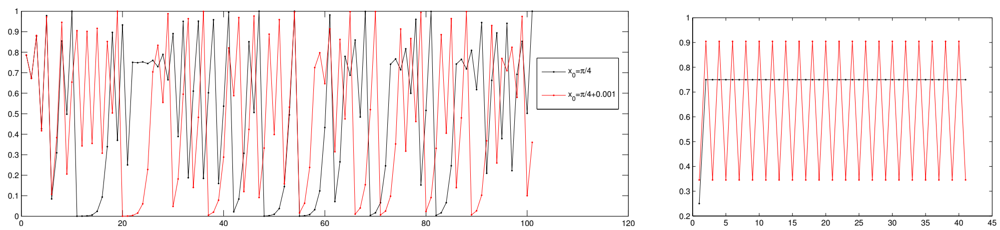
1 Trajectory v.s Density

Let consider a quadratic (or logistic) map

$$f(x) = \alpha x(x-1) \qquad x \in [0,1]. \tag{1}$$

For $\alpha = 4$, it is known to be chaotic. In fact, this map is homeomorphic to the tent map, of which periodic orbits are dense in $[0,1]$, through the change of variable $x = 0.5(1-\cos(\pi x))$. Figure 1 shows orbits of two different "typical" initial conditions that are slightly different. These typical trajectories look very irregular. In contrast, there are also "exceptional" initial conditions that do not exhibit irregular behaviors, e.g., the (stable) fixed point and period two points in Figure 1.



(a) (b)

Figure 1: (a) An illustration of the sensitive dependence of orbits of initial conditions using $x_0 = \pi/4$ and $x_0 = \pi/4 + 0.001$ (b) Exceptional initial conditions fails to exhibit the chaotic behavior

The study of individual orbits of a simple-looking map such as the quadratic map may be confounding. Alternative approach is to study the evolution of densities.

2 Frobenius-Perron Operator: Intuition

For simplicity, consider a map $F : [0,1] \circlearrowleft$ and choose a large number of initial states $\{x_0^1, \ldots, x_0^N\}$. To each of these points, we apply the map F to obtain $x_1^1 = F(x_0^1), \ldots, x_1^N = F(x_0^N)$. It is always true that for any subinterval $\Delta \subset [0,1]$,

$$x_1^j \in \Delta \qquad \text{if and only if } x_0^j \in F^{-1}(\Delta),$$

where $F^{-1}(\Delta) := \{x : F(x) \in \Delta\}$. It then follows that

$$1_\Delta(F(x)) = 1_{F^{-1}(\Delta)}(x).$$

Now, supposed that we have a density function $f_1(x)$ for the states $x_1^1, \dots, x_1^N$. Then we can show that

$$
\begin{aligned}
\int_\Delta f_1(u)du &\approx \frac{1}{N}\sum_{i=1}^{N} 1_\Delta(F(x_1^i)) \\
&= \sum_{j=1}^{N} 1_{F^{-1}(\Delta)}(x_0^j) \approx \int_{F^{-1}(\Delta)} f_0(u)du,
\end{aligned}
$$

where $f_0(x)$ is the density function of the initial states. This gives the relationship between f_0 and f_1 that is of our interest. In view of the operator theory, we may define $f_1 := Pf_0$ for some operator P defined on some appropriate function space, i.e.,

$$\int_\Delta Pf(u)du = \int_{F^{-1}(\Delta)} f(u)du. \tag{2}$$

If Δ is an interval, say $\Delta = [a, x]$, then $Pf(x)$ can be written explicitly by

$$Pf(x) = \frac{d}{dx}\int_{F^{-1}([a,x])} f(u)du. \tag{3}$$

The expressions (2) and (3) provide two equivalent definitions of the "Frobenius-Perron operator" (or "Transfer operator") P corresponding to the mapping F. There are still two more forms of the Frobenius-Perrorn operator,

$$Pf(x) = \sum_{y \in F^{-1}(x)} \frac{f(y)}{|F'(y)|}, \tag{4}$$

and

$$Pf(x) = \int f(y)\delta[x - F(y)]dy, \tag{5}$$

where $|F'(y)|$ is the Jacobian determinant. To see how we can arrive at the expression (4), we suppose that F is monotone, which implies that $F^{-1}([a, x]) = [F^{-1}(a), F^{-1}(x)]$. Then, we can rewrite (3) as

$$Pf(x) = \frac{d}{dx}\int_{F^{-1}(a)}^{F^{-1}(x)} f(u)du = f(F^{-1}(x))|\frac{d}{dx}[F^{-1}(x)]|. \tag{6}$$

If F is non-monotone, we can take the sum over all of the preimages $F^{-1}(x)$ to obtain the desired result. Finally, Eq. (6) comes about by using the delta function identity

$$\delta(g(x)) = \sum_{y^i : g(y^i)=0} \frac{\delta(y - y^i)}{|g'(y^i)|}. \tag{7}$$

If $P^n f \to f^*$ as $n \to \infty$, then we should find that $Pf^* = f^*$, i.e., f^* is a fixed point of f. This density f^* is called an "invariant density", which is very useful to study Ergodic theory and Dynamical Systems. Generally, however, the existence and uniqueness of the invariant density can be difficult to prove. In fact, even in one-dimensional chaotic maps, the invariant density may exhibit singularities at the critical points, which may be dense in the attracting interval (try to plot the histogram of $P^n f$ for the logistic map with $\alpha = 3.8$)!

When it is impossible to sensibly define the invariant density, we can take alternative approach by dealing with the invariant measure instead. Typically, we will deal with probability measures. We say that a probability measure μ is invariant if and only if

$$\mu(\Delta) = \mu(F^{-1}(\Delta)).$$

Note that we use the above definition in favor of $\mu(\Delta) = \mu(F(\Delta))$ since $F^{-1}(\Delta)$ is always measurable for a measurable set Δ. Furthermore, if $\mu(\Delta)$ is equivalent to the fraction of time that an orbit originating from almost every initial condition x_0 in a basin of attraction of F spends in Δ in the limit (the orbit length goes to infinity), then μ is called the "natural measure" of F. Therefore, if we know a natural measure μ on an attractor, we may say that the time-average of $f(x)$ over an orbit originating from a "typical" initial condition (i.e. all except a set of Lebesgue measure zero) in the basin of attractor equals to the spatial average based on μ,

$$\lim_{T\to\infty} \frac{1}{T} \sum_{n=0}^{T} f(F^n(x_0)) = \int f(x)\mu(dx). \tag{8}$$

More formally, the so-called "Birkhoff ergodic theorem" states that, for a finite measure space (X, Σ, μ), Eq. 8 holds for any integrable function f if μ is an ergodic invariant measure associated to F. Here, the mu is called "ergodic" w.r.t. F if every invariant set $A \in \Sigma$ is such that either $\mu(A) = 0$ or $mu(X) = 0$.

There are many notions in ergodic theory (e.g. ergodicity, mixing, and exactness) that can be used to characterize the degrees of irregular behaviors. The Frobenius-Perron operator and its adjoint (Koopman operator) are useful tools for these studies, which are beyond our interest here.

3 Approximation of Frobenius-Perron Operator: Ulam's method

We use what may be called the **Ulam-Galerkin method** - a specialized case of the Galerkin's method - to approximate the Frobenius-Perron operator. The approximation by **Galerkin's method** is based on the projection of the infinite dimensional linear space $L^1(M)$ with basis functions,

$$\{\phi_i(x)_{i=1}^{\infty}\} \subset L^1(M), \tag{9}$$

onto a finite-dimensional linear subspace with a subset of the basis functions,

$$\triangle_N = \text{span}\{\phi_i(x)\}_{i=1}^{N}. \tag{10}$$

For the Galerkin method, the projection

$$\Pi_N : L^1(M) \to \triangle_N, \tag{11}$$

maps an operator from the infinite-dimensional space to an operator of finite rank, an $N \times N$ matrix, by using the inner product,[1]

$$P_{i,j} = \langle P(\phi_i), \phi_j \rangle = \int_M P(\phi_i(x))\phi_j(x)dx. \tag{12}$$

The advantage of such a projection is that the action of the Markov operator which is initially a transfer operator in infinite dimensions reduces approximately to a Markov matrix on a finite dimensional vector space. Such is the usual goal of a Galerkin's method in PDE's, and similarly it is used here in the setting of transfer operators. Historically, Ulam's conjecture was proposed by S. Ulam in a broad collection of interesting open problems from applied mathematics including the problem of approximating Frobenius-Perron operators. His conjecture referred to both,

Ulam's Conjecture

1. A finite rank approximation of the Frobenius-Perron operator by Eq. (15) and,
2. The conjecture that the dominant eigenvector (corresponding to eigenvalue equal to 1 as is necessary for stochastic matrices) weakly approximates[2] the invariant distribution of the Frobenius-Perron operator.

It is straightforward to cast the Ulam's method as a Galerkin's method with a special choice of basis functions as follows. For the Ulam's method, the basis functions are chosen to be a family of characteristic functions,

$$\phi_i(x) = \chi_{B_i}(x) = 1 \text{ for } x \in B_i \text{ and zero otherwise.} \tag{13}$$

[1] Our use of the inner product structure requires the further assumption that the density functions are in the Hilbert space $L^2(M)$, rather than just the Banach space $L^1(M)$, which using the embedding $L^2(M) \hookrightarrow L^1(M)$, provided M is of finite measure.

[2] Weak approximation by functions may be defined as convergence of the functions under the integral relative to test functions. That is, if $\{f_n\}_{n=1}^{\infty} \in L^1(M)$, it is defined that $f_n \to^w f^*$ if $lim_{n\to\infty} \int_M |f^*(x) - f_n(x)|h(x)dx = 0$ for all $h \in L^\infty(M)$ which is referred to as the test function space.

We can then show that

$$
\begin{aligned}
P_{i,j} &= \int_M P(\chi_{B_i}(x))\chi_{B_j}(x)dx \\
&= \int_{B_j} P(\chi_{B_i}(x))dx \\
&= \int_{F^{-1}(B_j)} \chi_{B_i}(x)dx \qquad\qquad \texttt{using Eq. } (2) \\
&= m(B_i \cap F^{-1}(B_j)),
\end{aligned}
\tag{14}
$$

where m denotes the normalized Lebesgue measure on M and $\{B_i\}_{i=1}^N$ is a finite family of connected sets with nonempty and disjoint interiors that covers M. That is $M = \cup_{i=1}^N B_i$, and indexed in terms of nested refinements. After normalization so that $\sum_j P_{i,j} = 1$, the matrix approximation of the Frobenius-Perron operator has the form of

$$
P_{i,j} = \frac{m(B_i \cap F^{-1}(B_j))}{m(B_i)}. \tag{15}
$$

These $P_{i,j}$ can be interpreted as the ratio of the fraction of the box B_i that will be mapped inside the box B_j after one iteration of a map to the measure of B_i. By the construction, the matrix $\mathbf{P}$ is row-stochastic. Thus, it has a left eigenvector with the eigenvalue one. Simply put, this eigenvector characterizes the invariant measure of the Frobenius-Perron operator. Denote the normalized dominant left eigenvector by p, ($\sum p_i = 1$). We can approximate the invariant measure μ_N for $\{B_i\}_{i=1}^N$ by $\mu_N(B_i) = p_i$ and for any measurable set B the invariant measure can be approximated by

$$
\mu_N(B) = \sum \frac{m(B_i \cap B)}{m(B_i)} \mu_N(B_i).
$$