The Containment Problem, a general introduction and the particular case of Steiner configurations ideals

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Chapter 1

Inroduction and Symbolic Powers

1.1 Associated primes

Let R be a commutative ring with unity, and $\mathfrak{a},\mathfrak{b}$ two ideal, we say that the ideal

$$(\mathfrak{a} : \mathfrak{b}) = \{x \in R \,|\, x\mathfrak{b} \subseteq \mathfrak{a}\}$$

is the *ideal quotient*. For the case in which $\mathfrak{a}$ is the null ideal 0 we define the **annihilator** of $\mathfrak{b}$ as:

$$\operatorname{Ann}_R(\mathfrak{b}) = (0 : \mathfrak{b}) = \{x \in R \,|\, x\mathfrak{b} = 0\}$$

We can obviously omit the index R if it is clear by the context. In general given an R-module M and a set $S \subseteq M$ non empty we can define its annihilator as:

$$\operatorname{Ann}_R(S) = \{x \in R \,|\, xS = 0\} = \{x \in R \,|\, \forall s \in S \;\; xs = 0\}$$

Definiton 1.1.1 (Associated Prime)**.** Let M be an R-module. A prime ideal $\mathfrak{p} \subseteq R$ is an **associated prime** of M if there exists a non-zero element $a \in M$ such that $\mathfrak{p} = \operatorname{Ann}_R(a)$.
We define $\operatorname{Ass}_R(M)$ as the set of the associated primes of M.
For an ideal I we say that a prime is associated to I if it is associated to the R-module R/I.

Remark 1. Another name for associated ideal used by the Bourbaki group is *assasin*, a word play between associated and annihilator.

1.2 Primary decomposition

We would like to have some sort of factorization for the ideals of a ring, more general than the *unique factorization domains*, in fact this is useful only for principal ideals. With this objective **primary decomposition** was introduced.

Now I will recall some of the principal result on this topics, contained in [7, Section 7] and [1, Section 4 and Page 83]

Definiton 1.2.1. An ideal $\mathfrak{a}$ in a ring R is said primary if $R/\mathfrak{a}$ is different from zero and all its zerodivisors are nilpotent, otherwhise we can express this as:

$$fg \in \implies f \in \mathfrak{a} \text{ or } g^n \in \mathfrak{a} \text{ for some } n > 0$$

It is obvious that the radical of a primary ideal is a prime ideal, infact given $fg \in \mathrm{rad}(\mathfrak{a})$ we have $(fg)^m = f^m g^m \in \mathfrak{a}$ for $m > 0$, and so $f^m \in \mathfrak{a} \Rightarrow f \in \mathrm{rad}(\mathfrak{a})$ or exists $n > 0$ such that $g^{mn} \in \mathfrak{a} \Rightarrow g \in \mathrm{rad}(\mathfrak{a})$.

If $\mathfrak{a}$ is a primary ideal such that $\mathrm{rad}(\mathfrak{a}) = \mathfrak{p}$ we say that $\mathfrak{a}$ is $\mathfrak{p}$-primary.

Remarks 2.

1. The power of a prime ideal isn't always primary, for example if in $R = \mathbb{K}[x, y, z]/(xy - z^2)$ we consider the prime ideal $\mathfrak{p} = (x, z)$ (it is prime since $R/\mathfrak{p} \simeq \mathbb{K}[y]$ that is an integral domain) we have that y is a zero divisor in $R/\mathfrak{p}$ (since x is not zero and $yx = z^2 = 0$, since $z^2 \in \mathfrak{p}^2$) but it is not nilpotent since $y^k \notin \mathfrak{p}^2$ for all $k > 0$

We say tha an ideal $\mathfrak{a} \subseteq R$ has a **primary decomposition** if there exists a finite set of primary ideal $\{\mathfrak{q}_1, ..., \mathfrak{q}_n\}$ such that:

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

In general such structure does not exists, but for R noetherian we can prove, using Noetherian induction and the concept of irreducible ideal, that every proper ideal has a primary decomposition.

Definiton 1.2.2. We say that a proper ideal $\mathfrak{a}$ is irreducible if it cannot be written as a proper intersection of ideal, i.e. :

$$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \implies (\mathfrak{a} = \mathfrak{b} \text{ or } \mathfrak{a} = \mathfrak{c})$$

Lemma 1.2.3. *A proper ideal in a Noetherian ring R is always the intersection of a finite number of irreducible ideals.*

Proof. Let $\mathfrak{F}$ be the set of proper ideal such that the lemma is false. Let $\mathfrak{a}$ be a maximal ideal of $\mathfrak{F}$, since it cannot be irreducible there exists $\mathfrak{b}$, $\mathfrak{c}$ strictly greater than $\mathfrak{a}$ (so not in $\mathfrak{F}$) such that $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$. This is absurd and so $\mathfrak{F}$ is empty. □

Lemma 1.2.4. *In a Noetherian ring every irreducible ideal is primary*

Proof. Modulo working in the quotient ring we can assume to work with the zero ideal. So we assume that the ideal 0 is irreducible and we consider x, y such

that $xy = 0$ with $y \neq 0$, then x is a zerodivisor. So we have that $y \in \text{Ann}(x)$[1] and we consider the chain:

$$\text{Ann}(x) \subseteq \text{Ann}(x^2) \subseteq \ldots$$

And for the ascending chain condition there exists m with $\text{Ann}(x^m) = \text{Ann}(x^{m+1})$. Now consider $a \in (x^m) \cap (y)$, then $a = bx^m$ and $a = cy$, so since $y \in \text{Ann}(x)$ we have $0 = cyx = ax = bx^m x = bx^{m+1}$, so $b \in \text{Ann}(x^{m+1}) = \text{Ann}(x^m)$, then $a = bx^n = 0$. So $(x^m) \cap (y) = 0$ and since 0 is irreducible and $y \neq 0$ then $x^m = 0$. $\square$

Combining this two lemmas we have that the decomposition for Noetherian ring. In literature we say that a commutative ring is a **Lasker Ring** if every ideal has a primary decomposition, so we can state that:

Theorem 1.2.5 (Lasker-Noether)**.** *A Noetherian Ring is also a Lasker Ring*

Now we need to achive some kind of uniqueness. First of all we say that a decomposition $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is **minimal** if:

1. $\text{rad}(\mathfrak{q}_i)$ are all distinct

2. for all i we have $\mathfrak{q}_i \not\supseteq \bigcap_{j \neq i} \mathfrak{q}_j$

We can easly prove that from every decomposition we can obtain a minimal one using the following lemma:

Lemma 1.2.6. *If $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathfrak{p}$-primary then $\mathfrak{a} \cap \mathfrak{b}$ is $\mathfrak{p}$-primary*

Infact we can group the primaty ideal to get 1. and omit the superfluous terms to get 2.

So we have two theorem of uniqueness for the prime *associated*[2] to a particular decomposition.

Theorem 1.2.7 (First uniqueness theorem)**.** *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal with minimal decomposition $\bigcap_{i=1}^{n} \mathfrak{q}_i$, where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, then:*

$$\text{Ass}(R/\mathfrak{a}) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$$

and so the set of primes $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is uniquely determined by the ideal

This theorem show the strong ralation that we have between the associated prime ideal and the primary decomposition for Noetherian ring

Theorem 1.2.8 (Second uniqueness theorem)**.** *Let R be a ring and $\mathfrak{a}$ an ideal with minimal decomposition $\bigcap_{i=1}^{n} \mathfrak{q}_i$, where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, then if p_i is a minimal element of $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ $\mathfrak{q}_i$ is uniquely determined by the ideals $\mathfrak{a}$ and $\mathfrak{p}_i$. In particular if $\phi : R \to R_{\mathfrak{p}_i} = S^{-1}R$ is the canonical injection (where $S = R \setminus \mathfrak{p}_i$) we have*

$$\mathfrak{q}_i = \phi^{-1}(S^{-1}\mathfrak{a})$$

---

[1]For $\text{Ann}(x)$ we mean the annihilator of the principal ideal (x)

[2]not a random word

1.3   Sybolic power

Lets consider an homogeneous polynomial ring $k[x_0, ..., x_n]$, it is easy to se that if we consider a variety X with it's coordinate ring $R = k[X]$ and a point $p \in X$ (associated to the maximal ideal $\mathfrak{m}_p$) we have that:

$$\mathfrak{m}_p^n = \{f \in k[X] \text{ such that } f \text{ vanishes in } p \text{ with multiplicity } n\} \qquad (1.1)$$

For general ideal we don't have similiar properties for the normal power of ideal, so we define the **symbolic power**. First of all given a prime ideal $\mathfrak{p}$ in a Noetherian Ring R we can define the n-th symbolic power of $\mathfrak{p}$ as:

$$\mathfrak{p}^{(n)} = \{r \in R \text{ such that exists } s \in R \setminus \mathfrak{p} \text{ with } sr \in \mathfrak{p}^n\} \qquad (1.2)$$

This definition show clearly the idea between the symbolic power, but is not easy to work with. We can have another equivalent definition that use the localization on the prime ideal $R_\mathfrak{p}$. Infact we can see it as the contraction of $\mathfrak{p}^n R_\mathfrak{p}$ over R:

$$\mathfrak{p}^{(n)} = \mathfrak{p}^n R_\mathfrak{p} \cap R \qquad (1.3)$$

In general the generic and symbolic power are different concept. It is obvious that $\mathfrak{p}^n \subset \mathfrak{p}^{(n)}$ since $1 \notin \mathfrak{p}$. For the other direction we can costruct a counter example with the following proposition:

Proposition 1.3.1. $\mathfrak{p}^{(n)}$ *is the smallest $\mathfrak{p}$-primary ideal that contain $\mathfrak{p}^n$*

Proof.
Primary: If $xy \in \mathfrak{p}^{(n)}$ with $x \notin \mathfrak{p}^{(n)}$ we have that exists $s \notin \mathfrak{p}$ with $sxy \in \mathfrak{p}^n$. Suppose that $sy \notin \mathfrak{p}$, so $(sy)x \in \mathfrak{p}^n$ and then $x \in \mathfrak{p}^{(n)}$ that is absurd, so $sy \in \mathfrak{p} \Rightarrow (sy)^n \in \mathfrak{p}^n \Rightarrow s^n y^n \in \mathfrak{p}^n$. Since $\mathfrak{p}$ is prime $s^n \notin \mathfrak{p}$ and so $y^n \in \mathfrak{p}^{(n)}$.
p-primary: Infact $\mathfrak{p}^{(n)} \subset \mathfrak{p}$ and so $\mathrm{rad}(\mathfrak{p}^{(n)}) \subset \mathrm{rad}(\mathfrak{p}) = \mathfrak{p}$. Also if $x \in \mathfrak{p}$ we have $x^n \in \mathfrak{p}^n \subset \mathfrak{p}^{(n)}$ and so $x \in \mathfrak{p}^{(n)}$.
Minimal: If $\mathfrak{q}$ is $\mathfrak{p}$-primary and contains $\mathfrak{p}^n$, then for $r \in \mathfrak{p}^{(n)}$ there exists $s \notin \mathfrak{p} \supset \mathfrak{q}$ with $sr \in \mathfrak{p}^{(n)} \subset \mathfrak{q}$, and so since $s \notin \mathfrak{q}$ exists k such that $r^k \in \mathfrak{q}$. If $k = 1$ we have finished otherwhise we terminate by induction using $rr^{k-1} \in \mathfrak{q}$. $\square$

Using the same example from Remark 2 we can observe that necessarily $p^2 \neq \mathfrak{p}^{(2)}$ since the first one isn't prymary.

Remarks 3.   • The proposition 1.3.1 establish a new equivalent definition for the symbolic power, more in line to the use of this ideal in the Zarisky-Nakata Theorem.

• Using the properties of localization, like [1, Proposition 4.8] and working with the contraction we would have speed up the proof.

Now we can see the actual definition of this concept for a general ideal.

Definiton 1.3.2. Let R be a noetherian ring and I an ideal. Given an integer m we define the m**-th symbolic power** of I as:

$$I^{(m)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(R/I)} (I^m R_\mathfrak{p} \cap R) \qquad (1.4)$$

1.4 Zarisky-Nagata Theorem

Why do we study symbolic power? The Zarisky-Nagata Theorem give a geometric interpretation of its significance.

Theorem 1.4.1 (Zarisky-Nagata Theorem [8, 6]). *If $R = k[x_0, ..., x_n]$ is a polynomial ring and $\mathfrak{p}$ is a prime ideal then:*

$$\mathfrak{p}^{(m)} = \bigcap_{\substack{\mathfrak{m} \in \mathfrak{m} \, \mathrm{Spec}(R) \\ \mathfrak{p} \subset \mathfrak{m}}} \mathfrak{m}^n \tag{1.5}$$

Using the equation 1.1 we can see that in this case the n-th symbolic power of a prime ideal represents the ideal composed by all the polynomials vanishing on the variety with a multiplicity of n, also indicated with the notation:

$$I^{\langle n \rangle} = \{f \in R \text{ that vanishes on } \mathcal{V}(I) \text{ with multiplicity } n\} \tag{1.6}$$

1.5 Fat Points

Let's consider now an object of interest, that has particular relations with the symbolic powers.

Let k be a field and $\mathbb{P}^N$ the N-th projective space over k, consider now the distincts points $p_1, ..., p_k \in \mathbb{P}^N$ and some positive integers $m_1, ..., m_k$. If we consider the defining ideals $I(p_1), ..., I(p_k) \subset k[\mathbb{P}^N]$, representing the homogeneous polynomials vanishing on the point (before we have also used the notation $\mathfrak{m}_{p_i}$ that emphasise their role as maximal ideals) we can define another ideal:

$$I = \bigcap_{i=1}^{k} I(p_i)_i^m \subset k[\mathbb{P}^N] \tag{1.7}$$

Since is intersection of homogenous ideals I is also homogeneous and we can use it to define a 0 dimensional subscheme $Z \subset \mathbb{P}^N$, called **fat point subscheme** and indicated as

$$Z = m_1 p_1 + ... + m_k p_k$$

And we will denote I as $I(Z)$. This ideal represents the homogeneous polynomials that vanishes on p_i with multiplicity m_i for all $i = 1, ..., k$. The support of the scheme is the set of points $\{p_1, ..., p_k\}$.

The simbolic power of $I(Z)$ has the particular from:

$$I(Z)^{(m)} = \bigcap_{i=1}^{k} I(p_i)^{m_i m}$$

and so clearly represents the functions vanishing on p_i with multiplicity mm_i for all $i = 1, ..., k$.

> Potrebbe servirmi l'articolo [4] ma per ora non lo trovo (forse). Esiste un modo per provare che è 0 dimensionale?

1.6   Computation of Symbolic powers

1.7   Symbolic Powers and saturation of ideals

1.8   Relation between normal and symbolic power

> **Cosa trovata nell'intrdouduzione di [2] , sarebbe carino riprendere un discorso simile**
>
> Consider a homogeneous ideal I in a polynomial ring k[PN]. Taking powers of I is a natural algebraic construction, but it can be difficult to understand their structure geometrically (for example, knowing generators of Ir does not make it easy to know its primary decomposition). On the other hand, symbolic powers of I are more natural geometrically than algebraically. For example, if I is a radical ideal defining a finite set of points p 1, . . . , ps PN , then its mth symbolic power I(m) is generated by all forms vanishing to order at least m at each point pi, but it is not easy to write down specific generators for I(m), even if one has generators for I.
>
> A natural question that arises is the relation between the two powers.

Theorem 1.8.1. *If R is a Noetherian reduced ring then $I^r \subseteq I^{(m)}$ if and only if $r \geq m$*

Proof. Since if $I = 0$ it is obvious we assume it to be non zero.
The first implication is easy, infact for all $\mathfrak{p}$ we have that $I^m \subset I^m R_\mathfrak{p} \cap R$ since $I \subset \mathfrak{p}^3$ and $1 \notin \mathfrak{p}$.
Suppose now that $I^r \subseteq I^{(m)}$ and $r < m$. Consider an associated prime $\mathfrak{p}$, then if we consider the localization $I_\mathfrak{p}$ we have:

- $(I_\mathfrak{p})^r = (I^r)_\mathfrak{p}$ it is obvious

- $(I_\mathfrak{p})^m \supset (I^{(m)})_\mathfrak{p}$, because $I^{(m)} \subset I^m R_\mathfrak{p} \cap R = I_\mathfrak{p}^m \cap R$ and thus passing to the localization (again for the second term) we get the containment.

So composing this with the containment hypotesis (localized) we get $(I_\mathfrak{p})^r = (I^r)_\mathfrak{p} \subset (I^{(m)})_\mathfrak{p} \subset (I_\mathfrak{p})^m$. Since the other inclusion is obvious we have $I_\mathfrak{p}^m = I_\mathfrak{p}^r$, considering the intermediary power we have that $I_\mathfrak{p} I_\mathfrak{p}^r = I_\mathfrak{p}^r$, thus we can use the Nakayama Lemma:

Lemma 1.8.2 (Nakayama). *Given an ideal $\mathfrak{a}$ of a commutative ring with unity A and M a finetely-generated module over A with $IM = M$, then there exists a $x \in A$ such that $a \equiv 1 \mod I$ and $rM = 0$*

---

[3]So nothing become zero

In our case the ring is $R_{\mathfrak{p}}$, the ideal is $I_{\mathfrak{p}}$ and the module is $I_{\mathfrak{p}}^r$. So since R_p is a local ring with maximal ideal the localization of $\mathfrak{p}$, that is also the Jacobson Ideal, and $I_{\mathfrak{p}}^r$ is contained in it. So for the characterization of the Jacobson ideal the element x is invertible and then $I_{\mathfrak{p}}^r = 0$, but since we inherit that $R_{\mathfrak{p}}$ is a reduced ring this is possible only if $I_{\mathfrak{p}} = 0$. Since this is true for all the prime ideal (for the non associated one it is obvious since they intersect $R \setminus I$) and this is a local property ([1, Proposition 3.8]) we have $I = 0$, absurd.   □

Sadly the other direction of the containment isn't that easy, and it's an open question and in the last years was studied for some classes of ideals.

Chapter 2

The Containment Problem

As stated before the Containment Problem, is an open question in Algebraic Geometry and Commutative Algebra. The general form of the problem is:

Question 4. Given a Noetherian Ring R and an ideal I, for which m, r positive integers we have the containment:

$$I^{(m)} \subset I^r$$

The problem posed in Question 4 is quite general and does not seem to have a unique and simple answer (as for the inverse one of theorem 1.8.1). Usually we need to specify a particular ring and a particular ideal. Also we consider some precise pairs, like $3, 2$, or a subset given by a disequation.

Remark 5. If the containment holds for m, r it does also for all m', r with $m' \geq m$, since we have $I^{(m')} \subset I^{(m)} \subset I^r$

To better explain this let's see a celebrated result, showed in [5, 3]:

Theorem 2.0.1. *(Ein-Lazarsfeld-Smith, Hochster-Huneke) Let R be a regular ring and I a nonzero, radical ideal, then if h is the big height of I we have that for all $n \geq 0$ we have:*
$$I^{(hn)} \subset I^n$$

To understand this theorem we need two concepts. For a Noetherian local ring $(R, \mathfrak{m})$ we say that it is regular local ring if the minimal number of generators of the maximal ideal is equal to the dimension of R. Also Zarisky showed that a
In general we say that a Noetherian ring is **regular** if the localization at every prime ideal is a regular local ring. Also a geometrical interpretation of this definition is that for an affine variety V its ring of regular functions $\mathcal{O}_V$ is a regular ring if and only if V is a non singualr variety.

Instead to define the height of a prime ideal $(\mathrm{ht}(\mathfrak{p}))$ in a Noetherian Ring R is the supremum of the lenghts h of prime ideals chains descending from $\mathfrak{p}$:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq ... \subsetneq \mathfrak{p}_h = \mathfrak{p} \tag{2.1}$$

The concept of is equivalent to the codimension of the ideal $\mathfrak{p}$, that is the (Krull) dimension of the localization $R_{\mathfrak{p}}$ (looking at the definition it is easy to see that they are the same). Similarly we can define the coheight of $\mathfrak{p}$ as the dimension of the ideal $\mathfrak{p}$ (the supremum of the lenght of chanis ascending from $\mathfrak{p}$).

For a general ideal I we define the height of I as the minimum height of its prime ideals and the **big height of** I as the maximal height of its associated primes.

Given that $k[x_0, ..., x_N]$ is a regular ring (this is a consequence of the Hilbert's syzygy theorem **? ? ?**) and that obviously the big hieght of every homogeneous ideal I is less than the dimension of the ring (N) we have a more geometrically form of theorem 2.0.1, that states $I^{(Nm)} \subset I^m$ for all positive m.

Nell'atricolo di Szemberg nel teorema 1.3 non richiede che I sia radicale, mi sono perso qualcosa? Mentre lo chiedono nell'articolo di Ein quando fa il discorso per la dimensione della varietà n= dim(X)
Why associated?
Are them the minimal
And embedded primes from def 2.6 of Grifo symbolic power
Should I insert the definition of Krull dimension??

2.1   The Waldschmidt constant

Chapter 3

Some conjectures for the Containment Problem

Chapter 4

Steiner Configuration ideal

Interesting ideals, am example is the Fano Plane

Chapter 5

Coloration of Steiner Configuration ideal

Fun fuct, I can use the cp to check if there exists a colorarion

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