The Containment Problem, a general introduction and the particular case of Steiner configurations ideals

Giacomo Borin

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Chapter 1

Inroduction and Symbolic Powers

1.1 Associated primes

Let R be a commutative ring with unity, and $\mathfrak{a}$,$\mathfrak{b}$ two ideal, we say that the ideal

$$(\mathfrak{a} : \mathfrak{b}) = \{x \in R \,|\, x\mathfrak{b} \subseteq \mathfrak{a}\}$$

is the *ideal quotient*. For the case in which $\mathfrak{a}$ is the null ideal 0 we define the **annihilator** of $\mathfrak{b}$ as:

$$\mathrm{Ann}_R(\mathfrak{b}) = (0 : \mathfrak{b}) = \{x \in R \,|\, x\mathfrak{b} = 0\}$$

We can obviously omit the index R if it is clear by the context. In general given an R-module M and a set $S \subseteq M$ non empty we can define its annihilator as:

$$\mathrm{Ann}_R(S) = \{x \in R \,| xS = 0\} = \{x \in R \,|\, \forall s \in S \; xs = 0\}$$

Definiton 1.1.1 (Associated Prime)**.** Let M be an R-module. A prime ideal $\mathfrak{p} \subseteq R$ is an **associated prime** of M if there exists a non-zero element $a \in M$ such that $\mathfrak{p} = \mathrm{Ann}_R(a)$.
We define $\mathrm{Ass}_R(M)$ as the set of the associated primes of M.
For an ideal I we say that a prime is associated to I if it is associated to the R-module R/I.

Between the associated primes of an ideal we distinguish the minimal elements, that are called **isolated primes**, whilst the other one are said **embedded primes**.

We can define also the concept of minimal primes of ideal I, that are the minimal ones that contains I. In Noetherian rings these concepts are redundant, infact with the following proposition we have that minimal and isolated are equivalent.

Proposition 1.1.2. *For a Noetherian ring R, the minimal primes of R are among the associated primes of R*

Trova dove è spiegata bene
Controlla anche se la storiella dell'assassino è vera

Remark 1. Another name for associated ideal used by the Bourbaki group is *assasin*, a word play between associated and annihilator.

1.2 Primary decomposition

We would like to have some sort of factorization for the ideals of a ring, more general than the *unique factorization domains*, in fact this is useful only for principal ideals. With this objective **primary decomposition** was introduced.

Now I will recall some of the principal result on this topics, contained in [13, Section 7] and [1, Section 4 and Page 83]

Definiton 1.2.1. An ideal $\mathfrak{a}$ in a ring R is said primary if $R/\mathfrak{a}$ is different from zero and all its zerodivisors are nilpotent, otherwhise we can express this as:

$$fg \in \implies f \in \mathfrak{a} \text{ or } g^n \in \mathfrak{a} \text{ for some } n > 0$$

It is obvious that the radical of a primary ideal is a prime ideal, infact given $fg \in \mathrm{rad}(\mathfrak{a})$ we have $(fg)^m = f^m g^m \in \mathfrak{a}$ for $m > 0$, and so $f^m \in \mathfrak{a} \Rightarrow f \in \mathrm{rad}(\mathfrak{a})$ or exists $n > 0$ such that $g^{mn} \in \mathfrak{a} \Rightarrow g \in \mathrm{rad}(\mathfrak{a})$.

If $\mathfrak{a}$ is a primary ideal such that $\mathrm{rad}(\mathfrak{a}) = \mathfrak{p}$ we say that $\mathfrak{a}$ is $\mathfrak{p}$-primary.

Remarks 2.

1. The power of a prime ideal isn't always primary, for example if in $R = \mathbb{K}[x, y, z]/(xy - z^2)$ we consider the prime ideal $\mathfrak{p} = (x, z)$ (it is prime since $R/\mathfrak{p} \simeq \mathbb{K}[y]$ that is an integral domain) we have that y is a zero divisor in $R/\mathfrak{p}$ (since x is not zero and $yx = z^2 = 0$, since $z^2 \in \mathfrak{p}^2$) but it is not nilpotent since $y^k \notin \mathfrak{p}^2$ for all $k > 0$

We say tha an ideal $\mathfrak{a} \subseteq R$ has a **primary decomposition** if there exists a finite set of primary ideal $\{\mathfrak{q}_1, ..., \mathfrak{q}_n\}$ such that:

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

In general such structure does not exists, but for R noetherian we can prove, using Noetherian induction and the concept of irreducible ideal, that every proper ideal has a primary decomposition.

Definiton 1.2.2. We say that a proper ideal $\mathfrak{a}$ is irreducible if it cannot be written as a proper intersection of ideal, i.e. :

$$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \Longrightarrow (\mathfrak{a} = \mathfrak{b} \text{ or } \mathfrak{a} = \mathfrak{c})$$

Lemma 1.2.3. *A proper ideal in a Noetherian ring R is always the intersection of a finite number of irreducible ideals.*

Proof. Let $\mathfrak{F}$ be the set of proper ideal such that the lemma is false. Let $\mathfrak{a}$ be a maximal ideal of $\mathfrak{F}$, since it cannot be irreducible there exists $\mathfrak{b}$, $\mathfrak{c}$ strictly greater than $\mathfrak{a}$ (so not in $\mathfrak{F}$) such that $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$. This is absurd and so $\mathfrak{F}$ is empty. □

Lemma 1.2.4. *In a Noetherian ring every irreducible ideal is primary*

Proof. Modulo working in the quotient ring we can assume to work with the zero ideal. So we assume that the ideal 0 is irreducible and we consider x, y such that $xy = 0$ with $y \neq 0$, then x is a zerodivisor. So we have that $y \in \mathrm{Ann}(x)$[1] and we consider the chain:

$$\mathrm{Ann}(x) \subseteq \mathrm{Ann}(x^2) \subseteq ...$$

And for the ascending chain condition there exists m with $\mathrm{Ann}(x^m) = \mathrm{Ann}(x^{m+1})$. Now consider $a \in (x^m) \cap (y)$, then $a = bx^m$ and $a = cy$, so since $y \in \mathrm{Ann}(x)$ we have $0 = cyx = ax = bx^m x = bx^{m+1}$, so $b \in \mathrm{Ann}(x^{m+1}) = \mathrm{Ann}(x^m)$, then $a = bx^n = 0$. So $(x^m) \cap (y) = 0$ and since 0 is irreducible and $y \neq 0$ then $x^m = 0$. □

Combining this two lemmas we have that the decomposition for Noetherian ring. In literature we say that a commutative ring is a **Lasker Ring** if every ideal has a primary decomposition, so we can state that:

Theorem 1.2.5 (Lasker-Noether)**.** *A Noetherian Ring is also a Lasker Ring*

Now we need to achive some kind of uniqueness. First of all we say that a decomposition $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is **minimal** if:

1. $\mathrm{rad}(\mathfrak{q}_i)$ are all distinct
2. for all i we have $\mathfrak{q}_i \not\supseteq \bigcap_{j \neq i} \mathfrak{q}_j$

We can easly prove that from every decomposition we can obtain a minimal one using the following lemma:

Lemma 1.2.6. *If $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathfrak{p}$-primary then $\mathfrak{a} \cap \mathfrak{b}$ is $\mathfrak{p}$-primary*

Infact we can group the primaty ideal to get 1. and omit the superfluous terms to get 2.

So we have two theorem of uniqueness for the prime *associated*[2] to a particular decomposition.

[1] For $\mathrm{Ann}(x)$ we mean the annihilator of the principal ideal (x)

[2] not a random word

Theorem 1.2.7 (First uniqueness theorem)**.** *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal with minimal decomposition $\bigcap_{i=1}^n \mathfrak{q}_i$, where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, then:*

$$\mathrm{Ass}(R/\mathfrak{a}) = \{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$$

and so the set of primes $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ is uniquely determined by the ideal

This theorem show the strong ralation that we have between the associated prime ideal and the primary decomposition for Noetherian ring

Theorem 1.2.8 (Second uniqueness theorem)**.** *Let R be a ring and $\mathfrak{a}$ an ideal with minimal decomposition $\bigcap_{i=1}^n \mathfrak{q}_i$, where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, then if p_i is a minimal element of $\{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ $\mathfrak{q}_i$ is uniquely determined by the ideals $\mathfrak{a}$ and $\mathfrak{p}_i$. In particular if $\phi : R \to R_{\mathfrak{p}_i} = S^{-1}R$ is the canonical injection (where $S = R \setminus \mathfrak{p}_i$) we have*

$$\mathfrak{q}_i = \phi^{-1}(S^{-1}\mathfrak{a})$$

1.3 Sybolic power

Lets consider an homogeneous polynomial ring $k[x_0, ..., x_n]$, it is easy to se that if we consider a variety X with it's coordinate ring $R = k[X]$ and a point $p \in X$ (associated to the maximal ideal $\mathfrak{m}_p$) we have that:

$$\mathfrak{m}_p^n = \{f \in k[X] \text{ such that } f \text{ vanishes in } p \text{ with multiplicity } n\} \tag{1.1}$$

For general ideal we don't have similiar properties for the normal power of ideal, so we define the **symbolic power**. First of all given a prime ideal $\mathfrak{p}$ in a Noetherian Ring R we can define the n-th symbolic power of $\mathfrak{p}$ as:

$$\mathfrak{p}^{(n)} = \{r \in R \text{ such that exists } s \in R \setminus \mathfrak{p} \text{ with } sr \in \mathfrak{p}^n\} \tag{1.2}$$

This definition show clearly the idea between the symbolic power, but is not easy to work with. We can have another equivalent definition that use the localization on the prime ideal $R_\mathfrak{p}$. Infact we can see it as the contraction of $\mathfrak{p}^n R_\mathfrak{p}$ over R:

$$\mathfrak{p}^{(n)} = \mathfrak{p}^n R_\mathfrak{p} \cap R \tag{1.3}$$

In general the generic and symbolic power are different concept. It is obvious that $\mathfrak{p}^n \subset \mathfrak{p}^{(n)}$ since $1 \notin \mathfrak{p}$. For the other direction we can costruct a counter example with the following proposition:

Proposition 1.3.1. $\mathfrak{p}^{(n)}$ *is the smallest $\mathfrak{p}$-primary ideal that contain $\mathfrak{p}^n$*

Proof.
Primary: If $xy \in \mathfrak{p}^{(n)}$ with $x \notin \mathfrak{p}^{(n)}$ we have that exists $s \notin \mathfrak{p}$ with $sxy \in \mathfrak{p}^n$. Suppose that $sy \notin \mathfrak{p}$, so $(sy)x \in \mathfrak{p}^n$ and then $x \in \mathfrak{p}^{(n)}$ that is absurd, so $sy \in \mathfrak{p} \Rightarrow (sy)^n \in \mathfrak{p}^n \Rightarrow s^n y^n \in \mathfrak{p}^n$. Since $\mathfrak{p}$ is prime $s^n \notin \mathfrak{p}$ and so $y^n \in \mathfrak{p}^{(n)}$.

p-primary: Infact $\mathfrak{p}^{(n)} \subset \mathfrak{p}$ and so $\mathrm{rad}(\mathfrak{p}^{(n)}) \subset \mathrm{rad}(\mathfrak{p}) = \mathfrak{p}$. Also if $x \in \mathfrak{p}$ we have $x^n \in \mathfrak{p}^n \subset \mathfrak{p}^{(n)}$ and so $x \in \mathfrak{p}^{(n)}$.
Minimal: If $\mathfrak{q}$ is $\mathfrak{p}$-primary and contains $\mathfrak{p}^n$, then for $r \in \mathfrak{p}^{(n)}$ there exists $s \notin \mathfrak{p} \supset \mathfrak{q}$ with $sr \in \mathfrak{p}^{(n)} \subset \mathfrak{q}$, and so since $s \notin \mathfrak{q}$ exists k such that $r^k \in \mathfrak{q}$. If $k = 1$ we have finished otherwhise we terminate by induction using $rr^{k-1} \in \mathfrak{q}$. □

Using the same example from Remark 2 we can observe that necessary $p^2 \neq \mathfrak{p}^{(2)}$ since the first one isn't prymary.

Remarks 3.
- The proposition 1.3.1 establish a new equivalent definition for the symbolic power, more in line to the use of this ideal in the Zarisky-Nakata Theorem.
- Using the properties of localization, like [1, Proposition 4.8] and working with the contraction we would have speed up the proof.

Now we can see the actual definition of this concept for a general ideal.

Definiton 1.3.2. Let R be a noetherian ring and I an ideal. Given an integer m we define the **m-th symbolic power** of I as:

$$I^{(m)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(R/I)} (I^m R_{\mathfrak{p}} \cap R) \tag{1.4}$$

1.4 Zarisky-Nagata Theorem

Why do we study symbolic power? The Zarisky-Nagata Theorem give a geometric interpretation of its significance.

Theorem 1.4.1 (Zarisky-Nagata Theorem [15, 12])**.** *If $R = k[x_0, ..., x_n]$ is a polynomial ring and $\mathfrak{p}$ is a prime ideal then:*

$$\mathfrak{p}^{(m)} = \bigcap_{\substack{\mathfrak{m} \in \mathfrak{m}\,\mathrm{Spec}(R) \\ \mathfrak{p} \subset \mathfrak{m}}} \mathfrak{m}^n \tag{1.5}$$

Using the equation 1.1 we can see that in this case the n-th symbolic power of a prime ideal represents the ideal composed by all the polynomials vanishing on the variety with a multiplicity of n, also indicated with the notation:

$$I^{\langle n \rangle} = \{f \in R \text{ that vanishes on } \mathcal{V}(I) \text{ with multiplicity } n\} \tag{1.6}$$

1.5 Fat Points

Let's consider now an object of interest, that has particular relations with the symbolic powers.
Let k be a field and $\mathbb{P}^N$ the N-th projective space over k, consider now the distincts points $p_1, ..., p_k \in \mathbb{P}^N$ and some positive integers $m_1, ..., m_k$. If we consider the defining ideals $I(p_1), ..., I(p_k) \subset k[\mathbb{P}^N]$, representing the homogeneous

polynomials vanishing on the point (before we have also used the notation $\mathfrak{m}_{p_i}$ that emphasise their role as maximal ideals) we can define another ideal:

$$I = \bigcap_{i=1}^{k} I(p_i)_i^m \subset k[\mathbb{P}^N] \tag{1.7}$$

Since is intersection of homogenous ideals I is also homogeneous and we can use it to define a 0 dimensional subscheme $Z \subset \mathbb{P}^N$, called **fat point subscheme** and indicated as

$$Z = m_1 p_1 + ... + m_k p_k$$

And we will denote I as $I(Z)$. This ideal represents the homogeneous polynomials that vanishes on p_i with multiplicity m_i for all $i = 1, ..., k$. The support of the scheme is the set of points $\{p_1, ..., p_k\}$.
The simbolic power of $I(Z)$ has the particular from:

$$I(Z)^{(m)} = \bigcap_{i=1}^{k} I(p_i)^{m_i m}$$

and so clearly represents the functions vanishing on p_i with multiplicity mm_i for all $i = 1, ..., k$.

> Potrebbe servirmi l'articolo [8] ma per ora non lo trovo (forse). Esiste un modo per provare che è 0 dimensionale?

1.6 Computation of Symbolic powers

> Topics in commutative algebra: Symbolic Powers Eloìsa Grifo, 1.5
> Ben Drabkin, Eloísa Grifo, Alexandra Seceleanu, and Branden Stone. "Computations involving symbolic powers". In: *Journal of Software for Algebra and Geometry* 9 (Dec. 2017). DOI: 10.2140/jsag.2019.9.71
> Macaulay 2 package Symbolic Power

1.6.1 Symbolic Powers and saturation of ideals

1.7 Relation between normal and symbolic power

Cosa trovata nell'intrdouduzione di [3] , sarebbe carino riprendere un discorso simile
Consider a homogeneous ideal I in a polynomial ring k[PN]. Taking powers of I is a natural algebraic construction, but it can be difficult to understand their structure geometrically (for example, knowing generators of Ir does not make it easy to know its primary decomposition). On the other hand, symbolic powers of I are more natural geometrically than algebraically. For example, if I is a radical ideal defining a finite set of points p 1, . . . , ps PN , then its mth symbolic power I(m) is generated by all forms vanishing to order at least m at each point pi, but it is not easy to write down specific generators for I(m), even if one has generators for I.
A natural question that arises is the relation between the two powers.

Theorem 1.7.1. *If R is a Noetherian reduced ring then $I^r \subseteq I^{(m)}$ if and only if $r \geq m$*

Proof. Since if $I = 0$ it is obvious we assume it to be non zero.
The first implication is easy, infact for all $\mathfrak{p}$ we have that $I^m \subset I^m R_{\mathfrak{p}} \cap R$ since $I \subset \mathfrak{p}$[3] and $1 \notin \mathfrak{p}$.
Suppose now that $I^r \subseteq I^{(m)}$ and $r < m$. Consider an associated prime $\mathfrak{p}$, then if we consider the localization $I_{\mathfrak{p}}$ we have:

- $(I_{\mathfrak{p}})^r = (I^r)_{\mathfrak{p}}$ it is obvious
- $(I_{\mathfrak{p}})^m \supset (I^{(m)})_{\mathfrak{p}}$, because $I^{(m)} \subset I^m R_{\mathfrak{p}} \cap R = I^m_{\mathfrak{p}} \cap R$ and thus passing to the localization (again for the second term) we get the containment.

So composing this with the containment hypotesis (localized) we get $(I_{\mathfrak{p}})^r = (I^r)_{\mathfrak{p}} \subset (I^{(m)})_{\mathfrak{p}} \subset (I_{\mathfrak{p}})^m$. Since the other inclusion is obvious we have $I^m_{\mathfrak{p}} = I^r_{\mathfrak{p}}$, considering the intermediary power we have that $I_{\mathfrak{p}} I^r_{\mathfrak{p}} = I^r_{\mathfrak{p}}$, thus we can use the Nakayama Lemma:

Lemma 1.7.2 (Nakayama)**.** *Given an ideal I of a commutative ring with unity A and M a finetely-generated module over A with $IM = M$, then there exists a $x \in A$ such that $x \equiv 1 \mod I$ and $xM = 0$*

In our case the ring is $R_{\mathfrak{p}}$, the ideal is $I_{\mathfrak{p}}$ and the module is $I^r_{\mathfrak{p}}$. So since R_p is a local ring with maximal ideal the localization of $\mathfrak{p}$, that is also the Jacobson Ideal, and $I^r_{\mathfrak{p}}$ is contained in it. So for the characterization of the Jacobson ideal the element x is invertible and then $I^r_{\mathfrak{p}} = 0$, but since we inherit that $R_{\mathfrak{p}}$ is a

[3] So nothing become zero

reduced ring this is possible only if $I_{\mathfrak{p}} = 0$. Since this is true for all the prime ideal (for the non associated one it is obvious since they intersect $R \setminus I$) and this is a local property ([1, Proposition 3.8]) we have $I = 0$, absurd. $\square$

Sadly the other direction of the containment isn't that easy, and it's an open question and in the last years was studied for some classes of ideals.

Chapter 2

The Containment Problem

As stated before the Containment Problem, is an open question in Algebraic Geometry and Commutative Algebra. The general form of the problem is:

Question 4. Given a (**Noetherian ???**) Ring R and an ideal I, for which m, r positive integers we have the containment:

$$I^{(m)} \subset I^r$$

The problem posed in Question 4 is quite general and does not seem to have a unique and simple answer (as for the inverse one of theorem 1.7.1). Usually we need to specify a particular ring and a particular ideal. Also we consider some precise pairs, like $3, 2$, or a subset given by a disequation.

Remark 5. If the containment holds for m, r it does also for all m', r with $m' \geq m$, since we have $I^{(m')} \subset I^{(m)} \subseteq I^r$

To better explain this let's see a celebrated result, showed in [11, 7]:

Theorem 2.0.1. *(Ein-Lazarsfeld-Smith, Hochster-Huneke) Let R be a regular ring and I a nonzero, radical ideal, then if h is the big height of I we have that for all $n \geq 0$ we have:*

$$I^{(hn)} \subseteq I^n$$

To understand this theorem we need two concepts. For a Noetherian local ring $(R, \mathfrak{m})$ we say that it is regular local ring if the minimal number of generators of the maximal ideal is equal to the dimension of R. The name came from a Zarisky's result: for an algebraic variety a point p is non singular (regular) if and only if the ring of germs in p is regular ([16]). Is possible to see this in a more modern way , infact for lemma 1.7.2 (in a different form) is possible to show that R is regular if and only if $\dim(\mathfrak{m}/\mathfrak{m}^2) = \dim(R)$, and from algebraic geometry $\mathfrak{m}/\mathfrak{m}^2$ is the cotangent space of the point corresponding to $\mathfrak{m}$, so the tangent space has same dimension of the variety if and only if the localization

is regular.
In general we say that a Noetherian ring is **regular** if the localization at every prime ideal is a regular local ring. Also a geometrical interpretation of this definition is that for an affine variety V its ring of regular functions $\mathcal{O}_V$ is a regular ring if and only if V is a non singualr variety.

Instead to define the height of a prime ideal ($\mathrm{ht}(\mathfrak{p})$) in a Noetherian Ring R is the supremum of the lenghts h of prime ideals chains descending from $\mathfrak{p}$:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_h = \mathfrak{p} \tag{2.1}$$

The concept of is equivalent to the codimension of the ideal $\mathfrak{p}$, that is the (Krull) dimension of the localization $R_{\mathfrak{p}}$ (looking at the definition it is easy to see that they are the same). Similarly we can define the coheight of $\mathfrak{p}$ as the dimension of the ideal $\mathfrak{p}$ (the supremum of the lenght of chanis ascending from $\mathfrak{p}$).
For a general ideal I we define the height of I as the minimum height of its prime ideals (for proposition 1.1.2 we can consider only the associated ones) and the **big height of** I as the maximal height of its associated primes.
An example of a non trivial use of theorem 2.0.1 on a fat-point subscheme came directly from [7, p. 2.3]:

Example 6. Consider a reduced[1] fat point subscheme $Z = m_1p_1 + \ldots + m_kp_k$ (o more simply a finite set of points) in $\mathbb{P}^2$, since the subscheme has dimension 0 the ideal has big height 2, so we have $I^{(2m)} \subseteq I^m$ for $I = I(Z)$. Using Theorem 1.4.1 this implies that all F with multiplicity $\geq 2m$ (greater or equal than $2m_im$ for all the points p_i) stays in $I(Z)^m$.

Given that $k[x_0, \ldots, x_N]$ is a regular ring (this is a consequence of the Hilbert's syzygy theorem **? ? ?**) and that obviously the big hieght of every homogeneous ideal I is less than the dimension of the ring (N) we have a more geometrically form of theorem 2.0.1, that states $I^{(Nm)} \subseteq I^m$ for all positive m.

Nell'atricolo di Szemberg nel teorema 1.3 non richiede che I sia radicale (può essere che assuma essere l'ideale di uno schema ridotto), mi sono perso qualcosa? Mentre lo chiedono nell'articolo di Ein quando fa il discorso per la dimensione della varietà n= dim(X)
Why associated?
Are them the minimal
And embedded primes from def 2.6 of Grifo symbolic power
Should I insert the definition of Krull dimension??

2.1 The resurgence

To mesure the containment property we can some constants associated to the ideal I, one of them is the **resurgence**, proposed in [4]:

[1] A scheme is reduced if and only if the associated ideal is radical

Definiton 2.1.1. For a proper non-zero ideal I in a commutative ring R we define the resurgence of I as:

$$\rho(I) = \sup\left\{\frac{m}{r} \mid I^{(m)} \nsubseteq I^r\right\}$$

Bounding the resurgence of an ideal means finding a constant such that for $\frac{m}{r} > \rho$ mean that $I^{(m)} \subseteq I^r$ holds. For example using this quantity we can express the Theorem 2.0.1 (in a sligtly weaker version) as:

Theorem 2.1.2. *For a radical nonzero ideal in a regular ring $\rho(I) \leq h$ where h is the big height of I*

In general this is not an optimal bound and since it is difficult to directly evaluate $\rho(I)$ we can pose the question when the resurgence is strictly less than the big height.

Another constant, closely related to ρ, is the **asymptotich resurgence** (introduced in [9]), defined as:

Definiton 2.1.3. For a homogeneous nonzero proper ideal I of $k[x_0, ..., x_n]$ the asymptotic resurgence $\rho_a(I)$ is:

$$\rho_a(I) = \sup\left\{\frac{m}{r} \mid I^{(mt)} \nsubseteq I^{rt} \text{ for all } t \gg 0\right\}$$

2.2 The Waldschmidt constant

Definiton 2.2.1. Given an homogeneous ideal $I = \oplus_{d>0} I_d$ we can define the Waldschmidt constant as:

$$\hat{\alpha}(I) = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m}$$

where $\alpha(I)$ is the least degree of a generator of I, that is the smallest integer d such that $I_d \neq 0$.

This constant was introduced for the first time in the 1970' in [14]. And is of particular interest for ideal of fat points.

2.3 Some conjectures and questions for the Containment Problem

A possible question arises from Example 6: in this case we know (using $n = 1$) that $I^{(4)} \subseteq I^2$, but from several example arises the question:

Question 7 (Huneke). Let I be a saturated ideal of a reduced finite set of points in $\mathbb{P}^2$, does the containment:

$$I^{(3)} \subseteq I^2$$

hold?

Trovare esempi e controesempi
Contro: Fermat cofiguaration and others in [6]. Esempi: Star configurations of points [10].

Another good question is if it is possible to improve the result from theorem 2.0.1. Since there is no known example for which the bound is optimal a new conjecture have been posed:

Conjecture 2.3.1 (Harbourne). *Given a nonzero, proper, homogeneous, radical ideal $I \subset k[x_0, ..., x_n]$ with big height h, than for all $m > 0$:*

$$I^{(hm-h+1)} \subseteq I^m$$

Trying to solve this problem directly has been shown to be quite difficult, so there are several sharper version of the Conjecture 2.3.1, in particular the following one does not request the containment to hold in general, but only asymptotically:

Conjecture 2.3.2 (Stable Harbourne). *Given a nonzero, proper, homogeneous, radical ideal $I \subset k[x_0, ..., x_n]$ with big height h, than for all $m >> 0$:*

$$I^{(hm-h+1)} \subseteq I^m$$

Another way to modify the Harbourne Conjecture is to use the irrelevant ideal $\mathcal{M} = \langle x_0, ..., x_n \rangle$ (also said graded maximal ideal) :

Conjecture 2.3.3 (Stable Harbourne-Huneke). *Given a nonzero, proper, homogeneous, radical ideal $I \subset k[x_0, ..., x_n]$ with big height h, than for all $m >> 0$:*

- $I^{(hm)} \subset \mathcal{M}^{r(h-1)} I^r$
- $I^{(hm-h+1)} \subset \mathcal{M}^{(r-1)(h-1)} I^r$

One simple example of why do we use the graded maximal ideal is:

Proposition 2.3.4. *Given a $r > 0$ and a nonzero, proper, homogeneous ideal $I \subset k[x_0, ..., x_n]$, with k of characteristic 0 we have:*

$$I^{(r+1)} \subseteq \mathcal{M} I^{(r)}$$

Proof. This is a straight application of Euler identity for homegeneous polynomial:

$$(\deg F)F = \sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i}$$

, infact if $F \in I^{(r+1)}$ we have $\frac{\partial F}{\partial x_i} \in I^{(r)}$ for Zarisky-Nagata theorem (1.4.1) and the thesis follows. $\square$

Chapter 3

Steiner Configuration ideal

Introduzione sugli steiner configurations and their use (maybe)

A **Steiner system** (V, B) of type $S(t, n, v)$ is an *hypergraph* with $|V| = v$ and all the elements of B, called blocks, are n-subsets (of V) such that every t-tuple of elements in V is contained in only one block of B.
To be more clear we recall that an hypergraph (V, B) is a generalization of the normal graph, in which V is a finite set and B contains non-empty subset of V called hyperedges (a normal graph conatins only pairs) such that they cover V $(\bigcup_{H \in B} H = V)$.
Geometrically the blocks can be seen as linear subspace in a projective spce that contains points in V, in particular this interpretation is useful for **Steiner triple sistem**, that are Steiner system with $t = 2$ and $n = 3$, also indicated with $STS(v)$. Later we will use again algebraic geometry, but with a different approach.

Example 8. The most known example of Steiner is of type $STS(7)$ and, up to isomorphism, is the Fano Plane $(\mathbb{P}^3_{\mathbb{F}_2})$. It has as blocks all the lines (hypeplanes):

$$B := \{\{1, 2, 3\}, \{3, 4, 5\}, \{3, 6, 7\}, \{1, 4, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{1, 5, 6\}\}$$

.

```
%Bho␣non␣va
\begin{figure}
\includegraphics{images/fano_plane}
\caption{Fano␣plane}
\label{fig:fanoplane}
\end{figure}
```

In general the existence of a Stiner system depends on the parameters, for instance a for a Steiner Triple system ($t = 2, n = 3$) we need $v \equiv 1, 3 \mod 6$. There are not known sufficient existance conditions, but only necessary, for example if it exists a $S(t, n, v)$ Steiner system we need:

$$|B| = \frac{\binom{v}{t}}{\binom{n}{t}}$$

This is simply combinatorics, infact every t-tuple of vertices is contained in only one block and each one of these contains $\binom{n}{t}$ t-tuples.

3.1 An algebraic representation of Steiner systems

As said before is possible to use algebraic geometry to represents the Steiner systems, in particular the concept of star configuaration of points:

Definiton 3.1.1. A finite set of points $Z \subset \mathbb{P}^n$ is a **star configuaration of points** of degree $d \geq n$ if there exists d general hyperplanes such that the points of Z are exactly the ones that are intersection of n of these hyperplanes.

By general position we mean that any group of n hyperplanes intersect in only one point and there is no point belonging to more than n hyperplanes. It also used the notation d-star to emphasize the degree.

In our case we consider v-star configuartions in $\mathbb{P}^n$: let $\mathcal{H} = \{H_1, ..., H_v\}$ be general hyperplanes in $\mathbb{P}^n$, with H_i defined by the linear form l_i (a linear map from $\mathbb{P}^n$ to the field of scalars $\mathbb{C}$ **?** **?** **?**). Given an n-subset of V[1] $\sigma := \{\sigma_1, ..., \sigma_v\}$ we can associate to it a point $P_{\mathcal{H},\sigma} = \cap_{\sigma_i \in \sigma} H_{\sigma_i}$, that has as vanishing ideal $I_{P_{\mathcal{H},\sigma}} = \langle l_{\sigma_1}, ..., l_{\sigma_v} \rangle$.
Observe that in this case the vertices are represented by $(n-1)$-linear space and the blocks by points.
Also we can define:

Definiton 3.1.2. Given a finite set V and a collection of non empty subset $\mathcal{F}$ we can define, using the previous notation we can define the set of points:

$$X_{\mathcal{H},\mathcal{F}} := \bigcup_{\sigma \in \mathcal{F}} P_{\mathcal{H},\sigma} \tag{3.1}$$

and its defining ideal:

$$I_{X_{\mathcal{H},\mathcal{F}}} := \bigcap_{\sigma \in \mathcal{F}} I_{\mathcal{H},\sigma} \tag{3.2}$$

Please notice that these costructions are more general, so to obtain a Steiner Sytem we assign $\mathcal{F} = B$, obtaining $X_{\mathcal{H},B}$ and $I_{X_{\mathcal{H},B}}$. We call $X_{\mathcal{H},B}$ the

[1]Since V is finite we can index it using natual numbers and assume $V = \{1, ..., v\}$

Steiner configuration of points associated to the Steiner system (V, B) of type $S(t, n, v)$ with respect to $\mathcal{H}$.
Also we indicate $C_{(n,v)}$ as the family of all the n-subset of V and we can costruct the **Complement of a Steiner configuration** of points with respect to $\mathcal{H}$ as the scheme $X_{\mathcal{H},C_{(n,v)}\setminus B}$ (said C-Steiner and indicated X_C too). Now we obtain some interesting results for this particular scheme.

3.2 Containment problem for C-Steiner System

First of all we recall some results from [2] in partlicular the Theorem 3.9:

Theorem 3.2.1. *Consider a Steiner system (V, B) of type $S(t, n, v)$, let $X_C \subset \mathbb{P}^n$ be the corrispondent C-Steiner configuaration and I_{X_C} its ideal, then:*

1. $\alpha(I_{X_C}) = v - n$
2. $\alpha(I^{(q)}_{X_C}) = v - n + q$ *for* $q \in [2, n)$
3. $\alpha(I^{(m)}_{X_C}) = \alpha(I^{(q)}_{X_C}) + pv$ *where* $m = pn + q$ *and* $q \in [0, n)$

Spiega l'idea dietro la dimostrazione
Utilizzo simplicial complex
$I^{(m)}_{X_C}$ e $I^{(m)}_{\Delta_C}$ hanno le stesse invarianti omologiche
Appendice die Eisenbud

Remark 9. We can use the results from theorem 3.2.1 to get some situation in which the containment problem fails, infact for nonzero, proper, homogeneous ideals I and J is straightforward that if $\alpha(I) < \alpha(J)$ then $I \not\subseteq J$, infact $I_{\alpha(I)} \neq 0$ but $J_{\alpha(I)} = 0$.

Corollary 3.2.2. *In the same hypothesis of theorem 3.2.1 we have $I^{(m)}_{X_C} \not\subseteq I^d_{X_C}$ for any pair (m, d) such that:*

$$m \equiv 1 \mod n \text{ and } d > 1 + \frac{(m-1)v}{n(v-n)} \tag{3.3}$$

or

$$m \not\equiv 1 \mod n \text{ and } d > 1 + \frac{m-n}{n} + \frac{m}{v-n} \tag{3.4}$$

In particular if $v > 2n$ we have $I^{(n)}_{X_C} \not\subseteq I^2_{X_C}$

Proof. Using 1. from Theorem 3.2.1 and simple algebra we have $\alpha({I_{X_C}}^d) = d\alpha(I_{X_C}) = d(v - n)$, than from remark 9 is enought to prove:

$$\alpha(I^{(m)}_{X_C}) < d(v - n) \tag{3.5}$$

$m \equiv 1$: we have $m = pn + 1$ with p integer, so using 3. and 1. of theorem 3.2.1 for $q = 1$ we have $\alpha(I_{X_C}^{(m)}) = \alpha(I_{X_C}) + pv = (v - n) + \frac{m-1}{n}v$. Grouping by the factor $(n - v)$ and using the second part of 3.3 we get 3.5.

$m \not\equiv 0, 1$: we have $m = pn + q$ with $q = 0$ or $2 \leq q < n$ so using 3. and 2. of theorem 3.2.1 we get $\alpha(I_{X_C}^{(m)}) = \alpha(I_{X_C}^{(q)}) + pv = v - n + q + pv = (v - n) + m - pn + pv = (1 + p)(v - n) + m$. We can now simply osserve that $p = \frac{m-q}{n} \geq \frac{m-n}{n}$ and then group again by $v - n$ to use the second part of 3.4 hence we get 3.5.

$m \equiv 0$: we have $m = pn$, then 3.4 became with simple algebra $d > \frac{pv}{v-n}$, and hence for 3. of theorem 3.2.1 $\alpha(I_{X_C}^{(m)}) = pv < d(v - n)$, that satisfy 3.5.

In particular for $m = n$ and $v > 2n$ we have

$$1 + \frac{m - n}{n} + \frac{m}{v - n} = 1 + \frac{n}{v - n} = \frac{v}{v - n} <^2 2$$

and hence $I_{X_C}^{(n)} \not\subseteq I_{X_C}^2$

$\square$

Chapter 4

Coloration of Steiner Configuration ideal

Fun fuct, I can use the cp to check if there exists a colorarion

Bibliography

[1] Michael F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra. Student economy edition.* English. Student economy edition. Boulder: Westview Press, 2016, pp. ix + 128. ISBN: 978-0-8133-5018-9/print; 978-0-8133-4544-4/ebook.

[2] Edoardo Ballico, Giuseppe Favacchio, Elena Guardo, and Lorenzo Milazzo. "Steiner systems and configurations of points". In: *Designs, Codes and Cryptography* 89.2 (2021), pp. 199–219. DOI: 10.1007/s10623-020-00815-x. URL: https://doi.org/10.1007/s10623-020-00815-x.

[3] Cristiano Bocci and Brian Harbourne. "Comparing powers and symbolic powers of ideals". English. In: *J. Algebr. Geom.* 19.3 (2010), pp. 399–417. ISSN: 1056-3911; 1534-7486/e.

[4] Cristiano Bocci and Brian Harbourne. *The resurgence of ideals of points and the containment problem.* 2009. arXiv: 0906.4478 [math.AG].

[5] Ben Drabkin, Eloísa Grifo, Alexandra Seceleanu, and Branden Stone. "Computations involving symbolic powers". In: *Journal of Software for Algebra and Geometry* 9 (Dec. 2017). DOI: 10.2140/jsag.2019.9.71.

[6] Marcin Dumnicki, Tomasz Szemberg, and Halszka Tutaj-Gasińska. "Counterexamples to the $I^{(3)} \subseteq I^2$ containment". In: *Journal of Algebra* 393 (2013), 24–29. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2013.06.039. URL: http://dx.doi.org/10.1016/j.jalgebra.2013.06.039.

[7] Lawrence Ein, Robert Lazarsfeld, and Karen E. Smith. "Uniform bounds and symbolic powers on smooth varieties". In: *Inventiones mathematicae* 144.2 (2001), pp. 241–252. DOI: 10.1007/s002220100121. URL: https://doi.org/10.1007/s002220100121.

[8] Alessandro Gimigliano. *Our thin knowledge of fat points.* English. Curves Semin. at Queen's. Vol. VI, Kingston/Canada 1989, Queen's Pap. Pure Appl. Math. 83, Exposé B, 50 p. (1989). 1989.

[9] Elena Guardo, Brian Harbourne, and Adam Van Tuyl. *Asymptotic resurgences for ideals of positive dimensional subschemes of projective space.* 2012. arXiv: 1202.4370 [math.AG].

[10] Brian Harbourne and Craig Huneke. *Are symbolic powers highly evolved?* 2011. arXiv: 1103.5809 [math.AC].

[11] Melvin Hochster and Craig Huneke. "Comparison of symbolic and ordinary powers of ideals". In: *Inventiones mathematicae* 147.2 (2002), pp. 349–369. DOI: 10.1007/s002220100176. URL: https://doi.org/10.1007/s002220100176.

[12] Masayoshi Nagata. *Local rings*. English. Vol. 13. Interscience Publishers, New York, NY, 1962.

[13] Miles Reid. *Undergraduate commutative algebra*. English. Vol. 29. Cambridge: Cambridge Univ. Press, 1995, pp. xiii + 153. ISBN: 0-521-45889-7/pbk; 0-521-45255-4/hbk.

[14] Michel Waldschmidt. "Propriétés arithmétiques de fonctions de plusieurs variables (II)". In: *Séminaire Pierre Lelong (Analyse) Année 1975/76*. Ed. by Pierre Lelong. Berlin, Heidelberg: Springer Berlin Heidelberg, 1977, pp. 108–135. ISBN: 978-3-540-37341-4.

[15] Oscar Zariski. "A fundamental lemma from the theory of holomorphic functions on an algebraic variety". In: *Annali di Matematica Pura ed Applicata* 29.1 (1949), pp. 187–198. DOI: 10.1007/BF02413926. URL: https://doi.org/10.1007/BF02413926.

[16] Oscar Zariski. "Algebraic Varieties Over Ground Fields of Characteristics Zero". In: *American Journal of Mathematics* 62.1 (1940), pp. 187–221. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2371447.