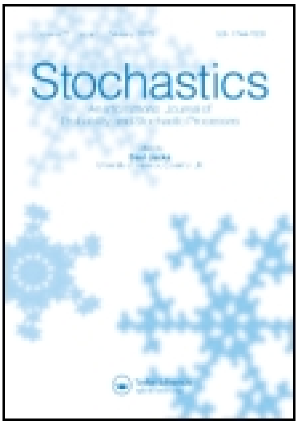

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Langevins stochastic differential equation extended by a time-delayed term

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LANGEVINS STOCHASTIC DIFFERENTIAL EQUATION EXTENDED BY A TIME-DELAYED TERM

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The stochastic differential equation $dX(t) = [aX(t) + bX(t-r)]\,dt + dW(t)$ is a generalization of Langevin's equation, which is obtained if $b = 0$. Necessary and sufficient conditions on a, b and r are given under which a stationary solution exists. In this case, it is unique and Gaussian. Its covariance function and its spectral density are studied. The covariance function tends to zero exponentially, and the exponential rate of convergence is non-smooth for certain values of a, b and r. Moreover, the covariance function oscillates around zero for certain parameters a, b, r.

KEY WORDS Linear stochastic difference-differential equations, Langevin-equation, time delay, stationary solutions, covariance function, spectral density.

1 INTRODUCTION

Suppose $W = (W(t), \mathscr{F}(t), t \geq 0)$ is a one-dimensional standard Wiener process, and Z is an $\mathscr{F}(0)$-measurable random variable, both defined on a probability space $(\Omega, \mathscr{F}, P)$ with the filtration $(\mathscr{F}(t), t \geq 0)$. Then for $a \in R$ the Langevin equation

$$dX(t) = aX(t)\,dt + dW(t), \quad t \geq 0, \tag{1.1}$$

$$X(0) = Z \tag{1.2}$$

has a unique solution given by

$$X(t) := Z\exp(at) + \int_0^t \exp(a(t-s))\,dW(s), \quad t \geq 0. \tag{1.3}$$

This process $X := (X(t), t \geq 0)$ is known as the Ornstein-Uhlenbeck velocity process, and it was often used for stochastic modelling in different sciences (see e.g. Ornstein, Uhlenbeck [17], Nelson [16], Lánský [12]).

What we are going to do in this note is to study the equation

$$dX(t) = [aX(t) + bX(t - r)]\,dt + dW(t), \quad t \geq 0 \tag{1.4a}$$

$$X(s) = Z(s), \quad s \in [-r, 0], \tag{1.4b}$$

where a, b, r are real constants, $r > 0$, and $(Z(s), s \in [-r, 0])$ is a given process. Differential equations, which include time-delayed terms, are useful for stochastic modelling e.g. in biological, biometrical or economical sciences (see Banks [2], Cushing [5] or Hale [7] for applications and further references).

We shall be concerned with the explicit solution of (1.4), analogously to (1.3) and with necessary and sufficient conditions on a, b and r such that a stationary solution X of (1.4a) exists. In this case we determine the spectral density which turns out to be a transcendent function (if $br \neq 0$). Moreover, we shall study the covariance function K of the stationary solution deriving a differential equation for K solving it on $[-r, r]$ and showing, that it tends to zero with exponential rate.

Contrary to the Ornstein-Uhlenbeck case the covariance function may oscillate around zero.

Equation (1.4) is a very special linear stochastic functional differential equation, and the results below can be extended partially to more general cases. Here we restrict ourselves to (1.4) because, firstly, several quantities related to (1.4) can be calculated explicitly, analogously to Langevin's equation. This does not seem to be possible in simple form even for the case

$$dX(t) = \sum_{i=0}^{N} a_i X(t - r_i)\,dt + dW(t), \quad t \geq 0 \tag{1.5}$$

where $0 = r_0 < r_1 < \cdots < r_N$; $a_i \in R$ are fixed. (For a special case see Bailey, Williams [1]). Secondly, the solutions of (1.4) already show typical effects due to the presence of time-delayed terms. Extensions of the results to higher dimensions and other driving terms than $W(t)$ are possible and will be presented elsewhere.

The following notations will be used: N is the set of nonnegative integers, R the real axis, $R_+ := [0, \infty)$, $R_2 := R \times R$, C the set of complex numbers, i the imaginary unit, $\mathbf{1}_A(\cdot)$ denotes the indicator function of the set A, ∂A the boundary of A, $\mathscr{L}^p(c, d)$ stands for the linear space of realvalued with respect to the Lebesgue-measure p-summable functions on $[c, d] \subset R$, $p \geq 1$, $\dot{x}$ denotes the derivative of the function $x(t)$. $N(\mu, \sigma^2)$ denotes the normal distribution with expectation μ and variation σ^2.

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2 DEFINITIONS AND RESULTS

2.1 *Solutions*

Assume $W = (W(t), \mathscr{F}(t), t \geq 0)$ is a real-valued standard Wiener process on a probability space $(\Omega, \mathscr{F}, P)$, $Z = (Z(t), t \in [-r, 0])$ is a real-valued stochastic process on the same space and let $Z(t)$ be $\mathscr{F}(0)$-measurable, $t \in [-r, 0]$. Such a process Z will be called an *initial process*. Suppose $a, b \in R$, $r > 0$. We consider the difference-differential equation

$$dX(t) = [aX(t) + bX(t-r)]\,dt + dW(t); \quad t \geq 0, \tag{2.1a}$$

with the initial condition

$$X(t) = Z(t), \quad t \in [-r, 0]. \tag{2.1b}$$

DEFINITION 2.1 A pathwise continuous stochastic process $X = (X(t),\ t \geq -r)$ on $(\Omega, \mathscr{F}, P)$ is called a solution of (2.1) if

i) $X(t)$ is $\mathscr{F}(t)$-measurable for every $t \geq 0$,
ii) $X(t) = Z(0) + \int_0^t (aX(s) + bX(s-r))\,ds + W(t) \quad P\text{-a.s.}, \quad t \geq 0,$
iii) X satisfies (2.1b).

A solution X is said to be unique if for every solution Y of (2.1) we have

$$P\left(\sup_{t \geq -r} |Y(t) - X(t)| > 0\right) = 0.$$

To solve the equation (2.1) firstly we have to consider the *deterministic equation* corresponding to (2.1):

$$\dot{x}(t) = ax(t) + bx(t-r), \quad t \geq 0, \tag{2.2a}$$

$$x(t) = g(t), \quad t \in [-r, 0], \tag{2.2b}$$

where g is a given function on $[-r, 0]$.

A function

$$x = x(t), \quad t \geq -r$$

is called a *solution of* (2.2), if it is absolutely continuous on $[0, \infty)$, satisfies (2.2a) (Lebesgue-almost everywhere) and (2.2b).

The equation (2.2) can be solved step by step on the intervals

$$[k \cdot r, (k+1)r], \quad k \geq 0,$$

provided that $g \in \mathscr{L}^1(-r, 0)$. In this way we get for

$$g(t) := \mathbf{1}_{\{0\}}(t), \quad t \in [-r, 0]$$

the so-called *fundamental solution* x_0 of (2.2):

$$x_0(t) = \sum_{k=0}^{[t/r]} (b^k/k!)(t - kr)^k \exp(a(t - kr)), \quad t \geq 0, \tag{2.3}$$

where $[t/r] := \max\{k \in N \mid k \leq t/r\}$.

For $g \in \mathscr{L}^1(-r, 0)$ the solution x_g of (2.2) is given by

$$x_g(t) := x_0(t) \cdot g(0) + b \int_{-r}^{0} x_0(t - s - r)g(s)\, ds, \quad t \geq 0. \tag{2.4}$$

Let us return to the stochastic equation (2.1). It holds

PROPOSITION 2.2 *Assume Z has continuous trajectories. Then the stochastic equation* (2.1) *has a unique solution. It is given by*

$$\begin{cases} X(t) = x_0(t)Z(0) + b \int_{-r}^{0} x_0(t - s - r)Z(s)\, ds + \int_0^t x_0(t - s)\, dW(s), & t \geq 0, \\ X(t) = Z(t), & t \in [-r, 0], \end{cases} \tag{2.5}$$

where x_0 denotes the fundamental solution of (2.2).

Proof Existence and uniqueness immediately follow by solving (2.1) using the step-method. The representation (2.5) is verified by inserting in (2.1). ■

2.2 *Stationary solutions*

We shall establish conditions under which a stationary solution of (2.1a) exists.

DEFINITION 2.3 A solution $U = (U(t), t \geq -r)$ of (2.1) is called a stationary solution if its finite-dimensional distributions are invariant under time translations, i.e.

$$P(U(t + t_k) \in A_k, k = 1, \ldots, n) = P(U(t_k) \in A_k, k = 1, \ldots, n)$$

for all $t > 0$, $n \geq 1$, $t_1, \ldots, t_n \geq -r$ and all Borel sets A_k, $k = 1, \ldots, n$.

We say (2.1a) has a stationary solution U if there is an initial process Z such that U is a stationary solution of (2.1) with $U(t) = Z(t)$, $t \in [-r, 0]$.

DEFINITION 2.4 A stationary solution is said to be uniquely determined if every two stationary solutions of (2.1a) have the same finite-dimensional distributions.

In the following we suppose that $W = (W(t), \mathscr{F}(t), t \geq 0)$ is extended to a Wiener process with R as time axis also denoted by $W = (W(t), \mathscr{F}(t), t \in R)$. This is no restriction of generality.

Before formulating conditions for the existence of stationary solutions we need some more knowledge on the deterministic equation (2.2) and related quantities, see e.g. Hale [7] for details.

The *characteristic function* $h(\cdot)$ of (2.1a) is defined by

$$h(\lambda) := \lambda - a - b\exp(-\lambda r), \quad \lambda \in C. \tag{2.6}$$

A *characteristic root* of (2.1a) is a solution of $h(\lambda) = 0$. Define Λ to be the set of all characteristic roots of (2.1a):

$$\Lambda := \{\lambda \in C \mid h(\lambda) = 0\}$$

and introduce the notation

$$v_0 = v_0(a, b, r) := \max\{\text{Re } \lambda \mid \lambda \in \Lambda\}. \tag{2.7}$$

Then we have the following result.

LEMMA 2.5 *Assume $a, b \in R$, $r > 0$ are fixed. Then it holds:*

i) *For every real c the set $\Lambda \cap \{\lambda \in K \mid \text{Re } \lambda > c\}$ is finite, in particular $v_0(a, b, r) < \infty$.*

ii) *For every $v > v_0$ there exist constants $K_j = K_j(v) > 0$, $j = 0; 1$, such that*

$$|x_0(t)| \leq K_0 \exp(v \cdot t), \quad t \geq 0, \tag{2.8a}$$

$$|\dot{x}_0(t)| \leq K_1 \exp(v \cdot t), \quad t \geq 0. \tag{2.8b}$$

Proof The proof of proposition (i) and of the inequality (2.8a) can be found in Hale [7] Chapter I. The inequality (2.8b) then follows immediately. □

COROLLARY 2.6 *For every $g \in \mathscr{L}^1(-r, 0)$ and every $v > v_0(a, b, r)$ there exists a constant $C = C(g, v) > 0$ such that*

$$|x_g(t)| \leq C \cdot \exp(v \cdot t); \quad t \geq 0.$$

Proof Apply (2.8a) and (2.4). ■

Define

$$S := \{(u, v) \in R_2 \colon u < 1,\ u + v < 0,\ -v < \xi \sin \xi + u \cos \xi\},$$

where $\xi = \xi(u)$ is the root of

$$\xi = u \cdot \tan \xi,\ 0 < \xi < \pi, \quad \text{if } u \neq 0,$$

and

$$\xi = \pi/2, \quad \text{if } u = 0.$$

The set $S \subseteq R^2$ can also be described in the following way:

$$S = \{(u, v) \in R^2 \colon u < 1,\ v \in (v_1(u), v_2(u))\}$$

where

$$v_1(u) := \begin{cases} -u/\cos\xi & \text{for} \quad u \neq 0, \\ -\pi/2 & \text{for} \quad u = 0, \end{cases}$$

($\xi = \xi(u)$ as above),

$$v_2(u) := -u.$$

The following proposition is essential for the sequel and a consequence of a well-known result for deterministic difference-differential equations.

PROPOSITION 2.7 *We have $v_0(a, b, r) < 0$ if and only if $(ar, br) \in S$.*

Proof Introduce

$$u := ar, \; v := br, \; \mu := \lambda r \tag{2.9a}$$

and note that λ is a root of $h(\lambda) = 0$ if and only if μ is a root of $\tilde{h} = 0$ with

$$\tilde{h}(\mu) := \mu - u - v\exp(-\mu), \quad \mu \in C. \tag{2.9b}$$

Defining

$$\tilde{v}_0(u, v) := \max\{\operatorname{Re}\mu : \tilde{h}(\mu) = 0\} \tag{2.9c}$$

we get:

$$v_0(a, b, r) = \tilde{v}_0(ar, br)/r. \tag{2.10}$$

Note, that $\tilde{h}$ is the characteristic function of

$$\dot{x}(t) = ux(t) + vx(t-1), \quad t \geq 0.$$

Now apply a result of Hayes [8] (see also Hale [7], p. 339), which says that $\tilde{v}_0(u, v) < 0$ if and only if $(u, v) \in S$. ■

PROPOSITION 2.8 *For the equation (2.1a) the following properties are equivalent:*

i) *There exists a stationary solution X,*

ii) *All characteristic root of (2.1a) have negative real part: $v_0(a, b, r) < 0$.*

iii) $(ar, br) \in S$,

iv) *The fundamental solution $x_0(\cdot)$ of (2.2) is square integrable:*

$$\sigma_0^2 := \int_0^\infty x_0^2(s)\, ds < \infty. \tag{2.11}$$

Proof The equivalence of (ii) and (iii) was established in Proposition 2.7. We show (ii)$\Leftrightarrow$(iv):

If v_0 is negative, then (2.11) follows from (2.8a) by choosing an $v \in (v_0, 0)$.

Assume (2.11) holds. Then every solution g of (2.2) with initial function $g \in \mathscr{L}^2(-r, 0)$ is square integrable over R_+. This follows from (2.4) using Schwartz' inequality. If $v_0 \geq 0$, then there exists a characteristic root λ_0 with Re $\lambda_0 \geq 0$. Obviously, $\varphi(t) := \exp(\lambda_0 t)$, $t \geq -r$ is a solution of (2.2) with $\varphi | [-r, 0] \in \mathscr{L}^2(-r, 0)$. But φ is not square integrable on R_+. Thus $v_0 < 0$ must hold.

It remains to show that (iv)$\Leftrightarrow$(i).

Suppose that (iv) holds. Then the following integrals exist by assumption

$$U_t := \int_{-\infty}^{t} x_0(t-s)\, dW_s, \quad t \in R.$$

Obviously $EU_t \equiv 0$. Calculating the characteristic function one can show that for all $t_1 < \cdots < t_n$ the random vector $(U(t_1), \ldots, U(t_n))$ is normally distributed with covariance matrix $G = (g_{ij})$ given by

$$g_{ij} = \int_0^\infty x_0(|t_i - t_j| + s)x_0(s)\, ds, \quad i, j = 1, \ldots, n. \tag{2.12}$$

In particular, U is continuous and stationary. That this process U satisfies the equation (2.1) is proved by inserting and using that $x_0(\cdot)$ is the fundamental solution of (2.2).

Thus (i) is valid.

Conversely, assume (i) holds and X is a stationary solution. In particular, X is continuous and has the representation (2.5). Thus, introducing X^0 by

$$X^0(t) := x_0(t)X(0) + b\int_{-r}^{0} x_0(t-s-r)X(s)\, ds, \quad t \geq 0, \tag{2.13}$$

we obtain

$$E\exp(i\lambda X(t)) = E\exp(i\lambda X^0(t)) \cdot \exp\left(-\lambda^2/2 \int_0^t x_0^2(s)\, ds\right), \quad t \geq 0,\ \lambda \in R. \tag{2.14}$$

(Use

$$\int_0^t x_0(t-s)\, dW_s \in N\left(0, \int_0^t x_0^2(s)\, ds\right)$$

and the independence of $(W(t), t \geq 0)$ and $\mathscr{F}(0)$.)

The left-hand side of (2.14) is independent of t by stationarity. Thus we get (2.11), i.e. (iv) and therefore $v_0 < 0$ by (ii). By (2.8) and the Corollary 2.6 it follows that

$$\lim_{t\to\infty} X^0(t) = 0 \quad P\text{-a.s.}$$

Thus $X(t) \in N(0, \sigma_0^2)$, in particular, X has finite second moments. ∎

Now it is easy to see from Proposition 2.2 that the covariance function $K(\cdot)$ of the stationary solution X is given by

$$\begin{aligned} K(u) &= EX(t)X(t+u) \\ &= EX^0(t)X^0(t+u) + \int_0^t x_0(t-s)x_0(t+u-s)\,ds, \quad u, t \geq 0. \end{aligned}$$

Consequently, we have

$$E(X_t^0)^2 + \int_0^t x_0^2(v)\,dv = K(0) < \infty, \quad t \geq 0.$$

In the following corollary the condition of stationarity will be presented in a more lucid form.

COROLLARY 2.9 *We have*:

i) *If*

$$a + b < 0 \quad \text{and} \quad a - b \leq 0 \tag{2.15}$$

then there exists a stationary solution of (2.1*a*) *regardless of the value of* $r > 0$.

ii) *If*

$$a + b \geq 0 \tag{2.16a}$$

or

$$ar \geq 1, \tag{2.16b}$$

then there does not exist a stationary solution of (2.1*a*) *regardless of* $r > 0$ *in the case* (2.16*a*) *or regardless of* $b \in R$ *in the case* (2.16*b*), *respectively.*

iii) *If*

$$a + b < 0 \quad \text{and} \quad a - b > 0, \tag{2.17}$$

then there exists a stationary solution of (2.1*a*) *if and only if*

$$r \in (0, r_0(a, b)) \tag{2.18}$$

where

$$r_0(a, b) := [\arccos(-a/b)]/(b^2 - a^2)^{1/2}$$

with $\arccos z \in [0, \pi]$ *for* $z \in [-1, 1]$.

Proof Obviously, (i) and (ii) follow from Proposition 2.8 and the definition of the set S. Let us prove (iii): Defining $r_0 = r_0(a, b) := \sup\{r > 0 : (ar, br) \in S\}$ we find that for fixed values a, b satisfying (2.17) it must hold $r_0 = (1/b)v_0 = (1/a)u_0$ where $v_0 = -u_0/\cos \xi_0 = -\xi_0/\sin \xi_0$, $\xi_0 = \xi_0(u_0)$ is the solution of $\xi_0 = u_0 \tan \xi_0$ with $\xi_0 \in (0, \pi)$ for $u_0 \neq 0$ and u_0 and v_0 satisfy $u_0/v_0 = a/b$.

Therefore,

$$\begin{aligned} r_0 &= -(1/b)(\xi_0/\sin \xi_0) = -(1/b)(\arccos(-a/b)/\sin(\arccos(-a/b))) \\ &= -(1/b)(\arccos(-a/b)/(1 - a^2/b^2)^{1/2} \end{aligned}$$

For $a = 0$, i.e. for $u_0 = 0$, we get $r_0 = -\pi/2b$. ■

Now we shall formulate a proposition crucial for the unicity of stationary solutions.

PROPOSITION 2.10 *Let Z be an initial process and X the corresponding solution of* (2.1). *If a stationary solution of* (2.1*a*) *exists, then the distribution of* $(X(t - t_1), \ldots, X(t + t_n))$ *where $n \in N$, $t_1, \ldots, t_n$ are fixed with $0 \leq t_1 < t_2 < \cdots < t_n$, tends for $t \to \infty$ to a zero mean normal distribution with the covariance matrix (g_{ij}) defined by* (2.12).

Proof From (2.5) it follows with the notation (2.13) that (note that X is nonstationary in general now):

$$X(t) = X^0(t) + G(t)$$

with

$$G(t) := \int_0^t x_0(t - s)\, dW_s, \quad t \geq 0.$$

Because of the existence of a stationary solution we have $v_0 < 0$ and as in the proof of Proposition 2.9 we get $X^0(t) \to 0$ a.s.

To show that the finite dimensional distributions of G tend to zero mean normal distributions with covariance matrix (g_{ij}) calculate the characteristic function $E \exp(i \sum_{k=1}^n \lambda_k \cdot G(t + t_k))$. ■

PROPOSITION 2.11 *Assume a stationary solution $V = (V(t), t \geq -r)$ of* (2.1*a*) *exists. Then*

i) *V is the unique stationary solution of* (2.1*a*) *and it is a zero mean Gaussian process with the covariance function $K(\cdot)$ given by*

$$K(t) := \int_0^\infty x_0(s + t)x_0(s)\, ds, \quad t \geq 0, \tag{2.19}$$

$$K(t) := K(-t), \quad t < 0.$$

ii) *V has a spectral density f related to K by*

$$K(t) = \int_R \exp(itu) f(u)\, du, \quad t \in R,$$

and given by

$$\begin{aligned} f(u) &= (1/2\pi)|h(iu)|^{-2} \\ &= (1/2\pi)((u + b \sin ur)^2 + (a + b \cos ur)^2)^{-1}, \quad u \in R. \end{aligned} \tag{2.20}$$

iii) *A version of V is given by $U = (U(t), t \geq -r)$ with*

$$U(t) := \int_{-\infty}^{t} x_0(t - s)\, dW(s), \quad t \geq -r.$$

Proof (i) follows at once from Proposition 2.10, and (iii) was shown in the proof of Proposition 2.8. Let us consider (ii).

Integrating (2.2a) we get for x_0

$$x_0(t) = 1 + a \int_0^t x_0(s)\, ds + b \int_0^t x_0(s - r)\, ds, \quad t \geq 0,$$

and thus

$$|x_0(t)| \leq 1 + (|a| + |b|) \int_0^t |x_0(s)|\, ds, \quad t \geq 0.$$

Applying a Gronwall-type lemma we obtain

$$|x_0(t)| \leq K \cdot \exp(c \cdot t), \quad t \geq 0,$$

with $K := 1 + |b|r$ and $c := |a| + |b|$.

Thus the Laplace-transform $L[x_0]$ of x_0 exists at least for Re $\lambda > c$. From (2.2) it easily follows

$$L[x_0](\lambda) = (\lambda - a - b \exp(-\lambda r))^{-1} = 1/h(\lambda) \quad \text{Re } \lambda > c. \tag{2.21}$$

Using (2.21) we get for the fundamental solution x_0:

$$x_0(t) = (1/2\pi) \int_R (\exp(itu)/h(iu))\, du, \quad t \geq 0. \tag{2.22}$$

Thus, $x_0(\cdot)$ and $x_0(\cdot + t)$ are the inverse Fourier transforms of $(2\pi)^{-1/2} h^{-1}(iu)$ and $(2\pi)^{-1/2} h^{-1}(-iu) \exp(-iut)$, $u \in R$, respectively. Applying the Parseval-equation to (2.19) we obtain (2.20) and therefore (ii). ∎

2.3 *The covariance function of the stationary solution*

Assume (2.1a) has a stationary solution, i.e. $(ar, br) \in S$, and let $K(\cdot)$ be its covariance function given by (2.19). We shall show that $K(\cdot)$ satisfies the difference-differential-equation (2.2a) and calculate it explicitly on $[-r, r]$. Then, using the step-method, K can be calculated principally on the whole real axis.

We shall treat $K(\cdot)$ on R_+ only. All formulas obtained below can be extended to $(-\infty, 0]$ by the property $K(-t) = K(t)$, $t \geq 0$.

LEMMA 2.12 *The covariance function $K(\cdot)$ of the stationary solution of (2.1a) has the following properties*:

i) *$K(\cdot)$ is continuously differentiable on $[0, \infty)$, where the derivative at zero is understood to be the right-hand side one.*

ii) *It holds*

$$\dot{K}(t) = aK(t) + bK(t-r), \quad t \geq 0. \tag{2.23}$$

iii) *We have*

$$2aK(0) + 2bK(r) = -1 \tag{2.24}$$

and

$$\dot{K}(0+) = -1/2. \tag{2.25}$$

iv) *K is twice continuously differentiable on $[0, r]$ and it holds*

$$\ddot{K}(t) = (a^2 - b^2)K(t), \quad t \in [0, r], \tag{2.26}$$

where $\ddot{K}(t)$ is defined at $t = 0$ and $t = r$ to be the right or left hand side derivative of $\dot{K}$, respectively.

Proof Using (2.11), the inequalities (2.8) and $v_0 < 0$ it follows from Lebesgue's dominating convergence principle that K is differentiable on R_+ with

$$\dot{K}(t) = \int_0^\infty \dot{x}_0(s+t)x_0(s)\, ds, \quad t \geq 0.$$

The continuity of $\dot{K}$ on $[0, \infty)$ follows from (2.8) similarly. Thus (i) is proved.

Now use that x_0 solves (2.2a), $x_0(u) = 0$ ($u \in [-r, 0)$) and $K(-u) = K(u)$ to get (2.23). Furthermore observe

$$\begin{aligned} bK(r) &= \int_0^\infty bx_0(s+r)x_0(s)\, ds = \int_0^\infty bx_0(s)x_0(s-r)\, ds \\ &= \int_0^\infty x_0(s)\dot{x}_0(s)\, ds - a\int_0^\infty x_0^2(s)\, ds \\ &= \lim_{s\to\infty} x_0^2(s)/2 - x_0^2(0)/2 - aK(0) = -1/2 - aK(0). \end{aligned}$$

Thus (2.24) holds. Inserting it in (2.23) and using the continuity of K and its symmetry we get (2.25).

From (2.23) it follows for $t \in [0, r]$

$$\dot{K}(t) = aK(t) + bK(r - t).$$

The right-hand side of this equation is continuous differentiable on $[0, r]$, where the derivatives at the boundaries $t = 0$ and $t = r$ are understood one-sided. Use (2.23) to obtain (2.26). ■

Now we are ready to calculate K explicity on $[0, r]$, Remark that $(ar, br) \in S$ by assumption.

PROPOSITION 2.13 *For $t \in [0, r]$ we have*

$$K(t) = \begin{cases} K(0) \cdot \cosh(lt) - (2l)^{-1} \sinh(lt), & |b| < -a \\ K(0) - t/2, & b = a \\ K(0) \cos(lt) - (2l)^{-1} \sin(lt), & b < -|a| \end{cases} \tag{2.27}$$

where $l := |a^2 - b^2|^{1/2}$ and

$$K(0) = \begin{cases} (b \sinh(lr) - l)/(2d(a + b\cosh(lr))), & |b| < -a \\ (br - 1)/4b, & b = a \\ (b \sin(lr) - l)/(2d(a + b\cos(lr))), & b < -|a|. \end{cases} \tag{2.28}$$

Proof Solve (2.26) with the conditions (2.24)–(2.25). ■

PROPOSITION 2.14 *If $(ar, br) \in S$ and $br \geq -\exp(ar - 1)$, then $K(\cdot)$ is strictly positive with*

$$\lim_{t \to \infty} t^{-1} \ln K(t) = v_0(a, b, r). \tag{2.29}$$

If $(ar, br) \in S$ and $br < -\exp(ar - 1)$, then $K(\cdot)$ oscillates around zero with

$$\overline{\lim_{t \to \infty}}\, t^{-1} \ln|K(t)| = v_0(a, b, r). \tag{2.30}$$

(Oscillation around zero means, that for every $t_0 > 0$ one can find $t_1, t_2 > t_0$ such that $K(t_1) < 0$ and $K(t_2) > 0$.)

Proof Because of (2.23), Proposition 3.2 and its proof we have that K is strictly positive if and only if x_0 is strictly positive. Now apply Proposition 3.2 again. ■

2.4 *The Stationary Solution For (ar, br) Near ∂S*

If $(ar, br) \in S$ then the stationary solution of (2.1a) has the variance $K(0)$, which depends on (a, b, r). It is easy to see that $K(0)$ tends to infinity if (ar, br) tends to a boundary point of S. Nevertheless, in some sense the normalized process $X(\cdot)[K(0)]^{-1/2}$ converges to a limit process. This is shown in the present chapter. Let $a, b, a^*, b^* \in R$, $r, r^* > 0$ such that $P := (ar, br) \in S$ and $P^* := (a^*r^*, b^*r^*) \in \partial S$, (the latter relation means $b^* = -a^*$ or $b^*r^* = v_1(a^*r^*)$). Denote by V the stationary solution of (2.1a) for the given coefficients (a, b, r), f its spectral density, $K(\cdot)$ its covariance function.

We want to study the behaviour of the stationary solution V if (a, b, r) tends to (a^*, b^*, r^*). Note that $(a^*r^*, b^*r^*) \notin S$, such that there does not correspond a stationary solution V^* to (a^*, b^*, r^*). Assume (a, b, r) tends to (a^*, b^*, r^*). Then

$$1/(2\pi f(u)) = ((u + b \sin ur)^2 + (a + b \cos ur)^2)$$

tends to

$$1/(2\pi f^*(u)) := ((u + b^* \sin ur^*)^2 + (a^* + b^* \cos ur^*)^2)$$

uniformly on all compact sets.

Note that $1/(2\pi f(u)) > 0$, $u \in R$, and that $1/(2\pi f^*(u))$ has zeros at $\pm u^*$ with

$$u^* := \begin{cases} 0 & \text{if } \ b^* = -a^* \\ \xi_0(a^*r^*)/r^* & \text{if } \ b^*r^* = v_1(a^*r^*) \end{cases},$$

where $\xi_0(\cdot)$ was defined in Chapter 2.2 above.

Now the following proposition is obvious:

PROPOSITION 2.15 *If (a, b, r) tends to (a^*, b^*, r^*) with $(ar, br) \in S$ and $(a^*r^*, b^*r^*) \in \partial S$ then $f(\cdot)$ tends to $f^*(\cdot)$ uniformly on compact subsets of $R\backslash\{u^*, -u^*\}$.*

COROLLARY 2.16 *It holds* $\lim_{(a,b,r)\to(a^*,b^*,r^*)} K(0) = \infty$.

Proof The assertion follows from $K(0) = \int_{-\infty}^{\infty} f(u)\,du$, $\int_{-\infty}^{\infty} f^*(u)\,du = \infty$, Proposition 2.15 and Fatou's lemma. □

Now introduce the stationary process Y by

$$Y(t) := (K(0))^{-1/2} X(t), \quad t \geq -r.$$

Then we have

$$dY(t) = [(aY(t) + bY(t - r))]\,dt + (K(0))^{-1/2}\,dW(t), \quad t \geq 0.$$

Its covariance function K_Y is given by

$$K_Y(t) = K(t)/K(0), \quad t \in R.$$

The Proposition 2.15, its Corollary 2.16, the Proposition 2.13 and formula (2.25) now imply immediately

PROPOSITION 2.17 *If (a, b, r) with $(ar, br) \in S$ tends to (a^*, b^*, r^*) with $(a^*r^*, b^*r^*) \in \partial S$ then for all $t \in R$*

$$K_Y(t) \to \begin{cases} \cosh u^*t \equiv 1 & \text{if} \quad b^* = -a^*, \\ \cos u^*t & \text{if} \quad b^*r^* = v_1(a^*r^*). \end{cases}$$

Remark 2.18 If $u^* \neq 0$, then there exists a zero mean (non Gaussian) weak stationary process Y^* having $\cos(u^*t)$ as its covariance function and satisfying the difference-differential equation (2.2a):

$$Y^*(t) := 2^{1/2} \cos(u^*t + \Phi) \quad t \in R,$$

where Φ is a uniform distributed on $[0, 2\pi]$ random variable.

Proof Because of

$$b^*r^* = -\xi(a^*r^*)/\sin \xi(a^*r^*) = -a^*r^*/\cos \xi(a^*r^*) \quad \text{and} \quad \xi = u^*r^*$$

we have

$$\begin{aligned} 2^{-1/2}r^*Y^*(t) &= -r^*u^* \sin(u^*r + \Phi) \\ &= -\xi(a^*r^*) \sin(u^*t + \Phi) \\ &= a^*r^* \cos(u^*t + \Phi) - a^*r^* \cos(u^*t + \Phi) + b^*r^* \sin \xi \sin(u^*t + \Phi) \\ &= a^*r^* \cos(u^*t + \Phi) + b^*r^* \cos(u^*(t - r^*) + \Phi), \quad t \geq 0. \end{aligned}$$

The proof that Y^* is a zero mean process with covariance function $\cos(u^*t)$ is an easy calculation. ■

3 APPENDIX

We shall summarize some more or less known facts on the deterministic equation (2.2), the fundamental solution x_0 and the function v_0. In particular, combining with Proposition 2.14 one gets some more information on the behaviour of the covariance function $K(\cdot)$.

3.1 Asymptotic Behaviour of the Solutions of (2.2)

Let us start with another representation of solutions of (2.2).

LEMMA 3.1 (Myškis [15], p. 101) *Every solution of* (2.2) *with initial function $g \in \mathscr{L}_1(-r, 0)$ has the following representation*

$$x_g(t) = \sum_{\operatorname{Re} \lambda_k \geq \gamma} \exp(\lambda_k t)(C_{0,k} + \cdots + C_{m_k - 1,k} t^{m_k - 1}) + o(\exp(\gamma t)) \tag{3.1}$$

where γ is an arbitrary real number, m_k denotes the multiplicity of the characteristic root λ_k of (2.2), $k \in N$.

This lemma will be used to prove the following

PROPOSITION 3.2 *For all $a, b \in R$, $r > 0$ it holds:*
If

$$br \geq -\exp(ar - 1) \tag{3.2}$$

then the fundamental solution x_0 of (2.2) *is strictly positive on* $[0, \infty)$, *and it holds*

$$\lim_{t \to \infty} t^{-1} \ln x_0(t) = v_0(a, b, r). \tag{3.3}$$

If

$$br < -\exp(ar - 1) \tag{3.4}$$

then x_0 oscillates around zero, i.e. for every $t_0 > 0$ one can find $t_1, t_2 > t_0$ such that $x_0(t_1) < 0$ and $x_0(t_2) > 0$, and it holds

$$\overline{\lim_{t \to \infty}} t^{-1} \ln|x_0(t)| = v_0(a, b, r). \tag{3.5}$$

Proof At first let us show that (3.5) holds.
Let us consider two difference-differential equations

$$\dot{x}(t) = ax(t) + bx(t - r), \quad t \geq 0, \tag{3.6}$$

$$\dot{x}(t) = \hat{a}x(t) + \hat{b}x(t - r), \quad t \geq 0, \tag{3.7}$$

where $\hat{a} := a - u$ and $\hat{b} := b \exp(-ur)$ with any real number u. Then, we can find the characteristic functions h and $\hat{h}$ for (3.6) and (3.7), respectively. It is easy to see that

$$h(\lambda) = \hat{h}(\lambda - u), \quad \lambda \in K, \tag{3.8}$$

$$x_0(t) = \exp(ut)\hat{x}_0(t), \quad t \geq 0. \tag{3.9}$$

where x_0 and $\hat{x}_0$ are the fundamental solutions corresponding to (3.6) and (3.7), respectively.

From (3.8) we get:

If λ_0 is a root of h with Re $\lambda_0 = z$, then $\lambda_0 - u$ is a root of $\hat{h}$ with $\text{Re}(\lambda_0 - u) = z - u$. Now we choose $u = v_0(a, b, r)$. Then,

$$\hat{v}_0 := \max\{\text{Re } \lambda : \hat{h}(\lambda) = 0\} = 0$$

and by using Lemma 3.1 we get

$$\hat{x}_0(t) = \sum_{\mathrm{Re}\,\lambda_k = 0} \exp(\lambda_k t)(C_{0,k} + \cdots + C_{m_k - 1,k} t^{m_k - 1}) + o(1).$$

Now we use (3.9) with $u = v_0$ and obtain for $t > 0$ with $x_0(t) \neq 0$

$$\begin{aligned} t^{-1} \ln|x_0(t)| &= t^{-1} \ln \exp(v_0 t)|\hat{x}_0(t)| \\ &= v_0 + t^{-1} \ln|\hat{x}_0(t)|. \end{aligned} \tag{3.10}$$

Therefore, $\overline{\lim}_{t\to\infty} t^{-1} \ln|x_0(t)| = v_0(a, b, r)$.

Thus we have (3.5).

If additionally x_0 is strictly positive then $\hat{x}_0$ is (up to an $o(1)$) a strictly positive polynomial, and thus (3.3) follows from (3.10). Consequently, it remains to show that (3.2) implies the strict positiveness of $x_0(\cdot)$ and that (3.4) implies the oscillation of $x_0(\cdot)$ around zero.

But this follows from the following lemma.

LEMMA 3.3 *Consider the equation*

$$\dot{x}(t) = qx(t-1), \quad t \geq 0. \tag{3.11}$$

Then it holds:

i) *The fundamental solution of* (3.11) *is positive if and only if* $q \geq -1/e$.

ii) *The fundamental solution of* (3.11) *oscillates around zero if and only if* $q < -1/e$.

Indeed, if x_0 is the fundamental solution of (2.2) then

$$\bar{x}(t) := \exp(-art)x_0(rt), \quad t \geq 0$$

is a solution of

$$\dot{\bar{x}}(t) = q\bar{x}(t-1), \quad t \geq 0$$

with $q := br\exp(-ar)$.

Lemma 3.3 is a very special case of a general property of certain functional-differential equation proved in Morgenthal [14]. For this particular case we shall give the following simple proof[1] for the sake of completeness (see also Ladas [11]).

Let $q \geq 0$. Then by the representation (2.3) we see that $x_0(t) > 0$ for all $t \geq 0$. Now let $q < 0$. Then we write instead of (3.11)

$$\dot{x}(t) = -px(t-1), \quad p > 0 \tag{3.12}$$

[1] Communicated to us by K. Morgenthal.

and prove

a) There exist positive solutions of (3.12) if and only if $p \leq 1/e$.

b) There exist positive solutions of (3.12) if and only if the fundamental solution of (3.12) is positive.

From (a) then it follows that all solutions of (3.12) oscillates around zero if and only if $p > 1/e$ and the lemma is proved.

Proof of (a) Assume, there exists a positive solution g of (3.12). Then $g(t) > 0$ for all $t \geq 0$ and $g'(t) < 0$ for $t \geq 1$. Therefore, there exists $\lim_{t \to \infty} g(t-1)/g(t) =: \alpha$ and it is easy to see that $1 \leq \alpha < \infty$. Let $\varepsilon > 0$. Then there is a real number Q such that

$$\alpha - \varepsilon \leq g(t-1)/g(t) \quad \text{and} \quad Q \leq t < \infty.$$

With (3.12) we get that

$$g'(t)/g(t) = -p(g(t-1)/g(t)) - p(\alpha - \varepsilon), \quad t \geq Q.$$

Integrating this inequality over the interval $[t-1, t]$ we find

$$\ln g(t) - g(t-1) \leq -p(\alpha - \varepsilon)$$

and therefore

$$\ln g(t-1)/g(t) \geq p(\alpha - \varepsilon), \quad t \geq Q.$$

Taking $\varepsilon \downarrow 0$ and $t \to \infty$ we get

$$\ln \alpha/\alpha \geq p.$$

Because of $\ln \alpha/\alpha \leq 1/e$, $\alpha \geq 1$, we have $p \leq 1/e$.

Assume now that $p \leq 1/e$. If there is a solution $x(t) = \exp(-\lambda t)$ with $\lambda \in R$ for the equation (3.12) then we have proved (a). Proceeding from $x(t) := \exp(-\lambda t)$ we find that there must be a real solution of $\lambda = p \exp \lambda$. And this is the case when $p \leq 1/e$.

Proof of (b) To prove this we use the following

LEMMA 3.4 *Let f and g be solutions of* (3.12) *with*

$$f(t) \leq g(t), \qquad -1 \leq t \leq 0,$$
$$f(0) = g(0),$$
$$g(t) > 0, \quad -1 \leq t < \infty.$$

Then, $f(t) \geq g(t)$ for $t \geq 0$.

The proof of this lemma is left to the reader (see also Kozakiewicz [10]).

Now, we can show (b): Assume, there exists a positive solution of (3.12). Then we can take it in that way that it is positive for all $t \geq -1$ (this can be done because the equation (3.12) is autonomous). Then, with Lemma 3.4 we find $x_0(t) > 0$ for all $t \geq 0$.

3.2 *The Function* $v_0(a, b, r)$

We have seen in Proposition 2.14 above that the number $v_0 = v_0(a, b, r)$ is connected with the asymptotic behaviour of the covariance function $K(\cdot)$ (more generally, with the solutions of (2.2a)). Thus it is of some interest to study its behaviour in more detail. We shall show, that the function v_0 is smooth outside the surface $F := \{(a, b, r) | br = -\exp(ar - 1)\}$ and that it has a certain singularity on this surface.

LEMMA 3.5 *Assume* $c \in R$ *and define* $\hat{h}(\cdot)$ *and* $\hat{v}_0(c)$ *by*

$$\hat{h}(z) := z - c \exp(-z), \quad z \in C,$$

$$\hat{v}_0(c) := \max\{\operatorname{Re} z : \hat{h}(z) = 0\}.$$

Then $\hat{v}_0(c) < \infty$, $c \in R$, *and we have*

$$v_0(a, b, r) = 1/r[\hat{v}_0(br \cdot \exp(-ar)) + ar]$$

Proof We have $h(\lambda) = 0$ if and only if $\hat{h}(z) = 0$ where $z = (\lambda - a)r$ and $c = br \exp(-ar)$.

Consequently, we have to study the function $\hat{v}_0$ of one real variable c only. This has been done by several authors, see e.g. Wright [20].

We shall present here without proof some properties of $\hat{v}_0$, partially known from Wright [20]. For details the reader is referred to Mensch [13]. Note that if a and r are fixed, then v_0 and $\hat{v}_0$ have very similar graphs as functions of br.

PROPOSITION 3.6 *The function*

$$\hat{v}_0(c) := \max\{\operatorname{Re} z | z = c \exp(-z), z \in C\}, \quad c \in R,$$

has the following properties:

i) $\hat{v}_0$ *is continuous on* R,

ii) *differentiable on* $R \backslash \{-\exp(-1)\}$,

iii) *strictly decreasing on* $(-\infty, -\exp(-1))$ *and strictly increasing on* $(-\exp(-1), \infty)$,

iv) $\hat{v}_0(-\pi/2) = \hat{v}_0(0) = 0$,

$$\hat{v}_0(-\exp(-1)) = -1, \ \hat{v}_0(x \exp x) = x, \ x \geq -1,$$

v) $\lim_{c \downarrow -\exp(-1)} \hat{v}_0'(c) = \infty$, $\lim_{c \uparrow -\exp(-1)} \hat{v}_0'(c) = -e$.

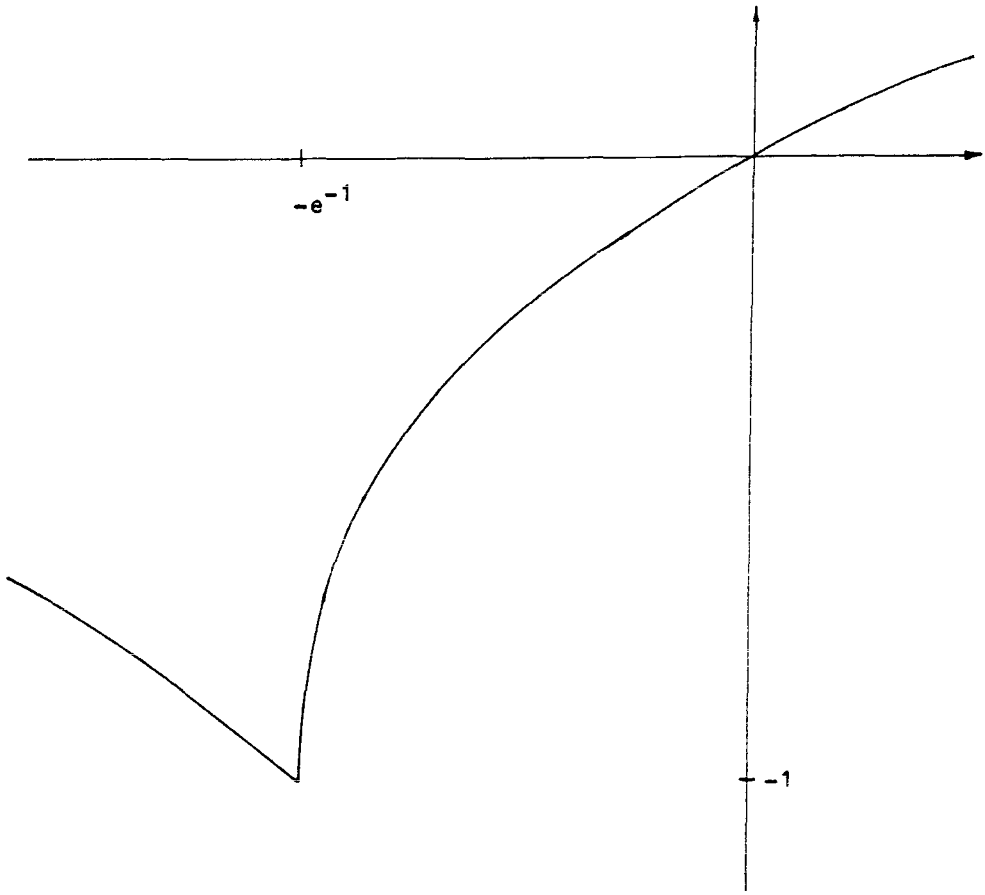

Figure 1 The function $\hat{v}_0(c)$

vi) *If* $|c| < \exp(-1)$ *then it holds*

$$\hat{v}_0(c) = \sum_{n=1}^{\infty} ((-1)^{n-1} n^{n-1}/n!)c^n. \tag{3.13}$$

The Figure 1 gives an imagination of the function $\hat{v}_0(c)$.

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