Politecnico di Torino

Department of Electronics and Telecommunications

Teoria dei Segnali

Esercitazione 2

Trasformata di Fourier

Esercizio 1

Calcolare la trasformata di Fourier dei seguenti segnali:

1. $s_1(t) = e^{-\alpha t} u(t), \alpha > 0$
2. $s_2(t) = e^{-2t+4} u(t-2)$
3. $s_3(t) = e^{-t/2} \cos(100\pi t) u(t)$
4. $s_4(t) = 10 \mathrm{sinc}(t)^2 \cos(300\pi t + \pi/6)$
5. $s_5(t) = \mathrm{tri}\left(\frac{t-1}{2}\right) e^{-j200\pi t}$

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Soluzione Esercizio 1

$$s_1(t) = e^{-\alpha t}u(t) \quad \alpha > 0$$

- Applicando la definizione di trasformata di Fourier:

$$S_1(f) = \int_{-\infty}^{+\infty} s_1(t) e^{-j2\pi ft} dt = \int_0^{+\infty} e^{-\alpha t} e^{-j2\pi ft} dt = \int_0^{+\infty} e^{-(\alpha + j2\pi f)t} dt =$$

$$= -\frac{1}{\alpha + j2\pi f} e^{-(\alpha + j2\pi f)t} \Big|_0^{+\infty} = \frac{1}{\alpha + j2\pi f}$$

$$\boxed{e^{-\alpha t}u(t) \rightarrow \frac{1}{\alpha + j2\pi f} \quad (\alpha > 0)}$$

Soluzione Esercizio 1

$$s_2(t) = e^{-2t+4}u(t-2) =$$

$$= e^{-2(t-2)}u(t-2) = s_1(t-2)\Big|_{\alpha=2}$$

- Dalla slide precedente ricaviamo che:

$$S_1(f)\Big|_{\alpha=2} = \frac{1}{2 + j2\pi f}$$

- Usando la proprietà della “traslazione nel tempo”:

$$S_2(f) = S_1(f)e^{-j4\pi f} = \frac{e^{-j4\pi f}}{2 + j2\pi f}$$

Soluzione Esercizio 1

$$s_3(t) = e^{-t/2}\cos(100\pi t)u(t) =$$

$$= s_1(t)\big|_{\alpha=\frac{1}{2}} \cdot \cos(100\pi t)$$

- Usando la proprietà del “prodotto”:

$$x(t)\cdot y(t) \;\rightarrow\; X(f) * Y(f)$$

si ottiene:

$$S_3(f) = S_1(f)\big|_{\alpha=\frac{1}{2}} * \frac{1}{2}\left[\delta(f-50)+\delta(f+50)\right] = \frac{1}{2}\left[S_1(f-50)\big|_{\alpha=\frac{1}{2}} + S_1(f+50)\big|_{\alpha=\frac{1}{2}}\right] =$$

$$= \frac{1}{2}\left[\frac{1}{1/2 + j2\pi(f-50)} + \frac{1}{1/2 + j2\pi(f+50)}\right] = \frac{1/2 + j2\pi f}{\left(1/2 + j2\pi f\right)^2 + (100\pi)^2}$$

Soluzione Esercizio 1

$$s_4(t) = 10\,\mathrm{sinc}(t)^2 \cos\left(300\pi t + \frac{\pi}{6}\right) =$$

$$= 10\,\mathrm{sinc}(t)^2 \cdot \frac{1}{2}\left(e^{j\left(2\pi \cdot f_0 t + \frac{\pi}{6}\right)} + e^{-j\left(2\pi \cdot f_0 t + \frac{\pi}{6}\right)}\right)$$

- Usando la proprietà del "prodotto":

$$S_4(f) = 10\,\mathrm{tri}(f) * \frac{1}{2}\left[\delta(f - f_0)e^{j\frac{\pi}{6}} + \delta(f + f_0)e^{-j\frac{\pi}{6}}\right] =$$

$$= 5\left[\mathrm{tri}(f - f_0)e^{+j\frac{\pi}{6}} + \mathrm{tri}(f + f_0)e^{-j\frac{\pi}{6}}\right]$$

Soluzione Esercizio 1

$$s_5(t) = \mathrm{tri}\left(\frac{t-1}{2}\right)e^{-j200\pi t}$$

$$\mathrm{tri}(t) \to \mathrm{sinc}^2(t)$$

$$\mathrm{tri}\left(\frac{t}{2}\right) \to 2\,\mathrm{sinc}^2(2f)$$

- Usando la proprietà della "traslazione nel tempo":

$$\mathrm{tri}\left(\frac{t-1}{2}\right) \to 2\,\mathrm{sinc}^2(2f)e^{-j2\pi f}$$

- Usando la proprietà della "traslazione in frequenza":

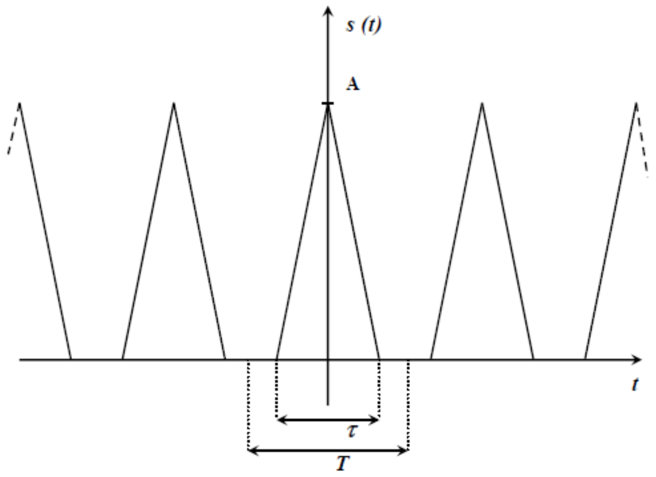
$$S_5(f) = 2\,\mathrm{sinc}^2(2f)e^{-j2\pi f}\Big|_{f+100} = 2\,\mathrm{sinc}^2(2f+200)e^{-j(2\pi f+200\pi)} =$$

$$= 2\,\mathrm{sinc}^2(2f+200)e^{-j2\pi f}$$

Esercizio 2

Dato il segnale $s(t)$ *onda triangolare* rappresentato in figura:

1. Calcolare la trasformata di Fourier
2. Calcolare i coefficienti della serie di Fourier



Soluzione Esercizio 2.1

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$$s(t) = \sum_{n=-\infty}^{+\infty} x(t - nT) \qquad x(t) = A\,\mathrm{tri}\left(\frac{2t}{\tau}\right)$$

- Dalla tavola delle trasformate:

$$S(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X\left(\frac{n}{T}\right) \delta\left(f - \frac{n}{T}\right)$$

$$\mathrm{tri}\left(\frac{t}{T}\right) \rightarrow T\,\mathrm{sinc}^2(f\,T) \quad \Longrightarrow \quad A\,\mathrm{tri}\left(\frac{2t}{\tau}\right) \rightarrow \frac{A\tau}{2} \mathrm{sinc}^2\left(f\,\frac{\tau}{2}\right)$$

- Sostituendo:

$$S(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} \frac{A\tau}{2} \mathrm{sinc}^2\left(\frac{n}{T}\,\frac{\tau}{2}\right) \delta\left(f - \frac{n}{T}\right) = \frac{A\tau}{2T} \sum_{n=-\infty}^{+\infty} \mathrm{sinc}^2\left(\frac{\tau}{2T} n\right) \delta\left(f - \frac{n}{T}\right)$$

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Soluzione Esercizio 2.2

$$x(t)=\sum_n \mu_n e^{j\frac{2\pi}{T}nt}$$

$$\mu_n=\frac{1}{T}\int_{-T/2}^{T/2}\underbrace{x(t)}_{\text{pari}}e^{-j\frac{2\pi}{T}nt}dt=\frac{1}{T}\int_{-\tau/2}^{\tau/2}\underbrace{A\,\mathrm{tri}\left(\frac{2t}{\tau}\right)}_{\text{pari}}e^{-j\frac{2\pi}{T}nt}dt=\frac{1}{T}\int_{-\tau/2}^{\tau/2}A\,\mathrm{tri}\left(\frac{2t}{\tau}\right)\left(\underbrace{\cos\left(\frac{2\pi}{T}nt\right)}_{\text{pari}}+j\underbrace{\sin\left(\frac{2\pi}{T}nt\right)}_{\text{dispari}}\right)dt=$$

$$=\frac{1}{T}\int_{-\tau/2}^{\tau/2}A\,\mathrm{tri}\left(\frac{2t}{\tau}\right)\cos\left(\frac{2\pi}{T}nt\right)dt+j\frac{1}{T}\int_{-\tau/2}^{\tau/2}A\,\mathrm{tri}\left(\frac{2t}{\tau}\right)\sin\left(\frac{2\pi}{T}nt\right)dt=\frac{1}{T}2\int_{0}^{\tau/2}A\left(1-\frac{2t}{\tau}\right)\cos\left(\frac{2\pi}{T}nt\right)dt=$$

$$=\frac{2A}{T\tau}\int_0^{\tau/2}(\tau-2t)\cos\left(\frac{2\pi}{T}nt\right)dt=\frac{2A}{T\tau}\left[\left.(\tau-2t)\frac{\sin\left(\frac{2\pi}{T}nt\right)}{\frac{2\pi}{T}n}\right|_0^{\tau/2}+\int_0^{\tau/2}2\frac{\sin\left(\frac{2\pi}{T}nt\right)}{\frac{2\pi}{T}n}dt\right]=-\frac{4A}{T\tau}\left.\frac{\cos\left(\frac{2\pi}{T}nt\right)}{\left(\frac{2\pi}{T}n\right)^2}\right|_0^{\tau/2}=$$

$$=\frac{A\tau}{T}\frac{1-\cos\left(\frac{\pi}{T}n\tau\right)}{\left(\frac{\pi}{T}n\tau\right)^2}=\frac{A\tau}{4T}\frac{2\sin^2\left(\frac{\pi}{2T}n\tau\right)}{\left(\frac{\pi}{2T}n\tau\right)^2}=\frac{A\tau}{2T}\mathrm{sinc}^2\left(\frac{\tau}{2T}n\right)$$

$$\boxed{x(t)=\frac{A\tau}{2T}\sum_n \mathrm{sinc}^2\left(\frac{\tau}{2T}n\right)e^{j\frac{2\pi}{T}nt}}$$

Soluzione Esercizio 2

TRASFORMATA DI FOURIER

$$\boxed{S(f)=\frac{A\tau}{2T}\sum_{n=-\infty}^{+\infty}\mathrm{sinc}^2\left(\frac{\tau}{2T}n\right)\delta\left(f-\frac{n}{T}\right)}$$

Stessi coefficienti

$$\boxed{s(t)=\frac{A\tau}{2T}\sum_{n}\mathrm{sinc}^2\left(\frac{\tau}{2T}n\right)e^{j\frac{2\pi}{T}nt}}$$

SERIE DI FOURIER

Esercizio 3

Determinare i valori di A ed F_A che rendono sempre valida la seguente uguaglianza:

$$\frac{\sin(2\pi F_1 t)}{2\pi F_1 t} * \frac{\sin(2\pi F_2 t)}{2\pi F_2 t} = A\frac{\sin(2\pi F_A t)}{2\pi F_A t}$$

Soluzione Esercizio 3

$$\frac{\sin(2\pi F_1 t)}{2\pi F_1 t} * \frac{\sin(2\pi F_2 t)}{2\pi F_2 t} = A\frac{\sin(2\pi F_A t)}{2\pi F_A t}$$

- Dalle tavole delle trasformate:

$$\frac{1}{T}\operatorname{sinc}\left(\frac{t}{T}\right) \rightarrow p_{1/T}(f) \rightarrow \operatorname{sinc}\left(\frac{t}{T}\right) \rightarrow T\, p_{1/T}(f)$$

$$\frac{\sin(2\pi F_1 t)}{2\pi F_1 t} = \operatorname{sinc}(2F_1 t) \rightarrow \frac{1}{2F_1} p_{2F_1}(f)$$

Soluzione Esercizio 3

$$\mathrm{sinc}(2F_1 t) * \mathrm{sinc}(2F_2 t) = A\,\mathrm{sinc}(2F_A t)$$

$$\Downarrow$$

$$\frac{1}{2F_1} p_{2F_1}(f) \frac{1}{2F_2} p_{2F_2}(f) = \frac{A}{2F_A} p_{2F_A}(f)$$

$$\frac{1}{4F_1F_2} p_{2\min\{F_1,F_2\}}(f) = \frac{A}{2F_A} p_{2F_A}(f)$$

$$\boxed{\begin{array}{l} F_A = \min\{F_1, F_2\} \\ \dfrac{1}{4F_1F_2} = \dfrac{A}{2F_A} \quad \rightarrow A = \dfrac{2F_A}{4F_1F_2} = \dfrac{\min\{F_1,F_2\}}{2F_1F_2} = \dfrac{1}{2\max\{F_1,F_2\}} \end{array}}$$

Esercizio 4

Calcolare l'energia dei seguenti segnali:

$$x_1(t) = 5\mathrm{sinc}(2t) \qquad x_2(t) = 2\mathrm{sinc}^2(t/2) \qquad x_3(t) = \mathrm{sinc}(2t)\mathrm{sinc}(3t)$$

Soluzione Esercizio 4.1

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$$x_1(t) = 5\,\text{sinc}(2t)$$

- Applichiamo il teorema di Parseval:

$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |X(f)|^2\, df$$

$$E(x_1) = \int_{-\infty}^{+\infty} |x_1(t)|^2\, dt = \int_{-\infty}^{+\infty} |X_1(f)|^2\, df =$$

$$= \int_{-\infty}^{+\infty} \left| 5\frac{1}{2} p_2(f) \right|^2 df = \int_{-1}^{1} \frac{25}{4}\, df = \frac{25}{2}$$

Soluzione Esercizio 4.2

$$x_2(t) = 2\,\mathrm{sinc}^2\left(\frac{t}{2}\right)$$

- Applichiamo il teorema di Parseval:

$$E(x_2) = \int_{-\infty}^{+\infty} |x_2(t)|^2\,dt = \int_{-\infty}^{+\infty} |X_2(f)|^2\,df = 4\int_{-\infty}^{+\infty} \left[2\,\mathrm{tri}(2f)\right]^2 df =$$

$$= 16\int_{-\frac{1}{2}}^{+\frac{1}{2}} (1-2|f|)^2\,df = 16\cdot 2\int_{0}^{+\frac{1}{2}} (1-2f)^2\,df =$$

$$= 32\int_{0}^{+\frac{1}{2}} \left(1-4f+4f^2\right)df = 32\left(f-4\frac{f^2}{2}+4\frac{f^3}{3}\Bigg|_0^{\frac{1}{2}}\right) = 32\left(\frac{1}{2}-\frac{4}{8}+\frac{4}{24}\right) = \frac{16}{3}$$

Soluzione Esercizio 4.3

$$x_3(t) = \text{sinc}(2t)\,\text{sinc}(3t)$$

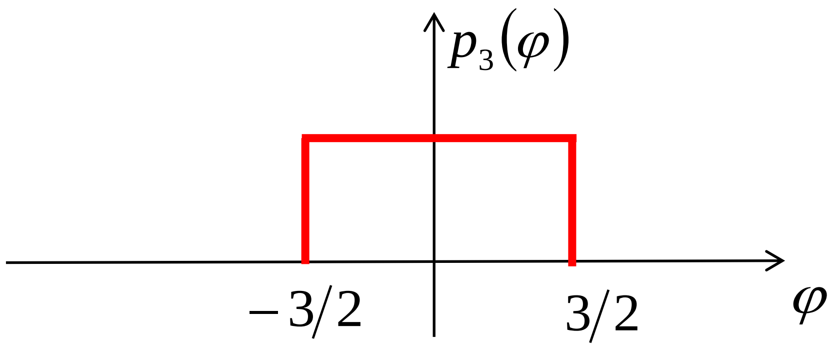
- Applichiamo il teorema di Parseval:

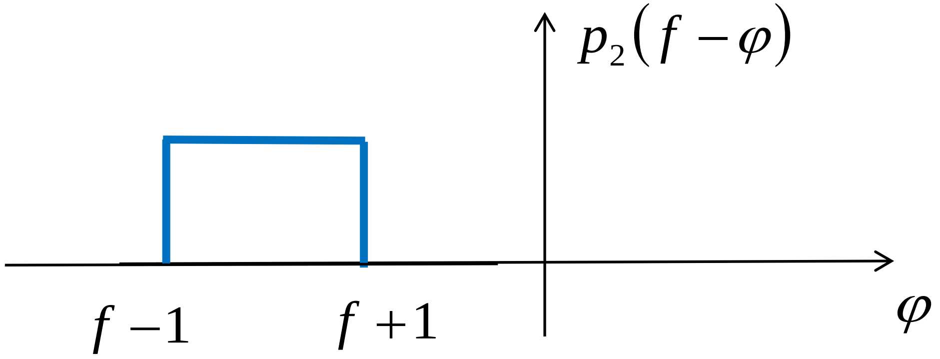
$$E(x_3) = \int_{-\infty}^{+\infty} |s_2(t)|^2 \, dt = \int_{-\infty}^{+\infty} |S_2(f)|^2 \, df$$

$$X_3(f) = \frac{1}{2} p_2(f) * \frac{1}{3} p_3(f) = \frac{1}{6} p_2(f) * p_3(f) =$$

$$= \frac{1}{6} \int_{-\infty}^{+\infty} p_3(\varphi)\, p_2(f-\varphi)\, d\varphi$$

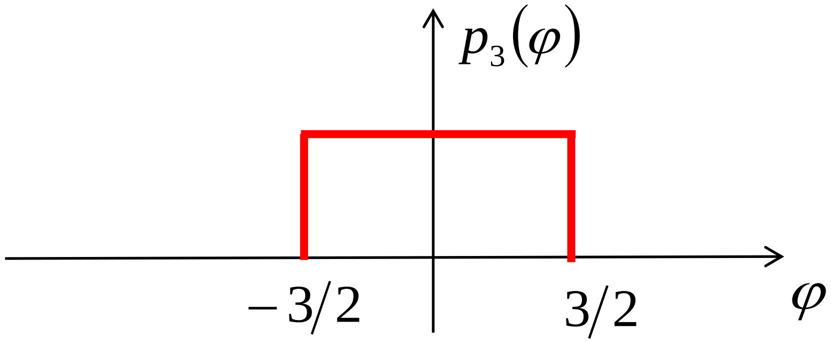
Soluzione Esercizio 4.3

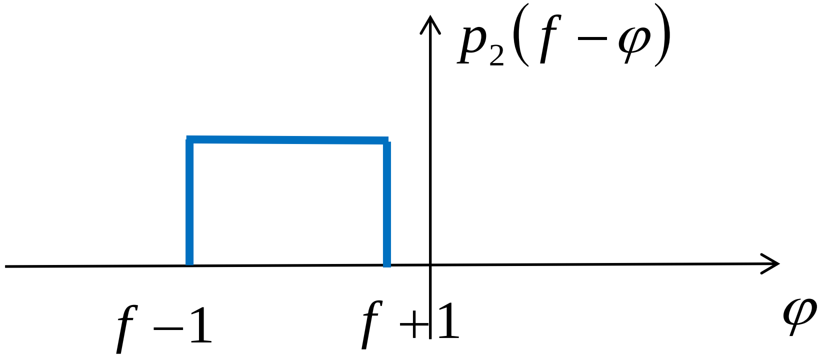




$$f < -\frac{5}{2} \text{ o } f > \frac{5}{2} \rightarrow S_2(f) = 0$$

Soluzione Esercizio 4.3



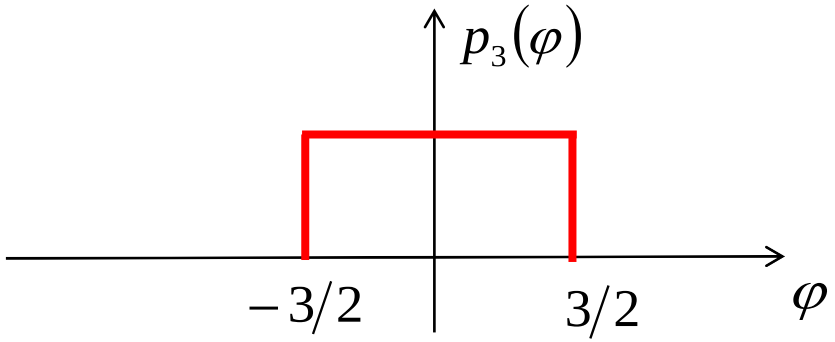


$$-\frac{5}{2} < f < -\frac{1}{2}$$

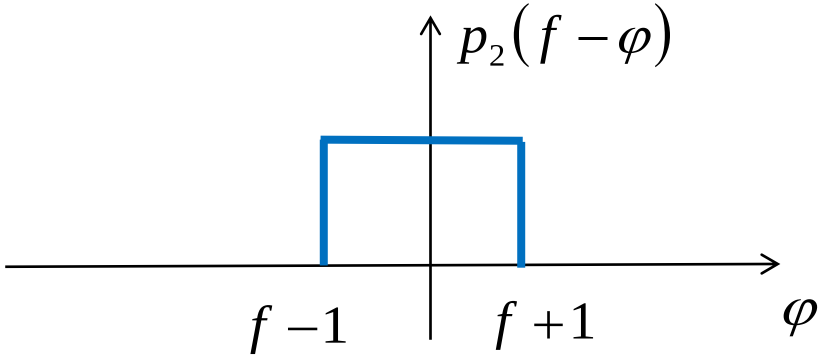
$$X_3(f) = \frac{1}{6}\int_{-\infty}^{+\infty} p_3(\phi)\cdot p_2(f-\phi)\,d\phi =$$

$$= \frac{1}{6}\int_{-\frac{3}{2}}^{f+1} 1\,d\phi = \frac{1}{6}\left(f+1+\frac{3}{2}\right) = \frac{f}{6}+\frac{5}{12}$$

Soluzione Esercizio 4.3



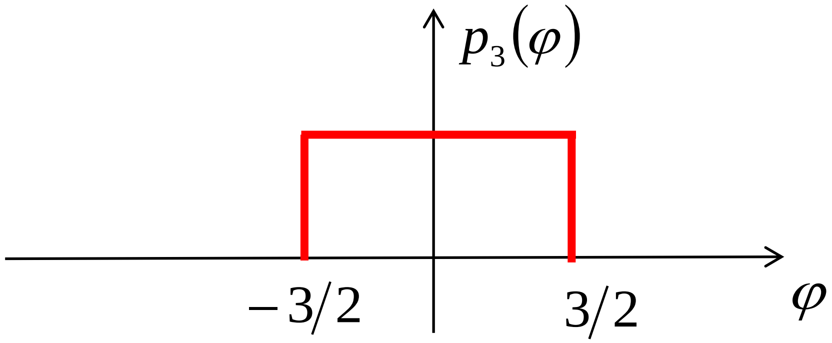
$$-\frac{1}{2} < f < \frac{1}{2}$$

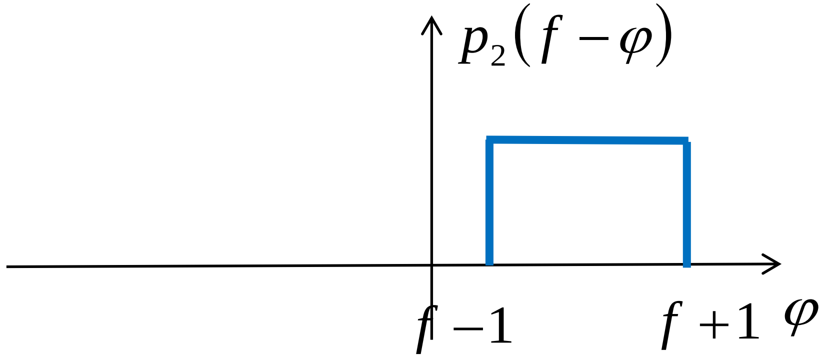


$$X_3(f) = \frac{1}{6}\int_{-\infty}^{+\infty} p_3(\phi)\cdot p_2(f-\phi)\,d\phi =$$

$$= \frac{1}{6}\int_{f-1}^{f+1} 1\,d\phi = \frac{1}{3}$$

Soluzione Esercizio 4.3





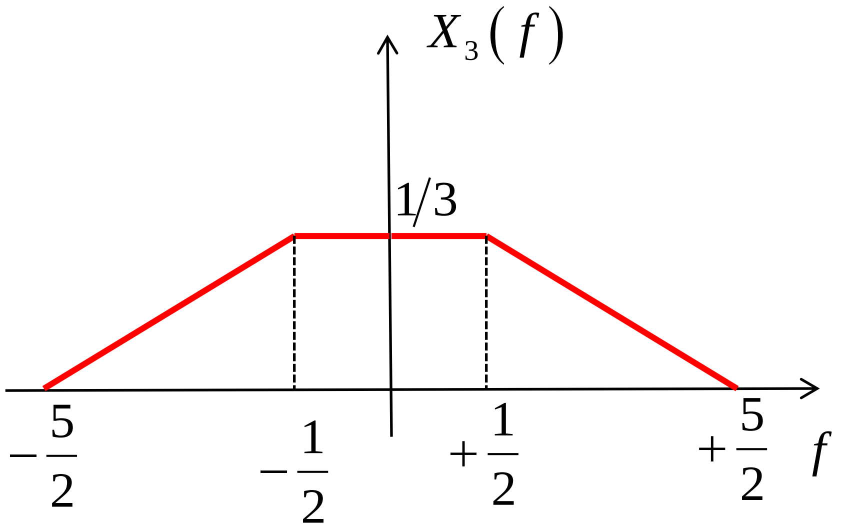
$$\frac{1}{2} < f < \frac{5}{2}$$

$$X_3(f) = \frac{1}{6}\int_{-\infty}^{+\infty} p_3(\phi)\cdot p_2(f-\phi)\,d\phi =$$

$$= \frac{1}{6}\int_{f-1}^{\frac{3}{2}} 1\,d\phi = \frac{1}{6}\left(\frac{3}{2} - f + 1\right) = -\frac{f}{6} + \frac{5}{12}$$

Soluzione Esercizio 4.3

$$X_3(f)=\frac{1}{6}\begin{cases}0 & |f|>\frac{5}{2}\\ f+\frac{5}{2} & -\frac{5}{2}<f<-\frac{1}{2}\\ 2 & -\frac{1}{2}<f<\frac{1}{2}\\ -f+\frac{5}{2} & \frac{1}{2}<f<\frac{5}{2}\end{cases}$$



$$E(x_3)=\int_{-\infty}^{+\infty}|X_3(f)|^2\,df=2\frac{1}{36}\int_{1/2}^{5/2}(-f+5/2)^2 df+2\frac{1}{36}\int_0^{1/2}4\ df=$$

$$=\frac{1}{18}\int_0^2\varphi^2 d\varphi+\frac{1}{9}=\frac{1}{18}\frac{\varphi^3}{3}\bigg|_0^2+\frac{1}{9}=\frac{1}{9}\frac{4}{3}+\frac{1}{9}=\frac{7}{27}$$