The Containment Problem

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Abstract

Stuff x 3

1 Inroduction

1.1 Associated primes

Let R be a commutative ring with unity, and $\mathfrak{a}$,$\mathfrak{b}$ two ideal, we say that the ideal

$$(\mathfrak{a} : \mathfrak{b}) = \{x \in R \,|\, x\mathfrak{b} \subseteq \mathfrak{a}\}$$

is the *ideal quotient*. For the case in which $\mathfrak{a}$ is the null ideal 0 we define the **annihilator** of $\mathfrak{b}$ as:

$$\mathrm{Ann}_R(\mathfrak{b}) = (0 : \mathfrak{b}) = \{x \in R \,|\, x\mathfrak{b} = 0\}$$

We can obviously omit the index R if it is clear by the context. In general given an R-module M and a set $S \subseteq M$ non empty we can define its annihilator as:

$$\mathrm{Ann}_R(S) = \{x \in R \,| xS = 0\} = \{x \in R \,|\, \forall s \in S \; xs = 0\}$$

Definiton 1.1 (Associated Prime)**.** Let M be an R-module. A prime ideal $\mathfrak{p} \subseteq R$ is an **associated prime** of M if there exists a non-zero element $a \in M$ such that $\mathfrak{p} = \mathrm{Ann}_R(a)$.
We define $\mathrm{Ass}_R(M)$ as the set of the associated primes of M.
For an ideal I we say that a prime is associated to I if it is associated to the R-module R/I.

Remark 1. Another name for associated ideal used by the Bourbaki group is *assasin*, a word play between associated and annihilator.

1.2 Primary decomposition

We would like to have some sort of factorization for the ideals of a ring, more general than the *unique factorization domains*, in fact this is useful only for principal ideals. With this objective **primary decomposition** was introduced.

Now I will recall some of the principal result on this topics, contained in [2, Section 7] and [1, Section 4 and Page 83]

Definiton 1.2. An ideal $\mathfrak{a}$ in a ring R is said primary if $R/\mathfrak{a}$ is different from zero and all its zerodivisors are nilpotent, otherwhise we can express this as:

$$fg \in \implies f \in \mathfrak{a} \text{ or } g^n \in \mathfrak{a} \text{ for some } n > 0$$

It is obvious that the radical of a primary ideal is a prime ideal, infact given $fg \in \mathrm{rad}(\mathfrak{a})$ we have $(fg)^m = f^m g^m \in \mathfrak{a}$ for $m > 0$, and so $f^m \in \mathfrak{a} \Rightarrow f \in \mathrm{rad}(\mathfrak{a})$ or exists $n > 0$ such that $g^{mn} \in \mathfrak{a} \Rightarrow g \in \mathrm{rad}(\mathfrak{a})$.

If $\mathfrak{a}$ is a primary ideal such that $\mathrm{rad}(\mathfrak{a}) = \mathfrak{p}$ we say that $\mathfrak{a}$ is $\mathfrak{p}$-primary.

Remarks 2.

1. The power of a prime ideal isn't always primary, for example if in $R = \mathbb{K}[x,y,z]/(xy - z^2)$ we consider the prime ideal $\mathfrak{p} = (x, z)$ (it is prime since $R/\mathfrak{p} \simeq \mathbb{K}[y]$ that is an integral domain) we have that y is a zero divisor in $R/\mathfrak{p}$ (since x is not zero and $yx = z^2 = 0$, since $z^2 \in \mathfrak{p}^2$) but it is not nilpotent since $y^k \notin \mathfrak{p}^2$ for all $k > 0$

We say tha an ideal $\mathfrak{a} \subseteq R$ has a **primary decomposition** if there exists a finite set of primary ideal $\{\mathfrak{q}_1, ..., \mathfrak{q}_n\}$ such that:

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

In general such structure does not exists, but for R noetherian we can prove, using Noetherian induction and the concept of irreducible ideal, that every proper ideal has a primary decomposition.

Definiton 1.3. We say that a proper ideal $\mathfrak{a}$ is irreducible if it cannot be written as a proper intersection of ideal, i.e. :

$$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \implies (\mathfrak{a} = \mathfrak{b} \text{ or } \mathfrak{a} = \mathfrak{c})$$

Lemma 1.4. *A proper ideal in a Noetherian ring R is always the intersection of a finite number of irreducible ideals.*

Proof. Let $\mathfrak{F}$ be the set of proper ideal such that the lemma is false. Let $\mathfrak{a}$ be a maximal ideal of $\mathfrak{F}$, since it cannot be irreducible there exists $\mathfrak{b}$, $\mathfrak{c}$ strictly greater than $\mathfrak{a}$ (so not in $\mathfrak{F}$) such that $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$. This is absurd and so $\mathfrak{F}$ is empty. □

Lemma 1.5. *In a Noetherian ring every irreducible ideal is primary*

Proof. Modulo working in the quotient ring we can assume to work with the zero ideal. So we assume that the ideal 0 is irreducible and we consider x, y such that $xy = 0$ with $y \neq 0$, then x is a zerodivisor. So we have that $y \in \mathrm{Ann}(x)$[1] and we consider the chain:

$$\mathrm{Ann}(x) \subseteq \mathrm{Ann}(x^2) \subseteq ...$$

[1]For $\mathrm{Ann}(x)$ we mean the annihilator of the principal ideal (x)

And for the ascending chain condition there exists m with $\mathrm{Ann}(x^m) = \mathrm{Ann}(x^{m+1})$. Now consider $a \in (x^m) \cap (y)$, then $a = bx^m$ and $a = cy$, so since $y \in \mathrm{Ann}(x)$ we have $0 = cyx = ax = bx^m x = bx^{m+1}$, so $b \in \mathrm{Ann}(x^{m+1}) = \mathrm{Ann}(x^m)$, then $a = bx^n = 0$. So $(x^m) \cap (y) = 0$ and since 0 is irreducible and $y \neq 0$ then $x^m = 0$. □

Combining this two lemmas we have that the decomposition for Noetherian ring. In literature we say that a commutative ring is a **Lasker Ring** if every ideal has a primary decomposition, so we can state that:

Theorem 1.6 (Lasker-Noether)**.** *A Noetherian Ring is also a Lasker Ring*

Now we need to achive some kind of uniqueness. First of all we say that a decomposition $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ is **minimal** if:

1. $\mathrm{rad}(\mathfrak{q}_i)$ are all distinct
2. for all i we have $\mathfrak{q}_i \not\supseteq \bigcap_{j \neq i} \mathfrak{q}_j$

We can easly prove that from every decomposition we can obtain a minimal one using the following lemma:

Lemma 1.7. *If $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathfrak{p}$-primary then $\mathfrak{a} \cap \mathfrak{b}$ is $\mathfrak{p}$-primary*

Infact we can group the primaty ideal to get 1. and omit the superfluous terms to get 2.

So we have two theorem of uniqueness for the prime *associated*[2] to a particular decomposition.

Theorem 1.8 (First uniqueness theorem)**.** *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal with minimal decomposition $\bigcap_{i=1}^n \mathfrak{q}_i$, where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, then:*

$$\mathrm{Ass}(R/\mathfrak{a}) = \{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$$

and so the set of primes $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ is uniquely determined by the ideal

This theorem show the strong ralation that we have between the associated prime ideal and the primary decomposition for Noetherian ring

Theorem 1.9 (Second uniqueness theorem)**.** *Let R be a ring and $\mathfrak{a}$ an ideal with minimal decomposition $\bigcap_{i=1}^n \mathfrak{q}_i$, where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, then if p_i is a minimal element of $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ $\mathfrak{q}_i$ is uniquely determined by the ideals $\mathfrak{a}$ and $\mathfrak{p}_i$. In particular if $\phi : R \to R_{\mathfrak{p}_i} = S^{-1}R$ is the canonical injection (where $S = R \setminus \mathfrak{p}_i$) we have*

$$\mathfrak{q}_i = \phi^{-1}(S^{-1}\mathfrak{a})$$

[2] not a random word

1.3 Sybolic power

Lets consider an homogeneous polynomial ring $k[x_0, ..., x_n]$, it is easy to se that if we consider a variety X with it's coordinate ring $R = k[X]$ and a point $p \in X$ (associated to the maximal ideal $\mathfrak{m}_p$) we have that:

$$\mathfrak{m}_p^n = \{f \in k[X] \text{ such that } f \text{ vanishes in } p \text{ with multiplicity } n\} \tag{1}$$

For general ideal we don't have similiar properties for the normal power of ideal, so we define the **symbolic power**. First of all given a prime ideal $\mathfrak{p}$ in a Noetherian Ring R we can define the n-th symbolic power of $\mathfrak{p}$ as:

$$\mathfrak{p}^{(n)} = \{r \in R \text{ such that exists } s \in R \setminus \mathfrak{p} \text{ with } sr \in \mathfrak{p}^n\} \tag{2}$$

This definition show clearly the idea between the symbolic power, but is not easy to work with. We can have another equivalent definition that use the localization on the prime ideal $R_\mathfrak{p}$. Infact we can see it as the contraction of $\mathfrak{p}^n R_\mathfrak{p}$ over R:

$$\mathfrak{p}^{(n)} = \mathfrak{p}^n R_\mathfrak{p} \cap R \tag{3}$$

In general the generic and symbolic power are different concept. It is obvious that $\mathfrak{p}^n \subset \mathfrak{p}^{(n)}$ since $1 \notin \mathfrak{p}$. For the other direction we can costruct a counter example with the following proposition:

Proposition 1.10. *$\mathfrak{p}^{(n)}$ is the smallest $\mathfrak{p}$-primary ideal that contain $\mathfrak{p}^n$*

Proof.
Primary: If $xy \in \mathfrak{p}^{(n)}$ with $x \notin \mathfrak{p}^{(n)}$ we have that exists $s \notin \mathfrak{p}$ with $sxy \in \mathfrak{p}^n$. Suppose that $sy \notin \mathfrak{p}$, so $(sy)x \in \mathfrak{p}^n$ and then $x \in \mathfrak{p}^{(n)}$ that is absurd, so $sy \in \mathfrak{p} \Rightarrow (sy)^n \in \mathfrak{p}^n \Rightarrow s^n y^n \in \mathfrak{p}^n$. Since $\mathfrak{p}$ is prime $s^n \notin \mathfrak{p}$ and so $y^n \in \mathfrak{p}^{(n)}$.
p-primary: Infact $\mathfrak{p}^{(n)} \subset \mathfrak{p}$ and so $\mathrm{rad}(\mathfrak{p}^{(n)}) \subset \mathrm{rad}(\mathfrak{p}) = \mathfrak{p}$. Also if $x \in \mathfrak{p}$ we have $x^n \in \mathfrak{p}^n \subset \mathfrak{p}^{(n)}$ and so $x \in \mathfrak{p}^{(n)}$.
Minimal: If $\mathfrak{q}$ is $\mathfrak{p}$-primary and contains $\mathfrak{p}^n$, then for $r \in \mathfrak{p}^{(n)}$ there exists $s \notin \mathfrak{p} \supset \mathfrak{q}$ with $sr \in \mathfrak{p}^{(n)} \subset \mathfrak{q}$, and so since $s \notin \mathfrak{q}$ exists k such that $r^k \in \mathfrak{q}$. If $k = 1$ we have finished otherwhise we terminate by induction using $rr^{k-1} \in \mathfrak{q}$. □

Using the same example from Remark 2 we can observe that necessarly $p^2 \neq \mathfrak{p}^{(2)}$ since the first one isn't prymary.

Remarks 3.
- The proposition 1.10 establish a new equivalent definition for the symbolic power, more in line to the use of this ideal in the Zarisky-Nakata Theorem.
- Using the properties of localization, like [1, Proposition 4.8] and working with the contraction we would have speed up the proof.

Now we can see the actual definition of this concept for a general ideal.

Definiton 1.11. Let R be a noetherian ring and I an ideal. Given an integer m we define the **m-th symbolic power** of I as:

$$I^{(m)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(R/I)} (I^m R_\mathfrak{p} \cap R) \tag{4}$$

1.4 Zarisky-Nagata Theorem

Why do we study symbolic power? The Zarisky-Nagata Theorem give a geometric interpretation of its significance.

Theorem 1.12 (Zarisky-Nagata Theorem)**.** *If $R = k[x_0, ..., x_n]$ is a polynomial ring and $\mathfrak{p}$ is a prime ideal then:*

$$\mathfrak{p}^{(m)} = \bigcap_{\substack{\mathfrak{m} \in \mathfrak{m}\,\mathrm{Spec}(R) \\ \mathfrak{p} \subset \mathfrak{m}}} \mathfrak{m}^n \tag{5}$$

Using the equation 1 we can see that in this case the n-th symbolic power of a prime ideal represents the ideal composed by all the polynomials vanishing on the variety with a multiplicity of n, also indicated with the notation:

$$I^{\langle n \rangle} = \{f \in R \text{ that vanishes on } \mathcal{V}(I) \text{ with multiplicity } n\} \tag{6}$$

1.5 $I^r \subseteq I^{(m)}$

2 The Containment Problem

2.1 The Waldschmidt constant

References

[1] Michael F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra. Student economy edition*. English. Student economy edition. Boulder: Westview Press, 2016, pp. ix + 128. ISBN: 978-0-8133-5018-9/print; 978-0-8133-4544-4/ebook.

[2] Miles Reid. *Undergraduate commutative algebra*. English. Vol. 29. Cambridge: Cambridge Univ. Press, 1995, pp. xiii + 153. ISBN: 0-521-45889-7/pbk; 0-521-45255-4/hbk.