Politecnico di Torino
Department of Electronics and Telecommunications

Teoria dei Segnali

Esercitazione 1

Energia e potenza media

Spazio dei segnali

Esercizio 1

Calcolare l'energia dei seguenti segnali:

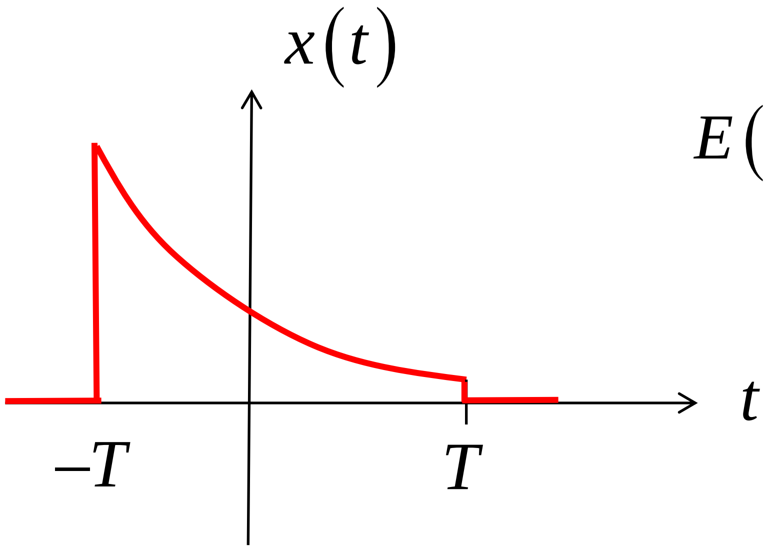
$$
\begin{aligned}
x_1(t) &= \mathrm{e}^{-\alpha t} p_{2T}(t), \;\; t \in \mathbb{R} \\
x_2(t) &= p_1\left(\tfrac{t-2}{4}\right) e^{-2t}, \;\; t \in \mathbb{R} \\
x_3(t) &= A\cos^2(2\pi f_0 t) p_{T_0}\left(t - \tfrac{T_0}{2}\right), \;\; t \in \mathbb{R}
\end{aligned}
$$

dove α, T, A e $f_0 = 1/T_0$ sono costanti reali positive e

$$
p_a(t) = \begin{cases} 1 & -a/2 \leq t \leq a/2, \\ 0 & \text{altrove.} \end{cases}
$$

Soluzione Esercizio 1a

$$x(t) = e^{-\alpha t} p_{2T}(t), \;\; t \in \mathbf{R}$$



$$E(x) = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-T}^{T} e^{-2\alpha t} dt = -\frac{1}{2\alpha} e^{-2\alpha t} \Big|_{-T}^{T} = \frac{e^{2\alpha T} - e^{-2\alpha T}}{2\alpha}$$

Es. 1a sotto forma di quiz a risposta multipla

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- L'energia del segnale $x(t) = e^{-\alpha t} p_{2T}(t)$, con $p_a(t) = \begin{cases} 1 & \text{se } -\frac{a}{2} \le t \le \frac{a}{2} \\ 0 & \text{altrove} \end{cases}$, vale:

  A) $E(x) = 0$

  B) $E(x) = \dfrac{e^{2\alpha T} - e^{-2\alpha T}}{2\alpha}$

  C) $E(x) = \infty$

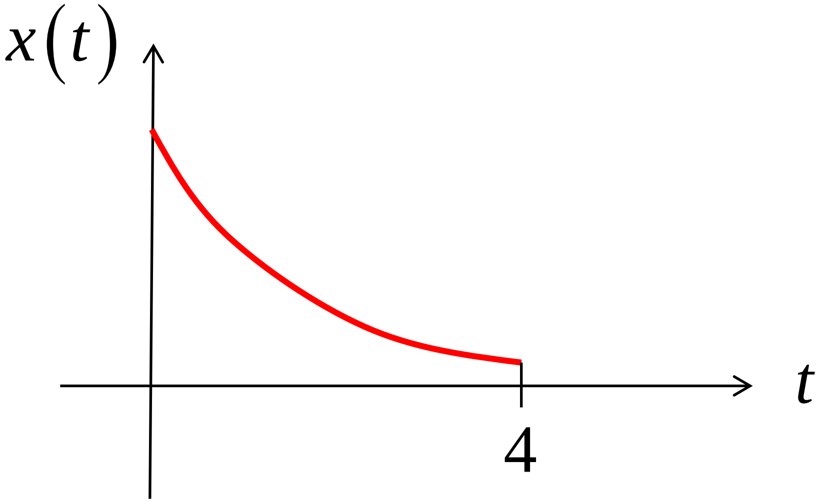
  D) $E(x) = \dfrac{e^{\alpha T} - e^{-\alpha T}}{\alpha}$

  E) $E(x) = \dfrac{1 - e^{-4\alpha T}}{2\alpha}$

  F) Nessuna delle altre risposte è corretta

Soluzione Esercizio 1b

$$x(t) = p_1\left(\frac{t-2}{4}\right)e^{-2t}\,,\;\; t \in \mathbf{R}$$



$$E(x) = \int_{-\infty}^{+\infty} \left|x(t)\right|^2 dt = \int_0^4 e^{-4t} dt = -\frac{1}{4}e^{-4t}\Big|_0^4 = \frac{1-e^{-16}}{4}$$

Soluzione Esercizio 1c

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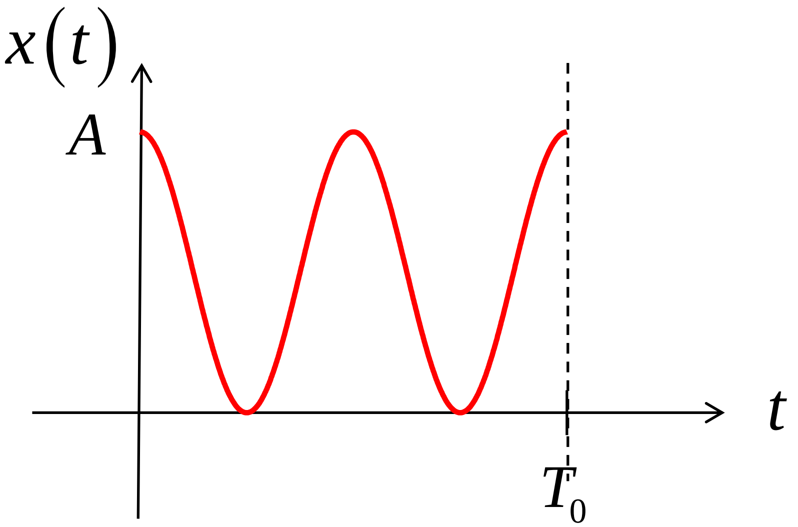
$$x(t) = A\cos^2(2\pi f_0 t)\, p_{T_0}\left(t - \frac{T_0}{2}\right) \qquad T_0 = \frac{1}{f_0}$$



$$E(x) = \int_{-\infty}^{+\infty} |x(t)|^2\, dt = A^2 \int_0^{T_0} \cos^4(2\pi f_0 t)\, dt = A^2 \int_0^{T_0} \cos^2(2\pi f_0 t) \cdot \cos^2(2\pi f_0 t)\, dt$$

$$= A^2 \int_0^{T_0} \left[\frac{1}{2} + \frac{1}{2}\cos(4\pi f_0 t)\right]^2 dt =$$

$$= A^2 \left[\int_0^{T_0} \frac{1}{4}\, dt + \int_0^{T_0} \frac{1}{2}\cos(4\pi f_0 t)\, dt + \int_0^{T_0} \frac{1}{4}\cos^2(4\pi f_0 t)\, dt\right] =$$

$$= \frac{A^2 T_0}{4} + 0 + A^2 \int_0^{T_0} \frac{1}{8}\, dt + A^2 \int_0^{T_0} \frac{1}{8}\cos(8\pi f_0 t)\, dt =$$

$$= \frac{A^2 T_0}{4} + \frac{A^2 T_0}{8} = \frac{3}{8} A^2 T_0$$

Esercizio 2

Calcolare la potenza media del seguente segnale:

$$x(t) = \sum_{n=-\infty}^{+\infty} \phi(t - 2nT_2)$$

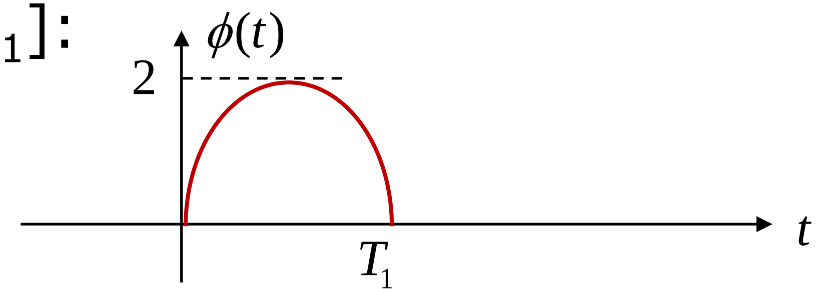
dove

$$\phi(t) = 2\sin\left(\frac{2\pi t}{2T_1}\right) p_{T_1}\left(t - \frac{T_1}{2}\right)$$

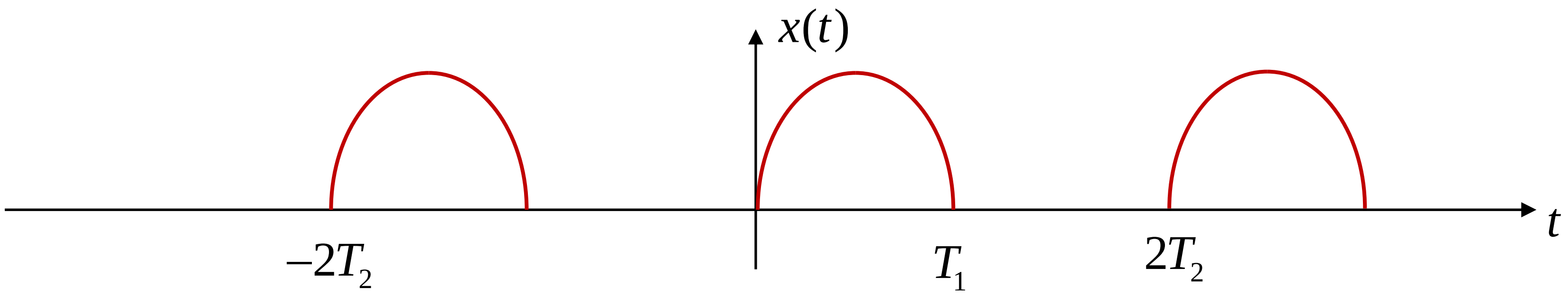
e T_1 e T_2 sono due costanti reali positive, con $T_1 < 2T_2$.

Soluzione Esercizio 2

- $x(t)$ è un segnale periodico di periodo $2T_2$.
- $\phi(t)$ è una funzione seno con periodo $2T_1$ moltiplicata per una funzione porta con supporto $[0,T_1]$:



- Siccome $T_1<2T_2$, le repliche del segnale non si sovrappongono:



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Soluzione Esercizio 2

- La potenza media di x(t) si può quindi calcolare come:

$$P_x = \frac{1}{2T_2}\int_0^{2T_2} |\phi(t)|^2\, dt = \frac{1}{2T_2}\int_0^{T_1} 4\sin^2\left(\frac{2\pi t}{2T_1}\right) dt =$$

Periodo

$$= \frac{4}{2T_2}\int_0^{T_1} \frac{1}{2}\left[1-\cancel{\cos\left(\frac{2\pi t}{T_1}\right)}\right] dt = \frac{T_1}{T_2}$$

=0
(coseno di periodo T_1 integrato su un periodo)

Es. 2 sotto forma di quiz a risposta multipla

- La potenza media del segnale $x(t) = \sum_{n=-\infty}^{+\infty} \phi(t - 2nT_2)$,

  dove $\phi(t) = 2\sin\left(\frac{2\pi t}{2T_1}\right) p_{T_1}\left(t - \frac{T_1}{2}\right)$ e $T_1 < 2T_2$, vale:

  A) 2

  B) $2T_1$

  C) T_1/T_2

  D) 0

  E) T_2/T_1

  F) Nessuna delle altre risposte è corretta

Esercizio 3

Calcolare la distanza Euclidea delle seguenti coppie di segnali:

a)

$$x_1(t) = \mathrm{e}^{-\alpha t} u(t) \quad \alpha > 0$$

$$x_2(t) = 2u(t)$$

b)

$$x_1(t) = \begin{cases} \left(\frac{t}{T}\right)^2 & \text{se } 0 \leq t \leq T, \\ 0 & \text{altrove} \end{cases}$$

$$x_2(t) = \begin{cases} -\frac{t}{T} & \text{se } 0 \leq t \leq T, \\ 0 & \text{altrove} \end{cases}$$

Soluzione Esercizio 3a

$$x_1(t) = e^{-\alpha t} u(t)$$
$$x_2(t) = 2u(t)$$

$$d^2(x_1, x_2) = \int_{-\infty}^{+\infty} \left| x_1(t) - x_2(t) \right|^2 dt = \int_{0}^{+\infty} \left(e^{-\alpha t} - 2 \right)^2 dt =$$

$$= \int_{0}^{+\infty} e^{-2\alpha t} dt - 4 \int_{0}^{+\infty} e^{-\alpha t} dt + \int_{0}^{+\infty} 4\, dt =$$

$$= \left. \frac{e^{-2\alpha t}}{-2\alpha} \right|_0^{+\infty} - 4 \left. \frac{e^{-\alpha t}}{-\alpha} \right|_0^{+\infty} + \infty = \frac{1}{2\alpha} + \frac{4}{\alpha} + \infty = \infty$$

Soluzione Esercizio 3b

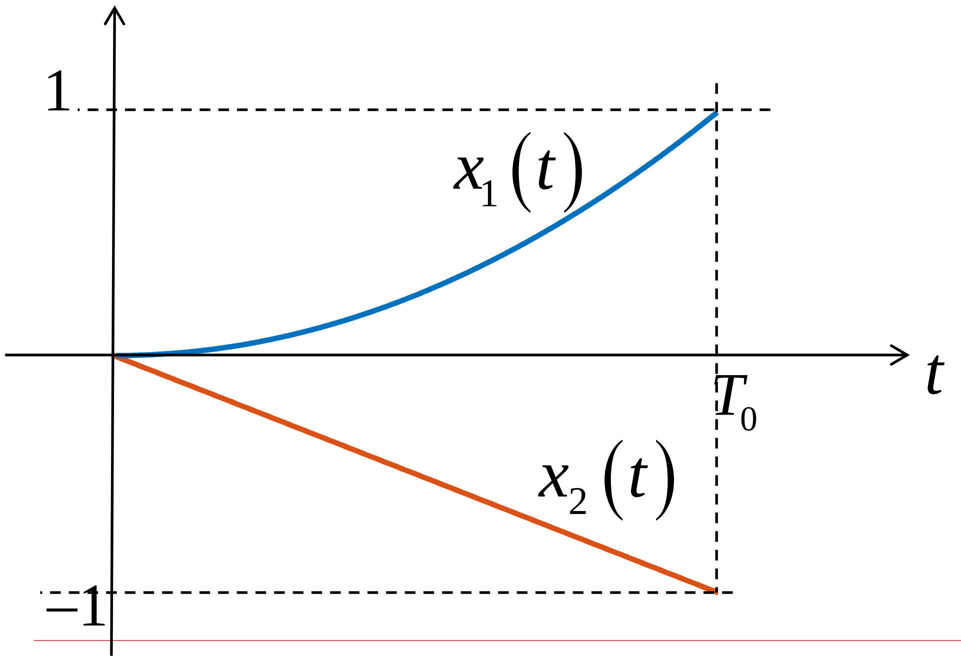
$$x_1(t) = \begin{cases} \left(\frac{t}{T}\right)^2 & \text{se } 0 \le t \le T, \\ 0 & \text{altrove} \end{cases}$$

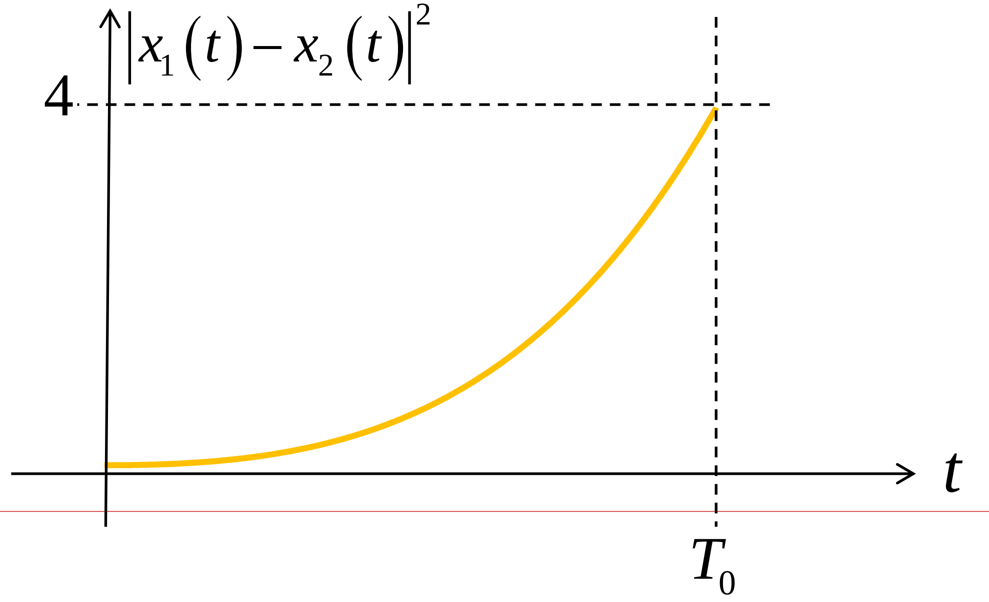
$$x_2(t) = \begin{cases} -\frac{t}{T} & \text{se } 0 \le t \le T, \\ 0 & \text{altrove} \end{cases}$$

$$d^2(x_1,x_2) = \int_{-\infty}^{+\infty} |x_1(t)-x_2(t)|^2\, dt = \int_0^T \left[\left(\frac{t}{T}\right)^2 - \left(-\frac{t}{T}\right)\right]^2 dt =$$

$$= \int_0^T \left(\frac{t}{T}\right)^4 dt + \int_0^T \left(\frac{t}{T}\right)^2 dt + 2\int_0^T \left(\frac{t}{T}\right)^3 dt =$$

$$= \left.\frac{t^5}{5T^4}\right|_0^T + \left.\frac{t^3}{3T^2}\right|_0^T + 2\left.\frac{t^4}{4T^3}\right|_0^T = \frac{T^5}{5T^4} + \frac{T^3}{3T^2} + 2\frac{T^4}{4T^3} = \frac{31}{30}T$$





Es. 3b sotto forma di quiz a risposta multipla

- Sia data la seguente coppia di segnali:

$$x_1(t) = \begin{cases} \left(\frac{t}{T}\right)^2 & \text{se } 0 \le t \le T, \\ 0 & \text{altrove} \end{cases} \qquad x_2(t) = \begin{cases} -\frac{t}{T} & \text{se } 0 \le t \le T, \\ 0 & \text{altrove} \end{cases}$$

La distanza euclidea tra $x_1(t)$ e $x_2(t)$ vale:

A) $d(x_1, x_2) = \sqrt{\frac{31}{30} T}$

B) $d(x_1, x_2) = \sqrt{\frac{1}{30} T}$

C) $d(x_1, x_2) = \infty$

D) $d(x_1, x_2) = \sqrt{\frac{5}{6} T}$

E) Nessuna delle altre risposte è corretta

Esercizio 4

Dato l'insieme di segnali ortonormali raffigurati in Figura 1, si sviluppi la funzione

$$z(t) = \frac{1}{2} + \cos\left(\frac{\pi t}{4}\right) + \sin(\pi t)$$

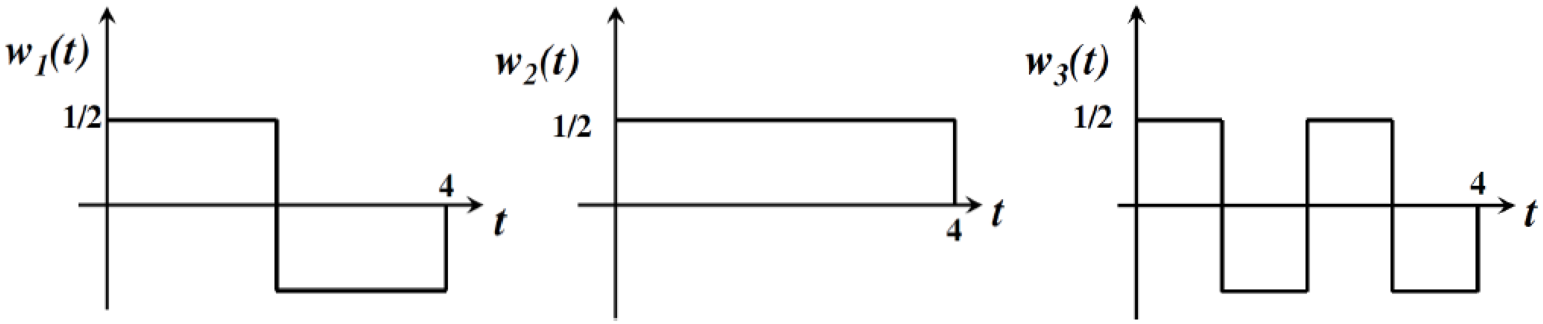


Figura 1: Esercizio 4

Soluzione Esercizio 4

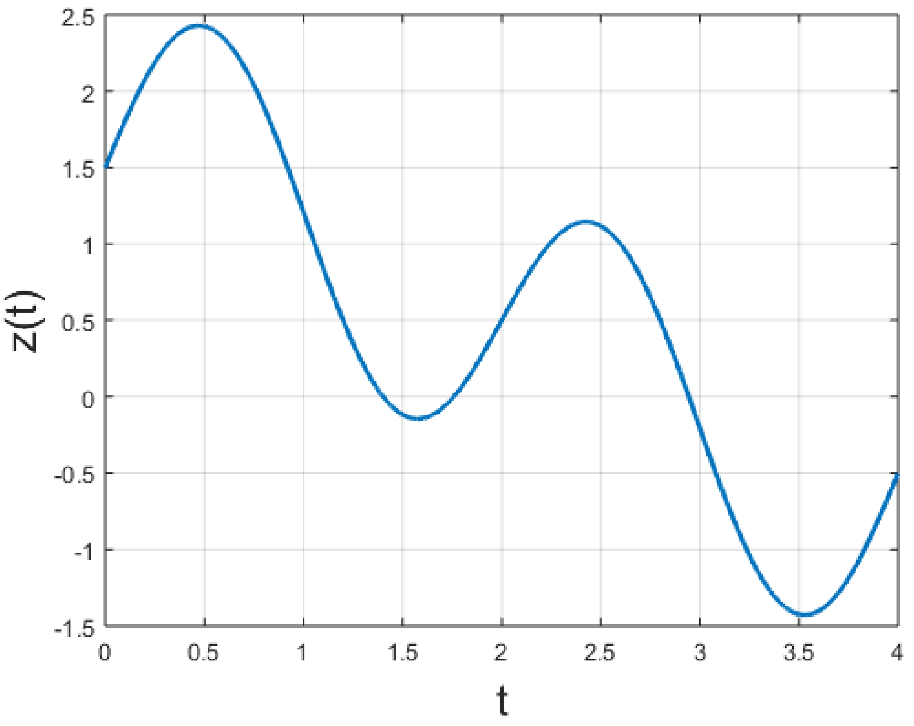
- Sviluppare una funzione z(t) su una base ortonormale $\{w_1(t), w_2(t), \ldots, w_n(t)\}$ significa scrivere z(t) come combinazione lineare dei segnali che compongono la base:

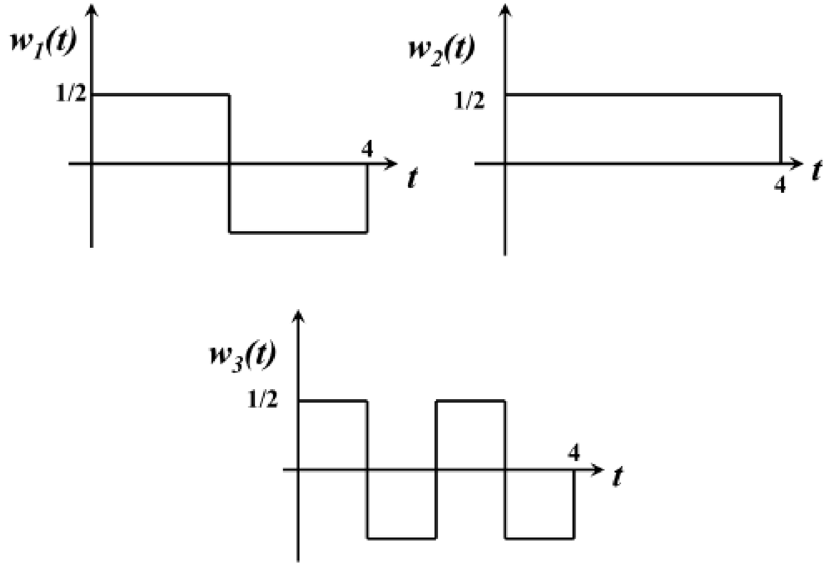
$$z(t) = \sum_{i=1}^{n} \alpha_i w_i(t)$$

- I coefficienti si ottengono tramite l'operazione di prodotto scalare:

$$\alpha_i = \langle z, w_i \rangle \quad i = 1, \ldots, n$$

Soluzione Esercizio 4

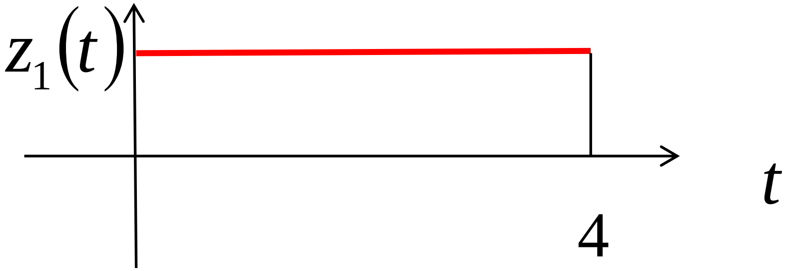




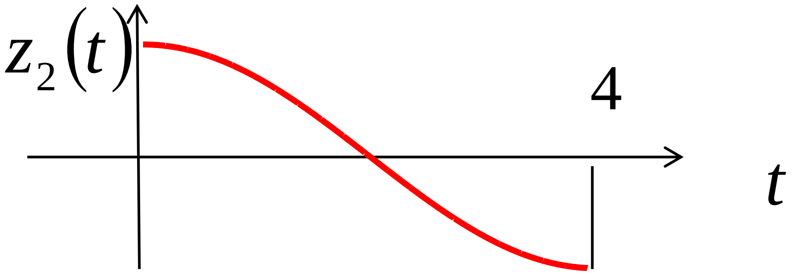
Soluzione Esercizio 4

$$z(t) = z_1(t) + z_2(t) + z_3(t)$$

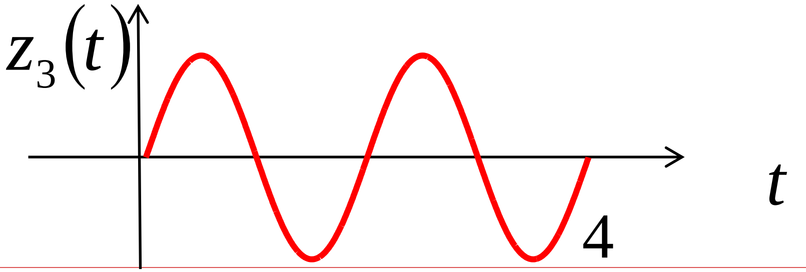
$$z_1(t) = \frac{1}{2}$$



$$z_2(t) = \cos\left(\frac{\pi t}{4}\right)$$



$$z_3(t) = \sin(\pi t)$$



Soluzione Esercizio 4

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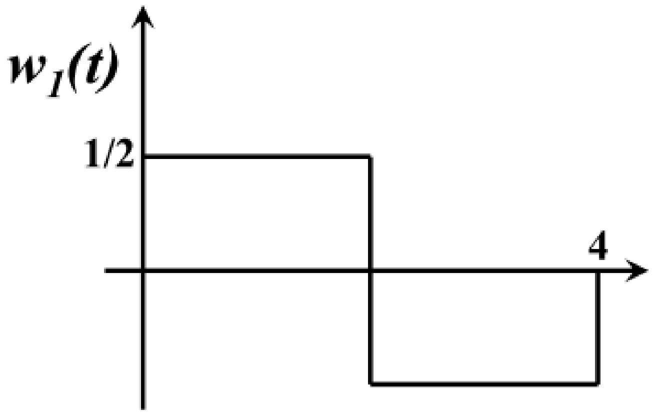
$$\langle z, w_1 \rangle = \langle z_1, w_1 \rangle + \langle z_2, w_1 \rangle + \langle z_3, w_1 \rangle$$

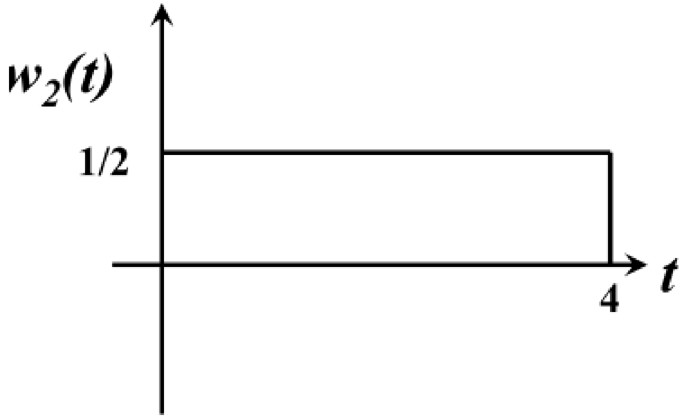


$$\langle z_1, w_1 \rangle = \langle z_3, w_1 \rangle = 0$$

$$\langle z_2, w_1 \rangle = 2\frac{1}{2}\int_0^2 \cos\left(\frac{\pi t}{4}\right) dt = \frac{4}{\pi}\sin\left(\frac{\pi t}{4}\right)\Bigg|_0^2 = \frac{4}{\pi}$$

Soluzione Esercizio 4

$$\langle z, w_2 \rangle = \langle z_1, w_2 \rangle + \langle z_2, w_2 \rangle + \langle z_3, w_2 \rangle$$



$$\langle z_1, w_2 \rangle = \int_0^4 \frac{1}{2}\frac{1}{2}\,dt = 1$$

$$\langle z_2, w_2 \rangle = \langle z_3, w_2 \rangle = 0$$

Soluzione Esercizio 4

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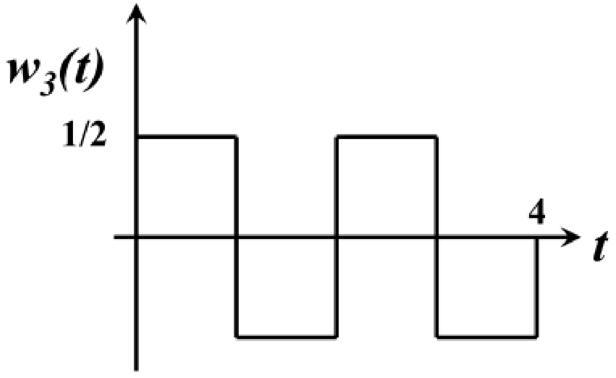
$$\langle z, w_3 \rangle = \langle z_1, w_3 \rangle + \langle z_2, w_3 \rangle + \langle z_3, w_3 \rangle$$



$$\langle z_1, w_3 \rangle = 0$$

$$\langle z_2, w_3 \rangle = \int_0^4 \cos\left(\frac{\pi t}{4}\right) w_3(t)\,dt = \frac{1}{2}\int_0^1 \cos\left(\frac{\pi t}{4}\right) dt +$$

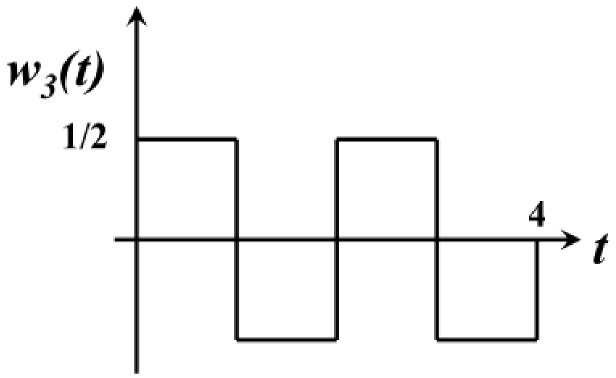
$$-\frac{1}{2}\int_1^2 \cos\left(\frac{\pi t}{4}\right) dt + \frac{1}{2}\int_2^3 \cos\left(\frac{\pi t}{4}\right) dt - \frac{1}{2}\int_3^4 \cos\left(\frac{\pi t}{4}\right) dt =$$

$$= \int_0^1 \cos\left(\frac{\pi t}{4}\right) dt - \int_1^2 \cos\left(\frac{\pi t}{4}\right) dt = \frac{4}{\pi}\sin\left(\frac{\pi t}{4}\right)\Bigg|_0^1 - \frac{4}{\pi}\sin\left(\frac{\pi t}{4}\right)\Bigg|_1^2 =$$

$$= \frac{4}{\pi}\sin\left(\frac{\pi}{4}\right) - \frac{4}{\pi}\sin\left(\frac{\pi}{2}\right) + \frac{4}{\pi}\sin\left(\frac{\pi}{4}\right) = \frac{8}{\pi}\frac{\sqrt{2}}{2} - \frac{4}{\pi} = \frac{4\sqrt{2}}{\pi} - \frac{4}{\pi}$$

Soluzione Esercizio 4

$$\langle z, w_3 \rangle = \langle z_1, w_3 \rangle + \langle z_2, w_3 \rangle + \langle z_3, w_3 \rangle$$



$$\langle z_3, w_3 \rangle = \int_0^4 \sin(\pi t) w_3(t) dt = \frac{1}{2}\int_0^1 \sin(\pi t) dt +$$

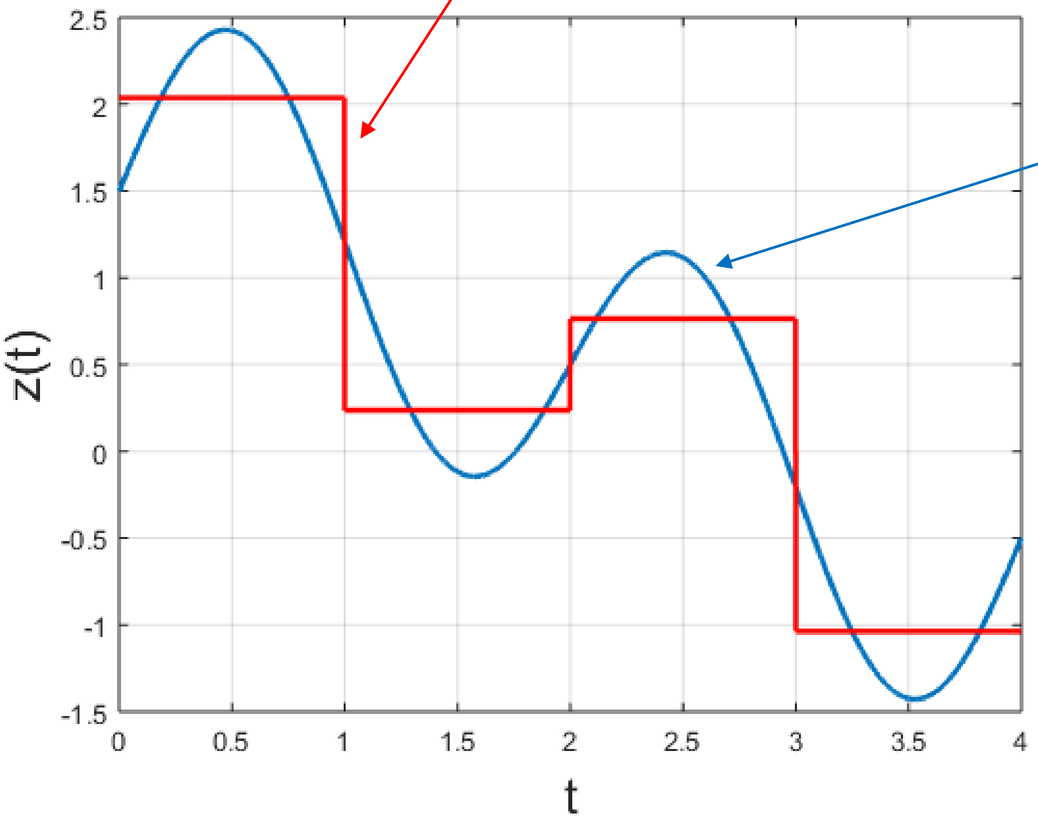
$$-\frac{1}{2}\int_1^2 \sin(\pi t) dt + \frac{1}{2}\int_2^3 \sin(\pi t) dt - \frac{1}{2}\int_3^4 \sin(\pi t) dt =$$

$$= 2\int_0^1 \sin(\pi t) dt = -\frac{2}{\pi}\cos(\pi t)\Big|_0^1 =$$

$$= -\frac{2}{\pi}[\cos(\pi) - \cos(0)] = \frac{4}{\pi}$$

Soluzione Esercizio 4

$$\hat{z}(t)=\sum_{i=1}^{n}\alpha_i w_i(t)=\frac{4}{\pi}w_1(t)+w_2(t)+\frac{4\sqrt{2}}{\pi}w_3(t)$$



$$z(t)=\frac{1}{2}+\cos\left(\frac{\pi t}{4}\right)+\sin(\pi t)$$

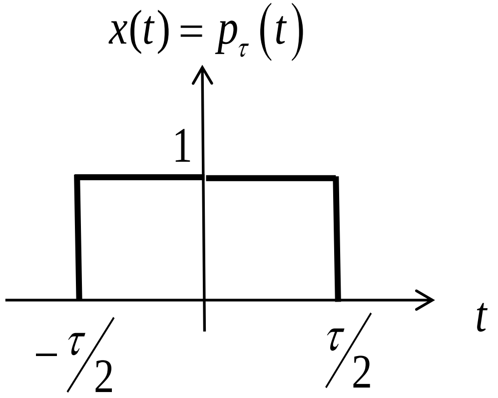
Esercizio 5

E' dato il segnale a energia finita $x(t) = p_\tau$, con $\tau < T$. Calcolarne lo sviluppo in serie di Fourier nell'intervallo $(-T/2, T/2)$, ossia:

$$x(t) = \sum_{n=-\infty}^{\infty} \mu_n e^{\jmath \frac{2\pi}{T} nt} \quad \text{con} \quad \mu_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-\jmath \frac{2\pi}{T} nt} dt$$

Soluzione Esercizio 5

$$x(t) = \sum_{n=-\infty}^{\infty} \mu_n e^{j\frac{2\pi}{T}nt} \qquad \mu_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi}{T}nt}\, dt$$



T = intervallo su cui è definito il segnale

$$\mu_n = \frac{1}{T}\int_{-T/2}^{T/2} p_\tau(t)\, e^{-j\frac{2\pi}{T}nt}\, dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} e^{-j\frac{2\pi}{T}nt}\, dt = \frac{1}{T}\left(\frac{T}{-j2\pi n}\right) e^{-j\frac{2\pi}{T}nt}\Big|_{-\tau/2}^{\tau/2} =$$

$$= \frac{1}{-j2\pi n}\left[e^{-j\frac{2\pi}{T}n\frac{\tau}{2}} - e^{j\frac{2\pi}{T}n\frac{\tau}{2}}\right] = \frac{1}{\pi n}\,\frac{e^{j\frac{2\pi}{T}n\frac{\tau}{2}} - e^{-j\frac{2\pi}{T}n\frac{\tau}{2}}}{2j} = \frac{\sin\left(n\pi\frac{\tau}{T}\right)}{n\pi} = \frac{\tau}{T}\,\frac{\sin\left(n\pi\frac{\tau}{T}\right)}{n\pi\frac{\tau}{T}} = \frac{\tau}{T}\,\mathrm{sinc}\left(\frac{\tau}{T}n\right)$$

$$\boxed{x(t) = \frac{\tau}{T}\sum_{n}\mathrm{sinc}\left(\frac{\tau}{T}n\right) e^{j\frac{2\pi}{T}nt}}$$

Esercizio 6

E' dato il segnale $f(t)$ ad energia finita in Figura 2. Calcolarne lo sviluppo in serie di Fourier nell'intervallo $(-3T, +3T)$.

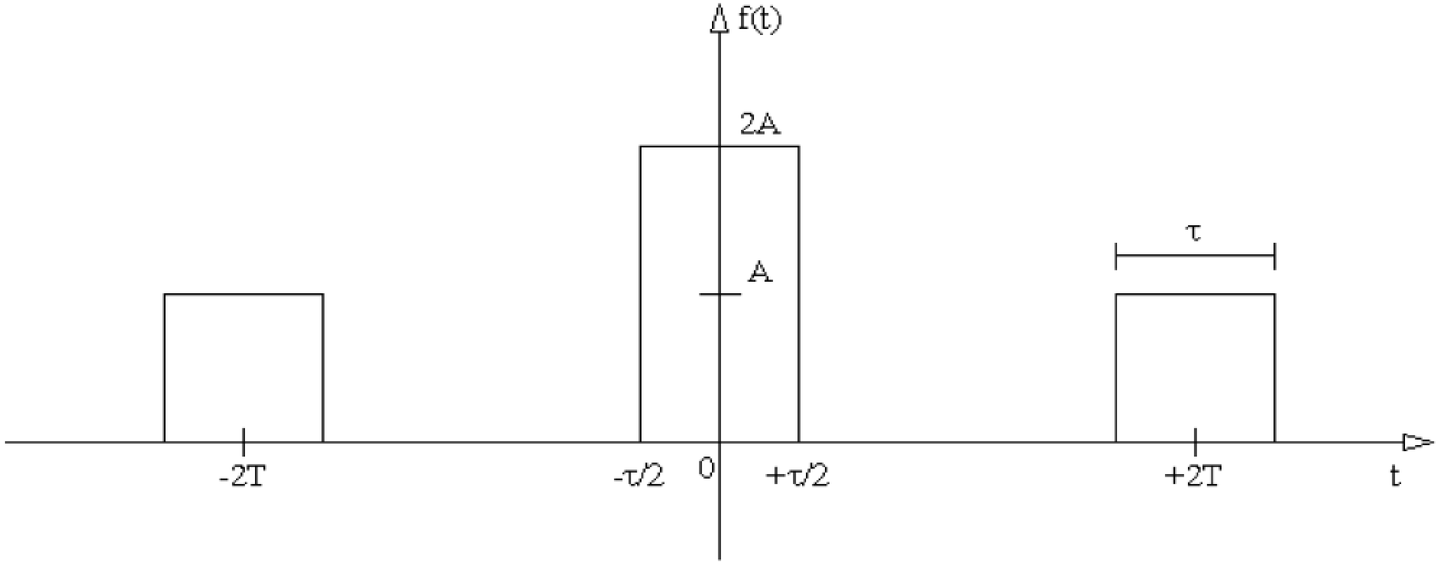
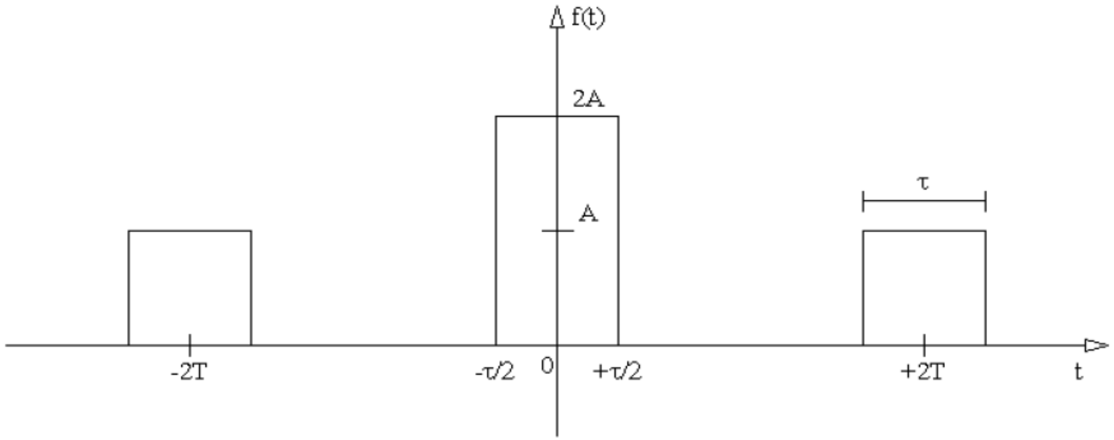


Figura 2: Esercizio 6

Serie di Fourier

$$x(t) = \sum_{n=-\infty}^{\infty} \mu_n e^{j\frac{2\pi}{T}nt} \qquad \mu_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi}{T}nt}\, dt$$

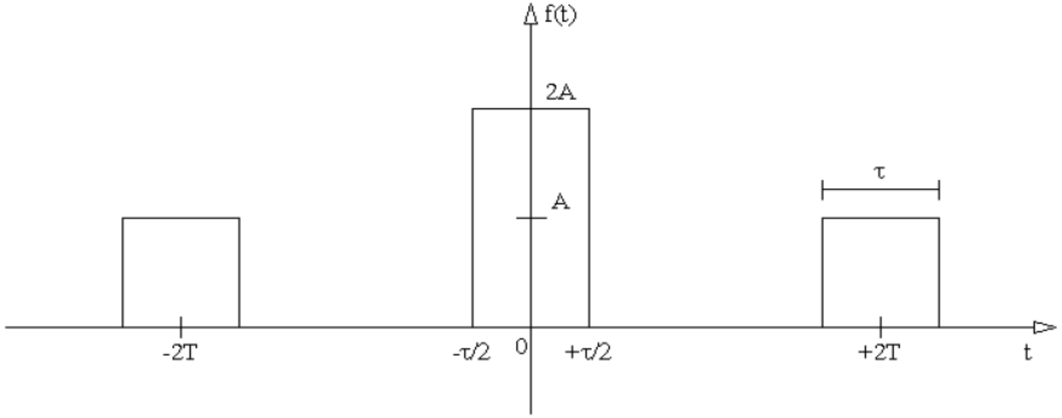
T = intervallo su cui è definito il segnale



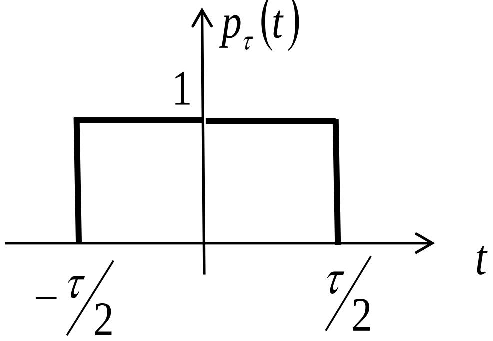
$$f(t) = A\left[2p_\tau(t) + p_\tau(t-2T) + p_\tau(t+2T)\right]$$

Per semplificare i conti, conviene sviluppare in serie di Fourier la funzione $p_\tau(t)$, e poi inserire lo sviluppo nell'espressione di f(t).

Soluzione Esercizio 6



- f(t) può essere espresso in termini del segnale elementare $p_\tau(t)$:



$$f(t) = A\left[2p_\tau(t) + p_\tau(t-2T) + p_\tau(t+2T)\right]$$

Soluzione Esercizio 6

$$p_\tau(t) = \sum_n \mu'_n e^{j\frac{2\pi}{6T}nt} = \sum_n \mu'_n e^{j\frac{\pi}{3T}nt}$$

$$\mu'_n = \frac{1}{6T}\int_{-3T}^{3T} p_\tau(t) e^{-j\frac{\pi}{3T}nt}\,dt = \frac{1}{6T}\int_{-\tau/2}^{\tau/2} e^{-j\frac{\pi}{3T}nt}\,dt = \frac{1}{6T}\left(\frac{3T}{-j\pi n}\right) e^{-j\frac{\pi}{3T}nt}\Bigg|_{-\tau/2}^{\tau/2} =$$

$$= \frac{1}{-j2\pi n}\left[e^{-j\frac{\pi}{3T}n\frac{\tau}{2}} - e^{j\frac{\pi}{3T}n\frac{\tau}{2}}\right] = \frac{1}{\pi n}\,\frac{e^{j\frac{\pi}{3T}n\frac{\tau}{2}} - e^{-j\frac{\pi}{3T}n\frac{\tau}{2}}}{2j} = \frac{\sin\left(n\pi\frac{\tau}{6T}\right)}{n\pi} =$$

$$= \frac{\tau}{6T}\,\frac{\sin\left(n\pi\frac{\tau}{6T}\right)}{n\pi\frac{\tau}{6T}} = \frac{\tau}{6T}\operatorname{sinc}\left(\frac{\tau}{6T}n\right)$$

$$\boxed{p_\tau(t) = \frac{\tau}{6T}\sum_n \operatorname{sinc}\left(\frac{\tau}{6T}n\right) e^{j\frac{\pi}{3T}nt}}$$

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$$f(t) = A\left[2p_\tau(t) + p_\tau(t-2T) + p_\tau(t+2T)\right] =$$

$$= 2A\left[\frac{\tau}{6T}\sum_n \mathrm{sinc}\left(n\frac{\tau}{6T}\right)e^{j\frac{\pi}{3T}nt}\right] + A\left[\frac{\tau}{6T}\sum_n \mathrm{sinc}\left(n\frac{\tau}{6T}\right)e^{j\frac{\pi}{3T}n(t-2T)}\right] + A\left[\frac{\tau}{6T}\sum_n \mathrm{sinc}\left(n\frac{\tau}{6T}\right)e^{j\frac{\pi}{3T}n(t+2T)}\right] =$$

$$= A\frac{\tau}{6T}\sum_n \mathrm{sinc}\left(n\frac{\tau}{6T}\right)e^{j\frac{\pi}{3T}nt}\left(2 + e^{-j\frac{\pi}{3T}n2T} + e^{+j\frac{\pi}{3T}n2T}\right) = \frac{A\tau}{6T}\sum_n \mathrm{sinc}\left(n\frac{\tau}{6T}\right)e^{j\frac{\pi}{3T}nt}\left(2 + 2\cos\left(\frac{2\pi}{3}n\right)\right)$$

$$f(t) = \sum_n \underbrace{\frac{A\tau}{3T}\mathrm{sinc}\left(n\frac{\tau}{6T}\right)\left(1+\cos\left(\frac{2\pi}{3}n\right)\right)}_{\mu_n} e^{j\frac{\pi}{3T}nt}$$