NISDE: Screenshots of Lecture 11

Lecture 11

Goals: ✓ Connection between SDEs and PDEs (Kolmogorov representation formula)
✓ bound of moments of exact and numerical solutions
✓ weak local error

6.2 Weak convergence analysis

Consider the (scalar) SDE:

$$\begin{cases} dX(t) = f(t,X(t))\,dt + g(t,X(t))\,dW(t), & 0 \le t \le T \\ X(0) = X_0 \end{cases}$$

Recall:

Polynomial growth: Let $\ell \in \mathbb{N}$.

$\phi : \mathbb{R}^n \to \mathbb{R}$ is in C_p^ℓ

if $\phi(x)$ is ℓ times differentiable

and there exist $m \in \mathbb{N}$, $C > 0$ s.t.

$$|\phi^{(k)}(x)| \le C(1 + |x|^m) \quad \text{for all } x \in \mathbb{R}^n, \; 1 \le k \le \ell$$

C_b^ℓ: Set of all functions that are ℓ times differentiable with bounded derivatives

$$C_b^\ell = \{ \phi : \phi(x) \ \ell \text{ times differentiable}, \ |\phi^{(k)}(x)| \le C, \ \forall x \in \mathbb{R}^n, \ 1 \le k \le \ell \}$$

Weak convergence:

A numerical method $\{X_n\}_{n\geq 1}$ is said to have weak order of convergence r, if there exists a constant C s.t.

$$|E[\phi(X_n)] - E[\phi(X(t_n))]| \leq C h^r$$

for any $t_n = nh \in [0,T]$ and all $\phi \in C_p^{2(r+1)}$
(h small enough)

A Connection between SDEs and PDEs

Consider a (one-dim.) SDE for $s \leq T$

$$(S) \begin{cases} dX(s) = f(s, X(s))\,ds + g(s, X(s))\,dW(s), & s \geq \xi \\ X(\xi) = x \end{cases}$$

where $x \in \mathbb{R}$

Assume that f and g satisfy the assumptions on existence & uniqueness of a solution and denote $X^{\xi,x}(s)$ the solution of this SDE.

Next define the differential operator for a smooth function $u(\xi,x) : [0,T] \times \mathbb{R} \to \mathbb{R}$

$$\mathcal{L} u(\xi,x) = f(\xi,x)\,\partial_x u(\xi,x) + \frac{1}{2} g^2(\xi,x)\,\partial_{xx} u(\xi,x)$$

Further consider the parabolic PDE for $s \leq T$

$$(P) \begin{cases} \partial_\xi u(\xi,x) + \mathcal{L} u(\xi,x) = 0, & 0 \leq \xi \leq s \\ u(s,x) = \varphi(x) \end{cases}$$

Backward Kolmogorov Equation

Classical result:

If $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous with $f(\xi,\cdot), g(\xi,\cdot) : \mathbb{R} \to \mathbb{R}$ in C_b^m uniformly in $\xi \in [0,T]$, $m \geq 2$ and $\varphi \in C_p^m$, $m \geq 2$.

Then there exits an unique solution $u(t,x)$ of (P) with $u(\xi,x)$, $\partial_\xi u(\xi,x)$, $\frac{\partial^\ell}{\partial x^\ell} u(\xi,x)$ continuous with polynomial growth $\ell \leq m$.

(without proof)

Thm 1: (Kolmogorov representation formula)

Assume that the assumptions of the classical result hold with $m \geq 2$.

Then the solution $u(\xi,x)$ of (P) has the representation

$$u(\xi,x) = E[\varphi(X^{\xi,x}(s))], \quad 0 \leq \xi \leq s$$

Proof: Applying the Itô formula yields

$$\begin{aligned} u(s, X^{\xi,x}(s)) &= u(\xi, \underbrace{X^{\xi,x}(\xi)}_{=x}) \\ &+ \int_\xi^s \underbrace{\left(\partial_\xi u(r, X^{\xi,x}(r)) + f(r, X^{\xi,x}(r))\, \partial_x u(r, X^{\xi,x}(r)) + \tfrac{1}{2} g^2(r, X^{\xi,x}(r))\, \partial_{xx} u(r, X^{\xi,x}(r))\right)}_{= \partial_\xi u + \mathcal{L}u = 0} dr \\ &+ \int_\xi^s g(r, X^{\xi,x}(r))\, \partial_x u(r, X^{\xi,x}(r))\, dW(r). \end{aligned}$$

Since u solves the PDE (P), we have

$$u(s, X^{\xi,x}(s)) - u(\xi,x) = \underbrace{\int_\xi^s g(r, X^{\xi,x}(r))\, \partial_x u(r, X^{\xi,x}(r))\, dW(r)}_{E[\cdots] = 0}$$

Applying the expectation and using the properties of Itô integrals

$$E[\underbrace{u(s, X^{\xi,x}(s))}_{= \varphi(X^{\xi,x}(s))}] = E[\underbrace{u(\xi,x)}_{\text{not stochastic}}] = u(\xi,x)$$

$$\Rightarrow \quad E[\varphi(X^{\xi,x}(s))] = u(\xi,x) \qquad \#$$

Rem.: Consider the SDE (S) with random initial value η assumed to be $\mathcal{F}_\xi$-measurable.

The same proof as above, using the Itô formula, gives:

$$u(s,X^{\xi,\eta}(s)) - u(\xi, \underbrace{X^{\xi,\eta}(\xi)}_{=\eta}) = \int_\xi^s g(r,X^{\xi,\eta}(r))\,\partial_x u(r,X^{\xi,\eta}(r))\,dW(r)$$

but next we have to take the conditional expectation:

$$E[u(s,X^{\xi,\eta}(s))\,|\,\eta] - \underbrace{E[u(\xi,\eta)\,|\,\eta]}_{\substack{=u(\xi,\eta) \\ \text{as it is } \eta\text{-measurable}}} = \underbrace{E\left[\int_\xi^s g(r,X^{\xi,\eta}(r))\,\partial_x u(r,X^{\xi,\eta}(r))\,dW(r)\,\Big|\,\eta\right]}_{\substack{=0 \\ \text{since } E[\int_a^b G(r)dW(r)\,|\,\mathcal{F}_a]=0}}$$

Hence

$$u(\xi,\eta) = E[u(s,X^{\xi,\eta}(s))\,|\,\eta]$$
$$= E[\varphi(X^{\xi,\eta}(s))\,|\,\eta].$$

Bound of moments of exact and numerical solutions

We have shown for the solution $X(t)$ of

$$\begin{cases} dX(t) = f(t,X(t))dt + g(t,X(t))dW(t), & 0 \le t \le T \\ X(0) = X_0 \end{cases}$$

that $E[\sup_{0\le t\le T} |X(t)|^2] \le C(1 + E[|X_0|^2])$

In fact, under the assumptions of existence & uniqueness, if $E[|X_0|^p] < \infty$ $p \ge 2$, then

$$E[\sup_{0\le t\le T} |X(t)|^p] \le C(1 + E[|X_0|^p]).$$

We want to establish a similar result for a numerical solution.

Lemma 1: Let $\{X_n\}_{n\geq 0}$ be a stochastic numerical method, $n=0,1,\dots,N$, $h=\frac{T}{N}$, $N\in\mathbb{N}^*$ with $\mathbb{E}[|X_0|^{2p}]<\infty$ $p\geq 1$.

Assume that

(i) $\mathbb{E}[X_{n+1}-X_n \mid X_n = x] \leq C(1+|x|)h$

(ii) $|X_{n+1}-X_n| \leq M_n(1+|X_n|)\sqrt{h}$,

where M_n is a r.v. indep. of X_n s.t. $\mathbb{E}[|M_n|^r]\leq C_r$ $\forall r\in\mathbb{N}$ with C_r a constant indep. of n,h.

Then there exists a constant C_{2p} indep. of h and n s.t.

$$\mathbb{E}[|X_n|^{2p}]\leq C_{2p} \quad \text{for } nh\leq T.$$

Proof: Set $X_{n+1} = X_n + \Delta X_n$.

We have

$$|X_{n+1}|^{2p} = |X_n+\Delta X_n|^{2p} = X_n^{2p} + \alpha_{2p}^1 X_n^{2p-1}\Delta X_n + \sum_{j=2}^{2p}\alpha_{2p}^j X_n^{2p-j}(\Delta X_n)^j$$

$$\Rightarrow \mathbb{E}[|X_{n+1}|^{2p}] = \mathbb{E}[|X_n|^{2p}] + \alpha_{2p}^1\mathbb{E}[X_n^{2p-1}\Delta X_n] + \sum_{j=2}^{2p}\alpha_{2p}^j\mathbb{E}[X_n^{2p-j}(\Delta X_n)^j]$$

$$
\begin{aligned}
(1)\quad \mathbb{E}[X_n^{2p-1}\Delta X_n] &= \mathbb{E}[\mathbb{E}[X_n^{2p-1}\Delta X_n \mid X_n]] \\
&= \mathbb{E}[X_n^{2p-1}\mathbb{E}[\Delta X_n\mid X_n]] \\
&\overset{(i)}{\leq} Ch\,\mathbb{E}[X_n^{2p-1}(1+|X_n|)] \\
&\leq Ch\left(\mathbb{E}[|X_n|^{2p}]+\mathbb{E}[|X_n|^{2p-1}]\right) \\
&\overset{\text{Hölder}}{\leq} Ch\left(\mathbb{E}[|X_n|^{2p}]+\left(\mathbb{E}[|X_n|^{2p}]\right)^{\frac{2p-1}{p}}\right) \\
&\leq 2Ch\left(1+\mathbb{E}[|X_n|^{2p}]\right)
\end{aligned}
$$

Since

$$\left(\mathbb{E}[|X_n|^{2p}]\right)^{\frac{2p-1}{p}} \leq \begin{cases} 1 & \text{if } \mathbb{E}[|X_n|^{2p}]\leq 1 \\ \mathbb{E}[|X_n|^{2p}] & \text{otherwise}\end{cases}$$

(2) $\mathbb{E}[X_n^{2p-j}(\Delta X_n)^j] \overset{(ii)}{\leq} \mathbb{E}[|X_n|^{2p-j} M_n^j (1+|X_n|)^j h^{j/2}]$

$= \mathbb{E}[|M_n|^j]\ \mathbb{E}[|X_n|^{2p-j}(1+|X_n|)^j h^{j/2}]$

$\leq C_j\, h\, \beta_{2p}\, (1+\mathbb{E}[|X_n|^{2p}])$

using similar calculations as in (1).

Finally,

$$\mathbb{E}[|X_{n+1}|^{2p}] \leq \hat{C}h\,(1+\mathbb{E}[|X_n|^{2p}]) + \mathbb{E}[|X_n|^{2p}].$$

Set $z_n = 1+\mathbb{E}[|X_n|^{2p}]$.

Then $z_{n+1} = 1+\mathbb{E}[|X_{n+1}|^{2p}]$

$\leq (1+\hat{C}h)(1+\mathbb{E}[|X_n|^{2p}])$

$= \underbrace{(1+\hat{C}h)}_{\leq e^{\hat{C}h}} z_n \leq e^{\hat{C}h} z_n$

Hence by iteration

$$z_n \leq e^{\hat{C}hn} z_0 \leq e^{\hat{C}T} z_0 \qquad \forall nh \in [0,T]$$

Therefore, $\mathbb{E}[|X_n|^{2p}] \leq e^{\hat{C}T}(1+\mathbb{E}[|X_0|^{2p}]) = C_{2p}.$

#

Example: The Euler-Maruyama method verifies the hypotheses of lemma 2.
(See exercises).

Weak local order

Let $X(s)$ be the solution of the SDE

$$\begin{cases} dX(s) = f(s,X(s))ds + g(s,X(s))dW(s), & 0\leq s\leq T \\ X(0) = X_0 \end{cases}$$

We make the usual assumption on f and g to have existence and uniqueness.

Further we assume sufficient smoothness (to apply successively the Itô formula).

Defn.: We say that a numerical method $\{X_n\}_{n\geq 0}$ has weak local order r, if for any function $\varphi \in C_p^{2r+2}$ there exist constants C, k s.t.

$$|E[\varphi(X(h))|X_0] - E[\varphi(X_1)|X_0]| \leq R$$

where $E[R] \leq C(1 + E[|X_0|^k])h^{r+1}$.

Rem.: If a numerical method has weak local order r, then we have

$$|E[\varphi(X(t_n))|X(t_{n-1}) = x] - E[\varphi(X_n)|X_{n-1} = x]| \leq R$$

with $E[R] \leq C(1 + E[|x|^k])h^{r+1}$.