Exercise 1. Central Limit Theorem

Objective: Consider the problem of throwing dice multiple times and calculating the resulting probability distribution. In this problem, N_{dice} dice are thrown N_{iter} times. In particular $N_{dice} = 1; 2; 6$ cases are to be considered. The goal is to calculate the probability distribution, average value and the variance.

1 Task

1.1 Figure 1

A die, thrown once generates an integer number in the range from 1 to 6.

Calculate the mean value μ,

$$\mu = \langle x \rangle = \frac{1}{6} \sum_{i=1}^{6} x_i \tag{1}$$

with $x_i = i$ and the variance σ

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 \tag{2}$$

where

$$\langle x^2 \rangle = \frac{1}{6} \sum_{i=1}^{6} x_i^2 \tag{3}$$

Use a random number generator to simulate dice throwing and calculate the resulting probability distribution p_i of obtaining each number i from 1 to 6. Use discrete normalization

$$\sum_i p_i = 1 \tag{4}$$

The random variable of interest is

$$x = \mathrm{rand}(6) \tag{5}$$

Generate N_{iter} random numbers and use them for the calculation of the probability distribution of finding $x = (1, ..., 6)$. Plot the resulting probability distribution function $p_i, i = 1, ..., 6$ and compare it with uniform distribution prediction.

1.2 Figure 2

Assume that one random event consists of throwing a die twice, $N_{dice} = 2$, and calculating the average value,

$$x = \frac{\mathrm{rand}(6) + \mathrm{rand}(6)}{2} \tag{6}$$

Calculate the probability distribution of the outcome $x = (1, 1.5, 2, 2.5, ..., 6)$ and show it in the figure.

1.3 Figure 3

Consider the case when one random event consists of throwing a die N times and calculating the average value

$$x = \frac{\sum\limits_{i=1}^{N_{dice}} \text{rand}(6)}{N_{dice}} \tag{7}$$

Calculate the probability distribution $p(x)$ and assume that for large N_{dice}, the spacing $dx = 1/N_{dice}$ between two allowed subsequent values of x is small and that the random value x can be considered a continuous variable. Use continuous variable convention to normalize the PDF,

$$\int p(x)dx = 1 \tag{8}$$

Compare the obtained result with the prediction of the Central Limit Theorem

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma'} \exp\left(-\frac{(x-\mu)^2}{2\sigma'^2}\right) \tag{9}$$

where the reduced variance is given by $\sigma' = \sigma/\sqrt{N_{dice}}$.

Find a large enough value of N_{dice} so that the Central Limit Theorem applies with a reasonable accuracy and show the resulting plot.

1.4 Error estimation

In this task, we aim to calculate the average value and to estimate the statistical error, associated with such estimation. Assume that single-die throwing is used to estimate the mean value

$$\mu = \langle x \rangle \approx \frac{\sum\limits_{i=1}^{N_{iter}} x_i}{N_{iter}} \tag{10}$$

and the variance

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 \approx \frac{\sum\limits_{i=1}^{N_{iter}} x_i^2}{N_{iter}} - \left(\frac{\sum\limits_{i=1}^{N_{iter}} x_i}{N_{iter}} \right)^2 . \tag{11}$$

Calculate the mean value by throwing the dice $N_{iter} = 10$ and $N_{iter} = 100$ times, compare the estimation of the mean value and the variance to the exact values, given by

$$\mu = \langle x \rangle = \frac{\sum\limits_{\ell=1}^{6} \ell}{6} \tag{12}$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{\sum\limits_{\ell=1}^{6} \ell^2}{6} - \left(\frac{\sum\limits_{\ell=1}^{6} \ell}{6} \right) . \tag{13}$$

Calculate the mean value by throwing the dice $N_{iter} = 10$ and $N_{iter} = 100$ times and estimate the statistical error by $\varepsilon = \sigma/\sqrt{N_{iter}}$ where the variance is Is the estimation of the statistical error, ε, similar to the actual error, i.e. the difference between $\mu(N_{iter})$ and the exact value μ?