The Containment Problem

a general introduction and the particular case for Steiner systems

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Abstract

In this presentation we view a brief introduction to the **Containment Problem**, an open problem in Algebraic Geometry and Commutative Algebra.
Later we will see some connections with Combinatorics and Graph Theory, in particular for Steiner Systems, mainly inspired by the article:

Edoardo Ballico, Giuseppe Favacchio, Elena Guardo, Lorenzo Milazzo, and Abu Chackalamannil Thomas. *Steiner Configurations ideals: containment and colouring*. 2021. arXiv: `2101.07168 [math.AC]`.

Contents

Introduction

Associated primes

Definiton (Associated Prime)

For an ideal I in a commutative ring R we say that a prime $\mathfrak{p}$ is an **associated prime** if it is associated to the R-module R/I, i.e. there exists a non zero element $a \in R$ such that is :

$$\mathfrak{p} = \mathrm{Ann}_R(a) = \{x \in R \text{ such that } xa \in I\}$$

We indicate the set of the associated primes to I as $\mathrm{Ass}_R(I)$

Primary decomposition

We say that an ideal $\mathfrak{a} \subseteq R$ has a **primary decomposition** if there exists a finite set of primary ideal $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ such that:

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

In a Noetherian Ring we have:

- Existence
- Uniqueness of the minimal primes $\mathfrak{p}_i = \mathrm{rad}(\mathfrak{q}_i)$, in particular they are the associated primes
- Uniqueness of the primary ideals $\mathfrak{q}_i$

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Normal powers

Given an homogeneous ideal $I = \langle f_1, \dots, f_k \rangle$ the n-th power of I is:

$$I^n = \langle \xi_1 \cdots \xi_n \,|\, \xi_i \in \{f_1, \dots, f_k\} \rangle$$

- Easy algebraic construction
- Unknown primary decomposition
- No clear geometric interpretation

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Symbolic powers

Given an ideal I in a Noetherian ring R the m-th symbolic power of i is:

$$I^{(m)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(R/I)} (I^m R_{\mathfrak{p}} \cap R)$$

- $I^m R_{\mathfrak{p}} \cap R = \{r \in R \text{ such that exists } s \in R \setminus \mathfrak{p} \text{ with } sr \in I^n\}$
- No easy set of generators
- Clear primary decomposition
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Zariski-Nagata Theorem

Theorem (Zariski-Nagata Theorem [Zar49; Nag62])

If $R = k[x_0, \dots, x_n]$ is a polynomial ring and $\mathfrak{p}$ is a prime ideal then:

$$\mathfrak{p}^{(m)} = \bigcap_{\substack{\mathfrak{m} \in \mathfrak{m}\,\mathrm{Spec}(R) \\ \mathfrak{p} \subset \mathfrak{m}}} \mathfrak{m}^n$$

So the symbolic power represents the polynomial vanishing with multiplicity m on the variety $\mathcal{V}(I)$:

$$I^{(m)} = I^{\langle m \rangle} = \{f \in R \text{ that vanishes on } \mathcal{V}(I) \text{ with multiplicity } m\}$$

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Containment

The natural question that arises is:

Question

Can we find a relation between I^n and $I^{(m)}$?

As a consequence of the Nakayama Lemma we have one direction:

Theorem

If R is a Noetherian reduced ring then $I^r \subseteq I^{(m)}$ if and only if $r \geq m$

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Question

Given a Noetherian Ring R and an ideal I, for which m, r positive integers we have the containment:

$$I^{(m)} \subset I^r$$

Containment and big height

Let's see a celebrated result, showed in [HH02; ELS01]:

Theorem

(Ein-Lazarsfeld-Smith, Hochster-Huneke) Let R be a regular ring and I a non-zero, radical ideal, then if h is the big height of I we have that for all $n \geq 0$ we have:

$$I^{(hn)} \subseteq I^n$$

- The height of a prime ideal $\mathfrak{p}$ is the supremum length of a descending chain of primes: $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_h = \mathfrak{p}$
- The big height is the maximum height of its associated primes

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An example of a non trivial use of the previous theorem on a reduced subscheme came directly from [ELS01, p. 2.3]:

Example

Consider a finite set of points Z in $\mathbb{P}^2$, since the subscheme has dimension 0 the ideal has big height 2, so we have $I^{(2m)} \subseteq I^m$ for $I = I(Z)$. So this implies that all F with multiplicity $\geq 2m$ on Z stays in $I(Z)^m$.

Open questions

Some important open questions for the Containment Problem are:

Question (Huneke)

Let I be a saturated ideal of a reduced finite set of points in $\mathbb{P}^2$, does the containment:

$$I^{(3)} \subseteq I^2$$

hold?

Conjecture (Harbourne)

Given a non-zero, proper, homogeneous, radical ideal $I \subset k[x_0, \ldots, x_n]$ with big height h, than for all $m > 0$:

$$I^{(hm-h+1)} \subseteq I^m$$

Colouring and containment

Hypergraph

An hypergraph is a pair (V, E) where V is a finite set of vertices and E contains non-empty subset of V called hyper edges

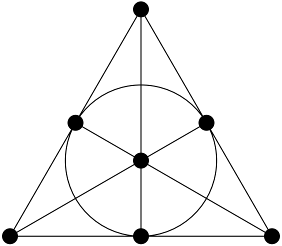
Definiton

A **Steiner system** (V, B) of type $S(t, n, v)$ is an *hypergraph* with $|V| = v$ and all the elements of B, called blocks, are n-subsets (of V) such that every t-tuple of elements in V is contained in only one block of B.

Fano plane

The most known example of Steiner is of type $S(2,3,7)$ and, up to isomorphism, is the Fano Plane ($\mathbb{P}^3_{\mathbb{F}_2}$). It has as blocks all the lines:

$$B := \{\{1,2,3\},\{3,4,5\},\{3,6,7\},\{1,4,7\},$$
$$\{2,4,6\},\{2,5,7\},\{1,5,6\}\}$$



Definiton

An m-colouring of the hypergraph $H = (V, E)$ is a partition in m subset of $V = U_1 \sqcup \ldots \sqcup U_m$ such that for every edge $\beta \in E$ we have $\beta \not\subseteq U_i$ for all $i = 1, \ldots, m$

A hypergraph H is m-colourable if there exists a proper m-colouring

Definiton

A hypergraph $H = (V, E)$ is said *c-coverable* if there exists a partition in c subset of $V = U_1 \sqcup \ldots \sqcup U_c$ such that every U_i is a **vertex cover**, which means that for all $\beta \in E$ we have $\beta \cap U_i \neq \emptyset$.

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Cover ideal

Definiton

Given an hypergraph $H = (V, E)$ we define equivalently the **cover ideal** as:

$$J(H) := \langle x_{j_1} \cdots x_{j_r} \mid \{x_{j_1}, ..., x_{j_r}\} \text{ is a vertex cover of } H \rangle = \bigcap_{\beta \in E} \mathfrak{p}_\beta$$

Where for every hyperedge $\beta = \{x_{i_1}, ..., x_{i_r}\} \in E$ then $\mathfrak{p}_\beta$ is the prime ideal $\langle x_{i_1}, ..., x_{i_r} \rangle$

Coverability and containment

For a hypergraph $H = (V, B)$ we define $\tau(H)$ as $\min_{\beta \in B}\{|\beta|\}$, so we obtain:

Theorem (Theorem 4.8 of [Bal+21])

Let $H = (V, B)$ be a hypergraph, if H is not d-coverable then $J(H)^{(\tau(H))} \not\subseteq J(H)^d$

For example for Steiner Systems we have:

Proposition (Proposition 4.9 of [Bal+21])

If $v > 3$ and $S = (V, B)$ is a Steiner Triple System $S(2, 3, v)$, then $J(S)^{(3)} \not\subseteq J(S)^2$

Colourability and containment

Theorem

Consider a simple hypergraph $H = (V, B)$, if we indicate $\tau = \tau(H)$ and the cover ideal $J = J(H)$, then for all $q < |V|$ if H is not q-colourable then we have: $J^{(\tau(q-1))} \not\subseteq J^q$

The proof rely on the two results:

Theorem (Theorem 3.2 of [FHV11])

Let $H = (V, E)$ be a simple hypergraph on $V = \{x_1, ..., x_v\}$, then for all $d > 0$ we have $(x_1 \cdots x_v)^{d-1} \in J(H)^d$ if and only if $d \geq \chi(H)$, where $\chi(H)$ is the minimum integer c such that H is c-colourable

Proposition

If I is a radical ideal in a polynomial ring we have:

$$I^{(m)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(R/I)} \mathfrak{p}^m \tag{1}$$

Edoardo Ballico, Giuseppe Favacchio, Elena Guardo, Lorenzo Milazzo, and Abu Chackalamannil Thomas. *Steiner Configurations ideals: containment and colouring*. 2021. arXiv: 2101.07168 [math.AC].

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