# The Containment Problem

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**Abstract**

Stuff x 3

# 1 Inroduction

## 1.1 Associated primes

Let $R$ be a commutative ring with unity, and $\mathfrak{a}$,$\mathfrak{b}$ two ideal, we say that the ideal

$$(\mathfrak{a} : \mathfrak{b}) = \{x \in R \,|\, x\mathfrak{b} \subseteq \mathfrak{a}\}$$

is the *ideal quotient*. For the case in which $\mathfrak{a}$ is the null ideal 0 we define the **annihilator** of $\mathfrak{b}$ as:

$$\mathrm{Ann}_R(\mathfrak{b}) = (0 : \mathfrak{b}) = \{x \in R \,|\, x\mathfrak{b} = 0\}$$

We can obviously omit the index $R$ if it is clear by the context. In general given an $R$-module $M$ and a set $S \subseteq M$ non empty we can define its annihilator as:

$$\mathrm{Ann}_R(S) = \{x \in R \,|xS = 0\} = \{x \in R \,|\, \forall s \in S\; xs = 0\}$$

**Definiton 1.1** (Associated Prime)**.** Let $M$ be an $R$-module. A prime ideal $\mathfrak{p} \subseteq R$ is an **associated prime** of $M$ if there exists a non-zero element $a \in M$ such that $\mathfrak{p} = \mathrm{Ann}_R(a)$.
We define $\mathrm{Ass}_R(M)$ as the set of the associated primes of $M$.
For an ideal $I$ we say that a prime is associated to $I$ if it is associated to the $R$-module $R/I$.

*Remark* 1. Another name for associated ideal used by the Bourbaki group is *assasin*, a word play between associated and annihilator.

## 1.2 Primary decomposition

We would like to have some sort of factorization for the ideals of a ring, more general than the *unique factorization domains*, in fact this is useful only for principal ideals. With this objective **primary decomposition** was introduced.

Now I will recall some of the principal result on this topics, contained in [2, Section 7] and [1, Section 4 and Page 83]

**Definiton 1.2.** An ideal $\mathfrak{a}$ in a ring $R$ is said primary if $R/\mathfrak{a}$ is different from zero and all its zerodivisors are nilpotent, otherwhise we can express this as:

$$fg \in \implies f \in \mathfrak{a} \text{ or } g^n \in \mathfrak{a} \text{ for some } n > 0$$

It is obvious that the radical of a primary ideal is a prime ideal, infact given $fg \in \operatorname{rad}(\mathfrak{a})$ we have $(fg)^m = f^m g^m \in \mathfrak{a}$ for $m > 0$, and so $f^m \in \mathfrak{a} \Rightarrow f \in \operatorname{rad}(\mathfrak{a})$ or exists $n > 0$ such that $g^{mn} \in \mathfrak{a} \Rightarrow g \in \operatorname{rad}(\mathfrak{a})$.

If $\mathfrak{a}$ is a primary ideal such that $\operatorname{rad}(\mathfrak{a}) = \mathfrak{p}$ we say that $\mathfrak{a}$ is $\mathfrak{p}$-primary.

*Remarks* 2.

1. The power of a prime ideal isn't always primary, for example if in $R = \mathbb{K}[x, y, z]/(xy - z^2)$ we consider the prime ideal $\mathfrak{p} = (x, z)$ (it is prime since $R/\mathfrak{p} \simeq \mathbb{K}[y]$ that is an integral domain) we have that $y$ is a zero divisor in $R/\mathfrak{p}$ (since $x$ is not zero and $yx = z^2 = 0$, since $z^2 \in \mathfrak{p}^2$) but it is not nilpotent since $y^k \notin \mathfrak{p}^2$ for all $k > 0$

We say tha an ideal $\mathfrak{a} \subseteq R$ has a **primary decomposition** if there exists a finite set of primary ideal $\{\mathfrak{q}_1, ..., \mathfrak{q}_n\}$ such that:

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

In general such structure does not exists, but for $R$ noetherian we can prove, using Noetherian induction and the concept of irreducible ideal, that every proper ideal has a primary decomposition.

**Definiton 1.3.** We say that a proper ideal $\mathfrak{a}$ is irreducible if it cannot be written as a proper intersection of ideal, i.e. :

$$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \implies (\mathfrak{a} = \mathfrak{b} \text{ or } \mathfrak{a} = \mathfrak{c})$$

**Lemma 1.4.** *A proper ideal in a Noetherian ring $R$ is always the intersection of a finite number of irreducible ideals.*

*Proof.* Let $\mathfrak{F}$ be the set of proper ideal such that the lemma is false. Let $\mathfrak{a}$ be a maximal ideal of $\mathfrak{F}$, since it cannot be irreducible there exists $\mathfrak{b}$, $\mathfrak{c}$ strictly greater than $\mathfrak{a}$ ( so not in $\mathfrak{F}$) such that $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$. This is absurd and so $\mathfrak{F}$ is empty. □

**Lemma 1.5.** *In a Noetherian ring every irreducible ideal is primary*

*Proof.* Modulo working in the quotient ring we can assume to work with the zero ideal. So we assume that the ideal 0 is irreducible and we consider $x$, $y$ such that $xy = 0$ with $y \neq 0$, then $x$ is a zerodivisor. So we have that $y \in \operatorname{Ann}(x)$[1] and we consider the chain:

$$\operatorname{Ann}(x) \subseteq \operatorname{Ann}(x^2) \subseteq ...$$

[1]For $\operatorname{Ann}(x)$ we mean the annihilator of the principal ideal $(x)$

And for the ascending chain condition there exists $m$ with $\mathrm{Ann}(x^m) = \mathrm{Ann}(x^{m+1})$. Now consider $a \in (x^m) \cap (y)$, then $a = bx^m$ and $a = cy$, so since $y \in \mathrm{Ann}(x)$ we have $0 = cyx = ax = bx^m x = bx^{m+1}$, so $b \in \mathrm{Ann}(x^{m+1}) = \mathrm{Ann}(x^m)$, then $a = bx^n = 0$. So $(x^m) \cap (y) = 0$ and since 0 is irreducible and $y \neq 0$ then $x^m = 0$. □

Combining this two lemmas we have that the decomposition for Noetherian ring. In literature we say that a commutative ring is a **Lasker Ring** if every ideal has a primary decomposition, so we can state that:

**Theorem 1.6** (Lasker-Noether). *A Noetherian Ring is also a Lasker Ring*

Now we need to achive some kind of uniqueness. First of all we say that a decomposition $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ is **minimal** if:

1. $\mathrm{rad}(\mathfrak{q}_i)$ are all distinct
2. for all $i$ we have $\mathfrak{q}_i \not\supseteq \bigcap_{j \neq i} \mathfrak{q}_j$

We can easly prove that from every decomposition we can obtain a minimal one using the following lemma:

**Lemma 1.7.** *If $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathfrak{p}$-primary then $\mathfrak{a} \cap \mathfrak{b}$ is $\mathfrak{p}$-primary*

Infact we can group the primaty ideal to get 1. and omit the superfluous terms to get 2.

So we have two theorem of uniqueness for the prime *associated*[2] to a particular decomposition.

**Theorem 1.8** (First uniqueness theorem). *Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal with minimal decomposition $\bigcap_{i=1}^n \mathfrak{q}_i$, where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, then:*

$$\mathrm{Ass}(R/\mathfrak{a}) = \{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$$

*and so the set of primes $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ is uniquely determined by the ideal*

This theorem show the strong ralation that we have between the associated prime ideal and the primary decomposition for Noetherian ring

**Theorem 1.9** (Second uniqueness theorem). *Let $R$ be a ring and $\mathfrak{a}$ an ideal with minimal decomposition $\bigcap_{i=1}^n \mathfrak{q}_i$, where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, then if $p_i$ is a minimal element of $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ $\mathfrak{q}_i$ is uniquely determined by the ideals $\mathfrak{a}$ and $\mathfrak{p}_i$. In particular if $\phi : R \to R_{\mathfrak{p}_i} = S^{-1}R$ is the canonical injection (where $S = R \setminus \mathfrak{p}_i$) we have*

$$\mathfrak{q}_i = \phi^{-1}(S^{-1}\mathfrak{a})$$

[2] not a random word

### 1.3 Sybolic power

Lets consider an homogeneous polynomial ring $k[x_0, ..., x_n]$, it is easy to se that if we consider a variety $X$ with it's coordinate ring $R = k[X]$ and a point $p \in X$ (associated to the maximal ideal $\mathfrak{m}_p$) we have that:

$$\mathfrak{m}^n = \{f \in k[X] \text{ such that } f \text{ vanishes in } p \text{ with multiplicity } n\}$$

**Definiton 1.10.** Let $R$ be a noetherian ring and $I$ an ideal. Given an integer $m$ we define the **$m$-th symbolic power** of $I$ as:

$$I^{(m)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(R/I)} (I^m R_{\mathfrak{p}} \cap R) \tag{1}$$

### 1.4 Zarisky-Nagata Theorem

Why do we studty symbolic power? The Zarisky-Nagata Theorem give a geometric interpretation of its significance.

### 1.5 $I^r \subseteq I^{(m)}$

## 2 The Containment Problem

### 2.1 The Waldschmidt constant

## References

[1] Michael F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra. Student economy edition*. English. Student economy edition. Boulder: Westview Press, 2016, pp. ix + 128. ISBN: 978-0-8133-5018-9/print; 978-0-8133-4544-4/ebook.

[2] Miles Reid. *Undergraduate commutative algebra*. English. Vol. 29. Cambridge: Cambridge Univ. Press, 1995, pp. xiii + 153. ISBN: 0-521-45889-7/pbk; 0-521-45255-4/hbk.