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On Superconvergence Techniques

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Abstract. A brief survey with a bibliography of superconvergence phenomena in finding a numerical solution of differential and integral equations is presented. A particular emphasis is laid on superconvergent schemes for elliptic problems in the plane employing the finite element method.

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1. Introduction

The purpose of this paper is to give a survey of the existing literature on superconvergence techniques for differential and integral equations. Especially, we shall concentrate (Section 2) on efficient superconvergent schemes for a class of important problems, viz. second-order elliptic boundary-value problems solved by the finite element (FE) method.

During the development of this method it has been found (see e.g. [17, 49, 73, 76, 102]) that the rate of convergence of FE approximations at some exceptional points of a domain exceeds the possible global rate. This phenomenon has come to be known as 'superconvergence'. Such points of exceptional accuracy of the derivatives of FE approximations have been observed, e.g., in [19, 50, 103, 169, 181, 197].

A systematic study of superconvergence phenomena seems to have its beginning in the seventies. Great effort has been focused on the superconvergence at nodal points and also at the Gauss–Legendre, Jacobi, and Lobatto points etc., see [76, 116, 36, 17]. However, at the present time, the term superconvergence is used in a much broader sense than before. One can recover the Galerkin solution or its derivatives by means of various post-processing techniques and produce an acceleration of convergence. This is also called superconvergence by many authors if the post-processing is *easily computable*. After such a post-processing, one can often get an increase of accuracy not only at some isolated points, but also in a subdomain (local superconvergence) or even in the whole domain (global superconvergence).

In Section 2 we introduce several superconvergent FE schemes for plane elliptic problems which will exemplify what has been done in this field. Further, we only mention superconvergence phenomena for two-point boundary-value problems and other related problems.

Section 3 is intended to be a brief survey of the extensive literature on superconvergence for parabolic and hyperbolic problems, integral equations, and other problems for which the Galerkin, collocation or least squares method are employed.

In order to help the reader, the bibliography is equipped with the *Mathematical Reviews* reference numbers.

2. Superconvergence Schemes for Elliptic Problems

2.1. THE AIM

The main aim of this section is to present several superconvergent FE schemes for a 2nd order elliptic boundary-value problem in the plane. For the sake of clarity, we demonstrate these schemes in their *simplest setting* and only for the Poisson equation

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{2.1}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with a Lipschitz boundary $\partial\Omega$ and $f \in L^2(\Omega)$. Furthermore, we mention superconvergence results for two-point boundary-value problems, for higher-order elliptic problems in the plane, and for elliptic systems. We close this section with a remark on the computation of the boundary flux.

2.2. PRELIMINARIES

Let us introduce several notations and definitions. The Euclidean norm is denoted by $|\cdot|$. We write $\Omega_0 \subset\subset \Omega$ if Ω_0 is a domain such that $\bar{\Omega}_0 \subset \Omega$. By $P_k(\Omega)$ we mean the space of polynomials of degree at most k. The usual norm and seminorm in the Sobolev space

$$(W_p^k(\Omega))^d, \; k \in \{0, 1, \dots\}, \quad d \in \{1, 2\}, p \in [1, \infty]$$

are denoted by $\|\cdot\|_{k,p,\Omega}$ and $|\cdot|_{k,p,\Omega}$, respectively. In particular, we write $H^k(\Omega) = W_p^k(\Omega) = W_p^k(\Omega)$ and $\|\cdot\|_k = \|\cdot\|_{k,\Omega} = \|\cdot\|_{k,p,\Omega}$ for $p = 2$. The space $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the $\|\cdot\|_1$-norm.

We denote by $\mathcal{T}_h$ a partition (triangulation) of the domain Ω in the usual sense [56, 169]. Up to the end of Subsection 2.3.7, the families $\{\mathcal{T}_h\}$ of partitions of $\bar{\Omega}$

are supposed to be strongly regular, i.e., any $K \in \mathcal{T}_h$ contains a circle of diameter Ch and is contained in a circle of diameter h, $0 < C < 1$ independent of h and K.*

We denote by $V_h \subset H_0^1(\Omega)$ a finite element space corresponding to $\mathcal{T}_h$. Throughout the Subsections 2.3 and 2.7, the notation u is used for the weak solution of (2.1) and u is supposed to be sufficiently smooth. By $u_h \in V_h$ we mean the Galerkin approximation of u, i.e.,

$$(\nabla u_h, \nabla v_h)_0 = (f, v_h)_0 \quad \forall v_h \in V_h,$$

where $(\cdot, \cdot)_0$ is the inner product in $(L^2(\Omega))^d$, $d = 1, 2$.

2.3. SUPERCONVERGENCE RESULTS FOR THE PROBLEM (2.1)

Superconvergence is mostly accepted to be the exhibition of exceptional rates of convergence of the approximate solution at certain points in the discretization whose location is *a priori* known. As the mesh is refined, the error at these points converges towards zero at rates superior to the optimal global rates. We roughly outline what we shall mean by superconvergence in this section.

The optimal global rate for the Galerkin method is usually of the form

$$\|Au - Au_h\|_{k,p,\Omega} \leq C_{k,p,\Omega}(u)\eta(h), \quad \text{as } h \to 0,$$

where $k \in \{0, 1, \ldots\}$, $p \in [1, \infty]$, $A = I$ (the identity operator) or $A = \nabla$, $C_{k,p,\Omega}(u)$ depends on some norm of u, and $\eta(h)$ (which is independent of u) has mostly the form $\eta(h) = h^\alpha$, $\alpha > 0$. Consider a linear continuous post-processing operator

$$\sim: Au_h \mapsto \widetilde{Au_h}. \tag{2.2}$$

The post-processing (2.2) is said to cause the *superconvergence* if

$$\|Au - \widetilde{Au_h}\| = o(\eta(h)) \quad \text{as } h \to 0.$$

Here, the norm $\|\cdot\|$ has to be close to the $\|\cdot\|_{k,p,\Omega}$-norm in some sense. For instance, $\|\cdot\|$ may be a discrete analog of the $\|\cdot\|_{k,p,\Omega}$-norm, or $\|\cdot\| = \|\cdot\|_{k,p,\Omega_0}$ for $\Omega_0 \subset\subset \Omega$, or $\|\cdot\| = \|\cdot\|_{k,p,\Omega}$, etc.

Early papers established superconvergence phenomena mainly when (2.2) was a restriction operator. Recently, however, there has been a growing literature on superconvergence, where the post-processing (2.2) is an averaging, convolution, or smoothing operator. We will illustrate various types of (2.2) in the following examples.

2.3.1. In the first place we introduce the superconvergence phenomenon at *nodal points*, which was analysed by Douglas, Dupont and Wheeler [76]. Here $A = I$ and the post-processing (2.2) is a restriction.

* Here and in what follows the letter C stands for a generic positive constant which may vary with context.

For a partition $0 = n_0 < n_1 \ldots < n_m = 1$, let $I_i = [n_{i-1}, n_i]$. Fix an integer $k \geq 3$ and define

$$S_h = \{s \in H_0^1((0,1)) \mid s|_{I_i} \in P_k(I_i), \quad i = 1, \ldots, m\} \tag{2.3}$$

and the finite element space on $\Omega = (0,1) \times (0,1)$ via the tensor product

$$V_h = S_h \otimes S_h .$$

Then*

$$\max_{x \in N_h} |u(x) - \tilde{u}_h(x)| \leq Ch^{k+2} \|u\|_{k+3},$$

where $N_h = \{(n_i, n_j)\}$ is the set of nodal points and $\tilde{u}_h(x) = u_h(x)$ for $x \in N_h$. This is a superconvergence result in the sense that the rate of convergence at nodes is greater than globally possible, that is

$$\|u - u_h\|_{0,\infty,\Omega} \leq Ch^{k+1}(\|u\|_{k+2,2,\Omega} + \|u\|_{k+1,\infty,\Omega}).$$

2.3.2. Next, we present the result of Zlámal [197], where $\sim$ is again a restriction operator but $A = \nabla$.

Let the domain $\bar{\Omega}$ be a finite union of rectangles with sides parallel to the coordinate axes. Partitions $\mathcal{T}_h$ are formed by rectangles and $V_h \subset H_0^1(\Omega)$ consists of continuous functions which are incomplete polynomials of the third degree on every $K \in \mathcal{T}_h$ (the two terms x_1^3 and x_2^3 are missing in the cubic polynomials). These polynomials of the so-called Serendipity family are uniquely determined by the values at the corners and at the midpoints of the sides, and it is well-known that

$$\|\nabla u - \nabla u_h\|_0 \leq Ch^2 \|u\|_3$$

is the best possible rate. Let us denote by G_h the set of all maps of the four *Gaussian points* $(\pm\sqrt{3}/3, \pm\sqrt{3}/3)$ of the square $\hat{K} = [-1,1] \times [-1,1]$ through one-to-one linear continuous mappings $F_K : \hat{K} \to K$, $K \in \mathcal{T}_h$. Then the arithmetic mean μ of the values $|\nabla u(x) - \widetilde{\nabla u_h}(x)|$, $x \in G_h$, (where $\widetilde{\nabla u_h}(x) = \nabla u_h(x)$, $x \in G_h$) is bounded by

$$\mu \leq Ch^3(|u|_3 + |u|_4)$$

or, equivalently,

$$h^2 \sum_{x \in G_h} |\nabla u(x) - \widetilde{\nabla u_h}(x)| \leq Ch^3(|u|_3 + |u|_4). \tag{2.4}$$

This estimate is valid even for a more general second-order elliptic equation with variable coefficients and also for the homogeneous Newton boundary condition

* In what follows all the statements hold only for a sufficiently small discretization parameter h.

[197]. In fact, Zlámal [198] proved more than (2.4), namely that

$$h\Big(\sum_{x\in G_h} |\nabla u(x) - \widetilde{\nabla u_h}(x)|^2\Big)^{1/2} = \mathcal{O}(h^3). \tag{2.5}$$

This is the discrete L^2-norm estimate. The bound (2.4) can be generalized to the one- and three-dimensional cases and for other elements of the Serendipity family; e.g., for rectangular bilinear elements, the sampling at centroids leads to the $\mathcal{O}(h^2)$ superconvergence. The generalization for curved isoparametric elements (including the reduced integration) can be found in Zlámal [198] and Lesaint and Zlámal [116]. Related papers include Zlámal [199, 200]. Some extensions of (2.4) under a weaker condition on the ellipticity can be found in Leyk [119], see also Chen [51, 53].

2.3.3. We introduce a simple averaging post-processing for the gradient ($A = \nabla$) suggested e.g. by Chen [50] or Lin *et al.* [123].

Let $\mathcal{T}_h$ consist of triangles and let

$$V_h = \{v_h \in H_0^1(\Omega) \mid v_h|_K \in P_1(K)\ \forall K \in \mathcal{T}_h\}. \tag{2.6}$$

Moreover, we assume that each $\mathcal{T}_h$ is uniform, i.e., any two adjacent triangles of $\mathcal{T}_h$ form a parallelogram.

Denote by M_h the set of the *midpoints* of all sides of the triangulation $\mathcal{T}_h$. As the gradient of $u_h \in V_h$ is constant on every $K \in \mathcal{T}_h$, one may define

$$\widetilde{\nabla u_h}(x) = \tfrac{1}{2}(\nabla u_h|_K + \nabla u_h|_{K'}), \quad x \in M_h \cap \Omega, \tag{2.7}$$

where $K, K' \in \mathcal{T}_h$ are those adjacent triangles for which $x \in K \cap K'$. Now if $u \in C^3(\bar{\Omega}) \cap H_0^1(\Omega)$ then

$$\max_{x\in M_h\cap\Omega} |\nabla u(x) - \widetilde{\nabla u_h}(x)| = \mathcal{O}(h^2|\ln h|). \tag{2.8}$$

Note that

$$\|\nabla u - \nabla u_h\|_{0,\infty,\Omega} = \mathcal{O}(h|\ln h|),$$

or even $\mathcal{O}(h)$ for convex polygons [151].

The bound (2.8) was derived (see [124]) also for quasi-uniform triangulations in which any two adjacent triangles of $\mathcal{T}_h$ form only an approximate parallelogram, i.e., it holds that

$$|l(S) - l(S')| \leq Ch^2,$$

where $l(S)$ and $l(S')$ are the lengths of the opposite sides of the parallelogram. For another type of triangulations (piecewise uniform) we further refer to Lin and Lü [122]. The $\mathcal{O}(h^2)$ superconvergence in the discrete L^2-norm (even for variable coefficients) is given by Andreev [2] and Levine [117, 118].

The averaging technique (2.7) is based on the fact that only the tangential component of ∇u_h is a superconvergent approximation to the tangential component of ∇u at the midpoints of sides [2, 118]. Also sampling at the two Gaussian points of each side of the triangular quadratic elements leads to the superconvergence of the tangential component of the gradient, see Andreev [3].

2.3.4. A superconvergent recovery of the gradient at centroids is proposed by Levine [118].

The space V_h is as defined in (2.6) over a quasi-uniform triangulation. Denote by C_h the set of *centroids* of all $K \in \mathcal{T}_h$. Then for $x \in C_h \cap \Omega_0$ we define

$$\widetilde{\nabla u_h}(x) = \tfrac{1}{6}\Big(3\nabla u_h|_K + \sum_{i=1}^{3} \nabla u_h|_{K_i}\Big), \tag{2.9}$$

where K_i are triangles adjacent to K, $x \in K$, and $\Omega_0 \subset\subset \Omega$ is fixed. Now we have the estimate

$$h\Big(\sum_{x\in C_h\cap\Omega_0} |\nabla u(x) - \widetilde{\nabla u_h}(x)|^2\Big)^{1/2} \leqslant Ch^2\|u\|_3,$$

whereas

$$\|\nabla u - \nabla u_h\|_0 \leqslant Ch\|u\|_2. \tag{2.10}$$

The $\mathcal{O}(h^2)$ superconvergence is also proved when an appropriate numerical quadrature (e.g., the centroid rule) is used.

The relation (2.9) can be interpreted as follows. We first recover the gradient at the midpoint of each side of a triangle (cf. (2.7)) and then average these three gradients to obtain an approximation to the gradient at the centroid. The $\mathcal{O}(h^2)$ superconvergence of the gradient at centroids, when the linear elements are used for solving a degenerated elliptic problem, is proved in El Hatri [92]. Obviously, we may uniquely determine a piecewise linear discontinuous field $\widetilde{\nabla u_h}$ on $\Omega_0 \subset \subset \Omega$, which fulfils (2.7). Then we can easily derive from (2.8), see Neittaanmäki and Křížek [140], that $\widetilde{\nabla u_h}$ recovers the gradient at any point of Ω_0, i.e.,

$$\|\nabla u - \widetilde{\nabla u_h}\|_{0,\infty,\Omega_0} = \mathcal{O}(h^2|\ln h|). \tag{2.11}$$

Analogously, the $\mathcal{O}(h^2)$ superconvergence can be obtained for the $\|\cdot\|_{0,\Omega_0}$-norm. Some extensions to a global superconvergence are discussed in the paper of Lin and Xu [126].

2.3.5. Another simple averaging technique has been analysed by Křížek and Neittaanmäki [108].

Again we assume that V_h is of the form (2.6). Thus, when one needs an approximation of ∇u at some *nodal point* $x \in N_h$, then it is quite natural to calculate an average of all the constant vectors $\nabla u_h|_K$, where $K \in \mathcal{T}_h$ are incident with x. This technique is, in fact, often used in practice, see, e.g., [93, 141, 180, 196]. Putting

$$\widetilde{\nabla u_h}(x) = \frac{1}{6} \sum_{K \cap \{x\} \neq \emptyset} \nabla u_h|_K, \quad \forall x \in N_h \cap \Omega, \tag{2.12}$$

we can uniquely define the continuous piecewise linear field $\widetilde{\nabla u_h}$ over a fixed $\Omega_0 \subset\subset \Omega$. Suppose now that Ω is a parallelogram and that $\mathcal{T}_h$ are uniform. Then (see [108]) the following local superconvergence estimate holds (cf. (2.10))

$$\|\nabla u - \widetilde{\nabla u_h}\|_{0,\Omega_0} \leq Ch^2 \|u\|_{3,\Omega}.$$

When $\partial\Omega$ is curved, the local $\mathcal{O}(h^{3/2})$ superconvergence can be achieved also for the Newton boundary condition and smooth coefficients [108]. If Ω is a polygonal domain covered by uniform triangulations we can also get (see Křížek and Neittaanmäki [110]) the global superconvergence of $\widetilde{\nabla u_h} \in W_h \times W_h$:

$$\|\nabla u - \widetilde{\nabla u_h}\|_{0,\Omega} \leq Ch^2 \|u\|_{3,\Omega}, \tag{2.13}$$

where

$$W_h = \{w_h \in H^1(\Omega) \mid w_h|_K \in P_1(K),\ \forall K \in \mathcal{T}_h\}. \tag{2.14}$$

However, this requires to define the averaged gradient at boundary nodes, for instance as

$$\widetilde{\nabla u_h}(x) = 0, \quad \text{for all vertices } x \text{ of } \partial\Omega,$$

$$\widetilde{\nabla u_h}(x) = \frac{1}{2}\left(\sum_{i=1}^{3} \nabla u_h|_{K_i} - \nabla u_h|_{K_0}\right) \tag{2.15}$$

for the other boundary nodes $x \in N_h \cap \partial\Omega$. Here K_i and K_3 form a parallelogram for every $i = 0, 1, 2$, and $K_1 \cap K_2 \cap K_3 = \{x\}$.

2.3.6. Next, we introduce the *smoothing technique* suggested by Oganesijan and Ruhovec [147], pp. 94 and 189.

Let the boundary $\partial\Omega$ be from the class C^3 and let V_h be again as in (2.6). Suppose that triangulations consist of right-angled isosceles triangles in every $\Omega_0 \subset\subset \Omega$ for a sufficiently small h. For details about the triangulation near the boundary $\partial\Omega$, we refer to [146, 147]. We set, see [147] (or [145], p. 148)

$$\tilde{u}_h(x) = \frac{1}{4} h^{-2} \int_{D_h} u_h(x+y)\,\mathrm{d}y,$$

where $D_h = (-h, h) \times (-h, h)$. Then for $\Omega_0 \subset\subset \Omega$, we have

$$\|u - \tilde{u}_h\|_{1,\Omega_0} \leq Ch^{3/2} \|u\|_{3,\Omega}.$$

By comparison with (2.10), we find that the above error bound is a superconvergent estimate for the gradient. For the global $\mathcal{O}(h^{3/2})$ superconvergence, it is necessary to extend u_h outside of Ω in an appropriate way (see [147], p. 21). Note that the averaged gradients in (2.11) and (2.13) are not potential fields in the contrary to the case introduced here.

2.3.7. The Galerkin solution can also be post-processed by a *convolution* with the kernel proposed by Bramble and Schatz [27].

For simplicity, assume that $\bar{\Omega}$ can be decomposed into a finite number of identical squares and let the nodal points of rectangular partitions $\mathcal{T}_h$ be of the form (ih, jh), where i, j are integers. For a fixed $k \geq 2$, define the space of the two-dimensional B-splines

$$V_h = \Big\{ v_h \in H_0^1(\Omega) \mid v_h(x_1, x_2) = \sum_{(ih,jh)\in\Omega} a_{ij} g_k(x_1/h - i) g_k(x_2/h - j) \Big\},$$

where $a_{ij} \in \mathbf{R}^1$ and g_k is the one-dimensional B-spline (of order $r = k+1$) given recurrently by the convolution

$$g_p = g_0 * g_{p-1}, \quad p = 1, 2, \dots, k,$$

g_0 being the characteristic function of the interval $[-\frac{1}{2}, \frac{1}{2}]$. Let us define $\tilde{u}_h$ via the convolution

$$\tilde{u}_h(x) = h^{-2} \int_{\mathbf{R}^2} \Big(\prod_{i=1}^{2} \sum_{j=1-k}^{k-1} k'_j g_{k-1}((x_i - y_i)/h - j) \Big) u_h(y)\, \mathrm{d}y, \tag{2.16}$$

where $k'_{-j} = k'_j = \frac{1}{2} k_j$ for $j = 1, \dots, k-1$, $k'_0 = k_0$, and where k_j, $j = 0, 1, \dots, k-1$, are determined as the unique solution of the linear algebraic system

$$\sum_{j=0}^{k-1} k_j \int_{\mathbf{R}^1} g_{k-1}(y)(y+j)^{2m}\, \mathrm{d}y = \delta_{0m}, \quad m = 0, 1, \dots, k-1.$$

Hence, k_j depend only on the choice of k (a table of the constants k'_j for $2 \leq k \leq 5$ can be found in [27], p. 110). In (2.16), $\tilde{u}_h(x)$ can be calculated analytically at any point $x \in \Omega_0 \subset\subset \Omega$, especially at nodes it is simple.

If $k \geq 2$ and $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ then we have the superconvergence estimates

$$\|u - \tilde{u}_h\|_{0,\Omega_0} \leq Ch^{2k}(\|u\|_{2k,\Omega_1} + \|u\|_{k+1,\Omega}),$$

$$\|u - \tilde{u}_h\|_{0,\infty,\Omega_0} \leq Ch^{2k}(\|u\|_{2k+2,\Omega_1} + \|u\|_{k+1,\Omega}),$$

whereas

$$\|u - u_h\|_0 = \mathcal{O}(h^{k+1}), \qquad \|u - u_h\|_{0,\infty,\Omega} = \mathcal{O}(h^{k+1}).$$

The above technique of [27] is presented for more general classes of splines in $\mathbf{R}^d$ and also for negative norms. Moreover, the authors show how to obtain the superconvergence up to the boundary when Ω is the unit square.

The generalization of the above superconvergent interior approximations also to derivatives of u is studied by Thomée [173]. For the post-processing by the convolution with the Bramble–Schatz kernel, see also Douglas [69, 70].

2.3.8. We are mostly far from superconvergence on general meshes. This seems to be an open problem for many schemes. However, the next averaging (smoothing) technique proposed by Louis [131] is applicable even for *irregular partitions* of a convex polygon $\bar{\Omega} \subset R^2$. We only assume that the finite-dimensional spaces $V_h \subset H_0^1(\Omega)$ have the following standard approximation property:

For any $v \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$, $k \geqslant 0$, there exists a $v_h \in V_h$ with

$$\|v - v_h\|_m \leqslant Ch^{k-m+1}\|v\|_{k+1}, \quad m \in \{0, 1\}.$$

Let $x \in \Omega$ and $r > 0$ be fixed with

$$U(x, r) = \{y \in \mathbf{R}^2 \,|\, |x - y| \leqslant r\} \subset \Omega.$$

Define the average of u_h by

$$\tilde{u}_h(x) = \int_{U(x,r)} (f(y)(\gamma(y-x) + \psi(y-x)) + u_h(y)\Delta\psi(y-x))\,\mathrm{d}y, \tag{2.17}$$

where

$$\gamma(z) = -\frac{1}{2\pi}\ln|z|, \quad z \in \mathbf{R}^2,$$

$$\psi(z) = -\frac{1}{8\pi r^4}(|z|^4 - 4r^2|z|^2 + (3 - 4\ln r)r^4), \quad z \in \mathbf{R}^2.$$

Then for $u \in H^{k+1}(\Omega)$, $k \geqslant 2$, it is

$$|u(x) - \tilde{u}_h(x)| = \mathcal{O}(h^{2k}). \tag{2.18}$$

Louis' method requires more effort than that of Bramble and Schatz (cf. (2.16)), since in (2.17) the domain of integration is independent of h. Moreover, (2.17) cannot generally be calculated analytically. So it is reasonable to use this procedure either only at points of special interest or at those points where other methods do not work (e.g., on irregular meshes). Louis [131] also gives a formula for averages of ∇u_h with the same accuracy $\mathcal{O}(h^{2k})$ like in (2.18).

2.3.9. Also a *least squares smoothing* proposed by Hinton and Campbell [93] can be performed to achieve a higher accuracy of ∇u_h. The field ∇u_h is discontinuous in general and thus a continuous field $\widetilde{\nabla u_h}$ may be computed from ∇u_h through the local or global L^2-least squares method. For instance, when V_h is given by (2.6), ∇u_h is piecewise constant and we can thus choose $\widetilde{\nabla u_h}$ in the class of continuous piecewise linear functions. Although the least squares method requires more arithmetic operations than (2.12) or (2.15) do, the numerical results are better, see [93]. Related works include Hinton and Owen [94], Hinton, Scott and Ricketts [95].

2.3.10. A number of papers is devoted to the *interior estimates* of higher accuracy, where (2.2) is the restriction $\widetilde{Au_h} = Au_h|_{\Omega_0}$, $\Omega_0 \subset\subset \Omega$. We mention Bramble and Thomée [28], Descloux [64], Haslinger [89], Nakao [136], Nitsche and Schatz [144], Schatz and Wahlbin [159].

2.3.11. An asymptotic *error expansion* of the piecewise linear approximation u_h from the space (2.6) on general triangulations of a convex domain is introduced in Lin and Wang [125]. In particular, when $\mathcal{T}_h$ are uniform we have, at any node $x \in N_h$,

$$u_h(x) = u(x) + h^2 e(u, x) + \mathcal{O}(h^4),$$

where $e(u, x)$ is independent of h. Moreover, if $\mathcal{T}_h$ consist of equilateral triangles, the function $e(u, x)$ vanishes (see Blum, Lin and Rannacher [22]) and we obtain the uniform $\mathcal{O}(h^4)$ superconvergence at nodes. Similar expansions (also for ∇u_h) are derived in [122–127] and they are a useful tool for examining superconvergence phenomena as they characterize all the behaviour of the finite element solution.

2.3.12. An interior nodal superconvergence for higher-order elements in R^d ($d \geq 2$) has been proved by Zhu [194] even for a general second-order elliptic equation with smooth coefficients. Under some monotonicity condition (see Chen [52]), several superconvergence phenomena for linear elliptic problems can be extended to nonlinear cases (including, e.g., the minimal surface problem). For other superconvergence results for second order problems, the reader is referred to Babuška, Izadpanah and Szabo[14], Bramble and Schatz [26], Dautov [59], Dautov and Lapin [61], Dautov, Lapin and Lyashko [63], Douglas and Milner [79], El Hatri [91,92], Korneev [107], Lin and Xu [126], Nakao [136, 138], Zhu [193, 195]. Related references further include, e.g., Arnold and Brezzi [4], Babuška and Miller [15], Carey and Oden [37], Mansfield [132], Thomée and Westergren [179], etc. Numerical tests can be found in [14, 15, 52, 93, 108–110, 116, 118, 131, 140, 198].

2.4. TWO-POINT BOUNDARY VALUE PROBLEMS

For completeness, let us further recall the superconvergence results for the two-point problem

$$-u'' + a(x)u' + b(x)u = f(x), \quad x \in (0, 1), \tag{2.19}$$

with the Dirichlet conditions $u(0) = u(1) = 0$. The functions a, b and f are supposed to be sufficiently smooth and let $f = 0$ imply $u = 0$. The Galerkin method chooses $u_h \in S_h$ (S_h given by (2.3)) such that, e.g., for a uniform partition

$$(u'_h, s'_h)_0 + (au'_h + bu_h, s_h)_0 = (f, s_h)_0, \quad \forall s_h \in S_h. \tag{2.20}$$

This method exhibits the $\mathcal{O}(h^{2k})$ superconvergence at nodal points, see

Douglas and Dupont [72] (or [73])

$$\max_{0 \leq i \leq m} |u(n_i) - u_h(n_i)| \leq Ch^{2k} \|u\|_{k+1}.$$

Moreover, on any segment I_i there are $k-1$ interior points (the Lobatto points), where $u - u_h$ is $\mathcal{O}(h^{k+2})$, i.e., one order better than the global estimate, see Bakker [17]. At the k Gauss-Legendre points of each I_i, the derivative u'_h has $\mathcal{O}(h^{k+1})$ convergence instead of $\mathcal{O}(h^k)$, see Lesaint and Zlámal [116]. A local $\mathcal{O}(h^{2k})$ accuracy for the first derivative can be obtained from a system which requires very little more computing than (2.20), see Douglas and Dupont [73]. In Dupont [80], a similar post-processing is performed to produce a superconvergence for both the values and the derivatives at any point of the interval. For a post-processing applied to the C^1-Galerkin approximation of (2.19), see Douglas and Dupont [71]. Other superconvergence results for the Galerkin method are described by Babuška and Miller [15], Chen [49], Douglas, Dupont and Wahlbin [74], Huang and Wu [101], Marshall [133], Wheeler [186]. In Volk [182], the author shows that the method of Sloan [164] may also be applied to linear boundary value problems for ordinary differential equations.

In the one-dimensional case similar to (2.1), it holds that $u(n_i) = u_h(n_i)$ when linear elements are used (see [169], p. 107), i.e., there is no discretization error at nodes for an arbitrary partition. Analogous results for higher-order one-dimensional problems can be found, e.g., in [189].

Finally, let us mention important papers on superconvergence for two-point problems using other than the Galerkin method. For the least-squares method, see Ascher [12], Locker and Prenter [129, 148]; for the method of moments, see Mock [134], Rachford and Wheeler [150]; for the collocation of collocation-Galerkin method, see Ascher and Weiss [13], Bakker [18], de Boor and Swartz [23, 25], Carey and Wheeler [38], Christiansen and Russell [54, 55], Diaz [65], de Hoog and Weiss [97], Houstis [99], Nakao [137], Pereyra and Sewell [149], Wheeler [188]. In [88], de Groen presents a finite element method with a large mesh-width for a stiff two point boundary value problem. Several papers given above contain also a superconvergence analysis for higher order problems. For more facts about superconvergence results in $\mathbf{R}^1$, we refer to the survey paper in two-point boundary problems written by Reddien [154], (or Reddien [153], Nitsche [143]).

2.5. HIGHER ORDER PROBLEMS IN THE PLANE

For superconvergence phenomena in higher-order (fourth-order) elliptic problems in $\mathbf{R}^2$ which are similar to those of second-order problems, we refer to Dautov [60], Dautov and Lapin [62], Korneev [107], Westergren [185], Zlámal [197].

2.6. ELLIPTIC SYSTEMS

The application of the technique (2.12) to general second-order elliptic systems (including, e.g., Lamé's equations of elasticity) with nonhomogeneous boundary conditions of the Dirichlet, Neumann or Newton type has been investigated by Hlaváček and Křížek [96]. When $\partial\Omega$ is smooth and $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega \subset R^2$ it holds (on certain other assumptions) that

$$\sum_{i=1}^{N} \|\nabla u^i - \widetilde{\nabla u_h^i}\|_{0,\Omega_0} \leq Ch^2 \sum_{i=1}^{N} (\|u^i\|_{3,\Omega_1} + \|u^i\|_{2,\Omega}),$$

where N is the number of equations, u^i and u_h^i is the ith component of the solution and its Galerkin approximation, respectively. Also some global superconvergent estimates are derived.

Further post-processings causing an interior superconvergence in R^d are analyzed in Westergren [185]. See also Chen [50] for the scheme (2.7).

2.7. BOUNDARY FLUX

We describe a method for finding a superconvergent approximation to the boundary flux $q = \nabla u|_{\partial\Omega} \cdot \nu$, where u is the weak solution of (2.1) and ν is the outward unit normal to $\partial\Omega$. The method was first proposed and analyzed by Douglas, Dupont and Wheeler [75] for rectangular elements. For simplicity, we restrict ourselves only to the usual linear elements, see, e.g., Glowinski [83], p. 398.

Let V_h and W_h be given by (2.6) and (2.14), respectively. We seek a function $\tilde{q}_h = w_h|_{\partial\Omega}$ (for some $w_h \in W_h$) which is a better approximation to q than $q_h = \nabla u_h|_{\partial\Omega} \cdot \nu$. By analogy to the Green formula, we may uniquely determine $\tilde{q}_h$ by

$$\int_{\partial\Omega} \tilde{q}_h v_h \, \mathrm{d}s = (\nabla u_h, \nabla v_h)_0 - (f, v_h)_0, \quad \forall v_h \in W_h.$$

Taking v_h as the usual Courant basis functions at boundary nodes, we obtain a system of algebraic equations with a sparse matrix. Solving this system requires only $\mathcal{O}(m)$ operations, where m is the number of boundary nodes.

As suggested in [110], the post-processing technique (2.15) can also be used for approximating q. Namely, define $\tilde{q}_h = \widetilde{\nabla u_h}|_{\partial\Omega} \cdot \nu$, where $\widetilde{\nabla u_h}$ is defined by (2.15)

Further superconvergence techniques for the boundary flux for second order problems in $\mathbf{R}^d$ $(d \geq 1)$ are characterized in Carey, Humphrey and Wheeler [36], King and Serbin [106], Louis [131], Wheeler [187].

3. Superconvergence Results for Other Problems

In this section we mention some superconvergence phenomena which were discovered for other than elliptic problems.

3.1. PARABOLIC PROBLEMS

Superconvergence results for parabolic problems have been reported in the text books of Fairweather [81], Fletcher [82] and Thomée [177]. Here, superconvergence phenomena analogous to that for elliptic problems can be expected. For convenience, consider the problem

$$\begin{aligned} &\frac{\partial u}{\partial t}-\Delta u=f, && \text{in } \Omega\times(0,T), \quad \Omega\subset\mathbf{R}^2, \quad T>0,\\ &u(x,0)=u^0(x), && \text{in } \Omega, \\ &u(x,t)=0 && \text{on } \partial\Omega\times(0,T). \end{aligned} \tag{3.1}$$

The superconvergence analysis of (3.1) with the usual Crank–Nicolson scheme was done by El Hatri [90], among others. For instance, biquadratic elements on uniform rectangular partitions $\mathcal{T}_h$ over $\bar{\Omega}$ give the accuracy $\mathcal{O}(h^3+\tau^2)$ (where τ is the time step) for the space gradient at the Gauss–Legendre points in the discrete L^2-norm. Thus, one sees that this is similar to (2.5). The method of [90] has also been considered for variable coefficients and higher-order elements (including numerical integration).

The superconvergence technique (2.7) for the problem (3.1) was justified by Andreev [2]. He used a semidiscrete Galerkin scheme (that is, the discretization in space only). Arnold and Douglas [5] also applied a semidiscrete scheme (with $\Omega\subset\mathbf{R}^1$) and obtained superconvergence results for a quasiparabolic problem at nodal points. See also Douglas [68] for a collocation method. With the help of the Laplace transform, an alternative proof of superconvergence is given in Bakker [16] and Adeboye [1].

In fact, the literature on superconvergence results for parabolic problems is very rich. Especially, let us recall the series of Thomée's works [170, 172, 174–177]. Furthermore, we mention the papers by Douglas [69, 70], Douglas, Dupont and Wheeler [75], Douglas, Ewing and Wheeler [77], Kendall and Wheeler [105], Lazarov, Andreev and El Hatri [114], Nakao [135, 139], Wheeler [188].

3.2. HYPERBOLIC PROBLEMS

Relatively few papers are devoted to superconvergence phenomena when solving hyperbolic problems. For the initial boundary-value problem for the first-order equations (particularly $\partial u/\partial t+\partial u/\partial x=0$), see Cullen [58], Dougalis [66], Houstis [98], Thomée [171], Thomée and Wendroff [178], Winther [191]. Second-order problems are treated in Dougalis [66], Dougalis and Serbin [67], Nakao [139]. Let us mention, for example, the result of [67], where the problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right)+b(x)u=f(x,t) \quad \text{in } (0,1)\times(0,T), \quad T\geqslant 0,\\ &u(x,0)=u^0(x), \quad 0\leqslant x\leqslant 1, \\ &\frac{\partial u}{\partial t}(x,0)=u_t^0(x), \quad 0\leqslant x\leqslant 1, \end{aligned} \tag{3.2}$$

with 1-periodicity of u^0, u_t^0, a, b and f in the x range, is considered. The functions a, b are supposed to be infinitely differentiable and $a(x) \geqslant \alpha > 0$, $b(x) \geqslant 0$. The Galerkin subspaces V_h consist of periodic B-splines of order $r = k + 1$ on uniform partitions of $[0, 1]$ ($k \geqslant 1$ is the degree of the polynomials). A semidiscrete approximation of (3.2) with a suitable choice of the initial conditions for the Galerkin equation leads to an increased accuracy of the approximate solution which is $\mathcal{O}(h^{2k})$-accurate at nodes whereas $\mathcal{O}(h^{k+1})$ is the optimum L^2-error. Analogous superconvergence results have been obtained also for a multistep fully discrete Galerkin approximation. The effect of numerical integration is analysed in [67] as well.

In Douglas and Gupta [78], the authors introduce a higher-order mixed finite element method to approximate the solution of wave propagation in a plane elastic medium. Estimates are given for difference quotients for a spatially periodic problem and superconvergence results of the same type as those of Bramble and Schatz [27] for Galerkin methods are derived.

3.3. SOME SPECIAL EQUATIONS

There are also superconvergence results for partial differential equations of various special types. For the Boussinesq equation, see Winther [192], for the Korteweg–de Vries equation, see Arnold and Winther [11], for the Sobolev equation, see Arnold, Douglas and Thomée [6]. An averaging technique for the Stokes problem can be found in Johnson and Pitkäranta [104]. A superconvergence result for the neutron transport equation has been established by Lesaint and Raviart [115]. We also mention the paper [142] of Neta and Victory, where superconvergence phenomena have been presented for cell-edge and cell-average fluxes.

3.4. INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

The investigation of superconvergence phenomena for integral equations had its origin about five years later than that for differential equations. Nevertheless, the existing literature is quite extensive.

Fredholm integral equations of the first and second kind that arise from elliptic and some more general problems have been analysed by Hsiao and Wendland [100], Arnold and Wendland [8, 10], Wendland *et al.* [57, 111, 168, 183, 184], see also Chandler [42] and Sloan and Thomée [166].

Superconvergence results for the integral equation of the first kind were obtained by Locker and Prenter [130] when they employed the least-squares method. The Fredholm integral equation of the second kind is the most studied case. Let us confine ourselves to a model example

$$u(x) - \int_0^1 \mathscr{k}(x, s) u(s)\, \mathrm{d}s = f(x), \quad x \in \Omega = (0, 1), \tag{3.3}$$

where u is sought for given f and $\mathscr{k}$. The Galerkin approximation of (3.3) consists in finding $u_h \in V_h$ such that

$$(u_h - \mathscr{K}u_h, v_h)_0 = (f, v_h)_0, \quad \forall v_h \in V_h,$$

where $\mathscr{K}$ is the integral operator occurring in (3.3) and V_h is a piecewise-polynomial space (see, e.g., (2.3)) based on polynomials of degree at most k.

The study of the post-processing

$$\tilde{u}_h = \mathscr{K}u_h + f \tag{3.4}$$

(which is proposed to be used also recurrently) is of particular interest here. In the most convenient case we have

$$\|u - \tilde{u}_h\|_{1,\infty,\Omega} = \mathcal{O}(h^{2k+2}),$$

whereas

$$\|u - u_h\|_{0,\infty,\Omega} = \mathcal{O}(h^{k+1}),$$

see Sloan and Thomée [166]. For other related works concerning the iterated Galerkin scheme (3.4), we refer to Chandler [39–41], Chatelin [43–45], Chatelin and Lebbar [47, 48], Graham [86], Hsiao and Wendland [100], Lin and Liu [120], Sloan [164, 165], Spence and Thomas [167]. Superconvergence properties of the Galerkin approximation were also studied by Richter [156].

For collocation methods, where the superconvergence at collocation points is obtained, we refer to Chatelin [46], and references therein. For superapproximation results on collocation methods, see Arnold and Wendland [8], Saranen and Wendland [157]. A comparison of Galerkin and collocation schemes and their iterated variants has been presented by Arnold and Wendland [9], Graham, Joe and Sloan [87].

In [10], Arnold and Wendland consider the convergence of spline collocation for strongly elliptic equations on curves. Their result include, among others, the first convergence proof of midpoints with piecewise constant functions, i.e., the panel method for solving systems of Cauchy singular integral equations.

Next, let us mention the works of Brunner [29–34], Brunner and Nørsett [35] which are concerned with superconvergence when solving Volterra integral equations of the first and second kind by collocation methods. Some special equations have been studied by Goldberg, Lea and Miel [85] (airfoil equation), Larsen and Nelson [113] (discrete-ordinate equations in slab geometry). More details about the convergence acceleration (superconvergence) for integral equations are included in the survey papers by Chandler [41], Goldberg [84], Sloan [163].

For integro-differential equations, see, e.g., Brunner [33], von Seggern [160, 161].

3.5. REMARKS

We close with a few comments about superconvergence in other fields. Some results for eigenvalue problems were obtained by de Boor and Swartz [24], Chatelin [43, 45, 46], Chatelin and Lebbar [48], Regińska [155], Schäfer [158], Sloan [162]. For the theory of spline approximation, see Beatson [20], Behforooz and Papamichael [21], for superapproximation, see Arnold and Saranen [7], Costabel, Stephan and Wendland [57], Stephan and Wendland [168], for variational inequalities, see Lapin [112], for the theory of optimal control, see Reddien [152], and, finally, for a general operator theory, see Lindberg [128], Winther [190].

References

1. Adeboye, K. R.: Superconvergence results for Galerkin method for parabolic initial value problems via Laplace transform, *Bull. Math. Soc. Sci. Math. R.S. Roumanie* (*N.S.*) **26** (1982), 115–127. MR 84h:65095.
2. Andreev, A. B.: Superconvergence of the gradient for linear triangle elements for elliptic and parabolic equations, *C.R. Acad. Bulgare Sci.* **37** (1984), 293–296. MR 85h:65230.
3. Andreev, A. B.: Error estimate of type superconvergence of the gradient for quadratic triangular elements, *C.R. Acad. Bulgare Sci.* **37** (1984), 1179–1182. MR 86f:65186.
4. Arnold, D. N. and Brezzi, F.: Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates, *RAIRO Modél. Math. Anal. Numér.* **19** (1985), 7–32.
5. Arnold, D. N. and Douglas, J. J.: Superconvergence of the Galerkin approximation of a quasilinear parabolic equation in a single space variable, *Calcolo* **16** (1979), 345–369. MR 83b:65118.
6. Arnold, D. N., Douglas, J. J., and Thomée, V.: Superconvergence of a finite element approximation to the solution of a Sobolev equation in a single space variable, *Math. Comp.* **36** (1981), 53–63. MR 82f:65108.
7. Arnold, D. N. and Saranen, J.: On the asymptotic convergence of spline collocation methods for partial differential equations, *SIAM J. Numer. Anal.* **21** (1984), 459–472.
8. Arnold, D. N. and Wendland, W. L.: On the asymptotic convergence of collocation methods, *Math. Comp.* **41** (1983), 349–381. MR 85h:65254.
9. Arnold, D. N. and Wendland, W. L.: Collocation versus Galerkin procedures for boundary integral methods, in *Boundary Element Methods in Engineering*, Springer-Verlag, Berlin, 1982, pp. 18–33. MR 85f:73074.
10. Arnold, D. N. and Wendland, W. L.: The convergence of spline collocation for strongly elliptic equations on curves, *Numer. Math.* **47** (1985), 317–341.
11. Arnold, D. N. and Winther, R.: A superconvergent finite element method for the Korteweg–de Vries equation, *Math. Comp.* **38** (1982), 23–36. MR 82m:65087.
12. Ascher, U.: Discrete least squares approximations for ordinary differential equations, *SIAM J. Numer. Anal.* **15** (1978), 478–496. MR 81e: 65043.
13. Ascher, U. and Weiss, R.: Collocation for singular perturbation. I. First order systems with constant coefficients, *SIAM J. Numer. Anal.* **20** (1983), 537–557. MR 85a:65113.
14. Babuška, I., Izadpanah, K., and Szabo, B.: The post-processing technique in the finite element method. The theory and experience, Technical Note BN-1016, Univ. of Maryland, 1984, 1–25.
15. Babuška, I. and Miller, A.: The post-processing in the finite element method, Parts I–II, *Internat. J. Numer. Methods Engrg.* **20** (1984), 1085–1109, 1111–1129.
16. Bakker, M.: On the numerical solution of parabolic equations in a single space variable by the continuous time Galerkin method, *SIAM J. Numer. Anal.* **17** (1980), 162–177. MR 81m:65167.

17. Bakker, M.: A note on C^0 Galerkin methods for two-point boundary problems, *Numer. Math.* **38** (1981/82), 447–453. MR 83e:65140.
18. Bakker, M.: One-dimensional Galerkin methods and superconvergence at interior nodal points, SIAM J. Numer. Anal. **21** (1984), 101–110. MR 85f:65080.
19. Barlow, J.: Optimal stress location in finite-element models, *Internat. J. Numer. Methods Engrg.* **10** (1976), 243–251.
20. Beatson, R.: Joint approximation of a function and its derivatives, in *Approximation Theory III*, (Proc. Conf., Univ. Texas, Austin, Tex., 1980), Academic Press, New York, 1980, pp. 199–206. MR 82c:41019.
21. Behforooz, G. H. and Papamichael, N.: Improved orders of approximation derived from interpolatory cubic splines. *BIT* **19** (1979), 19–26. MR 80e:41004.
22. Blum, H., Lin, Q., and Rannacher, R.: Asymptotic error expansion and Richardson extrapolation for linear finite elements, *Numer. Math.* **49** (1986), 11–38.
23. De Boor, C. and Swartz, B.: Collocation at Gaussian points, *SIAM J. Numer. Anal.* **10** (1973), 582–606. MR 51#9528.
24. De Boor, C. and Swartz, B.: Collocation approximation to eigenvalues of an ordinary differential equation: The principle of the thing, *Math. Comp.* **35** (1980), 679–694. MR 81k:65097.
25. De Boor, C. and Swartz, B.: Local piecewise polynomial projection methods for an ODE which give high-order convergence at knots, *Math. Comp.* **36** (1981), 21–33. MR 82f:65091.
26. Bramble, J. H. and Schatz, A. H.: Estimates for spline projection, *RAIRO Anal. Numér.* **10** (1976), 5–37. MR 55#9563.
27. Bramble, J. H. and Schatz, A. H.: Higher order local accuracy by averaging in the finite element method, *Math. Comp.* **31** (1977), 94–111. MR 55#4739.
28. Bramble, J. H., and Thomée, V.: Interior maximum norm estimates for some simple finite element methods, *RAIRO Anal. Numér.* **8** (1974), 5–18. MR 50#11808.
29. Brunner, H.: On superconvergence in collocation methods for Abel integral equations, *Numerical Mathematics and Computing* (Proc. Eighth Manitoba Conf., Winnipeg, 1978). Utilitas Mathematica Publishing, Inc., Winnipeg, Man, 1979, pp. 117–128. MR 80m: 45023.
30. Brunner, H.: Superconvergence of collocation methods for Volterra integral equations of the first kind, *Computing* **21** (1979), 151–157. MR 83a:65125.
31. Brunner, H.: A note on collocation methods for Volterra integral equations of the first kind, *Computing* **23** (1979), 179–187. MR 83b:65145.
32. Brunner, H.: Superconvergence in collocation and implicit Runge-Kutta methods for Volterra-type integral equations of the second kind, *Numerical Treatment of Integral Equations* (Workshop, Math. Res. Inst., Oberwolfach, 1979), Internat. Ser. Numer. Math., 53, Birkhäuser, Basel 1980, pp. 54–72. MR 81m:65194.
33. Brunner, H.: The application of the variation of constants formulas in the numerical analysis of integral and integro-differential equations, *Utilitas Math.* **19** (1981), 255–290. MR 83b:65146.
34. Brunner, H.: Iterated collocation methods and their discretizations for Volterra integral equations, *SIAM J. Numer. Anal.* **21** (1984), 1132–1145. MR 86d:65160.
35. Brunner, H. and Nørsett, S. P.: Superconvergence of collocation methods for Volterra and Abel integral equations of the second kind, *Numer. Math.* **36** (1981), 347–358. MR 83e:65202.
36. Carey, G. F., Humphrey, D., and Wheeler, M. F.: Galerkin and collocation-Galerkin methods with superconvergence and optimal fluxes, *Internat. J. Numer. Methods Engrg.* **17** (1981), 939–950. MR 82g:80008.
37. Carey, G. F. and Oden, J. T.: *Finite Elements. Vol. II: A second course*, The Texas Finite Element Series, Vol. 2, Prentice-Hall, Englewood Cliffs, New Jersey, 1983.
38. Carey, G. F. and Wheeler, M. F.: C^0-collocation-Galerkin methods, *Codes for Boundary-Value Problems in Ordinary Differential Equations* (Proc. Conf., Houston, 1978), LN in Comp. Sci., Vol. 76, Springer, Berlin, 1979, pp. 250–256. MR 82b:65074.
39. Chandler, G. A.: Global superconvergence of iterated Galerkin solutions for second kind integral equations, Technical Report, Australian Nat. Univ., Canberra, 1978.
40. Chandler, G. A.: Superconvergence of numerical solutions to second kind integral equations, PhD thesis, Australian Nat. Univ., Canberra, 1979.
41. Chandler, G. A.: Superconvergence for second kind integral equations, *The Application and*

Numerical Solution of Integral Equations (Proc. Conf., Canberra, 1978), Sijthoff & Noordhoff, Alphen-aan-den-Rijn, 1980, pp. 103–117. MR 81h:45027.

42. Chandler, G. A.: Galerkin's method for boundary integral equations on polygonal domains, *J. Austral. Math. Soc. Ser. B* **26** (1984), 1-13. MR 86a:65116.
43. Chatelin, F.: Sur les bornes d'erreur a posteriori pour les éléments propres d'opérateurs linéaires, *Numer. Math.* **32** (1979), 233-246. MR 81d:65027.
44. Chatelin, F.: Des resultats de superconvergence, Seminaire d'analyse numérique, No. 331, Univ. Scientifique et Medicale de Grenoble, Laboratoire IMAG, 1980.
45. Chatelin, F.: The spectral approximation of linear operators with applications to the computation of eigenelements of differential and integral operators, *SIAM Rev.* **23** (1981), 495–522. MR 83b:47023.
46. Chatelin, F.: *Spectral Approximation of Linear Operators*, Academic Press, New York, 1983.
47. Chatelin, F. and Lebbar, R.: The iterated projection solution for the Fredholm integral equation of second kind, *J. Austral. Math. Soc. Ser. B* **22** (1981), 439–451. MR 82h:65096.
48. Chatelin, F. and Lebbar, R.: Superconvergence results for the iterated projection method applied to a Fredholm integral equation of the second kind and the corresponding eigenvalue problem, *J. Integral Equations* **6** (1984), 71–91. MR 85i:65167.
49. Chen, C. M.: The good points of the approximation solution for Galerkin method for two-point boundary problem, *Numer. Math. J. Chinese Univ.* **1** (1979), 73–79. MR 82e:65085.
50. Chen, C. M.: Optimal points of the stresses for triangular linear element, *Numer. Math. J. Chinese Univ.* **2** (1980), 12–20. MR 83d:65279.
51. Chen, C. M.: Superconvergence of finite element solutions and their derivatives, *Numer. Math. J. Chinese Univ.* **3** (1981), 118–125. MR 82m:65100.
52. Chen, C. M.: Superconvergence of finite element approximations to nonlinear elliptic problems (*Proc. China–France Sympos. on Finite Element Methods, Beijing*, 1982), Gordon and Breach, New York, 1983, pp. 622–640. MR 85h:65285.
53. Chen, C. M.: Superconvergence of finite elements for nonlinear problems, *Numer. Math. J. Chinese Univ.* **4** (1982), 222–228. MR 83m:65083.
54. Christiansen, J. and Russell, R. D.: Adaptive mesh selection strategies for solving boundary value problems, *SIAM J. Numer. Anal.* **15** (1978), 59–80. MR 57#11071.
55. Christiansen, J. and Russell, R. D.: Error analysis for spline collocation methods with application to knot selection, *Math. Comp.* **32** (1978), 415–419. MR 58#13736.
56. Ciarlet, P. G.: *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, New York, Oxford, 1978. MR 58#25001.
57. Costabel, M., Stephan, E., and Wendland, W. L.: On the boundary integral equation of the first kind for the bi-Laplacian in a polygonal plane domain, *Ann. Scuola Norm. Sup. Pisa Cl. Sci., Ser. IV*, **10** (1983), 197-242. MR 85f:35074.
58. Cullen, M. J. P.: The use of quadratic finite element methods and irregular grids in the solution of hyperbolic problems, *J. Comput. Phys.* **45** (1982), 221–245. MR 83f:65152.
59. Dautov, R. Z.: Superconvergence of difference schemes for the third boundary value problem for quasilinear elliptic equations, *Čisl. Metody Meh. Splošn. Sredy* **11** (1980), 62–80. MR 84b:65102.
60. Dautov, R. Z.: Superconvergence of finite-element schemes with numerical integration for quasilinear fourth-order elliptic equations, *Differencial'nye Uravnenija* **18** (1982), 1172–1181, *Differential Equations* **18** (1982), 818–824. MR 83k:65080.
61. Dautov, R. Z. and Lapin, A. V.: Difference schemes of an arbitrary order of accuracy for quasilinear elliptic equations, *Izv. Vysš. Učebn. Zaved. Matematika* **209** (1979), 24–37. MR 81g:65128.
62. Dautov, R. Z. and Lapin, A. V.: Investigation of the convergence, in mesh norms, of finite-element-method schemes with numerical integration for fourth-order elliptic equations, *Differential Equations* **17** (1981), 807–817. MR 83h: 65122.
63. Dautov, R. Z., Lapin, A. V., and Lyashko, A. D.: Some mesh schemes for quasilinear elliptic equations, *Ž. Vyčisl. Mat. i Mat. Fiz.* **20** (1980), 334–339, *U.S.S.R. Computational Math. and Math. Phys.* **20** (1980), 62–78. MR 81j:65121.
64. Descloux, J.: Interior regularity and local convergence of Galerkin finite element ap-

proximations for elliptic equations, *Topics in Numerical Analysis*, II (Proc. Royal Irish Acad. Conf., Dublin, 1974), Academic Press, New York, 1975, pp. 24–41. MR 54#1658.

65. Diaz, J. C.: A collocation-Galerkin method for two-point boundary value problem using continuous piecewise polynomial spaces, *SIAM J. Numer. Anal.* **14** (1977), 844–858. MR 58#3481.
66. Dougalis, V. A.: Multistep-Galerkin methods for hyperbolic equations, *Math. Comp.* **33** (1979), 563–584. MR 81b:65081.
67. Dougalis, V. A. and Serbin, S. M.: On the superconvergence of Galerkin approximations to second-order hyperbolic equations. *SIAM J. Numer. Anal.* **17** (1980), 431–446. MR 81j:65106.
68. Douglas, J. J.: A superconvergence result for the approximate solution of the heat equation by a collocation method, *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations* (Proc. Sympos., Univ. Maryland, Baltimore, 1972). Academic Press, New York, 1972, pp. 475–490. MR 53#7063.
69. Douglas, J. J.: Improved accuracy through superconvergence in the pressure in the simulation of miscible displacement, *Computing Methods in Applied Sciences and Engineering* (Proc. Conf., Versailles, 1983), LN in Comp. Sci., North-Holland, 1984, pp. 633–638.
70. Douglas, J. J.: Superconvergence in the pressure in the simulation of miscible displacement, *SIAM J. Numer. Anal.* **22** (1985), 962–969. MR 86j:65129.
71. Douglas, J. J. and Dupont, T.: Superconvergence for Galerkin methods for the two-point boundary problem via local projections, *Numer. Math.* **21** (1973), 270–278. MR 48#10130.
72. Douglas, J. J. and Dupont, T.: Some superconvergence results for Galerkin methods for the approximate solution of two-point boundary value problems, *Topics in Numerical Analysis, II* (Proc. Royal Irish Acad. Conf., Dublin, 1972), Academic Press, London, 1973, pp. 89–93. MR 51#2295.
73. Douglas, J. J. and Dupont, T.: Galerkin approximations for the two-point boundary problem using continuous, piecewise polynomial spaces, *Numer. Math.* **22** (1974), 99–109. MR 50#15360.
74. Douglas, J. J., Dupont, T., and Wahlbin, L., Optimal L_∞ error estimates for Galerkin approximations to solutions of two-point boundary value problems, *Math. Comp.* **29** (1975), 475–483. MR 51#7298.
75. Douglas, J. J., Dupont, T., and Wheeler, M. F.: A Galerkin procedure for approximating the flux on the boundary for elliptic and parabolic boundary value problems, *RAIRO Anal. Numér.* **8** (1974), 47–59. MR 50#11811.
76. Douglas, J. J., Dupont, T., and Wheeler, M. F.: An L^∞ estimate and a superconvergence result for a Galerkin method for elliptic equations based on tensor products of piecewise polynomials, *RAIRO Anal. Numér.* **8** (1974), 61–66. MR 50#11812.
77. Douglas, J. J., Ewing, R. E., and Wheeler, M. F.: The approximation of the pressure by a mixed method in the simulation of miscible displacement, *RAIRO Anal. Numér.* **17** (1983), 17–33.
78. Douglas, J. J. and Gupta, C. P.: Superconvergence for a mixed finite element method for elastic wave propagation in a plane domain, *Numer. Math.* **49** (1986), 189–202.
79. Douglas, J. J. and Milner, F. A.: Interior and superconvergence estimates for mixed methods for second order elliptic problems, *RAIRO Modél. Math. Anal. Numér.* **19** (1985), 397–428.
80. Dupont, T.: A unified theory of superconvergence for Galerkin methods for two-point boundary problems, *SIAM J. Numer. Anal.* **13** (1976), 362–368. MR 53#12021.
81. Fairweather, G.: *Finite Element Galerkin Methods for Differential Equations*, Lecture Notes in Pure and Applied Mathematics, Vol. 34, Marcel Dekker, New York, Basel, 1978, MR 58#13781.
82. Fletcher, C. A. J.: *Computational Galerkin Methods*, Springer Series in Computational Physics, Springer-Verlag, New York, 1984. MR 85i:65003.
83. Glowinski, R.: *Numerical Methods for Nonlinear Variational Problems*, Springer Series in Computational Physics, Springer Verlag, New York, 1984.
84. Goldberg, M. A.: A survey of numerical methods for integral equations, *Solution Methods for Integral Equations*, Math. Concepts and Methods in Sci. and Engrg. Vol. 18. Plenum Press, New York, London, 1979, pp. 1–58. MR 80m:65090.
85. Goldberg, M. A., Lea, M., and Miel, G.: A superconvergence result for the generalized airfoil

equation with application to the flap problem, *J. Integral Equations* **5** (1983), 175–186. MR 84e:65127.

86. Graham, I. G.: Galerkin methods for second kind integral equations with singularities, *Math. Comp.* **39** (1982), 519–533. MR 84d:65090.
87. Graham, I. G., Joe, S., and Sloan, I. H.: Galerkin versus collocation for integral equations of the second kind, *IMA J. Numer. Anal.* **5** (1985), 358–369.
88. de Groen, P. P. N.: A finite element method with a large mesh-width for a stiff two-point boundary value problem, *J. Comput. Appl. Math.* **7** (1981), 3–15. MR 82d:65059.
89. Haslinger, J.: Elements finis et l'estimation de l'erreur interieur de la convergence, *Comment. Math. Univ. Carolin.* **15** (1974), 85–102. MR 49#1799.
90. El Hatri, M.: Superconvergence of the lumped mass approximation of parabolic equations, *C.R. Acad. Bulgare Sci.* **36** (1983), 575–578. MR 85h:65187.
91. El Hatri, M.: Superconvergence of axisymmetrical boundary-value problem, *C.R. Acad. Bulgare Sci.* **36** (1983), 1499–1502. MR 86e:65153.
92. El Hatri, M.: Superconvergence in the finite element method for a degenerated boundary value problem, *Constr. Theory of Functions* (Proc. Conf., Varna, 1984), Bulg. Acad. Sci., Sofia (1984), pp. 328–333.
93. Hinton, E. and Campbell, J. S.: Local and global smoothing of discontinuous finite element functions using a least squares method, *Internat. J. Numer. Methods Engrg.* **8** (1974), 461–480. MR 53#14859.
94. Hinton, E. and Owen, D. R. J.: *Finite Element Programming*, Academic Press, London, New York, San Francisco, 1977.
95. Hinton, E., Scott, F. C., and Ricketts, R. E.: Local least squares stress smoothing for parabolic isoparametric elements, *Internat. J. Numer. Methods Engrg.* **9** (1975), 235–238.
96. Hlaváček, I. and Křížek, M.: On superconvergent finite element scheme for elliptic systems, Parts I–III, *Apl. Mat.* **32** (1987), 131–153, 200–213.
97. De Hoog, F. R. and Weiss, R.: Collocation methods for singular boundary value problems, *SIAM J. Numer. Anal.* **15** (1978), 198–217. MR 57#8041.
98. Houstis, E. N.: Application of method of collocation on lines for solving nonlinear hyperbolic problems, *Math. Comp.* **31** (1977), 443–456. MR 56#1749.
99. Houstis, E. N.: A collocation method for systems of nonlinear ordinary differential equations, *J. Math. Anal. Appl.* **62** (1978), 24–37. MR 58#8295.
100. Hsiao, G. C. and Wendland, W. L.: The Aubin-Nitsche lemma for integral equations, *J. Integral Equations* **3** (1981), 299–315. MR 83j:45019.
101. Huang, D. and Wu, D.: The superconvergence of the spline finite element solution and its second order derivate for the two-point boundary problem of a fourth order differential equation, *J. Zhejiang Univ.* **3** (1982), 92–99.
102. Hulme, B. L.: One-step piecewise polynomial Galerkin methods for initial value problems, *Math. Comp.* **26** (1972), 415–426. MR 47#9834.
103. Irons, B. M. and Razzaque, A.: Experience with the patch test for convergence of finite elements, *Mathematical Foundations of the Finite Element Method with Applications to PDE*, Academic Press, New York, 1972, pp. 557–587. MR 54#11813.
104. Johnson, C. and Pitkäranta, J.: Analysis of some mixed finite element methods related to reduced integration, *Math. Comp.* **33** (1982), 375–400. MR 83d:65287.
105. Kendall, R. P. and Wheeler, M. F.: A Crank-Nicolson H^{-1}-Galerkin procedure for parabolic problems in a single space variable, *SIAM J. Numer. Anal.* **13** (1976), 861–876. MR 58#13785.
106. King, J. T. and Serbin, S. M.: Boundary flux estimates for elliptic problems by the perturbed variational method, *Computing* **16** (1976), 339–347. MR 54#6524.
107. Korneev, V. G.: Superconvergence of solutions of the finite element method in mesh norms, *Ž. Vyčisl. Mat. i Mat. Fiz.* **22** (1982), 1133–1148. MR 84b:65114.
108. Křížek, M. and Neittaanmäki, P.: Superconvergence phenomenon in the finite element method arising from averaging gradients, *Numer. Math.* **45** (1984), 105–116. MR 86c:65135.
109. Křížek, M. and Neittaanmäki, P.: Post-processing of a finite element scheme with linear elements, *Numerical Techniques in Continuum Mechanics* (Proc. 2nd GAMM-Seminar, Kiel, 1986). Notes on Numerical Fluid Mechanics, Vol. 16, 1987, pp. 69–83.

110. Křížek, M. and Neittaanmäki, P.: On a global superconvergence of the gradient of linear triangular elements, *J. Comput. Appl. Math.* (1987), (to appear).
111. Lamp, U., Schleicher, T., Stephan, E., and Wendland, W. L.: Theoretical and experimental asymptotic convergence of the boundary integral method for plane mixed boundary value problem, *Boundary Element Methods in Engineering* (Proc. Fourth Int. Semin., Southampton, 1982), Springer, Berlin, 1982, pp. 3–17. MR 85e: 3002.
112. Lapin, A. V.: Mesh schemes of high order of accuracy for some classes of variational inequalities, *Sov. J. Numer. Anal. Math. Modelling* **2** (1987), 37–55.
113. Larsen, E. W. and Nelson, P.: Finite-difference approximations and superconvergence for the discrete-ordinate equations in slab geometry, *SIAM J. Numer. Anal.* **19** (1982), 334–348. MR 83#82050.
114. Lazarov, R. D., Andreev, A. B., and El Hatri, M.: Superconvergence of the gradient in the finite element method for some elliptic and parabolic problems, *Variational-Difference Methods in Mathematical Physics*, Part 2 (Proc. Fifth Int. Conf., Moscow, 1983), Viniti, Moscow, 1984, pp. 13–25.
115. Lesaint, P. and Raviart, P. A.: On a finite element method for solving the neutron transport equation, *Mathematical Aspects of Finite Elements in PDE* (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, 1974). Univ. of Wisconsin, Madison, Academic Press, New York, 1974, pp. 89–123. MR 58#31918.
116. Lesaint, P. and Zlámal, M.: Superconvergence of the gradient of finite element solutions, *RAIRO Anal. Numér.* **13** (1979), 139–166. MR 80g:65112.
117. Levine, N.: Stress sampling points for linear triangles in the finite element method, Numerical approximation rep. IMA, *Numer. Anal.* **5** (1985), 407–427.
118. Levine, N.: Superconvergent recovery of the gradient from piecewise linear finite element method, Numerical analysis report 6/83, Univ. of Reading, 1983.
119. Leyk, Z.: Superconvergence in the finite element method, *Mat. Stos.* **20** (1982), 93–107. MR 85k:65090.
120. Lin, Q. and Liu, J. Q.: Extrapolation method for Fredholm integral equations with nonsmooth kernels, *Numer. Math.* **35** (1980), 459–464. MR 81m:65198.
121. Lin, Q. and Lü, T.: Asymptotic expansions for finite element eigenvalues and finite element solution (*Proc. Int. Conf., Bonn, 1983*), Math. Schrift. No. 158, Bonn, 1984, pp. 1–10.
122. Lin, Q. and Lü, T.: Asymptotic expansions for finite element approximation of elliptic problem on polygonal domains, *Computing Methods in Applied Sciences and Engineering* (Proc. Sixth Int. Conf. Versailles, 1983), LN in Comp. Sci., North-Holland, INRIA, 1984, pp. 317–321.
123. Lin, Q., Lü, T., and Shen, S.: Asymptotic expansion for finite element approximations, Research Report IMS-11, Inst. Math. Sci. Chengdu Branch of Acad. Sinica, 1983, pp. 1–6. MR 85c:65141.
124. Lin, Q., Lü, T., and Shen, S.: Maximum norm estimate, extrapolation and optimal point of stresses for the finite element methods on the strongly regular triangulation, *J. Comput. Math.* **1** (1983), 376–383.
125. Lin, Q. and Wang, J. P.: Some expansions of the finite element approximation Research Report IMS-15, Academia Sinica, Chengdu Branch, 1984, pp. 1–11. MR 86d:65148.
126. Lin, Q. and Xu, J. Ch.: Linear finite elements with high accuracy, *J. Comput. Math.* **3** (1985), 115–133.
127. Lin, Q. and Zhu, Q. D.: Asymptotic expansion for the derivative of finite elements, *J. Comput. Math.* **2** (1984), 361–363.
128. Lindberg, B.: Error estimation and iterative improvement for discretization algorithms, *BIT* **20** (1980), 486–500. MR 82j:65036.
129. Locker, J. and Prenter, P. M.: On least squares methods for linear two-point boundary value problems, *Functional Analysis Methods in Numerical Analysis* (Proc. Spec. Sess., AMS, St. Louis, 1977), LN in Math., Vol. 701, Springer, Berlin, 1979, pp. 149–168. MR 80j:65029.
130. Locker, J. and Prenter, P. M.: Regularization with differential operators. II: Weak least squares finite element solutions to first kind integral equations, *SIAM J. Numer. Anal.* **17** (1980), 247–267. MR 83j:65062b.
131. Louis, A.: Acceleration of convergence for finite element solutions of the Poisson equation, *Numer. Math.* **33** (1979), 43–53. MR 81i:65091.

132. Mansfield, L.: On mixed finite element methods for elliptic equations, *Comput. Math. Appl.* **7** (1981), 59–66. MR 82b:65133.

133. Marshall, R. S.: An alternative interpretation of superconvergence, *Internat. J. Numer. Methods Engrg.* **14** (1979), 1707–1709. MR 80m:65079.

134. Mock, M. S.: Projection methods with different trial and test spaces, *Math. Comp.* **30** (1976), 400–416. MR 54#11814.

135. Nakao, M.: Some superconvergence estimates for a collocation H^1-Galerkin method for parabolic problems, *Mem. Fac. Sci. Kyushu Univ. Ser. A* **35** (1981), 291–306. MR 83a:65111.

136. Nakao, M.: Interior estimates and superconvergence for H^{-1}-Galerkin method to elliptic equations, *Bull. Kyushu Inst. Tech. Math. Natur. Sci.* **30** (1983), 19–30. MR 85i:65154.

137. Nakao, M.: Superconvergence estimates at Jacobi points of the collocation-Galerkin method for two-point boundary value problems *J. Inform. Process.* **7** (1984), 31–34. MR 85m:65079.

138. Nakao, M.: Some superconvergence estimates for a Galerkin method for elliptic problems, *Bull. Kyushu Inst. Tech. Math. Natur. Sci.* **31** (1984), 49–58. MR 86e:65159.

139. Nakao, M.: Some superconvergence of Galerkin approximations for parabolic and hyperbolic problems in one space dimension, *Bull. Kyushu Inst. Tech. Math. Natur. Sci.* **32** (1985), 1–14. MR 86j:65123.

140. Neittaanmäki, P. and Křížek, M.: Superconvergence of the finite element schemes arising from the use of averaged gradients, *Accuracy Estimates and Adaptive Refinements in Finite Element Computations* (Proc. Int. Conf., Lisbon, 1984), pp. 169–178.

141. Neittaanmäki, P. and Saranen, J.: A modified least squares FE-method for ideal fluid flow problems, *J. Comput. Appl. Math.* **8** (1982), 165–169.

142. Neta, B. and Victory, H. D.: A new fourth-order finite-difference method for solving discrete-ordinates slab transport equations, *SIAM J. Numer. Ar ıl.* **20** (1983), 94–105. MR 84e:65082.

143. Nitsche, J. A.: L_∞-error analysis for finite elements, *Mathematics of Finite Elements and Applications, III* (Proc. Third MAFELAP Conf., Brunel Univ., Uxbridge, 1978), Academic Press, London, 1979, pp. 173–186. MR 81f:65084.

144. Nitsche, J.A. and Schatz, A. H.: Interior estimates for Ritz–Galerkin methods, *Math. Comp.* **28** (1974), 937–958. MR 51#9525.

145. Oganesjan, L. A., Rivkind, V. J., and Ruhovec, L. A.: Variation-difference methods for the solution of elliptic equations, Part I (*Proc. Sem., Issue 5, Vilnius, 1973*), Inst. of Phys. and Math., Vilnius, 1973, pp. 3–389, MR 57#18146.

146. Oganesjan, L. A. and Ruhovec, L. A.: An investigation of the rate of convergence of variational-difference schemes for second order elliptic equations in a two-dimensional region with smooth boundary, *Ž. Vyčisl. Mat. i Mat. Fiz.* **9** (1969), 1102–1120. MR 45#4665.

147. Oganesjan, L. A. and Ruhovec, L. A.: *Variational-Difference Methods for the Solution of Elliptic Equations*, Izd. Akad. Nauk Armjanskoi SSR, Jerevan, 1979. MR 82m:65105.

148. Prenter, P. M. and Locker, J.: Optimal L^2 and L^∞ error estimates for continuous and discrete least squares methods for boundary value problems, *SIAM J. Numer. Anal.* **15** (1978), 1151–1160. MR 80d:65097.

149. Pereyra, W. and Sewell, E. G.: Mesh selection for discrete solution of boundary value problems in ordinary differential equations, *Numer. Math.* **23** (1975), 261–268. MR 57#4527.

150. Rachford, H. H. and Wheeler, M. F.: An H^{-1}-Galerkin procedure for the two-point boundary value problem, *Mathematical Aspects of Finite Elements in Partial Diff. Equations*, Academic Press, New York, 1974, pp. 353–382. MR 50#1525.

151. Rannacher, R. and Scott, R.: Some optimal error estimates for piecewise linear finite element approximations, *Math. Comp.* **38** (1982), 437–445.

152. Reddien, G. W.: Collocation at Gauss points as a discretization in optimal control, *SIAM J. Control Optim.* **17** (1979), 298–306. MR 80b:93171.

153. Reddien, G. W.: Codes for boundary-value problems in ordinary differential equations (*Proc. Conf. Houston, 1978.*) LN in Comp. Sci., Vol. 76, Springer, Berlin, 1979, pp. 206–227.

154. Reddien, G. W.: Projection methods for two-point boundary value problems, *SIAM Rev.* **22** (1980), 156–171. MR 81e:65049.

155. Regińska, T.: Superconvergence of external approximation of eigenvalues of ordinary differential operators, *IMA J. Numer. Anal.* **6** (1986), 309–323.

156. Richter, G. R.: Superconvergence of piecewise polynomial Galerkin approximations, for Fredholm integral equations of the second kind, *Numer. Math.* **31** (1978/79), 63–70. MR 80a:65273.
157. Saranen, J. and Wendland, W. L.: On the asymptotic convergence of collocation methods with spline function of even degree, *Math. Comp.* **45** (1983), 91–108.
158. Schäfer, E.: Fehlerabschätzungen für Eigenwertnäherungen nach der Ersatzkernmethode bei Integralgleichungen, *Numer. Math.* **32** (1979), 281–290. MR 80e:65129.
159. Schatz, A. H. and Wahlbin, L. B.: Interior maximum norm estimates for finite element methods, *Math. Comp.* **31** (1977), 414–442. MR 55#4748.
160. Von Seggern, R.: Die Methode der finiten Elemente be der numerischen Behandlung linear Integrodifferentialgleichungen, Kernforschungsanlage Julich, Zentralinstitut für Angewandte Mathematik, Julich, 1979. MR 83e:65215.
161. Von Seggern, R.: Ein Superkonvergenzresultat bei Anwendung der Methode der finiten Elemente auf lineare Integrodifferentialgleichungen, *Z. Angew. Math. Mech.* **61** (1981), 320–321. MR 84e:45009.
162. Sloan, I. H.: Iterated Galerkin method for eigenvalue problems, *SIAM J. Numer. Anal.* **13** (1976), 753–760. MR 55#1727.
163. Sloan, I. H.: A review of numerical methods for Fredholm integral equations of the second kind, *Application and Numerical Solution of Integral Equations* (Proc. Sem., Australian Nat. Univ., Canberra, 1978), Sijthoff & Noordhoff, Alphen-aan-den-Rijn, 1980, pp. 51–74. MR 81h:65133.
164. Sloan, I. H.: Superconvergence and the Galerkin method for integral equations of the second kind, *Treatment of Integral Equations by Numerical Methods* (Proc. Symp., Durham, 1982). Academic Press, New York, London, 1982, pp. 197–207. MR 85e:65005.
165. Sloan, I. H.: Four variants of the Galerkin method for integral equations of the second kind, *IMA J. Numer. Anal.* **4** (1984), 9–17. MR 85h:65277.
166. Sloan, I. H. and Thomée, V.: Superconvergence of the Galerkin iterates for integral equations of the second kind, *J. Integral Equations* **9** (1985), 1–23. MR 86j:65184.
167. Spence, A. and Thomas, K. S.: On superconvergence properties of Galerkin's method for compact operator equations, *IMA J. Numer. Anal.* **3** (1983), 253–271. MR 85c:65074.
168. Stephan, E. and Wendland, W. L.: An augmented Galerkin procedure for the boundary integral method applied to two-dimensional screen and crack problems. *Applicable Anal.* **18** (1984), 183–220.
169. Strang, G. and Fix, G.: *An Analysis of the Finite Element Method*, Prentice Hall, Englewood Cliffs, New Jersey, 1973. MR 56#1747.
170. Thomée, V.: Spline approximation and difference schemes for the heat equation. *Math. Foundations of the Finite Element Method with Applications to PDE* (Proc. Sympos., Univ. of Maryland, 1972). Academic Press, New York, London, 1972, pp. 711–746. MR 49#11824.
171. Thomée, V.: Convergence estimates for semi-discrete Galerkin methods for initial-value problems, *Numerische, insbesondere approximations-theoretische Behandlung von Funktionalgleichungen*. LN in Math., Vol. 333, Springer Verlag, New York, 1973, pp. 243–262. MR 56#17147.
172. Thomée, V.: Some error estimates in Galerkin methods for parabolic equations, *Mathematical Aspects of Finite Element Methods* (Proc. Conf., Rome, 1975), LN in Math., Vol. 606, Springer-Verlag, Berlin, Heidelberg, New York, 1977, pp. 353–362. MR 58#31928.
173. Thomée, V.: High order local approximations to derivatives in the finite element method, *Math. Comp.* **31** (1977), pp. 652–660. MR 55#11572.
174. Thomée, V.: Galerkin-finite element method for parabolic equations, *Bull. Iranian Math. Soc.* **10** (1978), 3–17. MR 81b:65107.
175. Thomée, V.: Galerkin-finite element methods for parabolic equations (*Proc. Internat. Congress of Mathematicians, Helsinki, 1978*). Acad. Sci. Fennica, Helsinki, 1980, pp. 943–952. MR 81i:65100.
176. Thomée, V.: Negative norm estimates and superconvergence in Galerkin methods for parabolic problems, *Math. Comp.* **34** (1980), 93–113. MR 81a:65092.
177. Thomée, V.: *Galerkin Finite Element Methods for Parabolic Problems*, LN in Math., Vol. 1054, Springer-Verlag, Berlin, 1984. MR 86k:65006.

178. Thomée V. and Wendroff, B.: Convergence estimates for Galerkin method for variable coefficient initial value problems, *SIAM J. Numer. Anal.* **11** (1974), 1059–1068. MR 51#7309.
179. Thomée, V. and Westergren, B.: Elliptic difference equations and interior regularity, *Numer. Math.* **11** (1968), 196–210. MR 36#7347.
180. Vacek, J.: Dual variational principle for an elliptic partial differential equation, *Apl. Mat.* **21** (1976), 5–27. MR 54#716.
181. Veryard, O. A.: Problems associated with the convergence of isoparametric and miscoparametric finite elements, MSc thesis, Univ. of Wales, 1971.
182. Volk, W.: Globale Superconvergenz bei der Lösung von Differentialaufgaben, *Numer. Math.* **48** (1986), 617–625.
183. Wendland, W. L.: Boundary element methods and their asymptotic convergence, *Theoretical Acoustics and Numerical Techniques*, CISM Courses No. 277, Springer, Wien, New York, 1983, pp. 135–216.
184. Wendland, W. L., Stephan, E., and Hsiao, G. C: On the integral equation method for the plane mixed boundary value problem of the Laplacian, *Math. Methods Appl. Sci.* **1** (1979), 265–321. MR 82e:31003.
185. Westergren, B.: Interior estimates for elliptic systems of difference equations, PhD thesis, Univ. of Göteborg, 1982.
186. Wheeler, M. F.: An optimal L_∞ error estimate for Galerkin approximations to solutions of two-point boundary value problems, *SIAM J. Numer. Anal.* **10** (1973), 914–917. MR 49#8399.
187. Wheeler, M. F.: Galerkin procedure for estimating the flux for two-point boundary value problems, *SIAM J. Numer. Anal.* **11** (1974), 764–768. MR 52#4644.
188. Wheeler, M. F.: A C^0-collocation-finite element method for two-point boundary value problems and one space dimensional parabolic problems, *SIAM J. Numer. Anal.* **14** (1977), 71–90. MR 56#13667.
189. Wielgosz, C.: Exact results given by finite element methods in mechanics, *J. Méc. Appl.* **1** (1982), 323–329.
190. Winther, R.: Some superlinear convergent results for the conjugate gradient method, *SIAM J. Numer. Anal.* **17** (1980), 14–17. MR 81k: 65060.
191. Winther, R.: A stable finite element method for initial-boundary value problems for first-order hyperbolic systems, *Math. Comp.* **36** (1981), 65–86. MR 81m:65181.
192. Winther, R.: A finite element method for a version of the Boussinesq equation, *SIAM J. Numer. Anal.* **19** (1982), 561–570. MR 83f:65184.
193. Zhu, Q. D.: A superconvergence result for the finite element method, *Numer. Math. J. Chinese Univ.* **3** (1981), 50–55. MR 82f:65121.
194. Zhu, Q. D.: Natural inner superconvergence for the finite element method (*Proc. China–France Sympos. on Finite Element Methods, Beijing*, 1982), Gordon and Breach, New York, 1983, pp. 935–960. MR 85h:65253.
195. Zhu, Q. D.: Uniform superconvergence estimates of derivates for the finite element method, *Numer. Math. J. Chinese Univ.* **5** (1983), 311–318. MR 85k:65096.
196. Zienkiewicz, O. C. and Cheung, Y. K., *The Finite Element Method in Structural and Continuum Mechanics*, McGraw-Hill, London, 1967. MR 47#4518.
197. Zlámal, M.: Some superconvergence results in the finite element method, *Mathematical Aspects of Finite Element Methods* (Proc. Conf., Rome, 1975), LN in Math. Vol. 606, Springer-Verlag, Berlin, Heidelberg, New York, 1977, pp. 353–362. MR 58#8365.
198. Zlámal, M.: Superconvergence and reduced integration in the finite element method, *Math. Comp.* **32** (1978), 663–685. MR 58#13794.
199. Zlámal, M.: Superconvergence of gradients in the finite element method, *Variational-Difference Methods in Mathematical Physics* (Proc. Third. Conf., Novosibirsk, 1977), Akad. Nauk. SSSR Sibirsk. Otdel., Vychisl. Centr, Novosibirsk, 1978, pp. 15–22. MR 82e:65116.
200. Zlámal, M.: Superconvergence of the gradient of finite element solutions (*Wiss. Z. Hochschule Architektur Bauwesen Weimar*, 1979). Weimar, 1979, pp. 375–380.