Exercise 2. Random Walk solution of Laplace equation

Objective: Find electrostatic potential distribution inside of a square box or in a space with more complicated geometry using random walk method.

Theory

The problem under consideration is finding solution to Laplace equation in two dimensions in an area bounded by a square box, given the value of the electrostatic potential at all points at the boundary. The boundary is defined at the edges of the box and in addition, internal boundary of any shape can be specified inside of the box.

The electrostatic potential at all interior points satisfy the Laplace equation

$$\frac{\partial^2\psi}{\partial^2 x} + \frac{\partial^2\psi}{\partial^2 y} = 0$$

In absence of the boundary conditions the solution to the Laplace equation is given by a Gaussian distribution function. The Gaussian probability distribution as well arises in a Random Walk, thus it is possible to reformulate the problem of finding an electrostatic potential in a box with specified values of the potential at the edges as a random walk problem.

Problem 2.1

1. Consider a square box of size 1×1 where the left side has potential equal to $+10$ and the other sides (right,up and bottom) have potential equal to $+1$.
2. Divide the whole square into a mesh containing a certain number of equally spaced lattice sites. Typical numbers range from sides 30×30 to 100×100
3. Implement a discrete random walk algorithm for solving the Laplace equation.
4. solve the Laplace problem and report the dependence of the obtained potential on (x, y) with a heatmap, see Fig. for an example

Some results to compare with:

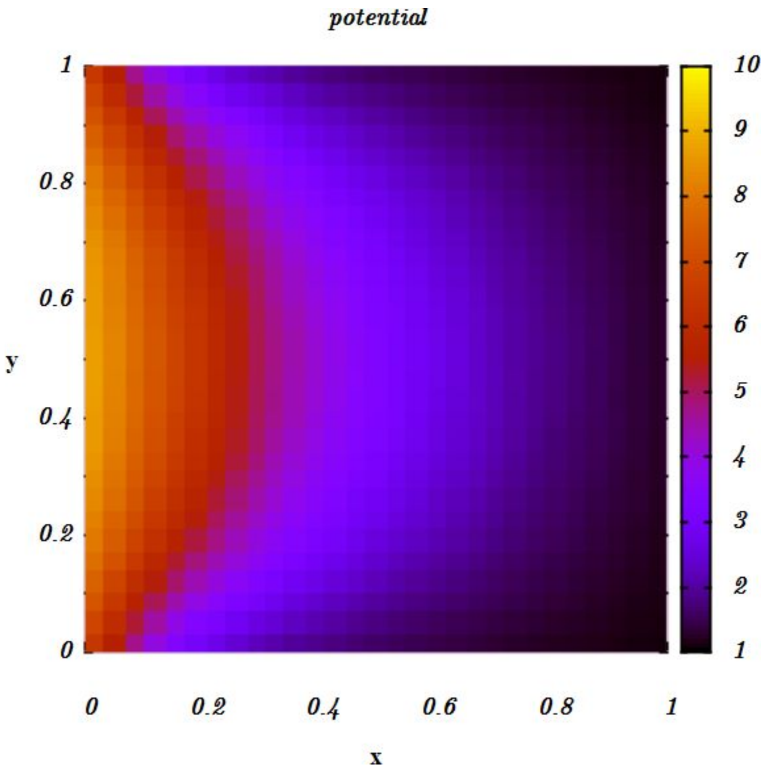

Figura 1: Example of the resulting potential on a 30×30 grid. The left edge has potential $+10$ and other edges of $+1$.

Discrete Random Walk algorithm

1. Begin at a point (x, y) where the value of the potential has to be estimated. Take a step in a random direction along x and y

2. Continue taking steps until the walker reaches the surface. Accumulate the value of the potential at the boundary $V_b(i)$

3. Repeat steps (1) and (2) M times and sum the potential found at the surface each time

4. The value of the potential $V(x, y)$ at the point (x, y) is then estimated according to $V(x,y) \approx \frac{1}{M}\sum_{i=1}^{M} V_b(i)$ where M is the total number of random walkers

Problem 2.2

1. Consider a two-dimensional model for a plane condenser composed of two finite-width plates of opposite potential. The voltage corresponds to the difference of the potentials on the plates while zero potential can be imposed at the edges of the box.

   find the potential as the function of the position, $V(x, y)$ and plot it with a heatmap figure, see Fig. for an example.

2. to do so, use Gaussian displacement (Wiener process) in a continuous space to generate a sufficiently large number of Random Walks and estimate the potential value according to the average over the boundary values, $V(x,y) \approx \frac{1}{M}\sum_{i=1}^{M} V_b(i)$ where M is the total number of random walkers

3. estimate the amplitude of the electric field $E(x, y)$. Its vector value corresponds to the antigradient of the potential field, $\vec{E} = -\nabla V$, and amplitude can be approximated as $E = \sqrt{\left[\frac{V(x+\Delta x,y)-V(x,y)}{\Delta x}\right]^2 + \left[\frac{V(x,y+\Delta y)-V(x,y)}{\Delta y}\right]^2}$. See Fig.

   NB1 Gaussian displacement should have zero expectation value, $\mu = 0$, while the typical displacement length σ should be small compared to the size of the condenser.

   NB2 Effects of the statistical noise is amplified in the calculation of the electric field, as difference of the potential is calculated.

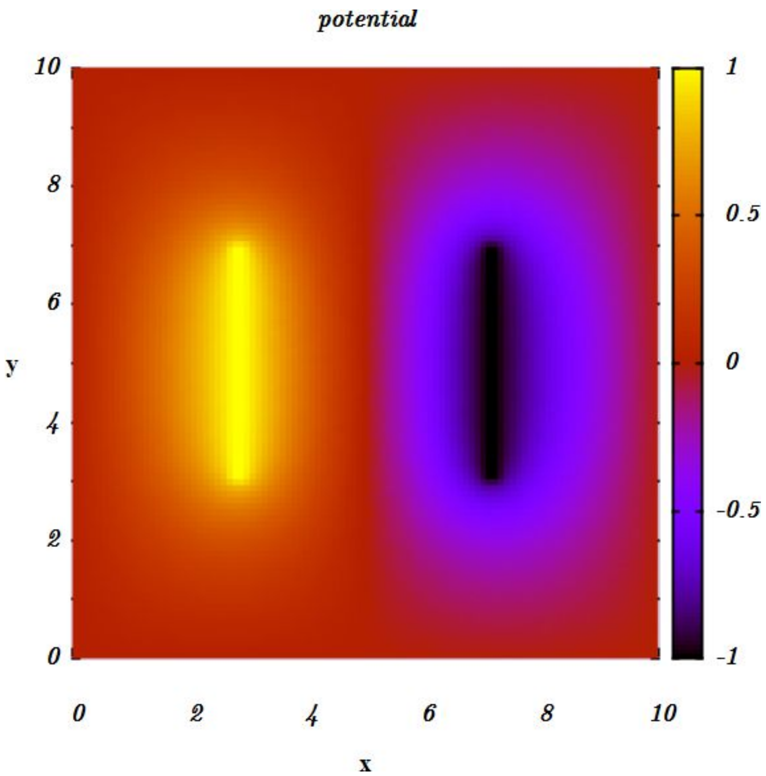

Figura 2: Example of electrostatic potential $V(x, y)$ in a plane condenser

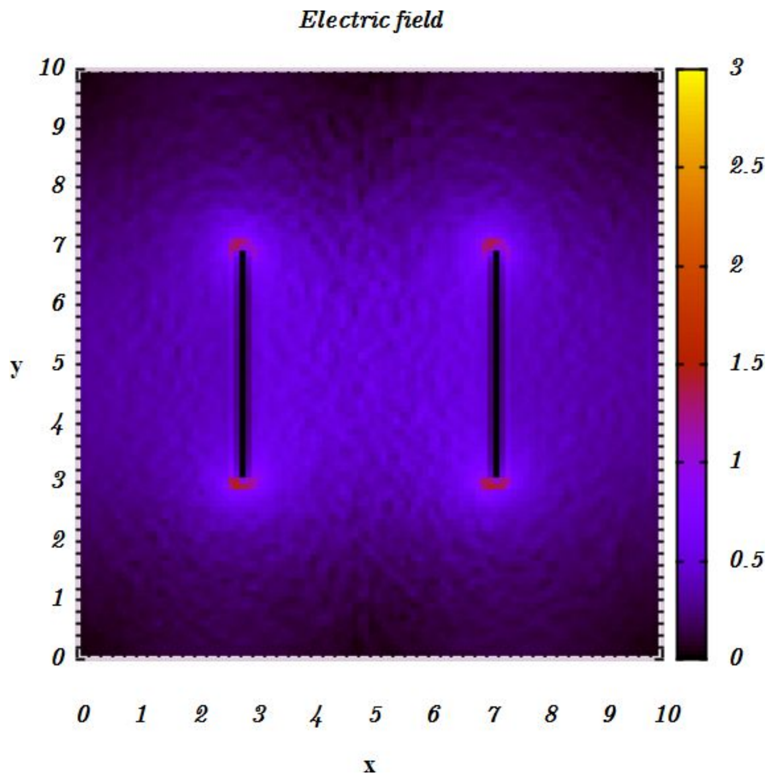

Figura 3: Example of the electric field $E(x, y)$ in a plane condenser