FWD Calibration and Monte Carlo simulation (Geometric)

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1 Spot Price

Our starting point in this framework was the assumption of a General Geometric model for the evolution of the electricity price. Therefore, we had that the following expression, where S_t represents the spot price:

$$ln(S_t) = ln(\Lambda_t) + \sum_{i=1}^{m} X_i(t) + \sum_{j=1}^{n} Y_i(t) \tag{1}$$

where respectively:

$$dX_i(t) = (\mu_i(t) - \alpha_i(t) * X_i(t))dt + \sum_{i=1}^{p} \sigma_{ik}(t)dB_k(t)$$
$$dY_i(t) = (\delta_i(t) - \beta_i(t) * Y_i(t))dt + \eta_i(t)dI_j(t)$$

In our case, we have p=m=n=1, $\mu(t) = 0, \delta(t) = 0$ and α, σ constant for the Gaussian OU and β, η constant for the additive OU, with $I(t)$ Variance Gamma process. So the expression becomes:

$$ln(S_t) = ln(\Lambda_t) + X(t) + Y(t)$$

where

$$dX(t) = -\alpha * X(t)dt + \sigma dB(t)$$
$$dY(t) = -\beta * Y(t)dt + \eta dI(t)$$
$$\Lambda_t = A * sin(2\pi t) + B + Ct$$

For X(t), we can find its solution thanks to

$$d(X_t * e^{\alpha t}) = e^{\alpha t}(-\alpha X_t dt + \sigma dB(t) + X_t \alpha e^{\alpha t} dt = e^{\alpha t}\sigma dB(t)$$

$$X_t * e^{\alpha t} = X_0 + \sigma \int_0^t e^{\alpha s} dB(s)$$

$$X_t = X_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dB(s) \tag{2}$$

With the same procedure we also get

$$Y_t = Y_0 e^{-\beta t} + \eta \int_0^t e^{-\beta(t-s)} dI(s) \tag{3}$$

We can get the Spot price by

$$S(t) = \Lambda(t) e^{X(t)+Y(t)} \tag{4}$$

Some **constraints** can be specified about the model parameters. If we want our model to have mean reverting behaviour, we need to put both α and β as > 0. In order to have statistical meaning, σ and η must be positive as they are volatilities. Moreover we require boundaries for the parameters A, B, C and η.

In order to have finite moments for the spot price, another condition that needs to be imposed is the *Condition G* of the Geometric Model, the existence of a C > 0 s.t.

$$\int_0^T \int_1^{+\infty} (e^{Cz} - 1) l(dt, dz) < +\infty$$

where $l(dt, dz)$ is the Lévy measure of $I(t)$.

2 Future Price under Q

We were now asked to write the expression of the forward price under the risk neutral measure using the Esscher trasform with $\tilde{\theta}_j, \hat{\theta}_k$ both equal to zero. In order to answer, we considered the Proposition 4.6 in *Benth et al.(2008)* [1] which states that under the condition:

$$sup_{0<s<\tau}|\eta e^{-\beta(\tau-s)|} \leq C$$

the forward price f(t,τ) is:

$$f(t,\tau) = \Lambda(\tau)\Theta(t,\tau,\theta(.))(e^{-\alpha(\tau-t)}X(t) + e^{-\beta(\tau-t)}Y(t)) \tag{5}$$

where $\Theta(t,\tau,\theta(.))$ is given as (with $\theta = 0$):

$$ln(\Theta(t,\tau,\theta(.))) = ln(\Theta(t,\tau,0)) = \Psi(t,\tau,-i\eta e^{-\beta(\tau-t)}) - \Psi(t,\tau,0) + \frac{1}{2}\int_t^\tau \sigma^2 e^{-2\alpha(\tau-u)}du \tag{6}$$

We know the integral can be analytically solved:

$$\frac{1}{2}\int_t^\tau \sigma^2 e^{-\alpha(\tau-u)}du = \frac{1}{2}[\frac{\sigma^2 e^{-2\alpha(\tau-u)}}{2\alpha u}]_t^T = \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha(\tau-t)})$$

and that Ψ is the characteristic exponent of the variance gamma:

$$\Psi(t,\tau,u) = -\frac{1}{\nu}ln(1 - i\delta\nu u + \frac{\sigma^2\nu u^2}{2}) \tag{7}$$

with ν the variance rate of the Gamma process. Finally we obtain this formula for the forward price:

$$f(t,\tau) = \Lambda(\tau)\Theta(t,\tau,0)e^{(e^{-\alpha(\tau-t)}X(t)+e^{-\beta(\tau-t)}Y(t))} \tag{8}$$

with the adjustment factor under the risk-neutral measure

$$\Theta(t,\tau,0) = exp(\Psi_{VG}(t,\tau,-i\eta e^{-\beta(\tau-t)} + \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha(\tau-t)})) \tag{9}$$

3 Swap Price

Following the procedure described in *Benth et al.(2008)* [1], the swap price at time t for a contract with a delivery period $[\tau_1, \tau_2]$ can be numerically estimated using a weighted sum over the forward prices evaluated at discrete monitoring dates within the delivery period.

For example, for the NOV4 swap the delivery period corresponds to the month of December 2024, and the settlement occurs on specific weekly observation dates, which are the Mondays within the period, plus the first and the last day of December; We can now compute the swap price as the sum of forwards weighted by the length of the time interval between two subsequent monitoring dates. Thus, the swap price $F(t,\tau_1,\tau_2)$ can be expressed as:

$$F(t,\tau_1,\tau_2) = \sum_{i=1}^{N} w(u_i,\tau_1,\tau_2) * f(t,u_i) \tag{10}$$

where:

- N is the total number of discrete monitoring dates within the delivery period.
- $f(t,u_i)$ is the forward price at time t for delivery at the future date u_i, given by equation 8 from the previous section.

4 Model Parameters Calibration

The calibration of the model parameters $\alpha, \beta, \sigma, \eta, A, B, C$ was performed to ensure that the theoretical swap prices obtained from the model match the market prices provided in `DATA_DEEEX.xlsx`. The MATLAB script responsible for this step is `GeometricModel-VG-MAIN.m`.

Given some ambiguity about whether market prices should be standardized prior to calibration, we chose to check out both approaches.

1. **Calibration with standardized prices**: Before the calibration, the market prices were transformed into zero mean and unit variance. Following the optimization process, the model prices were rescaled to correspond to the original distribution of market prices.

2. **Calibration without standardization**: The calibration was done according to raw market prices.

Figures 4 and 5 show how the two approaches differ in their behavior.

Calibration Method

The approach used for calibration involves minimizing the Mean Squared Error (MSE) between the market option prices and the model-generated option prices. This is achieved using MATLAB's `fmincon` function, which solves nonlinear optimization problems under constraints.

The objective function minimized is:

$$\text{Error} = \frac{1}{11} * \sum_{i=1}^{11} (\text{Model Price}_i - \text{Market Price}_i)^2 \tag{11}$$

where:

- Market Price_i represents the observed market prices of the i^{th} swap.
- Model Price_i refers to the i^{th} swap prices computed using the calibrated model.

Both calibration procedures gave visually equivalent outputs which are almost equivalent but standardizing the prices results in a lower error. This result implies that normalization might be a useful preprocessing step, particularly for datasets with high variability or many units of measurement.

Method	**RMSE**	**MAE**
Not Standardized	130.08	113.09
Standardized	51.72	41.13

Discussion of Results

Plots below compares the market pricing (blue) and model-generated option prices (red) after calibration. As previously said, the standardized calibration results in a better fitted model rather than the unstandardized one.

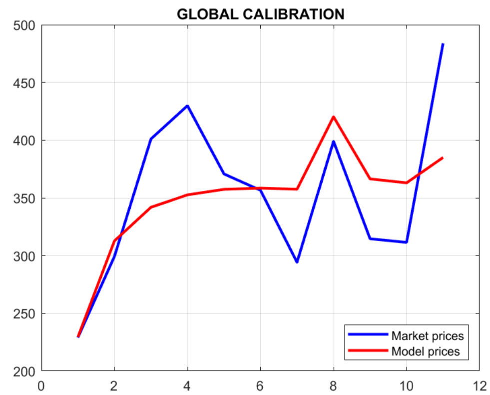

Figure 1: Standardized Calibration

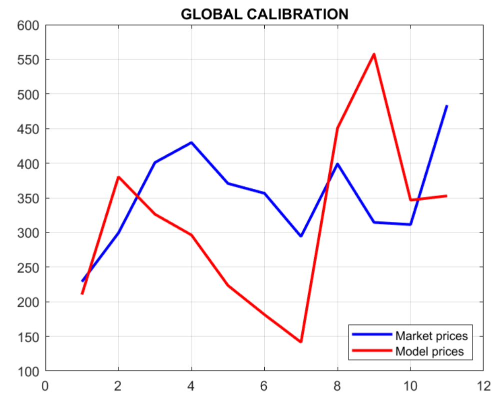

Figure 2: Unstandardized Calibration

α	σ	β	η	**A**	**B**	**C**	ν	μ_{VG}	σ_{VG}
1.1149	1.0268	1.1659	1.1371	0.5352	-3.8674	2.7329	1.0781	0.0577	1.1807

Table 1: Parameters for the standardized calibration

α	σ	β	η	**A**	**B**	**C**	ν	μ_{VG}	σ_{VG}
6.4498	6.7486	0.8624	2.8521	90.2073	-964.4412	1087.11078	0.00653	-2.03998	1.36157

Table 2: Parameters for the unstandardized calibration

Comment on the calibration results and speed

The unstardized calibration gave a Root Mean Squared Error of 130.08 , indicating a worse degree of precision in fitting the model to observed market prices than the standardized calibration. The calibrated parameters in Table 2 confirm the expected mean-reverting behavior of the underlying factors X and Y, which is compatible with energy price dynamics. The volatility parameters σ and η show significant uncertainty, in line with market behavior.

The calibration method needed a large amount of processing time. This is mostly due to the Monte Carlo simulations required to calculate theoretical prices. One possible method to improve calibration speed is to start with a fewer amount of Monte Carlo paths and gradually increase the precision as the optimization convergences.

Are the calibration results reliable?

The calibration results in Table 2 appear quite reliable even though Figure 2 shows an unoptimal calibration. This is mainly to the fact that swap market prices present high variance. This doesn't allow the model to learn efficiently the parameters. Something completely different happens when standardized prices are considered. As the variance between the data is slow, the calibration procedure better learns the relationship between them. Therefore the initial standardization can be useful, as it happens in other scientific fields when a minimization problem is involved. The main drawback of this approach is that parameters loose their financial meaning.

Further improvements could be made by increasing the calibration dataset dimension including data from different periods or market circumstances.

Proposals to improve the model parsimony

First of all, parameters related to the seasonality term $\Lambda(t)$ can be learned outside the calibration procedure since they are related only to the price of the energy spot. In this way we would reduce the complexity of the optimization procedure. Secondly, the choice of the source of randomness in the additive OU equation can play a huge role. If its distribution increases the model complexity without a significant improvement, a simpler and known distribution can be chosen in order to have also close formulas.

5 Implied Volatility Curve

After obtaining the calibrated parameters in the previous point, we proceeded to compute the implied volatility of the call options, as requested in Step 5 of the assignment. The idea was then to invert the prices in order to find the implied volatilities, using the Black and Scholes formula.

Methodology

The procedure followed in `GeometricModel_VG_main.m` is summarized as follows:

1. **Option Price Computation**: Using the calibrated parameters from the previous step, the script simulates the prices of call options written on futures for different strike prices. We considered the standardized parameters, as they yielded better performance.
2. **Implied Volatility Extraction**: For each strike price, the implied volatility was computed using MATLAB's `blsimpv` function. This function calculates the implied volatility from the call price under the Black-Scholes assumption.

The resulting implied volatility curve is shown below:

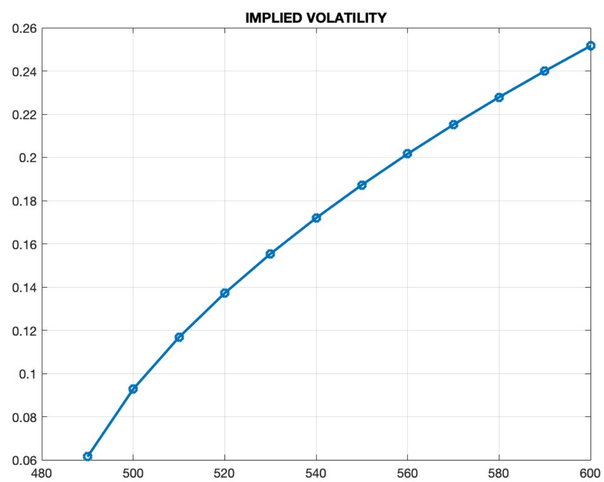

Figure 3: Implied volatility curve derived from the model.

Discussion of Results

As observed in Figure 6, the implied volatility curve does not exhibit the typical "smile" shape, but it presents an increasing pattern as we shift from at-the-money call options to out-of the money call options. This increase in volatility for an out-of-the-money call option is a typical phenomenon observed in the commodities market. An high volatility for a higher strike can be associated with an high demand for out-of-money calls.

5.1 Analytical Formula

In this case, as we do not the analytical expression of the probability distribution function, we do not have an analytical formula for the pricing of an option written on futures. This is why we have to base our analysis off simulations.

6 Procedure with only the Gaussian OU

The last point asked to recompute all the procedure described above using only the Gaussian OU. What happens basically is that the spot and the forward price becomes:

$$S_t = \Lambda(t)e^{X_t}$$

$$f(t,\tau) = \Lambda(\tau) * \frac{\sigma^2}{4\alpha}(1 - e^{-2\alpha(\tau-t)}) * e^{X_t e^{-\alpha(\tau-t)}}$$

with the swap price computed as before.

Also in this case the standardized and the unstandardized calibration was performed, with the first one which reports a better approximation.

Method	**RMSE**	**MAE**
Not Standardized	150.83	126.60
Standardized	55.52	47.73

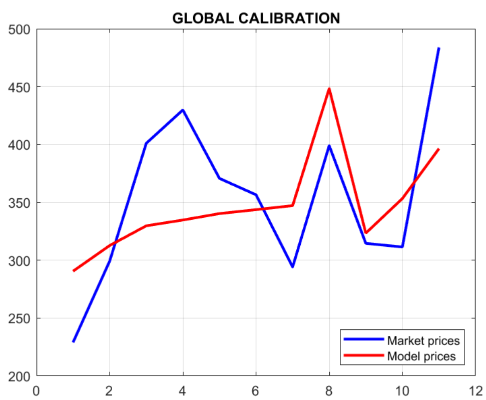

Figure 4: Standardized Calibration

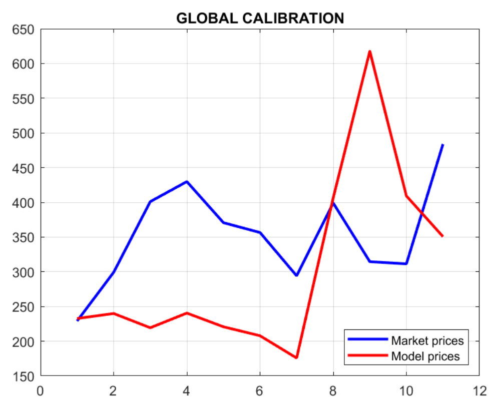

Figure 5: Unstandardized Calibration

α	σ	**A**	**B**	**C**
4.0160	14.0285	2.4816-06	-7.1193e-05	4.4572e-05

Table 3: Parameters for the standardized calibration

α	σ	**A**	**B**	**C**
0.9715	1.7112-08	337.7965	2597.6974	-635.5002

Table 4: Parameters for the unstandardized calibration

Observing the RMSE and the MAE, we can say that these calibrations performed worse than the ones in the previous model, indicating that the additive OU driven by a Variance Gamma plays a important role in the model. Also the parameters obtained by the optimization are not really accurate as in Table 4 we get a volatility almost null, indicating a almost-deterministic behavior which is distant from the reality of the market.

With this model we then computed the price of the call option on the 4Q25 swap contract and than the implied volatility curve was found.

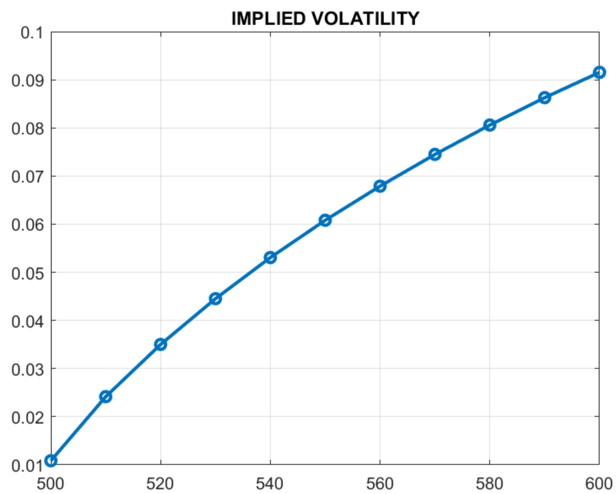

Figure 6: Implied volatility curve with Gaussian OU.

Due to the low value of the parameter σ, we get a truncated implied volatility curve, specifically at the "in-the-money" strikes. However, in the remaining part of the curve, it is possible to observe a similar shape with the curve of the previous model, where at higher strikes correspond higher implied volatilities.

6.1 Conclusion

This last point indicates how a single Gaussian OU process cannot be enough to model the electricity spot price. This one is severely affected by a lot of external events that can impact significantly the value in a really short time interval, therefore an additive OU equation driven by a jump process can be helpful. In conclusion a pure continuous process as a Brownian motion can't be enough for our purposes.

7 References

[1]: *F.E. Benth, J.S. Benth, S. Koekebakker, STOCHASTIC MODELLING of ELECTRICITY and RELATED MARKET, 2008*