Lecture 1 -Random processes

Random processes

Random variable is a quantity which takes different values with certain probabilities $p(x)$

Examples

discrete:

- flipping a coin $\{0, 1\}$ probability $1/2$
- dice $\{1, 2, \dots, 6\}$ probability $1/6$
- random generator $\{0, \dots, RAND\_MAX\}$
- number of students in a course

continuous:

- height and weight of students
- position of a bullet

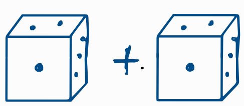
domain: set of accessible values

probability distribution function
- frequency with which the values appear

Discrete random variables

Example: throwing a dice twice

domain: $[2, 3, \dots, 12]$



probabilities:

$$[1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1] / 36$$

- $p_i \geq 0$ probability related to a given value must be positive o null
- $\sum_i p_i = 1$ sum of all probabilities must be unity, indicating certitude

Continuous random variables:

$x_i \to x$

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Density of probability distribution function (PDF) is defined as limit

$$p(x) = \lim_{\Delta x \to 0} \frac{P(\xi \in (x, x+\Delta x))}{\Delta x}$$

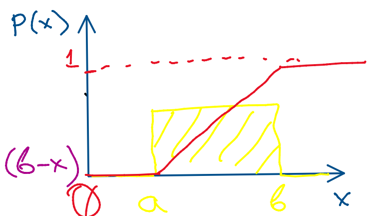
probability of having a value in interval $(x, x+\Delta x)$

Basic concepts

	Discrete	Continuous
domain	$\{x_1, x_2, \dots\}$	$[a, b]$
PDF	$p_i = Prob(X = x_i)$	$p(x) = Prob(x \leq X \leq x+dx]$
cumulative function	$P_i = Prob(X < x_i)$	$P(X) = \int_a^x p(x)\,dx$
positivity	$0 \leq p_i \leq 1$	$p(x) \geq 0$
bounds	$0 \leq P_i \leq 1$	$0 \leq P(X) \leq 1$
monotonicity	$P_i > P_j$ if $i > j$	$P(X) > P(Y)$ if $X > Y$
normalization	$\sum_{i=1}^{N} p_i = 1$	$\int_a^b p(x)dx = 1$
	$P_N = 1$	$P(b) = 1$

Examples of Distribution functions

Uniform distribution



domain $[a, b]$

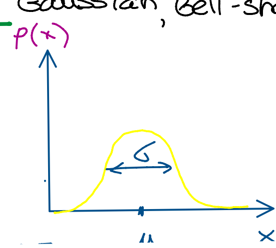
PDF $p(x) = \frac{1}{b-a} \vartheta(x-a) \cdot \vartheta(b-x)$

cumulative function $P(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x > b \end{cases}$

where step function $\vartheta(x) = \begin{cases} 0, & x < 0 \\ 0.5, & x = 0 \\ 1, & x > 0 \end{cases}$

Normal distribution

Gaussian, Bell-shape



$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}$$

μ, σ parameters

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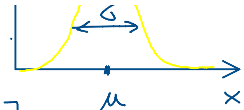

$$P(x) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$$

erf (...) error function

Expectation values

Let $p(x)$ be a PDF with domain $[a, b]$ and $f(x)$ some function defined in same domain

Expectation of $f(x)$ is

$$\langle f \rangle = \int_a^b f(x)\, p(x)\, dx$$

$\langle \dots \rangle$ - denotes the averaging

- In classical mechanics

$$\langle E \rangle = \frac{\int E(x)\, e^{-\frac{E(x)}{k_B T}}\, dx}{\int e^{-\frac{E(x)}{k_B T}}\, dx}$$

T - temperature

$p(x)$

Maxwell-Boltzmann distribution

- In quantum mechanics

$$\langle \psi | \hat{A} | \psi \rangle = \frac{\int \psi^*(x)\, \hat{A}\, \psi(x)\, dx}{\int \psi^*(x)\, \psi(x)\, dx}$$

$\hat{A}$ - operator

For local quantities

$$\langle \psi | \hat{A} | \psi \rangle = \int A(x) \underbrace{\frac{|\psi(x)|^2}{\int |\psi(x')|^2 dx'}}_{p(x)} dx$$

For non-local quantities (involving derivatives)

$$E_{loc}(x) = \frac{1}{\psi(x)} H\psi(x) \leftarrow \text{local energy}$$

$$\underset{\text{average energy}}{E} = \langle \psi | \hat{H} | \psi \rangle = \int E_{loc}(x) \underbrace{\frac{|\psi(x)|^2}{\int |\psi(x')|^2 dx'}}_{p(x)} dx$$

Moments of PDF $p(x)$

a) mean is the expectation of : $f(x) = x$

$$\mu(x) = \int_a^b x \cdot p(x)\, dx = \langle x \rangle$$

b) n^{th} moment $\mu'_n(x) = \langle x^n \rangle$

c) centered n^{th} moment $\mu_n(x) = \langle (x-\mu)^n \rangle$

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second centered moment:

d) variance $\sigma^2 \underset{def}{=} \langle (x-\mu)^2 \rangle$

$$\sigma^2 = \langle x^2 - 2x\mu + \mu^2 \rangle = \langle x^2 \rangle - 2\underbrace{\langle x \rangle}_{\mu}\mu + \mu^2 = \langle x^2 \rangle - \mu^2$$

$$\sigma^2 = \underbrace{\langle x^2 \rangle - \langle x \rangle^2}_{\text{calculate}}$$

Examples:

- uniform distribution $p(x) = \frac{1}{b-a} \vartheta(x-a)\, \vartheta(b-x)$

  mean: $\mu = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{x^2}{2}\Big|_a^b \frac{1}{b-a} = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}$

  - is the middle point between a and b

  variance: $\sigma^2 = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a} dx = \frac{(b-a)^2}{12}$

  - is proportional to the width $\sigma \sim (b-a)$

- Normal distribution $\mu;\ \sigma$

Central Limit Theorem (CLT)

- find distribution of N mean values $g(z)$
- with $z = \frac{x_1 + \ldots + x_N}{N}$

  where PDF of x is $p(x)$

$$g(z) = \int dx_1\, p(x_1) \ldots \int dx_N\, p(x_N) \cdot \delta\left(z - \frac{x_1 + x_2 + \ldots + x_N}{N}\right)$$

- Fourier Transform of a constant is a δ-function

$$\int e^{ikx} \frac{dk}{2\pi} = \delta(x)$$

$$\Rightarrow \delta\left(z - \frac{x_1 + \ldots + x_N}{N}\right) = \int e^{ik\left(z - \frac{x_1 + \ldots + x_N}{N}\right)} \frac{dk}{2\pi}$$

$$g(z) = \int dk\, e^{ikz} \int dx_1\, e^{-\frac{ikx_1}{N}} p(x_1) \ldots \int dx_N\, e^{-\frac{ikx_N}{N}} p(x_N)$$

$$g(z) = \int \frac{dk}{2\pi} e^{ikz} \cdot \underbrace{\int dx_1 e^{-\frac{ikx_1}{N}} p(x_1)}_{} \cdot \ldots \cdot \int dx_N e^{-\frac{ikx_N}{N}} p(x_N)$$

$$\langle e^{-\frac{ikx}{N}} \rangle \cdot \ldots \cdot \langle e^{-\frac{ikx}{N}} \rangle$$

$$\langle e^{-\frac{ikx}{N}} \rangle^N = \langle e^{-\frac{ikx}{N} + \frac{i\mu}{N}} \rangle^N \cdot \left(e^{-\frac{ik\mu}{N}} \right)^N$$

$$= \underbrace{\langle e^{-\frac{ik}{N}(x-\mu)} \rangle^N}_{} e^{-ik\mu}$$

$N \to$

$$\langle 1 - \frac{ik}{N}(x-\mu) + \frac{1}{2}\left(\frac{ik}{N}\right)^2 (x-\mu)^2 + \ldots \rangle^N$$

$$\Big(1 - \frac{ik}{N} \underbrace{(\underbrace{\langle x \rangle}_{\mu} - \mu)}_{\emptyset} - \frac{1}{2} \frac{k^2}{N^2} \underbrace{\langle (x-\mu)^2 \rangle}_{\sigma^2} + \ldots \Big)^N$$

$$\left(1 - \frac{1}{2} \frac{k^2 \sigma^2}{N^2} \right)^N \underset{N\to\infty}{\approx} \exp\left(-\frac{1}{2} \frac{k^2 \sigma^2}{N} \right)$$

* used $e^x = \lim_{N\to\infty} \left(1 + \frac{x}{N} \right)^N$

$$g(z) = \int \frac{dk}{2\pi} e^{ik(z-\mu)} e^{-\frac{1}{2}\frac{k^2\sigma^2}{N}}$$

$$\boxed{g(z) = \frac{1}{\sqrt{2\pi N \sigma}} \exp\left\{ -\frac{1}{2} \cdot \frac{(z-\mu)^2}{(\sigma/\sqrt{N})^2} \right\}}$$

Normal PDF with the same mean value μ
and a reduced variance $\sigma/\sqrt{N}$
where N is the number of values
used to calculate the average

<u>NB1</u> CLT applies $N \to \infty$

· exact when $p(x)$ is Normal

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NB2 can be used to provide confidence limit of an averaged quantity

$$\mu \pm \underbrace{\frac{\sigma}{\sqrt{N}}}_{\text{can be interpreted as statistical error}}$$

NB3 ways to decrease error

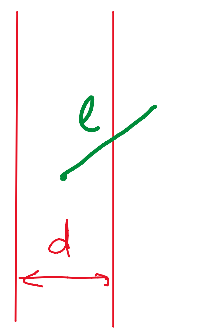
increase N:

by a factor of 2 : $N \to 4N$

Example

Buffon's method of estimation of π number

probability that a needle of length ℓ intersects a line is



$$p = \frac{2}{\pi} \cdot \frac{\ell}{d} \quad \text{- calculated}$$

$$p = \frac{N_{\text{intersections}}}{N}, \quad N \to \infty \quad \text{- measured in experiment}$$

$$\pi = \frac{2}{p}\frac{\ell}{d} = \frac{2\ell}{d} \cdot \frac{N}{N_{\text{intersections}}}$$

Marco Lazzarini 1901 $N = 3408$

$N_{\text{intersections}} = 1808$

$$\pi = \underbrace{3{,}141592}_{\text{correct}}9$$

6 digits!

Statistical error $\dfrac{\sigma}{\sqrt{N}} \sim \dfrac{1}{\sqrt{N}} \sim \dfrac{1}{\sqrt{3408}} \sim \dfrac{1}{\sqrt{3600}}$

2 digits $\sim \dfrac{1}{60} \sim 2\%$

Estimate: N needed for 6 digits?

$$\frac{1}{\sqrt{N}} \sim 10^{-6} \Rightarrow N \sim 10^{12}$$

<u>NB 5</u> Reducing variance σ is importantant for small statistical error bars $\sigma / \sqrt{N}$