}\xi_i(t)=0$ we get

$$
\xi_{0,i}(t)=\mathfrak c_i e^{-2\gamma t},\quad \mathfrak c_i=-\frac{\partial_{x,i}h_\gamma(0)\big(\int_{\mathbb R^3}5U(y)^4y_i\partial_{y_i}U(y)\,dy\big)}{4\gamma\Big(\int_{\mathbb R^3}|\partial_{y_i}U(y)|^2\,dy\Big)}. \tag{2.23}
$$

Also, we define $\mathfrak c:=(\mathfrak c_1,\mathfrak c_2,\mathfrak c_3)$.

Remark 2.3 (no local improvement in the radial case)**.** *In case $\Omega=B_1(0)$, searching $h_\gamma(r,0)$ solution to (2.12) in the radial form, we see that*

$$
\nabla_x R_\gamma(0)=2\nabla_{x_1}h_\gamma(0,0)=0,
$$

hence conditions (2.22) imply $\xi_0 = \underline{0}$, as expected. This implies that the local improvement ϕ_3, which in fact involves only non-zero modes, is null in the radial case.

With these choices for ϕ_3 and ξ_0 we conclude with the following expression of the error associated to the final ansatz u_3:

$$
\begin{aligned}
S[u_3] =&\mu^{-3/2}\nabla_y U(y)\cdot\Big[\dot\xi_1 + \Big(1-\mu^{-1}\mu_0\Big)\dot\xi_0\Big]\eta_l\\
&+5U(y)^4\Big[\mu^{-3/2}h_\gamma(x,\xi) - \mu^{-5/2}\mu_0\Big(\frac{1}{2}\mu_0 y\cdot\nabla_x R_\gamma(0)\Big)\Big]\eta_l\\
&+\Big[\mu^{-3/2}\nabla_y U(y)\cdot\dot\xi + 5U(y)^4\mu^{-3/2}h_\gamma(x,\xi)\Big](1-\eta_l)\\
&+\mu^{1/2}\dot\xi\cdot\nabla_x H_\gamma(x,\xi)\\
&+\mu^{-5/2}\mathcal{N}_3(y,t) + 5U(y)^4\mu^{-3/2}\Big(\frac{\mu_0}{\mu}\Big)^{1/2}J(x,t)\\
&-\Big\{\Big(-\frac{\dot\mu}{2\mu}\Big)\mu^{-1/2}\phi_3\eta_{l(t)} + \mu^{-1/2}\eta_{l(t)}\Big[\partial_t\phi_3 + \nabla_y\phi_3\cdot\Big(-\frac{\dot\mu}{\mu}y - \frac{\dot\xi}{\mu}\Big)\Big] + \mu^{-1/2}\phi_3\partial_t\eta\Big\}\\
&+2\mu^{-3/2}\nabla_y\phi_3\cdot\frac{y}{|y|}\Big(\frac{\eta'(|z_3|)}{\mu l}\Big) + \mu^{-1/2}\phi_3\Big(\frac{2}{|z_3|}\frac{\eta'(|z_3|)}{\mu^2 l^2} + \frac{\eta''(|z_3|)}{\mu^2 l^2}\Big).
\end{aligned}
$$

For later purpose, we split $S[u_3]$ in inner and outer error. At this stage, it is important to treat the terms involving directly $\dot\Lambda$ as part of the outer error, since, as we shall see, a priori these are the terms with less regularity. Let

$$S[u_3] = S_{\text{in}} + S_{\text{out}},$$

where

$$
\begin{aligned}
S_{\text{in}} :=&\mu^{-3/2}\Big(\frac{\mu_0}{\mu}\Big)^{1/2}5U(y)^4 J(x,t) + \mu^{-5/2}\mathcal{N}_3\\
&+\mu^{-3/2}\eta_l\Big(\dot\xi_1 + \Big(1-\mu^{-1}\mu_0\Big)\dot\xi_0\Big)\cdot\nabla_y U(y)\\
&+\mu^{-3/2}\eta_l 5U(y)^4\Big(h_\gamma(x,\xi) - \Big(\frac{\mu_0}{\mu}\Big)\Big(\frac{1}{2}\mu_0 y\cdot\nabla_x R_\gamma(0)\Big)\Big)
\end{aligned}
\tag{2.24}
$$

and

$$
\begin{aligned}
S_{\text{out}} :=&\mu^{-3/2}\Big[\nabla_y U(y)\cdot\dot\xi + 5U(y)^4 h_\gamma(x,\xi)\Big](1-\eta_l)\\
&+\mu^{1/2}\dot\xi\cdot\nabla_x H_\gamma(x,\xi)\\
&-\mu^{-1/2}\Big[(\gamma-\dot\Lambda)\eta_l(\phi_3 + 2y\cdot\nabla_y\phi_3) + \eta_l\Big(\partial_t\phi_3 - \mu^{-1}\dot\xi\cdot\nabla_y\phi_3\Big) + \phi_3\frac{\eta'(|z_3|)}{\mu l}\dot\xi\cdot\frac{z_3}{|z_3|}\Big]\\
&+2\mu^{-3/2}\nabla_y\phi_3\cdot\frac{y}{|y|}\Big(\frac{\eta'(|z_3|)}{\mu l}\Big) + \mu^{-1/2}\phi_3\Big(\frac{2}{|z_3|}\frac{\eta'(|z_3|)}{\mu^2 l^2} + \frac{\eta''(|z_3|)}{\mu^2 l^2}\Big).
\end{aligned}
\tag{2.25}
$$

2.4.1. *Size of S_{in}.* **??** We proceed with the estimate of S_{in}. Let

$$R(t) = \mu^{-\delta}, \tag{2.26}$$

for some $\delta > 0$. We need the following conditions on δ, l_0, l_1

$$\delta + l_1 < 1 \tag{2.27}$$

$$\delta \in \left(\frac{1-l_1}{2}, \frac{1+l_1}{6}\right) \tag{2.28}$$

$$l_1 \leq l_0, \tag{2.29}$$

$$k \geq -1 + 2\delta + l_1, \tag{2.30}$$

The condition (2.27) is used to get the estimate in the linear outer problem, and it is due to the fact that both the heat kernel p_t^Ω and the parameter $\mu_0(t)$ have an exponential decay for t large. To get the quadratic term $U^3\tilde{\phi}^2$ smaller than S_{in} in the inner problem we need the upper bound in (2.28). The lower bound is necessary to get a positive Hölder exponent in the regularity of $\dot{\Lambda}$. The last two conditions (2.29)-(2.30) ensure that the main term in S_{in} is given by the first term in (2.24). Thus, we fix the following values satisfying (2.27)-(2.28):

$$\delta = \frac{2}{9}, \quad l_1 = \frac{2}{3}. \tag{2.31}$$

Here and in what follows, we write $a \lesssim b$ if there exists a constant C, independent of t_0, such that $a \leq Cb$. If both the inequalities $a \lesssim b$ and $b \lesssim a$ hold we write $a \sim b$. Using (2.36) and (2.37) we estimate

$$\begin{aligned}\left|5U(y)^4\left(\frac{\mu_0}{\mu}\right)^{1/2}\mu^{-3/2}J(x,t)\right| &\lesssim \frac{\mu^{-3/2}}{1+|y|^4}\left(\mu^{l_1} + \frac{\mu}{1+|y|}\right),\\ &\lesssim \frac{\mu^{-3/2+l_1}}{1+|y|^4}\end{aligned}$$

and

$$\begin{aligned}|\mu^{-5/2}\mathcal{N}_3| &\lesssim \mu^{-1/2}U(y)^3(|H_\gamma(x,\xi)| + |J(x,t)|)^2\\ &\lesssim \frac{\mu^{-1/2}}{1+|y|^3}\left(\mu R + \mu^{l_1} + \frac{\mu}{1+|y|}\right)^2\\ &\lesssim \frac{\mu^{-1/2}}{1+|y|^4}\mu^{-\delta}(\mu^{1-\delta} + \mu^{l_1})^2\\ &\lesssim \frac{\mu^{-1/2-\delta+2\min\{1-\delta,l_1\}}}{1+|y|^4}\end{aligned}$$

Also,

$$\begin{aligned}|\mu^{-3/2}\eta_l\left(\dot{\xi}_1 + (1-\mu^{-1}\mu_0)\dot{\xi}_0\right)\cdot\nabla_y U| &\lesssim \eta_l\frac{\mu^{-3/2}}{1+|y|^4}R^2\left(|\dot{\xi}_1| + \mu^2\right)\\ &\lesssim \frac{\mu^{-3/2-2\delta+\min\{1+k,2\}}}{1+|y|^4}\end{aligned}$$

Now, we estimate the last term of S_{in} using expansion (2.19) and $\mu/\mu_0 = e^{2\Lambda}$ we get

$$|\mu^{-3/2}\eta_l U(y)^4\left(h_\gamma(x,\xi) - \left(\frac{\mu_0}{\mu}\right)\left(\frac{1}{2}\mu_0 y\cdot\nabla_x R_\gamma(0)\right)\right)| \lesssim \frac{\mu^{-3/2+\min\{1,l_0\}}}{1+|y|^4}.$$

Combining these estimates we get

$$|S_{\text{in}}| \lesssim \frac{1}{1+|y|^4}\Big[\mu^{-1/2-\delta+2\min\{1-\delta,l_1\}} + \mu^{-3/2-2\delta+\min\{1+k,2\}} + \mu^{-3/2+\min\{1,l_0,l_1\}}\Big]$$

and using the values (2.31) we get

$$\begin{aligned} |S_{\text{in}}| &\lesssim \frac{\mu^{-\frac{3}{2}+l_1}}{1+|y|^4} \\ &\lesssim \frac{\mu^{-5/6}}{1+|y|^4}. \end{aligned} \tag{2.32}$$

2.4.2. *Size of S_{out}.* For the first term in S_{out} we have

$$\begin{aligned} |(1-\eta_l)\nabla_y U\cdot\dot{\xi}| &\lesssim \mu^{3/2}(1-\eta_l) \\ |\mu^{-3/2}5U^4 h_\gamma(1-\eta_l)| &\lesssim \mu^{5/2}(1-\eta_l) \\ |\mu^{1/2}\dot{\xi}\cdot\nabla_{x_1}H_\gamma| &\lesssim \mu^{3/2} \end{aligned}$$

and using the estimate on $\phi_3, \nabla_y\phi_3, \partial_t\phi_3$ we get

$$|\mu^{-1/2}[(\gamma-\dot{\Lambda})\eta_l(\phi_3+2y\cdot\nabla_y\phi_3)+\eta_l\Big(\partial_t\phi_3-\mu^{-1}\dot{\xi}\cdot\nabla_y\phi_3\Big)+\phi_3\frac{\eta'(|z_3|)}{\mu l}\dot{\xi}\cdot\frac{z_3}{|z_3|}]| \lesssim \mu^{3/2}$$

Finally,

$$|2\mu^{-3/2}\nabla_y\phi_3\cdot\frac{y}{|y|}\left(\frac{\eta'(|z_3|)}{\mu l}\right)+\mu^{-1/2}\phi_3\left(\frac{2}{|z_3|}\frac{\eta'(|z_3|)}{\mu^2 l^2}+\frac{\eta''(|z_3|)}{\mu^2 l^2}\right)| \lesssim \mu^{3/2}.$$

We conclude that

$$|S_{\text{out}}| \lesssim \mu^{3/2}. \tag{2.33}$$

2.4.3. *Size of $S_{in}(1-\eta_R)$.* It remains to estimate the size of $S_{\text{in}}(1-\eta_R)$. We have

$$|(1-\eta_R)5U^4\mu^{-3/2}J(x,t)| \lesssim \frac{\mu^{-\frac{3}{2}+l_1+2\delta}}{1+|y|^2}(1-\eta_R)$$

Then,

$$\begin{aligned} |(1-\eta_R)\mu^{-5/2}\mathcal{N}_3| &\lesssim (1-\eta_R)\mu^{-1/2}\frac{1}{1+|y|^3}(|H_\gamma(x,\xi)|+|J(x,t)|)^2 \\ &\lesssim (1-\eta_R)\frac{\mu^{-1/2}R^{-1}}{1+|y|^2}\Big((\mu R)^2+\mu^{2l_1}\Big) \\ &\lesssim (1-\eta_R)\frac{1}{1+|y|^2}\Big[\mu^{3/2-\delta}+\mu^{-1/2+\delta+2l_1}\Big] \end{aligned}$$

Also,

$$\mu^{-3/2}\eta_l(\dot{\xi}_1+(1-\mu^{-1}\mu_0)\dot{\xi}_0)\cdot\nabla_y U(y)(1-\eta_R) \lesssim \frac{\mu^{\frac{1}{2}+\min\{0,k-\frac{1}{2}\}}}{1+|y|^2}(1-\eta_R)$$

and

$$\begin{aligned} |(1-\eta_R)\eta_l 5U(y)^4\mu^{-3/2}\Big(h_\gamma(x,\xi)-\Big(\frac{\mu_0}{\mu}\Big)\Big(\frac{1}{2}\mu_0 y\cdot\nabla_x R_\gamma(0)\Big)\Big)| &\lesssim \frac{R^{-2}\mu^{-1/2}}{1+|y|^2}(1-\eta_R) \\ &\lesssim \frac{\mu^{2\delta-\frac{1}{2}}}{1+|y|^2}(1-\eta_R) \end{aligned}$$

Combining these estimates we find

$$
\begin{aligned}
|S_{\text{in}}(1-\eta_R)| &\lesssim \mu^{-3/2+l_1+2\delta}\frac{1}{1+|y|^{2}}(1-\eta_R) \\
&\lesssim \mu^{-\frac{5}{6}+\frac{4}{9}}\frac{1}{1+|y|^{2}}(1-\eta_R)
\end{aligned} \tag{2.34}
$$

We conclude that

$$
|S_{\text{in}}(1-\eta_R)+S_{\text{out}}| \lesssim \mu^{-\frac{5}{6}+\frac{4}{9}}\frac{1}{1+|y|^{2}}(1-\eta_R)+\mu^{\frac{3}{2}}
$$

2.5. **Estimates of J_1, J_2 and ϕ_3.** The following lemma gives an estimate of $J_1[\dot\Lambda](x,t)$ in terms of $\dot\Lambda$. Observe that

$$
\lim_{t\to\infty}\left(\frac{\mu(t)}{\mu_0(t)}\right)^{1/2}\frac{\mu^2(t)-|x-\xi|}{\left(\mu^2+|x-\xi|^{2}\right)^{3/2}}+H_\gamma(x,\xi)=-\frac{1}{|x|}+H_\gamma(x,0).
$$

In order to control J_1 we will construct a supersolution using $\mathcal{J}$, that is the solution to

$$
\begin{aligned}
&\partial_t\mathcal{J}=\Delta_x\mathcal{J}+\gamma\mathcal{J}-\dot\Lambda(t)G_\gamma(x,0) \qquad \text{in } \Omega\times[t_0-1,\infty),\\
&\mathcal{J}(x,t)=0 \quad \text{on } \partial\Omega\times[t_0-1,\infty),\\
&\mathcal{J}(x,t_0-1)=0 \quad \text{in } \Omega.
\end{aligned} \tag{2.35}
$$

We define the L^∞-weighted space

$$
X_c:=\{f\in L^\infty(t_0-1,\infty):\|f\|_{\infty,c}<\infty\},
$$

where

$$
\|f\|_{\infty,c}:=\sup_{t>t_0-1}|f(t)\mu_0(t)^{-c}|\,.
$$

Lemma 2.2 (Estimate of J_1)**.** *Suppose $2\gamma l_1<\lambda_1-\gamma$ and*

$$
\|\dot\Lambda\|_{\infty,l_1}<\infty.
$$

Then we have

$$
\|J_1(\cdot,t)\|_{L^\infty(\Omega)}\lesssim \mu_0(t)^{l_1}\|\dot\Lambda\|_{\infty,l_1}, \tag{2.36}
$$

for $t\geq t_0$.

Since we have selected $l_1<1$ in (2.31), condition (1.6) guarantees that $2\gamma l_1<\lambda-\gamma$.

Proof. By parabolic comparison, it is enough to prove the bound for $\mathcal{J}$ defined as the solution to (2.35). Indeed, we have

$$
\left|\left(\frac{\mu}{\mu_0}\right)^{1/2}\dot\Lambda(t)\left(\frac{\mu^2-|x-\xi|^{2}}{\left(\mu^2+|x-\xi|^{2}\right)^{3/2}}+H_\gamma(x,\xi)\right)\right|\lesssim|\dot\Lambda(t)|\left|-\frac{1}{|x|}+H_\gamma(x,0)\right|.
$$

Consider

$$
\bar{\mathcal{J}}:=\|\dot\Lambda\|_{\infty,l_1}e^{-2\gamma l_1 t}f(x),
$$

where $f(x)$ solves

$$
\begin{aligned}
&-\Delta_x f(x)-\gamma(2l_1+1)f(x)=|G_\gamma(x,0)| \quad \text{in } \Omega,\\
&f(x)=0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

Then, $\bar{\mathcal{J}}$ satisfies

$$\begin{aligned}\partial_t \bar{\mathcal{J}} - \Delta \bar{\mathcal{J}} - \gamma \bar{\mathcal{J}} &= \|\dot{\Lambda}\|_{\infty,l_1} e^{-2\gamma l_1 t}[-\Delta_x f(x) - \gamma(2l_1+1)f(x)] \\ &\geq |\dot{\Lambda}(t)|\,|G_\gamma(x,0)|\,.\end{aligned}$$

Also, $\bar{\mathcal{J}}(x,t) = 0$ on $\partial\Omega \times [0,\infty)$ and $\bar{\mathcal{J}}(x,0) = \|\dot{\Lambda}\|_{\infty,l_1} f(x) \geq 0$ by the maximum principle since $\gamma(2l_1+1) < \lambda_1$. Thus, $\bar{\mathcal{J}}$ is a supersolution and for $t \in [t_0,\infty)$ we obtain

$$\begin{aligned}\|J_1(\cdot,t)\|_{L^\infty(\Omega)} &\lesssim \|\mathcal{J}(\cdot,t)\|_{L^\infty(\Omega)} \\ &\lesssim \|\dot{\Lambda}\|_{\infty,l_1} e^{-2\gamma l_1 t} \\ &\lesssim \|\dot{\Lambda}\|_{\infty,l_1} \mu_0(t)^{l_1}.\end{aligned}$$

□

Lemma 2.3 (Estimate of J_2)**.** *Let $J_2(x,t)$ be the unique solution to the problem*

$$\begin{aligned}\partial_t J_2 =& \Delta_x J_2 + \gamma J_2 - \left(\frac{\mu}{\mu_0}\right)^{\frac{1}{2}} \left[\gamma\left(\mu^{-1} 2 Z_4\left(\frac{x-\xi}{\mu}\right) + \frac{\alpha_3}{|x-\xi|}\right)\right. \\ &\left. + \mu^{-2} 5 U\left(\frac{x-\xi}{\mu}\right)^4 \theta_\gamma(x-\xi)\right] \quad \text{in } \Omega \times [t_0,\infty), \\ J_2(x,t) =& 0 \quad \text{on } \partial\Omega \times [t_0,\infty), \\ J_2(x,t_0) =& 0 \quad \text{in } \Omega.\end{aligned}$$

Suppose that $3\gamma < \lambda_1$. Then, there exists t_0 large such that

$$|J_2(x,t)| \lesssim \mu \frac{1}{(1+|y|^{1-\varepsilon})}, \tag{2.37}$$

for any $\varepsilon > 0$ and for all $(x,t) \in \Omega \times [t_0,\infty)$ where $y = \frac{x-\xi}{\mu}$.

Proof. Firstly, we observe that

$$\left|\frac{1-|y|^2}{(1+|y|^2)^{3/2}} + \frac{1}{|y|}\right| \lesssim \frac{1}{|y|\left(1+|y|^{2-\varepsilon}\right)}.$$

Also, by Taylor expanding the function θ_γ in (2.11) near the origin, we see that

$$\begin{aligned}|\mu^{-2} 5 U(y)^4 \theta_\gamma(\mu y)| &\lesssim \frac{\mu^{-1}}{1+|y|^4}|y| \\ &\lesssim \frac{\mu^{-1}}{|y|\left(1+|y|^{2-\epsilon}\right)},\end{aligned}$$

where $\epsilon > 0$ can be taken arbitrarily small. Thus, by parabolic comparison, it is enough to find $\bar{u}$ such that

$$\begin{aligned}&\partial_t \bar{u} \geq \Delta \bar{u} + \gamma \bar{u} + \mu^{-1} \frac{1}{|y|\,(1+|y|^{2-\varepsilon})} \quad \text{in } \Omega \times [t_0,\infty), \\ &\bar{u}(x,t) \geq 0 \quad \text{on } \partial\Omega \times [t_0,\infty), \\ &\bar{u}(x,t_0) \geq 0 \quad \text{in } \Omega.\end{aligned} \tag{2.38}$$

Let $\bar{v} := \mu^{-1}\bar{u}(x,t)$. We have

$$\begin{aligned}
\partial_t \bar{v} &= \partial_t\Big(\mu^{-1}u\Big) \\
&= \mu^{-1}\Big(u_t - \frac{\dot{\mu}}{\mu}u\Big) \\
&= \mu^{-1}\Big(u_t + (2\gamma - 2\dot{\Lambda})u\Big).
\end{aligned}$$

Thus, the problem for $\bar{v}$ becomes

$$\begin{aligned}
&\partial_t \bar{v} \geq \Delta_x \bar{v} + (3\gamma - 2\dot{\Lambda})\bar{v} + \frac{\mu^{-2}}{|y|\left(1+|y|^{2-\varepsilon}\right)} \quad \text{in } \Omega \times [t_0,\infty), \\
&\bar{v} \geq 0 \quad \text{on } \partial\Omega \times [t_0,\infty), \\
&\bar{v}(x,t_0) \geq 0 \quad \text{in } \Omega.
\end{aligned} \tag{2.39}$$

We look for $\bar{v}$ of the form

$$\bar{v}(x,t) = v_0\Big(\frac{x-\xi}{\mu}\Big)\eta\Big(\frac{x-\xi}{C_0}\Big) + v_1(x,t).$$

We need

$$\begin{aligned}
\partial_t v_1 - \Delta_x v_1 - (3\gamma - 2\dot{\Lambda})v_1 \geq & \eta\Big[-\partial_t v_0 + \mu^{-2}\Delta_y v_0 + (3\gamma - 2\dot{\Lambda})v_0 \\
& + \frac{\mu^{-2}}{|y|\,(1+|y|^{2-\varepsilon})}\Big] \\
& + (1-\eta)\frac{\mu^{-2}}{|y|\,(1+|y|^{2-\varepsilon})} + (\Delta_x\eta - \partial_t\eta)v_0 \\
& + 2\mu^{-1}\nabla_x\eta\cdot\nabla_y v_0,
\end{aligned} \tag{2.40}$$

and $v_1 \geq 0$ on $\partial\Omega \times [t_0,\infty)$ and $v_0(x,t_0) \geq 0$ in Ω. Without loss of generality let $\Omega \subset B_1$. Consider the positive radial solution $v_0(y,t)$ to

$$\begin{aligned}
&\Delta_y v_0 + 2\frac{1}{|y|\left(1+|y|^{2-\varepsilon}\right)} = 0 \quad \text{on } B_{\frac{1}{\mu(t)}} \\
&v_0 \equiv 0 \quad \text{on } \partial B_{\frac{1}{\mu(t)}},
\end{aligned}$$

given by the formula of variation of parameters

$$v_0(y) = 2\int_{|y|}^{\frac{1}{\mu(t)}} \frac{1}{\rho^2}\int_0^{\rho} \frac{s}{1+s^{2-\varepsilon}}\,ds\,d\rho$$

From this formula we obtain the following estimates in $(x,t) \in \Omega \times [t_0,\infty)$:

$$\begin{aligned}
|v_0(|y|,t)| &\lesssim \frac{1}{1+|y|^{1-\varepsilon}}, \\
|\partial_t v_0(y,t)| &\lesssim \frac{1}{1+|y|^{1-\varepsilon}}.
\end{aligned}$$

Thus, if $|x-\xi| < C_0$, for C_0 sufficiently small, then

$$\begin{aligned}-\partial_t v_0+&\mu^{-2}\Delta_y v_0 + (3\gamma-\dot\Lambda)v_0 + \frac{\mu^{-2}}{|y|\,(1+|y|^{2-\varepsilon})}\\ &= -\frac{\mu^{-2}}{|y|\,(1+|y|^{2-\varepsilon})} + O\left(\frac{1}{1+|y|^{1-\varepsilon}}\right) \le 0.\end{aligned}$$

Then, let v_1 be the solution to

$$\begin{aligned}\partial_t v_1 - \Delta_x v_1 - (3\gamma-\dot\Lambda)v_1 =& (1-\eta)\frac{\mu^{-2}}{|y|\,(1+|y|^{2-\varepsilon})} + (\Delta_x\eta - \partial_t\eta)v_0\\ &+ 2\mu^{-1}\nabla_x\eta\cdot\nabla_y v_0 \quad \text{in } \Omega\times[t_0,\infty),\end{aligned}$$

with

$$\begin{aligned}&v_1 = 0 \quad \text{on } \partial\Omega\times[t_0,\infty),\\ &v_1(x,t_0) = 0 \quad \text{in } \Omega.\end{aligned}$$

In the right-hand side we have

$$\begin{aligned}(1-\eta)\frac{\mu^{-2}}{|y|\,(1+|y|^{2-\varepsilon})} &\lesssim \mu^{1-\varepsilon},\\ |(\Delta_x - \partial_t\eta)v_0| &\lesssim \mu^{1-\varepsilon},\\ |2\mu^{-1}\nabla_x\eta\cdot\nabla_y v_0| &\lesssim \mu^{1-\varepsilon}.\end{aligned}$$

Since $3\gamma - 2\dot\Lambda(t) < \lambda_1$ provided that t_0 is sufficiently large, the comparison principle applies and we get $|v_0| \lesssim \mu^{1-\varepsilon}$. Thus, inequality (2.40) is verified. Also, we have $\bar v = v_1 \ge 0$ on $\partial\Omega\times[t_0,\infty)$ and $\psi_2(x,t_0) = \eta v_0(x,t_0) \ge 0$. Thus $\bar v$ satisfies (2.39) and hence $\bar u = \mu\bar v$ satisfies (2.38). Then, by parabolic comparison we get $|J_2| \lesssim |\bar u|$, hence we obtain (2.37). ☐

Lemma 2.4 (Estimate on ϕ_3)**.** *There exists a bounded solution to the problem*

$$\Delta_y\phi_3 + 5U(y)^4\phi_3(y,t) = -\mathcal{M}[\xi_0,\mu_0](y,t) \quad \textit{in } \mathbb{R}^3, \tag{2.41}$$

under the orthogonality conditions (2.22) on $\mathcal{M}[\xi_0,\mu_0](y,t)$. We have the following estimates on ϕ_3 and its derivatives:

$$|\phi_3(y,t)| + (1+|y|)|\nabla_y\phi_3(y,t)| + \partial_t\phi_3 \lesssim \mu^2(t)f(y,t), \tag{2.42}$$

where f is a smooth bounded function.

Proof. From the explicit form of the function $\mathcal{M}$ given in (2.20) we estimate its size as

$$|\mathcal{M}[\mu_0,\xi_0](y,t)| \le \mu^2\frac{1}{1+|y|^2}.$$

Let $\{\vartheta_m\}_{m=0}^{\infty}$ the orthonormal basis of $L^2(S^2)$ made up of spherical harmonics, namely the eigenfunctions of the problem

$$\Delta_{S^2}\vartheta_m + \lambda_m\vartheta_m = 0 \quad \text{in } S^2,$$

where $0 = \lambda_0 < \lambda_1 = \lambda_2 = \lambda_3 = 2 < \lambda_4 \le \dots$. We decompose

$$\mathcal{M}(y,t) = \sum_{i=1}^{\infty}\mathcal{M}_i(r,t)\vartheta_i(y/r), \quad \text{where} \quad r := |y|, \quad \mathcal{M}_i(r,t) := \int_{S^2}\mathcal{M}(r\theta,t)\vartheta_i(\theta)\,d\theta.$$

From (2.20) we see that $\mathcal{M}_i = 0$ for $i \geq 4$. Also, we decompose ϕ_3 as

$$\phi_3(y,t) = \sum_{i=1}^{\infty} \phi_{3,i}(r,t)\vartheta_i(y/r), \quad \text{where} \quad \phi_{3,i}(r,t) := \int_{S^2} \phi_3(r\theta,t)\vartheta_i(\theta)\,d\theta,$$

and, by (2.41), also ϕ_3 satisfies $\phi_{3,i}(y,t) = 0$ for $i \geq 4$. Similarly, we define

$$z_i(r) := \int_{S^2} Z_i(r\theta)\vartheta_i(\theta)\,d\theta.$$

The formula of variation of constants gives

$$\phi_{3,i}(r,t) = z_i(r)\int_0^r \frac{1}{\rho^2 z(\rho)^2}\mathcal{I}_i(\rho,t)\,d\rho,$$

where

$$\mathcal{I}_i(\rho,t) := \int_0^\rho \mathcal{M}_i(s,t)z_i(s)s^2\,ds.$$

Since

$$|\mathcal{M}_i(r,t)| \lesssim \frac{\mu^2(t)}{1+r^2}$$

and

$$|z_i(r)| \lesssim \frac{r}{(1+r^3)}$$

we deduce that

$$|\mathcal{I}_i(\rho,t)| \lesssim \mu^2(t)\rho^4 \quad \text{as} \quad \rho \to 0.$$

Also, by the orthogonality conditions (2.22) we have

$$\begin{aligned} |\mathcal{I}_i(\rho)| &= \left|\int_\rho^\infty \mathcal{M}_i(s,t)z_i(s)s^2\,ds\right| \\ &\lesssim \mu^2(t)\frac{1}{\rho} \quad \text{as} \quad \rho \to \infty. \end{aligned}$$

With these estimates we conclude

$$\begin{aligned} |\phi_3(r,t)| &\lesssim \frac{r}{1+r^3}\int_0^r \frac{(1+\rho^2)^3}{\rho^4}|\mathcal{I}(\rho,t)|\,d\rho \\ &\lesssim \mu^2(t)\frac{r}{1+r^3}\int_0^r \frac{(1+\rho^2)^3}{\rho^4}\frac{\rho^4}{1+\rho^5}\,d\rho \\ &\lesssim \mu^2(t). \end{aligned}$$

Similarly, taking the space and time derivatives of equation (2.41), we deduce the bounds on $\nabla_y\phi_3$ and $\partial_t\phi_3$. $\square$

We conclude this section by summarizing the key estimates on the size of the error $S[u_3]$.

Lemma 2.5. *Let $3\gamma < \lambda_1$, $\mu = \mu_0 e^{2\Lambda}$ and $\xi = \xi_0 + \xi_1$, where μ_0, ξ_0 are given by (2.5) and (2.23) respectively. Assume*

$$\begin{aligned} &|\Lambda(t)| \leq \mu_0(t)^{l_0}, \quad |\dot\Lambda(t)| \leq \mu_0(t)^{l_1}, \\ &R(t) = \mu^{-\delta}, \quad |\dot\xi_1(t)| \leq \mu_0^{1+k}, \end{aligned}$$

for positive constant δ, l_0, l_1, k satisfying (2.27),(2.28), (2.29) and (2.30). Then, setting $x = \mu y + \xi$, we have, for t_0 sufficiently large, the following estimate on the error function $S[u_3]$ holds:

$$S[u_3](y,t) = S_{in}(y,t)\eta_{R(t)}(|y|\,) + S_{in}(y,t)(1-\eta_{R(t)}(y)) + S_{out}(y,t),$$

where

$$|S_{in}(y,t)\eta_{R(t)}| \lesssim \mu^{-\frac{3}{2}+l_1}\frac{1}{1+|y|^4}, \tag{2.43}$$

$$|S_{out}(y,t)| \lesssim \mu^{3/2}, \tag{2.44}$$

$$|S_{in}(y,t)(1-\eta_{R(t)})| \lesssim \mu^{-3/2+l_1+2\delta}\frac{1}{1+|y|^2}. \tag{2.45}$$

The proofs of (2.43), (2.44) and (2.45) are given in sections 2.4.3, 2.4.2 and 2.4.3 respectively.

3. The Inner-outer scheme

We recall that our final purpose is to find an unbounded global in time solution u to (2.1) of the form

$$u = u_3 + \tilde{\phi}, \tag{3.1}$$

for a small perturbation $\tilde{\phi}$. The latter is constructed by means of the inner-gluing method. This consists in looking for a perturbation of the form

$$\tilde{\phi}(x,t) = \mu_0(t)^{1/2}\psi(x,t) + \eta_{R(t)}(|y|\,)\mu(t)^{-1/2}\phi(y,t), \tag{3.2}$$

where

$$\eta_R(|y|\,) = \left(\frac{|y|}{R(t)}\right), \quad y := y(x,t) := \frac{x-\xi(t)}{\mu(t)},$$

and $\eta(s)$ is a cut-off function with supp $(\eta) \subset [0,2]$ and $\eta \equiv 1$ in $[0,1]$. We have already chosen $R = R(t)$ in (2.26). In terms of $\tilde{\phi}$ the equation reads as

$$\begin{aligned}
0 = S[u] &= -\partial_t u + \Delta_x u + u^5 \\
&= \left(-\partial_t u_3 + \Delta_x u_3 + u_3^5\right) - \partial_t\tilde{\phi} + \Delta_x\tilde{\phi} + (u_3+\tilde{\phi})^4 - u_3^5 \\
&= S[u_3] - \partial_t\tilde{\phi} + \Delta_x\tilde{\phi} + 5u_3^4\tilde{\phi} + \mathcal{N}(u_3,\tilde{\phi})
\end{aligned}$$

where

$$\mathcal{N}(u_3,\tilde{\phi}) := (u_3+\tilde{\phi})^5 - u_3^5 - 5u_3^4\tilde{\phi}. \tag{3.3}$$

Hence the problem for $\tilde{\phi}$ is

$$\begin{aligned}
&\partial_t\tilde{\phi} = \Delta_x\tilde{\phi} + 5u_3^4\tilde{\phi} + S[u_3] + \mathcal{N}(u_3,\tilde{\phi}) \quad \text{in } \Omega\times[t_0,\infty), \\
&\tilde{\phi} = -u_3 \quad \text{on } \partial\Omega\times[t_0,\infty).
\end{aligned}$$

Now, the main idea is to split the problem for $\tilde{\phi}$ in a system for (ψ,ϕ), localizing the inner regime. We divide the error in

$$S[u_3] = S_{\text{in}}\eta_R + S_{\text{in}}(1-\eta_R) + S_{\text{out}},$$

where $S_{\rm in}, S_{\rm out}$ are defined in (2.24) and (2.25) respectively. Considering $\tilde\phi$ as in (3.2) we compute

$$
\begin{aligned}
\partial_t\tilde\phi =& \frac{\dot\mu_0}{2\mu_0}\mu_0^{1/2}\psi + \mu_0^{1/2}\partial_t\psi + \mu^{-1/2}\phi\partial_t\eta\Big(\frac{y(x,t)}{R(t)}\Big) - \frac{\dot\mu}{2\mu}\mu^{-1/2}\phi\eta_R \\
&+ \mu^{-1/2}(\partial_t\phi + \nabla_y\phi\cdot\partial_t y(x,t))\eta_R \\
=& -\gamma\mu_0^{1/2}\psi + \mu_0^{1/2}\partial_t\psi + \mu^{-1/2}\phi\Big[\nabla_z\eta\Big(\frac{y}{R}\Big)\cdot\Big(-\frac{\dot R}{R}\frac{y}{R} - \frac{\dot\mu}{\mu}\frac{y}{R} - \frac{\dot\xi}{\mu R}\Big)\Big] \\
&+ \Big(-\frac{\dot\mu}{2\mu}\Big)\mu^{-1/2}\phi\eta_R + \mu^{-1/2}\eta_R\Big(\partial_t\phi + \nabla_y\phi\cdot\Big(-\frac{\dot\mu}{\mu}y - \frac{\dot\xi}{\mu}\Big)\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_x\tilde\phi =& \mu_0^{1/2}\Delta_x\psi + \mu^{-1/2}\Delta_x\Big(\phi(y(x,t),t)\eta_{R(t)}(y(x,t))\Big) \\
=& \mu_0^{1/2}\Delta_x\psi + \mu^{-5/2}\eta_R(y)\Delta_y\phi(y,t) + \mu^{-1/2}\phi\Big(\frac{2}{|z|}\frac{\eta'(|z|)}{\mu^2R^2} + \frac{\eta''(|z|)}{\mu^2R^2}\Big) \\
&+ 2\mu^{-1/2}\frac{1}{\mu}\nabla_y\phi(y,t)\cdot\frac{z}{|z|}\frac{\eta'(|z|)}{\mu R},
\end{aligned}
$$

where $z := \frac{y}{R}$. We split

$$
5u_3^4\tilde\phi = 5u_3^4\mu_0^{1/2}\psi\eta_R + 5u_3^4\mu_0^{1/2}\psi(1-\eta_R) + 5u_3^4\mu^{-1/2}\phi\eta_R.
$$

Hence, the full equation reads as

$$
\begin{aligned}
-\gamma\mu_0^{1/2}\psi &+ \mu_0^{1/2}\partial_t\psi + \mu^{-1/2}\phi\partial_t\eta_R + \eta_R\mu^{-1/2}\partial_t\phi \\
&+ \eta_R\Big\{(\gamma-\dot\Lambda)\mu^{-1/2}(\phi + 2\nabla_y\phi\cdot y) - \mu^{-1/2}\nabla_y\phi\cdot\Big(\frac{\dot\xi}{\mu}\Big)\Big\} \\
=& \mu_0^{1/2}\Delta_x\psi + \mu^{-5/2}\eta_R\Delta_y\phi + \mu^{-1/2}\phi\Big(\frac{2}{|z|}\frac{\eta'(|z|)}{\mu^2R^2} + \frac{\eta''(|z|)}{\mu^2R^2}\Big) \\
&+ 2\mu^{-1/2}\frac{1}{\mu}\nabla_y\phi\cdot\frac{z}{|z|}\frac{\eta'(|z|)}{\mu R} \\
&+ 5u_3^4\mu_0^{1/2}\psi\eta_R + 5u_3^4\mu_0^{1/2}\psi(1-\eta_R) + 5u_3^4\mu^{-1/2}\phi\eta_R \\
&+ S_{\rm in}\eta_R + S_{\rm in}(1-\eta_R) + S_{\rm out} + \mathcal{N}(u_3,\tilde\phi)(1-\eta_R) + \mathcal{N}(u_3,\tilde\phi)\eta_R.
\end{aligned}
$$

We divide the full problem in a system. Firstly, we look for a solution ψ to

$$
\begin{aligned}
\mu_0^{1/2}\partial_t\psi =& \mu_0^{1/2}\Delta_x\psi + \gamma\mu_0^{1/2}\psi + 5u_3^4\mu_0^{1/2}\psi(1-\eta_R) + \mu^{-1/2}\phi\partial_t\eta_R \\
&+ \eta_R\Big\{(\gamma-\dot\Lambda)\mu^{-1/2}(\phi + 2\nabla_y\phi\cdot y) - \mu^{-1/2}\nabla_y\phi\cdot\Big(\frac{\dot\xi}{\mu}\Big)\Big\} \\
&+ \mu^{-1/2}\phi\Big(\frac{2}{|z|}\frac{\eta'(|z|)}{\mu^2R^2} + \frac{\eta''(|z|)}{\mu^2R^2}\Big) + 2\mu^{-1/2}\frac{1}{\mu}\nabla_y\phi\cdot\frac{z}{|z|}\frac{\eta'(|z|)}{\mu R} \\
&+ S_{\rm in}(1-\eta_R) + S_{\rm out} + \mathcal{N}(u_3,\tilde\phi)(1-\eta_R), \qquad \text{in } \Omega\times[t_0,\infty) \\
\psi(x,t) =& -\mu_0^{-1/2}u_3(x,t) \qquad \text{on } \partial\Omega\times[t_0,\infty).
\end{aligned}
$$

Thus, after dividing by $\mu_0^{1/2}$, ψ solves the **outer problem**

$$
\begin{aligned}
\partial_t\psi =&\Delta_x\psi + \gamma\psi + 5u_3^4\psi(1-\eta_R) + \mu^{-1}\left(\frac{\mu}{\mu_0}\right)^{1/2}\phi\partial_t\eta_R \qquad (3.4)\\
&+\mu^{-1}\left(\frac{\mu}{\mu_0}\right)^{1/2}\eta_R\left\{(\gamma-\dot\Lambda)(\phi+2\nabla_y\phi\cdot y)-\nabla_y\phi\cdot\left(\frac{\dot\xi}{\mu}\right)\right\}\\
&\mu^{-1}\left(\frac{\mu}{\mu_0}\right)^{1/2}\left(\phi\left(\frac{2}{|z|}\frac{\eta'(|z|)}{\mu^2R^2}+\frac{\eta''(|z|)}{\mu^2R^2}\right)+2\frac{\nabla_y\phi}{\mu}\cdot\frac{z}{|z|}\frac{\eta'(|z|)}{\mu R}\right)\\
&+\mu_0^{-1/2}S_{\text{in}}(1-\eta_R)+\mu_0^{-1/2}S_{\text{out}}+\mu_0^{-1/2}\mathcal{N}(u_3,\tilde\phi)(1-\eta_R), \qquad \text{in } \Omega\times[t_0,\infty)\\
\psi(x,t) =& -\mu_0^{-1/2}u_3(x,t) \qquad \text{on } \partial\Omega\times[t_0,\infty),
\end{aligned}
$$

Then, ϕ has to solve the problem

$$
\mu^{-1/2}\partial_t\phi = \mu^{-5/2}\Delta_y\phi + 5u_3^4\mu^{-1/2}\phi + 5u_3^4\mu_0^{1/2}\psi + S_{\text{in}} + \mathcal{N}(u_3,\tilde\phi) \qquad \text{in } B_{2R}(0)\times[t_0,\infty).
$$

Equivalently, multiplying by $\mu^{5/2}$, ϕ solves

$$
\begin{aligned}
\mu^2\partial_t\phi =&\Delta_y\phi + 5U^4\phi + 5U^4\left(\frac{\mu_0}{\mu}\right)^{1/2}\mu\psi(\mu y+\xi,t) + B_0[\phi+\mu\psi]\,(\mu y+\xi,t) \qquad (3.5)\\
&+\mu^{5/2}S_{\text{in}}(\mu y+\xi,t)+\mathcal{N}(\mu^{1/2}u_3,\mu^{1/2}\tilde\phi)(\mu y+\xi,t) \quad \text{in } B_{2R}(0)\times[t_0,\infty),
\end{aligned}
$$

where B_0 is the linear operator

$$
B_0[f] := 5\left[\left(U-\mu H_\gamma+\mu J[\dot\Lambda]+\mu^{-1/2}\phi_3(y,t)\eta_3\right)^4 - U^4\right]f, \qquad (3.6)
$$

3.0.1. *General strategy for solving the inner-outer system.* We now describe the method we use to solve the system (3.4)-(3.5).
Firstly, for fixed parameters $\Lambda,\dot\Lambda,\xi,\dot\xi$ and inner function ϕ in suitable weighted spaces, we solve problem (3.4) in $\psi=\psi[\Lambda,\dot\Lambda,\xi,\dot\xi,\phi]$. This is done in Section 4.
We insert such ψ in the inner problem. At this point we need to find $\Lambda,\dot\Lambda,\xi,\dot\xi$ and ϕ.
We make the change of variable $t(\tau)$ defined by the ODE

$$
\begin{aligned}
&\frac{dt(\tau)}{d\tau} = \mu^2(t(\tau))\\
&t(\tau_0) = t_0,
\end{aligned}
$$

that explicitly gives

$$
\begin{aligned}
\tau-\tau_0 &= \int_{t_0}^{t}\frac{1}{\mu(s)^2}\,ds\\
&= \int_{t_0}^{t}\frac{1}{\mu_0(s)^2}(1+o(1))\,ds\\
&= \frac{1}{4\gamma}\mu_0(t)^{-2}(1+o(1)).
\end{aligned}
$$

Expressing equation (3.5) in the new variables (y,τ) we get the **inner problem**

$$
\partial_\tau\phi = \Delta_y\phi + 5U^4\phi + H[\phi,\psi,\mu,\dot\Lambda,\xi,\dot\xi](y,\tau) \quad \text{in } B_{2R}\times[\tau_0,\infty), \qquad (3.7)
$$

where

$$H[\phi,\psi,\mu,\dot\mu,\xi,\dot\xi](y,\tau) := 5U(y)^4\mu\left(\frac{\mu_0}{\mu}\right)^{1/2}\psi(\mu y+\xi,t(\tau)) \tag{3.8}$$
$$+ B_0[\phi+\mu\psi]\,(\mu y+\xi,t(\tau)) + \mu^{5/2}S_{\rm in}(\mu y+\xi,t(\tau))$$
$$+ \mathcal{N}(\mu^{1/2}u_3,\mu^{1/2}\tilde\phi)(\mu y+\xi,t(\tau)).$$

Let Z_0 be the positive radially symmetric bounded eigenfunction associated to the only negative eigenvalue λ_0 of the problem

$$-\Delta_y\phi - 5U(y)^4\phi = \lambda_0\phi \quad \text{for} \quad \phi \quad \text{in } L^\infty(\mathbb{R}^3).$$

It is known that λ_0 is simple and

$$Z_0(y) \sim \frac{e^{-\sqrt{|\lambda_0|}|y|}}{|y|} \quad \text{as} \quad |y|\to\infty.$$

We solve (3.7) with a multiple of $Z_0(y)$ as initial datum, namely

$$\phi(\tau_0,y) = e_0 Z_0(y) \quad \text{in } B_{2R}, \tag{3.9}$$

for some constant $e_0 = e_0[H]$ to be found. Formally, this initial datum (3.9) allows ϕ to remain a small perturbation of the ansatz along its trajectory. Indeed, multiplying equation (3.7) by Z_0 and integrating by parts we obtain

$$\mu^2\partial_t p(t) + \lambda_0 p(t) = q(t),$$

where

$$p(t) = \int_{\mathbb{R}^3}\phi(y,t)Z_0(y)\,dy, \quad q(t) = \int_{\mathbb{R}^3} H(y,t)Z_0\,dy.$$

The general solution $p(t)$ is given by

$$p(t) = e^{|\lambda_0|\int_0^t \mu(s)^{-2}\,ds}\left(p(t_0) + \int_{t_0}^t \mu(s)^{-2}q(s)e^{-|\lambda_0|\mu(s)^{-2}}\,ds\right)$$

This shows that in order to get a decaying solution $p(t)$ (and hence $\phi(y,t)$), the following initial conditions should hold:

$$p(t_0) = \int_{\mathbb{R}^3}\phi(y,t_0)Z_0(y)\,dy = -\int_{t_0}^\infty \mu(s)^{-2}q(s)e^{-|\lambda_0|\mu(s)^{-2}}\,ds.$$

This argument formally suggests that, to avoid the instability caused by Z_0, the small initial value $\phi(y,t_0)$ should lie on a manifold locally described as a translation of the hyperplane orthogonal to $Z_0(y)$.

Another important observation is that, in order to solve the problem (3.7)-(3.9) we need to constrain the right-hand side H to be orthogonal to $\{Z_i\}_{i=1}^4$. Namely we need

$$\int_{B_{2R}} H(y,\tau)Z_i(y)\,dy = 0 \quad \text{for} \quad t\in[\tau_0,\infty) \quad \text{and} \quad i=1,2,3,4. \tag{3.10}$$

Indeed, the elliptic kernel generated by $\{Z_i\}_{i=1}^4$ is a subset of the kernel of the parabolic operator

$$\mu^2\partial_t\phi = \Delta_y\phi + 5U(y)^4\phi.$$

Hence we expect to have solvability of the inhomogeneous problem (3.7) with suitable space-time decay if the orthogonality conditions (3.10) are satisfied.

As we shall see in Section 5, the Condition (3.10) with index $i=4$ is equivalent to a nonlocal problem in Λ, for fixed ϕ,ξ. This operator turns out to be similar to a Caputo

$\frac{1}{2}$-derivative, and we develop a crucial invertibility theory in Section 8. In section 5 we solve (3.10) by fixed-point argument and hence we find Λ, ξ.
A main ingredient of the full proof is the linear theory for the inner problem developed in [5] and adapted in dimension 3 in [8].

3.0.2. *Statement of the linear estimate for the inner problem.* We recall the result on the linear theory in dimension 3, proved in [8]. To state the result we decompose a general function $h(\cdot,\tau) \in L^2(B_{2R})$ for any $\tau \in [\tau_0, \infty)$ in spherical modes. Let $\{\vartheta_m\}_{m=0}^{\infty}$ the orthonormal basis of $L^2(S^2)$ made up of spherical harmonics, namely the eigenfunctions of the problem

$$\Delta_{S^2}\vartheta_m + \lambda_m \vartheta_m = 0 \quad \text{in } S^2,$$

where $0 = \lambda_0 < \lambda_1 = \lambda_2 = \lambda_3 = 2 < \lambda_4 \le \dots$. We decompose h into the form

$$h(y,\tau) = \sum_{m=1}^{\infty} h_m(|y|,\tau)\vartheta_m\Big(\frac{y}{|y|}\Big), \quad h_j(|y|,\tau) = \int_{S^2} h(r\theta,\tau)\vartheta_m(\theta)\,d\theta.$$

Furthermore, we write $h = h^0 + h^1 + h^\perp$ where

$$h^0 = h_0(|y|,\tau), \quad h^1 = \sum_{m=1}^{3} h_m(|y|,\tau)\vartheta_m\Big(\frac{y}{|y|}\Big) \quad h^\perp = \sum_{m=4}^{\infty} h_m(|y|,\tau)\vartheta_m\Big(\frac{y}{|y|}\Big).$$

We solve the inner problem (3.13) for functions h in the space $X_{\nu,2+a}$ defined by

$$X_{\nu,2+a} := \{h \in L^\infty(B_{2R}\times[\tau_0,\infty)) : \|h\|_{\nu,2+a} < \infty\}, \tag{3.11}$$

whereν, a are positive constants and

$$\|h\|_{\nu,2+a} := \sup_{\tau>\tau_0, y\in B_{2R}} \tau^\nu (1+|y|^{2+a})|h(y,\tau)|.$$

We look for ϕ in the space of functions

$$X_* := \{\phi(y,t) \in L^\infty(\Omega\times[t_0,\infty)) : \|\phi\|_* < \infty\}$$

where

$$\|\phi\|_* := \sup_{\tau>\tau_0, y\in B_{2R}} \tau^\nu R(\tau)^{-3}\log^{-1}(R(\tau))(1+|y|^4)[|\phi(y,\tau)| + (1+|y|)|\nabla_y\phi(y,\tau)|] \tag{3.12}$$

$$+ \sup_{\tau, y\in B_{2R}, \tau_1,\tau_2\in[\tau,\tau+1]} \tau^\nu R(\tau)^{-3}\log^{-1}(R(\tau))(1+|y|^4)\frac{|\phi(y,\tau_1)-\phi(y,\tau_2)|}{|\tau_1-\tau_2|^{\frac{1}{2}+\varepsilon}}$$

$$+ \sup_{\tau>\tau_0, y\in B_{2R}, \tau_1,\tau_2\in[\tau,\tau+1]} \tau^\nu R(\tau)^{-3}\log^{-1}(R)(1+|y|^5)\frac{|\nabla_y\phi(y,\tau_1)-\nabla_y\phi(y,\tau_2)|}{|\tau_1-\tau_2|^{\frac{1}{2}+\varepsilon}}$$

Since in our problem h as in (3.7) decays as $\mu^{1+l_1}(1+|y|^4)^{-1} = \tau^{-1}(1+|y|^4)^{-1}$ we fix $\nu = \frac{1+l_1}{2}$.

Proposition 3.1. *Let ν and a be positive numbers. Then for all sufficiently large $R>0$ and any $h(y,\tau)$ with $\|h\|_{\nu,2+a} < \infty$ such that*

$$\int_{B_{2R}} h(y,\tau)Z_j(y)\,dy = 0 \quad \textit{in } [\tau_0,\infty), \quad \textit{for} \quad i = 1,2,3,4,$$

there exists $\phi[h]$ and $e_0[h]$ which solves

$$\partial_\tau \phi = \Delta_y \phi + 5U(y)^4\phi + h(y,\tau) \quad \text{in } B_{2R}\times(\tau_0,\infty) \tag{3.13}$$
$$\phi(y,\tau_0) = e_0 Z_0(y) \quad \text{in } B_{2R}.$$

They define linear operators of h that satisfy the estimates

$$|\phi(y,\tau)| + (1+|y|)|\nabla_y\phi(y,\tau)| \lesssim \tau^{-\nu}\Big[\frac{R^2\theta_0(R,a)}{1+|y|^3}\|h_0\|_{\nu,2+a} \tag{3.14}$$
$$+ \frac{R^3\theta_1(R,a)}{1+|y|^4}\|h_1\|_{\nu,2+a} + \frac{1}{1+|y|^a}\|h^\perp\|_{\nu,2+a}\Big],$$

and

$$|e_0[h]| \lesssim \|h\|_{\nu,2+a},$$

where

$$\theta_R^0(R,a) := \begin{cases} 1 & \text{if } a>2,\\ \log R & \text{if } a=2,\\ R^{2-a} & \text{if } a<2, \end{cases}, \quad \theta_R^1(R,a) := \begin{cases} 1 & \text{if } a>1,\\ \log R & \text{if } a=1,\\ R^{1-a} & \text{if } a<1. \end{cases}$$

In order to make the system for (ϕ,ψ) weakly coupled, ϕ needs to be small at distance $y\sim R$. For this reason we need to take $a>1$ in the statement of Proposition 3.1. This makes clear why we need to improve ansatz u_1 to u_3 in Section 2.
We apply the estimate above with constants $a=2$ and $\nu=1$ in the form

$$|\phi| + (1+|y|)|\nabla_y\phi(y,\tau)| \lesssim \|h\|_{\nu,4}\tau^{-\nu}\frac{R^3\ln(R)}{1+|y|^4} \tag{3.15}$$

and observe that

$$\|h\|_{\nu,4}\tau^{-\nu}\frac{R^3\ln(R)}{1+|y|^4} \lesssim \|h\|_{\nu,4}\begin{cases}\tau^{-\nu}R^{-1}\log R & \text{if } |y|\sim R,\\ \tau^{-\nu}R^3 & \text{if } |y|\sim 0.\end{cases}$$

Observe that from (3.15) and parabolic estimates we also have the bound on the Hölder seminorms in (3.12), thus

$$\|\phi\|_* \le C\|h\|_{\nu,4} \tag{3.16}$$

3.0.3. *Spaces for the parameters.* We introduce weighted Hölder spaces for the parameters. Let

$$X_{\sharp,a,b,\sigma} := \{\Lambda\in C(t_0,\infty) : \|\Lambda\|_{\sharp,a,b,\sigma} < \infty\},$$

where

$$\|\Lambda\|_{\sharp,a,b,\sigma} := \sup_{t>t_0}\Big\{\mu(t)^{-a}\|\Lambda\|_{\infty,[t,t+1]}\Big\} + \sup_{t>t_0}\Big\{\mu(t)^{-b}[\Lambda]_{0,\sigma,[t,t+1]}\Big\},$$

and

$$\|\Lambda\|_{\infty,[t,t+1]} = \sup_{s\in[t,t+1]}|\Lambda(s)|,$$
$$[\Lambda]_{0,\sigma,[t,t+1]} := \sup_{\substack{s_1,s_2\in[t,t+1]\\ s_1\neq s_2}}\frac{|\Lambda(s_1)-\Lambda(s_2)|}{|s_1-s_2|^\sigma}.$$

We also define $X_{\sharp,c,\sigma} := X_{\sharp,c,c,\sigma}$ and

$$\|h\|_{\sharp,c,\sigma} := \sup_{t>t_0}\mu(t)^{-c}\Big[\|h\|_{\infty,[t,t+1]} + [h]_{0,\sigma,[t,t+1]}\Big].$$

We look for Λ such that

$$\|\Lambda\|_{\sharp,l_0,\delta_0,\frac{1}{2}+\varepsilon} + \|\dot{\Lambda}\|_{\sharp,l_1,\delta_1,\varepsilon} < \mathfrak{b}_1, \tag{3.17}$$

for some positive constant $\varepsilon, \delta_0, \delta_1, l_0, l_1$ for $i = 1, 2$ to be chosen. We consider parameters ξ_1 such that

$$\|\xi_1\|_{\sharp,1+k,\frac{1}{2}+\varepsilon} + \|\dot{\xi}_1\|_{\sharp,1+k,\varepsilon} < \mathfrak{b}_2, \tag{3.18}$$

for some $k > 0$. The positive constants $\mathfrak{b}_i$, for $i = 1, 2$ will be selected as small as needed.

Choice of constants. Here we select the constants

$$l_0, l_1, \delta, \delta_0, \delta_1, \varepsilon, k, \beta,$$

which are sufficient to find the perturbation $\tilde{\phi}$ in (3.1) by the inner-outer gluing scheme. We choose

- $l_0 = l_1 + \delta = \frac{8}{9}$;
- $\delta_0 = l_1 + \delta - (1-\delta)(2\varepsilon) = \frac{8}{9} - \varepsilon_1$, where $\varepsilon_1 = \Big(1 - \frac{2}{3}\Big)2\varepsilon$;
- $l_1 = \frac{2}{3}$;
- $\delta_1 = l_1 + \delta - (1-\delta)(1+2\varepsilon) = \frac{1}{9} - \varepsilon_2$, where $\varepsilon_2 := (1-\delta)2\varepsilon$;
- $k = l_0$;
- $\varepsilon = \frac{1}{10}$;
- $\beta = \frac{1}{2} + l_1 + \delta$.

This choices are dictated by the following constraints, based on the estimate of the approximate solution, the characterization of the orthogonality conditions (6.1) and the estimates in Proposition 6.1:

- from the outer problem we know that $|\psi(x,t)| \lesssim \mu^{l_1} R^{-1}$. From the nonlocal equation (5.2) and $l_1 \leq l_0$ we need $|\Lambda| \lesssim |\psi(0,t)|$. This leads to the choice of l_0;
- in (2.31) we chose $\delta = 2/9$ and $l_1 = 2/3$ so that
$$\delta + l_1 < 1, \quad \text{and} \quad \delta \in \Big(\frac{1-l_1}{2}, \frac{1+l_1}{6}\Big).$$
- From estimate (6.4), equation (5.2) and the bound on the ε-Hölder seminorm of ψ we get
$$[\Lambda]_{\frac{1}{2}+\varepsilon,[t,t+1]} \lesssim [\psi(0,\tau)]_{\varepsilon,[t,t+1]} \lesssim \Big(\mu^{l_1} R^{-1}\Big)\frac{1}{(\mu R)^{2\varepsilon}} = \mu^{\delta_0},$$
which gives δ_0;
- similarly, from (4.13) the Hölder estimate on the outer solution gives
$$[\psi(0,\tau)]_{\frac{1}{2}+\varepsilon,[t,t+1]} \lesssim \mu^{l_1+\delta}(\mu R)^{-(1+2\varepsilon)}$$
and by equation (5.2) and estimate (6.5) we need $[\dot{\Lambda}]_{\frac{1}{2}+\varepsilon,[t,t+1]} \lesssim [\psi]_{\varepsilon,[t,t+1]}$. This leads to δ_1;
- From (5.5) we need $|\xi_1| + |\dot{\xi}_1| \lesssim \mu^{1+l_0}$, does the choice of k, which is consistent with (2.30).
- the constant ε is chosen small enough so that δ_1 is positive ($\varepsilon < 1/6$ is sufficient).

4. Solving the outer problem

We devote this section to solve the outer problem (3.4)

$$\begin{aligned}
&\partial_t\psi = \Delta_x\psi + \gamma\psi + V\psi + f[\psi,\phi,\Lambda,\dot\Lambda,\xi,\dot\xi](x,t), \quad \text{in } \Omega\times[t_0,\infty)\\
&\psi(x,t) = -\mu_0^{-1/2}u_3(x,t) \qquad \text{on } \partial\Omega\times[t_0,\infty).\\
&\psi(x,t_0) = \psi_0(x) \quad \text{in } \Omega,
\end{aligned}$$

where $\psi_0(x)$ is any suitable small initial condition,

$$\begin{aligned}
f(x,t) =&\mu^{-1}\left(\frac{\mu}{\mu_0}\right)^{1/2}\phi\partial_t\eta_R\\
&+\mu^{-1}\left(\frac{\mu}{\mu_0}\right)^{1/2}\eta_R\left\{(\gamma-\dot\Lambda)(\phi+2\nabla_y\phi\cdot y)-\nabla_y\phi\cdot\left(\frac{\dot\xi}{\mu}\right)\right\}\\
&\mu^{-1}\left(\frac{\mu}{\mu_0}\right)^{1/2}\left(\phi\left(\frac{2}{|z|}\frac{\eta'(|z|)}{\mu^2R^2}+\frac{\eta''(|z|)}{\mu^2R^2}\right)+2\frac{\nabla_y\phi}{\mu}\cdot\frac{z}{|z|}\frac{\eta'(|z|)}{\mu R}\right)\\
&+\mu_0^{-1/2}S_{\text{in}}(1-\eta_R)+\mu_0^{-1/2}S_{\text{out}}+\mu_0^{-1/2}\mathcal{N}(u_3,\tilde\phi)(1-\eta_R)
\end{aligned}$$

and

$$V(x,t) = 5u_3^4(1-\eta_R).$$

Observe that

$$|V(x,t)| = |5u_3^4(1-\eta_R)| \lesssim \frac{\mu^{-2}}{1+|y|^2}R^{-2}. \tag{4.1}$$

Let

$$\psi_1(x,t) := \mu_0(t)^{1/2}\psi(x,t).$$

Then, the problem for ψ_1 becomes

$$\begin{aligned}
&\partial_t\psi_1 = \Delta_x\psi_1 + V\psi_1 + F[\psi,\phi,\Lambda,\dot\Lambda,\xi,\dot\xi](x,t) \quad \text{in } \Omega\times[t_0,\infty), && (4.2)\\
&\psi_1(x,t) = g(x,t) \quad \text{on } \Omega\times[t_0,\infty),\\
&\psi_1(x,t_0) = \psi_1(x) \quad \text{in } \Omega
\end{aligned}$$

where

$$\begin{aligned}
&F(x,t) := \mu_0(t)^{1/2}f(x,t)\\
&g(x,t) := -u_3(x,t)\\
&\psi_1(x,t_0) := \mu_0(t_0)^{1/2}\psi_0(x).
\end{aligned}$$

In particular, in the proof of Proposition 4.1 we prove that for $\alpha>0$ sufficiently small

$$|F(x,t)| \lesssim \mu^{1/2+l_1+\delta-2\alpha}\left(\frac{\mu^{-2}}{1+|y|^{2+\alpha}}\right), \tag{4.3}$$

$$|g(x,t)| \lesssim \mu^{\frac{5}{2}}. \tag{4.4}$$

Thus, we firstly consider the linear version of (4.2). Assume

$$|F(x,t)| \le M\frac{\mu^{\beta-2}}{1+|y|^{\alpha+2}},$$

and define $\|F\|_{\beta-2,\alpha+2}$ for some $\beta,\alpha>0$ as the best constant $M>0$ for the inequality above.

Lemma 4.1. *Let F such that $\|F\|_{\beta-2,\alpha+2} < \infty$ for some constants β and $\alpha < 1$. Furthermore, assume that $\|e^{as}g(s)\|_{L^\infty(\partial\Omega\times(t_0,\infty))} < \infty$ for some $a > 0$ and $\|h\|_{L^\infty(\Omega)} < \infty$. Let $\psi_1[F,g,h]$ be the unique solution to*

$$\begin{aligned}
&\partial_t\psi_1 = \Delta\psi_1 + V\psi_1 + F(x,t) \quad \text{in } \Omega\times[t_0,\infty), \qquad (4.5)\\
&\psi_1(x,t) = g(x,t) \quad \text{on } \partial\Omega\times[t_0,\infty),\\
&\psi_1(x,t_0) = h(x) \quad \text{in } \Omega.
\end{aligned}$$

Then, for $\beta < 3/2$ and $b \in (0,\lambda_1)$ and $\tilde a \in (0,\min\{a,\lambda_1-\varepsilon\})$ for arbitrary $\varepsilon > 0$, we have

$$|\psi_1(x,t)| \lesssim \|F\|_{\beta-2,\alpha+2}\frac{\mu^\beta}{1+|y|^\alpha} + e^{-b(t-t_0)}\|h\|_{L^\infty(\Omega)} + e^{-\tilde a(t-t_0)}\|e^{as}g\|_{L^\infty(\partial\Omega\times(t_0,\infty)} \tag{4.6}$$

for all $x = \mu y + \xi \in \Omega$ and $t > t_0$. Furthermore, the following local estimate on the gradient holds:

$$|\nabla_x\psi_1(x,t)| \lesssim \|F\|_{\beta-2,\alpha+2}\frac{\mu^{\beta-1}}{1+|y|^{\alpha+1}} \quad \text{for} \quad |y| < R. \tag{4.7}$$

Also, in the same region, one has

$$[\psi_1]_{1+2\varepsilon,\frac12+\varepsilon}\Big[(\mu R)^{2+\alpha}\Big]^{\frac{1+2\varepsilon}{2}} + [\psi_1]_{2\varepsilon,\varepsilon}\Big[(\mu R)^{2+\alpha}\Big]^{\varepsilon} \lesssim \|F\|_{\beta-2,\alpha+2}\mu^\beta. \tag{4.8}$$

Proof. To prove (4.6) it is enough to find a supersolution to the problem

$$\begin{aligned}
&\partial_t\psi_2 = \Delta\psi_2 + \|F\|_{\beta-2,\alpha+2}\frac{\mu^{\beta-2}}{1+|y|^{2+\alpha}} \quad \text{in } \Omega\times[t_0,\infty),\\
&\psi_2 = g \quad \text{on } \partial\Omega\times[t_0,\infty),\\
&\psi_2 = h \quad \text{in } \Omega.
\end{aligned}$$

We use the notation $\psi_2 = \psi_2[F,g,h]$. Indeed, suppose that $\bar\psi_2$ is a supersolution to this problem. By (4.1) we have

$$|V\bar\psi_2| \lesssim \frac{\mu^{\beta-2}}{1+|y|^{2+\alpha}}R(t_0)^{-2},$$

and hence $\|V\bar\psi_2\|_{\beta-2,2+\alpha} < R(t_0)^{-2}$ for t_0 sufficiently large. Thus, we find that a large multiple of $\bar\psi_2$ works as supersolution of (4.5). Firstly, let $F, g \equiv 0$ and consider $\psi_2[0,0,h]$. Let $v_0(x)$ be the solution to

$$\begin{aligned}
-\Delta_x v_0 - bv_0 &= 0 \quad \text{in } \Omega,\\
v_0 &= 1 \quad \text{on } \partial\Omega,
\end{aligned}$$

for $b \in (0,\lambda_1)$ and define

$$\bar\psi_2 = \|h\|_\infty e^{-b(t-t_0)}v_0(x).$$

We claim that $\bar\psi_2$ is a supersolution for $\psi_2[0,0,h]$. Indeed, we have

$$\begin{aligned}
&\partial_t\bar\psi_2 - \Delta_x\bar\psi_2 = \|h\|_\infty e^{-b(t-t_0)}(-bv_0 - \Delta_x v_0) = 0 \quad \text{in } \Omega\times[t_0,\infty),\\
&\bar\psi_2(x,t) = \|h\|_\infty e^{-b(t-t_0)} \geq 0 \quad \text{on } \partial\Omega\times[t_0,\infty),\\
&\bar\psi_2(x,t_0) = \|h\|_\infty \geq h(x) \quad \text{in } \Omega,
\end{aligned}$$

where the last inequality is a consequence of the maximum principle applied to v_0. Secondly, we look for a supersolution to $\psi_2[0, g, 0]$. Let $v_1(x)$ the solution to

$$
\begin{aligned}
&-\Delta_x v_1 - \tilde{a} v_1 = 0 \quad \text{in } \Omega, \\
&v_1(x) = 1 \quad \text{on } \partial\Omega,
\end{aligned}
$$

where $\tilde{a} \in (0, \min\{a, \lambda_1 - \varepsilon\})$ and consider

$$
\bar{\psi}_2 = \|e^{as} g(s)\|_{L^\infty(\partial\Omega \times (t_0,\infty))} e^{-\tilde{a}(t-t_0)} v_1(x).
$$

We verify that

$$
\begin{aligned}
&\partial_t \bar{\psi}_2 - \Delta \bar{\psi}_2 = \|e^{as} g\|_\infty e^{-\tilde{a}(t-t_0)} (-\tilde{a} v_1 - \Delta v_1) = 0 \quad \text{in } \Omega \times [t_0, \infty), \\
&\bar{\psi}_2(x,t) = \|e^{as} g\|_\infty e^{-\tilde{a}(t-t_0)} \geq g(x,t) \quad \text{on } \partial\Omega \times [t_0, \infty), \\
&\bar{\psi}_2(x,t_0) = \|e^{as} g(s)\|_{L^\infty(\partial\Omega \times (t_0,\infty))} v_1(x) \geq 0 \quad \text{in } \Omega,
\end{aligned}
$$

where we use $\tilde{a} \leq a$ to get the second inequality and $\tilde{a} < \lambda_1$ to get the third one again by the maximum principle. It remains to find a supersolution for $\psi_2[F, 0, 0]$. Let $\psi_2[F, 0, 0] = e^{-c(t-t_0)} \psi_3$, where $c = 2\gamma\beta$ so that

$$
\partial_t \psi_3 = \Delta_x \psi_3 + c\psi_3 + \frac{\mu^{-2}}{1 + |y|^{2+\alpha}}.
$$

We find a bounded $\bar{\psi}_3$ supersolution in case $c < \lambda_1$, that is $3\gamma < \lambda_1$. Consider

$$
\bar{\psi}_3 = \psi_0\Big(\frac{x-\xi}{\mu}\Big) \eta\Big(\frac{x-\xi}{d}\Big) + \psi_1(x,t).
$$

We need

$$
\begin{aligned}
\partial_t \psi_1 - \Delta_x \psi_1 - c\psi \geq &\eta \left[-\partial_t \psi_0 + \mu^{-2} \Delta_y \psi_0 + c\psi_0 + \frac{\mu^{-2}}{1 + |y|^{2+\alpha}} \right] \\
&+ (1-\eta) \frac{\mu^{-2}}{1 + |y|^{2+\alpha}} + (\Delta_x \eta - \partial_t \eta)\psi_0 + 2\mu^{-1} \nabla_x \eta \cdot \nabla_y \psi_0,
\end{aligned} \tag{4.9}
$$

with $\bar{\psi}_3(x,t) \geq 0$ on $\partial\Omega \times [t_0, \infty)$ and initial datum $\bar{\psi}_3(x, t_0) \geq 0$. Suppose without loss of generality that $\Omega \subset B_1$ and take ψ_0 as the solution to

$$
\begin{aligned}
&\Delta_y \psi_0 = -\frac{2}{1 + |y|^{2+\alpha}} \quad \text{in } B_{\mu^{-1}} \\
&\psi_0 = 0 \quad \text{on } \partial B_{\mu^{-1}}.
\end{aligned}
$$

From the variation of parameters formula

$$
\psi_0(|y|) = 2 \int_{|y|}^{\mu^{-1}} \frac{1}{\rho^2} \int_0^\rho \frac{s^2}{1 + s^{2+\alpha}} \, ds \, d\rho,
$$

we find

$$
\begin{aligned}
|\psi_0| &\lesssim \frac{1}{1 + |y|^\alpha} \\
|\partial_t \psi_0| &= \partial_t \Big(\frac{1}{\mu}\Big) \mu^2 \int_0^{\mu^{-1}} \frac{s^2}{1 + s^{2+\alpha}} \, ds \lesssim \frac{1}{1 + |y|^\alpha},
\end{aligned}
$$

and, if $|x-\xi| < d$ for d fixed sufficiently small, we obtain

$$-\partial_t \psi_0 + \mu^{-2}\Delta_y \psi_0 + c\psi_0 + \frac{\mu^{-2}}{1+|y|^{2+\alpha}} = -\frac{\mu^{-2}}{1+|y|^{2+\alpha}} + O\Big(\frac{1}{1+|y|^{\alpha}}\Big) \leq 0.$$

Now, we take ψ_1 as the solution to

$$\begin{aligned}
&\partial_t \psi_1 - \Delta_x \psi_1 - c\psi_1 = (1-\eta)\frac{\mu^{-2}}{1+|y|^{2+\alpha}} + (\Delta_x \eta - \partial_t \eta)\psi_0 + 2\mu^{-1}\nabla_x \eta \cdot \nabla_y \psi_0 \\
&\psi_1 = 0 \quad \text{on } \partial\Omega \times [t_0, \infty), \\
&\psi_1(x, t_0) = 0 \quad \text{in } \Omega.
\end{aligned}$$

We estimate the right-hand side by

$$\begin{aligned}
&(1-\eta)\frac{\mu^{-2}}{1+|y|^{2+\alpha}} \lesssim \mu^{\alpha}, \\
&|(\Delta_x \eta - \partial_t \eta)\psi_0| \lesssim \mu^{\alpha}, \\
&|2\mu^{-1}\nabla_x \eta \cdot \nabla_y \psi_0| \lesssim \mu^{\alpha}.
\end{aligned}$$

Hence, by comparison principle using $c < \lambda_1$ we obtain a solution $|\bar{\psi}_3| \lesssim \mu^{\alpha}$. Thus, inequality (4.9) is satisfied. Also $\bar{\psi}_3 = 0$ on $\partial\Omega \times [t_0, \infty)$ and $\psi_3(x, t_0) = \eta\psi_0(x, t_0) \geq 0$. We conclude that $\bar{\psi}_3$ is a supersolution and the bound (4.6) is proven. Now, we prove the gradient estimate (4.7). Let

$$\psi_1(x,t) =: \tilde{\psi}\Big(\frac{x-\xi}{\mu}, \tau(t)\Big)$$

where $\dot{\tau}(t) = \mu(t)^{-2}$, that gives $\tau(t) \sim \mu^{-2}$. We can take $\tau(t_0) = 2$. Then, the equation for $\tilde{\psi}$ is

$$\partial_\tau \tilde{\psi} = \Delta_z \tilde{\psi} - \mu^2 \partial_t y \cdot \nabla_z \tilde{\psi} + \mu^2 V(\mu y + \xi, \tau(t))\tilde{\psi} + \mu^2 F(\mu y + \xi, \tau(t)).$$

Suppose that $|y| < \delta\mu^{-1}$ for δ sufficiently small. We have

$$\begin{aligned}
\mu^2 F(\mu y + \xi, \tau(t)) &\lesssim \mu^{\beta}\frac{\|F\|_{\beta-2,\alpha+2}}{1+|y|^{2+\alpha}} \\
\mu^2 V(\mu y + \xi, \tau(t)) &\lesssim \frac{R^{-2}}{1+|y|^{2}}.
\end{aligned}$$

We proved the L^∞-bound

$$\|\psi\|_{\beta,\alpha} \lesssim \|F\|_{\beta-2,\alpha+2},$$

whenever $\|F\|_{\beta-2,\alpha+2} < \infty$. We apply standard local parabolic estimates for the gradient: let $\sigma \in (0,1)$ and $\tau_1 \geq \tau(t_0) + 1$, then

$$\begin{aligned}
|\nabla_y \tilde{\psi}(\tau_1, \cdot)|_{\sigma, B_1(0)} + \|\nabla_y \tilde{\psi}(\tau_1, \cdot)\|_{L^\infty(B_1(0))} &\lesssim \|\tilde{\psi}\|_{L^\infty(B_2(0))\times(\tau_1-1,\tau_1)} + \|\tilde{f}\|_{L^\infty(B_2(0)\times(\tau_1-1,\tau_1))} \\
&\lesssim \mu(t(\tau_1-1))^{\beta}\|f\|_{\beta-2,\alpha+2} \\
&\lesssim \mu(t(\tau_1))^{\beta}\|f\|_{\beta-2,\alpha+2}.
\end{aligned}$$

In the original variables, for any $t \geq t_0 + 1$ we find

$$(R\mu)^{1+\sigma}[\nabla_x \psi(\cdot,t)]_{\eta, B_R(\xi)} + R\mu\|\nabla_x \psi(\cdot,t)\|_{L^\infty(B_R(\xi))} \lesssim \mu^{\beta}\|f\|_{\beta-2,\alpha+2}. \tag{4.10}$$

By analogue parabolic estimates using $\|\nabla_x \psi_0\|_\infty < \infty$ we can extend estimate (4.10) up to $t = t_0$, thus the proof of (4.7) is completed. Now we prove estimate (4.8). We consider the Hölder seminorm

$$[\psi_1] = \sup_{\substack{x_1 \neq x_2 \in \Omega, \\ t_1 \neq t_2 \in [t, t+1]}} \frac{|\psi_1(x_1, t_1) - \psi_1(x_2, t_2)|}{\left(|x_1 - x_2|^2 + |t_1 - t_2|\right)^{\frac{1+2\varepsilon}{2}}}.$$

We perform the change of variable

$$\psi_1(x, t) = \tilde{\psi}(z, \tau),$$

where $z := x - \xi/(R\mu)$ and τ satisfies

$$\frac{d\tau}{dt} = \frac{1}{(\mu_0(t)R(t))^{2+\alpha}},$$

that is

$$\begin{aligned} \tau - \tau_0 &= \int_{t_0}^{\infty} \frac{ds}{(\mu_0(t)R(t))^{2+\alpha}}\, ds \\ &= C(\mu_0 R)^{-(2+\alpha)}(1 + o(1)). \end{aligned}$$

The equation for $\tilde{\psi}$ is

$$\partial_\tau \tilde{\psi} = \Delta_z \tilde{\psi} + \tilde{V}\tilde{\psi} + (\mu R)^{2+\alpha} \frac{\mu^{\beta-2}}{1 + |zR|^{\alpha+2}}$$

where

$$\tilde{V}(z, \tau) = V\left(\frac{x - \xi}{\mu R}, \tau(t)\right)\mu^2 R^2.$$

We observe that the right-hand side is bounded by μ^β. Then, applying local parabolic estimate on $\tilde{\psi}$ we get

$$\begin{aligned} [\psi_1]_{1+2\varepsilon, \frac{1+2\varepsilon}{2}, \Omega \times [t, t+1]} &= \sup_{\substack{x_1 \neq x_2 \in \Omega, \\ t_1 \neq t_2 \in [t, t+1]}} \frac{|\psi_1(x_1, t_1) - \psi_1(x_2, t_2)|}{\left(|x_1 - x_2|^2 + |t_1 - t_2|\right)^{\frac{1+2\varepsilon}{2}}} \\ &= \frac{|\tilde{\psi}(z(x, \tau_1), \tau_1(t)) - \tilde{\psi}(z(x, t_2), \tau(t_2))|}{\left(|z_1 - z_2|^2 + |\tau_1 - \tau_2|\right)^{\frac{1+2\varepsilon}{2}}} \left| \frac{\left(|z_1 - z_2|^2 + |\tau_1 - \tau_2|\right)}{\left(|x_1 - x_2|^2 + |t_1 - t_2|\right)} \right|^{\frac{1+2\varepsilon}{2}} \\ &\lesssim [\tilde{\psi}]_{1+2\varepsilon, \frac{1+2\varepsilon}{2}} \frac{1}{[(\mu R)^{2+\alpha}]^{\frac{1+2\varepsilon}{2}}} \\ &\lesssim \|f\|_{\beta-2, \alpha+2} \mu^\beta \frac{1}{[(\mu R)^{2+\alpha}]^{\frac{1+2\varepsilon}{2}}}. \end{aligned}$$

The same computation with Hölder coefficient $(2\varepsilon, \varepsilon)$ gives

$$[\psi_1]_{2\varepsilon, \varepsilon} \lesssim \mu^\beta \frac{1}{[(\mu R)^{2+\alpha}]^\varepsilon}.$$

$\square$

Remark 4.1 (case $\alpha = 0$)**.** *If we let $\alpha = 0$ then a slight modification of this lemma is still true. More precisely, in estimates (4.6)-(4.8) we need to multiply the terms involving $\|F\|_{\beta-2,2}$ by $\log(e + |y|)$. Indeed, letting $\alpha = 0$ in the proof we obtain the* *bound* $\Psi_0(|y|) \lesssim \log(e+|y|)$ *which is given by a direct estimate of the variation of parameters formula.*

Let $\alpha > 0$ arbitrarily small. We introduce the following weighted norms for ψ:

$$\begin{aligned}\|\psi\|_{**} := &\sup_{x\in\Omega, t\in[t_0,\infty)} \Big[|\psi(x,t)|\,\mu^{2\alpha}(1+|y|^{\alpha})\mu^{-(l_1+\delta)}\Big] \\ &+ \sup_{x\in B_R, t\in[t_0,\infty)} \Big[|\nabla_x\psi(x,t)|\,\Big(1+|y|^{1+\alpha}\Big)\mu^{2\alpha+1}\mu^{-(l_1+\delta)}\Big] \\ &+ \sup_{t>t_0}\Big\{[\psi]_{1+2\varepsilon,\frac{1+2\varepsilon}{2},B_R\times[t,t+1]}(\mu R)^{1+2\varepsilon}\mu^{2\alpha}\mu^{-(l_1+\delta)}\Big\} \\ &+ \sup_{t>t_0}\Big\{[\psi]_{2\varepsilon,\varepsilon,B_R\times[t,t+1]}(\mu R)^{2\varepsilon}\mu^{2\alpha}\mu^{-(l_1+\delta)}\Big\},\end{aligned} \tag{4.11}$$

and define the space of function $X_{**} = \{\psi \in L^\infty(\Omega\times[t_0,\infty)) : \|\psi\|_{**} < \infty\}$. Now, we are ready to solve the outer problem (3.4) for ϕ such that

$$\|\phi\|_{*} < \mathfrak{b}, \tag{4.12}$$

for some parameters satisfying (3.17) and (3.18).

Proposition 4.1. *Assume that* Λ, ξ_1, ϕ *satisfy (3.17), (3.18) and (4.12) respectively. Also, suppose* $\psi_0 \in C^1(\bar\Omega)$ *such that*

$$\|\psi_0\|_{L^\infty} + \|\nabla\psi_0\|_{L^\infty(\Omega)} < e^{-\kappa t_0},$$

for some $\kappa > 0$*. Then, there exists* t_0 *large so that problem (3.4) has a unique solution* $\psi = \Psi[\Lambda,\dot\Lambda,\xi,\dot\xi,\phi]$*, and given* $\alpha > 0$ *sufficiently small, there exists* C_{**} *such that*

$$\|\psi\|_{**} \le e^{-\kappa t_0} C_{**}, \tag{4.13}$$

where $C_{**} = C_{**}(\mathfrak{b}_1,\mathfrak{b}_2)$ *and* $\mathfrak{b}_1, \mathfrak{b}_2$ *are the constants in (3.17) and (3.18) respectively.*

Proof. By a fixed point argument, we prove existence and uniqueness of ψ_1 solution of (4.2) in a space where estimate (4.13) holds. Thus, the same result applies to ψ using $\psi = e^{\gamma t}\psi_1$ and the relations (4.3), (4.4). Let T_1 the linear operator, defined by Lemma 4.1, such that, for all f, g, h with bounded norms $\|f\|_{\beta-2,\alpha+2}$, $\|e^{as}g\|_\infty$, $\|h\|_\infty$ respectively, $T[f,g,h] = \psi_1$ is the solution to (4.5).
Firstly, we define $\psi_B := T(0, -\mu_0^{1/2}u_3, \psi_0)$. From the definition of $u_3(x,t)$ we expand for $x \in \partial\Omega$ and t_0 large, to get

$$\begin{aligned} u_3(x,t) =&\mu^{-1/2}\frac{\alpha_3}{\Big(1+|\frac{x-\xi}{\mu}|^2\Big)^{1/2}} - \frac{\mu^{1/2}}{|x-\xi|} \\ =& \frac{\alpha_3\mu^{1/2}}{\Big(\mu^2+|x-\xi|^2\Big)^{1/2}} - \mu^{1/2}\frac{\alpha_3}{|x-\xi|} \\ =& \alpha_3\mu^{1/2}|x-\xi|^{-1}\left[\left(\frac{\mu^2+|x-\xi|^2}{|x-\xi|^2}\right)^{-1/2} - 1\right] \\ =& \mu^{5/2} f_B(x,t),\end{aligned}$$

for a smooth bounded function $f_B(x,t)$ on $\partial\Omega\times[t_0,\infty)$. Hence, Lemma 4.1 gives the bound

$$|\psi_B| \lesssim e^{-b(t-t_0)}\|\psi_0\|_{L^\infty} + e^{-a(t-t_0)}\|e^{as}u_3\|_{L^\infty(\partial\Omega\times[t_0,\infty))},$$

for any $b < \lambda_1$ and $a < \min\{5\gamma, \lambda_1 - \varepsilon\}$ for any $\varepsilon > 0$. We obtain a solution $\psi + \psi_B$ to (4.2) if ψ satisfies

$$\psi = \mathcal{A}(\psi), \quad \mathcal{A}(\psi) := T[F(\psi), 0, 0].$$

Let

$$\mathcal{B} = \{\psi : \|\psi\|_{**} \leq e^{-\kappa t_0} M\},$$

where M is a fixed large constant, independent of t and t_0. We prove that $\mathcal{A}(\psi) \in \mathcal{B}$ for any $\psi \in \mathcal{B}$. Firstly, we estimate $F(\psi)$ in the L^∞-norm. From definition (4.11) we apply Lemma (4.1) with $\beta = \frac{1}{2} + l_1 + \delta < 3/2$. We recall that $F = \mu_0^{1/2} f$ where

$$\begin{aligned}
f(x,t) &= \mu_0^{-1/2}\mathcal{N}(u_3, \tilde{\phi})(1-\eta_R) \\
&\quad + \mu^{-1}\Big(\frac{\mu}{\mu_0}\Big)^{1/2}\phi\partial_t\eta_R + \mu^{-1}\Big(\frac{\mu}{\mu_0}\Big)^{1/2}\eta_R\Big\{(\gamma - \dot{\Lambda})(\phi + 2\nabla_y\phi\cdot y) - \nabla_y\phi\cdot\Big(\frac{\dot{\xi}}{\mu}\Big)\Big\} \\
&\quad + \mu^{-1}\Big(\frac{\mu}{\mu_0}\Big)^{1/2}\Big(\phi\Big(\frac{2}{|z|}\frac{\eta'(|z|)}{\mu^2R^2} + \frac{\eta''(|z|)}{\mu^2R^2}\Big) + 2\frac{\nabla_y\phi}{\mu}\cdot\frac{z}{|z|}\frac{\eta'(|z|)}{\mu R}\Big) \\
&\quad + \mu_0^{-1/2}S_{\text{in}}(1-\eta_R) + \mu_0^{-1/2}S_{\text{out}}.
\end{aligned}$$

We have $\eta''(y/R) \neq 0$ and $\eta'(y/R) \neq 0$ only if $|y| \sim R$, hence we estimate

$$\begin{aligned}
\Big|\mu^{-1}\Big(\frac{\mu}{\mu_0}\Big)^{1/2}\phi\Delta\eta\Big| &\lesssim \mu^{-1}\|\phi\|_*\frac{\mu^{1+l_1}R^3\log(R)}{1+|y|^4}\frac{\mu^{-2}}{1+|y|^2}\eta''\Big(\Big|\frac{x-\xi}{\mu R}\Big|\Big) \\
&\lesssim e^{-\kappa t_0}\mu^{l_1+\delta}\log(R)\frac{\mu^{-2}}{1+|y|^2} \\
&\lesssim e^{-\kappa t_0}\mu^{l_1+\delta-2\alpha}\frac{\mu^{-2}}{1+|y|^{2+\alpha}}.
\end{aligned} \tag{4.14}$$

Using the bound on the gradient given in the definition of $\|\phi\|_*$ we obtain

$$\begin{aligned}
\Big|\mu^{-1}\Big(\frac{\mu}{\mu_0}\Big)^{1/2}\Big(2\frac{\nabla_y\phi}{\mu}\cdot\frac{z}{|z|}\frac{\eta'(|z|)}{\mu R}\Big)\Big| &\lesssim \|\phi\|_*\frac{\mu^{1+l_1}R^3\log(R)}{1+|y|^5}R\mu^{-1}\Big(\frac{|\eta'(|z|)|}{\mu^2R^2}\Big) \\
&\lesssim e^{-\kappa t_0}\mu^{l_1+\delta-2\alpha}\frac{\mu^{-2}}{1+|y|^{2+\alpha}}
\end{aligned}$$

Similarly, also using the bounds on $\dot{\Lambda}, \dot{\xi}$ we have

$$\begin{aligned}
\Big|\mu^{-1}\Big(\frac{\mu}{\mu_0}\Big)^{1/2}\phi\partial_t\eta_R\Big| &\lesssim \|\phi\|_*\mu^{l_1+\delta}\log(R)\Big|\eta'(|z|)\frac{z}{|z|}\cdot\Big(-\frac{\dot{\xi}}{\mu R} - z\frac{\partial_t(\mu R)}{\mu R}\Big)\Big| \\
&\lesssim \|\phi\|_*\mu^{l_1+\delta}\log(R)\mu^2R^2\frac{|\eta'(z)|}{\mu^2R^2} \\
&\lesssim e^{-\kappa t_0}\mu^{l_1+\delta-2\alpha}\frac{\mu^{-2}}{1+|y|^{2+\alpha}}.
\end{aligned}$$

Also,

$$\begin{aligned}\left|\mu^{-1}\Big(\frac{\mu}{\mu_0}\Big)^{1/2}\eta_R\bigg\{\Big(\gamma-\dot\Lambda\Big)(\phi+2\nabla_y\phi\cdot y)-\nabla_y\phi\cdot\Big(\frac{\dot\xi}{\mu}\Big)\bigg\}\right| &\lesssim \|\phi\|_*\frac{\mu^{l_1}R^3\log(R)}{1+|y|^4}\eta\\ &\lesssim \|\phi\|_*\mu^{2+l_1-3\delta}\log(R)\frac{\mu^{-2}}{1+|y|^{2+\alpha}}\\ &\lesssim e^{-\kappa t_0}\mu^{l_1+\delta-2\alpha}\frac{\mu^{-2}}{1+|y|^{2+\alpha}}\end{aligned}$$

Furthermore, using Lemma 2.5 we estimate

$$\begin{aligned}|\mu^{-1/2}S_{\rm out}| &\lesssim \mu\\ &\lesssim \mu^{3-\alpha}\frac{\mu^{-2}}{1+|y|^{2+\alpha}}\\ &\lesssim e^{-\kappa t_0}\mu^{l_1+\delta-2\alpha}\frac{\mu^{-2}}{1+|y|^{2+\alpha}},\end{aligned}$$

and

$$\begin{aligned}|\mu^{-1/2}S_{\rm in}(1-\eta_R)| &\lesssim \mu^{l_1+2\delta}\frac{\mu^{-2}}{1+|y|^2}\\ &\lesssim e^{-\kappa t_0}\mu^{l_1+2\delta-2\alpha}\frac{\mu^{-2}}{1+|y|^{2+\alpha}}.\end{aligned}$$

Finally, using (4.11)

$$\begin{aligned}|\mu_0^{-1/2}\mathcal{N}_3(u_3,\tilde\phi)(1-\eta_R)| &\lesssim \mu^{-1/2}u_3^3\Big(\mu^{-1/2}\phi\eta_R+\mu^{1/2}\psi\Big)^2(1-\eta_R) \qquad (4.15)\\ &\lesssim \frac{\mu^{-2}}{1+|y|^3}\Big(\mu^{-1}|\phi|^2\eta_R^2+|\psi|^2\mu\Big)(1-\eta_R)\\ &\lesssim \frac{\mu^{-2}}{1+|y|^3}\left(\mu^{-1}\|\phi\|_*^2\Big(\frac{\mu^{1+l_1}R^3\log(R)}{1+|y|^4}\Big)^2\eta^2+\|\psi\|_{**}^2\mu\frac{\mu^{2\beta}}{1+|y|^{2\alpha}}\right)(1-\eta_R)\\ &\lesssim \frac{\mu^{-2}}{1+|y|^{2+\alpha}}\Big(\|\phi\|_*^2\mu^{1+2l_1+2\delta}\log(R)^2+\|\psi\|_{**}^2\mu^{1+2\beta+2\alpha\delta}\Big)\\ &\lesssim e^{-\kappa t_0}\mu^{l_1+\delta-2\alpha}\frac{\mu^{-2}}{1+|y|^{2+\alpha}}\Big(\|\phi\|_*^2+\|\psi\|_{**}^2\Big)\end{aligned}$$

Summing up these estimates we conclude that for $\alpha,\kappa>0$ fixed sufficiently small

$$|f(x,t)|\lesssim e^{-\kappa t_0}\mu^{l_1+\delta-\alpha}\log(R)\frac{\mu^{-2}}{1+|y|^{2+\alpha}}.$$

Hence, we have

$$|F(x,t)|\lesssim e^{-\kappa t_0}\mu^{\frac12+l_1+\delta-2\alpha}\frac{\mu^{-2}}{1+|y|^{2+\alpha}},$$

and Lemma 4.1 gives

$$|T[F(\psi),0,0]|\lesssim e^{-\kappa t_0}\mu^{\frac12+l_1+\delta-2\alpha}\frac{1}{1+|y|^{\alpha}}.$$

Since $F \in L^\infty(\Omega \times [t_0, \infty))$, classic parabolic estimates give $\psi \in C^{1+\sigma, \frac{1+\sigma}{2}}(\Omega \times [t_0, \infty))$ for any $\sigma < 1$ and from Lemma 4.1 we get

$$\|\psi\|_{**} \le e^{-kt_0} M \tag{4.16}$$

for sufficiently large M. This proves $\mathcal{A}(\psi) \in \mathcal{B}$. Now, we claim that the map $\mathcal{A}(\psi)$ is a contraction, that is: there exists $\mathbf{c} < 1$ such that, for any $\psi_1, \psi_2 \in \mathcal{B}$,

$$\|\mathcal{A}(\psi_1) - \mathcal{A}(\psi_2)\|_{**} \le \mathbf{c}\|\psi_1 - \psi_2\|_{**}.$$

Since ψ appears in $F(\psi)$ only in the nonlinear term $\mathcal{N}$, we get

$$\begin{aligned}\mathcal{A}(\psi^1) - \mathcal{A}(\psi^2) = T\Big[&\mu_0^{-1/2}\mathcal{N}\Big(u_3, \mu_0^{1/2}\Big(\psi^1 + \psi_B\Big) + \mu^{-1/2}\phi\eta_R\Big) \\ &- \mu_0^{-1/2}\mathcal{N}\Big(u_3, \mu_0^{1/2}\Big(\psi^2 + \psi_B\Big) + \mu^{-1/2}\phi\eta_R\Big), 0, 0\Big].\end{aligned}$$

From definition (3.3) we write

$$\begin{aligned}&\mu_0^{-1/2}\Big[\mathcal{N}\Big(u_3, \mu_0^{1/2}\Big(\psi^1 + \psi_B\Big) + \mu^{-1/2}\phi\eta_R\Big) - \mathcal{N}\Big(u_3, \mu_0^{1/2}\Big(\psi^2 + \psi_B\Big) + \mu^{-1/2}\phi\eta_R\Big)\Big] \\ =&\mu_0^{-1/2}\Big[\Big(u_3 + \psi^1 + \mu^{-1/2}\phi\eta\Big)^5 - (u_3 + \psi^2 + \mu^{-1/2}\phi\eta)^5 - 5u_3^4(\psi^1 - \psi^2)\Big] \\ \\ =&\mu_0^{-1/2}\Big[\Big(u_3 + \mu_0^{1/2}\psi^1 + \mu^{-1/2}\phi\eta\Big)^5 - \Big(u_3 + \mu_0^{1/2}\psi^2 + \mu^{-1/2}\phi\eta\Big)^5 \\ &- 5(u_3 + \mu^{-1/2}\phi\eta)^4\mu_0^{1/2}\Big(\psi^1 - \psi^2\Big)\Big] \\ &+ 5\Big[(u_3 + \mu^{-1/2}\phi\eta)^4 - u_3^4\Big](\psi^1 - \psi^2) \\ =:&N_1 + N_2.\end{aligned}$$

Recalling that $\beta = \frac{1}{2} + l_1 + \delta$ we estimate

$$\begin{aligned}|N_1(x,t)| &\lesssim \mu^{-3/2}U^3\mu_0^{1/2}|\psi_1 - \psi_2|^2 \\ &\lesssim \mu^{-1}U^3|\psi_1 - \psi_2|^2, \\ &\lesssim \frac{\mu^{-1}}{1 + |y|^3}\frac{\mu^{2\beta}}{1 + |y|^{2\alpha}}\|\psi_1 - \psi_2\|_{**}^2 \\ &\lesssim e^{-\kappa t_0}\mu^{l_1 + \delta - 2\alpha}\frac{\mu^{-2}}{1 + |y|^{2+\alpha}}\|\psi_1 - \psi_2\|_{**}\end{aligned}$$

and

$$\begin{aligned}|N_2(x,t)| &\lesssim u_3^3\mu^{-1/2}\phi\eta\|\psi_1 - \psi_2\|_{**} \\ &\lesssim \frac{\mu^{-2}}{1 + |y|^3}\frac{\mu^{1+l_1}R^3\log(R)}{1 + |y|^4}\frac{\mu^\beta}{1 + |y|^\alpha}e^{-\kappa t_0}\|\psi_1 - \psi_2\|_{**} \\ &\lesssim e^{-\kappa t_0}\mu^{l_1 + \delta - 2\alpha}\frac{\mu^{-2}}{1 + |y|^{2+\alpha}}\|\psi_1 - \psi_2\|_{**},\end{aligned}$$

Finally, applying $T[\cdot, 0, 0]$ to $F(\psi)$ we obtain

$$|\mathcal{A}[\psi^1] - \mathcal{A}[\psi^2]| \lesssim e^{-\kappa t_0}\frac{\mu^{l_1 + \delta - 2\alpha}}{1 + |y|^\alpha}\|\psi_1 - \psi_2\|_{**}. \tag{4.17}$$

Arguing as in (4.16), from (4.17) and standard parabolic estimates we obtain

$$\|\mathcal{A}[\psi^1]-\mathcal{A}[\psi^2]\|_{**}\le \mathbf{c}\|\psi_1-\psi_2\|_{**},$$

with $\mathbf{c}<1$ if t_0 is taken sufficiently large. Applying the Banach fixed point theorem we get existence and uniqueness of ψ_1 and hence of $\psi=\Psi[\Lambda,\dot\Lambda,\xi,\dot\xi,\phi]$ with estimate (4.13) that is a consequence of estimates (4.6-(4.8). □

Remark 4.2 (Continuity with respect to the initial condition ψ_0)**.** *Given an initial datum ψ_0 Proposition 4.1 defines a solution to (3.4) $\psi=\Psi[\psi_0]$, from a small neighborhood of* 0 *in the $L^\infty(\Omega)$ space with the C^1-norm $\|\psi\|_\infty+\|\nabla\psi_0\|_\infty$ into the Banach space L^∞ with norm $\|\psi\|_{**}$ defined in (4.11). In fact, from the proof of Proposition 4.1 and the implicit function theorem applied to $\psi_0\mapsto\Psi[\psi_0]$ is a diffeomorphism and hence*

$$\|\Psi[\psi_0^1]-\Psi[\psi_0^2]\|_{**}\le c\Big[\|\psi_0^1-\psi_0^2\|_\infty+\|\nabla_x\psi_0^1-\nabla_x\psi_0^2\|_\infty\Big],$$

for some positive constant c.

The function $\psi=\Psi[\Lambda,\dot\Lambda,\xi,\dot\xi,\phi]$ depends continuously on the parameters $\Lambda,\dot\Lambda,\xi,\dot\xi,\phi$. To see this we argue similarly to [8, Proposition 4.3]. For example fix $\dot\Lambda,\xi,\dot\xi,\phi$ and consider

$$\bar\psi:=\psi^{(1)}-\psi^{(2)}\quad\text{where}\quad \psi^{(i)}=\Psi[\Lambda_i,\dot\Lambda,\xi,\dot\xi,\phi],\quad\text{for}\quad i=1,2$$

for Λ_1,Λ_2 satisfying (3.17). Then $\bar\psi$ solves

$$\partial_t\bar\psi=\Delta\bar\psi+V[\Lambda_1]\bar\psi+(V[\Lambda_1]-V[\Lambda_2])\psi^{(2)}+F[\Lambda_1]-F[\Lambda_2].$$

One can easily check each term in F and obtain

$$\|F[\Lambda_1]-F[\Lambda_2]\|_{\beta-2,\alpha+2}\le \mathbf{c}\|\Lambda_1-\Lambda_2\|_{l_0,\infty},$$

with $\mathbf{c}<1$ if t_0 is large enough. Also, using (4.1) we find that

$$\|V[\Lambda_1]-V[\Lambda_2]\|_{\beta-2,\alpha+2}\le \mathbf{c}\|\Lambda_1-\Lambda_2\|_{l_0,\infty}.$$

Then, arguing as in the proof of (4.6), a multiple of $\|\Lambda_1-\Lambda_2\|_{\sharp,l_0,\delta_0,\frac12+\varepsilon}\psi$, where ψ is the supersolution constructed in Lemma (4.1), is a supersolution for $\bar\psi$. Similarly, one obtain analogue estimates fixing the other parameters $\xi,\dot\Lambda,\dot\xi$. Let us consider all the parameters fixed except ϕ. We define $\bar\psi=\psi[\Lambda,\dot\Lambda,\xi,\dot\xi,\phi_1]-\psi[\Lambda,\dot\Lambda,\xi,\dot\xi,\phi_2]$, which satisfies the equation

$$\partial_t\bar\psi=\Delta\bar\psi+V\bar\psi+F[\phi_1]-F[\phi_2].$$

For instance, we estimate

$$\begin{aligned}\left|\mu^{-1}\Big(\frac{\mu}{\mu_0}\Big)^{1/2}(\phi_1-\phi_2)\partial_t\eta_R\right| &\lesssim[\mu(t_0)^\alpha\log(1/\mu(t_0))]\|\phi_1-\phi_2\|_*\mu^{l_1+\delta-2\alpha}\frac{\mu^{-2}}{1+|y|^{2+\alpha}}\\ &\lesssim \mathbf{c}\|\phi_1-\phi_2\|_*\mu^{l_1+\delta-2\alpha}\frac{\mu^{-2}}{1+|y|^{2+\alpha}},\end{aligned}$$

and, arguing as in (4.14)-(4.15), we obtain similar estimate on each term of $F[\phi_1]-F[\phi_2]$. Having the L^∞-bound, the estimate for the gradient and for the Hölder norm of $\bar\psi$ follow as in the proof of Lemma 4.1. We summarize the continuity of $\psi[\Lambda,\dot\Lambda,\xi,\dot\xi,\phi_1]$ with respect to the parameters in the following Proposition.

Proposition 4.2. *Under the same assumption of Proposition 4.1, the function* $\psi = \Psi[\Lambda, \dot\Lambda, \xi, \dot\xi, \phi]$ *is continuous with respect to the parameters* $\Lambda, \dot\Lambda, \xi, \dot\xi, \phi$. *Moreover the following estimate holds:*

$$\begin{aligned}\|\Psi[\Lambda^{(1)}, \dot\Lambda^{(1)},&\xi^{(1)}, \dot\xi^{(1)}, \phi^{(1)}] - \Psi[\Lambda^{(2)}, \dot\Lambda^{(2)}, \xi^{(2)}, \dot\xi^{(2)}, \phi^{(2)}]\|_{**} \\ &\le \mathbf{c}\{\|\Lambda^{(1)} - \Lambda^{(2)}\|_{\sharp, l_0, \delta_0, \frac12+\varepsilon} + \|\dot\Lambda^{(1)} - \dot\Lambda^{(2)}\|_{\sharp, l_1, \delta_1, \varepsilon} \\ &\quad + \|\xi_1^{(1)} - \xi_1^{(2)}\|_{\sharp, 1+l_0, \frac12+\varepsilon} + \|\dot\xi_1^{(1)} - \dot\xi_1^{(2)}\|_{\sharp, 1+l_0, \varepsilon} + \|\phi^{(1)} - \phi^{(2)}\|_{*}\}\end{aligned}$$

where $\mathbf{c} < 1$ *provided that* t_0 *is sufficiently large and the constants* $\mathfrak{b}_1, \mathfrak{b}_2$ *in (3.17),(3.18) are sufficiently small.*

5. Characterization of the orthogonality conditions (3.7)

Given the function $\psi = \Psi[\Lambda, \dot\Lambda, \xi, \dot\xi, \phi]$ provided by Proposition 4.1, we plug it in the inner problem for ϕ. From the linear theory stated in Proposition 3.1, the inner problem (3.7) with initial datum (3.13) can be solved if the orthogonality conditions

$$\int_{B_{2R}} H[\Lambda, \dot\Lambda, \xi, \dot\xi, \phi](y, t(\tau)) Z_i(y)\, dy = 0 \quad \text{for} \quad t > t_0, \quad \text{and} \quad i = 1, 2, 3, 4, \tag{5.1}$$

are satisfied. The aim of this section is to solve this system in Λ, ξ for any given $\phi \in X_*$. The next Lemma shows that the orthogonality condition with index $i = 4$ is equivalent to a nonlocal equation in the variable $\dot\Lambda$, for fixed ϕ, ξ.

Lemma 5.1. *Assume that* Λ, ξ, ϕ *satisfy (3.17),(3.18) and (3.12) respectively. Let* $\psi = \Psi[\Lambda, \dot\Lambda, \xi, \dot\xi, \phi]$ *be the solution to problem (3.4) given by Proposition 4.1. Then, the condition (5.1) with index* $i = 4$ *is equivalent to*

$$(1 + a[\dot\Lambda, \xi](t))\mathcal{J}[\dot\Lambda](0, t) = g(t) + G[\Lambda, \dot\Lambda, \xi, \dot\xi, \phi](t) \quad \textit{for} \quad t \in [t_0, \infty), \tag{5.2}$$

where $\mathcal{J}$ *is the solution to*

$$\begin{aligned}&\partial_t \mathcal{J} = \Delta \mathcal{J} + \gamma \mathcal{J} - \dot\Lambda(t) G_\gamma(x, 0) \quad \textit{in } \Omega \times [t_0 - 1, \infty), \\ &\mathcal{J}(x, t) = 0 \quad \textit{on } \partial\Omega \times [t_0 - 1, \infty), \\ &\mathcal{J}(x, t_0 - 1) = 0 \quad \textit{in } \Omega.\end{aligned}$$

The function a *is smooth and decays exponentially, with* $a[0, 0] = 0$. *The following estimate on* g *holds:*

$$\|g\|_{\sharp, l_0 - 2\alpha, \delta_1, \frac{1+2\varepsilon}{2}} + \|g\|_{\sharp, l_0 - 2\alpha, \delta_0, \varepsilon} \le e^{-\kappa t_0} C_0.$$

and

$$\begin{aligned}\|G[\Lambda, \dot\Lambda, \xi, \dot\xi, \phi]\|_{\sharp, l_0 - \alpha, \delta_1, \frac{1+2\varepsilon}{2}} + \|G[\Lambda, \dot\Lambda, \xi, \dot\xi, \phi]\|_{\sharp, l_0 - 2\alpha, \delta_0, \varepsilon} \le e^{-\kappa t_0}\{\|\Lambda\|_{\sharp, l_0, \delta_0, \varepsilon} \\ + \|\dot\Lambda\|_{\sharp, l_1, \delta_1, \varepsilon} + \|\xi_1\|_{\sharp, 1+l_0, \frac12+\varepsilon} + \|\dot\xi_1\|_{\sharp, 1+l_0, \varepsilon} + \|\phi\|_{*}\}.\end{aligned} \tag{5.3}$$

Furthermore, we have

$$\begin{aligned}\|G[\Lambda^{(1)}, \dot\Lambda^{(1)}, \xi^{(1)}, \dot\xi^{(1)}, \phi^{(1)}] - G[\Lambda^{(2)}, \dot\Lambda^{(2)}, \xi^{(2)}, \dot\xi^{(2)}, \phi^{(2)}]\|_{\sharp, l_0 - 2\alpha, \delta_1, \frac{1+2\varepsilon}{2}} \\ \le \mathbf{c}\{\|\Lambda^{(1)} - \Lambda^{(2)}\|_{\sharp, l_0, \delta_0, \frac{1+2\varepsilon}{2}} + \|\dot\Lambda^{(1)} - \dot\Lambda^{(2)}\|_{\sharp, l_1, \delta_1, \varepsilon} \\ + \|\xi_1^{(1)} - \xi_1^{(2)}\|_{\sharp, 1+l_0, \frac12+\varepsilon} + \|\dot\xi_1^{(1)} - \dot\xi_1^{(2)}\|_{\sharp, 1+l_0, \varepsilon} \\ + \|\phi^{(1)} - \phi^{(2)}\|_{*}\}\end{aligned} \tag{5.4}$$

with constant $\mathbf{c} < 1$ *provided that* t_0 *is fixed sufficiently large and* $\mathfrak{b}_i$ *small for* $i = 1, 2$.

Proof. We recall that

$$\begin{aligned}H[\phi,\psi,\mu,\dot\mu,\xi,\dot\xi](y,\tau):=&5U(y)^4\mu\Big(\frac{\mu_0}{\mu}\Big)^{1/2}\psi(\mu y+\xi,t(\tau))\\&+B_0[\phi+\mu\psi]\,(\mu y+\xi,t(\tau))+\mu^{5/2}S_{\rm in}(\mu y+\xi,t(\tau))\\&+\mathcal{N}(\mu^{1/2}u_3,\mu^{1/2}\tilde\phi)(\mu y+\xi,t(\tau)).\end{aligned}$$

Hence, (3.10) with index $i=4$ becomes

$$\begin{aligned}0=&\mu^{5/2}\int_{B_{2R}}Z_4(y)S_{\rm in}(y,t)\,dy+\mu\Big(\frac{\mu_0}{\mu}\Big)^{1/2}\int_{B_{2R}}Z_4(y)5U(y)^4\psi(\mu y+\xi,t)\,dy\\&+\int_{B_{2R}}Z_4(y)B_0[\phi+\mu\psi]\,(\mu y+\xi,t)\,dy+\int_{B_{2R}}Z_4(y)\mathcal{N}(\mu^{1/2}u_3,\mu^{1/2}\tilde\phi)(\mu y+\xi,t(\tau))\,dy\\=:&\sum_{j=1}^4 i_j(t).\end{aligned}$$

We follow the analogue [8, Lemma 5.1] to estimate the terms $i_j(t)$. Firstly, we analyze i_1. We have

$$\begin{aligned}i_1(t)=&\mu^{5/2}\int_{B_{2R}}S_{\rm in}(y,t)Z_4(y)\,dy\\=&\mu\Big(\frac{\mu_0}{\mu}\Big)^{1/2}\int_{B_{2R}}5U(y)^4J(\mu y+\xi,t)Z_4(y)\,dy\\&+\int_{B_{2R}}\mathcal{N}_3(y,t)Z_4(y)\,dy\\&+\mu\int_{B_{2R}}Z_4(y)5U(y)^4h_\gamma(\mu y+\xi,\xi)\,dy\\=:&a_1(t)+a_2(t)+a_3(t),\end{aligned}$$

where we used that the integral of $Z_4(y)U(y)^4\partial_{y_i}U(y)$ on $B_{2R}(0)$ is null by symmetry for $i=1,2,3$. Also,

$$\begin{aligned}\mu^{-1}\Big(\frac{\mu_0}{\mu}\Big)^{-1/2}a_1(t)=&\int_{B_{2R}}5U(y)^4Z_4(y)J[\dot\Lambda](\mu y+\xi,t)\,dy\\=&\mathcal{J}[\dot\Lambda](0,t)\int_{B_{2R}}5U(y)^4Z_4(y)\,dy\\&+\Big[J[\dot\Lambda](\xi(t),t)-\mathcal{J}[\dot\Lambda](0,t)\Big]\int_{B_{2R}}5U(y)^4Z_4(y)\,dy\\&+\int_{B_{2R}}5U(y)^4Z_4(y)[J[\dot\Lambda](\mu y+\xi,t)-J[\dot\Lambda](\xi,t)]\,dy\\=:&a_{11}(t)+a_{12}(t)+a_{13}(t).\end{aligned}$$

The main term of the left-hand side of (5.2) is given by

$$c_1^{-1}(1+O(R^{-2}))^{-1}a_{11}(t)=\mathcal{J}[\dot\Lambda](0,t),$$

where $c_1(1+O(R^{-2}))=\int_{B_{2R}}5U^4Z_4\,dy$. To analyze a_{12} one can proceed by estimating the right-hand side in $L^2(\Omega)$ of the equation for $w(x,t)=J(x,t)-\mathcal{J}(x,t)$ and using standard parabolic estimates we deduce

$$|a_{12}[\dot\Lambda](t)|\lesssim\mu^{l_1+\sigma},$$

for some $\sigma \in (0,1)$. To analyze a_{12} it is enough to proceed as in Appendix of [8] using the Duhamel's formula in $\mathbb{R}^3$ and then decomposing J as a sum of a solution on $\mathbb{R}^3$ and a smooth one in Ω with more decay, obtaining

$$|J[\dot{\Lambda}](\mu y+\xi,t)-J[\dot{\Lambda}](\xi,t)| = \mu^{l_1}|\mu y|^{\sigma}\Pi(t)\theta(|y|),$$

for some $\sigma \in (0,1)$ and bounded function Π, θ. Thus

$$a_{13}[\dot{\Lambda}](t) = \mu^{l_1+\sigma}\Pi_1[\dot{\Lambda}](t).$$

for some function $\Pi_1(t)$ in (t_0,∞) and constant $\sigma \in (0,1)$. Taking into account the behaviour of J_1, J_2 and ϕ_3 given in (2.36), (2.37) and (2.42) respectively, we have

$$\begin{aligned}
a_2 &= \int_{B_{2R}} Z_4(y)\mathcal{N}_3(y,t)\,dy \\
&= \int_{B_{2R}} Z_4\Big\{10\Big(U(y)+s(-\mu H_\gamma+\mu J+\mu^{-1/2}\phi_3\eta_l)\Big)^3\Big(-\mu H_\gamma+\mu J+\mu^{-1/2}\phi_3\eta_l\Big)^2\Big\}\,dy \\
&= \mu^2\int_{B_{2R}} 10Z_4(y)U(y)^3Q[\Lambda,\xi](y,t)\,dy,
\end{aligned}$$

for some bounded smooth function $Q[\Lambda,\xi](y,t)$ and constant $s \in (0,1)$.
Finally, Taylor expanding $h_\gamma(x,\xi)$ at $x=\xi$, we get

$$\begin{aligned}
a_3 =&\mu\int_{B_{2R}} Z_4(y)5U(y)^4h_\gamma(\mu y+\xi,\xi)\,dy \\
=&\mu R_\gamma(\xi)\int_{B_{2R}} Z_4(y)5U(y)^4\,dy \\
&+\mu^3\int_{B_{2R}} Z_4(y)5U(y)^4(y\cdot D_{xx}h_\gamma(\mu y^*(y)+\xi,\xi)\cdot y)\,dy \\
=&\mu^2\Pi_2[\Lambda,\xi](t),
\end{aligned}$$

for some $y^* \in \overline{[0,y]}$ and a smooth bounded function $\Pi_2(t)$. This gives the left-hand side in (5.2). Now, we look at i_2. We decompose

$$\begin{aligned}
\mu^{-\frac{1}{2}}\mu_0^{-\frac{1}{2}}i_2(t) =&\int_{B_{2R}} Z_4(y)5U(y)^4\psi[\Lambda,\xi,\dot{\Lambda},\dot{\xi},\phi](\mu y+\xi,t)\,dy \\
=&\psi[0,0,0,0,0](0,t)\int_{B_{2R}} Z_4(y)5U(y)^4\,dy \\
&+\int_{B_{2R}} Z_4(y)5U(y)^4\{\psi[0,0,0,0,0](\mu y+\xi,t)-\psi[0,0,0,0,0](0,t)\}\,dy \\
&+\int_{B_{2R}} Z_4(y)5U(y)^4\Big\{\psi[\Lambda,\xi,\dot{\Lambda},\dot{\xi},\phi](\mu y+\xi,t)-\psi[0,0,0,0,0](\mu y+\xi,t)\Big\}\,dy \\
=:&\, b_1(t)+b_2(t)+b_3(t),
\end{aligned}$$

The term

$$b_1(t) = \psi[0,0,0,0,0](0,t)\int_{B_{2R}} 5U(y)^4Z_4(y)\,dy,$$

is independent of parameters and, as a consequence of the Proposition (4.1), satisfies the estimate

$$\|b_1(t)\|_{\sharp,l_0-2\alpha,\delta_1,\frac{1}{2}+\varepsilon} \le C,$$
$$\|b_1(t)\|_{\sharp,l_0-2\alpha,\delta_0,\varepsilon} \le C.$$

Applying the mean value theorem in b_2 and using the gradient estimate of ψ we deduce that the same bound as b_1 hold. This gives the main term $b_1(t)+b_2(t)=g(t)$ in the right-hand side of (5.2). We analyze $b_3(t)$. We decompose

$$\begin{aligned}\psi[\Lambda,\xi,\dot\Lambda,\dot\xi,\phi]-\psi[0,0,0,0,0]=&\psi[\Lambda,\xi,\dot\Lambda,\dot\xi,\phi]-\psi[0,\xi,\dot\Lambda,\dot\xi,\phi]\\&+\psi[0,\xi,\dot\Lambda,\dot\xi,\phi]-\psi[0,0,\dot\Lambda,\dot\xi,\phi]\\&+\psi[0,0,\dot\xi,\dot\Lambda,\phi]-\psi[0,0,0,\dot\Lambda,\phi]\\&+\psi[0,0,0,\dot\Lambda,\phi]-\psi[0,0,0,0,\phi]\\&+\psi[0,0,0,0,\phi]-\psi[0,0,0,0,0]\\=&\sum_{j=1}^{4}c_j.\end{aligned}$$

By Proposition 4.2 applied to each c_j we obtain

$$\begin{aligned}\|b_3(t)\|_{\sharp,l_0-2\alpha,\delta_1,\frac{1+2\varepsilon}{2}}+\|b_3(t)\|_{\sharp,l_0-2\alpha,\delta_0,\varepsilon}\lesssim& e^{-\kappa t_0}\Big\{\|\Lambda\|_{\sharp,l_0,\delta_0,\frac12+\varepsilon}+\|\dot\Lambda\|_{\sharp,l_1,\delta_1,\varepsilon}\\&+\|\dot\xi_1\|_{\sharp,1+l_0,\varepsilon}+\|\xi_1\|_{\sharp,1+l_0,\frac{1+2\varepsilon}{2}}+\|\phi\|_{*}\Big\}.\end{aligned}$$

Also, again as a consequence of the Lipschitz estimates in ψ we have for example

$$\|b_3[\dot\Lambda_1]-b_3[\dot\Lambda_2]\|_{\sharp,l_0-2\alpha,\delta_1,\frac{1+2\varepsilon}{2}}\le \mathbf{c}\|\dot\Lambda_1-\dot\Lambda_2\|_{\sharp,l_1,\delta_1,\varepsilon},$$

for any $\dot\Lambda_1,\dot\Lambda_2\in X_{\sharp,l_1,\delta_1,\varepsilon}$ and fixed $\Lambda,\xi,\dot\Lambda,\dot\xi$ in the respective spaces. We analyze i_3. We recall that

$$B_0[\phi+\mu\psi]=5\Big[\Big(U-\mu H_\gamma+\mu J[\dot\Lambda]+\mu^{-1/2}\phi_3(y,t)\eta_3\Big)^4-U^4\Big][\phi+\mu\psi],$$

which is linear in ϕ,ψ and satisfies

$$|B_0[\phi+\mu\psi](\mu y+\xi,t)|\lesssim\frac{\mu}{1+|y|^3}\mu|\phi+\mu\psi|\,.$$

It follows that

$$\begin{aligned}|i_3(t)|\lesssim&\mu\|\phi\|_*\mu^{1+l_1}R^3\log(R)+\|\psi\|_{**}\mu^2\mu^{l_1+\delta-2\alpha}\\\lesssim&\|\phi\|_*\mu^{2+l_1-3\delta}+\|\psi\|_{**}\mu^{1+l_1+\delta-2\alpha}\\\lesssim&e^{-\kappa t_0}\mu^{l_0-2\alpha}.\end{aligned}$$

Then, the Hölder bounds on ψ and ϕ in the respective norms gives estimate (5.3) for i_3, and using Proposition 4.2 we also get the Lipschitz property (5.3) for i_3. Finally, we have

$$\begin{aligned}|\mathcal N(\mu^{1/2}u_3,\mu^{1/2}\tilde\phi)(\mu y+\xi,t(\tau))|&\lesssim\frac{1}{1+|y|^3}(\phi+\mu\psi)^2\\&\lesssim\frac{1}{1+|y|^3}\left(M\frac{\mu^{2(1+l_1)}R^6\log^2(R)}{1+|y|^8}+\mu^2\frac{\mu^{2(l_1+\delta-\alpha)}}{1+|y|^{2\alpha}}\right)\\&\lesssim e^{-\kappa t_0}\mu^{l_0-2\alpha},\end{aligned}$$

and (5.3)-(5.3) for i_4 follows arguing as for i_3. Summing up the estimates we obtain $G[\Lambda,\dot\Lambda,\xi,\dot\xi,\phi](t)=b_3+i_3+i_4$ as in (5.2) with the properties (5.3) and (5.4). $\square$

In the next Lemma we characterize the conditions

$$\int_{B_{2R}} Z_i(y)H[\Lambda,\dot\Lambda,\xi,\dot\xi,\phi](y,t)\,dy=0,\quad \text{for}\quad t\in(t_0,\infty)\quad\text{and}\quad i=1,2,3.$$

Since we proceed similarly to the proof of Lemma 5.1 recalling that $Z_i=\partial_{y_i}U(y)$ for $i=1,2,3$, we omit the proof.

Lemma 5.2. *The relation (5.1) for $i=1,2,3$ is equivalent to*

$$\dot\xi_{1,i}=\mathfrak{c}_i\mu_0(t)^{1+l_0}+\Theta_i[\Lambda,\xi,\phi](t) \tag{5.5}$$

for smooth bounded function $\Theta(t)$ which satisfies

$$\begin{aligned}\|\Theta[\Lambda,\dot\Lambda,\xi,\dot\xi,\phi]\|_{\sharp,1+l_0,\frac12+\varepsilon}+\|\Theta[\Lambda,\dot\Lambda,\xi,\dot\xi,\phi]\|_{\sharp,1+l_0,+\varepsilon}\le e^{-\kappa t_0}\{\|\Lambda\|_{\sharp,l_0,\delta_0,\varepsilon}\\ +\|\dot\Lambda\|_{\sharp,l_1,\delta_1,\varepsilon}+\|\xi_1\|_{\sharp,1+l_0,\frac12+\varepsilon}+\|\dot\xi_1\|_{\sharp,1+l_0,\varepsilon}+\|\phi\|_*\}.\end{aligned} \tag{5.6}$$

Furthermore, we have

$$\begin{aligned}&\|\Theta[\Lambda^{(1)},\dot\Lambda^{(1)},\xi^{(1)},\dot\xi^{(1)},\phi^{(1)}]-\Theta[\Lambda^{(2)},\dot\Lambda^{(2)},\xi^{(2)},\dot\xi^{(2)},\phi^{(2)}]\|_{\sharp,1+l_0,\frac12+\varepsilon}\\&\le\mathbf{c}\{\|\Lambda^{(1)}-\Lambda^{(2)}\|_{\sharp,l_0,\delta_0,\frac12+\varepsilon}+\|\dot\Lambda^{(1)}-\dot\Lambda^{(2)}\|_{\sharp,l_1,\delta_1,\varepsilon}\\&\quad+\|\xi_1^{(1)}-\xi_1^{(2)}\|_{\sharp,1+l_0,\frac12+\varepsilon}+\|\dot\xi_1^{(1)}-\dot\xi_1^{(2)}\|_{\sharp,1+l_0,\varepsilon}\\&\quad+\|\phi^{(1)}-\phi^{(2)}\|_*\},\end{aligned} \tag{5.7}$$

with constant $\mathbf{c}<1$ provided that t_0 is fixed sufficiently large and $\mathfrak{b}_i$ small for $i=1,2$.

6. Choice of parameters Λ,ξ

In the previous section we have proved that if $\phi\in X_*$ and Λ,ξ satisfying (3.17) and (3.18) then the system of orthogonality conditions

$$\int_{B_{2R}} H[\Lambda,\dot\Lambda,\xi,\dot\xi,\phi](y,t(\tau))Z_i(y)\,dy=0\quad\text{for}\quad t\in[t_0,\infty)\quad\text{and}\quad i=1,2,3,4,$$

is equivalent to the (nonlocal) system in $[t_0,\infty)$

$$\begin{cases}(1+a[\dot\Lambda,\xi](t))\mathcal{J}[\dot\Lambda](0,t)=g(t)+G[\Lambda,\dot\Lambda,\xi,\dot\xi,\phi](t),\\ \dot\xi_{1,i}=\mathfrak{c}_i\mu_0(t)^2\Big(1+\Theta_i[\Lambda,\dot\Lambda,\xi,\phi]\Big) \qquad\qquad \text{for}\quad i=1,2,3,\end{cases} \tag{6.1}$$

with g,G,a as in Lemma 5.1 and Θ_i as in Lemma 5.2. Next, we verify that this system is solvable for Λ,ξ satisfying (3.17),(3.18) respectively. This relies on the following crucial proposition, proved in section 8, on the solvability of the nonlocal operator $\mathcal{J}[\dot\Lambda](0,t)=g(t)$ for g as in (5.2).

Proposition 6.1. *Let $h:[t_0,\infty)\to\mathbb{R}$ a function satisfying $\|h\|_{\sharp,c_1,c_2,\varepsilon}<\infty$ for some constants $\varepsilon>0$ and c_1,c_2 such that*

$$0<c_2\le c_1<\frac{\lambda_1-\gamma}{2\gamma}. \tag{6.2}$$

Then there exists a function $\Lambda\in C^{\frac12+\varepsilon}(t_0-1,\infty)$ satisfying

$$\mathcal{J}[\dot\Lambda](0,t)=h(t)\quad in\ (t_0,\infty), \tag{6.3}$$

where $\mathcal{J}[\dot\Lambda]$ satisfies (2.35), and there exists a constant C_1 such that

$$\|\Lambda\|_{\sharp,c_1,c_2,\varepsilon+\frac12}\le C_1\|h\|_{\sharp,c_1,c_2,\varepsilon}. \tag{6.4}$$

Moreover, if $\|h\|_{\sharp,\frac{1}{2}+\varepsilon,c} < \infty$ *then* $\Lambda \in C^{1,\varepsilon}(t_0-1,\infty)$ *and there exists a constant* C_2 *such that*

$$\|\dot{\Lambda}\|_{\sharp,c_1,c_2,\varepsilon} \leq C_2 \|h\|_{\sharp,c_1,c_2\,\frac{1}{2}+\varepsilon}. \tag{6.5}$$

Thus, the linear operators

$$\begin{aligned} T_1 : & X_{\sharp,c_1,c_2,\varepsilon} \to X_{\sharp,c_1,c_2,\varepsilon+\frac{1}{2}} \\ & h(t) \mapsto \Lambda[h](t), \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} \hat{T}_1 := \frac{d}{dt} \circ T_1 : & X_{\sharp,c_1,c_2,\frac{1}{2}+\varepsilon} \to X_{\sharp,c_1,c_2,\varepsilon} \\ & h(t) \mapsto \dot{\Lambda}[h](t), \end{aligned} \tag{6.7}$$

are well-defined and continuous.

We are ready to solve the system (6.1) in Λ, ξ for fixed $\phi \in X_*$.

Proposition 6.2. *Suppose that* ϕ *satisfies (4.12). Then, there exist* $\Lambda = \Lambda[\phi](t)$ *and* $\xi = \xi[\phi](t)$ *to the nonlinear nonlocal ODE system (6.1) which satisfy (3.17) and* (3.18) *respectively. Moreover, they satisfy*

$$\begin{aligned} &\|\Lambda[\phi_1] - \Lambda[\phi_2]\|_{\sharp,l_0,\delta_0,\frac{1}{2}+\varepsilon} \leq \mathbf{c}\|\phi_1-\phi_2\|_*, \\ &\|\dot{\Lambda}[\phi_1] - \dot{\Lambda}[\phi_2]\|_{\sharp,l_1,\delta_1,\varepsilon} \leq \mathbf{c}\|\phi_1-\phi_2\|_*, \\ &\|\xi[\phi_1] - \xi[\phi_2]\|_{\sharp,1+l_0,\frac{1}{2}+\varepsilon} \leq \mathbf{c}\|\phi_1-\phi_2\|_*, \\ &\|\dot{\xi}[\phi_1] - \dot{\xi}[\phi_2]\|_{\sharp,1+l_0,\varepsilon} \leq \mathbf{c}\|\phi_1-\phi_2\|_*, \end{aligned} \tag{6.8}$$

with constant $\mathbf{c} < 1$ *provided that* t_0 *is fixed sufficiently large and* $\mathfrak{b}_i$ *small for* $i = 1, 2$.

Proof. Firstly, we observe that equation (5.2) can be rewritten as

$$\mathcal{J}[\dot{\Lambda}](0,t) = g_1(t) + G_1[\dot{\Lambda}, \Lambda, \dot{\xi}, \xi, \phi](t),$$

where

$$g_1(t) + G_1[\dot{\Lambda}, \Lambda, \dot{\xi}, \xi] = (1 + a[\dot{\Lambda}, \xi])^{-1}[g(t) + G[\dot{\Lambda}, \Lambda, \dot{\xi}, \xi(t)], \tag{6.9}$$

for new functions g_1, G_1 satisfying the same properties of g, G in Lemma 5.1. By Proposition 6.1 we reduce the equation for Λ to a fixed point problem

$$\dot{\Lambda}(t) = \mathcal{F}_1[\dot{\Lambda}](t), \quad \mathcal{F}_1[\dot{\Lambda}](t) = \hat{T}\Big[g_1(t) + G_1[\Lambda, \dot{\Lambda}, \xi, \dot{\xi}, \phi]\Big],$$

where $\hat{T}_1$ is defined in (6.7). Let

$$\dot{\Lambda}_0(t) := \hat{T}_1[g_1](t)$$

and define the operator $\mathcal{L}_1[h] := \hat{T}_1[h - g_1]$. We use the notation

$$\mathcal{L}_1[h] = \lambda[h](t) := \dot{\Lambda}[h](t) - \dot{\Lambda}_0(t),$$

for any $h \in X_{\sharp,l_1,\delta_1,\frac{1}{2}+\varepsilon}$. Observe that

$$\begin{aligned} |\dot{\Lambda}[h]| &= |\dot{\Lambda}_0| + |\lambda[h]| \\ &\lesssim \mu^{l_1}\|g\|_{\sharp,l_0-2\alpha,\delta_1,\frac{1}{2}+\varepsilon} + e^{-\kappa t_0}\mu^{l_1}\|h\|_{\sharp,l_1,\delta_1,\frac{1}{2}+\varepsilon} \end{aligned}$$

Given $h_j \in X_{\sharp,1+l_0,\frac{1}{2}+\varepsilon}$ we consider the solution to the ODE

$$\dot{\xi}_{1,j} = \mathfrak{c}_j \mu_0(t)^{1+l_0} + h_j(t), \tag{6.10}$$

given explicitly by

$$\xi_{1,j}[h](t) = \mathfrak{c}_j \int_t^\infty \mu_0(s)^{1+l_0}\, ds + \int_t^\infty h(s)\, ds := \Upsilon_j + \int_t^\infty h(s)\, ds.$$

In particular

$$|\xi_{1,j}(t)| \lesssim \mu_0(t)^{1+l_0} + \mu_0(t)^{1+l_0} \|h\|_{1+l_0,\infty}, \qquad |\dot{\xi}_{1,j}(t)| \lesssim \mu_0(t)^{1+l_0} \|h\|_{1+l_0,\infty}$$

We define the vector

$$\Xi(t) := \dot{\xi} - \dot{\Upsilon} = h(t),$$

where $h = (h_1, h_2, h_3)$ satisfies $\|h_i\|_{\sharp,1+l_0,\frac{1}{2}+\varepsilon} < \infty$ for $i = 1, 2, 3$.

$$\|h\|_{\sharp,1+l_0,\frac{1}{2}+\varepsilon} := \max_{i=1,2,3} \Big\{ \|h_i\|_{\sharp,1+l_0,\frac{1}{2}+\varepsilon} \Big\}.$$

Let $\mathcal{L}_2$ the linear operator defined as $\mathcal{L}_2[h] = \Xi$ by relation (6.10) for $i = 1, 2, 3$. We observe that $(\dot{\Lambda}, \dot{\xi})$ is a solution to (6.1) if (λ, Ξ) is a fixed point

$$(\Lambda, \Xi) = \mathcal{A}(\Lambda, \Xi),$$

where $\mathcal{A}$ is the operator

$$\mathcal{A}(\lambda, \Xi) := (\mathcal{A}_1[\lambda, \Xi], \mathcal{A}_2(\lambda, \Xi)) := \Big(\hat{T}_1[\hat{G}_1[\lambda, \Xi, \phi]], \mathcal{L}_2[\hat{\Theta}[\lambda, \Xi, \phi]] \Big),$$

where

$$\hat{G}_1(\lambda, \Xi, \phi) := G_1\Big[\Lambda_0(t) + \int_t^\infty \lambda(s)\, ds, \Upsilon + \int_t^\infty \Xi(s)\, ds, \phi\Big],$$

$$\hat{\Theta}(\lambda, \Xi, \phi) := \Theta\Big[\Lambda_0(t) + \int_t^\infty \lambda(s)\, ds, \Upsilon(t) + \int_t^\infty \Xi(s)\, ds, \phi\Big],$$

with G_1 and Θ defined in (6.9) and (5.5). We show that there exists a unique fixed point $\lambda = \lambda[\phi], \Xi[\phi]$ in

$$\mathcal{B} = \{(\lambda, \Xi) \in (L^\infty(t_0, \infty))^4 : \|\lambda\|_{\sharp,l_1,\delta_1,\varepsilon} + \|\Xi\|_{\sharp,1+l_0,\frac{1}{2}+\varepsilon} \le e^{-\kappa t_0} L\}$$

for some L fixed large. Indeed, estimates (6.5) and (5.3) give

$$\begin{aligned} \|\mathcal{A}_1[\lambda, \Xi]\|_{\sharp,l_1,\delta_1,\varepsilon} &\le C_2 \|\hat{G}_1[\lambda, \Xi, \phi]\|_{\sharp,l_1,\delta_1,\frac{1}{2}+\varepsilon} \\ &\le C_2 e^{-\kappa t_0} \Big\{ \|\lambda\|_{\sharp,l_1,\delta_1,\varepsilon} + \|\Xi\|_{\sharp,1+l_0,\frac{1}{2}+\varepsilon} + \|\phi\|_* \Big\}. \end{aligned}$$

Also, from (5.6)

$$\begin{aligned} \|\mathcal{A}_2[\lambda, \Xi]\|_{\sharp,l_0,\delta_0,\frac{1}{2}+\varepsilon} &\le \|\Theta[\lambda, \Xi, \phi]\| \\ &\le C e^{-\kappa t_0} \Big\{ \|\lambda\|_{\sharp,l_1,\delta_1,\varepsilon} + \|\Xi\|_{\sharp,1+l_0,\frac{1}{2}+\varepsilon} + \|\phi\|_* \Big\}. \end{aligned}$$

Then, we have to verify that $\mathcal{A}$ is a contraction. For instance, we have

$$\begin{aligned} \|\mathcal{A}_1[\lambda_1, \Xi] - \mathcal{A}_1[\lambda_2, \Xi]\|_{\sharp,l_1,\delta_1,\varepsilon} &= \|\hat{T}_1[\hat{G}_1[\lambda_1, \Theta, \phi] - \hat{G}_1[\lambda_2, \Theta, \phi]]\|_{\sharp,l_1,\delta_1,\frac{1}{2}+\varepsilon} \\ &\le C_2 \|\hat{G}_1[\lambda_1, \Theta, \phi] - \hat{G}_1[\lambda_2, \Theta, \phi]\|_{\sharp,l_0-2\alpha,\delta_1,\frac{1}{2}+\varepsilon} \\ &\le C_2 \mathbf{c} \|\lambda_1 - \lambda_2\|_{\sharp,l_1,\delta_1,\varepsilon} \end{aligned}$$

where $C_2, \mathbf{c}$ is the constant appearing in (6.5) and (5.4) respectively. Since $\mathbf{c}$ can be as small as required provided that t_0 is fixed sufficiently large, we obtain that $\mathcal{A}_1$ is a contraction map. By means of the Lipschitz property of $\hat{\Theta}$ in (5.7) we can estimate $\mathcal{A}_2[\lambda_1, \Xi_1] - \mathcal{A}_2[\lambda_2, \Xi]$ similarly. Finally, using the estimates on $\hat{G}, \hat{\Theta}$ with respect to Ξ, we get

$$\|\mathcal{A}(\lambda_1, \Xi_1) - \mathcal{A}(\lambda_1, \Xi_1)\| \leq \mathbf{c}\Big[\|\lambda_1 - \lambda_2\|_{\sharp, l_1, \delta_1, \varepsilon} + \|\Xi_1 - \Xi_2\|_{\sharp, l_1, \delta_1, \frac{1}{2}+\varepsilon}\Big].$$

As a consequence of the Banach fixed point theorem, provided that L and t_0 are fixed large, the map $\mathcal{A}$ has a unique fixed point (λ, Ξ) in the space $\mathcal{B}$. Observe that

$$\Lambda[h](t) = -\int_t^\infty \hat{T}_1[h](s)\, ds = T_1[h],$$

where T_1 is defined in (6.6), satisfies (3.17) thanks to (6.4). Also, the components of vector $\xi_1 = \int_t^\infty \Xi(s)\, ds$ satisfy (3.18). This proves the existence of a solution (Λ, ξ) to the system (6.1) satisfying (3.17)-(3.18). With similar estimates on $\lambda[\phi_1] - \lambda[\phi_2]$ and $\Xi[\phi_1] - \Xi[\phi_2]$ using (5.4),(5.7) relations (6.8) follow. □

We observe from the proof that $\hat{T}_1$, like an half-fractional derivative, loses 1/2-Hölder exponent but we regain it through g, G_1 as a consequence of estimates on ψ from Proposition 4.1. This is the main reason to put all the terms of $S[u_3]$ involving directly $\dot{\mu}$ in the outer error (2.25). Indeed, to get $\dot{\Lambda} \in C^\varepsilon$ it is crucial to allow H in (3.8) (and hence S_{in} in (2.24)) to depend on $\dot{\Lambda}$ only indirectly through $\psi[\dot{\Lambda}]$ or $J_1[\dot{\Lambda}]$.

Remark 6.1. *By remark 4.2 the outer solution $\psi = \Psi[\psi_0]$ is smooth as a function of the initial datum ψ_0, provided that $\|\psi_0\|_\infty + \|\nabla\psi_0\|_\infty$ is sufficiently small. Thus, also the parameters $\Lambda[\psi_0], \xi[\psi_0]$ found in the previous preposition depend smoothly on ψ_0, and from the proof we also obtain*

$$\|\Lambda[\psi_0^1] - \Lambda[\psi_0^2]\|_\infty \lesssim \|\psi_0^1 - \psi_0^2\|_\infty,$$
$$\|\xi_1[\psi_0^1] - \xi_1[\psi_0^2]\|_\infty \lesssim \|\psi_0^1 - \psi_0^2\|_\infty.$$

7. Final argument: solving the inner problem

This section provides the final step in the proof of Theorem 1. At this point, given ϕ satisfying (4.12), we have a solution $\psi = \Psi[\Lambda[\phi], \xi[\phi], \phi]$ of the outer problem (3.4) and parameters $\Lambda[\phi], \xi[\phi]$ such that the orthogonality conditions (5.1) are satisfied. Thus, to get a solution

$$u = u_3 + \tilde{\phi}(x, t),$$

where $\tilde{\phi}$ is defined in (3.2), we need to prove the existence of ϕ such that $\|\phi\|_* < \infty$.

Proof of Theorem 1. We make a fixed point argument using the linear estimate (3.15). Proposition 3.1 well defines a linear operator $\mathcal{T} : h \mapsto (\phi[h], e[h])$ and it is continuous between the L^∞-weighted space described in (3.15). Thus, the solution ϕ to the nonlinear inner problem satisfies

$$\phi = \mathcal{A}_{\text{in}}(\phi), \quad \text{where} \quad \mathcal{A}_{\text{in}}(\phi) := \mathcal{T}(H[\phi]). \tag{7.1}$$

We claim that $\mathcal{A}_{\text{in}}$ has a unique fixed point in the space

$$\mathcal{B} = \{\phi \in L^\infty(B_{2R}) : \|\phi\|_* \leq B\},$$

for some fixed constant $\mathfrak{b}$ large. Firstly, we prove

$$|H[\Lambda, \xi, \dot{\Lambda}, \dot{\xi}](y, t)| \lesssim e^{-\kappa t_0} \frac{\mu^{1+l_1}}{1 + |y|^4}.$$

We recall that

$$\begin{aligned}H[\phi,\psi,\mu,\dot\mu,\xi,\dot\xi](y,\tau) :=& 5U(y)^4\mu\left(\frac{\mu_0}{\mu}\right)^{1/2}\psi(\mu y+\xi,t(\tau))\\&+B_0[\phi+\mu\psi]\,(\mu y+\xi,t(\tau))+\mu^{5/2}S_{\rm in}(\mu y+\xi,t(\tau))\\&+\mathcal N(\mu^{1/2}u_3,\mu^{1/2}\tilde\phi)(\mu y+\xi,t(\tau)).\end{aligned}$$

Using the estimate on ψ from Proposition 4.1, we have

$$|5U(y)^4\mu\left(\frac{\mu_0}{\mu}\right)^{1/2}\psi(\mu y+\xi,t(\tau))|\lesssim e^{-\kappa t_0}\frac{\mu^{1+l_1+\delta}}{1+|y|^{4+\alpha}}$$

and from (3.6) we get

$$\begin{aligned}|B_0[\phi+\mu\psi]\,(\mu y+\xi,t(\tau))| &\lesssim \frac{1}{1+|y|^3}|\mu H_\gamma+\mu J+\mu^{-1/2}\phi_3\eta_3|\,(\phi+\mu\psi)\\&\lesssim \frac{\mu}{1+|y|^3}\left(Me^{-\kappa t_0}\frac{\mu^{1+l_1}R^3\log(R)}{1+|y|^4}+\mu\frac{\mu^{l_1+\delta}}{1+|y|^\alpha}\right)\\&\lesssim e^{-\kappa t_0}\frac{\mu^{1+l_1}}{1+|y|^4}.\end{aligned}$$

Recalling the estimates on ϕ at $y\sim 0$ and $y\sim R$ given by the norm (3.12), using that $R=\mu^{-\delta}$ with δ satisfying (2.28) we deduce

$$\begin{aligned}|\mathcal N(\mu^{1/2}u_3,\mu^{1/2}\tilde\phi)(\mu y+\xi,t(\tau))| &\lesssim \frac{1}{1+|y|^3}(\phi+\mu\psi)^2\\&\lesssim \frac{1}{1+|y|^3}\left(M\frac{\mu^{2(1+l_1)}R^6\log^2(R)}{1+|y|^8}+\mu^2\frac{\mu^{2(l_1+\delta-\alpha)}}{1+|y|^{2\alpha}}\right)\\&\lesssim e^{-\kappa t_0}\frac{\mu^{1+l_1}}{1+|y|^4}.\end{aligned}$$

By Lemma 2.5 we have the main error

$$|\mu^{5/2}S_{\rm in}(\mu y+\xi,t(\tau))|\lesssim\frac{\mu^{1+l_1}}{1+|y|^4}.$$

Thus, provided that t_0 is large enough, we have

$$\|\mathcal T[H]\|_*\le C\|H\|_{\nu,4}<B,$$

for B chosen large, where C is the constant in (3.16). This proves $\mathcal A_{\rm in}(\phi)\in\mathcal B$. Now, we need to prove that for $\phi^{(1)},\phi^{(2)}\in\mathcal B$ we have

$$|H[\phi^{(1)}]-H[\phi^{(2)}]|\lesssim \mathbf c\|\phi^{(1)}-\phi^{(2)}\|_*\frac{\mu^{1+l_1}}{1+|y|^4},$$

for some $\mathbf c<1$. This is a consequence of Propositions 4.2 and 6.2. Indeed, for instance we get

$$\begin{aligned}&5U(y)^4\mu_0\Big|e^{\Lambda[\phi^{(1)}]}\psi[\phi^{(1)}]-e^{\Lambda[\phi^{(2)}]}\psi[\phi^{(2)}]\Big|\\&\quad=5U(y)^4\mu_0\Big|\Big[e^{\Lambda[\phi^{(1)}]}-e^{\Lambda[\phi^{(2)}]}\Big]\psi[\phi^{(1)}]+e^{\Lambda[\phi^{(2)}]}\Big[\psi[\phi^{(1)}]-\psi[\phi^{(2)}]\Big]\Big|\\&\quad\lesssim \mathbf c\|\phi^{(1)}-\phi^{(2)}\|_*\frac{\mu^{1+l_1}}{1+|y|^4},\end{aligned}$$

and similarly we get the same control on the other terms of $H[\phi^{(1)}] - H[\phi^{(2)}]$. Finally, since the operator $\mathcal{T} : X_{\nu,4} \to X_*$ is continuous, where $X_{\nu,4}$ is defined in (3.11) for $a = 2$, by composition with $H : X_* \to X_{\nu,4}$ we obtain for B_1

$$\|\mathcal{A}_{\text{in}}[\phi^{(1)}] - \mathcal{A}_{\text{in}}[\phi^{(2)}]\|_* \le \mathbf{c}\|\phi^{(1)} - \phi^{(2)}\|_*,$$

provided that t_0 is fixed sufficiently large. Thus, $\mathcal{A}_{\text{in}} : \mathcal{B} \to \mathcal{B}$ is a contraction map and by Banach fixed point theorem we obtain the existence and uniqueness of $\phi \in X_*$ such that (7.1) holds. Finally, we recall that the constant $e_0 = e_0[H]$ in the initial condition $\phi(y, t_0) = e_0 Z_0(y)$ is a linear operator of H. The existence of ϕ immediately defines e_0. This completes the proof of the existence of $u = u_3 + \tilde{\phi}$ in Theorem 1, with the bubbling profile centered in $x = 0 \in \Omega$ and parameters satisfying (1.7). □

Remark 7.1 (continuity of (ϕ, e_0) with respect to ψ_0)**.** *We found the inner perturbation ϕ and its initial datum $\phi(y, t_0) = e_0 Z_0(y)$ based on the existence of the outer solution $\Psi[\phi]$ given by Proposition 4.1, which in fact can be found for any initial condition $\psi_0 \in C^1(\bar{\Omega})$. Furthermore, as a consequence of the continuity of $\Psi[\psi_0]$ and $\Lambda[\psi_0], \xi[\psi_0]$ found in remark 4.2 and 6.1 we obtain*

$$|e_0[\psi_0^1] - e_0[\psi_0^2]| \lesssim \Big[\|\psi_0^1 - \psi_0^2\|_{L^\infty(\Omega)} - \|\nabla\psi_0^1 - \nabla\psi_0^2\|\Big].$$

Since we know that $\Lambda, \dot{\Lambda}, \xi, \dot{\xi}, \psi$ depends smoothly on ψ_0, by the implicit function theorem, we deduced that map $\psi_0 \mapsto (\phi[\psi_0], e_0[\psi_0])$ is C^1 with respect to $\psi_0 \in C^1(\bar{\Omega})$. This allows to prove the 1-codimensional stability of Corollary 1.1. We omit the proof since it is identical to that one of [5].

8. Invertibility theory for the nonlocal linear problem

In this section we prove Proposition 6.1. We deduce the result by Laplace transform method combined with asymptotic estimates of the heat kernel p_t^Ω associated to Ω. It turns out that the operator $\mathcal{J}[\dot{\Lambda}]$ in (2.35) is similar to a half-fractional integral of $\dot{\Lambda}$. Thus, roughly speaking, we expect the inverse operator to behave as a fractional derivative of order 1/2. In fact, the Proposition 6.1 can be seen as a precise statement of this idea.

For later purpose we recall some facts about the Dirichlet heat kernel. For the definition and properties we follow [12,18]. A function $p_t^\Omega(x,y)$ continuous on $\bar{\Omega}\times\bar{\Omega}\times\mathbb{R}^+$, C^2 in x and C^1 in t is called Dirichlet heat kernel for the problem

$$\begin{aligned}
&\partial_t u(x,t) = \Delta u(x,t) \quad \text{in } \Omega \times \mathbb{R}^+,\\
&u(x,t) = 0 \quad \text{on } \partial\Omega \times [0,\infty),\\
&u(x,0) = u_0(x) \quad \text{in } \Omega,
\end{aligned}$$

if, for any $y \in \Omega$, satisfies

$$\begin{aligned}
&\partial_t p_t^\Omega(x,y) = \Delta_x p_t^\Omega(x,y) \quad \text{in } \Omega \times \mathbb{R}^+,\\
&p_t^\Omega(x,y) = 0 \quad \text{in } \Omega,
\end{aligned}$$

and

$$\lim_{t\to 0^+} \int_\Omega p_t^\Omega(x,y) u_0(y)\, dy = u_0(x),$$

uniformly for every function $u_0 \in C_0(\bar{\Omega})$. The existence of the Dirichlet heat kernel is a classical result by Levi [24]. It has the following basic properties:

- $p_t^\Omega(x,y) \geq 0$, $p_t^\Omega(x,y) = p_t^\Omega(y,x)$ and $p_t^\Omega(x,y,t) = 0$ if $x \in \partial\Omega$;
- for any $y \in \Omega$ the function $p_t^\Omega(x,y) \in C^\infty(\mathbb{R}^+ \times \Omega)$;
- it satisfies $\partial_t p_t^\Omega(x,y) = \Delta_x p_t^\Omega(x,y)$ for $(x,y,t) \in \Omega \times \Omega \times \mathbb{R}^+$ and ;
- $p_t^\Omega(x,y) = 0$ for $x \in \partial\Omega$ or $y \in \partial\Omega$;

Also, from [18, Theorem 10.13] and its proof, the heat kernel $p_t^\Omega(x,y)$ admits the expansion

$$p_t^\Omega(x,y) = \sum_{k\geq 1} e^{-\lambda_k t}\phi_k(x)\phi_k(y), \tag{8.1}$$

where λ_k is the k-th Dirichlet eigenvalue of $-\Delta$ on Ω and ϕ_k the corresponding eigenfunction and also for $t \geq 1$ and $n \geq 1$ (see [18, Remark 10.15])

$$\sum_{k=n}^{\infty} \sup_{x,y\in\Omega} |\phi_k(x)\phi_k(y)| < \infty \tag{8.2}$$

The series (8.1) converges absolutely and uniformly in $[\varepsilon,\infty] \times \Omega \times \Omega$ for any $\varepsilon > 0$, as well as in the topology of $C^\infty(\mathbb{R}^+ \times \Omega \times \Omega)$.

Before starting the proof of Proposition 6.1, we recall an estimate on the short time behaviour of the heat kernel $p_\tau^\Omega(x,y)$ due to Varadhan [33, Theorem 4.9]. We will use it in the following form as in Hsu [22, Corollary 1.6].

Lemma 8.1. *Let $\varepsilon > 0$ fixed such that $B_\varepsilon(0) \subset \Omega$. Then for $y \in B_\varepsilon(0)$ we have*

$$p_\tau^{\mathbb{R}^3}(0,y)(1 - e^{-\frac{\delta^2}{4\tau}}) \leq p_\tau^\Omega(0,y), \tag{8.3}$$

where $\delta < \delta_0$ is independent of y and

$$\delta_0 := d(\partial\Omega, \partial B_\varepsilon) = \min_{a\in\partial\Omega, b\in\partial B_\varepsilon(0)} |a-b| > 0.$$

Proof. Recall the useful identities in [33, p. 675]

$$\limsup_{\tau\to 0} 4\tau \log\Big(p_\tau^{\mathbb{R}^3}(x,y) - p_\tau^\Omega(x,y)\Big) \leq -d_{\partial\Omega}(x,y)^2, \tag{8.4}$$

$$\lim_{\tau\to 0} 4\tau \log\Big(p_t^{\mathbb{R}^3}(x,y)\Big) = -d(x,y)^2, \tag{8.5}$$

where

$$d_{\partial\Omega}(x,y) := \inf_{z\in\partial\Omega}\{d(x,z) + d(z,y)\}.$$

From (8.4) there exists τ_0 such that for $\tau < \tau_0$ we have

$$p_\tau^{\mathbb{R}^3}(x,y) - e^{-\frac{d_{\partial\Omega}(x,y)^2}{4\tau}} \leq p_\tau^\Omega(x,y),$$

for all $x,y \in \Omega$. In particular, fix $x = 0$ and consider $y \in B_\varepsilon(0) \subset \Omega$ for a small $\varepsilon > 0$. Then we have

$$d_{\partial\Omega}(0,y) \geq \varepsilon + \delta_0.$$

Thus for $y \in B_\varepsilon(0)$

$$e^{-\frac{d^2_{\partial\Omega}(0,y)}{4\tau}} \leq e^{-\frac{\varepsilon^2+\delta_0^2}{4\tau}} \leq e^{-\frac{d(0,y)^2}{4\tau}} e^{-\frac{\delta_0^2}{4\tau}},$$

and (8.5) says

$$p_\tau^{\mathbb{R}^3}(0,y) = e^{-\frac{d^2(0,y)}{4\tau}(1+o(1))} \quad \text{as} \quad \tau \to 0^+.$$

Thus, we have for $\tau < \tau_0$ small and $y \in B_\varepsilon(0)$

$$p_\tau^{\mathbb{R}^3}(0,y)(1 - e^{-\frac{\delta^2}{4\tau}}) \le p_\tau^{\Omega}(0,y), \tag{8.6}$$

for any $\delta < \delta_0$ independent of y. $\square$

We mention that the uniform bound (8.6) holds for y ranging in any convex subset of the domain, see [22, p.374-375]. Also, for any $\tau > 0$ and $x, y \in \Omega$ we have the upper bound

$$p_\tau^{\Omega}(x,y) \le p_\tau^{\mathbb{R}^3}(x,y), \tag{8.7}$$

as a consequence of the maximum principle. Thus, Varadhan's estimate (8.3) is a precise statement about the idea that for small times the heat kernel "does not feel the boundary". We refer to Kac [23] and Dodziuk [12] for statements of the same flavor. In the proof of Proposition 6.1 we need the following lemma.

Lemma 8.2. *Define the function*

$$I(\tau) := \int_\Omega p_\tau^{\Omega}(0,y) G_\gamma(y,0)\, dy,$$

where $p_\tau^{\Omega}(x,y)$ denotes the Dirichlet heat kernel associated to Ω and $G_\gamma(x,y)$ the Green function of the operator $-\Delta - \gamma$ on Ω. Then $I(\tau)$ has the following asymptotic behavior:

$$I(\tau) = \begin{cases} O\Big(e^{-\lambda_1 \tau}\Big) & \text{for} \quad \tau \to \infty, \\ c_{1,*}\sqrt{\tau} + c_{2,*}\sqrt{\tau} + c_{3,*}\tau + O(\tau^{3/2}) & \text{for} \quad \tau \to 0^+, \end{cases} \tag{8.8}$$

for some constant $c_{i,*}$ for $i = 1, 2, 3$.

Proof. **Step 1** (Asymptotic for $t \to \infty$). We recall that the heat kernel $p_\tau^{\Omega}(x,y)$ admits the series expansion (8.1) which converges absolutely and uniformly in the domain $[\varepsilon, \infty) \times \Omega \times \Omega$ for any $\varepsilon > 0$, as well as in the topology $C^\infty(\mathbb{R}^+ \times \Omega \times \Omega)$. By the uniform convergence with respect to $y \in \Omega$ we obtain for $\tau > 0$

$$\begin{aligned} I(\tau) &= \int_\Omega \sum_{k=1}^{\infty} e^{-\lambda_k \tau} \phi_k(0)\phi_k(y) G_\gamma(y,0)\, dy \\ &= \sum_{k=1}^{\infty} e^{-\lambda_k \tau} \int_\Omega \phi_k(y) G_\gamma(y,0)\, dy. \end{aligned} \tag{8.9}$$

Multiplying equation (2.7) by ϕ_k and integrating by parts we get

$$\begin{aligned} -\lambda_k \int_\Omega G_\gamma(x,0)\phi_k(x)\, dx &= \int_\Omega G_\gamma(x,0)\Delta\phi_k(x) \\ &= \int_\Omega \phi_k(x)\Delta G_\gamma(x,0)\, dx \\ &= -\gamma \int_\Omega G_\gamma(x,0)\phi_k(x)\, dx - c_3 \int_\Omega \phi_k(x)\delta_0(x)\, dx \\ &= -\gamma \int_\Omega G_\gamma(x,0)\phi_k(x)\, dx - c_3\phi_k(0), \end{aligned}$$

that gives

$$\int_\Omega G_\gamma(x,0)\phi_k(x)\, dx = c_3 \frac{\phi_k(0)^2}{\lambda_k - \gamma}. \tag{8.10}$$

We plug (8.10) into (8.9). Finally, from (8.2) we obtain the asymptotic behaviour (8.8) for $\tau \to \infty$.

Step 2 (Asymptotic for $t \to 0^+$). Firstly, we split

$$\begin{aligned} I(\tau) &= \int_\Omega p_\tau^\Omega(0,y)\frac{\alpha_3}{|y|}\,dy + \int_\Omega p_\tau^\Omega(0,y)H_\gamma(y,0)\,dy \\ &=: I_1(\tau) + I_2(\tau). \end{aligned}$$

We analyze $I_1(\tau)$. For the region $B_\varepsilon(0)$ we invoke Varadhan's estimate (8.6) and we obtain

$$\begin{aligned} \int_{B_\varepsilon(0)} \frac{p_\tau^\Omega(0,y)}{|y|}\,dy &\geq \int_{B_\varepsilon} \frac{e^{-\frac{|y|^2}{4\tau}}}{[4\pi\tau]^{3/2}}\frac{1}{|y|}\,dy(1-e^{-\frac{\varepsilon^2}{\tau}}) \\ &= 4\pi\int_0^\varepsilon \frac{e^{-\frac{\rho^2}{4\tau}}}{[4\pi\tau]^{3/2}}\rho\,d\rho(1-e^{-\frac{\varepsilon^2}{\tau}}) \\ &= \frac{1}{\sqrt{4\pi\tau}}\int_0^{\frac{\varepsilon}{2\sqrt{\tau}}} e^{-r^2}r\,dr(1-e^{-\frac{\varepsilon^2}{\tau}}) \\ &= \frac{1}{\sqrt{4\pi\tau}}\left(\frac{1-e^{-\frac{\varepsilon^2}{4\tau}}}{2}\right)\left(1-e^{-\frac{\varepsilon^2}{\tau}}\right) \\ &= \frac{1}{4\sqrt{\pi\tau}} + O\left(\frac{e^{-\frac{c}{\tau}}}{\sqrt{\tau}}\right) \end{aligned}$$

for some $c > 0$, and using (8.7)

$$\begin{aligned} \int_{B_\varepsilon(0)} \frac{p_\tau^\Omega(0,y)}{|y|}\,dy &\leq \int_{B_\varepsilon(0)} \frac{p_\tau^{\mathbb{R}^3}(0,y)}{|y|}\,dy \\ &\leq \frac{1}{4\sqrt{\pi\tau}} + O\left(\frac{e^{-\frac{c}{\tau}}}{\sqrt{\tau}}\right). \end{aligned}$$

From these bounds we conclude

$$\int_{B_\varepsilon(0)} \frac{p_\tau^\Omega(0,y)}{|y|}\,dy = \frac{1}{4\sqrt{\pi\tau}} + O\left(\frac{e^{-\frac{c}{\tau}}}{\sqrt{\tau}}\right).$$

In the region $\Omega \setminus B_\varepsilon(0)$ by (8.7) we get

$$\begin{aligned} \int_{\Omega\setminus B_\varepsilon(0)} \frac{p_\tau^\Omega(0,y)}{|y|}\,dy &\leq \tau^{-3/2}\int_\varepsilon^1 e^{-\frac{\rho^2}{cs}}\rho\,d\rho \\ &= \tau^{-1/2}\int_{\frac{\varepsilon}{\sqrt{s}}}^{\frac{1}{\sqrt{s}}} e^{-r^2}r\,dr = O\left(\frac{e^{-\frac{c}{\tau}}}{\sqrt{\tau}}\right). \end{aligned}$$

We conclude that

$$\begin{aligned} I_1(\tau) &= \alpha_3\int_{\Omega\setminus B_\varepsilon(0)} \frac{p_\tau^\Omega(0,y)}{|y|}\,dy + \alpha_3\int_{B_\varepsilon(0)} \frac{p_\tau^\Omega(0,y)}{|y|}\,dy \\ &= \frac{c_{1,*}}{\sqrt{\tau}} + O\left(\frac{e^{-\frac{c}{\tau}}}{\sqrt{\tau}}\right) \quad \text{as} \quad \tau \to 0^+, \quad c_{1,*} = \frac{\alpha_3}{4\sqrt{\pi\tau}}. \end{aligned}$$

Now, we estimate the term $I_2(\tau)$. We treat it similarly to $I_1(\tau)$ but we get a lower order term in the expansion since $H_\gamma(y,0)$ is not singular. We use decomposition (2.10) for

$H_\gamma(y,0)$ and we consider the integral over $B_\varepsilon(0)$. Using the cosine expansion we get

$$\theta_\gamma(y,0) = \alpha_3\frac{\gamma}{2}|y| + O(|y|^3).$$

Thus, we compute the integral associated to the first term with Varadhan's estimate (8.3) and the upper bound (8.7):

$$\begin{aligned}\int_{B_\varepsilon(0)} p_\tau^\Omega(y,0)\frac{1-\cos(\sqrt{\gamma}|y|)}{|y|}\,dy &= \alpha_3\frac{\gamma}{2}\int_{B_\varepsilon(0)}\frac{e^{-\frac{|y|^2}{4\tau}}}{[4\pi\tau]^{3/2}}|y|\,dy\Big(1+o\Big(e^{-\frac{c}{\tau}}\Big)\Big) \qquad (8.11)\\ &= 4\pi\alpha_3\frac{\gamma}{2}\int_0^\varepsilon\frac{e^{-\frac{\rho^2}{4\tau}}}{[4\pi\tau]^{3/2}}\rho^3\,d\rho\Big(1+o\Big(e^{-\frac{c}{\tau}}\Big)\Big)\\ &= 4\pi\alpha_3\sqrt{\tau}\frac{\gamma}{2}\int_0^{\frac{\varepsilon}{2\sqrt{\tau}}} e^{-r^2}r^3\,dr\Big(1+o\Big(e^{-\frac{c}{\tau}}\Big)\Big)\\ &= c_{2,*}\sqrt{\tau}\Big(1+o\Big(e^{-\frac{c}{\tau}}\Big)\Big),\end{aligned}$$

for an explicit constant $c_{2,*}$. The same computation on the remainder $O\Big(|y|^3\Big)$ gives a term of order $O\Big(\tau^{3/2}\Big)$. Another Taylor expansion at $y=0$ gives

$$h_\gamma(y,0) = \nabla_y h_\gamma(0,0)\cdot y + \frac{1}{2}y\cdot D_{yy}h_\gamma(0,0)\cdot y + O(|y|^3),$$

where $D_{yy}h_\gamma(0,0)$ denotes the Hessian of $h_\gamma(\cdot,0)$ evaluated in $y=0$. Integrating the first term on $B_\varepsilon(0)$ against $p_t^\Omega(0,y)$ and using (8.3)-(8.7) wee see by symmetry of the integrand $p_t^{\mathbb{R}^3}(0,y)\nabla_y h_\gamma(0,0)\cdot y$ that the integral gives an exponentially decaying term. The second term in (8.12) can be treated similarly to (8.11) and gives a term of order $c_{3,*}\tau(1+o(1))$ for some explicit constant $c_{3,*}$. The integral of $p_\tau^\Omega(y,0)H_\gamma(y,0)$ on the complement can be treated as before and gives an exponentially decay term for $\tau\to 0$. Thus we conclude that

$$I_2(\tau) = c_{2,*}\sqrt{\tau} + c_{3,*}\tau + O\Big(\tau^{3/2}\Big) \quad \text{as} \quad \tau\to 0^+.$$

We conclude that $I(\tau) = I_1(\tau)+I_2(\tau)$ has the asymptotic (8.8) for $\tau\to 0^+$. □

We start here the main proof of Proposition 6.1.

Proof of Proposition 6.1. Firstly, we observe that $J(0,t_0) = h(t_0)$ is in general not compatible with a null initial condition. For this reason it is natural to solve the problem for $\mathcal{J}$ starting from $t = t_0 - 1$. We look for $\Lambda(t)$ for $t\in(t_0-1,\infty)$. The function $\mathcal{J}$ is a solution to the problem

$$\begin{aligned}&\partial_t\mathcal{J} = \Delta_x\mathcal{J} + \gamma\mathcal{J} - \dot{\Lambda}(t)G_\gamma(x,0) \quad \text{in } \Omega\times(t_0-1,\infty),\\ &\mathcal{J}(x,t)\equiv 0 \quad \text{on } \partial\Omega\times(t_0-1,\infty),\end{aligned}$$

such that

$$\mathcal{J}(0,t) = h^*(t) \quad \text{in } (t_0,\infty),$$

where

$$h^*(t) = \begin{cases} h(t) & t\in[t_0,\infty),\\ h_{\text{ext}}(t) & t\in[t_0-1,t_0),\end{cases} \qquad (8.12)$$

and

$$h_{\mathrm{ext}}(t) = \eta(t)h(t_0),$$

where η is a smooth such that $\eta(t_0-1)=0$, $\eta(t_0)=1$ and

$$|\eta(t_0-\nu)h(t_0) - h(t_0+\nu)| \le [h]_{\varepsilon,[t_0,t_0+1]}\nu^{\varepsilon},$$

for any $\nu \le 1$. This choice gives an extension $h^*(t) \in C^{\varepsilon}$ with

$$\|h^*\|_{\sharp,c_1,c_2,(t_0-1,\infty)} \lesssim \|h\|_{\sharp,c_1,c_2,(t_0,\infty)}. \tag{8.13}$$

Let $s := t-(t_0-1)$ and for $s \in (0,\infty)$ define

$$\begin{aligned} \mathcal{J}_0(x,s) &:= e^{-\gamma s}\mathcal{J}(x, s+(t_0-1)), \\ \beta(s) &:= -\Lambda(s+(t_0-1)), \\ h_0^*(s) &:= h^*(s+(t_0-1)). \end{aligned} \tag{8.14}$$

The function $\mathcal{J}_0$ is a solution to

$$\begin{aligned} &\partial_s \mathcal{J}_0(x,s) = \Delta_x \mathcal{J}_0 + e^{-\gamma s}\dot{\beta}(s)G_\gamma(x,0) \quad \text{in } \Omega\times(0,\infty) \\ &\mathcal{J}_0(x,s) = 0 \quad \text{on } \partial\Omega\times(0,\infty), \end{aligned}$$

such that

$$\mathcal{J}_0[\dot{\beta}](0,s) = h_0^*(s)e^{-\gamma s} \quad \text{in } (0,\infty). \tag{8.15}$$

Imposing the initial condition $\mathcal{J}(x,t_0) \equiv 0$ in Ω, that is $\mathcal{J}_0(x,0) \equiv 0$, by Duhamel's formula we have

$$\mathcal{J}_0[\dot{\beta}](0,s) = \int_0^s e^{-\gamma(s-\tau)}\dot{\beta}(s-\tau)I(\tau)\,d\tau, \tag{8.16}$$

where

$$I(\tau) := \int_\Omega p_\tau^\Omega(0,y)G_\gamma(y,0)\,dy,$$

and $p_\tau^\Omega(x,y)$ denotes the heat kernel associated to Ω. The asymptotic behaviour of $I(\tau)$ is given by Lemma 8.2. We denote the Laplace transform of a function f as

$$\tilde{f}(\xi) := \int_0^\infty e^{-\xi s}f(s)\,ds.$$

We refer to the book [13] by Doetsch for classic properties of the Laplace transform. Applying the Laplace transform to (8.16) and using (8.15) we obtain

$$\begin{aligned} \tilde{h}_0^*(\xi+\gamma) &= \tilde{\dot{\beta}}(\xi+\gamma)\tilde{I}(\xi) \\ &= \Big[(\xi+\gamma)\tilde{\beta}(\xi+\gamma) - \beta(0)\Big]\tilde{I}(\xi), \end{aligned}$$

and hence

$$\tilde{\beta}(\xi+\gamma) = \frac{\beta(0)}{\xi+\gamma} + \tilde{h}_0^*(\xi+\gamma)\tilde{\sigma}(\xi), \tag{8.17}$$

where

$$\tilde{\sigma}(\xi) := \frac{1}{(\xi+\gamma)\tilde{I}(\xi)}.$$

By definition we have

$$\tilde{I}(\xi) = \int_0^\infty e^{-\xi s} I(s)\, ds,$$

that is well defined and analytic in the right-half plane $\operatorname{Re}\xi > -\lambda_1$ thanks to Lemma 8.2. By expansion (8.8) we have

$$|e^{-\xi s} I(s)| \lesssim g(s), \quad g(s) = \begin{cases} \frac{1}{\sqrt{s}} & \text{for} \quad s \to 0^+, \\ e^{-(\lambda_1 + \operatorname{Re}\xi)s} & \text{for} \quad s \to +\infty, \end{cases}$$

and g is integrable in $\mathbb{R}^+$ if $\operatorname{Re}\{\xi\} > -\lambda_1$. Thus, using (8.9), in any half plane $\operatorname{Re}\xi \geq c$ where $c > -\lambda_1$ the dominated convergence theorem applies to get

$$\begin{aligned} \tilde{I}(\xi) &= \int_0^\infty e^{-\xi s} I(s)\, ds \\ &= \int_0^\infty e^{-\xi s} \sum_{k=1}^\infty \frac{\phi_k(0)^2}{\lambda_k - \gamma} e^{-\lambda_k s}\, ds \\ &= \sum_{k=1}^\infty \frac{\phi_k(0)^2}{\lambda_k - \gamma} \int_0^\infty e^{-\xi s} e^{-\lambda_k s}\, ds \\ &= \sum_{k=1}^\infty \frac{\phi_k(0)^2}{\lambda_k - \gamma} \frac{1}{\lambda_k + \xi} \end{aligned}$$

At this point we can extend $\tilde{I}(\xi)$ analytically from $\{\xi \in \mathbb{C} : \xi > -\lambda_1\}$ to $\mathbb{C} \setminus \{-\lambda_k\}_{k=1}^\infty$. Let $\xi = a + ib$ and rewrite the series as

$$\begin{aligned} \tilde{I}(\xi) =& \sum_{k=1}^\infty \frac{\phi_k(0)^2}{\lambda_k - \gamma} \frac{1}{\lambda_k + a + ib} \\ =& \sum_{k=1}^\infty \frac{\phi_k(0)^2}{\lambda_k - \gamma} \frac{\lambda_k + a}{(\lambda_k + a)^2 + b^2} - ib \sum_{k=1}^\infty \frac{\phi_k(0)^2}{\lambda_k - \gamma} \frac{1}{(\lambda_k + a)^2 + b^2}. \end{aligned}$$

Since the coefficients of the series are positive, $\tilde{I}(\xi) = 0$ implies $b = 0$. Plugging $b = 0$ into the first series we obtain that a root $\xi = a$ of $\tilde{I}$ satisfies

$$\sum_{k=1}^\infty \frac{\phi_k(0)^2}{\lambda_k - \gamma} \frac{1}{\lambda_k + a} = 0.$$

Hence, we deduce that the set of zeros of $\tilde{I}$ is given by a sequence $\{-a_k\}_{k=1}^\infty$ where $a_k \in (\lambda_k, \lambda_{k+1})$. In particular,

$$\tilde{I}(\xi) \neq 0 \quad \text{for} \quad \operatorname{Re}\xi > -\lambda_1. \tag{8.18}$$

By standard argument [13, Theorem 33.7] on the Laplace transform, using (8.8), we have

$$\tilde{I}(\xi) = c_{1,*}\sqrt{\pi}\xi^{-1/2} + c_{2,*}\frac{\sqrt{\pi}}{2}\xi^{-3/2} + c_{3,*}\xi^{-2} + O(\xi^{-5/2}) \quad \text{as} \quad |\xi| \to \infty,$$

in the half-plane $\operatorname{Re}\xi > -\lambda_1$. Thus, in the same half-plane we have

$$\begin{aligned} \tilde{\sigma}(\xi) =& \frac{1}{(\xi + \gamma)\tilde{I}(\xi)} \\ =& d_{1,*}\xi^{-1/2} + d_{2,*}\xi^{-3/2} + d_{3,*}\xi^{-2} + O(\xi^{-5/2}) \quad \text{as} \quad |\xi| \to \infty. \end{aligned} \tag{8.19}$$

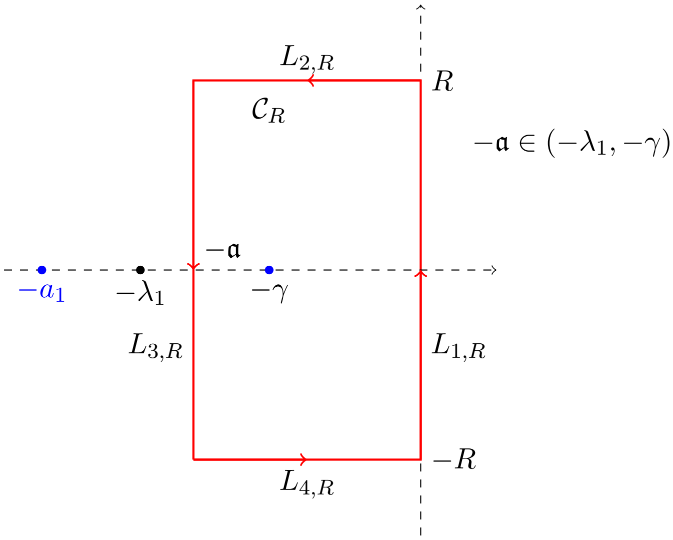

Figure 1. *Contour integral $\mathcal{C}_R$.*

As a consequence of (8.18), $\tilde{\sigma}(\xi)$ has a unique singularity at $\xi = -\gamma$ in the half-plane of convergence. By [13, Theorem 28.3] the function $\tilde{\sigma}(\xi)$ can be represented as a Laplace transform of a function.[1] Finally, we compute the inverse Laplace transform by means of the Residue theorem defining the rectangular contour integral $\mathcal{C}_R$ in Figure 1 as suggested in the proof of [13, Theorem 35.1]. For later purpose we observe that, looking at the contour integral $\mathcal{C}_R$, the constant $\mathfrak{a} \in (\gamma, \lambda_1)$ can be taken arbitrarily close to λ_1. An application of the Riemann-Lebesgue Lemma (as in [13, p.237]) implies

$$\lim_{R\to\infty} \int_{L_{2,R}} e^{\xi\tau} F(\xi)\, d\xi = 0,$$
$$\lim_{R\to\infty} \int_{L_{4,R}} e^{\xi\tau} F(\xi)\, d\xi = 0.$$

Since

$$\sigma(\tau) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{L_{1,R}} e^{\xi\tau} F(\xi)\, d\xi$$

we obtain

$$\sigma(t) = \operatorname{Res}\Big(e^{\xi t}\tilde{\sigma}(\xi), -\gamma\Big)e^{-\gamma t} + \lim_{R\to\infty} \frac{1}{2\pi i} \int_{-\mathfrak{a}-iR}^{-\mathfrak{a}+iR} e^{\xi t}\tilde{\sigma}(\xi)\, d\xi. \tag{8.20}$$

We easily compute

$$\operatorname{Res}\Big(e^{\xi\tau}\tilde{\sigma}(\xi), -\gamma\Big) = \lim_{\xi\to-\gamma} (\xi+\gamma)\frac{1}{(\xi+\gamma)\tilde{I}(\xi)} =: c_\infty.$$

[1] We cannot have an estimate directly on $\dot{\beta}$ at this point. Indeed, $(\tilde{I}(\xi))^{-1}$ is not a Laplace transform of a function since diverges as $|\xi| \to \infty$. However, it still can be represented as the Laplace transform of a distribution, see [13, Theorem 29.3].

Now, we analyze the integral (8.20). We decompose

$$\lim_{R\to\infty}\int_{-\mathfrak{a}-iR}^{-\mathfrak{a}+iR} e^{\xi\tau}\tilde{\sigma}(\xi)\,d\xi = ie^{-\mathfrak{a}\tau}\int_{-R}^{R} e^{iy\tau}\Big[\tilde{\sigma}(-\mathfrak{a}+iy) - \frac{d_{1,*}}{\sqrt{-\mathfrak{a}+iy}} - \frac{d_{2,*}}{(-\mathfrak{a}+iy)^{3/2}}$$
$$- \frac{d_{3,*}}{(-\mathfrak{a}+iy)^2}\Big]\,dy$$
$$+ ie^{-\mathfrak{a}\tau}\int_{-iR}^{iR} e^{\xi t}\frac{d_{1,*}}{\sqrt{-\mathfrak{a}+iy}}\,dy$$
$$+ ie^{-\mathfrak{a}\tau}\int_{-iR}^{iR} \frac{d_{2,*}}{(-\mathfrak{a}+iy)^{3/2}}\,dy$$
$$+ ie^{-\mathfrak{a}\tau}\int_{-iR}^{iR} \frac{d_{3,*}}{(-\mathfrak{a}+iy)^{2}}\,dy$$

It is easy to see (by means of another contour to avoid the standard branch) that, up to constants, the last three integral are respectively the inverse Laplace transform of $\xi^{-1/2}, \xi^{-3/2}, \xi^{-2}$. The integral

$$R(\tau) := \int_{-R}^{R} e^{iy\tau}\Big[\tilde{\sigma}(-\mathfrak{a}+iy) - \frac{d_{1,*}}{\sqrt{-\mathfrak{a}+iy}} - \frac{d_{2,*}}{(-\mathfrak{a}+iy)^{3/2}} - \frac{d_{3,*}}{(-\mathfrak{a}+iy)^2}\Big]\,dy$$

is absolutely convergent thanks to the second order expansion of $\tilde{\sigma}(\xi)$. In fact, obtaining the absolute convergence of $R(\tau)$ (and $R'(\tau)$) is the main reason to use the sharp Varadhan's estimate on the heat kernel p_t^{Ω}. Thus, from (8.20) we obtain

$$\sigma(\tau) = c_\infty e^{-\gamma\tau} + e^{-\mathfrak{a}\tau}\Big[\frac{C_{1,*}}{\sqrt{\tau}} + C_{2,*}\sqrt{\tau} + C_{3,*}\tau + R(\tau)\Big],$$

for some constants $c_\infty, C_{i,*}$ for $i = 1,2,3$, where $R(\tau)$ is bounded. This gives the asymptotic behaviour

$$\sigma(\tau) = \begin{cases} c_\infty e^{-\gamma\tau} + O(e^{-\mathfrak{a}\tau}) & \text{for} \quad \tau\to\infty, \\ \frac{C_{1,*}^{-1}}{\sqrt{\tau}} + c_\infty + O(\sqrt{\tau}) & \text{for} \quad \tau\to 0^+, \end{cases}$$

for any $\mathfrak{a}\in(\gamma,\lambda_1)$. For later purposes, we observe that $\sigma(\tau)$ is differentiable. Indeed, differentiating $R(\tau)$, we still obtain an absolutely convergent integral thanks to the full expansion (8.19), and an application of the dominated convergence theorem gives $\sigma\in C^1$ with

$$\sigma'(\tau) = \begin{cases} -\gamma c_\infty e^{-\gamma\tau} + O(e^{-\mathfrak{a}\tau}) & \text{for} \quad \tau\to\infty, \\ -(2C_{1,*})^{-1}\tau^{-3/2}(1+O(\tau)) & \text{for} \quad \tau\to 0^+, \end{cases}$$

From (8.17), taking the inverse Laplace transform of both sides, we get

$$\beta(s)e^{-\gamma s} = \beta(0)e^{-\gamma s} + \int_0^s e^{-\gamma(s-\tau)}h_0^*(s-\tau)\sigma(\tau)d\tau,$$

that is

$$\beta(s) = \beta(0) + \int_0^s e^{\gamma\tau}\sigma(\tau)h_0^*(s-\tau)\,d\tau.$$

Proof of (6.4). We rewrite this formula as

$$\begin{aligned}\beta(s) =&\beta(0) + c_\infty \int_0^s h_0^*(\tau)\, d\tau + \int_0^s h_0^*(\tau)\Big[e^{\gamma(s-\tau)}\sigma(s-\tau) - c_\infty\Big]\, d\tau\\ =&\Big[\beta(0) + c_\infty \int_0^\infty h_0^*(\tau)\, d\tau\Big] - c_\infty \int_s^\infty h_0^*(\tau)\, d\tau\\ &+ \int_0^s h_0^*(\tau)\Big[e^{\gamma(s-\tau)}\sigma(s-\tau) - c_\infty\Big]\, d\tau.\end{aligned}$$

We choose $\beta(0) = -c_\infty \int_0^\infty h_0^*(\tau) d\tau$. We reduced the problem to estimate

$$\begin{aligned}\beta_1(s) &:= -c_\infty \int_s^\infty h_0^*(\tau)\, d\tau,\\ \beta_2(s) &:= \int_0^s h_0^*(\tau)\Big[e^{\gamma(s-\tau)}\sigma(s-\tau) - c_\infty\Big]\, d\tau.\end{aligned}$$

We recall that the extension $h_0^*(s)$ has been selected so that (8.13) holds. Here and in what follows, without losing in generality we assume the same value $c = c_i$ for $i = 1, 2$. When we estimate the L^∞ norm of β we will only use the L^∞ norm of h_0^* and hence we get the same L^∞-weight constant c_1. Instead, when we estimate the $C^{1/2+\varepsilon}$ we need both the L^∞ and C^ε norms of h_0^*, thus we will get the same C^ε-weight constant $c_2 = \min\{c_1, c_2\}$. Thus, conditionally to $c_i < (\lambda_1 - \gamma)/(2\gamma)$, the weight constant c_i with $i = 1, 2$ for β and h_0^* are respectively the same. We proceed with the L^∞ estimate of β. We have

$$\begin{aligned}|\beta_1(s)| &\lesssim \|h_0^*\|_{\sharp,c,\varepsilon} \int_s^\infty e^{-2\gamma c\tau}\, d\tau\\ &\lesssim \|h_0^*\|_{\sharp,c,\varepsilon}\mu_0(s)^c,\end{aligned}$$

Using hypothesis (6.2) and selecting $\mathfrak{a}$ close enough to λ_1 so that

$$c < \mathfrak{a} < \frac{\lambda_1 - \gamma}{2\gamma}, \tag{8.21}$$

we get

$$\begin{aligned}|\beta_2(s)| &\lesssim \|h_0^*\|_{\sharp,c,\varepsilon} \int_0^s e^{-2\gamma c\tau} e^{-\mathfrak{a}(s-\tau)}\, ds\\ &\lesssim \|h_0^*\|_{\sharp,c,\varepsilon} e^{-\min\{2\gamma c,\mathfrak{a}\}s}\\ &\lesssim \|h_0^*\|_{\sharp,c,\varepsilon}\mu_0(s)^c.\end{aligned}$$

Combining the bounds on β_1 and β_2 we obtain

$$|\beta(s)| \lesssim \|h_0^*\|_{\sharp,c,\varepsilon}\mu_0(s)^c. \tag{8.22}$$

Now we estimate the $(1/2+\varepsilon)$-Hölder seminorm. In the following it is enough to assume $\eta \in (0, 1)$. We have

$$\begin{aligned}|\beta_1(s) - \beta_1(s-\eta)| &\le \left|\int_{s-\eta}^s h_0^*(\tau)\, d\tau\right|\\ &\le \|h_0^*\|_{\infty,c}\mu_0(s)^c|\eta|\\ &\le \|h_0^*\|_{\infty,c}\mu_0(s)^c|\eta|^{\frac{1}{2}+\varepsilon}\end{aligned} \tag{8.23}$$

Let

$$l(\tau) := e^{\gamma\tau}\sigma(\tau) - c_\infty.$$

Following the classical fractional integral estimate of Hardy and Littlewood [21, Theorem 14], we decompose

$$\begin{aligned}
\beta_2(s)-\beta_2(s-\eta)=&\int_0^s h_0^*(s-\tau)l(\tau)\,d\tau-\int_0^{s-\eta} h_0^*(s-\eta-\tau)l(\tau)\,d\tau\\
=&h_0^*(s)\int_0^s l(\tau)\,d\tau-\int_0^s [h_0^*(s)-h_0^*(s-\tau)]l(\tau)\,d\tau\\
&-h_0^*(s)\int_0^{s-\eta} l(\tau)\,d\tau-\int_0^{s-\eta} [h_0^*(s-\eta-\tau)-h_0^*(s)]l(\tau)\,d\tau\\
=&h_0^*(s)\int_{s-\eta}^s l(\tau)\,d\tau-\int_0^\eta [h_0^*(s)-h_0^*(s-\tau)]l(\tau)\,d\tau\\
&-\int_\eta^s [h_0^*(s)-h_0^*(s-\tau)](l(\tau)-l(\tau-\eta))\,d\tau\\
=:&A_1(s,\eta)+A_2(s,\eta)+A_3(s,\eta).
\end{aligned}$$

For $s-\eta\in(\eta,1)$ we have

$$\begin{aligned}
|A_1|&\lesssim|h_0^*(s)|\int_{s-\eta}^s\frac{1}{\sqrt{\tau}}\,d\tau\\
&\lesssim|h_0^*(s)|\left(s^{1/2}-(s-\eta)^{1/2}\right)\\
&\lesssim[h_0^*]_{0,\varepsilon,[s,s+1]}s^{\varepsilon-\frac{1}{2}}\eta\\
&\lesssim\|h_0^*\|_{\sharp,c,\varepsilon}\mu(s)^c\eta^{\varepsilon+\frac{1}{2}}.
\end{aligned}$$

For $s-\eta\geq 1$ we get

$$\begin{aligned}
|A_1|&\leq|h_0^*(s)|\int_{s-\eta}^s l(\tau)\,d\tau\\
&\lesssim|h_0^*(s)|\int_{s-\eta}^s e^{-\mathfrak{a}\tau}\,d\tau\\
&\lesssim|h_0^*(s)|\,\eta\\
&\lesssim\|h_0^*\|_{\sharp,c,\varepsilon}\mu(s)^c\eta^{\frac{1}{2}+\varepsilon}.
\end{aligned}$$

For $s-\eta\in(0,\eta)$ we obtain

$$\begin{aligned}
|A_1|&\lesssim|h_0^*(s)|\int_{s-\eta}^s\frac{1}{\sqrt{\tau}}\,d\tau\\
&\lesssim[h_0^*]_{0,\varepsilon,[s-\eta,s-\eta+1]}|s-\eta|^{\varepsilon}\eta^{\frac{1}{2}}\\
&\lesssim\|h_0^*\|_{\sharp,c,\varepsilon}\mu(s)^c\eta^{\frac{1}{2}+\varepsilon}.
\end{aligned}$$

Now we estimate A_2. We have

$$\begin{aligned}
|A_2|&\leq\|h_0^*\|_{\sharp,c,\varepsilon}\mu(s)^c\int_0^\eta|\tau|^{\varepsilon}|l(\tau)|\,d\tau\\
&\lesssim\|h_0^*\|_{\sharp,c,\varepsilon}\mu(s)^c\int_0^\eta\tau^{\varepsilon}\frac{1}{\sqrt{\tau}}\,d\tau\\
&\lesssim\|h_0^*\|_{\sharp,c,\varepsilon}\mu(s)^c\eta^{\frac{1}{2}+\varepsilon}.
\end{aligned}$$

Finally, we estimate A_3. Using the L^∞ norm of h_0^* for $\tau > 1$ and C^ε seminorm for $\tau < 1$ we obtain

$$\begin{aligned}
|A_3| \lesssim & \int_\eta^s |h_0^*(s) - h_0^*(s-\tau)|\,|l(\tau) - l(\tau-\eta)|\, d\tau \\
\lesssim & \|h_0^*\|_{\sharp,c,\varepsilon} \int_\eta^s |\tau|^{\varepsilon} |l(\tau) - l(\tau-\eta)|\, d\tau \\
\lesssim & \|h_0^*\|_{\sharp,c,\varepsilon} \int_\eta^s |\tau|^{\varepsilon} [\tau^{-1/2} - (\tau-\eta)^{-1/2}]\, d\tau \\
\lesssim & \|h_0^*\|_{\sharp,c,\varepsilon} \eta \int_\eta^s |\tau|^{\varepsilon} \tau^{-3/2}\, d\tau \\
\lesssim & \|h_0^*\|_{\sharp,c,\varepsilon} \eta^{\frac{1}{2}+\varepsilon} \\
\lesssim & \|h_0^*\|_{\sharp,c,\varepsilon} \mu(s-1)^c \eta^{\frac{1}{2}+\varepsilon} \\
\lesssim & \|h_0^*\|_{\sharp,c,\varepsilon} \mu(s)^c \eta^{\frac{1}{2}+\varepsilon},
\end{aligned}$$

Combining the bounds on A_1, A_2, A_3 and we obtain

$$|\beta_2(s) - \beta_2(s-\eta)| \lesssim \|h_0^*\|_{\sharp,c,\varepsilon} \mu(s)^c |\eta|^{\frac{1}{2}+\varepsilon}. \tag{8.24}$$

Finally, from (8.22), (8.23) and (8.24) we obtain

$$\|\beta\|_{\sharp,c,\frac{1}{2}+\varepsilon} \lesssim \|h_0^*\|_{\sharp,c,\varepsilon}$$

Going back to the original variable t using (8.14), we obtain

$$\|\Lambda\|_{\sharp,c,\frac{1}{2}+\varepsilon} \lesssim \|h_0^*\|_{\sharp,c,\varepsilon},$$

and recalling (8.13) the proof of (6.4) is complete. □

We proceed to prove the second part of Proposition 6.1: in case $h \in X_{\sharp,c,\frac{1}{2}+\varepsilon}$, then Λ is differentiable and $\dot{\Lambda} \in X_{\sharp,c,\varepsilon}$.

Proof of (6.5). In the same notation of the previous lemma, we need to prove that $\beta_1(s), \beta_2(s)$ are differentiable and estimate the derivatives. Since

$$\beta_1(s) := -\int_s^\infty h_0^*(\tau)\, d\tau,$$

we clearly have $\beta_1(s) \in C^1(0,\infty)$ and $\beta_1'(s) = c_\infty h(s) \in X_{\sharp,c,\frac{1}{2}+\varepsilon}$ by hypothesis. To analyze β_2, following [21, Theorem 19], we introduce for any $\epsilon \geq 0$ the function

$$\beta_{2,\epsilon}(s) = \int_0^{s-\epsilon} h_0^*(\tau) l(s-\tau)\, d\tau,$$

so that $\beta_{2,0}(s) = \beta_2(s)$. Since $\sigma(\tau) \in C^1$, we can differentiate $\beta_{2,\epsilon}(s)$ to obtain

$$\begin{aligned}
\beta_{2,\epsilon}'(s) =& h_0^*(s-\epsilon) l(\epsilon) + \int_0^{s-\epsilon} h_0^*(\tau) l'(s-\tau)\, d\tau \\
=& -[h_0^*(s) - h_0^*(s-\epsilon)] l(\epsilon) + l(s-\epsilon) h_0^*(s) \\
&+ \int_0^{s-\epsilon} [h_0^*(\tau) - h_0^*(s)] l'(s-\tau)\, d\tau.
\end{aligned}$$

Observe that we can choose the extension h_0^* such that $h_0^*(s) = o(s^{1/2})$ for $s \to 0$. Since $h_0^* \in X_{\sharp,c,\frac{1}{2}+\varepsilon}$, when $\epsilon \to 0$ the right-hand side tends uniformly to

$$l(s) h_0^*(s) + g(s),$$

where

$$g(s) := \int_0^s [h_0^*(\tau) - h_0^*(s)] l'(s-\tau)\, ds.$$

By hypothesis and the choice of the extension we have $l(s)h_0^*(s) \in X_{\sharp,c,\frac{1}{2}+\varepsilon}$. Also, the function $g(s)$ is continuous since $h_0^*(s) \in C^{\frac{1}{2}+\varepsilon}$.

$$\begin{aligned}
\beta_2(s_1) - \beta_2(s_2) &= \lim_{\epsilon \to 0} (\beta_{2,\epsilon}(s_1) - \beta_{2,\epsilon}(s_2)) \\
&= \lim_{\epsilon \to 0} \int_{s_1}^{s_2} \beta'_{2,\epsilon}(\tau)\, d\tau \\
&= \int_{s_1}^{s_2} l(\tau)h_0^*(\tau) + g(\tau)\, d\tau,
\end{aligned}$$

hence

$$l(s)h_0^*(s) + g(s) = \beta_2'(s).$$

It remains to prove that $g(s) \in X_{\sharp,c,\varepsilon}$. Using the asymptotic of $\sigma'(t)$ and the assumption (6.2) with $\mathfrak{a}$ as in (8.21) we have

$$\begin{aligned}
|g(s)| \lesssim & [h]_{0,\frac{1}{2}+\varepsilon,[s-1,s]} \int_{s-1}^{s} l'(s-\tau)|s-\tau|^{\frac{1}{2}+\varepsilon}\, d\tau \\
& + \|h\|_{\sharp,c,\frac{1}{2}+\varepsilon} \int_0^{s-1} l'(s-\tau)\mu(\tau)^c\, d\tau \\
\lesssim & \|h\|_{\sharp,c,\frac{1}{2}+\varepsilon} \left[\mu(s)^c \int_0^1 |w|^{-1+\varepsilon}\, dw + \int_0^s e^{-2\gamma c \tau} e^{-\mathfrak{a}(s-\tau)} \right] d\tau \\
\lesssim & \|h\|_{\sharp,c,\frac{1}{2}+\varepsilon} \mu(s)^c
\end{aligned} \tag{8.25}$$

We write

$$\begin{aligned}
g(s-\eta) - g(s) =& \int_0^s [h(s) - h(\tau)] l'(s-\tau)\, d\tau - \int_0^{s-\eta} [h(s-\eta) - h(\tau)] l'(s-\eta-\tau)\, d\tau \\
=& \int_0^s [h(s) - h(s-u)] l'(u)\, du - \int_\eta^s [h(s-\eta) - h(s-u)] l'(u-\eta)\, du \\
=& - \int_\eta^s [h(s-\eta) - h(s-u)][l'(u-\eta) - l'(u)]\, du \\
& + \int_\eta^s [h(s) - h(s-\eta)] l'(u)\, du + \int_0^\eta [h(s) - h(s-u)] l'(u)\, du \\
=:& B_1(s,\eta) + B_2(s,\eta) + B_3(s,\eta)
\end{aligned}$$

Using again assumption (6.2) we get

$$\begin{aligned}
|B_1| \lesssim & \|h_0^*\|_{0,\frac{1}{2}+\varepsilon,[s-1,s]} \int_\eta^1 |u-\eta|^{\frac{1}{2}+\varepsilon} |(u-\eta)^{-3/2} - u^{-3/2}|\, du \\
& + \|h\|_{\sharp,c,\frac{1}{2}+\varepsilon} \int_1^s \mu(s-u)\eta \frac{e^{-\mathfrak{a}(u-\eta)} - e^{-\mathfrak{a}u}}{\eta}\, du \\
& \lesssim \|h_0^*\|_{\sharp,c,\frac{1}{2}+\varepsilon} \mu(s)^c \eta^\varepsilon.
\end{aligned}$$

Also

$$\begin{aligned}
|B_2| &\lesssim |h_0^*(s) - h_0^*(s-\eta)|\, \eta^{-1/2} \\
&\lesssim \|h_0^*\|_{\sharp,c,\frac{1}{2}+\varepsilon} \mu(s)^c \eta^\epsilon,
\end{aligned}$$

and

$$
\begin{aligned}
|B_3| &\lesssim \|h_0^*\|_{\sharp,c,\frac{1}{2}+\varepsilon}\mu(s)^c \int_0^\eta u^{-1+\varepsilon}\,du \\
&\lesssim \|h_0^*\|_{\sharp,c,\frac{1}{2}+\varepsilon}\mu(s)^c\eta^\varepsilon.
\end{aligned}
$$

This proves

$$|g(s)-g(s-\eta)| \lesssim \mu(s)^c\|h_0^*\|_{\sharp,c,\frac{1}{2}+\varepsilon}|\eta|^{\varepsilon}.$$

Combining it with (8.25) we obtain

$$\|g\|_{\sharp,c,\varepsilon} \lesssim \|h_0^*\|_{\sharp,c,\frac{1}{2}+\varepsilon}.$$

Summing up the estimates for $\beta_1'(s)$ and $\beta_2'(s) = l(s)h_0^*(s) + g(s)$ we obtain

$$\|\beta'(s)\|_{\sharp,c,\varepsilon} \lesssim \|h_0^*\|_{\sharp,c,\frac{1}{2}+\varepsilon}.$$

Finally, in the original variable t, using (8.14) and (8.13), we obtain the bound (6.5). □

Remark 8.1 (the initial datum $J_1(x,t_0)$)**.** *From the proof of Proposition 6.1 we have $\mathcal{J}(t_0,x) = \int_0^1 h^*(s)I(x,\tau-s)\,ds$ where h_0^* is an arbitrary smooth function with $h_0^*(t) = o(t^{1/2})$ for $t\to 0$ and $h_0^*(1) = h(t_0)$, connecting to $h(t)$ at $t=t_0$ to maintain the C^ε regularity of h. We observe by estimate (2.2) that*

$$\|J_1(\cdot,t_0)\|_{L^\infty(\Omega)} \lesssim \|\mathcal{J}[\dot\Lambda](\cdot,t_0)\|_{L^\infty(\Omega)} \lesssim |\dot\Lambda(t_0)| \lesssim \mu_0(t_0)^{l_1}.$$

Thus, our initial datum remains positive provided that t_0 is fixed sufficiently large.

Appendix A. Properties of the Robin function $H_\gamma(x,x)$.

In this appendix we prove some properties of the Robin function that we use in our construction. We recall that the Green function associated to the operator $-\Delta-\gamma$ satisfies

$$
\begin{aligned}
&-\Delta_x G_\gamma(x,y) - \gamma G_\gamma(x,y) = 4\pi\alpha_3\delta(x-y) \quad \text{in } \Omega, \\
&G(\cdot,y) = 0 \quad \text{on } \partial\Omega.
\end{aligned}
\tag{A.1}
$$

As usual, we split

$$G_\gamma(x,y) = \Gamma(x-y) - H_\gamma(x,y) \quad \text{where} \quad \Gamma(x) = \frac{\alpha_3}{|x|},$$

where the regular part $H_\gamma(x,y)$ satisfies

$$
\begin{aligned}
&-\Delta_x H_\gamma(x,y) - \gamma H_\gamma(x,y) = -\gamma\Gamma(x-y) \quad \text{in } \Omega, \\
&H_\gamma(\cdot,y) = \Gamma(\cdot - y) \quad \text{on } \partial\Omega,
\end{aligned}
$$

for any fixed $y\in\Omega$. We recall (from [7] and reference therein) the following properties of $R_\gamma(x) := H_\gamma(x,x)$:

(1) $R_\gamma(x) \in C^\infty(\Omega)$
(2) $\partial_\gamma R_\gamma(x) < 0$ and belongs to $C^\infty(\Omega)$.
(3) $R_0(x)$ satisfies

$$R_0(x) = \frac{1}{2d(x,\partial\Omega)}(1+o(1)) \qquad \text{as} \quad d(x,\partial\Omega)\to 0.$$

(4) for each $\gamma\in(0,\lambda_1)$ fixed, $H_\gamma(x,x)\to+\infty$ as $x\to\partial\Omega$.

Lemma A.1 (Behavior near the first eigenvalue). *The function $H_\gamma(x,y)$ satisfies*

$$H_\gamma(x,y) \sim -\frac{\omega_3\alpha_3}{\lambda_1-\gamma}\phi_1(y)\phi_1(x) \qquad \text{as} \quad \gamma \nearrow \lambda_1,$$

where $\phi_1(x)$ is the first Dirichlet eigenfunction of $-\Delta$ in Ω with $\|\phi_1\|_2 = 1$ and $\phi_1(x) > 0$ in Ω.

Proof. We decompose

$$H_\gamma(x,y) = \alpha(y)\phi_1(x) + H_0(x,y) + h_{\perp,\gamma}(x,y) \tag{A.2}$$

where

$$\alpha(y) := \int_\Omega (H_\gamma(x,y) - H_0(x,y))\phi_1(x)\,dx,$$

and H_0 satisfies

$$\Delta_x H_0(x,y) = 0 \quad \text{in } \Omega, \quad H_0(x,y) = \frac{\alpha_3}{|x-y|} \quad \text{on } \partial\Omega.$$

Thus, for any fixed $y \in \Omega$, $h_{\perp,\gamma}(x,y)$ is the solution to

$$\begin{aligned} &\Delta_x h_{\perp,\gamma}(x,y) + \gamma h_{\perp,\gamma}(x,y) = \gamma G_0(x,y) + \alpha(y)(\lambda_1-\gamma)\phi_1(x) \quad \text{in } \Omega \\ &h_{\perp,\gamma} = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{A.3}$$

By the definition of $\alpha(y)$ we have

$$\begin{aligned} \int_\Omega h_{\perp,\gamma}(x,y)\phi_1(x)\,dx &= \int_\Omega (H_\gamma(x,y) - H_0(x,y))\phi_1(x)\,dx - \alpha(y)\|\phi_1\|_2^2 \\ &= 0. \end{aligned} \tag{A.4}$$

Testing (A.1) against ϕ_1 we get

$$\int_\Omega G_\gamma(x,y)\phi_1(x)\,dx = \frac{\alpha_3 4\pi\phi(y)}{\lambda_1-\gamma}.$$

Also, testing (A.3) against ϕ_1 and using (A.4) we obtain

$$\begin{aligned} 0 &= (-\lambda_1+\gamma)\int_\Omega h_{\perp,\gamma}(x,y)\phi_1(x)\,dx \\ &= \gamma\int_\Omega \phi_1(x)G_0(x,y)\,dx + \alpha(y)(\lambda_1-\gamma). \end{aligned}$$

Thus, we have

$$\begin{aligned} \alpha(y) &= -\frac{\gamma}{\lambda_1-\gamma}\int_\Omega G_0(x,y)\phi_1(x)\,dx \\ &= -\frac{\gamma}{\lambda_1}\frac{4\pi\alpha_3\phi_1(y)}{\lambda_1-\gamma}, \end{aligned} \tag{A.5}$$

and plugging (A.5) in (A.2) we obtain

$$H_\gamma(x,y) = -\frac{\gamma}{\lambda_1}\frac{4\pi\alpha_3}{\lambda_1-\gamma}\phi_1(y)\phi_1(x) + H_0(x,y) + h_{\perp,\gamma}(x,y). \tag{A.6}$$

We notice that only the first and last term in the right-hand side depends on γ. Hence we just need to prove that $h_{\perp,\gamma}(x,y)$ is bounded as $\gamma \to \lambda_1^-$. This is a consequence of the Poincaré inequality for functions orthogonal to ϕ_1. Indeed, expanding $h_{\perp,\gamma}$ in the L^2-basis made of Laplacian eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ we get

$$\begin{aligned}
\|\nabla h_{\perp,\gamma}(\cdot,y)\|_2^2 &= \int_\Omega h_{\perp,\gamma}(x,y)(-\Delta_x h_{\perp,\gamma}(x,y))\,dx \\
&= \int_\Omega \left(\sum_{k\geq 2}\alpha_k(y)\phi_k(x)\right)\left(\sum_{k\geq 2}\alpha_k(y)\phi_k(x)\lambda_k\right)dx \\
&= \sum_{k\geq 2}\alpha_k(y)^2\lambda_k \geq \lambda_2\|h_{\perp,\gamma}(\cdot,y)\|_2^2,
\end{aligned}$$

where

$$\alpha_k(y) = \int_\Omega \phi_k(x)h_{\perp,\gamma}(x,y)\,dx.$$

In particular $\alpha_1(y) = \alpha(y)$. Now, testing equation (A.3) against $h_{\perp,\gamma}$ and using Cauchy–Schwarz inequality we get

$$\begin{aligned}
(\lambda_2-\gamma)\|h_{\perp,\gamma}(\cdot,y)\|_2^2 &\leq \|\nabla h_{\perp,\gamma}(\cdot,y)\|_2^2 - \gamma\|h_{\perp,\gamma}(\cdot,y)\|_2^2 \\
&= \gamma\int_\Omega G_0(x,y)h_{\perp,\gamma}(x,y)\,dx \\
&\leq \gamma\|G_0(\cdot,y)\|_2\|h_{\perp,\gamma}(\cdot,y)\|_2.
\end{aligned}$$

We conclude that

$$\begin{aligned}
\|h_{\perp,\gamma}(\cdot,y)\|_2 &\leq \frac{\gamma}{\lambda_2-\gamma}\|G_0(\cdot,y)\|_2 \\
&\leq \frac{\lambda_1}{\lambda_2-\lambda_1}\|G_0(\cdot,y)\|_2 \\
&\leq C_\Omega,
\end{aligned}$$

for some constant C_Ω independent of y and γ. By standard elliptic estimates we get

$$\|h_{\perp,\gamma}(\cdot,y)\|_\infty \leq K_\Omega,$$

with K independent of y and γ. This concludes the proof. □

The following lemma gives the asymptotic of $\gamma^*(x)$ as x approaches the boundary $\partial\Omega$.

Lemma A.2. *The unique number $\gamma^*(x) \in (0,\lambda_1)$ defined by the relation*

$$H_{\gamma^*}(x,x) = 0$$

satisfies

$$\gamma^*(x) \sim \lambda_1 - 8\pi\big[\partial_\nu\phi_1(x')\big]^2 d(x,\partial\Omega)^3 \quad \text{as} \quad x\to x'\in\partial\Omega, \tag{A.7}$$

and $d(x,\partial\Omega) = |x-x'|$.

Proof. We divide the proof in two steps. Given $x\in\Omega$, consider the set

$$D_x := \{x'\in\partial\Omega : |x-x'| = d(x,\partial\Omega)\}.$$

If D_x is not a singleton we choose the unique $x' = (x_1', x_2', x_3') \in D_x$ with the property $x \prec y$ for all $y' = (y_1', y_2', y_3') \in D_x$, where, by definition, $x\prec y$ holds if

$$\begin{aligned}
&x_1' < y_1', \quad \text{or} \\
&x_1' = y_1' \quad \text{and} \quad x_2' < y_2', \quad \text{or} \\
&x_1' = y_1', \quad x_2' = y_2' \quad \text{and} \quad x_3' \leq y_3'.
\end{aligned}$$

This defines $x' := x'(x)$ uniquely.

Step 1. Firstly we prove (A.7) for domains such that, for all $x \in \Omega$, the reflection point $x''(x) := 2x'(x) - x$ satisfies

$$x'' \notin \Omega. \tag{P}$$

We decompose

$$H_\gamma(x,y) = \frac{\alpha_3}{|x''-y|} + F(x,y), \tag{A.8}$$

where F satisfies

$$\begin{aligned} &\Delta_x F + \gamma F = \gamma\alpha_3 g_1(x,y) \quad \text{in } \Omega,\\ &F(x,y) = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

and

$$g_1(x,y) = \frac{1}{|x-y|} - \frac{1}{|x''-y|}$$

We write

$$F(x,y) = \alpha(y)\phi_1(x) + w_\perp(x,y)$$

and select $\alpha(y)$ so that $\int_\Omega w_\perp(x,y)\phi_1(x)\,dx = 0$. By decomposition (A.8) and (2.8) we obtain

$$\begin{aligned} \alpha(y) &= \int_\Omega (F(x,y) - w_\perp(x,y))\phi_1(x)\,dx \\ &= \int_\Omega \left(H_\gamma(x,y) - \frac{\alpha_3}{|x-y''|} \right)\phi_1(x)\,dx \\ &= \int_\Omega g_1(x,y)\phi_1(x)\,dx - \int_\Omega G_\gamma(x,y)\phi_1(x)\,dx \\ &= \int_\Omega g_1(x,y)\phi_1(x)\,dx + \frac{4\pi\alpha_3\phi_1(y)}{\gamma-\lambda_1} \end{aligned}$$

The equation for $w_\perp$ is

$$\Delta w_\perp + \gamma w_\perp = \alpha(y)(\lambda_1-\gamma)\phi_1 + \gamma\alpha_3 g_1(x,y).$$

Multiplying this equation by $w_\perp$ and integrating by parts we get

$$\|\nabla w(\cdot,y)\|_2^2 - \gamma\|w(\cdot,y)\|_2^2 = -\gamma\int_\Omega g_1(x,y)w(x,y)\,dx.$$

Using the improved Poincaré inequality we have

$$(\lambda_2-\gamma)\|w_\perp(\cdot,y)\|_2^2 \le \|\nabla w_\perp(\cdot,y)\| - \gamma\|w_\perp(\cdot,y)\|_2^2,$$

and by Cauchy–Schwarz we obtain

$$\|w(\cdot,y)\|_2 \le \frac{\gamma}{\lambda_2-\gamma}\|g_1(\cdot,y)\|_2 < \frac{\lambda_1}{\lambda_2-\lambda_1}\|g_1(\cdot,y)\|_2. \tag{A.9}$$

Now, we want to estimate uniformly in y the right-hand side of

$$H_\gamma(x,y) = \frac{\alpha_3}{|x''-y|} + \phi_1(x)\int_\Omega g_1(z,y)\phi_1(z)\,dz - \frac{4\pi\alpha_3\phi_1(y)\phi_1(x)}{\gamma-\lambda_1} + w_\perp(x,y).$$

We can suppose $O \in \Omega$. Let $M = 2\mathrm{diam}(\Omega)$ we have

$$0 < \int_\Omega \frac{1}{|x-y|^2}\,dx \le \int_{B_{M(\Omega)}(y)} \frac{1}{|x-y|^2}\,dx \le C_\Omega.$$

Let $\Omega'' = \{x'' \in \mathbb{R}^3 : x'' = x''(x) \text{ for some } x \in \Omega\}$. We have

$$0 < \int_\Omega \frac{1}{|x'' - y|^2}\, dx \le \int_{\Omega'' \cup \Omega} \frac{1}{|x - y|^2} \le \int_{B_{M_2}} \le C_\Omega,$$

where $M_2 = 2\,\mathrm{diam}(\Omega'' \cup \Omega)$ hence we get

$$\sup_{y \in \Omega} \|g_1(\cdot, y)\|_2 < C_\Omega.$$

We combine this bound with (A.9) to get

$$\|w_\perp(\cdot, y)\|_2 \le K_\Omega$$

with K independent of y and by standard elliptic estimates we get $\sup_{y\in\Omega} \|w(\cdot, y)\|_\infty < K$. We conclude that

$$H_\gamma(x, y) = \frac{\alpha_3}{|x'' - y|} + \frac{4\pi\alpha_3\phi_1(y)\phi_1(x)}{\gamma - \lambda_1} + \phi_1(x)B(y) + w_\perp(x, y) \tag{A.10}$$

where

$$B(y) := \int_\Omega g_1(z, y)\phi_1(z)\, dz,$$

with $w_\perp(x, y)$ bounded in $\Omega \times \Omega$. Also we notice that

$$0 < \int_\Omega \phi_1(z) \frac{1}{|z - y|}\, dz \le \|\phi_1\|_\infty \int_{B_{M(\Omega)}} \frac{1}{|z - y|}\, dz \le C_\Omega,$$

and

$$0 < \int_\Omega \frac{\phi_1(x)}{|x''(x) - y|}\, dx \le \|\phi_1\|_\infty \int_{B_{M''}} \frac{1}{|x - y|}\, dx \le C_\Omega.$$

This proves the boundedness of $B(y)$. Now, the equation for $\gamma^*(x)$ reads as

$$0 = \frac{\alpha_3}{d(x, \partial\Omega)} + \frac{4\pi\alpha_3\phi_1(x)^2}{\gamma^* - \lambda_1} + \phi_1(x)B(x) + w_\perp(x, x).$$

Let $c := |\partial_\nu\phi_1(x')|$. We expand $\phi_1(x)$ at $x' \in \partial\Omega$ to get

$$\frac{8\pi c^2 d(x, \partial\Omega)^3}{\lambda_1 - \gamma^*} = \Big[1 + 2cd(x, \partial\Omega)^2 B(x) + 2d(x, \partial\Omega)w(x, x)\Big](1 + O(d(x, \partial\Omega)))$$

Since $B(x)$ and $w(x, x)$ are bounded, we conclude that

$$\frac{8\pi c^2 d(x, \partial\Omega)^3}{\lambda_1 - \gamma^*} \sim 1 \qquad \text{as} \quad x \to x' \in \partial\Omega. \tag{A.11}$$

Step 2. Now, we modify the method in Step 1 to obtain an expansion similar to (A.10) and conclude that (A.7) is true for general smooth bounded domains. Let $y \in \Omega_{\epsilon/4}$. Now we prove (A.7) for all smooth domains Ω. Fix $\epsilon = \epsilon(\Omega) > 0$ so small that the set $\Omega_\epsilon := \{x \in \Omega : d(x, \partial\Omega) < \epsilon\}$ possesses the property (P) and let η_ϵ be a smooth cut-off function with $\mathrm{supp}(\eta_\epsilon) \subset \Omega_\epsilon$ and $\eta_\epsilon(x) \equiv 1$ for $x \in \Omega_{\epsilon/2}$. We write

$$\begin{aligned} H_\gamma(x, y) =& \eta_\epsilon(x)\eta_\epsilon(y)H_\gamma(x, y) + (1 - \eta_\epsilon(x)\eta_\epsilon(y))H_\gamma(x, y) \\ =& \frac{\alpha_3}{|x'' - y|}\eta_\epsilon(x)\eta_\epsilon(y) + F_2(x, y) \end{aligned}$$

where

$$F_2(x, y) = \eta_\epsilon(x)\eta_\epsilon(y)F(x, y) + (1 - \eta_\epsilon(x)\eta_\epsilon(y))H_\gamma(x, y).$$

We notice that $\eta_\epsilon \eta_\epsilon F$, where F is defined in Step 1, is well-defined in Ω thanks to the cut-off functions. The problem for F_2 is

$$\begin{aligned}&\Delta_x F_2(x,y) + \gamma F_2(x,y) = \alpha_3 g_2(x,y) \quad \text{in } \Omega,\\ &F_2(x,y) = 0 \quad \text{on } \partial\Omega,\end{aligned}$$

where

$$g_2(x,y) := g_{2,1}(x,y) + g_{2,2}(x,y) + g_{2,3}(x,y) + g_{2,4}(x,y),$$

and

$$\begin{aligned}g_{2,1}(x,y) &:= \frac{\gamma}{|x-y|},\\ g_{2,2}(x,y) &:= -\gamma\frac{\eta_\epsilon(x)\eta_\epsilon(y)}{|x''-y|},\\ g_{2,3}(x,y) &:= 2\eta_\epsilon(y)\frac{\operatorname{div}\eta_\epsilon(x)}{|x''-y|^3},\\ g_{2,4}(x,y) &:= -\eta_\epsilon(y)\frac{\Delta_x \eta(x)}{|x''-y|}.\end{aligned}$$

We decompose

$$F_2(x,y) = \beta(y)\phi_1(x) + w_2(x,y),$$

where β is chosen such that $\int w_2(x,y)\phi_1(x)\,dx = 0$, that gives

$$\begin{aligned}\beta(y) = \int_\Omega F_2(x,y)\phi(x)\,dx &= \int_\Omega \phi_1(x)\left[-G_\gamma(x,y) - \frac{\eta_\epsilon(x)\eta_\epsilon(y)}{|x''-y|} + \frac{\alpha_3}{|x-y|}\right]dx\\ &= \frac{4\pi\phi_1(y)}{-\lambda_1+\gamma}\int_\Omega \frac{\alpha_3\phi_1(x)}{|x-y|}\,dx - \eta_\epsilon(y)\int_{\Omega_\epsilon}\frac{\alpha_3\eta_\epsilon(x)}{|x''-y|}.\end{aligned}$$

Next we prove that $w_2(x,y)$ is uniformly bounded in $\Omega\times\Omega$. Using the improved Poincaré inequality and standard elliptic estimates as in Step 1, we reduce the problem to estimate the L^2-norm of $g(\cdot,y)$ uniformly in $y\in\Omega_{\epsilon/4}$. We have

$$\begin{aligned}\|g_{2,1}\|_2^2 &= \gamma\int_\Omega \frac{1}{|x-y|^2}\,dx \le \int_{B_M}\frac{1}{|x-y|^2}\,dx \le C_\Omega,\\ \|g_{2,2}\|_2^2 &\le \gamma\int_{\Omega_\epsilon}\frac{1}{|x''-y|}\,dx \le \int_{\Omega''}\frac{1}{|x-y|}\,dx \le \int_{B_{M_2}}\frac{1}{|x-y|^2} \le C_\Omega,\\ \|g_{2,4}\|_2^2 &\le \int_{\Omega_\epsilon\setminus\Omega_{\epsilon/2}}\frac{|\Delta\eta_\epsilon(x)|}{|x''-y|^2} \le C\epsilon^{-2}\|g_{2,2}\|_2^2\\ \|g_{2,3}\|_2^2 &\le C\int_{\Omega_\epsilon\setminus\Omega_{\epsilon/2}}\frac{1}{|x''-y|^4} \le C_\Omega\epsilon^{-4}|\Omega|\,.\end{aligned}$$

Since ϵ depends only on Ω we obtain

$$\|g_2(\cdot,y)\|_2^2 < C_{\Omega,\epsilon}.$$

Now we prove that $B_\epsilon(y)\le C_\Omega$. Indeed, we have

$$B_\epsilon(y) := +B_{2,\epsilon} = \underbrace{\int_\Omega \frac{1}{|z-y|}\phi_1(z)\,dz}_{B_1} - \underbrace{\int_\Omega \frac{\eta_{\epsilon(z)\eta_\epsilon(y)}}{|z''-y|}\phi_1(z)\,dz,}_{B_{2,\epsilon}}$$

and

$$
\begin{aligned}
|B_1| &\leq \int_\Omega \frac{\phi_1(z)}{|z-y|}\,dz \leq \|\phi_1\|_\infty C_\Omega \\
|B_{2,\epsilon}| &\leq \eta_\epsilon(y) \int_{\Omega_\epsilon} \frac{\phi_1(z)}{|z''-y|}\,dz \leq \|\phi_1\|_\infty \int_{B_{M_2}} \frac{1}{|z-y|}\,dz \leq C_\Omega.
\end{aligned}
$$

Finally, the equation for $\gamma^*(x)$ is

$$
0 = H_{\gamma^*}(x,x) = \frac{1}{2d(x,\partial\Omega)} + \frac{4\pi\phi_1(x)^2}{-\lambda_1+\gamma^*} + B_\epsilon(x)\phi_1(x) + w_2(x,x),
$$

and from the boundedness of $B_\epsilon(x)$ and $w_2(x,x)$ we obtain (A.11). □

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2.2. Conclusions

We provided the first rigorous proof of nonradial threshold solution to the critical heat equation. If we choose the domain to be a ball or a cube, we have examples of infinite blow-up and the solution becomes unbounded in an arbitrarily fixed point of an open set, relatively far from the boundary. In general, our construction is valid in any domain such that $3\mu_{\mathrm{BN}}(\Omega) < \lambda_1(\Omega)$. One of the most surprising feature is the non trivial generalization of the blow-up rate from the radial to the nonradial situation. Indeed, we discovered that the rate of blow up depends on the position of the blow-up point inside the domain. With respect to what happens in higher dimension $n \geq 5$, this is a new phenomenon. Heuristically, it is suggested by the explicit computations on the ball and, more in general, by the behaviour of the number $\gamma(q)$ near the boundary, that larger is the distance between the blow-up point and the boundary, slower is the blow-up growth of the threshold solution.

Adapting the parabolic gluing method in [7] required three main ingredients. Firstly, an educated guess was found by means of an heuristic argument. In correcting the first ansatz we exploit the importance of the regular part of the Green function associated to the elliptic operator $-\Delta - \gamma$ in Ω. Secondly, we needed the solvability of a nonlocal equation, which we proved for any domains by means of an inverse Laplace transform argument using the asymptotic properties of the heat kernel associated to Ω. Lastly, we realized that a loss of regularity emerges when we solve the nonlocal equation. A first approach to a nonlocal operator in this type of problems can be found in [9], where $\Omega = \mathbb{R}^3$ allows an explicit computation. However, a loss of regularity needs to be address. For this reason we argued by fixed point arguments in suitable weighted-C^α spaces instead than the usual weighted-L^∞ setting.

In the next chapter we present the first steps towards an answer to the analogue conjecture in dimension 4.

Chapter 3

Further work and Outlook

3.1. The nonradial construction in the 4 dimensional case

In this section we start the construction of a 4 dimensional infinite time blow-up solution to the critical heat equation. This is a work in progress with Juan Dávila and Manuel del Pino. Firstly, we make an educated guess, that is

$$u_0 = \mu^{-1} U\left(\frac{x-\xi}{\mu}\right) - \mu H(x,\xi),$$

where H is the regular part of the Green function associated to $-\Delta$. Contrary to the 3 dimensional case, we assume that $\dot{\mu}/\mu \to 0$, as suggested by the radial analysis in [14]. It turns out that a nonlocal correction is needed to make one of the orthogonality conditions in the inner problem satisfied. Arguing formally, we obtain the main order equation for the nonlocal operator. At this point, arguing rigorously, we deduce that the expected main order for $\mu(t)$ is given by

$$\mu_0(t) = e^{-k\sqrt{t}}, \quad \text{where} \quad k = \left(\sqrt{2}H(q,q)\right)^{1/2}.$$

Then, by means of an explicit computation we find k for $\Omega = B_R(0)$ and $q = 0$, and we deduce the blow-up rate associated to the radial solution found in Galaktionov and King [14]. Lastly, we modify [7, Lemma 7.2] to get a coercivity estimate on the quadratic form associated to $-\Delta - pU^{p-1}$, where p is the critical exponent. This is essential to recover the linear estimate for the inner problem in dimension 4. Our analysis suggests two fundamental features similar to the case $n = 3$ and in contrast with higher dimension:

- the asymptotic behaviour of the threshold solution strongly depends on the location of the blow-up point;
- the second order in the asymptotic expansion of the blow up is controlled by a nonlocal operator, qualitatively different from the 3 dimensional analogue, and similar to the nonlocal operator treated in [8].

There are two main differences between the 4D and 3D cases.

- The blow-up rate in 4D is sub-exponential, unlike the exponential blow-up observed in 3D. Consequently, there is no resonance effect between $\mu(t)$ and the Dirichlet Heat Kernel. Therefore, we conjecture that in 4D, there are no analytical constraints regarding the location of blow-up points within Ω. This leads to the second difference.

- In 4D, we can select a point near the boundary, unlike in 3D. This allows us to naturally detect the multispike scenario by choosing points close to the boundary and sufficiently far apart, following the same condition as in dimensions $n \geq 5$.

3.1.1 First approximate solution

Let $\Omega \subset \mathbb{R}^4$ a smooth bounded domain. We look for a positive infinite-time blow-up solution to the problem

$$\begin{aligned} &\partial_t u = \Delta u + u^3 \quad \text{in } \Omega \times \mathbb{R}^+, \\ &u(x,t) = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ &u(x,0) = u_0(x) \quad \text{in } \Omega. \end{aligned} \tag{3.1}$$

Here the exponent $p := (n+2)/(n-2) = 3$ is the critical one in dimension $n = 4$. For this equation the energy

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{4}\int_\Omega |u|^4 \, dx$$

is a Lyapunov functional. Indeed, integrating by parts

$$\begin{aligned} \partial_t E(u(\cdot,t)) &= \frac{1}{2}\int_\Omega 2\nabla u \cdot \nabla u_t \, dx - \int_\Omega u^3 u_t \, dx \\ &= -\int_\Omega u_t\big(\Delta u + u^3\big) \, dx \\ &= -\int_\Omega u_t^2 \, dx, \end{aligned}$$

we see that $E(u(\cdot,t))$ is non-increasing in time. Without loss of generality, we construct a solution that blows up at the origin $q = 0 \in \Omega$. Our building blocks are the scaled Talenti bubble

$$U_{\mu(t),\xi(t)}(x) = \mu(t)^{-1} U\left(\frac{x - \xi(t)}{\mu(t)}\right),$$

where

$$U(x) = \frac{\alpha_4}{1 + |x|^2}, \quad \alpha_n = [n(n-2)]^{\frac{n-2}{4}}.$$

The family of functions $U_{\mu,\xi}$, for constants $\mu \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^4$, forms the set of positive solution of the Yamabe equation

$$\Delta U + U^3 = 0 \quad \text{in } \mathbb{R}^4.$$

Similarly to the 3 dimensional case, we look for a solution of the form

$$u_1(x,t) \approx U_{\mu(t),\xi(t)}(x).$$

Supposing $\mu(t), \xi(t) \to 0$ as $t \to \infty$ we have that $U_{\mu(t),\xi(t)}(x)$ is concentrating around $x = \xi$ and becomes uniformly small away from it. For this reason, we should have

$$\begin{aligned}
\partial_t u_1 - \Delta u_1 &= u_1(x,t)^3 \\
&\approx \delta_0(x-\xi) \int_{\mathbb{R}^4} \left(\mu^{-1} U\left(\frac{x-\xi}{\mu} \right) \right)^3 dx \\
&= \delta_0(x-\xi)\mu\alpha_4\omega_4,
\end{aligned}$$

where $\omega_4 = 2\pi^2$ is the area of the unit sphere S^3 and $\alpha_4 = \sqrt{8}$. We write the parameter μ in form

$$\mu(t) = b\mu_0(t)(1+o(1)),$$

for some function $\mu_0(t)$. The equation for

$$v(x,t) = \mu_0^{-1} u_1(x,t)$$

becomes

$$v_t \approx \Delta v - \mu_0^{-1}\dot{\mu}_0 v + a_4\omega_4\delta_\xi(x) \quad \text{in } \Omega \times \mathbb{R}^+.$$

Contrary to the 3 dimensional problem, a priori we assume

$$\mu_0(t)^{-1}\dot{\mu}_0 \to 0.$$

Since the problem is translation invariant, if we find a solution $u(x,t)$ starting with a large initial time t_0, then $u_0(x, t-t_0)$ is a solution to the original problem (3.1). For this reason we we will choose t_0 fixed as large as needed. This assumption on $\mu(t)$ is suggested by the radial case, where the blow-up rate is sub-exponential. Thus, we get

$$\begin{aligned}
&v_t \approx \Delta v + \alpha_4\omega_4\delta_0(x-\xi) \quad \text{in } \Omega \times \mathbb{R}^+, \\
&v = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+.
\end{aligned}$$

This means that far away from $x = 0$, we should have

$$v(x,t) \approx G(x,\xi),$$

that in terms of u_1 means

$$u_1(x,t) \approx \mu\frac{\alpha_4}{|x|^2} - \mu H(x,\xi).$$

These formal arguments suggest our first ansatz

$$
\begin{aligned}
u_1(x,t) &= \mu^{-1} U\left(\frac{x-\xi(t)}{\mu}\right) - \mu H(x,\xi) \\
&= \mu^{-1} \frac{\alpha_4}{1+\left|\frac{x-\xi}{\mu}\right|^2} - \mu H(x,\xi),
\end{aligned}
$$

and $H(x,y)$ is the regular part of the Green function and for $y \in \Omega$ satisfies

$$
\begin{aligned}
&\Delta_x H(x,y) = 0 \quad \text{in } \Omega \\
&H(x,y) = \frac{\alpha_4}{|x-y|^2} \quad \text{on } \partial\Omega.
\end{aligned}
$$

We compute the error associated to the ansatz u_1, that is

$$
S[u_1] := -\partial_t u_1 + \Delta u_1 + u_1^3.
$$

In the following we use the scaled variable

$$
y := y(x,t) := \frac{x-\xi(t)}{\mu(t)}.
$$

We have

$$
\begin{aligned}
\partial_t u_1 = &- \mu^{-2}\dot{\mu} U(y) - \mu^{-2}\dot{\mu} y \cdot \nabla_y U(y) \\
&- \mu^{-2}\dot{\xi} \cdot \nabla_y U(y) - \dot{\mu} H(x,0) - \mu\dot{\xi} \cdot \nabla_{x_2} H(x,\xi),
\end{aligned}
$$

and, using the equations for $U(x)$ and H we get

$$
\begin{aligned}
\Delta u_1 =& \mu^{-1} \Delta_x U\left(\frac{x-\xi(t)}{\mu}\right) - \mu \Delta_x H(x,0) \\
=& \mu^{-3} \Delta_y U(y) \\
=& - \mu^{-3} U(y)^3.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
S[u_1] =& \mu^{-2}\dot{\mu} Z_5(y) + \mu^{-2}\dot{\xi} \cdot \nabla_y U(y) + \mu\dot{\xi} \cdot \nabla_{x_2} H(x,\xi) + \dot{\mu} H(x,\xi) \\
&+ \mu^{-3}\left[(U(y) - \mu^2 H(x,\xi))^3 - U(y)^3\right],
\end{aligned} \tag{3.2}
$$

where

$$
Z_5 := \alpha_4 \frac{1-|y|^2}{\left(1+|y|^2\right)^2}.
$$

For $|\mu y| < \frac{1}{2}$ we Taylor expand

$$H(\mu y + \xi, \xi) = R(\xi) + \mu y \cdot \nabla_{x_1} H(\xi, \xi) + \frac{1}{2}\mu^2 y^2 : D_{x_1 x_1} H(\bar{x}, \xi), \tag{3.3}$$

for some $\bar{x} \in [\xi, x]$, where $R(\xi) := H(\xi, \xi)$ denotes the Robin function, that is the diagonal of the regular part of the Green function. Expanding the Robin function we have

$$R(\xi) = R(0) + \xi \cdot \nabla R(\xi^*),$$

for some $\xi^* \in [0, \xi]$. Looking close to $x = 0$, say $|x - \xi| < 1/2$, we estimate the error function by

$$\begin{aligned} S[u_1] =& \mu^{-2}\dot{\mu} Z_5(y) - 3\mu^{-1} U(y)^2 R(0) \\ & - 3\mu^{-1} U(y)^2 \left[\xi \cdot \nabla R(\xi^*) + \mu y \cdot \nabla_{x_1} H(\xi, \xi) + \frac{1}{2}\mu^2 y^2 : D_{x_1 x_1} H(\bar{x}, \xi) \right] \\ & + \mu^{-2} \dot{\xi} \cdot \big(\nabla_y U(y) + \mu^3 \nabla_{x_2} H(x, \xi) \big) + \dot{\mu} H(x, \xi) \\ & + \mu^{-3} \big[(U(y) - \mu^2 H(x, \xi))^3 - U(y)^3 + 3\mu^2 U(y)^2 H(x, \xi) \big] \end{aligned} \tag{3.4}$$

Our exact solution will have the profile

$$u = u_1 + \tilde{\phi},$$

with $\tilde{\phi}$ smaller compared to u_1. It is natural to look for perturbation in the same scale of the Talenti bubble

$$\tilde{\phi}(x, t) = \mu^{-1} \phi\left(\frac{x - \xi}{\mu}, t \right),$$

the equation for $\phi(y, t)$ reads as

$$\begin{aligned} 0 =& \mu^3 S(u_1 + \tilde{\phi}) \\ =& - \mu^2 \partial_t \phi + \mu \dot{\xi} \cdot \nabla_y \phi + \dot{\mu} \mu (\phi + \nabla_y \phi \cdot y) \\ & + \Delta_y \phi + 3 \big[U(y) - \mu^2 H(x, 0) \big]^2 \phi + \mu^3 S[u_1] + \mu^3 N(u_1, \tilde{\phi}) \\ =& \Delta_y \phi + 3U(y)^2 \phi + \mu^3 S[u_1] + A[\phi]. \end{aligned}$$

where

$$N(u_1, \tilde{\phi}) = (u_1 + \tilde{\phi})^3 - u_1^3 - 3u_1^2 \tilde{\phi},$$

and

$$\begin{aligned} A[\phi] = &-\mu^2\partial_t\phi + \mu\dot{\xi}\cdot\nabla_y\phi + \dot{\mu}\mu(\phi + \nabla_y\phi\cdot y) \\ &+ \Big[3\big[U(y) - \mu^2 H(x,0)\big]^2 - 3U^2\Big]\phi + \mu^3 N(u_1,\tilde{\phi}). \end{aligned}$$

We assume that $\phi(y,t)$ decays in y and, for t large, the terms in $A[\phi]$ are small compared to the others. Hence the function ϕ can be approximated by the solution $\phi_0(y,t)$ to the problem

$$\begin{aligned} &\Delta_y\phi_0 + 3U(y)^2\phi_0 = -E_0(y,t) \quad \text{in } \mathbb{R}^4, \\ &\phi_0(y,t) \to 0 \quad \text{as} \quad |y| \to 0. \end{aligned}$$

It is known that all the bounded solutions are spanned by $\{Z_i\}_{i=1}^5$ where

$$Z_i(y) = \partial_{y_i}U(y) \quad \text{for} \quad i = 1,2,3,4, \quad Z_5(y) = U(y) + \nabla_y U(y)\cdot y.$$

The problem above can be solved if and only if

$$\int_{\mathbb{R}^4} E_0(y,t)Z_i(y)\,dx = 0 \quad \forall t > t_0 \quad \text{and} \quad i = 1,2,3,4,5.$$

In case $i = 1,2,3,4$ the symmetry of the integrand gives the desired orthogonality. However, since $Z_5 \notin L^2(\mathbb{R}^4)$, the condition with index $i = 5$ does not make sense at this stage. Indeed, considering the first two terms in (3.4) as leading terms, we should have

$$0 \approx \mu\dot{\mu}\int_{\mathbb{R}^4} Z_5(y)^2\,dy - \mu^2 R(0)\int_{\mathbb{R}^4} 3U(y)^2 Z_5(y)\,dy,$$

but $Z_5(y) \notin L^2(\mathbb{R}^4)$. In the next section we modify our ansatz to overcome this difficulty. This is the ultimate reason for the condition $n \geq 5$ in [7].

3.1.2 Nonlocal improvement of the approximation

As in the 3 dimensional case we can get rid of this term by adding a nonlocal term in the approximate solution. In order to make the error associated to the ansatz u_1 smaller for $|y|$ large, we modify u_1 by defining

$$\begin{aligned} u_2(x,t) :&= u_1(x,t) + \mu(t)J(x,t) \\ &= \mu^{-1}U(y) - \mu(t)H(x,\xi) + \mu(t)J(x,t). \end{aligned}$$

The idea is to remove the slowing-decay term through a linear parabolic equation satisfied by $J(x,t)$. We define

$$\lambda(t) := \ln\left(\frac{1}{\mu(t)}\right),$$

so that

$$\dot{\lambda} = -\frac{\dot{\mu}(t)}{\mu(t)}, \quad \text{and} \quad \mu(t) = e^{-\lambda(t)}.$$

We choose $J(x,t)$ as the solution to

$$\begin{aligned}
&\partial_t J = \Delta_x J + \dot{\lambda}(t) \frac{\alpha_4}{\left(\mu^2(t) + |x-\xi(t)|^2\right)} \quad \text{in } \Omega \times (t_0, \infty),\\
&J(x,t) = 0 \quad \text{on } \partial\Omega \times (t_0,\infty),\\
&J(x,0) = 0 \quad \text{in } \Omega.
\end{aligned}$$

We will approximate $J(x,t)$ with the solution to the Cauchy problem $J_{0,\mathbb{R}^4}$

$$\begin{aligned}
&\partial_t J_{0,\mathbb{R}^4} = \Delta_x J_{0,\mathbb{R}^4} + \dot{\lambda}(t) \frac{\alpha_4}{\left(\mu^2(t) + |x|^2\right)} \quad \text{in } \mathbb{R}^4 \times (t_0, \infty),\\
&J_{0,\mathbb{R}^4}(x,t_0) = 0 \quad \text{in } \mathbb{R}^4,
\end{aligned} \tag{3.5}$$

which can be express by the Duhamel formula. We split

$$\begin{aligned}
Z_5(y) &= \alpha_4 \frac{2-(1+|y|)^2}{\left(1+|y|^2\right)^2}\\
&= \underbrace{\frac{2\alpha_4}{\left(1+|y|^2\right)^2}}_{\in L^2(\mathbb{R}^4)} - \underbrace{\frac{\alpha_4}{1+|y|^2}}_{\notin L^2(\mathbb{R}^4)} = \frac{2}{\alpha_4} U(y)^2 - U(y).
\end{aligned}$$

We compute the error associated to u_2. Using expansion (3.2) we get

$$\begin{aligned}
S[u_2] =& \left\{ S[u_1] + \frac{\dot{\mu}}{\mu^2} \frac{\alpha_4}{1+|y|^2} \right\} + \mu \left\{ -\partial_t J + \Delta_x J - \frac{\dot{\mu}}{\mu^3} \frac{\alpha_4}{1+|y|^2} \right\} + u_2^3 - u_1^3\\
=& \mu^{-2} \dot{\mu} \frac{2\alpha_4}{\left(1+|y|^2\right)^2} + \mu^{-2} \dot{\xi} \cdot \nabla_y U(y) + \mu \dot{\xi} \cdot \nabla_{x_2} H(x,\xi) + \dot{\mu} H(x,\xi)\\
&+ \mu^{-3} \left[(U(y) - \mu^2 H(x,\xi) + \mu^2 J)^3 - U(y)^3 \right].
\end{aligned}$$

Now, we look at the new error close to the blow up point $x = 0$. Let $|x - \xi(t)| < 1/2$ for $t > t_0$ sufficiently large, then from (3.4) we have

$$
\begin{aligned}
S[u_2] =& \mu^{-2}\dot{\mu}\frac{2\alpha_4}{\left(1+|y|^2\right)^2} - 3\mu^{-1}U(y)^2R(0) - 3\mu^{-1}U(y)^2J(x,t) \\
& - 3\mu^{-1}U(y)^2\left[\xi \cdot \nabla R(\xi^*) + \mu y \cdot \nabla_{x_1} H(\xi,\xi) + \frac{1}{2}\mu^2 y^2 : D_{x_1x_1}H(\bar{x},\xi)\right] \\
& + \mu^{-2}\dot{\xi} \cdot \left(\nabla_y U(y) + \mu^3 \nabla_{x_2} H(x,\xi)\right) + \dot{\mu}H(x,\xi) \\
& + \mu^{-3}\left[(U(y) - \mu^2 H(x,\xi) + \mu^2 J)^3 - U(y)^3 + 3\mu^2 U(y)^2 (H(x,\xi) + J(x,t))\right]
\end{aligned}
$$

As in the 3 dimensional case we need to separate the main inner error. We decompose the error as

$$
S[u_2] = S_{\text{in}} + S_{\text{out}}
$$

where

$$
\begin{aligned}
S_{\text{in}} =& \mu^{-2}\dot{\mu}\frac{2\alpha_4}{\left(1+|y|^2\right)^2} + \mu^{-2}\dot{\xi} \cdot \nabla_y U(y) \\
& - 3\mu^2 U(y)^2 H(x,\xi) + 3\mu^2 U(y)^2 J(x,t) + \mathcal{N}_2,
\end{aligned}
$$

where $\mathcal{N}_2$ denotes the nonlinear term

$$
\mathcal{N}_2 = \mu^{-3}\left[(U(y) - \mu^2 H(x,\xi) + \mu^2 J)^3 - U(y)^3 + 3\mu^2 U(y)^2 (H(x,\xi) - J(x,t))\right].
$$

If we perform the same formal argument with this new error S_{in} we see that

$$
\begin{aligned}
0 =& - 3\mu^{-1}R(0)\int_{\mathbb{R}^4} U(y)^2 Z_5(y)\,dy - 3\mu^{-1}J_{\mathbb{R}^4}(0,t)\int_{\mathbb{R}^4} U(y)^2 Z_5(y)\,dy \\
& - \dot{\lambda}(t)\mu^{-1}\int_{\mathbb{R}^4}\frac{2\alpha_4}{(1+|y|^2)^2} Z_5(y)\,dy \\
& - 3\mu^{-1}\int_{\mathbb{R}^4} U(y)^2 Z_5(y)\left[J(\mu y + \xi, t) - J_{0,\mathbb{R}^4}(0,t)\right]dy \\
& + 3\mu^{-1}\int_{\mathbb{R}^4} U(y)^2 Z_5(y)[H(\mu y + \xi, t) - R(0)]\,dy + \int_{\mathbb{R}^4} \mathcal{N}_2(y,t) Z_5(y)\,dy \\
=& - 3\mu^{-1}c_1[R(0) + J_{\mathbb{R}^4}(0,t)] + \mu^{-1}\mathcal{R}[\dot{\lambda}](t),
\end{aligned}
$$

where $c_1 := \int_{\mathbb{R}^4} U(y)^2 Z_5(y)\,dy$ and

$$\begin{aligned}
\mathcal{R}[\dot\lambda](t) = &-\dot\lambda(t)\int_{\mathbb{R}^4}\frac{2\alpha_4}{(1+|y|^2)^2}Z_5(y)\,dy\\
&-3\int_{\mathbb{R}^4}U(y)^2Z_5(y)\big[J(\mu y+\xi,t)-J_{0,\mathbb{R}^4}(0,t)\big]\,dy\\
&+3\int_{\mathbb{R}^4}U(y)^2Z_5(y)[H(\mu y+\xi,t)-R(0)]\,dy+\mu\int_{\mathbb{R}^4}\mathcal{N}_2(y,t)Z_5(y)\,dy
\end{aligned}$$

Taking into account the expansion (3.3) and the decay of $\dot\lambda$, it is natural to assume that for $t>t_0$ large the equation at the main order is

$$0=-3c_1\mu^{-1}\Big[R(0)+\mathcal{R}[\dot\lambda](t)+J_{0,\mathbb{R}^4}(0,t)\Big],$$

where $\mathcal{R}[\dot\lambda](t)$ is a lower order remainder which decays in time. Hence, we should find λ such that

$$J_{0,\mathbb{R}^4}[\dot\lambda](0,t)=R(0)+\mathcal{R}[\dot\lambda](t).$$

This equation is a constraint on the evolution of $J_{0,\mathbb{R}^4}[\dot\lambda]$ at the origin. In the next section we show that there is a choice of $\dot\lambda$ such that this equation is satisfied at the main order.

3.1.3 The blow-up growth of the threshold solution

As the heuristic argument suggests, we need to find λ such that

$$J_{0,\mathbb{R}^4}[\dot\lambda](0,t)=R(0)+\mathcal{R}[\dot\lambda](t).$$

where $\mathcal{R}[\dot\lambda](t)$ decays. The following lemma shows that, neglecting the remainder term $\mathcal{R}[\dot\lambda](t)$, we can approximately solve $J_{0,\mathbb{R}^4}[\dot\lambda](0,t)=R(0)$ up to an error of size $O(t^{-1/2}\ln(t))$ if we choose $\lambda=k\sqrt{t-t_0}$ for some k. In this way we find the main order term of $\mu(t)$, and since

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}=u(\xi,t)\sim\alpha_4\mu(t)^{-1},$$

we obtain the infinite blow-up rate of the expected global solution.

Lemma 3.1.1. *Let $J_{\mathbb{R}^4,0}[\dot\lambda_0](0,t)$ the solution to the Cauchy problem*

$$\begin{aligned}
&\partial_t J_{0,\mathbb{R}^4}=\Delta_x J_{0,\mathbb{R}^4}+\dot\lambda(t)\frac{\alpha_4}{\mu^2(t)+|x|^2}\quad \text{in } \mathbb{R}^4\times(t_0,\infty),\\
&J_{0,\mathbb{R}^4}(x,t_0)=0\quad \text{in } \mathbb{R}^4.
\end{aligned}$$

given by the Duhamel formula. Let

$$\lambda_0(t) = k\sqrt{t-t_0}, \quad \mu_0(t) = e^{-\lambda_0(t)} \quad k = \left(\sqrt{2}H(0,0)\right)^{1/2}. \tag{3.6}$$

Then, there exists a constant $C_1 > 0$ sufficiently large, such that for $t > t_0 + 1$

$$H(0,0) - C_1^{-1}t^{-1/2} \le J_{\mathbb{R}^4,0}[\dot{\lambda}_0](0,t) \le H(0,0) + C_1 t^{-1/2}\ln(t)$$

Proof. We split the proof in two steps, respectively the upper and lower bound for $J[\dot{\lambda}_0](0,t)$. After a time-translation from t to $t-t_0$ we reduce to the problem with initial time 0.

Step 1 (Upper bound). Firstly, we show the upper bound. Using spherical coordinates we have

$$\begin{aligned}
J_{\mathbb{R}^4}[\dot{\lambda}_0](0,t) &= \alpha_4 \int_0^t \frac{\dot{\lambda}_0(s)}{[4\pi(t-s)]^2} \int_{\mathbb{R}^4} \frac{e^{-\frac{|y|^2}{4(t-s)}}}{\mu_0(s)^2 + |y|^2}\, dy\, ds \\
&= \alpha_4 \frac{k}{2} \int_0^t \frac{s^{-1/2}}{[4\pi(t-s)]^2} \int_{\mathbb{R}^4} \frac{e^{-\frac{|y|^2}{4(t-s)}}}{\mu_0(s)^2 + |y|^2}\, dy\, ds \\
&= \alpha_4 \frac{k}{2} 2\pi^2 \int_0^t \frac{s^{-1/2}}{[4\pi(t-s)]^2} \int_0^\infty \frac{e^{-\frac{\rho^2}{4(t-s)}}}{\mu(s)^2 + \rho^2} \rho^3\, d\rho\, ds \\
&= \alpha_4 \frac{k}{2} 2\pi^2 \int_0^t \frac{s^{-1/2}}{[4\pi(t-s)]^2} \int_0^\infty \frac{e^{-r^2}}{\mu(s)^2 + 4(t-s)r^2} r^3\, dr \left[2\sqrt{t-s}\right]^4 ds \\
&= \alpha_4 k \int_0^t s^{-1/2} \int_0^\infty \frac{e^{-r^2} r^3}{\mu(s)^2 + 4(t-s)r^2}\, dr\, ds \\
&= \alpha_4 k \int_0^t \frac{s^{-1/2}}{4(t-s)} \int_0^\infty \frac{e^{-r^2} r^3}{\tilde{r}(t,s)^2 + r^2}\, dr\, ds, \quad \tilde{r} := \tilde{r}(t,s) := \frac{\mu(s)}{2\sqrt{t-s}},
\end{aligned}$$

where in the fourth identity we used the change of variable $\rho = 2\sqrt{t-s}r$. Now, we have

$$\int_0^\infty \frac{e^{-r^2} r^3}{\tilde{r}^2 + r^2}\, dr = \int_0^{\tilde{r}} \frac{e^{-r^2} r^3}{\tilde{r}^2 + r^2}\, dr + \int_{\tilde{r}}^\infty \frac{e^{-r^2} r^3}{\tilde{r}^2 + r^2}\, dr = A_1(t,s) + A_2(t,s).$$

We estimate

$$\begin{aligned}
A_1(t,s) &= \int_0^{\tilde{r}} \frac{e^{-r^2} r^3}{\tilde{r}^2 + r^2}\, dr \\
&\le \int_0^{\tilde{r}} e^{-r^2} r^3\, dr \tilde{r}^{-2} = \frac{1}{2}\left[1 - e^{-\tilde{r}^2}(1+\tilde{r}^2)\right]\tilde{r}^{-2},
\end{aligned}$$

and

$$A_2(t,s) = \int_{\tilde r}^{\infty} \frac{e^{-r^2} r^3}{\tilde r^2 + r^2}\,dr \le \int_{\tilde r}^{\infty} e^{-r^2} r\,dr = \frac{1}{2} e^{-\tilde r^2}.$$

This gives

$$\begin{aligned}
J[\dot\lambda_0](0,t) \le & \alpha_4 k \int_0^t \frac{s^{-1/2}}{4(t-s)} \frac{1}{2}\Big[1 - e^{-\tilde r^2}(1+\tilde r^2)\Big] \frac{4(t-s)}{\mu(s)^2}\,ds \\
& + \alpha_4 k \int_0^t \frac{s^{-1/2}}{4(t-s)} \frac{1}{2} e^{-\frac{\mu(s)^2}{4(t-s)}}\,ds \\
= & \alpha_4 \frac{k}{2} \int_0^t s^{-1/2} \mu(s)^{-2} \Big[1 - e^{-\tilde r^2}(1+\tilde r^2)\Big]\,ds + \alpha_4 \frac{k}{8} \int_0^t \frac{s^{-1/2}}{(t-s)} e^{-\frac{\mu(s)^2}{4(t-s)}}\,ds \\
=: & B_1(t) + B_2(t).
\end{aligned}$$

We show that $B_1(t) = O\big(t^{-1/2}\ln(t)\big)$. By definition of $\tilde r$ we have

$$\tilde r(t,s) < 1 \Leftrightarrow \frac{\mu(s)^2}{4} < t - s \Leftrightarrow s + \frac{\mu(s)^2}{4} < t$$

Let $s^*(t)$ such that

$$s^*(t) + \frac{\mu(s^*(t))^2}{4} = t,$$

which exists and is unique for t large enough. Since $\mu(t)$ is positive we have $s^*(t) < t$ which, combined with the decreasing monotonicity of $\mu(t)$, gives

$$t = s^*(t) + \frac{\mu(s^*(t))^2}{4} > s^*(t) + \frac{\mu(t)^2}{4}.$$

Summarizing we have

$$s^*(t) \in \left(t - \frac{\mu(t)^2}{4}, t\right).$$

We observe that if $s \le s^*(t)$ then we have

$$\tilde r(s,t) \le 1.$$

Indeed, if $s < 1$ we have

$$\tilde r(s,t) = \frac{\mu(s)^2}{4(t-s)} \le \frac{\mu(0)^2}{4(t-1)} < 1,$$

for t large enough. Also, if $s \in [1, s^*(t)]$ then the function $s + \frac{\mu(s)^2}{4}$ is increasing and we deduce

$$s+\frac{\mu(s)^2}{4}\leq s^*(t)+\frac{\mu(s^*(t))}{4}=t,$$

that means $\tilde{r}(t,s)\leq 1$. Also, for t large enough and $s>s^*(t)$ the monotonicity of $s+\frac{\mu(s)^2}{4}$ gives

$$s+\frac{\mu(s)^2}{4}>s^*(t)+\frac{\mu(s^*(t))}{4}=t$$

that means $\tilde{r}(t,s)>1$. We split

$$\begin{aligned}B_1(t)=&\alpha_4\frac{k}{2}\int_0^{s^*(t)}s^{-1/2}\mu(s)^{-2}\Big[1-e^{-\tilde{r}(t,s)^2}(1+\tilde{r}(t,s)^2)\Big]\,ds\\&+\alpha_4\frac{k}{2}\int_{s^*(t)}^{t}s^{-1/2}\mu(s)^{-2}\Big[1-e^{-\tilde{r}(t,s)^2}(1+\tilde{r}(t,s)^2)\Big]\,ds\\=&B_{1,0}(t)+B_{1,1}(t).\end{aligned}$$

We estimate $B_{1,1}$ as

$$\begin{aligned}B_{1,1}(t)&\leq\alpha_4\frac{k}{2}\int_{s^*(t)}^{t}s^{-1/2}\mu(s)^{-2}\,ds\leq\alpha_4\frac{k}{2}\int_{t-\frac{\mu(t)^2}{4}}^{t}s^{-1/2}\mu(s)^{-2}\,ds\\&\leq\alpha_4\frac{k}{2}\left(t-\frac{\mu(t)}{4}\right)^{-1/2}\mu\left(t-\frac{\mu(t)^2}{4}\right)^{-2}\frac{\mu(t)^2}{4}\\&=\alpha_4\frac{k}{8}t^{-1/2}(1+o(1)).\end{aligned}$$

We split $B_{1,0}$ as

$$\begin{aligned}B_{1,0}=&\alpha_4\frac{k}{2}\int_0^{t-1}s^{-1/2}\mu(s)^{-2}\Big[1-e^{-\tilde{r}^2}(1+\tilde{r}^2)\Big]\,ds\\&+\alpha_4\frac{k}{2}\int_{t-1}^{s^*(t)}s^{-1/2}\mu(s)^{-2}\Big[1-e^{-\tilde{r}^2}(1+\tilde{r}^2)\Big]\,ds\\=:&\,B_{1,0,0}(t)+B_{1,0,1}(t).\end{aligned}$$

Using the inequality

$$1-e^{-\tilde{r}^2}(1+\tilde{r}^2)\leq\frac{\tilde{r}^4}{2},$$

we get

$$\begin{aligned}
B_{1,0,0}(t) &\lesssim \int_0^{t-1} s^{-1/2}\frac{\mu(s)^2}{(t-s)^2}\,ds\\
&= \int_0^{1-t^{-1}} \frac{t^{1/2}z^{-1/2}}{t^2}\frac{\mu(tz)^2}{(1-z)^2}\,dz\\
&\lesssim t^{-3/2}\int_0^{1-t^{-1}} \frac{z^{-1/2}}{(1-z)^2}\,dz\\
&\lesssim t^{-3/2}[1+t]\\
&\lesssim t^{-1/2}.
\end{aligned}$$

We estimate

$$\begin{aligned}
B_{1,0,1}(t) &\lesssim \int_{t-1}^{s^*(t)} s^{-1/2}\mu(s)^{-2}\frac{\mu(s)^4}{(t-s)^2}\,ds\\
&\lesssim \mu(t-1)^2\int_{t-1}^{t-\frac{\mu(s^*(t))}{4}} \frac{s^{-1/2}}{(t-s)^2}\,ds\\
&= \mu(t-1)^2 t^{-3/2}\int_{1-t^{-1}}^{1-\frac{\mu(s^*(t))^2}{4t}} \frac{z^{-1/2}}{(1-z)^2}\,dz\\
&= \mu(t-1)^2 t^{-3/2}\int_{\frac{\mu(s^*(t))^2}{4t}}^{1/t} \frac{(1-w)^{-1/2}}{w^2}\,dw\\
&\lesssim \mu(t-1)^2 t^{-3/2}\big(1-t^{-1}\big)^{-1/2}\left(\frac{\mu(s^*(t))^2}{t}\right)^{-1}\\
&\lesssim t^{-1/2}\frac{\mu(t-1)^2}{\mu(s^*(t))^2}\\
&\lesssim t^{-1/2}(1+o(1)),
\end{aligned}$$

where we used the change of variables $s = tz$ in the third line and $z = 1-w$ in the fourth one. Now, we estimate $B_2(t)$. Let

$$\begin{aligned}
B_2(t) &= \frac{\alpha_4 k}{8}\int_0^{t-1} \frac{s^{-1/2}}{(t-s)}e^{-\frac{\mu(s)^2}{4(t-s)}}\,ds + \frac{\alpha_4 k}{8}\int_{t-1}^{t} \frac{s^{-1/2}}{t-s}e^{-\frac{\mu(s)^2}{4(t-s)}}\,ds\\
&:= B_{2,1}(t) + B_{2,2}(t).
\end{aligned}$$

We show that $B_{2,1}(t) \lesssim t^{-1/2}\ln(t)$. Indeed, we have

$$\begin{aligned}
B_{2,1}(t) &= \frac{\alpha_4 k}{8}\int_0^{t/2} \frac{s^{-1/2}}{(t-s)}e^{-\frac{\mu(s)^2}{4(t-s)}}\,ds + \frac{\alpha_4 k}{8}\int_{t/2}^{t-1} \frac{s^{-1/2}}{(t-s)}e^{-\frac{\mu(s)^2}{4(t-s)}}\,ds\\
&:= B_{2,1,1}(t) + B_{2,1,2}(t),
\end{aligned}$$

where

$$
\begin{aligned}
B_{2,1,1}(t) &\lesssim t^{-1}\int_0^{t/2} s^{-1/2} e^{-\frac{\mu(s)^2}{4(t-s)}}\,ds \\
&\lesssim t^{-1}\int_0^{t/2} s^{-1/2}\,ds \lesssim t^{-1/2},
\end{aligned}
$$

and

$$
\begin{aligned}
B_{2,1,2}(t) &\lesssim t^{-1/2}\int_{t/2}^{t-1} \frac{e^{-\frac{\mu(s)^2}{4(t-s)}}}{t-s}\,ds \\
&\lesssim t^{-1/2}\int_{t/2}^{t-1} \frac{1}{t-s}\,ds \lesssim t^{-1/2}\ln(t).
\end{aligned}
$$

Next, we bound the main term $B_{2,2}(t)$. We have

$$
\begin{aligned}
B_{2,2}(t) &= \frac{\alpha_4 k}{8} t^{-1/2}\int_{1-t^{-1}}^{t} \frac{z^{-1/2}}{(1-z)} e^{-\frac{\mu(tz)^2}{4t(1-z)}}\,dz \\
&\le \frac{\alpha_4 k}{8} t^{-1/2}\big(1-t^{-1}\big)^{-1/2}\int_{1-t^{-1}}^{1} \frac{e^{-\frac{\mu(t)^2}{4t(1-z)}}}{(1-z)}\,dz.
\end{aligned}
$$

Using the change of variable $\alpha(t)(1-z)^{-1} = w$, where $\alpha(t) = \frac{\mu(t)}{4t}$ we get

$$
\int_{1-t^{-1}}^{1} \frac{1}{1-z} e^{-\frac{\alpha(t)}{1-z}}\,dz = \int_{t\alpha(t)}^{\infty} \frac{e^{-w}}{w}\,dw,
$$

hence we conclude that

$$
\begin{aligned}
B_{2,2}(t) &\le \frac{\alpha_4 k}{8} t^{-1/2}\ln\big(\mu^{-2}(t)\big) + O(t^{-1/2}) \\
&\le H(0,0) + O(t^{-1/2}).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
J_{0,\mathbb{R}^4}[\dot\lambda_0](0,t) &\le B_1(t) + B_2(t) \\
&= B_{1,0,0}(t) + B_{1,0,1}(t) + B_{1,1}(t) + B_{2,1}(t) + B_{2,2}(t) \\
&\le B_{2,2}(t) + O(t^{-1/2}\ln(t)) \\
&\le H(0,0) + O(t^{-1/2}\ln(t)).
\end{aligned}
$$

as $t \to \infty$.

Step 2 (Lower bound). We decompose $J[\dot\lambda_0](0,t)$ and estimate

$$
\begin{aligned}
J[\dot\lambda_0](0,t) &= \alpha_4 k \int_0^t \frac{s^{-1/2}}{4(t-s)} \int_0^\infty \frac{e^{-r^2} r^3}{\tilde r^2 + r^2}\, dr\, ds \\
&\geq \alpha_4 k \int_{t-1}^t \frac{s^{-1/2}}{4(t-s)} \int_{\tilde r}^\infty \frac{e^{-r^2} r^2}{\tilde r^2 + r^2}\, dr.
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
\alpha_4 k \int_{t-1}^t \frac{s^{-1/2}}{4(t-s)} \int_{\tilde r}^\infty \frac{e^{-r^2} r^2}{\tilde r^2 + r^2}\, dr =& \alpha_4 k \int_{t-1}^t \frac{s^{-1/2}}{4(t-s)} \int_{\tilde r}^\infty e^{-r^2} r\, dr\, ds \\
&- \alpha_4 k \int_{t-1}^t \frac{s^{-1/2}}{4(t-s)} \tilde r^2 \int_{\tilde r}^\infty \frac{e^{-r^2} r}{(\tilde r^2 + r^2)}\, dr\, ds \\
=& \frac{\alpha_4 k}{8} \int_{t-1}^t \frac{s^{-1/2}}{(t-s)} e^{-\frac{\mu(s)^2}{4(t-s)}}\, ds \\
&- \alpha_4 k \int_{t-1}^t \frac{s^{-1/2}}{4(t-s)} \tilde r^2 \int_{\tilde r}^\infty \frac{e^{-r^2} r}{(\tilde r^2 + r^2)}\, dr\, ds \\
=:& \, B_{22}(t) - B_3(t).
\end{aligned}
$$

We prove that $B_3(t) \leq C t^{-1/2}$ for t large enough. We estimate

$$
B_3 \leq \alpha_4 k \int_{t-1}^t \frac{s^{-1/2}}{4(t-s)} \tilde r^2 \int_{\tilde r}^\infty \frac{e^{-\tilde r^2}}{r}\, dr\, ds.
$$

We use the following bounds:

$$
\int_{\tilde r}^\infty \frac{e^{-r^2}}{r}\, dr \leq \begin{cases} \ln\left(\frac{1}{\tilde r}\right) + c_1 & \tilde r \leq 1, \quad c_1 = \int_1^\infty \frac{e^{-\tilde r}}{r}\, dr, \\ (2\tilde r^2)^{-1} e^{-\tilde r^2} & \tilde r > 1, \end{cases}
$$

Thus, using $\tilde r^2[-\ln(\tilde r) + c_1] \leq \tilde r^{3/2}$ for $\tilde r \in (0,1]$ we obtain

$$
\begin{aligned}
B_3(t) \leq& \alpha_4 k \int_{t-1}^{s^*(t)} \frac{s^{-1/2}}{4(t-s)} \tilde r^2 \left[\ln\left(\frac{1}{\tilde r}\right) + c_1\right] ds \\
&+ \alpha_4 \frac{k}{2} \int_{s^*(t)}^t \frac{s^{-1/2}}{4(t-s)} e^{-\tilde r^2}\, ds \\
\leq& \alpha_4 k \int_{t-1}^{s^*(t)} \frac{s^{-1/2}}{4(t-s)} \tilde r^{3/2}\, ds \\
&+ \frac{\alpha_4 k}{2} \int_{s^*(t)}^t \frac{s^{-1/2}}{4(t-s)} e^{-\frac{\mu(s)^2}{4(t-s)}}\, ds \\
=:& \, B_{3,1}(t) + B_{3,2}(t).
\end{aligned}
$$

We estimate $B_{3,2}$. We have

$$\begin{aligned}
B_{3,1} &\lesssim \int_{t-1}^{s^*(t)} \frac{s^{-1/2}}{(t-s)^{1+\frac{3}{4}}}\mu(s)^{3/2}\,ds \\
&\lesssim \mu(t-1)^{3/2}(t-1)^{-1/2}\int_{t-1}^{s^*(t)} \frac{1}{(t-s)^{1+\frac{3}{4}}}\,ds \\
&\lesssim t^{-1/2}\mu(t-1)^{3/2}\mu(s^*(t))^{3/2} \\
&\lesssim t^{-1/2}.
\end{aligned}$$

This shows that $B_{3,1}(t) \leq Ct^{-1/2}$. Now, we have

$$\begin{aligned}
B_{3,2}(t) &\lesssim \int_{t-\mu(t)^2/4}^{t} \frac{s^{-1/2}}{(t-s)}e^{-\frac{\mu(s)^2}{4t}}\,ds \\
&\lesssim t^{-1/2}\int_{1-\alpha(t)}^{1} \frac{e^{-\frac{\alpha(t)}{(1-z)}}}{(1-z)}\,dz \\
&\lesssim t^{-1/2}\int_{1}^{\infty} \frac{e^{-w}}{w}\,dw \\
&\lesssim t^{-1/2}.
\end{aligned}$$

Thus, we have

$$J_{0,\mathbb{R}^4}[\dot{\lambda}_0](0,t) \geq B_{22}(t) + O(t^{-1/2}).$$

We need to estimate $B_{22}(t)$ from below. We have

$$\begin{aligned}
B_{22}(t) &= \frac{\alpha_4 k}{8}\int_{t-1}^{t} \frac{s^{-1/2}}{(t-s)}e^{-\frac{\mu(s)^2}{4(t-s)}}\,ds \\
&= \frac{\alpha_4 k}{8}t^{-1/2}\int_{1-t^{-1}}^{1} \frac{z^{-1/2}}{(1-z)}e^{-\frac{\mu(tz)}{4(t-s)}}\,ds \\
&\geq \frac{\alpha_4 k}{8}t^{-1/2}\int_{1-t^{-1}}^{1} \frac{z^{-1/2}}{1-z}e^{-\frac{\mu(t-1)^2}{4t(1-z)}}\,dz.
\end{aligned}$$

Now, letting

$$\beta(t) := \frac{\mu(t-1)^2}{4t},$$

we use the change of variable $\beta(t)(1-z)^{-1} = w$ to get

$$\begin{aligned}
\frac{\alpha_4 k}{8} t^{-1/2} \int_{1-t^{-1}}^{1} \frac{z^{-1/2}}{1-z} e^{-\frac{\mu(t-1)^2}{4t(1-z)}} &= \frac{\alpha_4 k}{8} t^{-1/2} \int_{t\beta(t)}^{\infty} \frac{e^{-w}}{w}\, dw \\
&\geq \frac{\alpha_4 k}{8} t^{-1/2} [\ln(t\beta(t))] - Ct^{-1/2} \\
&\geq \frac{\alpha_4 k}{8} t^{-1/2} \big[\ln\big(\mu(t-1)^{-2}\big)\big] + O(t^{-1/2}) \\
&= \frac{\alpha_4 k}{8} 2k\big(1-t^{-1}\big)^{1/2} + O(t^{-1/2}) \\
&= \frac{\alpha_4 k^2}{4} + O(t^{-1/2}) \\
&= H(0,0) + O(t^{-1/2}).
\end{aligned}$$

Combining the lower and the upper bound we conclude that

$$H(0,0) - C_1^{-1} t^{-1/2} \leq J_{\mathbb{R}^4,0}[\dot\lambda_0](0,t) \leq H(0,0) + C_1 t^{-1/2} \ln(t)$$

for a sufficiently large positive constant C_1. $\square$

3.1.4 The blow-up growth in the radial case

The radial solution to

$$\begin{aligned}
\Delta_x H(x,0) &= 0 \quad \text{in } B_R(0) \\
H(x,0) &= \frac{\alpha_4}{|x|^2} \quad \text{on } \partial B_R(0),
\end{aligned}$$

it is given by $h(|x|) = H(x,0)$ which solves

$$\begin{aligned}
&h''(r) + \frac{3}{r} h'(r) = 0 \quad \text{in } [0,R] \\
&\partial_r h(0) = 0, \quad h(R) = \frac{\alpha_4}{R^2}.
\end{aligned}$$

The solution to this problem is the constant function

$$h(r) = \frac{\alpha_4}{|R|^2},$$

and hence $H(x,0) = \alpha_4 R^{-2}$ for all $x \in B_R(0)$. Thus, plugging the value $H(0,0) = \alpha_4 R^{-2}$ in (3.6) we deduce

$$\ln\left(\frac{1}{\mu(t)}\right) = \frac{2}{R}\sqrt{t}(1+o(1)),$$

that is exactly the asymptotic behaviour found by Galaktionov and King in [14].

3.1.5 The coercivity of the quadratic form

Here we modify [7, Lemma 7.2] to get the same result in dimension 4. This lemma establishes a weak coercivity estimate for the quadratic form

$$\begin{aligned} Q: \quad & H_R \to \mathbb{R} \\ & \phi \mapsto Q(\phi) := \int_{B_{2R}(0)} |\nabla \phi(y)|^2 - 5U(y)^4 |\phi(y)|^2 dy \end{aligned}$$

where

$$H_R := \left\{ \phi \in H_0^1(B_{2R}) : \phi \text{ is radial}, \quad \int_{B_{2R}(0)} Z_0 \phi \, dx = 0 \right\}. \tag{3.7}$$

The result in dimension 3 is already present in [9]. Compared to the higher dimensions we obtain weaker bounds for $n = 3, 4$ due to the unbounded growth of the $L^2(B_R)$-norm of Z_{n+1} as $R \to \infty$. Roughly speaking, the next Lemma gives an approximated estimate from below for the second eigenvalue λ_{2,B_R} on $B_{2,R}$ of the linear operator

$$-\Delta - pU^{p-1}.$$

Lemma 3.1.2. *There exist positive constants γ_n, R_n, depending only on n, such that the estimate*

$$Q(\phi, \phi) \geq \begin{cases} \dfrac{\gamma_3}{R^2} \|\phi\|^2_{L^2(B_R)} & \textit{if} \quad n = 3, \\ \dfrac{\gamma_4}{R^2 \ln(R)} \|\phi\|^2_{L^2(B_R)} & \textit{if} \quad n = 4, \\ \dfrac{\gamma_n}{R^{n-2}} \|\phi\|^2_{L^2(B_R)} & \textit{if} \quad n \geq 5, \end{cases} \tag{3.8}$$

holds for all $\phi \in H_R$ and $R > R_n$.

If we could replace Z_0, that is the eigenfunction associated to the negative eigenvalue for this operator on $\mathbb{R}^n$, with the analogue eigenfunction for the operator on B_{2R}, we would get a lower bound on the second eigenvalue $\lambda_{2,R}$.

Proof. We consider the subspace $H_R \subset H_0^1(B_R)$ defined in (3.7) and let

$$\lambda_R := \inf_{\substack{\phi \in H_R \\ \|\phi\|_{L^2} = 1}} Q(\phi, \phi).$$

Since the quadratic form Q is coercive and convex in ∇u the value λ_R is achieved. Let $\phi_R \in H_R$ with $\|\phi_R\|_2 = 1$ such that

$$\lambda_R = Q(\phi_R, \phi_R).$$

The function ϕ_R satisfies

$$L_0[\phi] := \Delta\phi_R + pU^{p-1}\phi_R = \lambda_R\phi_R - c_0 Z_0 \quad \text{in } B_R, \tag{3.9}$$
$$\phi_R \in H_R,$$

that, in terms of ψ_R, means

$$\mathcal{L}[\psi_R] = \psi_R'' + \frac{n-2}{r}\psi_R' + pU^{p-1}\psi_R = h_R(r),$$
$$\partial_r\psi_R(0) = 0, \quad \psi_R(R) = 0,$$

where $h_R(r) = -\lambda_R\psi_R(r) + c_R Z_0(r)$ for a suitable Lagrangian multiplier c_R. The constant λ_R is non-negative. Indeed, Consider the radial problem

$$\mathcal{L}[\Phi] + \lambda\Phi = 0 \quad \Phi'(0) = 0, \quad \lim_{r\to\infty}\Phi(r) = 0,$$

where

$$\mathcal{L}[\Phi] = \Phi'' + \frac{n-1}{r}\Phi' + pU^{p-1}\Phi.$$

The kernel contains Z_{n+1} which changes sign once. Hence, from the maximum principle we obtain that the quadratic form associated to the operator in $\mathbb{R}^n$ cannot be negative. Using the Rayleigh quotient characterization of the eigenvalues we see that $\lambda_\infty \geq 0$. If we suppose that $\lambda_R < 0$, we can extend ϕ_R trivially outside B_{2R} and such extension makes the quadratic form on $\mathbb{R}^n$ negative, but this is not possible. Hence $\lambda_R \geq 0$.
Let $\eta_R(s)$ to be a cut-off function such that $\eta_R(s) = 1$ for $s \leq R/4$, $s = 0$ for $s \geq R/2$. Testing equation (3.9) against Z_0 and integrating by parts we get $c_R = O\big(e^{-\sigma R}\big)$ for some $\sigma > 0$. Let $\tilde{Z}(r)$ be a second solution to $L_0[Z] = 0$ linearly independent of Z_{n+1}, such that the Wronskian is normalized as

$$\begin{aligned} W(Z_{n+1}, \tilde{Z}) &= \tilde{Z}'(r)Z_{n+1}(r) - \tilde{Z}(r)Z_{n+1}'(r) \\ &= \frac{1}{r^{n-1}}. \end{aligned}$$

We have $\tilde{Z} \sim c_0 r^{2-n}$ as $r \to 0$ and $\tilde{Z} \sim c_\infty$ as $r \to 0$ for some non-zero constants c_0, c_∞. The formula of variation of parameters gives the solution

$$\begin{aligned} \psi_R(r) =& \tilde{Z}(r)\int_0^r h_R(s)Z_{n+1}(s)s^{n-1}\,ds \\ &+ Z_{n+1}(r)\int_r^{2R} h_R(s)\tilde{Z}(s)s^{n-1}ds - D_R Z_{n+1}(r), \end{aligned} \tag{3.10}$$

where

$$D_R := Z_{n+1}(2R)^{-1}\tilde{Z}(2R)\int_0^{2R} h_R(s)Z_{n+1}(s)s^{n-1}\,ds.$$

We want to estimate $\|\phi_R\|_{L^2(B_{2R})}$ by means of (3.10) where $\phi_R(x) = \psi_R(r)$, however $\tilde{Z}(r)^2$ is not integrable in $r = 0$ when $n \geq 4$, hence we estimate its L^2-norm on the annulus $\mathcal{A}_R := B_{2R}(0) \setminus B_{R^{-1}}(0)$. We have

$$\left|\int_0^r h_R(s)Z_{n+1}(s)s^{n-1}ds\right| \lesssim (\lambda_R + e^{-\sigma R})\|Z_{n+1}\|_{B_{2R}},$$

and

$$\|D_R Z_{n+1}\|_{L^2(\mathcal{A}_R)} \lesssim R^{n-2}(\lambda_R + e^{-\sigma R})\|Z_{n+1}\|^2_{L^2(B_{2R})}.$$

Also, for $r \in (R^{-1}, 2R)$ we have

$$\left|\int_r^{2R} \tilde{Z}(s)s^{n-1}\,ds\right| \leq \|h\|_{L^2(B_{2R})}\|\tilde{Z}\|_{L^2(\mathcal{A}_R)}.$$

We estimate

$$\begin{aligned}\|\tilde{Z}\|_{\,L^2(\mathcal{A}_R)} &\lesssim \left(\int_{\frac{1}{R}}^1 r^{2(2-n)}r^{n-1}\,dr + \int_1^R r^{n-1}\,dr\right)^{\frac{1}{2}}\\ &\lesssim R^{\frac{n}{2}}.\end{aligned}$$

Since Z_{n+1} is given explicitly, it is easy to compute

$$\|Z_{n+1}\|^2_{L^2(B_R)} \sim \begin{cases} \sqrt{3}\pi\left[R - \dfrac{5\pi}{4} + \dfrac{5}{R}\right] & \text{if} \quad n = 3,\\ 16\pi^2\left[\ln(R) - \dfrac{7}{6} + \dfrac{3}{R^2}\right] & \text{if} \quad n = 4,\\ \dfrac{\alpha_n^2\omega_n(n-2)^2}{4}\left[c_n - \dfrac{R^{4-n}}{n-4}\right] & \text{if} \quad n \geq 5,\end{cases}$$

for $R \to \infty$, where

$$c_n := \int_0^\infty \frac{(r^2-1)^2}{(r^2+1)^n}r^{n-1}\,dr.$$

Combining all these estimates we get

$$
\begin{aligned}
\|\phi_R\|_{L^2(\mathcal{A}_R)} \lesssim & \|h_R\|_{L^2(B_R)} \big(\|\tilde{Z}\|_{L^2(\mathcal{A}_R)} \|Z_{n+1}\|_{L^2(B_{2R})} \\
& + \|\tilde{Z}\|_{L^2(\mathcal{A}_R)} \|Z_{n+1}\|_{L^2(B_{2R})} + R^{n-2} \|Z_{n+1}\|^2_{L^2(B_{2R})} \big) \\
\lesssim & \big(\lambda_R + e^{-\sigma R}\big) \Big(R^{n/2} \|Z_{n+1}\|_{L^2(B_{2R})^2} + \|Z_{n+1}\|^2_{L^2(B_{2R})^2} R^{n-2} \Big) \\
\lesssim & \big(\lambda_R + e^{-\sigma R}\big) \|Z_{n+1}\|^2_{L^2(B_{2R})^2} R^{n-2}
\end{aligned}
$$

Now, from standard elliptic estimates (see [17]) using the equation for ϕ and the L^2-estimate we deduce the L^∞-bound $\|\phi_R\|_{L^\infty(B_1)} \leq c$ for some constant c independent of R and $\|\phi_R\|_{L^2(B_{2R})} = 1$ we get

$$
\begin{aligned}
\left(1 - \frac{c^2}{R^n}\right)^{1/2} & \leq \|\phi_R\|_{L^2(\mathcal{A}_R)} \\
& \leq \|Z_{n+1}\|^2_{L^2(B_{2R})} (\lambda_R + e^{-\sigma R}) R^{\max\{\frac{n}{2}, n-2\}}.
\end{aligned}
$$

This implies that, in dimension $n \geq 5$, $n = 4$ and $n = 3$, λ_R cannot be $o\big(R^{2-n}\big)$, $o\big(R^{-2}\ln(R)\big)$ and $o\big(R^{-2}\big)$ as $R \to +\infty$ respectively. This proves the existence of constants γ_n such that the estimates (3.8) hold for all R sufficiently large. $\square$

Thus, defining

$$
\theta^*_{R,n} = \begin{cases} R^{2-n} & \text{if} \quad n \geq 5, \\ R^2 \log(R) & \text{if} \quad n = 4, \\ R^2 & \text{if} \quad n = 3, \end{cases}
$$

the linear estimate for the inner problem in [7], also recalled in chapter 2, generalizes to any dimension in

$$
\begin{aligned}
|\phi(y,\tau)| + (1+|y|)|\nabla_y \phi(y,\tau)| \lesssim \tau^{-\nu} \Bigg[& \frac{\theta^*_{R,n}\theta_0(R,a)}{1+|y|^3} \|h_0\|_{\nu,2+a} \\
& + \frac{R^3 \theta_1(R,a)}{1+|y|^4} \|h_1\|_{\nu,2+a} + \frac{1}{1+|y|^a} \|h^\perp\|_{\nu,2+a} \Bigg],
\end{aligned}
$$

where $\nu > 0$ and

$$
\theta^0_R(R,a) := \begin{cases} 1 & \text{if } a > 2, \\ \log R & \text{if } a = 2, \\ R^{2-a} & \text{if } a < 2, \end{cases}, \quad \theta^1_R(R,a) := \begin{cases} 1 & \text{if } a > 1, \\ \log R & \text{if } a = 1, \\ R^{1-a} & \text{if } a < 1. \end{cases}
$$

In particular in dimension 4, we expect to use the estimate with $a = 1$ and $n = 4$, that gives

$$|\phi(y,\tau)| \lesssim \tau^{-\nu}\left[\frac{R^4\ln(R)}{1+|y|^5}\|h\|_{\nu,2+a}\right].$$

3.2. Outlook

Concerning the main result in chapter 2, it would be very interesting to understand if the analytical assumption $3\mu_{BN}(\Omega) < \lambda_1(\Omega)$ is necessary for the existence of an infinite blow-up solution in Ω. If the result is valid without such condition then

- any domain Ω admits a positive infinite time blow-up solution with a single spike at any fixed $q \in \Omega$;
- it becomes natural to construct multi-spike threshold solutions.

For $n \geq 5$ multispike solutions exist if the points are sufficiently close the boundary and relatively far from each other. This is a natural requirement in order to successfully treat the error terms associated to the interactions between bubbles. In contrast, the condition $3\gamma(q) < \lambda_1$ implies that the blow-up point is far from the boundary. Thus, it would be very interesting to understand if multi-spike solutions exist.

In section 3.1 we have started a program for the analogue construction in the 4 dimensional case. A natural step towards the full solution is adapting the parabolic gluing scheme to rigorously prove the existence of the perturbation. Since in this situation the expected blow-up rate is not exponential, we believe that the result does not require any analytical constraint on the location of blow up points inside Ω, and thus the multispike scenario should be naturally detected.

Finally, it would be interesting to find sign-changing solutions in low dimension. The inspiring work [10] for $n \geq 5$ presents the first example of sign-changing unbounded global solution and suggests a suitable ansatz.

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