Willingness to bet and Wealth Effects: A Preferential Approach

Giacomo Cattelan
Università Bocconi

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Abstract

A different definition of comparative uncertainty aversion is introduced, to address some issues of the defintion of Ghirardato and Marinacci. In particular, the aim is to describe different attitudes toward ambiguity in the presence of different degrees of risk aversion. A general characterization is provided for a large class of preferences: monotone and continuous preferences which satisfy risk independence. Then, in this light, attitudes toward uncertainty determined by wealth effects are studied from a preferential viewpoint.

1 Introduction

The main body of research in applied economics focuses on economic agents that are Expected Utility maximizers, a concept developed in economic theory by the milestone work of von Neumann-Morgenstern [25]. The results obtained through this theory in the years are impressive, and for this reason it now pervades all aspects of Economics. However, beginning with Ellsberg's famous thought experiment and the seminal work of David Schmeidler, several choice models have been proposed in the large literature on choice under uncertainty that deals with ambiguity, that is, with Ellsberg-type phenomena. At the same time, many papers have investigated the economic consequences of *ambiguity aversion*.

To fix ideas, it is convenient to introduce clearly the terms that will be used throughout this work, even though informally. The formal definitions will be presented in the next sections.

Uncertainty is the situation in which a Decision Maker (DM for short from here onwards) does not know for sure the outcome of her actions, since they are subject to forces out of her control. It is made up of two main components, risk and ambiguity. Risk is the form of uncertainty whose outcomes are governed by a known objective probability distribution.

Typical forms of risk that are presented as examples are roulette wheels, coin tossing or dice rolling. Attitude toward risk has been fully characterized in the von Neumann-Morgenstern framework, and are pervasive of all main research branches in economics. On the other hand, ambiguity is that form of uncertainty that stems from the fact that the agent does not have any form of quantifiable knowledge on the outcomes. This is due to the fact that the outcome itself depends on something called state of the world, i.e. the intrinsic features of the environment in which the agent has to act and whose physical process is totally unknown to her. This means that not only the DM may not know the probability distribution of the states of the world, but she may not even recognize that such a probability distribution exists. Therefore ambiguity is not just a concept to overcome the very strong assumption of agent knowing the distribution of the problem he is facing; instead it allows to take into account the possibility that the agent does not even think in probabilistic terms. This distinction between ambiguity and risk was first introduced by Frank Knight [18] in 1921, and for this reason ambiguity is also known, especially in applied literature, as *Knightian uncertainty.*

Together with problems of representation of preferences under uncertainty, an important issue to be settled in the discipline was of course to define ambiguity aversion, both in relative and in absolute terms. In the former case, two preferences, $\succsim_1$ and $\succsim_2$, are compared, in order to say which one of the two is more ambiguity averse than the other. The latter deals with a generic concept of ambiguity aversion. In the field of Economic Theory the debate is still ongoing on which is the most appealing definition. One of the most widely accepted definition is the one given by Ghirardato and Marinacci [11], according to which a preference relation $\succsim_1$ is more ambiguity averse than another $\succsim_2$ if and only if

$$f \succsim_1 x \Longrightarrow f \succsim_2 x$$

where f is an Anscombe-Aumann act and x is a lottery with known distribution.

Although very suitable in a vast range of economic situation, and very successful from a theoretic point of view, such definition fails to disentangle risk and ambiguity. In fact, it requires that the two preferences share the same risk attitude, so that, once such attitude is fixed, it is possible to perform the analysis on ambiguity.

The obvious question to ask, then, is what happens if risk attitudes are left varying instead of being fixed equal. Therefore, a definition that can be proposed is one that compare purely ambiguous acts, that is acts for which all uncertainty is resolved once the state of the world is revealed and that thus are riskless (Savage acts in Economic theoretic jargon), with purely risky acts, i.e. constant acts.

Organization of the Work The following work proposes to investigate the consequences of such new definition, which will be called *willingness to bet.* Furthermore, the analysis is extended from an inter-personal setting to an intra-personal setting, trying to understand absolute and relative attitudes to wealth. This part follows the work of Cerreia-Vioglio et al. [5].

Therefore, Section 2 is dedicated to a brief and informal introduction of the definitions of comparative ambiguity aversion of Epstein [8] and Ghirardato and Marinacci [11]. In Section 3, the mathematical foundations of decision theory are introduced formally. Section 4 deals with the definition of willingness to bet, and the main results on the features of the representations are derived. Lastly, in Section 5, wealth effects on the willingness to bet are investigated.

2 Comparative Ambiguity Aversion

The first definitions of ambiguity aversion were linked to the earlier representation of preferences that agreed with Ellsberg type phenomena, in particular the CEU representation proposed in Schmeidler [24]. In fact, ambiguity aversion was defined exactly by the convexity of the capacity ν, specifically for all $A, B \in \Sigma$,

$$\nu(A \cap B) + \nu(A \cup B) \geq \nu(A) + \nu(B).$$

Furthermore, the degree of ambiguity aversion was measured as the difference between the left and the right hand side of the inequality above. This mathematical peculiarity of capacities and Choquet integrals was the most appealing feature of the CEU representation.

However, there are many examples of rankings of acts, even within the simplest Ellsberg urn experiment framework, which can be represented by CEU and not by SEU, but whose capacity fails to satisfy convexity (see Epstein [8] for such examples). In particular, it is shown that convexity of the capacity is sufficient but not necessary for an agent to be ambiguity averse (expression that, for the moment, refers to the ability to explain Ellsberg paradox). Furthermore, being this definition restricted to the CEU preferences, it did not encompass all the others representations that followed through the years, based on different axioms.

On the other hand, axiomatic foundation, which implies behavioral foundation, was exactly what was needed to generalize the concept of ambiguity aversion. This comes directly from the praxis established in the theory of risk à la von Neumann Morgenstern. In fact, in the context of risk, $\succsim_1$ is more risk averse than $\succsim_2$ if $x \succsim_1 y \implies x \succsim_2 y$ for all lotteries x, y such that x is riskier than y.

The main papers that dealt with the axiomatization of ambiguity aversion for a generic set of preferences are those by Epstein [8] and Ghirardato and Marinacci [11]. Both papers follow the path of categorizing acts as unambiguous acts and ambiguous acts, so that one can say that $\succsim_1$ is more *uncertainty* averse than $\succsim_2$ if and only if

$$f^A \succsim_1 f^U \Longrightarrow f^A \succsim_2 f^U$$

where f^A is the ambiguous act and f^U is the unambiguous act. Of course, the way one defines unambiguous acts has a great impact on the definition of relative ambiguity aversion and on its consequences, and the discrepancy between the two approaches can be found exactly on such definition. Note that the main difference with risk is that lotteries can be objectively recognized as riskier or less risky by direct probabilistic computation, whereas the same is not true for the ambiguity case.

Moreover, as emphasized above, this is a generic definition of uncertainty aversion, which of course contains both ambiguity and risk aversion. For example, an individual could be ambiguity neutral (even though such concept is yet to be formally defined, but the term is used for sake of discussion) and risk averse, and this will make her uncertainty averse in general. Therefore, it is also important to control also for risk attitudes, to arrive to a meaningful definition of ambiguity aversion.

Probabilistic Sophistication as Ambiguity Neutrality Epstein decided to conduct his study in a Savage set up, therefore ruling out risk and assuming that uncertainty is completely resolved once the state of the world s has been revealed. In this way it is possible to disentangle risk and ambiguity aversion, since the set up allows only for ambiguity, and so uncertainty aversion and ambiguity aversion coincide. In such framework, a collection of unambiguous events in Σ is defined, and such collection is required to be a λ-system. Unambiguous acts are thus defined as all acts measurable with respect to such collection.

It is possible to show that ambiguity neutral preferences, i.e. the preferences for which the implication in the definition is symmetric, are equivalent to what Machina and Schmidler [21] call *probabilistically sophisticated*, under appropriate conditions for the class of unambiguous events. In the framework where X is the set of possible outcomes (notice that since the current set up is the Savage set up, X does not need to have any particular structure), an agent is said to be probabilistically sophisticated if there exists a finitely additive probability measure μ on Σ, and a functional $T : \Delta(X) \to \mathbb{R}$ such that acts are evaluated according to

$$V(f) = T(P(\mu, f)),$$

where $P(\mu, f)$ is the distribution on X induced by μ and act f. It can be thought as some sort of generalization of the SEU, and the two concepts coincide if such functional is the expectation operator.

As said above, Epstein cleans out risk attitude from the picture considering a Savage set up. However, he argues that his framework encompasses also risk analysis: in fact, on the set of unambiguous acts, all preferences agree with the probabilistically sophisticated preference, so that on those acts the DM is ambiguity neutral and can be treated as a DM under risk.

Epstein's definition has the main flaw of requiring a class of unambiguous events that is exogenously defined, thus requiring an extraneous device to be effective. Furthermore, and maybe more importantly, the identification of the ambiguity neutral preferences with the probabilistically sophisticated ones can lead to many counterintuitve examples in the simplest of the settings, like the Ellsberg experiments. These examples have been pointed out in Ghirardato and Marinacci [11], that thus have tried to give a different definition of comparative ambiguity aversion, with lighter assumptions but the same explanatory power.

Ghirardato and Marinacci Ghirardato and Marinacci [11] prefer to work in the Anscombe-Aumann set up, thus imposing a proper topological structure on X, and allowing it to be the set of lotteries on a set of prizes. First of all, they make the intuitive assumption that the only acts that can be regarded as unambiguous are constant acts, i.e. acts that yield the same lottery for every state of the world. Secondly, it is assumed that SEU preferences are the only ambiguity neutral, so that SEU preferences are their benchmark to define absolute ambiguity aversion. This latter assumption stems from the criticisms moved to the approach used by Epstein of indicating the probabilistically sophisticated preferences, which of course is a larger class that contains the SEU, as the ambiguity neutral benchmark. In fact, such approach, as stated above, leads to contradictions that make also preferences that are obviously ambiguity non-neutral (like the CEU) look like ambiguity neutral, if analyzed with the definition that Ghirardato and Marinacci propose.

The richer structure of the set of consequences X of the Anscombe-Aumann set up may complicate the definition, since ambiguity and risk attitudes coexist in the same framework. Therefore, comparative uncertainty aversion is defined as

$$f \succsim_1 x \Longrightarrow f \succsim_2 x.$$

This definition allows, for each of the representations introduced above, to keep the risk attitude fixed, that is, the two agents share the same risk attitude. In particular, von Neumann-Morgenstern utility index of one agent is an affine transformation of the utility of the other, and so the definition for comparative uncertainty aversion can be used also to characterize comparative ambiguity aversion and viceversa, since the only source of uncertainty that the two agents perceive as different is ambiguity.

Such simple but powerful definition yields many useful results in comparing ambiguity attitudes. In fact, checking that two particular preferences satisfy the implication above can be computationally demanding. However, it is possible to characterize such attitude in terms of a component of the mathematical representation of preferences.

Ghirardato and Marinacci definition is the most widely used and plays a pivotal role in Uncertainty Economics, for its ability to translate in a tractable way in the mathematical representation of preferences. However, it sparked a great debate for its main flaws, first and foremost the fact that risk attitude is not allowed to change when analyzing two different individuals. In fact, even if it is a strength of the definition, because it allows to investigate just ambiguity ignoring risk, it fails to provide a meaningful analysis of uncertainty as a whole.

One of the ways in which such goal can be achieved is to compare preferences on the choice between purely ambiguous acts, that means acts that maps states into constant outcomes, with purely risky acts, that is the constant acts used also by Ghirardato and Marinacci. Note that purely ambiguous acts have the same features of acts in the Savage setting, but they are used in an Anscombe-Aumann setting.

In this way, the current work hopes to overcome the problem of fixing risk attitudes to compare ambiguity attitudes. However, it is clear that, in this way, the focus is not on ambiguity anymore, since preferences will differ both in risk and ambiguity attitudes. Therefore, it is convenient to give a different definition to the newly introduced notion. The name *willingness to bet* has been chosen since betting can be viewed as a purely ambiguous act, that is a gamble whose uncertainty is resolved once the state of the world has been revealed.

2.1 Wealth Effects: a preferential viewpoint

The framework of comparative attitudes toward uncertainty as presented in the previous Section is thought to be applied in an inter-personal situation. That means that the two preferences to be compared are the preferences of two distinct agents, and all results that arise are to be applied to such agents. However, following Cerreia-Vioglio et al. [5], it is possible, and very convenient from a theoretical perspective, to apply such comparative analysis to the preferences of a single agent that change according to the agent's wealth level. This is done to study the effects that a shift in wealth has on the agent's ranking of acts. This means moving from an inter-personal comparison of preferences to an intra-personal one.

However, to perform the same type of analysis introduced in the previous subsection, it is important to maintain a preferential viewpoint, something that is not always used in the literature. For this reason, Cerreia-Vioglio et al. [5] introduce the concept of "translation of preferences", i.e. they define, for some $\succsim$ and for every level of wealth $w \in \mathbb{R}$, a different preference $\succsim^w$ that ranks acts "shifting" their monetary outcomes of the same amount of the wealth that those preferences are representing. In this way it is possible to compare preferences corresponding to different amount of wealth $w' > w$, labelled $\succsim^{w'}$ and $\succsim^w$, in the same way one compares preferences of two different agents $\succsim_1$ and $\succsim_2$.

Cerreia-Vioglio et al. [5] use this type of approach to characterize changing in absolute and relative ambiguity aversion, as wealth level increases. Following closely their paper, the current work will try to give a characterization of the preference operator corresponding to decreasing, increasing or constant absolute willingness to bet, keeping in mind the results obtained from its definition.

3 Mathematical Preliminaries

The environment considered is a generalized version of the Anscombe and Aumann [1] setup. There is a nonempty set S of *states of the world*, an algebra Σ of subsets of S called *events*, and a nonempty convex set X of *consequences*. We denote by $\mathcal{F}$ the set of all *acts*: functions $f : S \to X$ that are Σ-measurable.

Given any $x \in X$, define $x \in \mathcal{F}$ to be the constant act that takes value x. Thus, with a slight abuse of notation, X is identified with the subset of constant acts in $\mathcal{F}$. Since X is nonempty and convex, it is possible to define a mixture operation over $\mathcal{F}$. For each $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, the act $\alpha f + (1-\alpha)g \in \mathcal{F}$ is defined to be such that $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s) \in X$ for all $s \in S$. A binary relation $\succsim$ on $\mathcal{F}$ is called *preference*, according to which acts in $\mathcal{F}$ are ordered. Thus, for each $f \in \mathcal{F}$ we denote by $x_f \in X$ a certainty equivalent of f, that is, $x_f \sim f$.

Given a function $u : X \to \mathbb{R}$, let $\operatorname{Im} u$ be the set $u(X)$. Furthermore, let $A \subseteq \mathbb{R}$, then $B_0(\Sigma, A)$ indicates the set of all functions $\varphi : S \to \mathbb{R}$ which are Σ-measurable and have values in A. Observe that $u \circ f \in B_0(\Sigma, \operatorname{Im} u)$ when $f \in \mathcal{F}$.

The most commonly accepted classes of preferences $\succsim$ on $\mathcal{F}$ in uncertainty economics all rely on a set of common axioms, discussed in the original papers as well as in Gilboa and Marinacci [13].

Axiom A. 1 (Weak Order) *$\succsim$ is nontrivial, complete, and transitive.*

Axiom A. 2 (Monotonicity) *If $f, g \in \mathcal{F}$ and $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.*

Axiom A. 3 (Continuity) *If $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f + (1-\alpha)g \succsim h\}$ and $\{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha)g\}$ are closed.*

Axiom A. 4 (Risk Independence) *If $x, y, z \in X$ and $\alpha \in (0,1)$,*

$$x \sim y \Longrightarrow \alpha x + (1-\alpha) z \sim \alpha y + (1-\alpha) z.$$

Axiom A. 5 (Convexity) *If $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$,*

$$f \sim g \Longrightarrow \alpha f + (1-\alpha) g \succsim f.$$

Axiom A. 6 (Weak C-Independence) *If $f, g \in \mathcal{F}$, $x, y \in X$, and $\alpha \in (0,1)$,*

$$\alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x \Longrightarrow \alpha f + (1-\alpha)y \succsim \alpha g + (1-\alpha)y.$$

Axiom A. 7 (C-Independence) *If $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in (0,1)$,*

$$f \succsim g \iff \alpha f + (1-\alpha)x \succsim \alpha g + (1-\alpha)x.$$

Axiom A. 8 (Unboundedness) *There exist x and y in X such that $x \succ y$ and for each $\alpha \in (0,1)$ there exists $z \in X$ that satisfies either*

$$y \succ \alpha z + (1-\alpha)x$$

or

$$\alpha z + (1-\alpha)y \succ x.$$

Each axiom corresponds to a different behavioral interpretation. In fact, Convexity states that the agent prefers mixing acts to single acts, meaning that she is better off hedging; C-Independence and Weak C-Independence are two forms of independence from mixing acts with constant acts, and the basic idea is that mixing acts with constant acts does not change the preference of the agent.

Theorem 1 (Omnibus) *A preference $\succsim$ on $\mathcal{F}$ satisfies Weak Order, Monotonicity, Continuity, and Risk Independence if and only if there exist a nonconstant and affine function $u : X \to \mathbb{R}$ and a normalized, monotone, and continuous functional $I : B_0(\Sigma, \operatorname{Im} u) \to \mathbb{R}$ such that the criterion $V : \mathcal{F} \to \mathbb{R}$, given by*

$$V(f) = I(u(f)) \qquad \forall f \in \mathcal{F} \tag{1}$$

represents $\succsim$. The function u is cardinally unique and, given u, I is the unique normalized, monotone, and continuous functional satisfying (1). In this case, we say that $\succsim$ is a rational preference*. Furthermore, a rational preference satisfies:*

(i) C-Independence if and only if I is constant linear; in this case, we say that $\succsim$ is an invariant biseparable preference*.*

(ii) Convexity if and only if I is quasiconcave; in this case, we say that $\succsim$ is an uncertainty averse preference*.*

(iii) Convexity and Weak C-Independence if and only if I is quasiconcave and constant additive; in this case, we say that $\succsim$ is a variational preference*.*

(iv) *Convexity and C-Independence if and only if I is quasiconcave and constant linear; in this case, we say that $\succsim$ is a* maxmin preference.

(v) *Unboundedness if and only if* $\operatorname{Im} u$ *is unbounded.*

In order to link the generic mathematical structure defined above with more common economic situations, it will be assumed from now on that the set of consequences X is the set of simple monetary lotteries, that is either $X = \Delta_0(\mathbb{R})$ or $X = \Delta_0(\mathbb{R}_{++})$, where

$$\Delta_0(\mathbb{R}) = \left\{ x \in [0,1]^{\mathbb{R}} : x(c) \neq 0 \text{ for finitely many } c \in \mathbb{R} \text{ and } \sum_{c \in \mathbb{R}} x(c) = 1 \right\}.$$

Thus, for a given $c \in \mathbb{R}$ and $x \in X$, $x(c)$ indicates the probability assigned to c by lottery x, whereas for a generic $f \in \mathcal{F}$, the value $f(c \mid s)$ indicates the probability assigned to c by act f under state $s \in S$. In such monetary framework, note that given f, x_f is a lottery that, received with certainty in each state s, is indifferent to f. Thus, x_f is a *risky* prospect, independent of the realization on S.

It is useful to spend some more words on the von Neumann-Morgenstern utility index, in order to have a regular function to work with. In fact, on the top of the axioms listed above, it is possible to introduce other two behavioral assumptions, called Regularity axioms.

Axiom A. 9 (Certainty Equivalence) *For every $x \in X$ there exists a unique $r \in \mathbb{R}$ such that $x \sim \delta_r$.*

Axiom A. 10 (Dominance) *Let $x, y \in \Delta_0(\mathbb{R})$. Then, $x([r, +\infty)) \geq y([r, +\infty))$ for every $r \in \mathbb{R} \implies x \succsim y$.*

Remember that δ_r indicates the Dirac distribution, i.e. the distribution that assigns all the probability mass to r. With these new axioms, which date back to the von Neumann-Morgenstern work, it is possible to characterize the utility function. In fact, the following theorem, essentially due to Kreps [19] holds.

Theorem 2 (Utility Regulairty) *If a preference $\succsim$ is rational (i.e. it satisfies Weak Order, Monotonicity, Continuity and Risk Independence), it satisfies also Certainty Equivalence and Dominance if and only if the utility function $v : \mathbb{R} \to \mathbb{R}$ is strictly increasing and and continuous.*

Notice that for a generic act $f \in \mathcal{F}$, the affine utility $u(f)$ is a map $u(f) : S \to \mathbb{R}$, given by, for every $s \in S$,

$$u(f)(s) = u(f(s)) = \int_{\mathbb{R}} v df(\cdot \mid s).$$

Since acts map states in finite lotteries only, the last equation can be rewritten eliminating the integral, and yielding the more compact form

$$u(f)(s) = \sum_{c\in\mathbb{R}} v\,(c)\, f(c \mid s).$$

Lastly, such function is measurable, as already observed above, i.e. $u(f) \in B_0(S, \Sigma, v(\mathbb{R}))$.

4 Willingness to bet

As stated in the introduction, I want to restrict the definition of ambiguity aversion of Ghirardato and Marinacci to purely ambiguous acts, i.e. acts for which all uncertainty is resolved once the state of the world is revealed. Notice that this is exactly the definition of acts in the Savage set up.

Let $\mathcal{G} \subseteq \mathcal{F}$ such that

$$\mathcal{G} = \left\{ g \in \mathcal{F} : \forall s \in S\ \exists c(s) \in \mathbb{R},\, g(s) = \delta_{c(s)} \right\}.$$

Purely ambiguous acts map states into degenerate lotteries that are distributed as a Dirac on a specific outcome, which is monetary in the present case. With a slight abuse of notation, $g(s)$ will be used both for the value of g at state s, that is $c(s)$, and the act evaluated at s, so that the affine von Neumann-Morgenstern index becomes $u(g(s)) = \sum_{c\in\mathbb{R}} g(c|s)v(c) = \delta_{c(s)}v(c) = v(c(s)) = v(g(s))$.

Let's now use this subset of acts do define formally the willingness to bet introduced at the beginning.

Definition 1 *We say that $\succsim_2$ is more willing to bet than $\succsim_1$ if given any $g \in \mathcal{G}$ and $x \in \Delta_0(\mathbb{R})$, $g \succsim_1 x \implies g \succsim_2 x$.*

As already said, this definition is not fixing risk attitudes of the DMs to compare just their ambiguity attitudes, but it is comparing uncertainty on both dimensions. This justifies from an intuitive point of view the next Proposition.

Proposition 1 *Given two preferences $\succsim_1$and $\succsim_2$, and their representations (u_1, I_1) and (u_2, I_2) respectively, the following two conditions are equivalent:*

i) $\succsim_2$ is more willing to bet than $\succsim_1$

ii) $\succsim_2$ is more risk averse than $\succsim_1$and, given the convex, increasing and continuous function $\phi : \operatorname{Im} v_2 \to \operatorname{Im} v_1$ defined as $\phi = v_1 \circ v_2^{-1}$, it holds that

$$I_1(\varphi) \leq \phi\left(I_2\left(\phi^{-1}\left(\varphi\right)\right)\right) \qquad \forall\varphi \in B_0\left(S, \Sigma, v_1\left(\mathbb{R}\right)\right). \tag{2}$$

Notice that the inequality 2 can be restated with the lighter notation

$$I_1(\phi(\varphi)) \leq \phi(I_2(\varphi)).$$

This last formula will be useful in later Sections.

Of course, willingness to bet and ambiguity aversion in the sense of Ghirardato and Marinacci [9] are related. In fact, if different risk attitudes are removed from the picture, it is easy to see, like in the next Proposition, that the two definitions coincide.

Proposition 2 *If $\succsim_1$ is more risk averse than $\succsim_2$ and $\succsim_1$ is less willing to bet than $\succsim_2$ then $\succsim_1$ is more ambiguity averse than $\succsim_2$.*

Dropping the assumption that the attitude toward risk is the same, however, does not guarantee that the two definitions to coincide. In fact, the simple fact that the two preferences do not share the same risk attitude rules out the possibility that they can be compared in the context of ambiguity in the sense of Ghirardato and Marinacci.

5 Wealth Effects

5.1 Induced Preferences

Let $w \in \mathbb{R}$, and consider affine maps $^w : X \to X$, that is, $(\alpha x + (1-\alpha) y)^w = \alpha x^w + (1-\alpha) y^w$ for all $x, y \in X$ and $\alpha \in [0,1]$. It is possible to extend these maps to $\mathcal{F}$ by defining $f \mapsto f^w$ where is defined as $f^w(s) = f(s)^w$ for all $s \in S$. Attention will be focused on sets X of monetary simple lotteries and affine maps induced by wealth shifts that are additive, that is

$$x^w(c) = x(c-w) \qquad \forall c \in \mathbb{R}.$$

A preference $\succsim$ on $\mathcal{F}$ induces, through an affine and bijective transformation ^{w} on X, a preference $\succsim^w$ on $\mathcal{F}$ given by

$$f \succsim^w g \Longleftrightarrow f^w \succsim g^w.$$

Cerreia-Vioglio et al. proved that the original preference transmits some of its properties to the induced preference (Cerreia-Vioglio et al. [5], Proposition 1).

Proposition 3 (Cerreia-Vioglio et al.) *Let $\succsim$ be a preference on $\mathcal{F}$ and $^w : X \to X$ an affine bijection. Then:*

(i) If $\succsim$ is a rational preference, so is $\succsim^w$.

(ii) If $\succsim$ is an uncertainty averse preference, so is $\succsim^w$.

To understand the reason behind the analysis of such transformations, it is enough to interpret w as the wealth level of the agent. Therefore, x^w is representing the actual lottery that an agent with wealth w is facing when she chooses x. Thus, map ^{w} is "shifting" the prizes of the lotteries along $\mathbb{R}$. Furthermore, $\succsim^w$ can be interpreted as the preference of an agent with original preferences $\succsim$ and endowed with wealth w.

Consider a rational preference $\succsim$ with canonical representation (u, I), where u has von Neumann-Morgenstern utility $v : \mathbb{R} \to \mathbb{R}$. Fix $w \in \mathbb{R}$. Define $v_w : \mathbb{R} \to \mathbb{R}$ to be such that $v_w(c) = v(c + w)$ for all $c \in \mathbb{R}$ and $u_w : \Delta_0(\mathbb{R}) \to \mathbb{R}$ to be the associated expected utility Note that $\operatorname{Im} u_w = \operatorname{Im} v_w \subseteq \operatorname{Im} v = \operatorname{Im} u$ for all $w \in \mathbb{R}$.

The next proposition puts together the representation theorem with the preferential approach to wealth level.

Proposition 4 *Assume $\succsim$ is a rational preference represented by (u, I). Then, for any w, rational preference $\succsim^w$ is represented by (u_w, I).*

The implication of Proposition 4 is pivotal. In fact, since by Proposition 3 $\succsim^w$ is rational, it can be represented, by Theorem 1, by the pair (u_w, I_w). It follows that DM ranks acts according to the functional

$$V_w(f) = I_w\left(u_w\left(f\right)\right) = I_w\left(u\left(f^w\right)\right) \qquad \forall f \in \mathcal{F}.$$

However, Proposition 4 shows that operator I is not dependent by the wealth level.

Now that the general framework for the preferential approach to studying wealth effects on agents' choices has been presented, changes in willingness to bet due to different wealth levels are studied more in detail.

5.2 Willingness to bet and Wealth

Definition 2 *A DM is Increasing (resp. Decreasing) Absolute Willing to Bet (IAWB, resp. DAWB) if $\succsim^{w'}$ is more (resp. less) willing to bet than $\succsim^w$ provided $w' > w$.*

As a direct consequence of Proposition 1, we can state the following.

Proposition 5 *Let $\succsim$ be a rational preference represented by (u, I), with a von Neumann-Morgenstern utility function v. Then, the following conditions hold:*

i) $\succsim$ is IAWB if and only if it is IARA and

$$I\left(v\left(v^{-1}\left(\varphi\right) - k\right)\right) \leq v\left(v^{-1}\left(I\left(\varphi\right)\right) - k\right) \qquad \forall k > 0, \varphi \in B_0(S, \Sigma);$$

ii) $\succsim$ is DAWB if and only if it is DARA and

$$I\left(v\left(v^{-1}\left(\varphi\right) + k\right)\right) \leq v\left(v^{-1}\left(I\left(\varphi\right)\right) + k\right) \qquad \forall k > 0, \varphi \in B_0(S, \Sigma);$$

iii) if $\succsim$ is CARA, it is DAWB if and only if it is Increasing Absolute Ambiguity Aversion (IAAA), whereas it is IAWB if and only if it is Decreasing Absolute Ambiguity Aversion (DAAA).

In particular, for the last case of the previous proposition the analysis of this kind of preferences have been fully investigated by Cerreia-Vioglio et al.[5].

Let, for a given act $f \in \mathcal{F}$, $c(f^k)$ be that number such that $\delta_{c(f^k)} \sim f^k$. A preference $\succsim$ is said to be wealth subadditive if $c(f^k) \leq c(f)+k$, and wealth superaddtive if $c(f^k) \geq c(f)+k$. An immediate and interesting corollary of the result above is the following.

Corollary 1 *Let $\succsim$ be a rational preference. Then:*

i) if $\succsim$ is DARA and $c(f^k) \leq c(f) + k$, then it is DAWB;

ii) if $\succsim$ is IARA and $c(f^k) \geq c(f) + k$, then it is IAWB.

Corollary 1 is interesting from an experimental point of view. In fact, it is testable whether an individual is DARA or IARA, but most importantly it is testable whether it is wealth subadditive or wealth superadditive. Corollary 1 allows to determine whether a preference is DAWB or IAWB, starting from the testable conditions on absolute risk attitude and the effect of wealth on the certain equivalents.

It is commonly assumed in economics, since the seminal work of Arrow [2, p. 96], that individuals display decreasing absolute risk aversion: the richer they are, the more they want to risk. Such assumption seems intuitively appealing, even though experimental economics provides a massive quantity of evidence that it may be flawed. However, taking for granted such statement, as done in many branches of economic research, an economy should be populated by agents that are DARA, which would mean, provided the conditions of Proposition 5 hold, agents are also decreasing absolute willing to bet (DAWB). This may go against the common sense that sees people more keen to bet when they become richer, but indeed may be heuristically explained by intrinsic features of wealthier people: caution is what made them wealthy, therefore they stick to their cautious decision rules, but when probabilistic computation arises, they are willing to embark in risky enterprises. Another possible explanation of these characteristics of economic agents would be that the poorer the agent, the more she hopes to improve its condition in a purely ambiguous bet rather than in a lottery whose outcome are already known, and thus leaves no room for hope.

Of course this is a non-scientific approach to the issue, since people's attitudes are being discussed from a purely intuitive point of view. A formal experimental design would be needed to falsify all the previous argument. Anyway, such discussion highlights the need for understanding the behavior of willingness to bet subject to wealth changes, focusing on the case in which DM is DARA and DAWB.

To better understand the mathematical behavior of the von Neumann Morgenstern utility function under wealth shifts, Consider the following Lemma, from Cerreia-Vioglio et al.[5].

Lemma 1 *Let $\succsim$ be a rational preference. Then:*

i) If $\succsim$ is DARA,

$$\frac{\phi_{w,w'}(u) - \phi_{w,w'}(t)}{u - t} \leq 1 \qquad \forall u, t \in \operatorname{Im} v_w$$

and, in particular, if $\phi_{w,w'}$ is differentiable, its derivative $\phi'_{w,w'}(t) \in (0,1)$ for every w, w';

ii) If $\succsim$ is IARA, it holds that

$$\frac{\phi_{w,w'}(u) - \phi_{w,w'}(t)}{u - t} \geq 1 \qquad \forall u, t \in \operatorname{Im} v_w$$

and, in particular, if $\phi_{w,w'}$ is differentiable, $\phi'_{w,w'}(t) \geq 1$ for every w, w'.

Another important result, which is strictly mathematical, but which is also very useful for the following analysis of preferences, is the following.

Lemma 2 *Let $D \subseteq \mathbb{R}$, and $\phi : D \to \mathbb{R}$ an increasing, continuous, convex but not affine function. Then there exists a point $c \in D$ such that, for some $\beta \in \partial\phi(c)$ it holds that*

$$\phi(t) > \phi(c) + \beta(t - c) \qquad \forall t \neq c.$$

The next result links absolute willingness to bet with absolute ambiguity attitudes for a wide range of preferences, namely those whose representation functional I is translation invariant and subhomogeneous. In particular, translation invariance means that $I(\varphi + k) = I(\varphi) + k$ for all $k \in \mathbb{R}$, whereas subhomogeneity means $I(\lambda\varphi) \geq \lambda I(\varphi)$ for all $\lambda \in (0,1)$.

Proposition 6 *Let $\succsim$ be a rational preference such that I is strictly monotone, translation invariant and subhomogeneous. Then, if is $\succsim$ DAWB, then $\succsim$ is CARA and increasing absolute ambiguity averse.*

The conditions imposed by the hypotheses of Proposition 6 on operator I are valid for an important class of preferences introduced at the beginning, that is Variational Preferences:

$$\begin{aligned}\min_{p \in \Delta} \left\{ \int (\varphi + k) dp + c(p) \right\} &= \min_{p \in \Delta} \left\{ \int \varphi dp + k + c(p) \right\} \\ &= \min_{p \in \Delta} \left\{ \int \varphi dp + c(p) \right\} + k \qquad \forall k \in \mathbb{R}, \varphi \in B_0(S, \Sigma),\end{aligned}$$

and

$$\min_{p\in\Delta}\left\{\int \lambda\varphi dp + c(p)\right\} \geq \min_{p\in\Delta}\left\{\lambda\int \varphi dp + \lambda c(p)\right\}$$
$$= \lambda \min_{p\in\Delta}\left\{\int \varphi dp + c(p)\right\} \qquad \forall\lambda\in(0,1), \varphi\in B_0(S,\Sigma).$$

Apart from being a quite general class, in the sense that they encompass also invariant biseparable preferences, the advantage of Variational preferences is their ever growing application, especially in intertemporal analysis in Finance and Financial Macroeconomics. In fact, Hansen and Sargent [15] work on robust DSGE models is built on a version of Variational Preferences.

Notice that the specular version of Proposition 6 can be stated in the following way.

Proposition 7 *Let $\succsim$ be a rational preference such that I is invariant biseparable and strictly monotone. Then, if is $\succsim$ IAWB, then $\succsim$ is CARA and decreasing absolute ambiguity averse.*

The proof will not be explicitly written, as it is almost identical to the one of Proposition 6. However it must be noted that now the stronger requirement of invariant biseparability is being imposed. Invariant biseparability guarantees that $I(\lambda\varphi + k) = \lambda I(\varphi) + k$ for all $\lambda, k \in \mathbb{R}_+$ and all $\varphi \in B_0(S,\Sigma)$. This is due to the fact that, since the incremental ratios of the function ϕ for a IARA preference (linked to the IAWB) are greater than 1, subhomogeneity is not enough to disprove equality 5, for the sake of contradiction. The importance of such result is remarkable anyway: invariant biseparable preferences encompass SEU, CEU and MEU preferences, and they are all of which are widely used in many applications.

6 Conclusion

The understanding of comparative attitude toward uncertainty is pivotal in many branches of applied economics, for two main reasons. First of all, it allows for comparative statics in an economy populated by heterogeneous individuals. For example, the general equilibrium of a market with uncertainty and with multiple agents, all of whom have different attitudes toward uncertainty, may depend heavily on how different such attitudes are. In this regard, the first part of this paper tries to tackle exactly this problem. Secondly, it allows for a better comprehension on how choices are shaped when agents increase or decrease their wealth, and here the reference to the second part of the present work is clear. It is enough to consider the relationship between wealth and, for example, portfolio allocation problems and insurance demand.

Therefore, it is clear that studies in this field are ever more essential, especially in an historical period in which the classical paradigm of risk analysis has failed to predict and prevent certain disruptive phenomena. This work's aim is to make some small steps forward in understanding the subject of comparative uncertainty aversion. This has been done by trying to weaken the dominant definition of Ghirardato and Marinacci, who already made a great job in expanding Epstein's ideas, and then applying the new concepts to the analysis of wealth effects. The most staggering results are probably those that bridge the definition of willingness to bet and comparative ambiguity aversion, especially because such results deal with classes of preferences that are ever more widespread in many applications.

In conclusion, I hope that this work can be considered as a possible road toward a more complete understanding of choices under uncertainty.

7 Appendix: Proofs

Proof of Theorem 2. By the von Neumann Morgenstern Theorem, for rational preferences the existence of the function v in the statement is proven.

Let $c' > c$. For $\delta_x \in \Delta_0(\mathbb{R})$,

$$\delta_x([r,+\infty)) = \begin{cases} 0 & r > x \\ 1 & r \leq x \end{cases}.$$

Therefore, if $r \geq c'$, then $\delta_{c'}([r,+\infty)) = \delta_c([r,+\infty)) = 0$, if $r < c$ then $\delta_{c'}([r,+\infty)) = \delta_c([r,+\infty)) = 1$, and if $c \leq r < c'$ then $1 = \delta_{c'}([r,+\infty)) > \delta_c([r,+\infty)) = 0$. It is possible to conclude that $\delta_{c'}([r,+\infty)) \geq \delta_c([r,+\infty))$ for all $r \in \mathbb{R}$, which implies, by Dominance Axiom, that $\delta_{c'} \succsim \delta_c$. But this, in turn, implies that $c' > c \implies v(c') \geq v(c)$, thus v is increasing.

Assume v is not strictly increasing. Then, there exist c, c' such that $v(c) = v(c')$. But then, consider the lottery $l = \frac{1}{2}\delta_c + \frac{1}{2}\delta_{c'}$, which yields a utility $\frac{1}{2}v(c) + \frac{1}{2}v(c') = v(c) = v(c')$. But then

$$\delta_c \sim \frac{1}{2}\delta_c + \frac{1}{2}\delta_{c'} \sim \delta_{c'}$$

which means that the lottery has more than one certain equivalent, contradicting Certainty Equivalence Axiom. Therefore, function v must be strictly increasing.

Next, assume that v is not continuous. Since it is increasing, it implies that there is a countable set of discontinuity points, and that for a given discontinuity point z it holds that

$$v_-(z) = \lim_{c \to z^-} v(c) < \lim_{c \to z^+} v(c) = v_+(z).$$

Take a lottery $x \in X$ such that $u(x) = k$ with $k \in (v_-(z), v_+(z))$. This means that there is no $r \in \mathbb{R}$ such that $v(r) = u(x)$, meaning that there exists no $r \in \mathbb{R}$ such that $x \sim \delta_r$. This fact contradicts the Certainty Equivalence Axiom.

In conclusion, $v : \mathbb{R} \to \mathbb{R}$ must be increasing and continuous. ∎

Proof of Proposition 1. $i) \implies ii)$. Set $\delta_c \in \mathcal{G}$ as the constant act that assigns probability 1 to $c \in \mathbb{R}$. for every sate $s \in S$. Then we can say that $\delta_c \succsim_1 x \implies \delta_c \succsim_2 x$ for every constant act (i.e. every lottery) x. But this implies $x \succ_2 \delta_c \implies x \succ_1 \delta_c$, which is exactly the definition of more risk aversion.

The latter fact implies that $v_1 = \phi \circ v_2$ for some $\phi : \operatorname{Im} v_2 \to \operatorname{Im} v_1$ convex. Since both functions v_2 and v_1 are increasing and continuous, so it is ϕ.

Now, let $\varphi \in B_0(S, \Sigma, v(\mathbb{R}))$, then we have an AA act $f \in \mathcal{F}$ such that $\varphi = u_1(f)$. Fix a state $s \in S$, and take the certain equivalent of the lottery $f(s)$ defined by $c_f(s) = v_1^{-1}(u_1(f(s)))$. Define now the purely ambiguous act $g(s) = c_f(s)$ for every $s \in S$. It is clear that $g(s) \sim_1 f(s)$ for every $s \in S$, so, by monotonicity axiom, we can state that $g \sim_1 f$.

Take now the certain equivalent of act g defined by $c(g) = v_1^{-1}(I_1(v_1(g))$, which can then be considered a degenerate lottery that assigns probability 1 to $c(g) \in \mathbb{R}$. By hypothesis it is true that

$$g \sim_1 \delta_{c(g)} \implies g \succsim_2 \delta_{c(g)}$$

which implies, from the point of view of representations, that

$$I_1(\varphi) = I_1(u_1(f)) = I_1(v_1(g)) = v_1(c(g)) = \phi^{-1}(v_2(c_g))$$

and that

$$v_2(c(g)) \leq I_2(v(g)) = I_2(\phi^{-1}(v_1(g))) = I_2\left(\phi^{-1}(\varphi)\right).$$

Combining the two relations one obtains

$$\phi^{-1}(I_1(\varphi)) \leq I_2\left(\phi^{-1}(\varphi)\right)$$

and since ϕ is strictly increasing, as has been noted before, we can invert it and obtain

$$I_1(\varphi) \leq \phi\left(I_2\left(\phi^{-1}(\varphi)\right)\right).$$

ii) $\implies$ *i)*. Assume $\succsim_2$ is more risk averse than $\succsim_1$ and therefore there exists a convex function $\phi : \operatorname{Im} v_2 \to \operatorname{Im} v_1$ such that $v_1 = \phi \circ v_2$. Let $x \in X$ and $g \in \mathcal{F}_0$ such that $g \succsim_1 x$. Then, this implies

$$u_1(x) \leq I_1(v_1(g)) \leq \phi\left(I_2\left(\phi^{-1}(v_1(g))\right)\right) = \phi(I_2(v_2(g))).$$

Since ϕ and ϕ^{-1} are strictly increasing, it is possible to state that

$$\phi^{-1}(u_1(x)) \leq I_2(v_2(g)).$$

However, notice that

$$\begin{aligned} u_2(x) &= \sum_{c\in\mathbb{R}} v_2(c)x(c) = \sum_{c\in\mathbb{R}} \phi^{-1}(v_1(c))\,x(c) \\ &\leq \phi^{-1}\left(\sum_{c\in\mathbb{R}} v_1(c)x(c)\right) = \phi^{-1}(u_1(x)) \end{aligned}$$

where the inequality follows from Jensen's, since ϕ^{-1} is concave.

Thus, all the above imply

$$u_2(x) \leq I_2(v_2(g))$$

which in turn implies that $g \succsim_2 x$.

We have thus proven that

$$g \succsim_1 x \implies g \succsim_2 x.$$

■

Proof of Proposition 2. By Proposition 1 it must be true that $\succsim_2$ is more risk averse than $\succsim_1$, so, if $\succsim_1$ is more risk averse than $\succsim_2$, the two preferences share the same attitude toward risk, so that we can conclude that ϕ is an affine function. Normalizing, it is possible to impose that ϕ is the identity, so $v_1 = v_2 = v$ and $u_1 = u_2 = u$.

Thus, since by previous proposition we have

$$I_1(\varphi) \leq I_2(\varphi),$$

suppose $f \in \mathcal{F}$ and $f \succsim_1 x$. Then

$$u(x) \leq I_1(u(f)) \leq I_2(u(f))$$

which implies $f \succsim_2 x$. It is possible to conclude that $f \succsim_1 x \implies f \succsim_2 x$. ■

Proof of Proposition 4. Since by hypothesis $\succsim$ is rational, it can be represented, by Theorem 1, by (u, I). So, we have that

$$\begin{aligned}
&f \succsim^w g \iff f^w \succsim g^w \iff I(u(f^w)) \geq I(u(g^w)) \\
&\iff I(\int_{\mathbb{R}} v df^w) \geq I(\int_{\mathbb{R}} v dg^w) \\
&\iff I(\sum_{c\in\mathbb{R}} v(c) f(c-w|\cdot)) \geq I(\sum_{c\in\mathbb{R}} v(c) g(c-w|\cdot)) \\
&\iff I(\sum_{t\in\mathbb{R}} v(t+w) f(t|\cdot)) \geq I(\sum_{t\in\mathbb{R}} v(t+w) g(t|\cdot)) \\
&\iff I(\sum_{t\in\mathbb{R}} v_w(t) f(t|\cdot)) \geq I(\sum_{t\in\mathbb{R}} v_w(t) g(t|\cdot)) \\
&\iff I(\int_{\mathbb{R}} v_w df) \geq I(\int_{\mathbb{R}} v_w dg) \\
&\iff I(u_w(f)) \geq I(u_w(g)) \quad \forall f, g \in \mathcal{F}.
\end{aligned}$$

Therefore, we can state that $\succsim^w$ is represented by (u_w, I). ■

Proof of Proposition 5. i) "Only if": assume $\succsim$ is IAWB. This implies that $\succsim^{w'}$ is more willing to bet than $\succsim^w$, given $w' > w \geq 0$. Then $\succsim^{w'}$ is more risk averse than $\succsim^w$, by Proposition 1, which in turn implies that $\succsim$ is IARA.

Assume $w = w' + k$ for $k > 0$. Thus, consider the function $\phi_{w,w'} = v_{w'} \circ v_w^{-1}$, which is convex by the fact that $\succsim$ is IARA, for any w, w'. The explicit form of such function is given by

$$\begin{aligned}\phi_{w,w'}(t) &= \left(v_{w'} \circ v_w^{-1}\right)(t) = v_{w'}(v_w^{-1}(t)) \\ &= v(v_w^{-1}(t) + w') \\ &= v(v^{-1}(t) - w + w') \qquad \forall w > w' \geq 0, t \in \operatorname{Im} v.\end{aligned}$$

The above yields

$$\phi_{w,w'}(t) = v(v^{-1}(t) - k) \qquad \forall k > 0, t \in \operatorname{Im} v. \tag{3}$$

Thus, remember that by Proposition 1, the following relationship holds

$$I_w(\phi_{w,w'}(\varphi)) \leq \phi_{w,w'}(I_{w'}(\varphi)) \qquad \forall w' > w \geq 0, \varphi \in B_0(S, \Sigma)$$

which becomes, thanks to Equation 3 and Proposition 4,

$$I\left(v\left(v^{-1}\left(\varphi\right) - k\right)\right) \leq v\left(v^{-1}\left(I\left(\varphi\right)\right) - k\right) \qquad \forall k > 0, \varphi \in B_0(S, \Sigma)$$

The "if" part is trivial provided Proposition 1.

ii) "Only if": assume $\succsim$ is DAWB. This implies that $\succsim^w$ is more willing to bet than $\succsim^{w'}$, given $w' > w \geq 0$. Then $\succsim^w$ is more risk averse than $\succsim^{w'}$, by Proposition 1, which in turn implies that $\succsim$ is DARA.

Again, let $w' = w + k$. Thus, consider the function $\phi_{w,w'} = v_{w'} \circ v_w^{-1}$, which is convex by the fact that $\succsim$ is DARA, for any w, w'. The explicit form of such function is given by

$$\begin{aligned}\phi_{w,w'}(t) &= \left(v_{w'} \circ v_w^{-1}\right)(t) = v_{w'}(v_w^{-1}(t)) \\ &= v(v_w^{-1}(t) + w') \\ &= v(v^{-1}(t) - w + w') \qquad \forall w' > w \geq 0, t \in \operatorname{Im} v\end{aligned}$$

Then, letting again $w' = w + k$,

$$\phi_{w,w'}(t) = v(v^{-1}(t) + k) \qquad \forall k > 0, t \in \operatorname{Im} v. \tag{4}$$

Thus, remember that by 1, the following relationship holds

$$I_{w'}(\phi_{w,w'}(\varphi)) \leq \phi_{w,w'}(I_w(\varphi)) \qquad \forall w' > w \geq 0, \varphi \in B_0(S, \Sigma)$$

which becomes, thanks to Equation 4 and Proposition 4,

$$I\left(v\left(v^{-1}\left(\varphi\right) + k\right)\right) \leq v\left(v^{-1}\left(I\left(\varphi\right)\right) + k\right) \qquad \forall k > 0, \varphi \in B_0(S, \Sigma)$$

As above, the "if" part is trivial.

iii) Assume $\succsim$ is CARA. This means that $\succsim^{w'}$ and $\succsim^{w}$ share the same risk attitude, thus, by Proposition 2, the definition of comparative willingness to bet and comparative ambiguity aversion coincide.

In particular, if $\succsim$ is IAWB, $\succsim^{w'}$ is more willing to bet than $\succsim^{w}$ for all $w' > w \geq 0$, which implies $\succsim^{w'}$ is less ambiguity averse than $\succsim^{w}$ for all $w' > w \geq 0$, which in turn implies that $\succsim$ is decreasing absolute ambiguity averse (DAAA).

On the other hand, if $\succsim$ is DAWB, $\succsim^{w'}$ is less willing to bet than $\succsim^{w}$ for all $w' > w \geq 0$, which implies $\succsim^{w'}$ is more ambiguity averse than $\succsim^{w}$ for all $w' > w \geq 0$, which in turn implies that $\succsim$ is increasing absolute ambiguity averse (IAAA).

In both cases, the converse is trivially true: if $\succsim$ is DAAA (resp. IAAA) it has to be CARA. Furthermore, $\succsim^{w'}$ is less (resp. more) ambiguity averse than $\succsim^{w}$, it means that $\succsim^{w'}$ is more (resp. less) willing to bet than $\succsim^{w}$ with $w' > w \geq 0$. This also implies that $\succsim$ is IAWB (resp. DAWB). ■

Proof of Corollary 1. i) Let, for any act $f \in \mathcal{F}$ the function g be defined by $g(s) = v^{-1}(u(f(s))$ for every $s \in S$. Then, clearly, we have that $f(s) \sim g(s)$ for all $s \in S$. However, since $\succsim$ is DARA, the indifference relation above implies, for a certain $k > 0$, $f(s) \succsim^{k} g(s)$ for all $s \in S$, which then, by monotonicity of $\succsim$, means $f \succsim^{k} g$. But then,

$$\begin{aligned} f \succsim^{k} g &\Longleftrightarrow f^{k} \succsim g^{k} \\ &\Longleftrightarrow \delta_{c(f^{k})} \succsim g^{k} \\ &\Longleftrightarrow v(c(f^{k})) \geq I(v(g^{k})) \\ &\Longleftrightarrow v(c(f^{k})) \geq I(v(g+k)). \end{aligned}$$

Now, let $\varphi = u(f)$, it is clear that $c(f) = v^{-1}(I(\varphi))$ and $g = v^{-1}(\varphi)$. But then, by hypothesis, $c(f^{k}) \leq c(f) + k = v^{-1}(I(\varphi)) + k$. Thus,

$$v(v^{-1}(I(\varphi)) + k) \geq v(c(f^{k})) \geq I\left(v\left(v^{-1}(\varphi) + k\right)\right)$$

which implies, by Proposition 5 point ii), that $\succsim$ is DAWB.

ii) Fix $k > 0$. Let, for any act $f \in \mathcal{F}$ the function g be defined by $g(s) = v^{-1}(u_k(f(s)) - k$ for every $s \in S$. Then, clearly, we have that $f(s) \sim^{k} g(s)$ for all $s \in S$. However, since $\succsim$ is IARA, the indifference relation above implies $f(s) \succsim g(s)$ for all $s \in S$, which then, by monotonicity of $\succsim$, means $f \succsim g$. But then,

$$\delta_{c(f)} \sim f \succsim g \Longleftrightarrow v(c(f)) \geq I(v(g))$$

Now, let $\varphi = u_k(f)$, it is clear that $c(f^{k}) = v^{-1}(I(u(f^{k})) = v^{-1}(I(u_k(f)) = v^{-1}(I(\varphi))$ and $g = v^{-1}(\varphi)$. But then, by hypothesis, $c(f^{k}) \geq c(f) + k$, so $v^{-1}(I(\varphi)) - k = c(f^{k}) - k \geq c(f)$

Thus,

$$v(v^{-1}(I(\varphi)) - k) \geq v(c(f)) \geq I\left(v\left(v^{-1}(\varphi) - k\right)\right).$$

which implies, by Proposition 5 point i), that $\succsim$ is IAWB. ■

Proof of Lemma 2. Assume first that ϕ is non differentiable for some point $c \in D$. Then, we have that $\partial\phi(c) = [\phi'_-(c), \phi'_+(c)]$. Take $\beta \in (\phi'_-(c), \phi'_+(c))$, then, for $t < c$,

$$\phi(c) + \beta(t - c) < \phi(c) + \phi'_-(c)(t - c) \leq \phi(t).$$

Similarly,

$$\phi(c) + \beta(t - c) < \phi(c) + \phi'_+(c)(t - c) \leq \phi(t).$$

Assume now that ϕ is differentiable: its derivative is increasing and, since it is not affine, there are $x < y$ such that $\phi'(x) < \phi'(y)$. However, ϕ' is continuous, which means that there is $(a, b) \subseteq (x, y)$ such that the derivative is strictly increasing on that interval, which thus implies that ϕ is strictly convex on (a, b). Take $c \in (a, b)$, since ϕ' is continuous and strictly increasing on that interval, we have that $\phi'(c) \in (\phi'(a), \phi'(b))$. Suppose $t \in [a, b]$. If $t > c$

$$\begin{aligned}\phi(t) &= \frac{\phi(t) - \phi(c)}{t - c}(t - c) + \phi(c) \\ &> \phi'(c)(t - c) + \phi(c),\end{aligned}$$

and similarly for $t < c$

$$\begin{aligned}\phi(t) &= \frac{\phi(c) - \phi(t)}{c - t}(t - c) + \phi(c) \\ &> \phi'(c)(t - c) + \phi(c).\end{aligned}$$

Take now $t \notin (a, b)$. For $t > b$,

$$\begin{aligned}\phi(t) &\geq \phi(b) + \phi'(b)(t - b) \\ &\geq \phi(c) + \phi'(c)(b - c) + \phi'(b)(t - b) \\ &> \phi(c) + \phi'(c)(b - c) + \phi'(c)(t - b) \\ &= \phi(c) + \phi'(c)(t - c).\end{aligned}$$

Similarly, for $t < a$

$$\begin{aligned}\phi(t) &\geq \phi(a) + \phi'(a)(t-a) \\ &\geq \phi(c) + \phi'(c)(a-c) + \phi'(b)(t-a) \\ &> \phi(c) + \phi'(c)(a-c) + \phi'(c)(t-a) \\ &= \phi(c) + \phi'(c)(t-c).\end{aligned}$$

In all cases, the statement is true. ■

Proof of Proposition 6. It can be proven (see Cerreia-Vioglio et al. [5]) that if I is strictly monotone, translation invariant and subhomogeneous., the operator I satisfies the Jensen inequality, i.e. for all convex functions ϕ it holds

$$I(\phi(\varphi)) \geq \phi(I(\varphi)) \qquad \forall \varphi \in B_0(S, \Sigma, u(X)).$$

However, since $\succsim$ is DAWB, for $w' > w \geq 0$ it holds, with $\phi_{w,w'}$ convex by Proposition 1, that

$$I(\phi_{w,w'}(\varphi)) \leq \phi_{w,w'}(I(\varphi)) \qquad \forall \varphi \in B_0(S, \Sigma, u(X)).$$

This means that for all $w' > w \geq 0$, if I is strictly monotone, translation invariant and subhomogeneous and DAWB,

$$I(\phi_{w,w'}(\varphi)) = \phi_{w,w'}(I(\varphi)) \qquad \forall \varphi \in B_0(S, \Sigma, u(X)). \tag{5}$$

Fix $w' > w$, and assume $\phi_{w,w'}$ is convex but not affine. Then, by Lemma 2, there exists a point $c \in \operatorname{Im} v_w$ such that, for some $\beta \in \partial \phi_{w,w'}(c)$, it holds that

$$\phi_{w,w'}(t) > \phi_{w,w'}(c) + \beta(t-c) \qquad \forall t \neq c.$$

Notice that the above implies, for $t > c$,

$$\beta < \frac{\phi_{w,w'}(t) - \phi_{w,w'}(c)}{t-c} \in (0,1]$$

by Lemma 1.

Now, let $\varphi \in B_0(S, \Sigma, u(X))$ such that $I(\varphi) = c$ and $\varphi(s) \neq c$ for every $s \in S$. Such function exists, as it would be enough to take proper $a > c > b$, $\varphi(s) = a\mathbb{I}_A(s) + b\mathbb{I}_{A^c}(s)$, where $\mathbb{I}$ is the indicator function and $A \in \Sigma$ an event in the algebra of S. Then, by what has been observed above,

$$\phi_{w,w'}(\varphi) > \phi_{w,w'}(c) + \beta(\varphi - c) \qquad \forall s \in S$$

By strict monotonicity, translation invariance, subhomogeneity and normalization of the operator I, we get

$$\begin{aligned} I(\phi_{w,w'}(\varphi)) &> I(\phi_{w,w'}(c) + \beta(\varphi - c)) \\ &= \phi_{w,w'}(c) + I(\beta(\varphi - c)) \\ &\geq \phi_{w,w'}(c) + \beta(I(\varphi - c)) \\ &= \phi_{w,w'}(c) + \beta(I(\varphi) - c) = \phi_{w,w'}(I(\varphi)) \end{aligned}$$

contradicting the equality 5.

This means that $\phi_{w,w'}$ is affine for any $w' > w \geq 0$, which also implies that $\succsim^{w'}$ and $\succsim^{w}$ share the same risk attitude for any $w' > w \geq 0$, showing that $\succsim$ is CARA. By Proposition 5 point iii), this means that, being $\succsim$ DAWB, $\succsim$ is IAAA. ■

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