Exercise : Impulse optimization

This exercise deals with a simple optimization problem :

the preparation and optimal impulse in a one - dimensional vertical movement, corresponding to a vertical jump from a standing position, where the aim is to reach a fixed bar some distance above.

The movement must follow the guiding movement indicated in the figure as closely as possible. The vertical axis x represents the hip height of person executing the jump, the horizontal axis representing time.

Hip heigth[t]

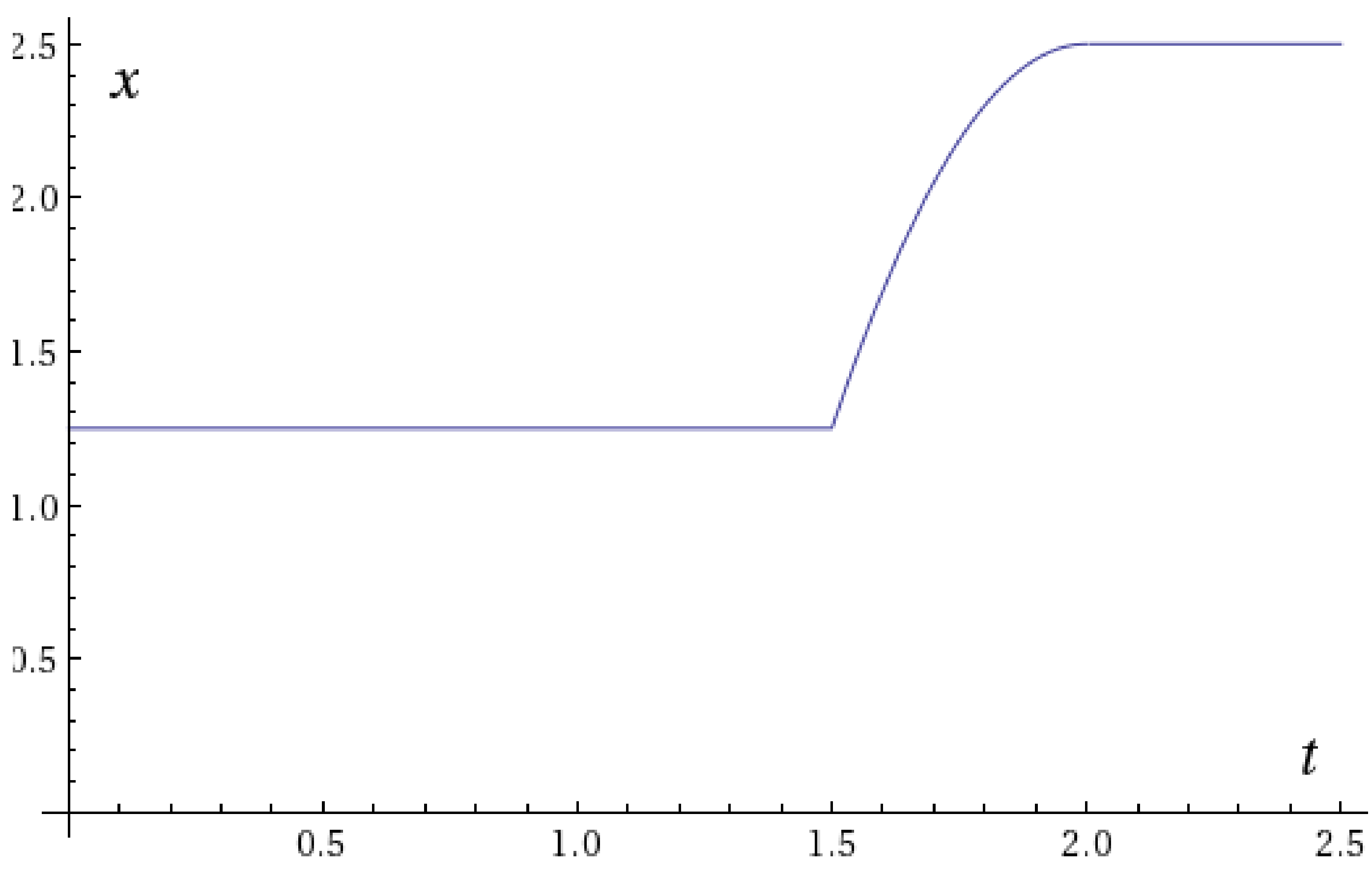
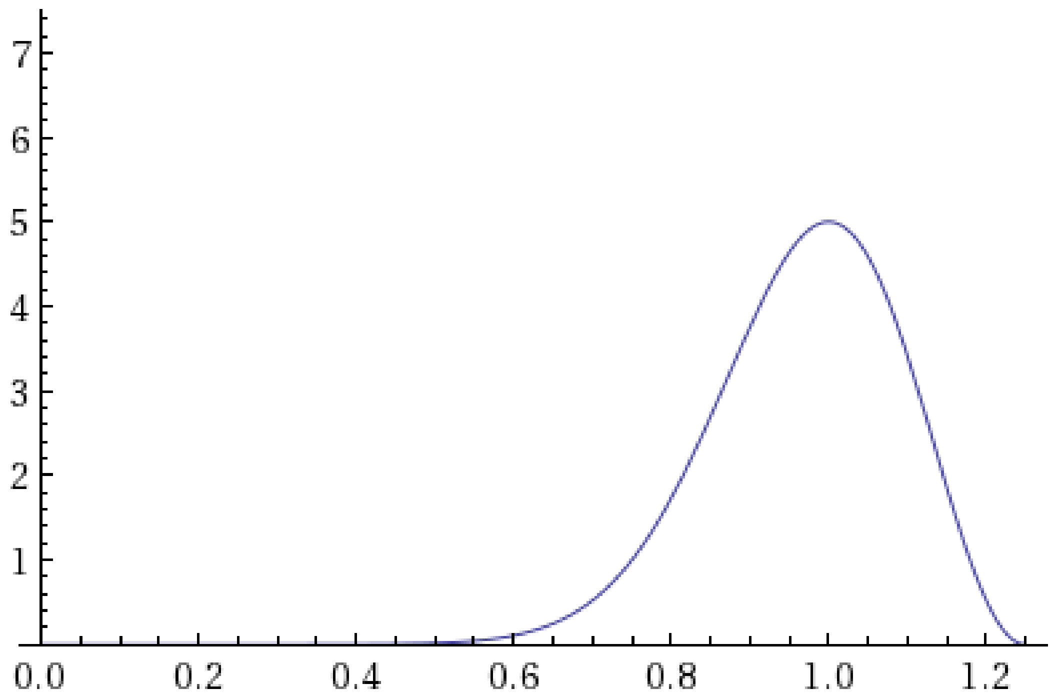


Fig 1

We will use a model for the maximum vertical impulse force, which states that when the person is in contact with the ground the maximum vertical force is given by

$$\text{fmax}[x] = F0 \left(\frac{L - x}{xc}\right)^2 \text{Exp}\left[1 - ((L - x)/xc)^2\right]$$



The maximum corresponds to $x = L - xc$ $(\text{fmax}[x = L - xc] = F0)$

and diminishes when the legs are progressively extended $x \to L = 1.25$

or totaly flexed $x \to 0.5$

Data :

The model data is $\{L = 1.25,\ xc = 0.25,\ F0 = 75,\ g = 10\}$.

The impulse force can take any value $f \in [0, \ fmax[x]]$ and the resulting acceleration is $\alpha = f - g$:

(we use as unit of mass the jumping person mass, so that $M = 1$)

If the jumping player tries to apply a vertical force f, the physical model will filter out unphysical values. Large positive values are limited by the mechanical limits modeled via fmax[x], whilst negative values are simply not possible. The resulting vertical acceleration α in response to a desired force f is then $\alpha Phys[f, x] = f_{Phys} - g$, given by the constraints

$$\alpha Phys[f, x] = \begin{cases} -g & \text{if } f \leq 0 \\ fmax[x] - g & \text{if } x < L \;\&\&\; f > fmax[x] \\ -g & \text{if } x > L \\ f - g & \text{Otherwise (unconstrained case)} \end{cases}$$

Given the above restrictions, we want to follow as closely as possible the guiding movement represented in fig. 1

and described by Guiding Movement :

$$xg[t] = \begin{cases} L = 1.25 & t < 1.5 \\ 2.5 - 5\,(2 - t)^2 & 1.5 \le t \le 2 \\ h = 2.5 & t > T = 2 \end{cases}$$

$$vg[t] = \begin{cases} 10\,(2 - t) & 1.5 < t < 2 \\ 0 & t > T = 2 \end{cases}$$

We want to find out the optimal physical trajectory which :

a) respects the physical limitations for $\alpha = \alpha\mathrm{Phys}[f, x]$. Starting from $x[t = 0] = L$, it is clear that the physical trajectory will need first to lower x, then take impulse and finally execute the jump.

b) Minimizes the quantity

$$\text{Penalty} = \int_0^T \left(\frac{1}{2}\,(x[t] - xg[t])^2 + \frac{1}{2}\,(v[t] - vg[t])^2 + \frac{\gamma}{2}\,(\alpha[t])^2 \right) dt +$$

$$+ \frac{\Gamma}{2}\,\left((x[T = 2] - h)^2 + v[T = 2]^2 \right)$$

and therefore maximizes the payoff

$$\text{Payoff} = -\int_0^T \left(\frac{1}{2} (x[t] - xg[t])^2 + \frac{1}{2} (v[t] - vg[t])^2 + \frac{\Upsilon}{2} (\alpha[t])^2 \right) dt - $$

$$- \frac{\Gamma}{2} \left((x[T = 2] - h)^2 + v[T = 2]^2 \right)$$

$$\boxed{\text{Take} \quad \gamma = 1 \qquad \Gamma = 250\,000}$$

The system is described as follows :

$$X \equiv \begin{pmatrix} x \\ v \end{pmatrix} \qquad \dot{X} = \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ a \end{pmatrix} \equiv F \qquad (a = \alpha);$$

$$r = -\frac{1}{2} (x - x^g)^2 - \frac{1}{2} (v - v^g)^2 - \frac{\Upsilon}{2} \alpha^2$$

$$P = \begin{pmatrix} p1 \\ p2 \end{pmatrix}$$

The Hamiltonian control is

$$H = F . P + r \to \quad v\, p1 + \alpha\, p2 - \frac{1}{2} (x - x^g)^2 - \frac{1}{2} (v - v^g)^2 - \frac{\Upsilon}{2} \alpha^2$$

We know that the optimal α maximizes H, $\partial_\alpha H = 0$:

$$\partial_\alpha H = 0 \quad \rightarrow \quad \boxed{\alpha = p2/\gamma \qquad f_{imp} = p2/\gamma + g}$$

where p1 and p2 are functions of t whose evolution is dictated by optimization control theory :

$$\boxed{\dot{X}[t] = \partial_P H \qquad \dot{P}[t] = -\partial_X H \qquad \partial_\alpha H = 0}$$

so that the optimal evolution is

$$\dot{p}_1 = -\frac{\partial H}{\partial x} = x - x^g \qquad \dot{x} = v$$

$$\dot{p}_2 = -\frac{\partial H}{\partial v} = -p_1 + \left(v - v^g\right) \qquad \dot{v} = \alpha$$

$\alpha = p_2/\gamma \qquad f = \alpha + g \quad \rightarrow \quad \alpha = \alpha\mathrm{Phys}[f, x]$ due to the model constrains.

MC Procedure : consider as optimization parameters $\{t_0, \lambda 1, \lambda 2\}$,

with some initial unknown values for $\lambda 1$ = p1[0], $\lambda 2$ = p2[0], and

t_0 being the instant when the player starts de movement ($t_0 \in [0, 1.5]$)

and $\lambda 1$ = p1[t0], $\lambda 2$ = p2[t0] the initial values for {p1, p2}. The

initial values for {x, v} are known to be {L, 0}.

* Use a time step such that the interval t0 < t < tf = T = 2 is divided

in (say N_b = 1000) bins and use this time step both to solve the differential

eequations describing the dynamics, and to compute the integral involved in

the Penalty function as a simple discrete sum.

* Find a reasonable set of parameters $\{t_0, \lambda 1, \lambda 2\}$ such that the trajectory
has the three phases (initially at rest, impulsing phase, and jump). In order
to do this estimation, consider that the optimal values should provide $x[t] \sim x^g[t]$
and $v[t] \sim v^g[t]$ and assuming this is the case, $\alpha[t] = p2[t]/\gamma$ depends solely
on $\{\lambda 1, \lambda 2\}$. For the 3 phases of the movement to exist, the sign and approximate
value of $\{t_0, \lambda 1, \lambda 2\}$ can be guessed.

* Using the function Penalty[t_0, λ1, λ2],
write a simple Monte Carlo code which generates random trials

$\{t_0, \lambda 1, \lambda 2\}' \leftarrow \{t_0, \lambda 1, \lambda 2\} + \{\Delta t_0, \Delta\lambda 1, \Delta\lambda 2\}$

and updates the values only if

Penalty[t_0, λ1, λ2]' < Penalty[t_0, λ1, λ2]

and reject the trial otherwise.

Report the results ot the optimal trajectory:

* Physical trajectory vs. guiding trajectory
* Minimum Penalty value found
* Optionally: discuss weather the optimization hamiltonian is constant (Non constancy being and indication that the constraints are acting, preventing a further maximization of the payoff functional).