# 1

# Introduction

We begin by showing how densities may arise from the operation of a one-dimensional discrete time system and how the study of such systems can be facilitated by the use of densities.

If a given system operates on a density as an initial condition, rather than on a single point, then successive densities are given by a linear integral operator, known as the Frobenius–Perron operator. Our main objective in this chapter is to offer an intuitive interpretation of the Frobenius–Perron operator. We make no attempt to be mathematically precise in either our language or our arguments.

The precise definition of the Frobenius–Perron operator is left to Chapter 3, while the measure-theoretic background necessary for this definition is presented in Chapter 2.

## 1.1 A simple system generating a density of states

One of the most studied systems capable of generating a density of states is that defined by the quadratic map

$$S(x) = \alpha x(1 - x) \qquad \text{for } 0 \leqslant x \leqslant 1. \tag{1.1.1}$$

We assume that $\alpha = 4$ so $S$ maps the closed unit interval $[0, 1]$ onto itself. This is also expressed by the saying that the **state** (or **phase**) **space** of the system is $[0, 1]$. The graph of this transformation is shown in Figure 1.1.1a.

Having defined $S$ we may pick an initial point $x^0 \in [0, 1]$ so that the successive states of our system at times $1, 2, \ldots$ are given by the trajectory

$$x^0, S(x^0), S^2(x^0) = S(S(x^0)), \ldots. \tag{1.1.2}$$

A typical trajectory corresponding to a given initial state is shown in Figure 1.1.1b. It is visibly erratic or chaotic, as is the case for almost all $x^0$. What is even worse is that the trajectory is significantly altered by a slight change in the initial state, as shown in Figure 1.1.1c for an initial state differing by $10^{-3}$ from that used to generate Figure 1.1.1b. Thus we are seemingly faced with a real problem in characterizing systems with behaviors like that of (1.1.1).

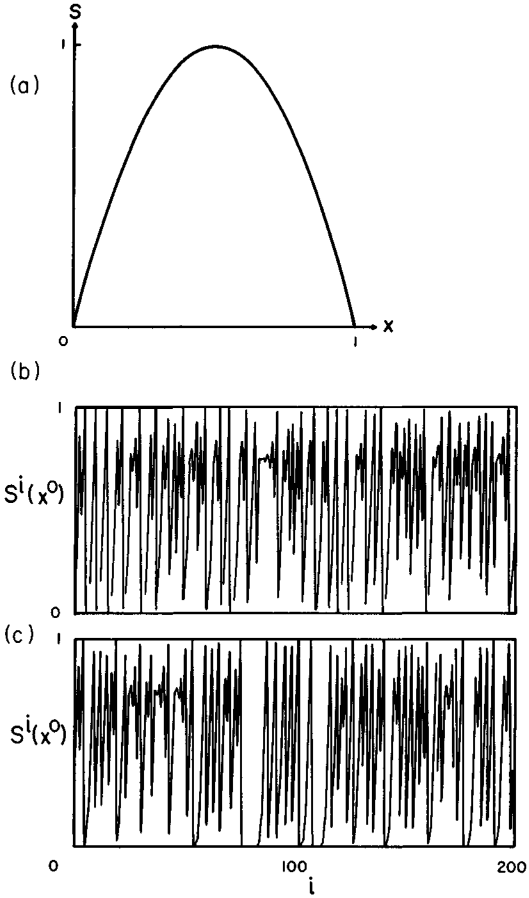


Figure 1.1.1. The quadratic transformation (1.1.1) with $\alpha = 4$ is shown in (a). In (b) we show the trajectory (1.1.2) determined by (1.1.1) with $x^0 = \pi/10$. Panel (c) illustrates the sensitive dependence of trajectories on initial conditions by using $x^0 = (\pi/10) + 0.001$. In (b) and (c), successive points on the trajectories have been connected by lines for clarity of presentation.

By taking a clue from other areas, we might construct a histogram to display the frequency with which states along a trajectory fall into given regions of the state space. This is done in the following way. Imagine that we divide the state space $[0, 1]$ into $n$ discrete nonintersecting intervals so the $i$th interval is (we neglect the end point 1)

$$[(i-1)/n, i/n) \qquad i = 1, \ldots, n.$$

Next we pick an initial system state $x^0$ and calculate a long trajectory

$$x^0, S(x^0), S^2(x^0), \ldots, S^N(x^0)$$

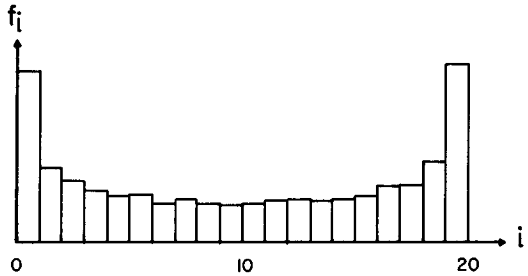


Figure 1.1.2. The histogram constructed according to equation (1.1.3) with $n = 20$, $N = 5000$, and $x^0 = \pi/10$.

of length $N$ where $N >> n$. Then it is straightforward to determine the fraction, call it $f_i$, of the $N$-system states that is in the $i$th interval from

$$f_i = \frac{n}{N}\,[\text{number of } S^j(x^0) \in [(i-1)/n,\ i/n),\ j = 1, \ldots, N]. \tag{1.1.3}$$

We have carried out this procedure for the initial state used to generate the trajectory of Figure 1.1.1b by taking $n = 20$ and using a trajectory of length $N = 5000$. The result is shown in Figure 1.1.2. There is a surprising symmetry in the result, for the states are clearly most concentrated near 0 and 1 with a minimum at $\frac{1}{2}$. Repeating this process for other initial states leads, in general, to the same result. Thus, in spite of the sensitivity of trajectories to initial states, this is not *usually* reflected in the distribution of states within long trajectories.

However, for certain select initial states, different behaviors may occur. For some initial conditions the trajectory might arrive at one of the fixed points of equation (1.1.1), that is, a point $x_*$ satisfying

$$x_* = S(x_*).$$

(For the quadratic map with $\alpha = 4$ there are two fixed points, $x_* = 0$ and $x_* = \frac{3}{4}$.) If this happens the trajectory will then have the constant value $x_*$ forever after, as illustrated in Figure 1.1.3a. Alternately, for some other initial states the trajectory might become periodic (see Figure 1.1.3b) and also fail to exhibit the irregular behavior of Figures 1.1.1b and c. The worst part about these exceptional behaviors is that we have no a priori way of predicting which initial states will lead to them.

In the next section we illustrate an alternative approach to avoid these problems.

***Remark 1.1.1.*** Map (1.1.1) has attracted the attention of many mathematicians. Ulam and von Neumann [1947] examined the case when $\alpha = 4$, whereas Ruelle

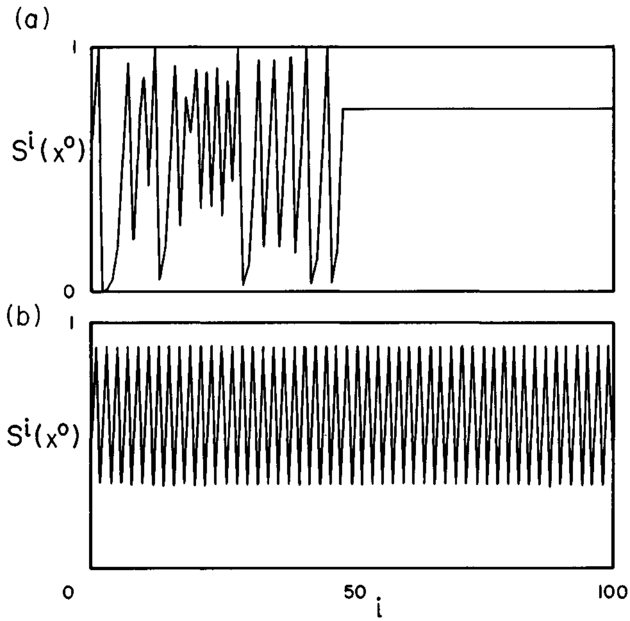


Figure 1.1.3. Exceptional initial conditions may confound the study of transformations via trajectories. In (a) we show how an initial condition on the quadratic transformation (1.1.1) with $\alpha = 4$ can lead to a fixed point $x_*$ of $S$. In (b) we see that another initial condition leads to a period 2 trajectory, although all other characteristics of $S$ are the same.

[1977], Jakobson [1978], Pianigiani [1979], Collet and Eckmann [1980] and Misiurewicz [1981] have studied its properties for values of $\alpha < 4$. May [1974], Smale and Williams [1976], and Lasota and Mackey [1980], among others, have examined the applicability of (1.1.1) and similar maps to biological population growth problems. Interesting properties related to the existence of periodic orbits in the transformation (1.1.1) follow from the classical results of Šarkovskiĭ [1964]. □

## 1.2 The evolution of densities: an intuitive point of view

The problems that we pointed out in the previous section can be partially circumvented by abandoning the study of individual trajectories in favor of an examination of the flow of densities. In this section we give a heuristic introduction to this concept.

Again we assume that we have a transformation $S\colon [0,1] \to [0,1]$ (a shorthand way of saying $S$ maps $[0,1]$ onto itself) and pick a large number $N$ of initial states

$$x_1^0, x_2^0, \ldots, x_N^0.$$

To each of these states we apply the map $S$, thereby obtaining $N$ new states denoted by

$$x_1^1 = S(x_1^0), x_2^1 = S(x_2^0), \ldots, x_N^1 = S(x_N^0).$$

To define what we mean by the densities of the initial and final states, it is helpful to introduce the concept of the **characteristic** (or **indicator**) **function** for a set $\Delta$. This is simply defined by

$$1_\Delta(x) = \begin{cases} 1 & \text{if } x \in \Delta \\ 0 & \text{if } x \notin \Delta . \end{cases}$$

Loosely speaking, we say that a function $f_0(x)$ is the **density function** for the initial states $x_1^0, \ldots, x_N^0$ if, for every (not too small) interval $\Delta_0 \subset [0,1]$, we have

$$\int_{\Delta_0} f_0(u)\,du \simeq \frac{1}{N}\sum_{j=1}^{N} 1_{\Delta_0}(x_j^0). \tag{1.2.1}$$

Likewise, the density function $f_1(x)$ for the states $x_1^1, \ldots, x_N^1$ satisfies, for $\Delta \subset [0,1]$,

$$\int_{\Delta} f_1(u)\,du \simeq \frac{1}{N}\sum_{j=1}^{N} 1_{\Delta}(x_j^1). \tag{1.2.2}$$

We want to find a relationship between $f_1$ and $f_0$.

To do this it is necessary to introduce the notion of the **counterimage** of an interval $\Delta \subset [0,1]$ under the operation of the map $S$. This is the set of all points that will be in $\Delta$ after one application of $S$, or

$$S^{-1}(\Delta) = \{x\colon S(x) \in \Delta\}.$$

As illustrated in Figure 1.2.1, for the quadratic map considered in Section 1.1, the counterimage of an interval will be the union of two intervals.

Now note that for any $\Delta \subset [0,1]$

$$x_j^1 \in \Delta \qquad \text{if and only if } x_j^0 \in S^{-1}(\Delta).$$

We thus have the very useful relation

$$1_\Delta(S(x)) = 1_{S^{-1}(\Delta)}(x). \tag{1.2.3}$$

With (1.2.3) we may rewrite equation (1.2.2) as

$$\int_{\Delta} f_1(u)\,du \simeq \frac{1}{N}\sum_{j=1}^{N} 1_{S^{-1}(\Delta)}(x_j^0) \tag{1.2.4}$$

Because $\Delta_0$ and $\Delta$ have been arbitrary up to this point, we simply pick $\Delta_0 = S^{-1}(\Delta)$. With this choice the right-hand sides of (1.2.1) and (1.2.4) are

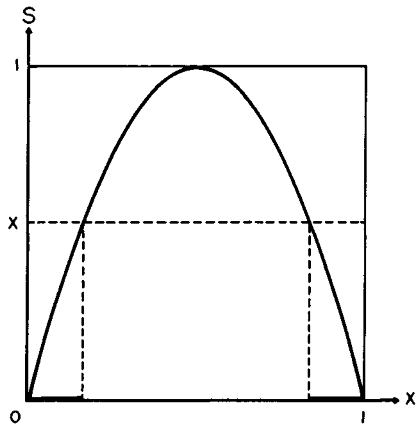


Figure 1.2.1. The counterimage of the set $[0,x]$ under the quadratic transformation consists of the union of the two sets denoted by the heavy lines on the $x$-axis.

equal and therefore

$$\int_{\Delta} f_1(u)\,du = \int_{S^{-1}(\Delta)} f_0(u)\,du. \tag{1.2.5}$$

This is the relationship that we sought between $f_0$ and $f_1$, and it tells us how a density of initial states $f_0$ will be transformed by a given map $S$ into a new density $f_1$.

If $\Delta$ is an interval, say $\Delta = [a,x]$, then we can obtain an explicit representation for $f_1$. In this case, equation (1.2.5) becomes

$$\int_a^x f_1(u)\,du = \int_{S^{-1}([a,x])} f_0(u)\,du,$$

and differentiating with respect to $x$ gives

$$f_1(x) = \frac{d}{dx}\int_{S^{-1}([a,x])} f_0(u)\,du. \tag{1.2.6}$$

It is clear that $f_1$ will depend on $f_0$. This is usually indicated by writing $f_1 = Pf_0$, so that (1.2.6) becomes

$$Pf(x) = \frac{d}{dx}\int_{S^{-1}([a,x])} f(u)\,du \tag{1.2.7}$$

(we have dropped the subscript on $f_0$ as it is arbitrary). Equation (1.2.7) explicitly defines the **Frobenius–Perron operator** $P$ corresponding to the transformation $S$; it is very useful for studying the evolution of densities.

To illustrate the utility of (1.2.7) and, incidentally, the Frobenius–Perron operator concept, we return to the quadratic map $S(x) = 4x(1-x)$ of the preceding section. To apply (1.2.7) it is obvious that we need an analytic formula for the counterimage of the interval $[0,x]$. Reference to Figure 1.2.1 shows that the end points of the two intervals constituting $S^{-1}([0,x])$ are very simply calculated by solving a quadratic equation. Thus

$$S^{-1}([0,x]) = [0, \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-x}\,] \cup [\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-x}\,, 1].$$

With this, equation (1.2.7) becomes

$$Pf(x) = \frac{d}{dx}\int_0^{1/2-1/2\sqrt{1-x}} f(u)\,du + \frac{d}{dx}\int_{1/2+1/2\sqrt{1-x}}^{1} f(u)\,du,$$

or, after carrying out the indicated differentiation,

$$Pf(x) = \frac{1}{4\sqrt{1-x}}\{f(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-x}\,) + f(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1-x}\,)\}. \tag{1.2.8}$$

This equation is an explicit formula for the Frobenius–Perron operator corresponding to the quadratic transformation and will tell us how $S$ transforms a given density $f$ into a new density $Pf$. Clearly the relationship can be used in an iterative fashion.

To see how this equation works, pick an initial density $f(x) \equiv 1$ for $x \in [0,1]$. Then, since both terms inside the braces in (1.2.8) are constant, a simple calculation gives

$$Pf(x) = \frac{1}{2\sqrt{1-x}}. \tag{1.2.9}$$

Now substitute this expression for $Pf$ in place of $f$ on the right-hand side of (1.2.8) to give

$$\begin{aligned}P(Pf(x)) &= P^2f(x)\\ &= \frac{1}{4\sqrt{1-x}}\left\{\frac{1}{2\sqrt{1-\frac{1}{2}+\frac{1}{2}\sqrt{1-x}}} + \frac{1}{2\sqrt{1-\frac{1}{2}-\frac{1}{2}\sqrt{1-x}}}\right\}\\ &= \frac{\sqrt{2}}{8\sqrt{1-x}}\left\{\frac{1}{\sqrt{1+\sqrt{1-x}}} + \frac{1}{\sqrt{1-\sqrt{1-x}}}\right\}.\end{aligned} \tag{1.2.10}$$

In Figure 1.2.2 we have plotted $f(x) \equiv 1$, $Pf(x)$ given by (1.2.9), and $P^2f(x)$ given by (1.2.10) to show how rapidly they seem to approach a limiting density. Actually, this limiting density is given by

$$f_*(x) = \frac{1}{\pi\sqrt{x(1-x)}}. \tag{1.2.11}$$

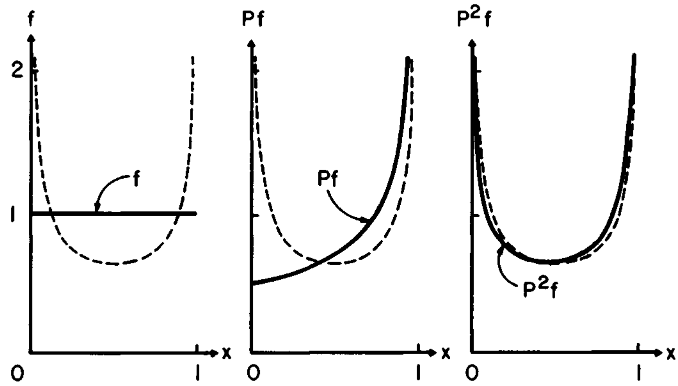


Figure 1.2.2. The evolution of the constant density $f(x) = 1$, $x \in [0,1]$, by the Frobenius–Perron operator corresponding to the quadratic transformation. Compare the rapid and regular approach of $P^nf$ to the density given in equation (1.2.11) (shown as a dashed line) with the sustained irregularity shown by the trajectories in Figure 1.1.1.

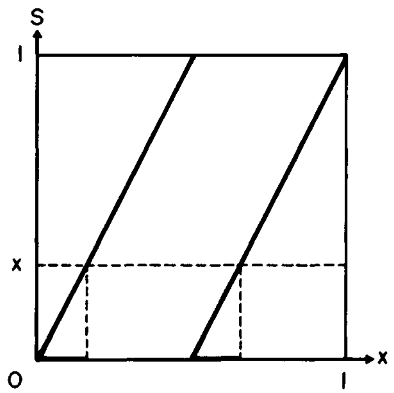


Figure 1.2.3. The dyadic transformation is a special case of the $r$-adic transformation. The heavy lines along the $x$-axis mark the two components of the counterimage of the interval $[0,x]$.

If $f_*$ is really the ultimate limit of $P^nf$ as $n \to \infty$, then we should find that $Pf_* \equiv f_*$ when we substitute into equation (1.2.8) for the Frobenius–Perron operator. A few elementary calculations confirm this. Note also the close similarity between the graph of $f_*$ in Figure 1.2.2 and the histogram of Figure 1.1.2. Later we will show that for the quadratic map the density of states along a trajectory approaches the same unique limiting density $f_*$ as the iterates of densities approach.

***Example 1.2.1.*** Consider the transformation $S\colon [0,1] \to [0,1]$ given by

$$S(x) = rx \qquad (\text{mod } 1)\,, \tag{1.2.12}$$

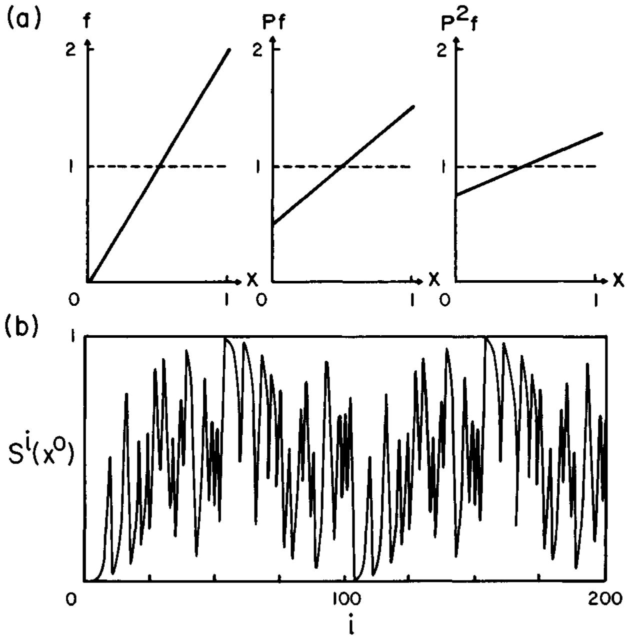


Figure 1.2.4. Dynamics of the dyadic transformation. (a) With an initial density $f(x) = 2x$, $x \in [0,1]$, successive applications of the Frobenius–Perron operator corresponding to the dyadic transformation result in densities approaching $f_* \equiv 1$, $x \in [0,1]$. (b) A trajectory calculated from the dyadic transformation with $x^0 \cong 0.0005$. Compare the irregularity of this trajectory with the smooth approach of the densities in (a) to a limit.

where $r$ is an integer. This expression is customarily called the ***r*-adic transformation** and is illustrated in Figure 1.2.3 for $r = 2$ (the **dyadic transformation**).

Pick an interval $[0,x] \subset [0,1]$ so that the counterimage of $[0,x]$ under $S$ is given by

$$S^{-1}([0,x]) = \bigcup_{i=0}^{r-1} \left[\frac{i}{r}, \frac{i}{r} + \frac{x}{r}\right]$$

and the Frobenius–Perron operator is thus

$$Pf(x) = \frac{d}{dx} \sum_{i=0}^{r-1} \int_{i/r}^{i/r+x/r} f(u)\,du = \frac{1}{r} \sum_{i=0}^{r-1} f\left(\frac{i}{r} + \frac{x}{r}\right). \tag{1.2.13}$$

This formula for the Frobenius–Perron operator corresponding to the $r$-adic transformation (1.2.12) shows again that densities $f$ will be rapidly smoothed by $P$, as can be seen in Figure 1.2.4a for an initial density $f(x) = 2x$, $x \in [0,1]$. It is clear that the density $P^n f(x)$ rapidly approaches the constant distribution $f_*(x) \equiv 1$, $x \in [0,1]$. Indeed, it is trivial to show that $P1 \equiv 1$. This behavior should be contrasted with that of a typical trajectory (Figure 1.2.4b). □

## 1.3 Trajectories versus densities

In closing this chapter we offer a qualitative examination of the behavior of two transformations from both the flow of trajectories and densities viewpoints.

Let $R$ denote the entire real line, that is, $R = \{x: -\infty < x < \infty\}$, and consider the transformation $S: R \to R$ defined by

$$S(x) = \alpha x, \qquad \alpha > 0. \tag{1.3.1}$$

Our study of transformations confined to the unit interval of Section 1.2 does not affect expression (1.2.7) for the Frobenius–Perron operator. Thus (1.3.1) has the associated Frobenius–Perron operator

$$Pf(x) = (1/\alpha)f(x/\alpha).$$

We first examine the behavior of $S$ for $\alpha > 1$. Since $S^n(x) = \alpha^n x$, we see that, for $\alpha > 1$,

$$\lim_{n\to\infty} |S^n(x)| = \infty, \qquad x \neq 0,$$

and thus the iterates $S^n(x)$ escape from any bounded interval.

This behavior is in total agreement with the behavior deduced from the flow of densities. To see this note that

$$P^n f(x) = (1/\alpha^n)f(x/\alpha^n).$$

By the qualitative definition of the Frobenius–Perron operator of the previous section, we have, for any bounded interval $[-A, A] \subset R$,

$$\int_{-A}^{A} P^n f(x)\,dx = \int_{-A/\alpha^n}^{A/\alpha^n} f(x)\,dx.$$

Since $\alpha > 1$,

$$\lim_{n\to\infty} \int_{-A}^{A} P^n f(x)\,dx = 0$$

and so, under the operation of $S$, densities are reduced to zero on every finite interval when $\alpha > 1$.

Conversely, for $\alpha < 1$,

$$\lim_{n\to\infty} |S^n(x)| = 0$$

for every $x \in R$, and therefore all trajectories converge to zero. Furthermore, for every neighborhood $(-\varepsilon, \varepsilon)$ of zero, we have

$$\lim_{n\to\infty} \int_{-\varepsilon}^{\varepsilon} P^n f(x)\,dx = \lim_{n\to\infty} \int_{-\varepsilon/\alpha^n}^{\varepsilon/\alpha^n} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1,$$

so in this case all densities are concentrated in an arbitrarily small neighborhood

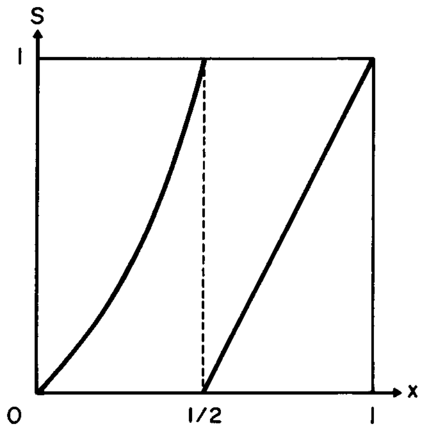


Figure 1.3.1. The transformation $S(x)$, defined by equation (1.3.2), has a single weak repelling point at $x = 0$.

of zero. Thus, again, the behaviors of trajectories and densities seem to be in accord.

However, it is not always the case that the behavior of trajectories and densities seem to be in agreement. This may be simply illustrated by what we call the **paradox of the weak repellor.** In Remark 6.2.1 we consider the transformation $S\colon [0,1] \to [0,1]$ defined by

$$S(x) = \begin{cases} x/(1-x) & \text{for } x \in [0, \frac{1}{2}] \\ 2x - 1 & \text{for } x \in (\frac{1}{2}, 1], \end{cases} \tag{1.3.2}$$

(see Figure 1.3.1). There we prove that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \int_\varepsilon^1 P^n f(x)\, dx = 0 .$$

Thus, since $P^n f$ is a density,

$$\lim_{n\to\infty} \int_0^\varepsilon P^n f(x)\, dx = 1 ,$$

and all densities are concentrated in an arbitrarily small neighborhood of zero. This behavior is graphically illustrated in Figure 1.3.2a.

If one picks an initial point $x_0 > 0$ very close to zero (see Figure 1.3.2b), then, as long as $S^n(x_0) \in (0, \frac{1}{2}]$, we have

$$S^n(x_0) = x_0/(1 - nx_0) \geqslant \alpha^n x_0$$

where $\alpha = 1/(1 - x_0) > 1$. Thus initially, for small $x_0$, this transformation behaves much like transformation (1.3.1), and the behavior of the trajectory near

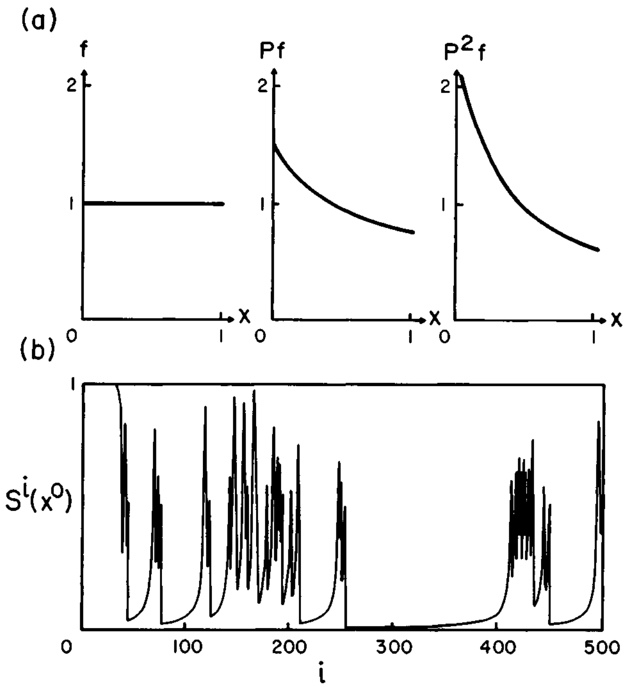


Figure 1.3.2. Dynamics of the weak repellor defined by (1.3.2). (a) The evolution $P^n f$ of an initial distribution $f(x) = 1$, $x \in [0, 1]$. (b) The trajectory originating from an initial point $x^0 \cong 0.25$.

zero apparently contradicts that expected from the behavior of the densities.

This paradox is more apparent than real and may be easily understood. First, note that even though all trajectories are repelled from zero (zero is a repellor), once a trajectory is ejected from $(0, \frac{1}{2}]$ it is quickly reinjected into $(0, \frac{1}{2}]$ from $(\frac{1}{2}, 1]$. Thus zero is a "weak repellor." The second essential point to note is that the speed with which any trajectory leaves a small neighborhood of zero is small; it is given by

$$S^n(x_0) - S^{n-1}(x_0) = \frac{x_0^2}{(1 - nx_0)[1 - (n - 1)x_0]}.$$

Thus, starting with many initial points, as $n$ increases we will see the progressive accumulation of more and more points near zero. This is precisely the behavior predicted by examining the flow of densities.

Although our comments in this chapter lack mathematical rigor, they offer some insight into the power of looking at the evolution of densities under the operation of deterministic transformations. The next two chapters are devoted to introducing the mathematical concepts required for a precise treatment of this problem.