CHAPTER 5

APPLICATION OF THE FINITE ELEMENT METHOD TO SOME NONLINEAR PROBLEMS

Introduction

Consider the minimization problem: Find $u \in U \subset V$ such that $J(u) = \inf_{v \in U} J(v)$, with a functional J of the form $J(v) = F(v) - f(v)$, $f \in V'$. There are two ways in which this problem can become nonlinear (the nonlinearity is as usual that of the mapping $f \to u$): Either the functional is quadratic, i.e., $F(v) = \frac{1}{2}a(v, v)$ but the set U is not a vector space, or the functional is not quadratic, in which case the problem is nonlinear even if the set V is a vector space.

In the first case, we have shown (Section 1.1) that when U is a closed convex subset of the space V the minimization problem is equivalent to a set of *variational inequalities*. Several important physical problems correspond to this modeling, in particular the *obstacle problem*, which corresponds to the following data:

$$V = H_0^1(\Omega), \quad \Omega \subset \mathbf{R}^2,$$

$$U = \{v \in H_0^1(\Omega);\ v \geq \chi \text{ a.e. in } \Omega\},$$

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \qquad f(v) = \int_\Omega fv \,\mathrm{d}x.$$

In Section 5.1, we consider the finite element approximation of this problem. Following an analysis of R.S. Falk, we show (Theorem 5.1.2) that the discrete solution u_h obtained with triangles of type (1) satisfies

$$\|u - u_h\|_{1,\Omega} = O(h).$$

This result is itself a consequence of an abstract error estimate (Theorem 5.1.1) valid for a general class of variational inequalities.

In the second case, i.e., of non quadratic functionals, there are almost as many problems as there are non quadratic functionals and, consequently, no general theory is available. We have nevertheless considered two significant examples.

The first one, considered in Section 5.2, is the *minimal surface problem*, which consists in minimizing the functional

$$J(v) = \int_\Omega \sqrt{1 + \|\nabla v\|^2}\, dx, \quad \Omega \subset \mathbf{R}^2,$$

over all functions v in the convex set

$$U = \{v \in H^1(\Omega);\ v = u_0 \quad \text{on} \quad \Gamma\}, \quad u_0 \in H^1(\Omega).$$

Following a recent paper of C. Johnson and V. Thomée, we show (Theorem 5.2.2) that

$$\|u - u_h\|_{1,\Omega} = O(h),$$

where u_h is the discrete solution again obtained through the use of triangles of type (1).

The second problem, studied in Section 5.3, consists in minimizing the functional

$$J(v) = \frac{1}{p}\int_\Omega \|\nabla v\|^p\, dx - f(v), \quad f \in V',$$

over the space

$$V = W_0^{1,p}(\Omega), \quad p \geq 2.$$

We show in particular that this problem has a unique solution (Theorem 5.3.1), which is also solution of the equation

$$J'(u) = Au - f = 0,$$

where the operator $A: V \to V'$ is an instance of so-called *strongly monotone operators*, i.e., which satisfy inequalities of the form

$$(Au - Av)(u - v) \geq \chi(\|u - v\|)\|u - v\|,$$

for some function $\chi: [0,+\infty[\to [0,+\infty[$ such that $\lim_{t\to\infty} \chi(t) = \infty$.

Following a recent work by R. Glowinski and A. Marrocco, we next consider a finite element approximation of this problem (for $n = 2$) using again triangles of type (1). We then prove the following convergence results (Theorems 5.3.2 and 5.3.5):

$$\lim_{h\to 0} \|u - u_h\|_{1,p,\Omega} = 0 \quad \text{if } u \in W_0^{1,p}(\Omega),$$
$$\|u - u_h\|_{1,p,\Omega} = O(h^{1/(p-1)}) \quad \text{if } u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega).$$

The last error estimate is itself a corollary of an abstract error estimate valid for strongly monotone operators in general (Theorem 5.3.4).

Perhaps the most striking feature of nonlinear problems, by contrast with linear problems, is that their solutions *cannot be very smooth over the whole set* $\bar{\Omega}$ even if the data are very smooth. For example, the solution of the membrane problem is in general "only" in the space $H^2(\Omega)$, whatever the smoothness of the data χ, f and Γ. Consequently, *finite element "of low degree" (of the local polynomial spaces* P_K*) are sufficient for all practical purposes, a fact amply confirmed by numerical evidence.*

Finally, we mention that the three sections in this chapter can be read independently of each other.

5.1. The obstacle problem

Variational formulation of the obstacle problem

The *obstacle problem* consists in finding the equilibrium position of an elastic membrane, with tension τ, which

(i) passes through a curve Γ, i.e., the boundary of an open set Ω of the "horizontal" plane of coordinates (x_1, x_2),

(ii) is subjected to the action of a "vertical" force of density $F = \tau f$,

(iii) must lie over an "obstacle" which is represented by a function $\chi: \bar{\Omega} \to \mathbf{R}$, as illustrated in Fig. 5.1.1.

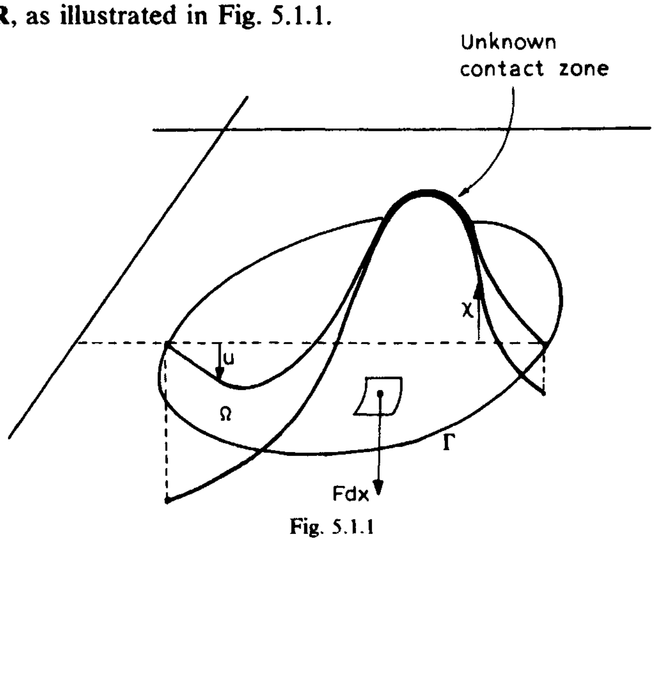

Fig. 5.1.1

Thus this is another *membrane problem* which, following the example given in Section 1.2, is associated with the following data:

$$\begin{cases} V = H_0^1(\Omega), \quad n = 2, \\ U = \{v \in H_0^1(\Omega), \quad v \geqslant \chi \text{ a.e. in } \Omega\}, \\ a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \\ f(v) = \int_\Omega fv \, \mathrm{d}x. \end{cases} \tag{5.1.1}$$

Throughout this section, we shall make the following assumptions on the functions χ and f:

$$\chi \in H^2(\Omega), \quad \chi \leqslant 0 \quad \text{on} \quad \Gamma, \quad f \in L^2(\Omega). \tag{5.1.2}$$

The set U, which is not empty by virtue of the second assumption of (5.1.2), is easily seen to be convex. To show that it is closed, it suffices to notice that every convergent sequence in the space $L^2(\Omega)$ contains an a.e. pointwise convergent subsequence.

Thus we may apply Theorem 1.1.1: There exists a unique function $u \in U$ which minimizes the membrane energy

$$J: v \to J(v) = \frac{1}{2} \int_\Omega \|\nabla v\|^2 \, \mathrm{d}x - \int_\Omega fv \, \mathrm{d}x \tag{5.1.3}$$

over the set U, and it is also the unique solution of the variational inequalities

$$\forall v \in U, \quad \int_\Omega \nabla u \cdot \nabla(v - u) \, \mathrm{d}x \geqslant \int_\Omega f(v - u) \, \mathrm{d}x, \tag{5.1.4}$$

by Theorem 1.1.2.

This variational problem corresponds to the formal solution of a boundary value problem. See Exercise 5.1.1.

One should notice that *the region where the membrane touches the obstacle, i.e., the set $\{x \in \Omega;\ u(x) = \chi(x)\}$, is not known in advance.*

By contrast with the linear membrane problem of Section 1.2, *the solution of the obstacle problem is not smooth in general, even if the data are very smooth.* To be convinced of this phenomenon, consider the one-dimensional analog with $f = 0$, as shown in Fig. 5.1.2. In this case, the solution is affine in the region where it does not touch the obstacle and consequently, whatever the smoothness of the function χ, the

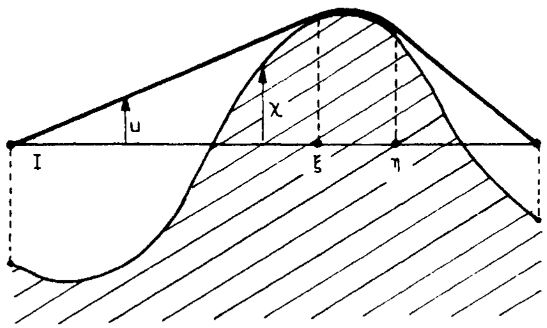

Fig. 5.1.2.

second derivative of u has discontinuities at points such as ξ and η. Therefore the solution u is "only" in the space $H^2(I)$.

These results carry over to the 2-dimensional case, but they are of course much less easy to prove. For example, it is known that if the function χ satisfies the assumptions of (5.1.2), $f = 0$, and $\bar{\Omega}$ is a convex polygon, the solution u belongs to the space $H_0^1(\Omega) \cap H^2(\Omega)$. If the set $\bar{\Omega}$ is convex with a boundary of class $\mathscr{C}^2$ and assumptions (5.1.2) hold then we have again $u \in H_0^1(\Omega) \cap H^2(\Omega)$. In both cases, the norm $\|u\|_{2,\Omega}$ can be estimated in terms of the norms $\|\chi\|_{2,\Omega}$ and $|f|_{0,\Omega}$ of the data. These results are proved in Brezis & Stampacchia (1968) and Lewy & Stampacchia (1969).

An abstract error estimate for variational inequalities

We next consider the approximation of such a problem. Following an analysis due to R.S. Falk, we shall first prove an abstract error estimate (Theorem 5.1.1) which is valid for a general class of approximation schemes for variational inequalities of the form (5.1.5) below, and then we shall apply this result to a particular finite element method, well adapted to the present problem (Theorem 5.1.2).

The *abstract setting* is the following: Let V be a Hilbert space, with norm $\|\cdot\|$, let $a(\cdot,\cdot)$: $V \times V \to \mathbf{R}$ be a continuous, symmetric and V-elliptic bilinear form (with the usual V-ellipticity and continuity constants α and M), let f: $V \to \mathbf{R}$ be a continuous linear form, and let U be a non empty closed convex subset of V. Then there is a unique element u which satisfies (cf. Theorem 1.1.2).

$$u \in U \quad \text{and} \quad \forall v \in U, \quad a(u, v-u) \geqslant f(v-u). \tag{5.1.5}$$

Let then V_h be a finite-dimensional subspace of the space V and let U_h be a non empty closed convex subset of V_h. Observe that, in general, *the set U_h is not a subset of U.*

Then, quite naturally, the *discrete problem* consists in finding an element u_h such that

$$u_h \in U_h \quad \text{and} \quad \forall v_h \in U_h, \quad a(u_h, v_h - u_h) \geq f(v_h - u_h), \tag{5.1.6}$$

and, again, this abstract variational problem has a unique solution u_h.

In the proof of the next theorem, we shall need the mapping $A \in \mathscr{L}(V; V')$ defined by the relations

$$\forall v, w \in V, \quad Av(w) = a(v, w), \tag{5.1.7}$$

and which we already used in the proof of Theorem 1.1.3. Notice that in the present situation we do *not* have $Au = f$ in general, as in the case of the linear problem ($U = V$). Also, we shall consider a Hilbert space H, with norm $|\cdot|$ and inner product $(\cdot\,.\,\cdot)$, such that

$$\bar{V} = H \quad \text{and} \quad V \hookrightarrow H. \tag{5.1.8}$$

The space H will be identified with its dual, so that it may be in turn identified with a subspace of the dual space of V, as we showed in Section 3.2.

We are now in a position to prove an *abstract error estimate* in the norm $\|\cdot\|$.

Theorem 5.1.1. *Assume that*

$$(Au - f) \in H. \tag{5.1.9}$$

Then there exists a constant C independent of the subspace V_h and of the set U_h such that

$$\|u - u_h\| \leq C\Big(\inf_{v_h \in U_h} \{\|u - v_h\|^2 + |Au - f|\,|u - v_h|\} + \\ + |Au - f| \inf_{v \in U} |u_h - v|\Big)^{1/2}. \tag{5.1.10}$$

Proof. We have

$$\begin{aligned} \alpha\|u - u_h\|^2 &\leq a(u - u_h, u - u_h) \\ &= a(u, u) + a(u_h, u_h) - a(u, u_h) - a(u_h, u), \end{aligned}$$

and, using (5.1.5) and (5.1.6),

$$\forall v \in U, \quad a(u, u) \leq a(u, v) + f(u - v),$$
$$\forall v_h \in U_h, \quad a(u_h, u_h) \leq a(u_h, v_h) + f(u_h - v_h).$$

Therefore we deduce that, for all $v \in U$ and all $v_h \in U_h$,

$$\begin{aligned} \alpha\|u - u_h\|^2 &\leq a(u, v - u_h) + a(u_h, v_h - u) + f(u - v) + f(u_h - v_h) \\ &= a(u, v - u_h) - f(v - u_h) + a(u, v_h - u) - f(v_h - u) \\ &\qquad + a(u_h - u, v_h - u) \\ &= (f - Au, u - v_h) + (f - Au, u_h - v) \\ &\qquad + a(u - u_h, u - v_h). \end{aligned}$$

We thus have, for all $v \in U$ and all $v_h \in U_h$,

$$\alpha\|u - u_h\|^2 \leq |f - Au|(|u - v_h| + |u_h - v|) + M\|u - u_h\| \, \|u - v_h\|.$$

Since

$$\|u - u_h\| \, \|u - v_h\| \leq \frac{1}{2}\left(\frac{\alpha}{M}\|u - u_h\|^2 + \frac{M}{\alpha}\|u - v_h\|^2\right),$$

we obtain, upon combining the two previous inequalities,

$$\frac{\alpha}{2}\|u - u_h\|^2 \leq |f - Au|(|u - v_h| + |u_h - v|) + \frac{M^2}{2\alpha}\|u - v_h\|^2, \quad (5.1.11)$$

from which inequality (5.1.10) follows. □

Remark 5.1.1. Several comments are in order about this theorem:

(i) The proof has been given in such a way that it includes the case where the bilinear form is not symmetric.

(ii) If $U = V$ then $Au - f = 0$, so that, with the obvious choice $U_h = V_h$, the error estimate of (5.1.10) reduces to the familiar error estimate of Céa's lemma.

(iii) If the inclusion $U_h \subset U$ holds, then of course the term $\inf_{v \in U} |u_h - v|$ (which can be expected to be the harder to evaluate) vanishes in the error estimate. For such an example, see Exercise 5.1.3. This is not the case, however, of the finite element approximation of the obstacle problem which we shall describe below.

(iv) Also, had we not introduced the space H in our argument, we would have found, instead of inequality (5.1.11), the inequality

$$\frac{\alpha}{2}\|u-u_h\|^2 \leq \|f-Au\|^\star(\|u-v_h\|+\|u_h-v\|)+\frac{M^2}{2\alpha}\|u-v_h\|^2,$$

where $\|\cdot\|^\star$ denotes as usual the norm of the dual space of V. However, this last inequality is likely to yield a poorer order of convergence, since the term $\inf_{v_h\in U_h}\{\|u-v_h\|^2+|Au-f|\|u-v_h|\}$ can be anticipated to be of a higher order than the term $\inf_{v_h\in U_h}\{\|u-v_h\|^2+\|Au-f\|^\star\|u-v_h\|\}$. This will be confirmed in the proof of Theorem 5.1.2. □

Finite element approximation with triangles of type (1). Estimate of the error $\|u-u_h\|_{1,\Omega}$

Let us return to the obstacle problem. For simplicity, we shall assume that the set $\bar{\Omega}$ is a polygon, leaving the case of a curved boundary as a problem (Exercise 5.1.2). With a triangulation $\mathcal{T}_h$ of the set $\bar{\Omega}=\bigcup_{K\in\mathcal{T}_h}K$, we associate the finite element space X_h whose generic finite element is the triangle of type (1) and we let as usual

$$X_{0h}=\{v_h\in X_h;\ v_h=0 \quad \text{on} \quad \Gamma\}. \tag{5.1.12}$$

Letting $\mathcal{N}_h$ denote the set of the nodes of the space X_h, i.e., the set of all the vertices, we let

$$V_h=X_{0h}, \tag{5.1.13}$$

$$U_h=\{v_h\in V_h;\ \forall b\in\mathcal{N}_h,\ v_h(b)\geq\chi(b)\} \tag{5.1.14}$$

(as an element of the space $H^2(\Omega)$, the function χ is in the space $\mathscr{C}^0(\bar{\Omega})$ and, therefore, its point values are well defined).

Notice that *the set U_h is not in general contained in the set U*, as the one-dimensional case considered in Fig. 5.1.3 exemplifies.

Let us now apply the abstract error estimate of Theorem 5.1.1.

Theorem 5.1.2. *Assume that the solution u is in the space $H^2(\Omega)$. Then, given a regular family of triangulations, there exists a constant $C(u,f,\chi)$ independent of h such that*

$$\|u-u_h\|_{1,\Omega}\leq C(u,f,\chi)h. \tag{5.1.15}$$

Proof. We shall let

$$H=L^2(\Omega),$$

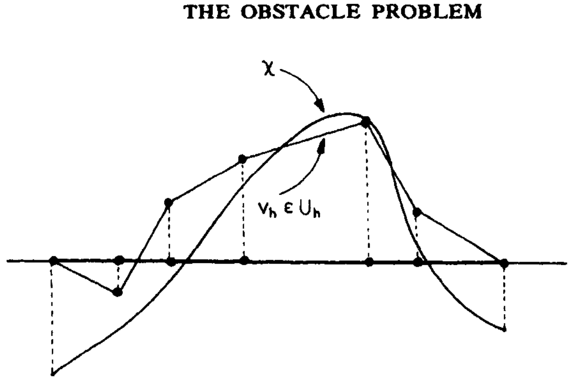

Fig. 5.1.3

so that we need to verify that $Au \in L^2(\Omega)$($f \in L^2(\Omega)$ by assumption). Since the solution u is assumed to be in the space $H^2(\Omega)$, we have

$$\forall v \in V, \quad Au(v) = \int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega \Delta u \, v \, dx$$

and thus

$$\forall v \in V, \ |Au(v)| \leq |\Delta u|_{0,\Omega} |v|_{0,\Omega},$$

so that Au is indeed an element of the space H.

Let $\Pi_h u$ denote as usual the X_h-interpolant of the function u, which is in the space X_{0h}. Since

$$\forall b \in \mathcal{N}_h, \ \Pi_h u(b) = u(b) \geq \chi(b),$$

it is also an element of the set U_h. Thus,

$$\begin{aligned} &\inf_{v_h \in U_h} \{\|u - v_h\|_{1,\Omega}^2 + |Au - f|_{0,\Omega} |u - v_h|_{0,\Omega}\} \leq \\ &\quad \leq \|u - \Pi_h u\|_{1,\Omega}^2 + (|\Delta u|_{0,\Omega} + |f|_{0,\Omega}) |u - \Pi_h u|_{0,\Omega} \\ &\quad \leq C(|u|_{2,\Omega}^2 + (|\Delta u|_{0,\Omega} + |f|_{0,\Omega}) |u|_{2,\Omega}) |u|_{2,\Omega}) h^2. \qquad (5.1.16) \end{aligned}$$

In order to evaluate the term $\inf_{v \in U} |u_h - v|_{0,\Omega}$, it is convenient to introduce the function (Fig. 5.1.4)

$$u_h^* = \max\{u_h, \chi\},$$

so that the inequality $u_h^* \geq \chi$ holds in Ω. Both functions u_h and χ being in the space $H^1(\Omega)$, it follows that their maximum u_h^* is also in $H^1(\Omega)$ (this is a non-trivial fact, whose proof may be found in p. 169 of LEWY & STAMPACCHIA (1969)). Finally, the condition $\chi \leq 0$ on Γ implies

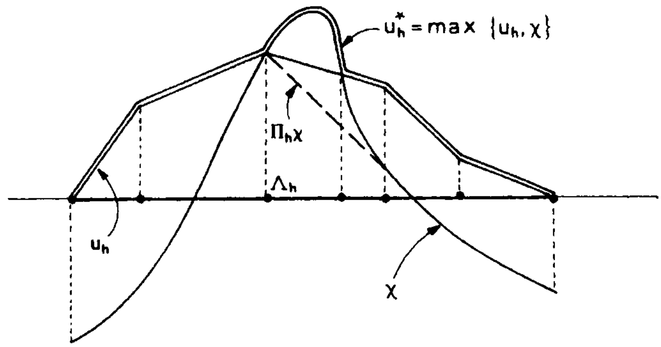

Fig. 5.1.4

that $u_h^* \in H_0^1(\Omega)$. Thus the function u_h^* is an element of the set U. Let

$$\Lambda_h = \{x \in \Omega;\ u_h < \chi\},$$

so that

$$|u_h - u_h^*|_{0,\Omega}^2 = \int_{\Lambda_h} |u_h - \chi|^2\,\mathrm{d}x,$$

since $u_h - u_h^* = 0$ on $\Omega - \Lambda_h$. Let us introduce the X_h-interpolant $\Pi_h\chi$ of the function χ. Since

$$\forall b \in \mathcal{N}_h, \quad u_h(b) \geq \chi(b) = \Pi_h\chi(b),$$

it follows that

$$u_h - \Pi_h\chi \geq 0 \quad \text{in} \quad \Omega.$$

Consequently,

$$\forall x \in \Lambda_h, \quad 0 < |(\chi - u_h)(x)| = (\chi - u_h)(x) \leq (\chi - \Pi_h\chi)(x) = |(\chi - \Pi_h\chi)(x)|,$$

and thus,

$$|u_h - u_h^*|_{0,\Omega}^2 = \int_{\Lambda_h} |u_h - \chi|^2\,\mathrm{d}x \leq \int_{\Lambda_h} |\chi - \Pi_h\chi|^2\,\mathrm{d}x \leq |\chi - \Pi_h\chi|_{0,\Omega}^2.$$

Therefore, we obtain

$$\inf_{v\in U} |u_h - v|_{0,\Omega} \leq |u_h - u_h^*|_{0,\Omega} \leq |\chi - \Pi_h\chi|_{0,\Omega} \leq C|\chi|_{2,\Omega}h^2, \tag{5.1.17}$$

and the conclusion follows from inequalities (5.1.16) and (5.1.17). □

Exercises

5.1.1. Show that the solution of the variational problem associated with the data (5.1.1) corresponds to the formal solution of the following boundary value problem:

$$\begin{cases} -\Delta u \geqslant f & \text{in } \Omega, \\ u \geqslant \chi & \text{in } \Omega, \\ (-\Delta u - f)(u-\chi) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \\ u = \chi & \text{on } \Gamma^*, \\ \partial_\nu u = \partial_\nu \chi & \text{on } \Gamma^*, \end{cases}$$

where Γ^* is the "interface" between the sets $\{x \in \Omega;\ u(x) = \chi(x)\}$ and $\{x \in \Omega;\ u(x) > \chi(x)\}$ and ∂_ν is the normal derivative operator along Γ^*. Notice that the set Γ^* is an unknown of the problem: This is why such a problem is also called a *free surface problem.*

5.1.2. Show that the error estimate (5.1.15) of Theorem 5.1.2 holds unchanged in the following situation: The set Ω is convex with a sufficiently smooth boundary, so that $u \in H^2(\Omega)$. Then we let $\bar{\Omega}_h = \bigcup_{K \in \mathcal{T}_h} K$ denote a triangulation made up of triangles, in such a way that all the vertices of $\mathcal{T}_h$ which are on the boundary of the set Ω_h are also on Γ (Fig. 5.1.5).

With such a triangulation, we associate the finite element space X_h whose generic element is the triangle of type (1) and we let X_{0h} denote as usual the subspace of X_h whose functions vanish on the boundary of the set Ω_h. The space V_h then consists of the functions in the space X_{0h} prolongated by zero on the set $\bar{\Omega} - \bar{\Omega}_h$ (thus, the functions in the space V_h are defined over the set $\bar{\Omega}$).

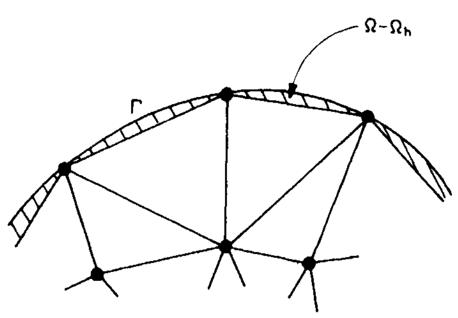

Fig. 5.1.5

[Hint: To prove the analog of inequality (5.1.16), show that, if $u \in H^2(\Omega) \cap H_0^1(\Omega)$, then

$$\|u\|_{m,\Omega-\Omega_h} \leq Ch^{2-m}\|u\|_{2,\Omega}, \quad m = 0, 1.$$

To prove the analog of inequality (5.1.17), assume for simplicity that $\chi = 0$ on Γ, and show that

$$|\max\{0, \chi\}|_{0,\Omega-\Omega_h} \leq |\chi|_{0,\Omega-\Omega_h} \leq Ch^2\|\chi\|_{2,\Omega}.]$$

5.1.3. Another problem which is modeled by variational inequalities is the *elastic-plastic torsion problem*, which arises in the following situation: Consider a cylindrical thin rod with a simply connected cross section $\bar{\Omega} \subset \mathbf{R}^2$, subjected to a torsion around the axis supporting the vector e_3. The torsion angle τ per unit length is assumed to be constant throughout the length of the rod (cf. Fig. 5.1.6, where the vertical scale should be considerably increased).

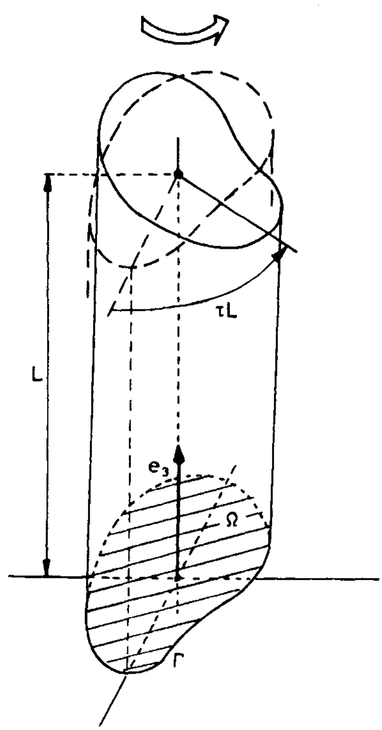

Fig. 5.1.6

Let us first assume that we are in the domain of validity of *linear* elasticity. Then certain simplifying assumptions (the weight of the rod is neglected among other things) imply that the components σ_{11}, σ_{12}, σ_{22} and σ_{33} of the stress tensor vanish everywhere in the rod, while the components σ_{13} and σ_{23} are functions of x_1, x_2 only, and are such that

$$\sigma_{13} = 2\mu\tau\partial_2 u, \quad \sigma_{23} = -2\mu\tau\partial_1 u,$$

where μ is the second Lamé coefficient of the constitutive material of the rod, and the *stress function* u satisfies

$$-\Delta u = 1 \quad \text{in} \quad \Omega \quad \text{and} \quad u = 0 \quad \text{on} \quad \Gamma.$$

Therefore the function u minimizes the functional

$$J: v \to J(v) = \frac{1}{2}\int_\Omega \|\nabla v\|^2 \,\mathrm{d}x - \int_\Omega fv \,\mathrm{d}x, \quad \text{with} \quad f = 1,$$

over the space $H_0^1(\Omega)$.

If we take into account the *plasticity* of the material, then the stresses cannot take arbitrary large values. A particular mathematical representation of this effect, known as the *von Mises criterion*, reduces in this case to the condition that the quantity $(|\sigma_{13}|^2 + |\sigma_{23}|^2)^{1/2}$, and consequently the norm $\|\nabla u\|$, cannot exceed a certain constant. Notice, however, that contrary to the linear case, it is not straightforward to recuperate the displacement field from the knowledge of the stress field, as shown by the discussion in DUVAUT & LIONS (1972, Chapter 5, Section 6).

Therefore this problem corresponds to the following data (where, for definiteness, the upper bound on $\|\nabla u\|$ has been set equal to one):

$$\begin{cases} V = H_0^1(\Omega), \qquad n = 2, \\ U = \{v \in H_0^1(\Omega);\ \|\nabla v\| \leq 1 \text{ a.e. in } \Omega\}, \\ a(u, v) = \displaystyle\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \\ f(v) = \displaystyle\int_\Omega fv \,\mathrm{d}x, \quad f \in L^2(\Omega), \end{cases}$$

with $f = 1$ in this case.

(i) Show that U is a non empty closed convex subset of the space V and, consequently, that the variational problem associated with the above data has a unique solution u (which can be shown to be in the space $W^{2,p}(\Omega) \cap H_0^1(\Omega)$ for all $p < \infty$ if $f \in L^\infty(\Omega)$ and the boundary Γ is smooth enough; cf. BREZIS & STAMPACCHIA (1968)).

Show that this problem amounts to formally solving the following boundary value problem:

$$\begin{cases} -\Delta u \leq f \quad \text{in} \quad \Omega, \\ \|\nabla u\| \leq 1 \quad \text{in} \quad \Omega, \\ (-\Delta u - f)(1 - \|\nabla u\|) = 0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \Gamma. \end{cases}$$

(ii) Consider the one-dimensional analogue of this problem and its finite element approximation, with

$$X_{0h} = \{v_h \in \mathscr{C}^0(\bar{I});\ v_{h|\bar{I}_i} \in P_1(\bar{I}_i),\ 1 \leq i \leq M,\ v_h(0) = v_h(1) = 0\},$$

where $\bar{I} = \bigcup_{i=1}^M \bar{I}_i$ is a partition of the interval $\bar{I} = [0, 1]$ with

$$\bar{I}_i = [x_{i-1}, x_i],\ 1 \leq i \leq M,\ h = \max_{1 \leq i \leq M} |x_i - x_{i-1}|,$$

and

$$U_h = \{v_h \in X_{0h};\ |v_h'| \leq 1 \text{ a.e.} \quad \text{in} \quad I\}.$$

Derive the error estimate

$$\|u - u_h\|_{1,I} = \mathrm{O}(h).$$

(iii) Returning to a two-dimensional polygonal set $\bar{\Omega}$, let

$$U_h = \{v_h \in X_{0h};\ \|\nabla v_h\| \leq \text{ a.e.} \quad \text{in} \quad \Omega\},$$

the space X_{0h} being defined as in (5.1.12).

Show that the X_h-interpolant of a function $v \in U$ is not necessarily contained in the set U_h.

Assume that the solution u belongs to the space $W^{2,p}(\Omega)$ for some $p \in]2, \infty]$. Then show that there exist appropriate quantities $\epsilon(h) > 0$ with $\lim_{h \to 0} \epsilon(h) = 0$ such that the functions $(1 + \epsilon(h))^{-1} \Pi_h u$ belong to the set U_h. Using this result, show that

$$\|u - u_h\|_{1,\Omega} = \mathrm{O}(h^{1/2 - 1/p}).$$

5.2. The minimal surface problem

A formulation of the minimal surface problem

Let Ω be a bounded open subset of the plane $\mathbf{R}^2$, and let u_0 be a function given on the boundary Γ of the set Ω.

The *minimal surface problem* consists in finding a function u which minimizes the functional

$$J: v \to J(v) = \int_\Omega \sqrt{1 + \|\nabla v\|^2}\, dx \tag{5.2.1}$$

over an appropriate space of functions which equal u_0 on Γ. In other words, among all surfaces given by an equation $x_3 = v(x_1, x_2)$, $x = (x_1, x_2) \in \Omega$ (for which the area can be defined) and which pass through a given curve of the form $x_3 = u_0(x_1, x_2)$, $(x_1, x_2) \in \Gamma$, one looks for a surface whose area is minimal.

The mathematical analysis of this problem is not easy. In particular, it is not straightforward to decide which function space is more appropriate to insure existence and uniqueness of a solution. However, we shall not go here into such matters, refering instead the reader to the section "Bibliography and Comments" for additional information. See also Exercise 5.2.1.

In this section, we shall make the following hypotheses: The set Ω is *convex* and has a Lipschitz-continuous boundary, and the function u_0 is the trace over Γ of a function (still denoted u_0) of the space $H^2(\Omega)$. Then, for our subsequent analysis, it will be convenient to consider that the minimal surface problem consists in finding a function u such that

$$u \in U \quad \text{and} \quad J(u) = \inf_{v \in U} J(v), \tag{5.2.2}$$

where

$$U = \{v \in H^1(\Omega);\ (v - u_0) \in H_0^1(\Omega)\}. \tag{5.2.3}$$

Remark 5.2.1. The functional J of (5.2.1) is defined over any Sobolev space $W^{1,p}(\Omega)$, $1 \leq p \leq \infty$. One reason for the present choice $p = 2$ is that it is easily seen that the functional J is differentiable over the space $H^1(\Omega)$, as we next show. □

For notational simplicity in the subsequent computations, it will be convenient to introduce the function

$$f: x = (x_1, x_2) \in \mathbf{R}^2 \to f(x) = \sqrt{1 + \|x\|^2} = \sqrt{1 + x_1^2 + x_2^2}. \tag{5.2.4}$$

Then, for all points $x \in \mathbf{R}^2$ and all vectors $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$,

$$\sum_{i,j=1}^{2} \partial_{ij} f(x) \xi_i \xi_j = \frac{\|\xi\|^2 + (x_2\xi_1 - x_1\xi_2)^2}{(1 + \|x\|^2)^{3/2}} \leq \|\xi\|^2, \tag{5.2.5}$$

so that, for all v, $w \in H^1(\Omega)$,

$$J(v + w) - J(v) = \int_\Omega \frac{\nabla v \cdot \nabla w}{f(\nabla v)} \mathrm{d}x + \mathcal{R}(v, w),$$

where

$$\mathcal{R}(v, w) \leq \frac{1}{2} \int_\Omega \|\nabla w\|^2 \mathrm{d}x \leq \frac{1}{2} \|w\|_{1,\Omega}^2.$$

Therefore, the functional J is differentiable over the space $H^1(\Omega)$, and its derivative is given by

$$\forall v, w \in H^1(\Omega),\ J'(v)w = \int_\Omega \frac{\nabla v \cdot \nabla w}{f(\nabla v)} \mathrm{d}x. \tag{5.2.6}$$

We also record the following result which has already been proved (cf. (1.1.9)):

Let V be a normed vector space, let U be a convex subset of V, let $J: V \to \mathbf{R}$ be a functional and, finally, let u be a point of the set U such that $J(u) = \inf_{v \in U} J(v)$ and such that the functional J is differentiable at the point u. Then the inequalities

$$\forall v \in U,\ J'(u)(v - u) \geq 0 \tag{5.2.7}$$

hold.

Finite element approximation with triangles of type (1). *Estimate of the error* $\|u - u_h\|_{1,\Omega_h}$

We next define the *discrete problem*: Let $\mathcal{T}_h$ be a triangulation made up of triangles K, $K \in \mathcal{T}_h$, in such a way that all the vertices situated on the boundary Γ_h of the set

$$\bar{\Omega}_h = \bigcup_{K \in \mathcal{T}_h} K \tag{5.2.8}$$

also belong to the boundary Γ (cf. Fig. 5.2.1). Notice that the inclusion $\bar{\Omega}_h \subset \bar{\Omega}$ holds, by virtue of the assumption of convexity for the set Ω.

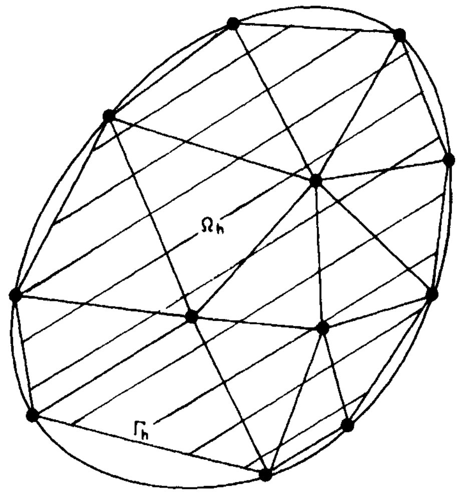

Fig. 5.2.1

With such a triangulation, we associate the finite element space X_h whose generic finite element is the triangle of type (1). The functions in the space X_h are therefore defined over the set $\bar{\Omega}_h$.

Letting $\mathcal{N}_h$ denote the set of nodes of the space X_h (which coincides with the set of vertices in this particular instance), we let

$$U_h = \{v_h \in X_h;\ \forall b \in \mathcal{N}_h \cap \Gamma,\quad v_h(b) = u_0(b)\} \tag{5.2.9}$$

(recall that, by assumption, $u_0 \in H^2(\Omega)$ and that $H^2(\Omega) \hookrightarrow \mathscr{C}^0(\bar{\Omega})$).

Then the discrete problem consists in finding a function u_h *such that*

$$u_h \in U_h \quad \text{and} \quad J_h(u_h) = \inf_{v_h \in U_h} J_h(v_h), \tag{5.2.10}$$

where

$$J_h(v_h) = \int_{\Omega_h} \sqrt{1 + \|\nabla v_h\|^2}\, \mathrm{d}x. \tag{5.2.11}$$

As usual, our first task is to examine the questions of existence and uniqueness for the discrete problem.

Theorem 5.2.1. *The discrete problem* (5.2.10) *has one and only one solution.*

Proof. If we define the norm

$$v_h \in X_h \to \|v_h\|_h = \max_{x \in \bar{\Omega}_h} |v_h(x)|,$$

it easily follows that

$$v_h \in U_h \quad \text{and} \quad \|v_h\|_h \to \infty \Rightarrow \lim J_h(v_h) = \infty. \tag{5.2.12}$$

To see this, it suffices to observe that

$$v_h \in U_h \quad \text{and} \quad \|v_h\|_h \to \infty \Rightarrow \max_{K \in \mathcal{T}_h} \|\nabla v_h|_K\| \to \infty$$

(argue by contradiction) and then to observe that

$$J_h(v_h) \geq \left(\sqrt{1 + \max_{K \in \mathcal{T}_h} \|\nabla v_h|_K\|^2}\right) \min_{K \in \mathcal{T}_h} \operatorname{meas}(K),$$

since the gradient of each function $v_h \in X_h$ is constant over each triangle K.

Let then $\bar{v}_h$ denote a fixed function in the set U_h. Condition (5.2.12) implies that there exists a number r such that

$$v_h \in U_h \quad \text{and} \quad \|v_h\|_h > r \Rightarrow J_h(\bar{v}_h) < J_h(v_h).$$

Therefore the solutions of the minimization problem (5.2.10) coincide with the solutions of an analogous minimization problem, with the set U_h replaced by the set

$$\tilde{U}_h = U_h \cap \{v_h \in X_h;\ \|v_h\|_h \leq r\}.$$

Since the set $\tilde{U}_h$ is now compact, we have shown that problem (5.2.10) has at least one solution.

Let us next turn to the question of uniqueness. It follows from the equality of (5.2.5) that the function f defined in (5.2.4) is strictly convex. This will in turn imply that the function J_h: $U \to \mathbf{R}$ is also strictly convex: To prove this, let v_h and w_h be two distinct elements of the set U_h, let

$$\tilde{\Omega}_h = \left\{x \in \bigcup_{K \in \mathcal{T}_h} \mathring{K};\ \nabla v_h(x) \neq \nabla w_h(x)\right\},$$

and let θ be a given number in the interval $]0, 1[$. Then

$$J_h(\theta v_h + (1-\theta) w_h) = \int_{\Omega_h} f(\theta \nabla v_h(x) + (1-\theta) \nabla w_h(x))\, \mathrm{d}x,$$

and the assertion follows by using the relations

$$\forall x \in \Omega_h - \bar{\tilde{\Omega}}_h, \quad f(\theta \nabla v_h(x) + (1-\theta)\nabla w_h(x)) = \theta f(\nabla v_h(x)) + (1-\theta) f(\nabla w_h(x)),$$

$$\forall x \in \tilde{\Omega}_h, \quad f(\theta \nabla v_h(x) + (1-\theta)\nabla w_h(x)) < \theta f(\nabla v_h(x)) + (1-\theta) f(\nabla w_h(x)),$$

$$\operatorname{meas}(\tilde{\Omega}_h) > 0.$$

Since the set U_h is convex, the minimization problem (5.2.10) has a unique solution. □

Remark 5.2.2. The same argument shows that the minimization problem (5.2.2) has at most one solution. □

We next obtain an error estimate in the norm $\|\cdot\|_{1,\Omega_h}$.

Theorem 5.2.2. *Assume that the solution u of the minimization problem (5.2.2) exists and is in the space $H^2(\Omega) \cap W^{1,\infty}(\Omega)$. Then, given a regular family of triangulations, there exists a constant $C(u)$ independent of h such that*

$$\|u - u_h\|_{1,\Omega_h} \leq C(u)h. \tag{5.2.13}$$

Proof. In what follows, the notation $C(u)$ stands for various constants solely dependent upon the solution u. For clarity, the proof has been subdivided in five steps. The first four steps consist in establishing that $|u - u_h|_{1,\Omega_h} = O(h)$.

(i) Let us first record some *relations which are consequences of the minimizing properties of the functions u_h and u.*

Using (5.2.7), we know that

$$\forall v_h \in U_h, \quad J_h'(u_h)(v_h - u_h) \geq 0.$$

But in view of the particular form of the set U_h (cf. (5.2.9)), these inequalities are equivalent to the equations

$$\forall w_h \in X_{0h}, \quad J_h'(u_h) w_h = 0,$$

where, as usual,

$$X_{0h} = \{v_h \in X_h;\ v_h = 0 \quad \text{on} \quad \Gamma_h\}. \tag{5.2.14}$$

By a computation similar to that which led to (5.2.6), we deduce that

$$\forall w_h \in X_{0h}, \quad J_h'(u_h)w_h = \int_{\Omega_h} \frac{\nabla u_h \cdot \nabla w_h}{\sqrt{1+\|\nabla u_h\|^2}}\,\mathrm{d}x. \tag{5.2.15}$$

Using again (5.2.7) and the particular form of the set U (cf. (5.2.3)), we see that

$$\forall w \in H_0^1(\Omega), \quad J'(u)w = 0,$$

and therefore, by (5.2.6),

$$\forall w_h \in X_{0h}, \quad \int_{\Omega_h} \frac{\nabla u \cdot \nabla w_h}{\sqrt{1+\|\nabla u\|^2}}\,\mathrm{d}x = 0. \tag{5.2.16}$$

Clearly, this application of (5.2.6) supposes that each function $w_h \in X_{0h}$ be identified with its extension to the space $H_0^1(\Omega)$ obtained by prolongating it by zero on the set $\Omega - \Omega_h$.

(ii) Let us next show that, with the assumption that the solution u belongs to the space $H^2(\Omega) \cap W^{1,\infty}(\Omega)$, *there exists a constant* $C(u)$ *such that the quantity*

$$\Delta_h = \left(\int_{\Omega_h} \frac{\|\nabla(u-u_h)\|^2}{\sqrt{1+\|\nabla u_h\|^2}}\,\mathrm{d}x\right)^{1/2} \tag{5.2.17}$$

satisfies an inequality of the form

$$\Delta_h \leq C(u)h. \tag{5.2.18}$$

Let v_h be an arbitrary function in the set U_h, so that the function $w_h = v_h - u_h$ belongs to the space X_{0h}. Then, using relations (5.2.15) and (5.2.16) established in step (i), we can write ($f(x) = \sqrt{1+\|x\|^2}$; cf. (5.2.4)):

$$\Delta_h^2 = \int_{\Omega_h} \frac{\nabla(u-u_h)\cdot\nabla(u-v_h)}{f(\nabla u_h)}\,\mathrm{d}x + \\ + \int_{\Omega_h} \left(\frac{1}{f(\nabla u_h)} - \frac{1}{f(\nabla u)}\right)\nabla u \cdot \nabla w_h\,\mathrm{d}x. \tag{5.2.19}$$

The first integral can be bounded as follows:

$$\left|\int_{\Omega_h} \frac{\nabla(u-u_h)\cdot\nabla(u-v_h)}{f(\nabla u_h)}\,\mathrm{d}x\right| \leq \int_{\Omega_h} \left(\frac{\|\nabla(u-u_h)\|}{\sqrt{f(\nabla u_h)}}\right)\|\nabla(u-v_h)\|\,\mathrm{d}x \\ \leq \Delta_h |u-v_h|_{1,\Omega_h}. \tag{5.2.20}$$

In order to get an estimate for the second integral, we observe that

$$\frac{1}{f(\nabla u_h)} - \frac{1}{f(\nabla u)} = \frac{\nabla(u-u_h)\cdot\nabla(u+u_h)}{f(\nabla u_h)f(\nabla u)(f(\nabla u_h)+f(\nabla u))},$$

and thus,

$$\left|\frac{1}{f(\nabla u_h)} - \frac{1}{f(\nabla u)}\right| \leq \frac{\|\nabla(u-u_h)\|}{f(\nabla u_h)f(\nabla u)}.$$

Therefore,

$$\begin{aligned}
&\left|\int_{\Omega_h}\left(\frac{1}{f(\nabla u_h)} - \frac{1}{f(\nabla u)}\right)\nabla u\cdot\nabla w_h \,\mathrm{d}x\right| \leq \\
&\quad\leq \int_{\Omega_h}\frac{\|\nabla u\|}{f(\nabla u)}\frac{\|\nabla(u-u_h)\|}{\sqrt{f(\nabla u_h)}}\frac{\|\nabla w_h\|}{\sqrt{f(\nabla u_h)}}\,\mathrm{d}x \\
&\quad\leq \gamma(u)\Delta_h\left(\int_{\Omega_h}\frac{\|\nabla w_h\|^2}{f(\nabla u_h)}\,\mathrm{d}x\right)^{1/2} \\
&\quad\leq \gamma(u)\Delta_h\left(\Delta_h + \left(\int_{\Omega_h}\frac{\|\nabla(u-v_h)\|^2}{f(\nabla u_h)}\,\mathrm{d}x\right)^{1/2}\right) \\
&\quad\leq \gamma(u)\Delta_h(\Delta_h + |u-v_h|_{1,\Omega_h}),
\end{aligned} \tag{5.2.21}$$

where ($u \in W^{1,\infty}(\Omega)$ by assumption)

$$\gamma(u) = \left|\frac{\|\nabla u\|}{\sqrt{1+\|\nabla u\|^2}}\right|_{0,\infty,\Omega}. \tag{5.2.22}$$

Combining relations (5.2.19), (5.2.20) and (5.2.21), we obtain

$$\forall v_h \in U_h, \quad \Delta_h \leq \gamma(u)\Delta_h + (1+\gamma(u))|u-v_h|_{1,\Omega_h}.$$

Since the constant $\gamma(u)$ of (5.2.22) is strictly less than one, it follows that

$$\Delta_h = \left(\int_{\Omega_h}\frac{\|\nabla(u-u_h)\|^2}{\sqrt{1+\|\nabla u_h\|^2}}\,\mathrm{d}x\right)^{1/2} \leq C(u)\inf_{v_h\in U_h}|u-v_h|_{1,\Omega_h}, \tag{5.2.23}$$

with $C(u) = (1+\gamma(u))/(1-\gamma(u))$.

Since the function u belongs to the space $H^2(\Omega)$ by assumption, its X_h-interpolant is well defined, and it belongs to the set U_h. Thus

$$\inf_{v_h\in U_h}|u-v_h|_{1,\Omega_h} \leq |u|_{2,\Omega}h, \tag{5.2.24}$$

and inequality (5.2.18) is a consequence of inequalities (5.2.23) and (5.2.24).

(iii) Let us show that

$$|u_h|_{1,\infty,\Omega_h} \leq C(u). \tag{5.2.25}$$

Let K be an arbitrary triangle in the triangulation. Using step (ii) (cf. inequality (5.2.18)), one obtains

$$\begin{aligned}\left(\int_K \frac{\|\nabla u_h\|^2}{\sqrt{1+\|\nabla u_h\|^2}}\,\mathrm{d}x\right)^{1/2} &\leq \Delta_h + \left(\int_K \frac{\|\nabla u\|^2}{\sqrt{1+\|\nabla u_h\|^2}}\,\mathrm{d}x\right)^{1/2} \\ &\leq C(u)h + |u|_{1,\infty,\Omega}(\mathrm{meas}(K))^{1/2} \leq C(u)h.\end{aligned} \tag{5.2.26}$$

Because the restriction $\nabla u_h|_K$ is constant over the triangle K, we may write

$$\begin{aligned}\int_K \frac{\|\nabla u_h\|^2}{\sqrt{1+\|\nabla u_h\|^2}}\,\mathrm{d}x &= \frac{\|\nabla u_h|_K\|^2}{\sqrt{1+\|\nabla u_h|_K\|^2}}\,\mathrm{meas}(K) \\ &\geq C\frac{\|\nabla u_{h|K}\|^2}{\sqrt{1+\|\nabla u_{h|K}\|^2}}\,h^2,\end{aligned} \tag{5.2.27}$$

for some constant C independent of h. Then the conjunction of inequalities (5.2.26) and (5.2.27) implies that

$$\forall K \in \mathcal{T}_h, \quad \forall h, \quad \frac{\|\nabla u_h|_K\|^2}{\sqrt{1+\|\nabla u_h|_K\|^2}} \leq C(u).$$

Therefore, the norms $\|\nabla u_{h|K}\|$ are bounded independently of $K \in \mathcal{T}_h$ and $h(\lim_{x\to\infty}(x^2/\sqrt{1+x^2}) = \infty)$ and thus property (5.2.25) is proved.

(iv) Combining steps (ii) and (iii), we obtain

$$\begin{aligned}|u-u_h|_{1,\Omega_h} &= \left(\int_{\Omega_h} \frac{\|\nabla(u-u_h)\|^2}{\sqrt{1+\|\nabla u_h\|^2}}\sqrt{1+\|\nabla u_h\|^2}\,\mathrm{d}x\right)^{1/2} \\ &\leq \left(\max_{K\in\mathcal{T}_h}\sqrt{1+\|\nabla u_h|_K\|^2}\right)^{1/2}\Delta_h \leq C(u)h.\end{aligned} \tag{5.2.28}$$

(v) Let us add triangles $K \in \mathcal{T}_h^\partial$ to each triangulation $\mathcal{T}_h$ as indicated in Fig. 5.2.2, i.e., in such a way that, for all h,

$$\bar{\Omega} \subset \bar{\Omega}_h^* = \bar{\Omega}_h \cup \Big(\bigcup_{K\in\mathcal{T}_h^\partial} K\Big),$$

so that the triangulations $\mathcal{T}_h \cup \mathcal{T}_h^\partial$ again constitute a regular family (such a construction is certainly possible).

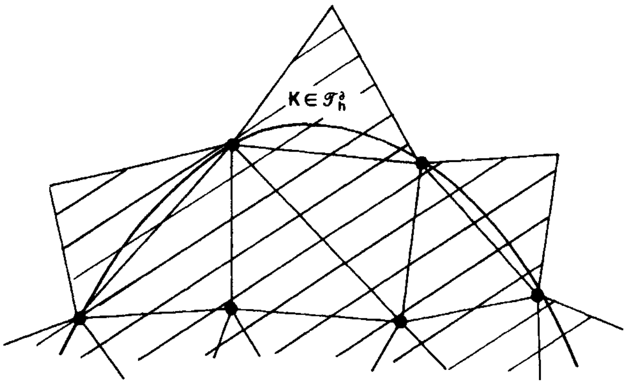

Fig. 5.2.2

Because the boundary Γ is Lipschitz-continuous, there exists (cf. LIONS (1962, Chapter 2) or NEČAS (1967, Chapter 2)) an *extension operator* E: $H^2(\Omega) \to H^2(\mathbf{R}^n)$, i.e., such that for all $v \in H^2(\Omega)$, the function $Ev \in H^2(\mathbf{R}^n)$ satisfies $Ev|_\Omega = v$ and, besides, this operator is continuous: There exists a constant $C(\Omega)$ such that

$$\forall v \in H^2(\Omega), \quad \|Ev\|_{2,\mathbf{R}^n} \leq C(\Omega)\|v\|_{2,\Omega}. \tag{5.2.29}$$

Let then $Eu = u^*$. We define an extension u_h^*: $\bar{\Omega}_h^* \to \mathbf{R}$ of the function u_h by letting

$$\begin{cases} u_h^* = u_h \quad \text{on} \quad \bar{\Omega}_h, \\ \forall K \in \mathcal{T}_h^\partial, \quad u_h^* = \Pi_K(u^*), \end{cases} \tag{5.2.30}$$

where Π_K denotes the $P_1(K)$-interpolant associated with triangles of type (1). Observe that, since the function u_h belongs to the set U_h as defined in (5.2.9), the function u_h^* is continuous over the set $\bar{\Omega}_h^*$ by virtue of the second condition (5.2.30) and thus, it is in the space $H^1(\Omega_h^*)$.

Finally, we shall use the following inequality, due to Friedrichs (cf. NEČAS (1967), Theorem 1.9): There exists a constant $C(\Omega)$ such that

$$\forall v \in H^1(\Omega), \quad \|v\|_{1,\Omega} \leq C(\Omega)(|v|_{1,\Omega} + \|v\|_{L^2(\Gamma)}).$$

Let then $v = u^* - u_h^*$ in this inequality. We obtain, upon combining with inequality (5.2.28),

$$\begin{aligned}\|u-u_h\|_{1,\Omega_h} &\leq \|u^*-u_h^*\|_{1,\Omega}\\ &\leq C(\Omega)(|u-u_h|_{1,\Omega_h}+|u-u_h^*|_{1,\Omega-\Omega_h}+\|u-u_h^*\|_{L^2(\Gamma)})\\ &\leq C(\Omega,u)h+C(\Omega)(|u^*-u_h^*|_{1,\Omega_h^*-\Omega_h}+\|u^*-u_h^*\|_{L^2(\Gamma)}).\end{aligned} \tag{5.2.31}$$

Using inequality (5.2.29), we get

$$|u^*-u_h^*|_{1,\Omega_h^*-\Omega_h}\leq Ch|u^*|_{2,\Omega_h^*-\Omega_h}\leq Ch\|u\|_{2,\Omega}, \tag{5.2.32}$$

$$\begin{aligned}|u^*-u_h^*|_{0,\infty,\Omega_h^*-\Omega_h} &= \max_{K\in\mathcal{T}_h^2}|u^*-u_h^*|_{0,\infty,K}\\ &\leq Ch\max_{K\in\mathcal{T}_h^2}|u^*|_{2,K}\leq Ch|u^*|_{2,\Omega_h^*-\Omega_h}\\ &\leq Ch\|u\|_{2,\Omega},\end{aligned} \tag{5.2.33}$$

through two applications of Theorem 3.1.5. Inequality (5.2.33) implies that

$$\|u^*-u_h^*\|_{L^2(\Gamma)}\leq |u^*-u_h^*|_{0,\infty,\Omega_h^*-\Omega_h}\left(\int_\Gamma d\gamma\right)^{1/2}\leq Ch\|u\|_{2,\Omega}, \tag{5.2.34}$$

and inequality (5.2.13) follows from inequalities (5.2.31), (5.2.32) and (5.2.34). □

Exercises

5.2.1. Let $\Omega=\{x\in\mathbf{R}^2;\ 1<\|x\|<2\}$, $u_0=\gamma$ for $\|x\|=1$ and $u_0=0$ for $\|x\|=2$, where γ is a constant. Show that the associated minimal surface problem has a solution if γ is smaller than a quantity γ^* while there is no solution if $\gamma>\gamma^*$ (cf. Fig. 5.2.3).
[Hint: Reduce this problem to a minimization problem for functions in one variable.]

This is a very simple example of a general phenomenon that R. Témam has analyzed through the introduction of "generalized solutions" (cf. the section "Bibliography and Comments").

5.2.2. (i) Show that the minimal surface problem amounts to formally solving a boundary value problem of the form

$$(*)\qquad \begin{cases}-\sum_{i,j=1}^{2}\partial_j(a_{ij}(\nabla u)\partial_i u)=0 & \text{in}\quad \Omega,\\ u=u_0 \quad \text{sur} & \Gamma,\end{cases}$$

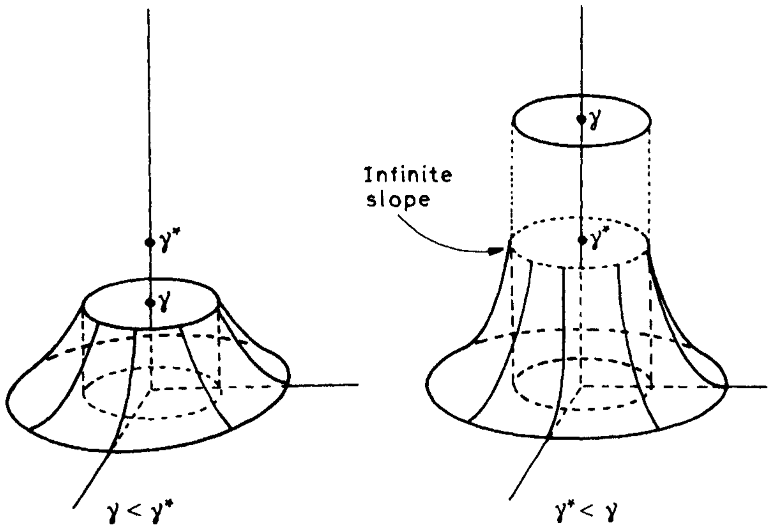

Fig. 5.2.3

i.e., a nonhomogeneous Dirichlet problem for the nonlinear operator

$$u \to -\sum_{i,j=1}^{2} \partial_j(a_{ij}(\nabla u)\partial_i u)$$

and that this operator satisfies an ellipticity condition in the sense that, for any smooth enough function u,

$$\exists \beta(u) > 0, \quad \forall \xi_i, \quad i = 1, 2, \sum_{i,j=1}^{2} a_{ij}(\nabla u)\xi_i\xi_j \geqslant \beta(u) \sum_{i=1}^{2} \xi_i^2.$$

However, the constant $\beta(u)$ cannot be bounded below away from zero independently of u.

(ii) Show that, for smooth functions, the boundary value problem (*) can also be written

$$\begin{cases} (1 + (\partial_2 u)^2)\partial_{11} u - 2\partial_1 u \partial_2 u \partial_{12} u + (1 + (\partial_1 u)^2)\partial_{22} u = 0 & \text{in} \quad \Omega, \\ u = u_0 & \text{on} \quad \Gamma. \end{cases}$$

5.3. Nonlinear problems of monotone type

A minimization problem over the space $W_0^{1,p}(\Omega)$, $2 \leq p$, and its finite element approximation with n-simplices of type (1)

Let there be given a convex open subset Ω of $\mathbf{R}^n$ and let p be a number such that

$$2 \leq p \tag{5.3.1}$$

(for the case where $1 < p < 2$, see Exercise 5.3.2). We consider the *minimization problem*: Find a function u such that

$$u \in W_0^{1,p}(\Omega) \quad \text{and} \quad J(u) = \inf_{v \in W_0^{1,p}(\Omega)} J(v) \tag{5.3.2}$$

where the functional J is given by

$$J(v) = \frac{1}{p}\int_\Omega \|\nabla v\|^p \, dx - f(v), \tag{5.3.3}$$

for some given element f of the dual space of the space $W_0^{1,p}(\Omega)$. We use the standard notation

$$\|\nabla v\| = \left(\sum_{i=1}^n (\partial_i v)^2\right)^{1/2}.$$

For computational convenience, we shall consider throughout this section that the space $W_0^{1,p}(\Omega)$ is equipped with the norm

$$v \to \|v\| = \left(\int_\Omega \|\nabla v\|^p \, dx\right)^{1/p}, \tag{5.3.4}$$

which is clearly equivalent to the standard semi-norm $|\cdot|_{1,p,\Omega}$, itself a norm equivalent to the norm $\|\cdot\|_{1,p,\Omega}$ over the space $W_0^{1,p}(\Omega)$. Finally, we shall use the notation $\|\cdot\|^*$ for the norm in the dual space $(W_0^{1,p}(\Omega))'$ of the space $W_0^{1,p}(\Omega)$.

Remark 5.3.1. For $p = 2$, this minimization problem reduces to the familiar homogeneous Dirichlet problem $-\Delta u = f$ in Ω, $u = 0$ on Γ. □

Our proof of the existence of a solution of the minimization problem (5.3.2) (cf. Theorem 5.3.1) uses the simplest finite element approximation of this problem, which we now proceed to describe: We consider triangulations $\mathcal{T}_h$ made up of n-simplices $K \in \mathcal{T}_h$, in such a way that all the vertices situated on the boundary Γ_h of the set $\bar{\Omega}_h = \bigcup_{K\in\mathcal{T}_h} K$ also belong to the boundary Γ of the set Ω (a similar situation was considered in the previous section; see in particular Fig. 5.2.1 for $n = 2$). Then with each such triangulation, we associate the finite element space X_h whose generic finite element is the n-simplex of type (1) (notice that the functions in the space X_h are defined only on the set $\bar{\Omega}_h$), and we let as usual

$$X_{0h} = \{v_h \in X_h;\ v_{h|\Gamma_h} = 0\}.$$

Then we denote by V_h the space formed by the extensions of the functions of the space X_{0h} which vanish over the set $\bar{\Omega} - \bar{\Omega}_h$. In fact we shall not distinguish between the functions in X_{0h} and their corresponding extensions in the space V_h.

Notice that, because the set $\bar{\Omega}$ was assumed to be convex, the inclusion

$$V_h \subset W_0^{1,p}(\Omega) \tag{5.3.5}$$

and the relations

$$v \in \mathscr{C}^0(\bar{\Omega}) \quad \text{and} \quad v = 0 \quad \text{on} \quad \Gamma \Rightarrow \Pi_h v \in V_h \tag{5.3.6}$$

hold.

Then the *discrete problem* consists in finding a function u_h such that

$$u_h \in V_h \quad \text{and} \quad J(u_h) = \inf_{v_h \in V_h} J(v_h), \tag{5.3.7}$$

where the functional J is defined as in (5.3.3).

Theorem 5.3.1. *The minimization problems* (5.3.2) *and* (5.3.7) *both have one and only one solution. Their respective solutions* $u \in W_0^{1,p}(\Omega)$ *and* $u_h \in V_h$ *are also the unique solutions of the variational equations*

$$\forall v \in W_0^{1,p}(\Omega), \quad \int_\Omega \|\nabla u\|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x = f(v), \tag{5.3.8}$$

$$\forall v_h \in V_h, \quad \int_\Omega \|\nabla u_h\|^{p-2}\nabla u_h \cdot \nabla v_h \,\mathrm{d}x = f(v_h), \tag{5.3.9}$$

respectively.

Proof. We begin by proving several properties of the functional J of (5.3.3).

(i) Since

$$J(v) = \frac{1}{p}\|v\|^p - f(v) \geq \frac{1}{p}\|v\|^p - \|f\|^*\|v\|,$$

we deduce that

$$\lim_{\|v\| \to \infty} J(v) = \infty. \tag{5.3.10}$$

(ii) Let us next establish the *strict convexity* of the functional J. The functional f being convex, it suffices to establish the strict convexity of the mapping

$$v \in W_0^{1,p}(\Omega) \to \int_\Omega F(\nabla v(x))\,dx, \quad \text{with} \quad F: \xi \in \mathbf{R}^n \to \frac{1}{p}\|\xi\|^p. \tag{5.3.11}$$

Let u and v be two different elements in the space $W_0^{1,p}(\Omega)$ such that

$$\text{meas}\,\tilde{\Omega} > 0, \quad \text{where} \quad \tilde{\Omega} = \{x \in \Omega;\, \nabla u \neq \nabla v\},$$

and let $\theta \in\,]0, 1[$ be given. Then write

$$\int_\Omega F(\theta\nabla u + (1-\theta)\nabla v)\,dx = \int_{\tilde{\Omega}} F(\theta\nabla u + (1-\theta)\nabla v)\,dx + \int_{\Omega - \tilde{\Omega}} F(\theta\nabla u + (1-\theta)\nabla v)\,dx,$$

so that the conclusion follows by making use of the strict convexity of the mapping F (which is itself a straightforward consequence of the strict convexity of the mapping $t \in \mathbf{R} \to |t|^p$). For a similar argument, see the proof of Theorem 5.2.1.

Notice at this stage that the property of strict convexity implies the *uniqueness* of the solution of both minimization problems (5.3.2) and (5.3.7).

(iii) We then show that *the functional J is differentiable*, and in so doing, we compute its derivative. Clearly, it suffices to examine the differentiability properties of the mapping considered in (5.3.11). We first observe that the mapping F is twice differentiable, with

$$\partial_i F(\xi) = \|\xi\|^{p-2}\xi_i, \quad 1 \leq i \leq n,$$

$$\partial_{ij} F(\xi) = (p-2)\|\xi\|^{p-4}\xi_i\xi_j + \|\xi\|^{p-2}\delta_{ij}, \quad 1 \leq i, j \leq n.$$

Consequently, we can write

$$F(\xi+\eta)-F(\xi)=\|\xi\|^{p-2}\xi\cdot\eta+R(\xi,\eta),$$

with

$$|R(\xi,\eta)|\leq C(p)(\|\xi\|+\|\eta\|)^{p-2}\|\eta\|^2,$$

and thus

$$\int_\Omega F(\nabla(u+v)(x))\,dx-\int_\Omega F(\nabla u(x))\,dx=$$
$$=\int_\Omega \|\nabla u\|^{p-2}\nabla u\cdot\nabla v\,dx+\mathcal{R}(u,v),$$

with

$$|\mathcal{R}(u,v)|\leq C(p)\int_\Omega(\|\nabla u\|+\|\nabla v\|)^{p-2}\|\nabla v\|^2\,dx.$$

On the one hand, we have

$$\left|\int_\Omega \|\nabla u\|^{p-2}\nabla u\cdot\nabla v\,dx\right|\leq\|u\|^{p-1}\|v\|,$$

and thus the linear mapping

$$v\in W_0^{1,p}(\Omega)\to\int_\Omega\|\nabla u\|^{p-2}\nabla u\cdot\nabla v\,dx$$

is continuous for a fixed $u\in W_0^{1,p}(\Omega)$. On the other, we have

$$\int_\Omega(\|\nabla u\|+\|\nabla v\|)^{p-2}\|\nabla v\|^2\,dx\leq(\|u\|+\|v\|)^{p-2}\|v\|^2,$$

and thus the mapping of (5.3.11) is differentiable. Let us then record for future uses the expression of the derivative of the mapping J:

$$\forall v\in W_0^{1,p}(\Omega),\quad J'(u)v=\int_\Omega\|\nabla u\|^{p-2}\nabla u\cdot\nabla v\,dx-f(v).\quad(5.3.12)$$

This shows in particular that the solutions u and u_h of the minimization problems (5.3.2) and (5.3.7) (assuming at this stage their existence) must satisfy relations (5.3.8) and (5.3.9), respectively. In view of the strict convexity of the functional J (step (ii)), these relations are also sufficient for the existence of a unique minimum.

(iv) We next show that *the approximate minimization problem* (5.3.7) *always has a solution*: This is simply a consequence of the strict convexity of the functional J (step (ii)) and of the property

$$\lim_{\substack{\|v_h\|\to\infty \\ v_h\in V_h}} J(v_h) = \infty$$

(step (i)) (the argument has already been given in the proof of Theorem 5.2.1 and shall not be repeated here).

We also remark that *the discrete solutions u_h are bounded independently of the subspace V_h*: Letting $v_h = u_h$ in (5.3.9), we obtain $\|u_h\|^p = f(u_h)$ and thus,

$$\|u_h\| \leqslant (\|f\|^*)^{1/(p-1)}. \tag{5.3.13}$$

(v) We are now in a position to show the *existence of a solution of the minimization problem* (5.3.2): We consider from now on a family V_h of finite element spaces (of the type described at the beginning of this section) associated with a regular family of triangulations.

The space $W_0^{1,p}(\Omega)$ being reflexive, the uniform boundedness of the discrete solutions u_h, as shown in (5.3.13), implies that there exists a sequence $(u_{h_k})_{k=1}^{\infty}$ which weakly converges to some element $u \in W_0^{1,p}(\Omega)$.

Let then ϕ be an arbitrary function in the space $\mathcal{D}(\Omega)$. By definition of the discrete problems, we have, in particular

$$\forall k \geqslant 1, \quad J(u_{h_k}) \leqslant J(\Pi_{h_k}\phi).$$

Since the functional J is continuous and convex, it is weakly lower semicontinuous. Consequently,

$$J(u) \leqslant \varliminf_{k\to\infty} J(u_{h_k}) \leqslant \varliminf_{k\to\infty} J(\Pi_{h_k}\phi). \tag{5.3.14}$$

Because the support of the function ϕ is a compact subset of the set Ω, it is easily seen that there exists an integer k_0 such that

$$k \geqslant k_0 \Rightarrow \operatorname{supp}\phi \subset \bar{\Omega}_{h_k}.$$

Using Theorem 3.1.6, we obtain for any $k \geqslant k_0$,

$$\|\Pi_{h_k}\phi - \phi\|_{1,p,\Omega} \leqslant C h_k (\operatorname{meas}\Omega)^{1/p} |\phi|_{2,\infty,\Omega},$$

and therefore,

$$\lim_{k\to\infty} \|\Pi_{h_k}\phi - \phi\| = 0.$$

This last relation and the continuity of the functional J imply that

$$\lim_{k\to\infty} J(\Pi_{h_k}\phi) = J(\phi). \tag{5.3.15}$$

Combining (5.3.14) and (5.3.15), we have thus proved that

$$\forall \phi \in \mathscr{D}(\Omega), \quad J(u) \leq J(\phi).$$

The space $\mathscr{D}(\Omega)$ being dense in the space $W_0^{1,p}(\Omega)$, we deduce that

$$\forall v \in W_0^{1,p}(\Omega), \quad J(u) \leq J(v),$$

and therefore the function u is the (unique as observed in step (ii)) solution of the minimization problem (5.3.2). □

Remark 5.3.2. From relations (5.3.8), it is immediately seen that the minimization problem (5.3.2) is formally equivalent to the homogeneous Dirichlet problem

$$\begin{cases} -\sum_{i=1}^{n} \partial_i(\|\nabla u\|^{p-2}\partial_i u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma, \end{cases}$$

where the operator $u \to -\sum_{i=1}^{n} \partial_i(\|\nabla u\|^{p-2}\partial_i u)$ is nonlinear for $p > 2$. □

Sufficient condition for $\lim_{h\to 0} \|u - u_h\|_{1,p,\Omega} = 0$

Using the last part of the proof of the above theorem, we are in addition able to prove the convergence of the discrete solutions towards the solution u, as we now show.

Theorem 5.3.2. *Let there be given a family of finite element spaces as previously described, i.e., made up of n-simplices of type* (1), *associated with a regular family of triangulations. Then with the sole assumption that the solution u is in the space* $W_0^{1,p}(\Omega)$ *we have*

$$\lim_{h\to 0} \|u - u_h\|_{1,p,\Omega} = 0. \tag{5.3.16}$$

Proof. We continue the argument used in part (v) of the proof of the previous theorem. Since the weak limit u is unique, we deduce that the whole family (u_h) weakly converges to the solution u. Thus,

$$f(u) = \lim_{h\to 0} f(u_h).$$

On the other hand, we have

$$\forall \phi \in \mathcal{D}(\Omega), \quad \lim_{h\to 0} \sup J(u_h) \le \lim_{h\to 0} \sup J(\Pi_h \phi) = \lim_{h\to 0} J(\Pi_h \phi) = J(\phi).$$

Since the functions ϕ can be chosen arbitrarily close to the solution u (in the norm of the space $W_0^{1,p}(\Omega)$), we deduce from the above relations that

$$J(u) = \lim_{h\to 0} J(u_h),$$

i.e., in view of the expression for the functional J, that

$$\|u\| = \lim_{h\to 0} \|u_h\|, \tag{5.3.17}$$

since $\lim_{h\to 0} f(u_h) = f(u)$.

The space $W_0^{1,p}(\Omega)$ being uniformly convex, the weak convergence and the convergence (5.3.17) imply the convergence in the norm. □

The equivalent problem $Au = f$. Two properties of the operator A

In order to have an approach similar to that of the linear case, let us introduce, for any function $u \in W_0^{1,p}(\Omega)$, the element $Au \in (W_0^{1,p}(\Omega))'$ defined by (cf. the proof of Theorem 5.3.1)

$$\forall v \in W_0^{1,p}(\Omega), \quad Au(v) = \int_\Omega \|\nabla u\|^{p-2} \nabla u \cdot \nabla v \,\mathrm{d}x. \tag{5.3.18}$$

Notice that the element Au is nothing but the derivative of the mapping of (5.3.11), so that relation (5.3.12) may be equivalently written as

$$J'(u) = Au - f. \tag{5.3.19}$$

In other words, the original minimization problem (5.3.2) is equivalent to the solution of the (nonlinear if $p > 2$) equation $Au = f$. Our next task is to establish (cf. Theorem 5.3.3) two properties of the operator

$$A: W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))'$$

defined in (5.3.18) and whose bijectivity has been proved in Theorem 5.3.1. The first property (cf. (5.3.20)) is a generalization of the usual ellipticity condition in the linear case, while the second property (cf. (5.3.21)) is a generalization of the continuity of the operator A in the linear case (cf. the inequality $\|A\|_{\mathcal{L}(V;V')} \le M$ established in (1.1.21)). In order to simplify the exposition, we shall henceforth assume that $n = 2$ (the extension to higher dimensions is indeed possible, but at the expense of additional technicalities).

Theorem 5.3.3. *For a given number p in the interval* $[2, \infty[$, *let* $A: W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))'$ *be the operator as defined in* (5.3.18). *Then,*

$$\exists \alpha > 0, \quad \forall u, v \in W_0^{1,p}(\Omega), \quad \alpha \|u - v\|^p \leq (Au - Av)(u - v), \tag{5.3.20}$$

$$\exists M > 0, \quad \forall u, v \in W_0^{1,p}(\Omega), \quad \|Au - Av\|^* \leq \\ \leq M(\|u\| + \|v\|)^{p-2}\|u - v\|. \tag{5.3.21}$$

Proof. Let us introduce the auxiliary function

$$\phi: (\xi, \eta) \in \mathcal{O} = \{(\xi, \eta) \in \mathbf{R}^2 \times \mathbf{R}^2;\ \xi \neq \eta\} \to \\ \to \phi(\xi, \eta) = \frac{(\|\xi\|^{p-2}\xi - \|\eta\|^{p-2}\eta) \cdot (\xi - \eta)}{\|\xi - \eta\|^p}, \tag{5.3.22}$$

where $\cdot$ denotes as usual the Euclidean inner-product in the space $\mathbf{R}^2$. We shall show that

$$\exists \alpha > 0, \quad \forall (\xi, \eta) \in \mathcal{O}, \quad \alpha \leq \phi(\xi, \eta), \tag{5.3.23}$$

a property which is easily seen to imply inequality (5.3.20). First, we notice that since

$$\forall \eta \neq 0, \quad \phi(0, \eta) = 1, \tag{5.3.24}$$

it suffices to consider the case where $\xi \neq 0$. Next, we prove that

$$\forall (\xi, \eta) \in \mathcal{O}, \quad \phi(\xi, \eta) > 0. \tag{5.3.25}$$

This follows from the relations

$$\begin{aligned}
&(\|\xi\|^{p-2}\xi - \|\eta\|^{p-2}\eta) \cdot (\xi - \eta) = \\
&\quad = \|\xi\|^p - (\|\xi\|^{p-2} + \|\eta\|^{p-2})(\xi \cdot \eta) + \|\eta\|^p \\
&\quad \geq \|\xi\|^p - \|\xi\|^{p-1}\|\eta\| - \|\eta\|^{p-1}\|\xi\| + \|\eta\|^p \\
&\quad = (\|\xi\|^{p-1} - \|\eta\|^{p-1})(\|\xi\| - \|\eta\|) \\
&\quad > 0 \quad \text{unless} \quad \|\xi\| = \|\eta\|.
\end{aligned}$$

Since the penultimate inequality is an equality if and only if $\eta = \mu\xi$ for some $\mu \in \mathbf{R}$, the only remaining case is that where $\eta = -\xi$. But then

$$(\|\xi\|^{p-2}\xi - \|\eta\|^{p-2}\eta)\cdot(\xi-\eta) = 4\|\xi\|^p > 0.$$

Finally, we observe that we may restrict ourselves to the case where $\xi = \bar{\xi} = (1,0)$ since $\phi(\lambda\xi, \lambda\eta) = \phi(\xi, \eta)$ for all $\lambda > 0$ on the one hand and since the Euclidean inner product is invariant through rotations around the origin on the other. Because

$$\lim_{\|\eta\|\to\infty} \phi(\bar{\xi}, \eta) = 1, \tag{5.3.26}$$

it remains to study the behavior of the function $\eta = (\eta_1, \eta_2) \in (\mathbf{R}^2 - \bar{\xi}) \to \phi(\bar{\xi}, \eta)$ in the neighborhood of the point $\bar{\xi}$. For this purpose, let

$$\eta_1 = 1 + \rho\cos\theta, \quad \eta_2 = \rho\sin\theta.$$

Then a simple computation shows that

$$\phi(\bar{\xi}, \eta) = \frac{1 + (p-2)\cos^2\theta + \epsilon(\rho, \theta)}{\rho^{p-2}},$$

with $\lim_{\rho\to 0} \epsilon(\rho, \theta) = 0$ uniformly with respect to $\theta \in [0, 2\pi[$. Therefore,

$$\lim_{\eta\to\bar{\xi}} \phi(\bar{\xi}, \eta) = \begin{cases} 1 & \text{if} \quad p = 2, \\ \infty & \text{if} \quad p > 2, \end{cases} \tag{5.3.27}$$

and relation (5.3.23) follows from the conjunction of relations (5.3.24) through (5.2.27).

To prove the second relation (5.3.21), we introduce the auxiliary function

$$\psi: (\xi, \eta) \in \mathcal{O} = \{(\xi, \eta) \in \mathbf{R}^2 \times \mathbf{R}^2, x \neq y\} \to$$

$$\to \psi(\xi, \eta) = \frac{\|\,\|\eta\|^{p-2}\eta - \|\xi\|^{p-2}\xi\|}{\|\eta - \xi\|(\|\eta\| + \|\xi\|)^{p-2}}, \tag{5.3.28}$$

and we shall show that

$$\exists M > 0, \quad \forall (\xi, \eta) \in \mathcal{O}, \quad \psi(\xi, \eta) \leq M. \tag{5.3.29}$$

Since

$$\forall \eta \neq 0, \quad \psi(0, \eta) = 1, \tag{5.3.30}$$

we may assume that $\xi \neq 0$. In fact, it suffices to consider the case where $\xi = \bar{\xi} = (1,0)$ since $\psi(\lambda\xi, \lambda\eta) = \psi(\xi, \eta)$ for all $\lambda > 0$ on the one hand, and since the Euclidean norm is invariant through rotations around the origin on the other. We also have

$$\lim_{\|\eta\|\to\infty} \psi(\bar{\xi}, \eta) = 1. \tag{5.3.31}$$

To study the behavior of the function $\eta = (\eta_1, \eta_2) \in (\mathbf{R}^2 - \bar{\xi}) \to \psi(\bar{\xi}, \eta)$ in the neighborhood of the point $\bar{\xi}$, we let $\eta_1 = 1 + \rho \cos\theta$, $\eta_2 = \rho \sin\theta$ as before. In this fashion we obtain

$$\psi(\bar{\xi}, \eta) = 2^{2-p}(1 + p(p-2)\cos^2\theta)^{1/2} + \epsilon(\rho, \theta),$$

with $\lim_{\rho\to 0} \epsilon(\rho, \theta) = 0$ uniformly with respect to $\theta \in [0, 2\pi[$ and therefore,

$$\limsup_{\eta\to\bar{\xi}} \psi(\bar{\xi}, \eta) < \infty. \tag{5.3.32}$$

Then relation (5.3.29) follows from relations (5.3.30) to (5.3.32). As a consequence, we have

$$\forall \xi, \eta \in \mathbf{R}^2, \quad \| \|\eta\|^{p-2}\eta - \|\xi\|^{p-2}\xi\| \leq M\|\eta - \xi\|(\|\eta\| + \|\xi\|)^{p-2}. \tag{5.3.33}$$

To prove inequality (5.3.21), we shall use the characterization

$$\|Au - Av\|^* = \sup_{w\in V} \frac{|(Au - Av)w|}{\|w\|}. \tag{5.3.34}$$

By making use of inequality (5.3.33), we infer that

$$\begin{aligned}
|(Au - Av)(w)| &= \left|\int_\Omega (\|\nabla u\|^{p-2}\nabla u - \|\nabla v\|^{p-2}\nabla v)\cdot \nabla w \,dx\right| \\
&\leq \int_\Omega \| \|\nabla u\|^{p-2}\nabla u - \|\nabla v\|^{p-2}\nabla v\| \, \|\nabla w\| \,dx \\
&\leq M\int_\Omega \|\nabla(u - v)\|(\|\nabla u\| + \|\nabla v\|)^{p-2}\|\nabla w\| \,dx \\
&\leq M\|u - v\|\left\{\int_\Omega (\|\nabla u\| + \|\nabla v\|)^p \,dx\right\}^{(p-2)/p}\|w\| \\
&\leq M\|u - v\|(\|u\| + \|v\|)^{p-2}\|w\|,
\end{aligned}$$

and inequality (5.3.21) follows from the above inequality coupled with characterization (5.3.34). □

Strongly monotone operators. Abstract error estimate

We are now in a position to describe an *abstract setting* particularly appropriate for this type of problem and its approximation: We are given a (generally nonlinear) mapping

$$A: V \to V'$$

acting from a space V, with norm $\|\cdot\|$, into its dual space V', with norm $\|\cdot\|^*$, which possesses the two following properties:

(i) The mapping A is *strongly monotone*, i.e., there exists a strictly increasing function $\chi: [0, +\infty[\to \mathbf{R}$ such that

$$\chi(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} \chi(t) = \infty, \tag{5.3.35}$$

$$\forall u, v \in V, \quad (Au - Av)(u - v) \geq \chi(\|u - v\|)\|u - v\|. \tag{5.3.36}$$

In particular, the operator A as defined in (5.3.18) is strongly monotone, with (cf. Theorem 5.3.3)

$$\chi(t) = \alpha t^{p-1}. \tag{5.3.37}$$

(ii) The mapping A is *Lipschitz-continuous for bounded arguments* in the sense that, for any ball $B(0; r) = \{v \in V; \|v\| \leq r\}$, there exists a constant $\Gamma(r)$ such that

$$\forall u, v \in B(0; r), \quad \|Au - Av\|^* \leq \Gamma(r)\|u - v\|. \tag{5.3.38}$$

Thus, the operator A as defined in (5.3.18) is Lipschitz-continuous for bounded arguments, with (cf. Theorem 5.3,3)

$$\Gamma(r) = M(2r)^{p-2}. \tag{5.3.39}$$

Let there be given an element $f \in V'$. For operators which satisfy assumptions (i) and (ii), we are able to obtain in the next theorem an abstract estimate for the error $\|u - u_h\|$, where u and u_h are respectively the solutions of the equations

$$\forall v \in V, \quad Au(v) = f(v) \quad (\text{equivalently, } \; Au = f), \tag{5.3.40}$$

$$\forall v_h \in V_h, \quad Au_h(v_h) = f(v_h), \tag{5.3.41}$$

where V_h is a (finite-dimensional in practice) subspace of the space V (we showed in Theorem 5.3.1 that, with the operator A of (5.3.18), problems (5.3.40) and (5.3.41) have solutions; for general existence results, see "Bibliography and Comments").

Theorem 5.3.4. *Let there be given a mapping $A: V \to V'$ which is strongly monotone and Lipschitz-continuous for bounded arguments.*

Then there exists a constant C independent of the subspace V_h such that

$$\chi(\|u-u_h\|) \leq C \inf_{v_h \in V_h} \|u-v_h\|. \tag{5.3.42}$$

Proof. To begin with, we show that the assumption of strong monotonicity for the operator A implies that *the same a priori bound holds for both solutions u and u_h*: The conjunction of inequality (5.3.36) and relations (5.3.40) implies that

$$\begin{aligned}\chi(\|u\|)\|u\| &\leq (Au - AO)u \\ &= f(u)-(AO)u \leq (\|f\|^* + \|AO\|^*)\|u\|,\end{aligned}$$

and a similar inequality holds with u replaced by u_h. Therefore, the function χ being strictly increasing with $\chi(0)=0$ and $\lim_{t\to\infty}\chi(t)=\infty$ by assumption, we have

$$\|u\|, \quad \|u_h\| \leq \chi^{-1}(\|f\|^* + \|AO\|^*). \tag{5.3.43}$$

Next, let v_h be an arbitrary element in the space V_h. Using the inclusion $V_h \subset V$ and relations (5.3.40) and (5.3.41), we obtain $(Au - Au_h)w_h = 0$ for all $w_h \in V_h$ so that, in particular,

$$(Au - Au_h)(u_h - v_h) = 0.$$

Combining the above equations with inequalities (5.3.36), (5.3.38) and the *a priori* bound (5.3.43), we obtain

$$\begin{aligned}\chi(\|u-u_h\|)\|u-u_h\| &\leq (Au-Au_h)(u-u_h) \\ &= (Au-Au_h)(u-v_h) \\ &\leq \|Au-Au_h\|^*\|u-v_h\| \\ &\leq \Gamma(\chi^{-1}(\|f\|^*+\|AO\|^*))\|u-u_h\|\,\|u-v_h\|,\end{aligned}$$

and thus inequality (5.3.42) is proved, with $C = \Gamma(\chi^{-1}(\|f\|^* + \|AO\|^*))$. □

Remark 5.3.3. The abstract error estimate of the previous theorem is another generalization of Céa's lemma, since in the linear case one has $\chi(t) = \alpha t$. □

Remark 5.3.4. In the particular case of the operator A of (5.3.18), we have $AO = O$, so that with the function χ of (5.3.37), we obtain

$$\|u\|, \quad \|u_h\| \leq \left(\frac{\|f\|^*}{\alpha}\right)^{1/(p-1)}$$

If we argue as in part (iv) of the proof of Theorem 5.3.1, however, we obtain the improved *a priori* bound

$$\|u\|,\quad \|u_h\| \le (\|f\|^*)^{1/(p-1)}. \qquad \square$$

Estimate of the error $\|u - u_h\|_{1,p,\Omega}$

Let us now return to the minimization problem (5.3.2) and its finite element approximation as described at the beginning of this section. For simplicity, *we shall assume that the set* $\bar{\Omega}$ *is polygonal.* Then we get as an application of Theorem 5.3.4:

Theorem 5.3.5. *Let there be given a family of finite element spaces made up of triangles of type* (1), *associated with a regular family of triangulations. Then, if the solution* $u \in W_0^{1,p}(\Omega)$ *of the minimization problem* (5.3.2) *is in the space* $W^{2,p}(\Omega)$, *there exists a constant* $C(\|f\|^*, |u|_{2,p,\Omega})$ *such that*

$$\|u - u_h\|_{1,p,\Omega} \le C(\|f\|^*, |u|_{2,p,\Omega})h^{1/(p-1)}. \tag{5.3.44}$$

Proof. Since $AO = O$, the constant which appears in inequality (5.3.42) is a function of $\|f\|^*$ only. Next, for some constants C independent of the subspace V_h, we have

$$\inf_{v_h \in V_h} \|u - v_h\| \le C|u - \Pi_h u|_{1,p,\Omega} \le Ch|u|_{2,p,\Omega}.$$

It then remains to apply inequality (5.3.42) with the function $\chi(t) = \alpha t^{p-1}$. $\square$

One should be aware that the above error estimate may be somehow illusive in that the solution u need not be in the space $W^{2,p}(\Omega)$ even with very smooth data (cf. Exercise 5.3.1). This is why it was worth proving convergence with the minimal assumption that $u \in W_0^{1,p}(\Omega)$ (Theorem 5.3.2). This is also why we did not consider the (otherwise straightforward) case where the generic finite element in the spaces V_h would be for example the triangle of type (k).

Exercises

5.3.1. Following GLOWINSKI & MARROCCO (1975), consider the one-dimensional analog of the minimization problem (5.3.2), where

$$\begin{cases} \Omega =]-1, +1[, \\ J(v) = \dfrac{1}{p}\displaystyle\int_\Omega |v'|^p \,\mathrm{d}x - \gamma\int_\Omega v \,\mathrm{d}x, \quad \gamma \in \mathbf{R}. \end{cases}$$

Show that the unique solution $u \in W_0^{1,p}(\Omega)$ of this problem is given by

$$u(x) = \left(1 - \frac{1}{p}\right)\gamma^{1/(p-1)}(1 - |x|^{p/(p-1)}),$$

and that

$$u \in W^{2,p}(\Omega) \quad \text{if} \quad 1 < p < \frac{3+\sqrt{5}}{2},$$

$$u \notin W^{2,p}(\Omega) \quad \text{if} \quad \frac{3+\sqrt{5}}{2} \leq p.$$

5.3.2. The object of this problem is to study the minimization problem (5.3.2) (with the functional J as in (5.3.3)) when $1 < p < 2$.

(i) Let V be a reflexive Banach space, and let $J: V \to \mathbf{R}$ be a continuous and convex functional such that $\lim_{\|v\|\to\infty} J(v) = \infty$. Show that there exists at least one element $u \in V$ such that $J(u) = \inf_{v\in V} J(v)$ (cf. CÉA (1971) or LIONS (1968, 1969)).

(ii) Deduce from this result the existence of a unique solution of the minimization problem (5.3.2). Show that this problem is equivalent to solving the equation $Au = f$, where the mapping $A: V \to V'$ is defined as in (5.3.18) and $V = W_0^{1,p}(\Omega)$.

(iii) Following GLOWINSKI & MARROCCO (1975), show that

$$\exists \alpha > 0, \quad \forall u, v \in V, \quad \alpha\|u - v\|^2 \leq (\|u\| + \|v\|)^{2-p}(Au - Av)(u - v),$$

$$\exists M > 0, \quad \forall u, v \in V, \quad \|Au - Av\|^* \leq M\|u - v\|^{p-1}.$$

(iv) Deduce from (iii) that (GLOWINSKI & MARROCCO (1975))

$$\|u - u_h\|_{1,p,\Omega} \leq C(\|f\|^*, |u|_{2,p,\Omega})h^{1/(3-p)},$$

if $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$.

Bibliography and comments

5.1. The content of this section is based on the analysis of FALK (1974). The abstract error estimate of Theorem 5.1.1 has been independently rediscovered by ROUX (1976) in the study of the numerical approximation of a two-dimensional compressible flow problem which can be reduced to variational inequalities, using the method of BREZIS & STAMPACCHIA (1973). Incidentally, the functional setting for this problem is interesting in itself in that the corresponding space V is a weighted Sobolev space, and the domain of definition of its functions is unbounded. The same problem is similarly studied via variational inequalities by CIAVALDINI & TOURNEMINE (1977), who have extended the abstract error estimate of Theorem 5.1.1 so as to include the case where the bilinear and linear forms are approximated (through the process of numerical integration).

FALK (1975) has extended his results to the case of a non convex domain with a smooth boundary. For additional results concerning the approximation of the obstacle problem, see MOSCO & STRANG (1974), MOSCO & SCARPINI (1975). BREZZI, HAGER & RAVIART (1977) have given another proof of Theorem 5.1.2. They have also shown that $\|u - u_h\|_{1,\Omega} = O(h^{(3/2)-\epsilon})$, $\epsilon > 0$ arbitrarily small, when triangles of type (2) are used. NATTERER (1976) has studied the error in the norm $|\cdot|_{0,\Omega}$, using an argument based on the Aubin–Nitsche lemma. For another approach, see BERGER (1976). Finally, NITSCHE (1977) has been able to apply his method of weighted norms to this problem. In this fashion, he obtains an estimate of the form

$$|u - u_h^*|_{0,\infty,\Omega} \leq Ch^2|\ln h|(\|u\|_{2,\infty,\Omega} + \|\psi\|_{2,\infty,\Omega}).$$

However, the corresponding discrete solution u_h^* is found in the subset $U_h^* = U \cap X_{0h}$, instead of the present subset U_h.

FRÉMOND (1971a, 1972) has given a thorough treatment of the related problem of an elastic body lying on a support, the contact surface being unknown.

The elastic-plastic torsion problem (Exercise 5.1.3) is extensively studied in LANCHON (1972). Using techniques from duality theory, FALK & MERCIER (1977) have recently constructed a finite element method which yields directly an approximation of the stresses σ_{13} and σ_{23} with an $O(h)$ convergence in the norm $|\cdot|_{0,\Omega}$. In fact their formulation is more appropriate for this type of problem, where a direct knowledge of the stresses is more important than a knowledge of the stress function. For related results, see MERCIER (1975a, 1975b), GABAY & MERCIER (1976), and BREZZI, JOHNSON & MERCIER (1977), where elasto-plastic plates are considered.

A third type of problem which reduces to variational inequalities occurs with sets U of the form

$$U = \{v \in H^1(\Omega);\ v \geqslant \psi \text{ a.e. on } \Gamma\}.$$

Such problems with unilateral constraints occur in particular in elasticity, where they are known as *Signorini problems* (cf. Exercise 1.2.5). A finite element approximation of such problems is studied in SCARPINI & VIVALDI (1977).

An extension of the present setting consists in looking for the solution u of variational inequalities of the form (see DUVAUT & LIONS (1972)):

$$\forall v \in U,\quad a(u, v-u) + j(v) - j(u) \geqslant f(v-u),$$

where $j: V \to \mathbf{R}$ is a *non differentiable* functional. Such problems are found in particular in the study of *Bingham flows*, with $j(v) = \int_\Omega \|\nabla v\|\, dx$. Their finite element approximations have been analyzed in BRISTEAU (1975, Chapter 2), FORTIN (1972a), GLOWINSKI (1975).

An extensive treatment of variational inequalities and of their approximations is found in GLOWINSKI, LIONS & TRÉMOLIÈRES (1976a, 1976b). The reader who is also interested in the actual solution of the corresponding discrete problems should consult GLOWINSKI (1976b).

A crucial generalization consists in considering the *quasi-variational inequalities* introduced by BENSOUSSAN & LIONS (1973, 1974): Instead of a fixed set U, one considers a family $(U(v))_{v \in V}$ of nonempty closed convex subsets of V and one looks for an element u such that

$$u \in U(u) \quad \text{and} \quad \forall v \in U(u),\quad a(u, v-u) \geqslant f(v-u).$$

Introductions to such "quasi-variational" problems are given in LIONS (1975a, 1975b). A much more complete treatment is given in LIONS (1976).

A variety of *free surface problems* can be reduced to quasi-variational inequalities. In particular, problems of *flows through porous media* can be reduced to variational inequalities or quasi-variational inequalities, by a method due to BAIOCCHI (1971, 1972, 1974, 1975). See also BAIOCCHI, COMINCIOLI, MAGENES & POZZI (1973). Such problems may be also reduced to *optimal domain problems* as in BÉGIS & GLOWINSKI (1974, 1975), CÉA, GIOAN & MICHEL (1974).

5.2. There exist several approaches for analyzing the minimal surface problem. When $n = 2$, there is always a solution in the classical sense (i.e., of the associated boundary value problem; cf. Exercise 5.2.2) for continuous boundary data (RADÓ (1930)), when the set Ω is convex. In higher dimensions, JENKINS & SERRIN (1968) have shown that, rather than convexity, it is the positivity of the mean curvature of the boundary which insures existence of a unique solution in the classical sense (for sufficiently smooth boundary and boundary data). They also proved that, if the mean curvature of the boundary is not everywhere positive, there exist smooth boundary data for which the Dirichlet problem has no solution. MORREY (1966, Theorem 4.2.1) has shown that if the set Ω is strictly convex with a sufficiently smooth boundary and if the function u_0 belongs to the space $W^{2,q}(\Omega)$ for some $q > 2$ and satisfies a "bounded slope condition", then the minimization problem (5.2.2) has a unique solution in the space $W^{2,q}(\Omega)$. For an extensive treatment of the minimal surface problem, see the monumental work of J.C.C. NITSCHE (1975).

TÉMAM (1971) (see also EKELAND & TÉMAM (1974, Chapter V)) has extended the notion of solution so as to get existence for arbitrary bounded open sets Ω, when there is no solution in a more traditional sense (cf. Exercise 5.2.1). The main result is the following:

If $u_0 \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, there exists a *generalized solution* $\tilde{u} \in W^{1,1}(\Omega)$ (unique up to a constant additive factor) in the following sense:

(i) It is analytic in Ω and solution of the associated partial differential equation.

(ii) Any sequence (v_k) with $v_k \in W^{1,1}(\Omega)$ and $\lim_{k\to\infty} J(v_k) = \inf\{J(v);\ (v-u_0) \in W_0^{1,1}(\Omega)\}$ is such that $\lim_{k\to\infty} v_k = \tilde{u}$ in the space $L^1(\Omega)/\mathbf{R}$ and $\lim_{k\to\infty} |v_k - \tilde{u}|_{1,\tilde{\Omega}} = 0$ for any open set $\tilde{\Omega}$ with $\bar{\tilde{\Omega}} \subset \Omega$.

(iii) If there exists a point $x_0 \in \Gamma$ such that $\limsup_{\{x\to x_0,\ x\in\Omega\}} \|\nabla u(x)\| < \infty$, then the generalized solution is unique, and $\tilde{u} = u_0$ on the set $\{x \in \Gamma;\ \limsup_{\{y\to x,\ y\in\Omega\}} \|\nabla u(y)\| < \infty\}$. For recent developments of R. Témam's analysis, see LICHNEWSKY (1974a, 1974b).

There are relatively few references on the application of the finite element method to this problem. Let us first quote HINATA, SHIMASAKI & KIYONO (1974) where only numerical results are presented. The proof that $|u - u_h|_{1,\Omega_h} = O(h)$, i.e., the four first steps of the proof of Theorem 5.2.2, as well as the proof of Theorem 5.2.1, are given in JOHNSON & THOMÉE (1975). Using an adaptation of the Aubin–Nitsche lemma, C. Johnson and V. Thomée have in addition shown that, if $u \in W^{2,q}(\Omega)$ for some $q > 2$ and if u_0 is sufficiently smooth, then for any p with $1 \leqslant p < 2$, one has $|u - u_h|_{0,p,\Omega_h} = O(h^2)$.

More recently, RANNACHER (1977) has completed the results of C.

Johnson and V. Thomée by showing, under the same assumptions, that $|u - u_h|_{0,\Omega_h} = O(h^2)$. Especially, R. Rannacher has been able to adapt the method of weighted norms of J.A. Nitsche described in Section 3.3 so as to derive the error estimate

$$|u - u_h|_{0,\infty,\Omega_h} = O(h^{2-(2/q)}|\ln h|),$$

assuming $u \in W^{2,q}(\Omega)$ for some q with $2 < q \leq \infty$. See also FREHSE & RANNACHER (1976). For similar, but weaker, results, see MITTELMANN (1977). Finally, JOURON (1975) has made an interesting study of the approximation of the generalized solution in the sense of R. Témam.

5.3. We have followed in this section a paper of GLOWINSKI & MARROCCO (1975), where the case $1 < p < 2$ is also treated along the lines indicated in Exercise 5.3.2 (problems of this last type arise, with more complicated boundary conditions however, in the modeling of strains in ice; in this respect, see the thorough study of PÉLISSIER (1975)). Actual methods for solving the discrete problems are described and studied in the above paper by R. Glowinski and A. Marrocco.

In CÉA (1971) and LIONS (1968, 1969), several general theorems concerning the existence of solutions for a problem of the form $\inf_{v \in V} J(v)$ are proved for general functionals J. Another approach for obtaining existence results is to use the theory of monotone operators: A mapping $A: V \to V'$ is said to be *monotone* if $(Au - Av)(u - v) \geq 0$ for all $u, v \in V$. BROWDER (1965) and LERAY & LIONS (1965) have proved: Let V be a reflexive Banach space and let A be a monotone operator such that

(i) there exists a strictly increasing function χ with $\lim_{t\to\infty} \chi(t) = \infty$ such that $Av(v) \geq \chi(\|v\|)\|v\|$ for all $v \in V$ (a property implied by the definition of strongly monotone operators as given in the text),

(ii) given any finite-dimensional subspace W of the space V and given any sequence of elements $w_k \in W$ which converges to $w \in W$, one has $\lim_{k\to\infty} Aw_k(v) = Aw(v)$ for all $v \in V$ (a property implied by the Lipschitz-continuity for bounded arguments).

Then the mapping $A: V \to V'$ is a bijection. Therefore this result provides a more general method for proving existence.

One of the first systematic treatments of variational approximations of nonlinear problems of monotone type appears to be that of CIARLET (1966). This work was then extended in several directions, in CIARLET, SCHULTZ & VARGA (1967, 1968a, 1968c, 1969), CIARLET, NATTERER & VARGA (1970), MOCK (1975), NOOR & WHITEMAN (1976), SCHULTZ (1969a, 1971), LOUIS (1976). See also MELKES (1970) for an independent work. In particular, Theorem 5.3.4 is adapted from Theorem 2.1 of CIARLET, SCHULTZ & VARGA (1969). In this paper, one considers monotone operator equations of the general form

$$\forall v \in V, \quad \sum_{|\alpha| \leq m} \int_\Omega A_\alpha(x, u, Du, \ldots, D^m u)\partial^\alpha v \,\mathrm{d}x = 0.$$

These contain as special case the equations $J'(u)v = 0$ associated with functionals of the type

$$v \in H_0^1(\Omega) \to J(v) = \frac{1}{2}\int_\Omega \|\nabla v\|^2 \,\mathrm{d}x - \int_\Omega \left\{\int_0^{v(x)} f(x, \eta)\,\mathrm{d}\eta\right\} \mathrm{d}x,$$

which correspond to *nonlinear Dirichlet problems* of the form $-\Delta u = f(x, u)$ in Ω, $u = 0$ on Γ. General approximate methods for problems with monotone operators are studied in BRÉZIS & SIBONY (1968). Techniques from duality theory can be applied to such problems, as in BERCOVIER (1976), SCHEURER (1977).

Additional bibliography and comments

Other nonlinear problems

We continue this review by mentioning nonlinear problems of various type, some of which are reminiscent of the problems considered in Chapter 5.

For example, the nonlinear boundary value problem

$$\begin{cases} -\sum_{i=1}^{2} \partial_i(a(\|\nabla u\|)\partial_i u) = 0 & \text{in} \quad \Omega \subset \mathbf{R}^2, \\ u = 0 & \text{on} \quad \Gamma, \end{cases}$$

where $a(r)$ is a strictly increasing function of its argument r, describes the magnetic state in the cross-section of an alternator. It is thoroughly studied, as well as its finite element approximation, in GLOWINSKI & MARROCCO (1974).

NITSCHE (1976c) and FREHSE & RANNACHER (1977) have applied the method of weighted norms of J.A. Nitsche (cf. Section 3.3) to nonlinear problems of the form

$$\begin{cases} -\sum_{i=1}^{n} \partial_i(a(x, u)\partial_i u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma, \end{cases}$$

and

$$\begin{cases} -\sum_{i=1}^{n} \partial_i a_i(x, u, \nabla u) + a_0(x, u, \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases}$$

respectively. See also DOUGLAS & DUPONT (1975) for another approach.

The method of weighted norms has also been applied to problems where the solution $u \in H_0^1(\Omega)$, $\Omega \subset \mathbf{R}^2$, minimizes an integral of the form $\int_\Omega F(x, v, \nabla v)\, dx$ over the space $H_0^1(\Omega)$. For uniformly convex functions $F(x, \xi, \cdot)$ (this is not the case of the minimal surface problem), FREHSE (1976) shows that $|u - u_h|_{0,\infty,\Omega_h} = O(h^2|\ln h|)$ for the piecewise linear approximations.

A wide class of nonlinear problems arises in nonlinear elasticity, particularly in the study of large strains. For a thorough treatment of the application of the finite element method to such problems, see the book of ODEN (1972a), and also ODEN (1973b, 1976b), CAREY (1974).

Problems in which the solution must satisfy various equality and inequality constraints arise in water pollution control. Their numerical solution, which combines finite element methods and linear programming methods, is considered by FUTAGAMI (1976). A challenging domain of study is the approximation of *bifurcation problems* (which arise in particular in elasticity). In this direction, we mention the pioneering work of KIKUCHI (1976b), who considers the problem

$$\begin{cases} -\Delta u = \lambda u - u^3 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases}$$

The Navier–Stokes problem

For large gradients of the velocity, a new term has to be added in the partial differential equation of the Stokes problem (cf. the section "Additional Bibliography and Comments" of Chapter 4), a process which gives rise to the *Navier–Stokes problem*:

$$\begin{cases} -\nu\Delta \boldsymbol{u} + \sum_{i=1}^{n} u_i\partial_i \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma. \end{cases}$$

One can show (cf. LADYŽENSKAJA (1963), LIONS (1969)) that if the ratio $|f|_{0,\Omega}/\nu^2$ is small enough, the associated variational problem (whose derivation follows the same line as for the Stokes problem) has one and only one solution in the space $V \times \{L^2(\Omega)/P_0(\Omega)\}$.

JAMET & RAVIART (1974) have extended to this problem the analysis which CROUZEIX & RAVIART (1973) developed for the Stokes problem, even adding the effect of numerical integration (which is especially needed for the computation of the integrals associated with the nonlinear term $\Sigma_{i=1}^n u_i\partial_i u$). Their results are related to those of FORTIN (1972a), who was the first to mathematically analyze the finite element approximation of the Navier-Stokes problem.

Further references are BERCOVIER (1976), GIRAULT (1976b), OSBORN (1976a) for the finite element approximation of the associated eigenvalue problem, TÉMAM & THOMASSET (1976), THOMASSET (1974) and especially, the extensive treatments given by TÉMAM (1973, 1977).

For a reference in the Engineering literature, see TAYLOR & HOOD (1973), GARTLING & BECKER (1976) where infinite domains are also considered.