UNIVERSITÀ DI TRENTO

The Containment Problem

a general introduction and the particular case for Steiner systems

Giacomo Borin

Università di Trento

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Abstract

In this presentation we view a brief introduction to the **Containment Problem**, an open problem in Algebraic Geometry and Commutative Algebra.
Later we will see some connections with the colouration of an hypergraph (from Combinatorics and Graph Theory), in particular for Steiner Systems, mainly inspired by the article:

Edoardo Ballico, Giuseppe Favacchio, Elena Guardo, Lorenzo Milazzo, and Abu Chackalamannil Thomas. *Steiner Configurations ideals: containment and colouring*. 2021. arXiv: 2101.07168 [math.AC].

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Introduction

Primary decomposition

We say that an ideal $\mathfrak{a} \subseteq R$ has a **primary decomposition** if there exists a finite set of primary ideal $\{\mathfrak{q}_1, ..., \mathfrak{q}_n\}$ such that:

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

In a Noetherian Ring we have:

- Existence
- Uniqueness of the minimal primes $\mathfrak{p}_i = \mathrm{rad}(\mathfrak{q}_i)$, in particular they are the associated primes $\mathrm{Ass}(R/I)$
- Uniqueness of the primary ideals $\mathfrak{q}_i$

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Normal powers

Given an homogeneous ideal $I = \langle f_1, \dots, f_k \rangle$ the n-th power of I is:

$$I^n = \langle \xi_1 \cdots \xi_n \,|\, \xi_i \in \{f_1, \dots, f_k\} \rangle$$

- Easy algebraic construction
- Unknown primary decomposition
- No clear geometric interpretation

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Symbolic powers

Given an ideal I in a Noetherian ring R the m-th symbolic power of i is:

$$I^{(m)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(R/I)} (I^m R_{\mathfrak{p}} \cap R)$$

- $I^m R_{\mathfrak{p}} \cap R = \{r \in R$ such that exists $s \in R \setminus \mathfrak{p}$ with $sr \in I^n\}$ is the minimal $\mathfrak{p}$-primary ideal that contains I^m
- Clear primary decomposition
- No easy set of generators
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Theorem (Zariski-Nagata Theorem [Zar49; Nag62])

If $R = k[x_0, \dots, x_n]$ is a polynomial ring and $\mathfrak{p}$ is a prime ideal then:

$$\mathfrak{p}^{(m)} = \bigcap_{\substack{\mathfrak{m} \in \mathfrak{m}\,\mathrm{Spec}(R) \\ \mathfrak{p} \subset \mathfrak{m}}} \mathfrak{m}^n$$

So the symbolic power represents the polynomial vanishing with multiplicity m on the variety $\mathcal{V}(I)$:

$$I^{(m)} = I^{\langle m \rangle} = \{f \in R \text{ that vanishes on } \mathcal{V}(I) \text{ with multiplicity } m\}$$

Zariski-Nagata Theorem

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Containment

The natural question that arises is:

Question

Can we find a relation between I^n and $I^{(m)}$?

As a consequence of the Nakayama Lemma we have one direction:

Theorem

If R is a Noetherian reduced ring then $I^r \subseteq I^{(m)}$ if and only if $r \geq m$

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The Containment Problem

Question

Given a Noetherian Ring R and an ideal I, for which m, r positive integers we have the containment:

$$I^{(m)} \subset I^r$$

Let's see a celebrated result, showed in [HH02; ELS01]:

Theorem

(Ein-Lazarsfeld-Smith, Hochster-Huneke) Let R be a regular ring and I a non-zero, radical ideal, then if h is the big height of I we have that for all $n \geq 0$ we have:

$$I^{(hn)} \subseteq I^n$$

- The height of a prime ideal $\mathfrak{p}$ is the supremum length of a descending chain of primes: $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_h = \mathfrak{p}$
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An example of a non trivial use of the previous theorem on a reduced subscheme came directly from [ELS01, p. 2.3]:

Example

Consider a finite set of points Z in $\mathbb{P}^2$, since the subscheme has dimension 0 the ideal has big height 2, so we have $I^{(2m)} \subseteq I^m$ for $I = I(Z)$. So this implies that all F with multiplicity $\geq 2m$ on Z stays in $I(Z)^m$.

Open questions

Some important open questions for the Containment Problem are:

Question (Huneke)

Let I be a saturated ideal of a reduced finite set of points in $\mathbb{P}^2$, does the containment:

$$I^{(3)} \subseteq I^2$$

hold?

Conjecture (Harbourne, 2009)

Given a non-zero, proper, homogeneous, radical ideal $I \subset k[x_0, \ldots, x_n]$ with big height h, than for all $m > 0$:

$$I^{(hm-h+1)} \subseteq I^m$$

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Asymptotic conjecture

Harbourne Conjecture was proven to be false for several ideal and arbitrary $m > 0$, but there are not counterexample for:

Conjecture (Stable Harbourne, 2013)

Given a non-zero, proper, homogeneous, radical ideal $I \subset k[x_0, ..., x_n]$ with big height h, than for all $m \gg 0$:

$$I^{(hm-h+1)} \subseteq I^m$$

Colouring and containment

An hypergraph is a pair (V, E) where V is a finite set of vertices and E contains non-empty subset of V called hyper edges

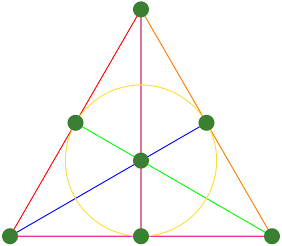
Definiton

A **Steiner system** (V, B) of type $S(t, n, v)$ is an *hypergraph* with $|V| = v$ and all the elements of B, called blocks, are n-subsets (of V) such that every t-tuple of elements in V is contained in only one block of B.

Fano plane

The most known example of Steiner is of type $S(2,3,7)$ and, up to isomorphism, is the Fano Plane ($\mathbb{P}^3_{\mathbb{F}_2}$). It has as blocks all the lines:

$$B := \{\{1,2,3\},\{3,4,5\},\{3,6,7\},\{1,4,7\},\\ \{2,4,6\},\{2,5,7\},\{1,5,6\}\}$$



Definiton

An m-colouring of the hypergraph $H = (V, E)$ is a partition in m subset of $V = U_1 \sqcup \ldots \sqcup U_m$ such that for every edge $\beta \in E$ we have $\beta \not\subseteq U_i$ for all $i = 1, \ldots, m$. A hypergraph H is m-colourable if there exists a proper m-colouring.

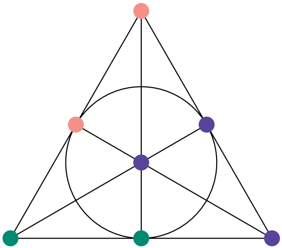


Figure: A 3-colouring for the Fano Plane

Coverability

Definiton

A hypergraph $H = (V, E)$ is said *c-coverable* if there exists a partition in c subset of $V = U_1 \sqcup \ldots \sqcup U_c$ such that every U_i is a **vertex cover**, which means that for all $\beta \in E$ we have $\beta \cap U_i \neq \emptyset$.

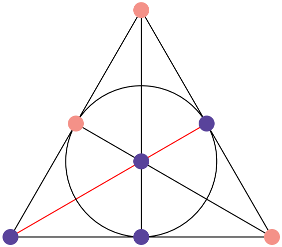


Figure: Steiner System $S(2, 3, v)$ for $v > 3$ are not 2-coverable

For an hypergraph $H = (V, E)$ with $V = \{x_1, ..., x_v\}$ consider the polynomial ring $k[V] = k[x_1, ..., x_v]$:

Definiton

The **cover ideal** of the hypergraph H in $k[V]$ is:

$$J(H) := \langle x_{j_1} \cdots x_{j_r} \mid \{x_{j_1}, ..., x_{j_r}\} \text{ is a vertex cover of } H \rangle = \bigcap_{\beta \in E} \mathfrak{p}_\beta$$

Where for every hyperedge $\beta = \{x_{i_1}, ..., x_{i_r}\} \in E$ then $\mathfrak{p}_\beta$ is the prime ideal $\langle x_{i_1}, ..., x_{i_r} \rangle$

For a hypergraph $H = (V, B)$ we define $\tau(H)$ as $\min_{\beta \in B}\{|\beta|\}$, so we obtain:

Theorem (Theorem 4.8 of [Bal+21])

Let $H = (V, B)$ be a hypergraph, if H is not d-coverable then

$$J(H)^{(\tau(H))} \not\subseteq J(H)^d$$

For example for Steiner Systems we have:

Proposition (Proposition 4.9 of [Bal+21])

If $v > 3$ and $S = (V, B)$ is a Steiner Triple System $S(2, 3, v)$, then $J(S)^{(3)} \not\subseteq J(S)^2$

Theorem

Consider a simple hypergraph $H = (V, B)$, if we indicate $\tau = \tau(H)$ and the cover ideal $J = J(H)$, then for all $q \leq |V|$ if H is not q-colourable then we have:

$$J^{(\tau(q-1))} \not\subseteq J^q$$

The proof rely on the two results:

Theorem (Theorem 3.2 of [FHV11])

Let $H = (V, E)$ be a simple hypergraph on $V = \{x_1, ..., x_v\}$, then for all $d > 0$ we have $(x_1 \cdots x_v)^{d-1} \in J(H)^d$ if and only if $d \geq \chi(H)$, where $\chi(H)$ is the minimum integer c such that H is c-colourable

Proposition

If I is a radical ideal in a polynomial ring we have:

$$I^{(m)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(R/I)} \mathfrak{p}^m \tag{1}$$

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