Threshold signatures from different group actions

Giacomo Borin
2025.04.30 - SQIparty - Lleida SPAIN

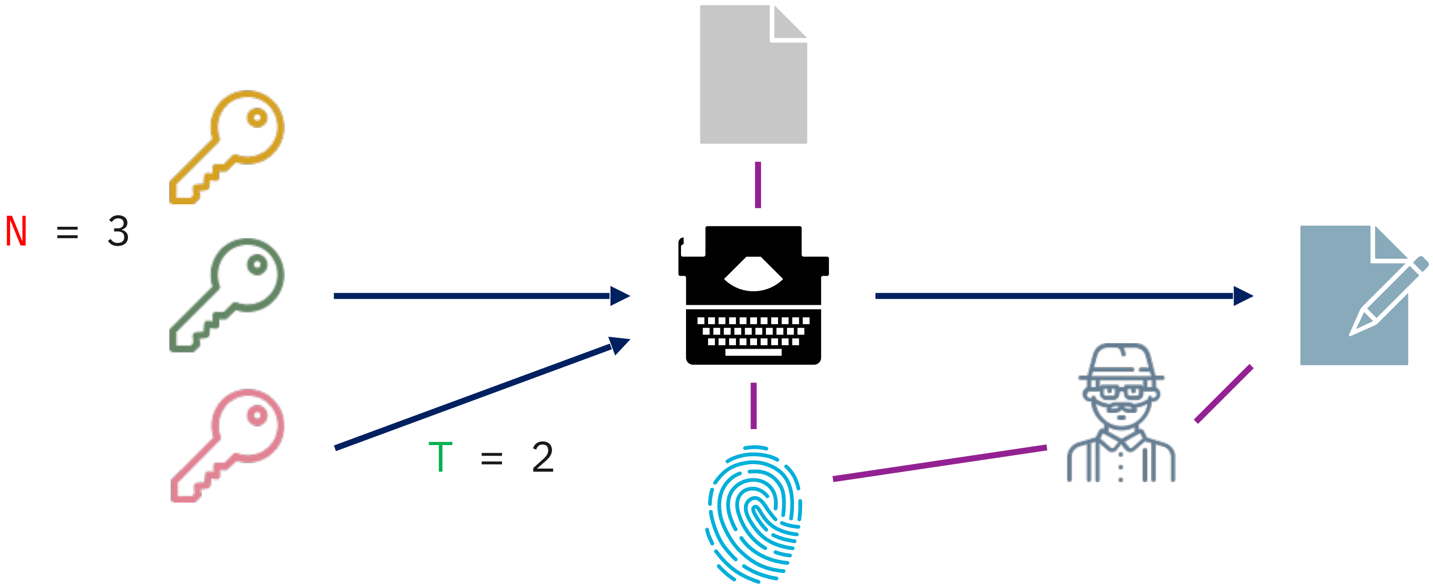
Universität Zürich[UZH]

IBM

- Introduction of different *group actions*
- N-out-of-N case
- Active security
- T-out-of-N case
- Few words on open problems and DKG

(Threshold) Signatures

An (T,N)-threshold digital signature scheme is a protocol where any subset of at least T out of N key owners can sign an agreed message, but not one of less than T.



Cryptographic Group Actions - Definitions

$$\boxed{G} \times X \to X$$

$$(g, x) \mapsto \boxed{g \star x}$$

Needs to be a group

Composable:
$g_1 \star (g_2 \star x) = (g_1 g_2) \star x$

- Effective, i.e. we can efficiently:
  - compute, sample & canonically represent elements in G
  - compute the action of all the elements of G
- Cryptographic:
  - Vectorization: given x, y it is hard to find g s.t. $g \star x = y$
  - Parallelisation: given $x, y = g \star x, z = h \star x$ and w it is hard to say if $w = (gh) \star x$

Cryptographic Group Actions - Instantiations

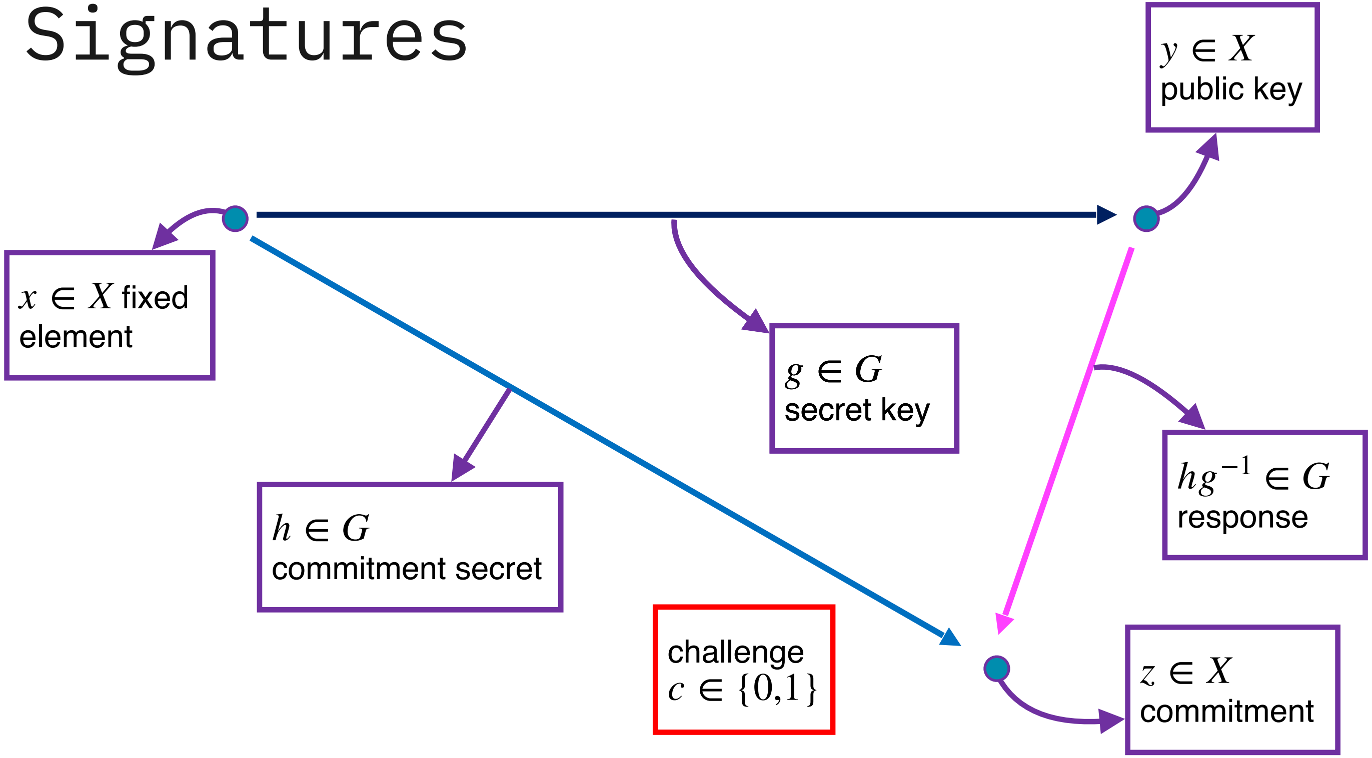
$$\boxed{G} \times X \to X$$
$$(g, x) \mapsto g \star x$$

Which kind of?

- Isomorphism problems from Tensors/Coding Theory (1)+ > **Non-Abelian**
- Class Group acting on Oriented Supersingular Elliptic Curves:
  - CSI-FiSh (2) > **Cyclic** > We can work with $\mathbb{Z}/\#G\mathbb{Z}$
  - PEGASIS (3) > **Abelian**

(1) Barenghi A, Biasse JF, Persichetti E, Santini P. LESS-FM: fine-tuning signatures from the code equivalence problem.
(2) Beullens W, Kleinjung T, Vercauteren F. CSI-FiSh: efficient isogeny based signatures through class group computations.
(3) Dartois P, Eriksen JK, Fouotsa TB, Le Merdy AH, Invernizzi R, Robert D, Rueger R, Vercauteren F, Wesolowski B. PEGASIS: Practical Effective Class Group Action using 4-Dimensional Isogenies.

Signatures and Threshold Signatures



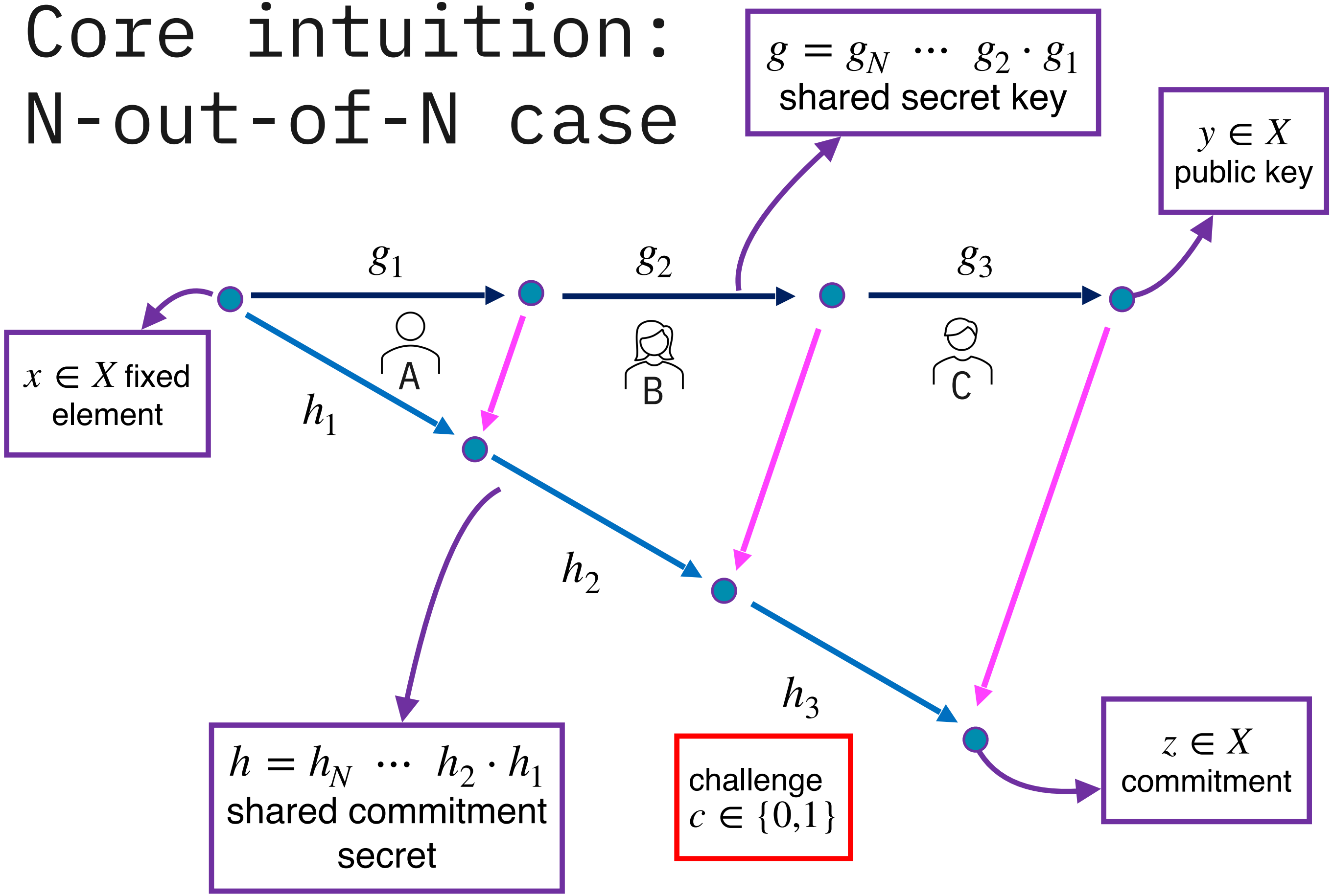
-Repeat λ times;

-With Fiat-Shamir transform you get a signature;

-Boneh et.al. (2): you need to do that at least λ group actions.

(1) De Feo L, Galbraith SD. SeaSign: compact isogeny signatures from class group actions
(2) Boneh D, Guan J, Zhandry M. A lower bound on the length of signatures based on group actions and generic isogenies.

Core intuition: N-out-of-N case



- the intermediate pks are in relation given by:

$$y_{i+1} = g_{i+1} \star y_i$$

- in the abelian case we can compress the response phase to one round

- the hard part is the sharing of the secret, not the commitment

How to make this secure against active attackers?

(1) Cozzo D, Smart NP. Sashimi: cutting up CSI-FiSh secret keys to produce an actively secure distributed signing protocol.
(2) Battagliola M, Borin G, Meneghetti A, Persichetti E. Cutting the grass: Threshold group action signature schemes.

- In an active scenario the last user can always perform a basic version of the ROS attack;
- *Solution from (1)*:
  - Add a ZKPoK for <u>every action performed</u> in commitment generation;
  - **Con**: Very inefficient (*memo: Boneh et.al. result*);
  - **Pro**: Simple and imply adaptive security.
- *Solution from (2)*:
  - use secure randomness + verify all intermediate signatures
  - **Pro**: Much more efficient;
  - **Con**: Requires to know all intermediate public keys.

	Passive, Non-Abelian	Passive, Abelian	Active, with ZKPs	Active, with Secure Randomness
# Rounds	N + N	N + 1	N + 1 + 1	N + N + 1
Complexity	O(N λ)	O(N λ)	O(N λ^2)	O(N λ)
Share size	O(λ)	O(λ)	O(λ)	O(N λ)

Cozzo D, Giunta E. Round-robin is optimal: lower bounds for group action based protocols.

How to make this for T-out-of-N ? Cyclic Case

De Feo L, Meyer M. Threshold schemes from isogeny assumptions

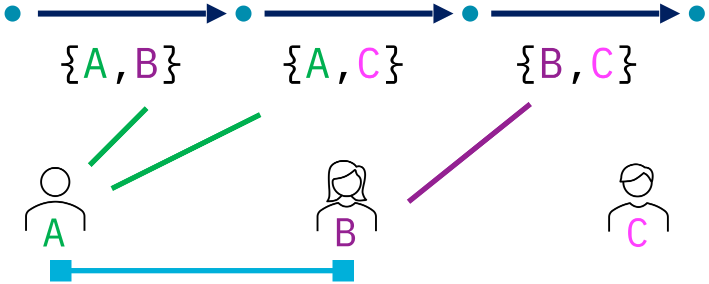
Shamir Secret Sharing

- **Idea**: each authorised subset of parties L can write the secret as a linear combination of their shares $s = \lambda_{S,1}s_1 + \cdots + \lambda_{S,T}s_T$, then $y = [\lambda_{S,1}s_1]\cdots[\lambda_{S,T}s_T]\, x$
- **Problem 1**: requires G to be a ring with division, but $\#G$ is composite,
  - **Remark**: the denominator abs is bounded by N
  - **Solution**: modify the generator so that $N \leq$ all prime factors of $\#G$;
- **Problem 2**: still requires T rounds.
- **Problem 3**: ZKPs becomes much more complicated (PVP)

How to make this for T-out-of-N ? Non-Abelian Case

Replicated Secret Sharing

- Idea: increase (exponentially) the number of secrets and assign the knowledge to multiple parties;
- Example: 2-out-of-3 users:



Battagliola M, Borin G, Meneghetti A, Persichetti E. Cutting the grass: Threshold group action signature schemes.

How to make this for T-out-of-N ? Non-Abelian Case

(1) Desmedt Y, Di Crescenzo G, Burmester M. Multiplicative non-abelian sharing schemes and their application to threshold cryptography.
(2) Battagliola M, Borin G, Di Crescenzo G, Meneghetti A, Persichetti E. Enhancing Threshold Group Action Signature Schemes: Adaptive Security and Scalability Improvements.

‘Vandermonde’ Secret Sharing

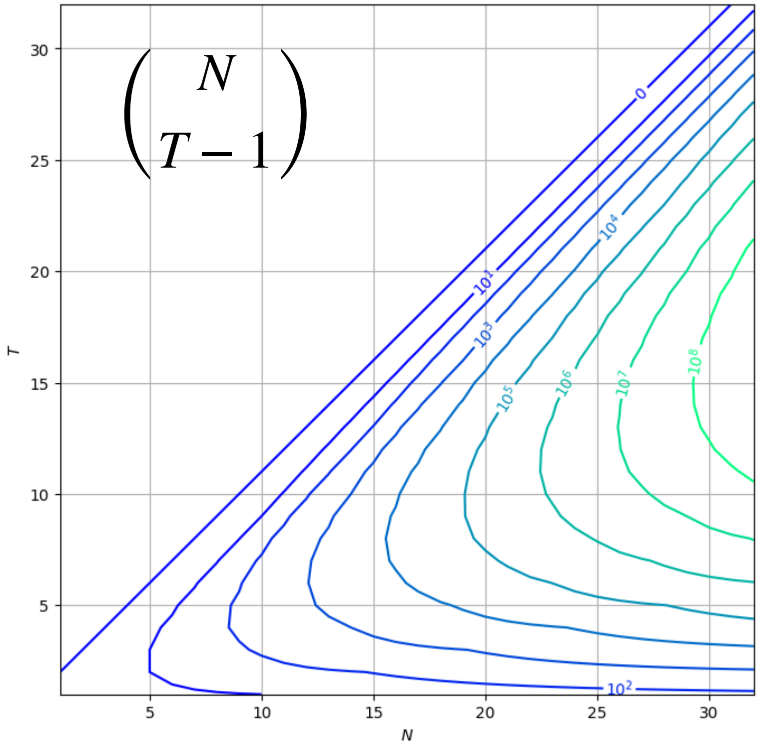
- Recursive idea, use algorithmically the Vandermonde inequality:

$$\binom{N}{T} = \sum_{k=0}^{T} \binom{b}{k} \cdot \binom{N-b}{T-k}$$

- Recursive evaluation of T-out-of-N:
  - If $T = 1$ or $T = N$ share the secret in the ‘obvious way’
  - If $T \leq 0$ or $T > N$ ignore the sharing
  - Otherwise:
    - divide in two groups of size $\approx N/2$
    - for each k do a k-out-of-$N/2$ and $T-k$-out-of-$N/2$ sharing

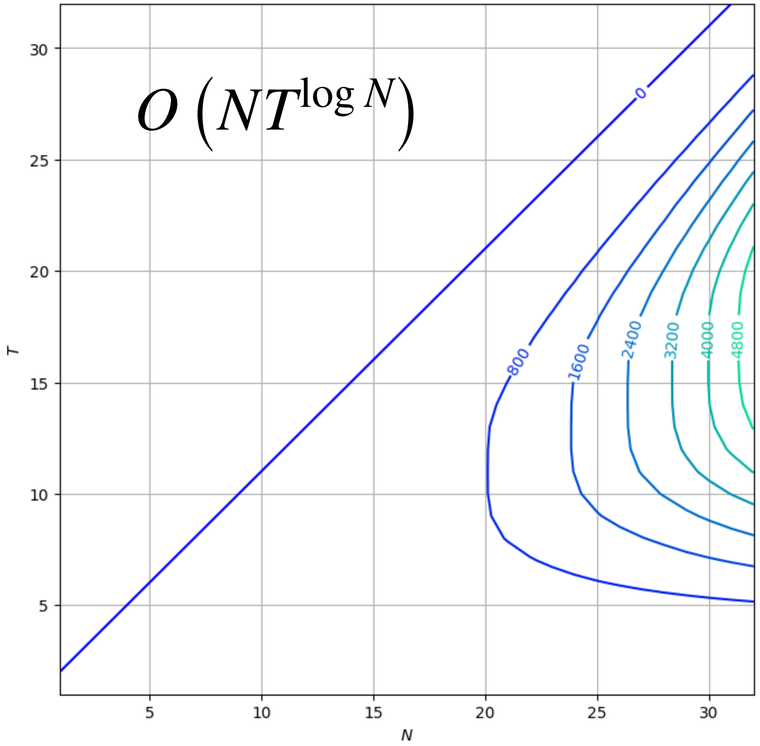
Replicated Secret Sharing

- Less efficient, but simpler



'Vandermonde' Secret Sharing

- More complicated, but efficient



How to make this for T-out-of-N ? Abelian Case (open)

- **Problem**: no field like structure (since $\#G$ is unknown):

$$\lambda_{S,i} = \frac{\prod_{j \in S} j}{\boxed{\prod_{j \in S} (j - i)}}$$

I need to make sense of this division!

- **Note**: this is the same problem they had with RSA.
- **Solution 1a**: work on $\mathbb{Z}$ and use LISS, not compatible with PVP.
- **Solution 1b**: multiply by $N!$ so we are in $\mathbb{Z}$ (compatible with PVP?)
- **Solution 2**: use previous Vandermonde Sharing:
  - Active security with ZKPs or with Secure Randomness

	Shamir	Replicated	Vandermonde	Vandermonde
	Cyclic	Non-Abelian	Non-Abelian	Abelian
# Rounds	2T + 1	$2\binom{N}{T-1}+1$	2T + 1	T + 2
Signing Complexity	O(N λ^2)	$O\left(\binom{N}{T-1}\lambda\right)$	O(T λ)	O(T λ)
Share size	O(1)	$O\left(\binom{N}{T-1}\lambda\right)$	$O(NT^{\log N}\lambda)$	$O(NT^{\log N}\lambda)$

How to distribute the generation of the key? (open)

(1) Atapoor S, Baghery K, Cozzo D, Pedersen R. CSI-SharK: CSI-FiSh with sharing-friendly keys.
(2) Frixons P, Gilchrist V, Kutas P, Merz SP, Petit C. Another Look at the Quantum Security of the Vectorization Problem with Shifted Inputs.
(3) Cozzo D, Smart NP. Sashimi: cutting up CSI-FiSh secret keys to produce an actively secure distributed signing protocol.

- **Option 1**: CSI-SharK (1) introduces PVP,
  - Requires an assumption that is *quantumly broken* (2).
- **Option 2**: Sashimi (3) approach in the KeyGen,
  - Working, but the number of iterations is $\binom{N}{T-1}$,
  - Assumption *not secure for the non-abelian case*,
  - **Option 2b**: Extractable ZKPoK with commitment at the start to the secret.
- **Option 3** *[OPEN]*: can we have DKG for the Vandermonde Sharing?

Thanks