Preface

This book is about densities. In the history of science, the concept of densities emerged only recently as attempts were made to provide unifying descriptions of phenomena that appeared to be statistical in nature. Thus, for example, the introduction of the Maxwellian velocity distribution rapidly led to a unification of dilute gas theory; quantum mechanics developed from attempts to justify Planck's ad hoc derivation of the equation for the density of blackbody radiation; and the field of human demography grew rapidly after the introduction of the Gompertzian age distribution.

From these and many other examples, as well as the formal development of probability and statistics, we have come to associate the appearance of densities with the description of large systems containing inherent elements of uncertainty. Viewed from this perspective one might find it surprising to pose the question: "What is the smallest number of elements that a system must have, and how much uncertainty must exist, before a description in terms of densities becomes useful and/or necessary?" The answer is surprising, and runs counter to the intuition of many. A one-dimensional system containing only one object whose dynamics are completely deterministic (no uncertainty) can generate a density of states! This fact has only become apparent in the past half-century due to the pioneering work of E. Borel [1909], A. Rényi [1957], and S. Ulam and J. von Neumann. These results, however, are not generally known outside that small group of mathematicians working in ergodic theory.

The past few years have witnessed an explosive growth in interest in physical, biological, and economic systems that could be profitably studied using densities. Due to the general inaccessibility of the mathematical literature to the non-mathematician, there has been little diffusion of the concepts and techniques from ergodic theory into the study of these "chaotic" systems. This book attempts to bridge that gap.

Here we give a unified treatment of a variety of mathematical systems generating densities, ranging from one-dimensional discrete time transformations through continuous time systems described by integro-partial-differential equations. We have drawn examples from a variety of the sciences to illustrate the utility of the techniques we present. Although the range of these examples is not encyclopedic, we feel that the ideas presented here may prove useful in a number of the applied sciences.

This book was organized and written to be accessible to scientists with a knowledge of advanced calculus and differential equations. In various places, basic concepts from measure theory, ergodic theory, the geometry of manifolds, partial differential equations, probability theory and Markov processes, and stochastic integrals and differential equations are introduced. This material is presented only as needed, rather than as a discrete unit at the beginning of the book where we felt it would form an almost insurmountable hurdle to all but the most persistent. However, in spite of our presentation of all the necessary concepts, we have not attempted to offer a compendium of the existing mathematical literature.

The one mathematical technique that touches every area dealt with is the use of the lower-bound function (first introduced in Chapter 5) for proving the existence and uniqueness of densities evolving under the action of a variety of systems. This, we feel, offers some partial unification of results from different parts of applied ergodic theory.

The first time an important concept is presented, its name is given in bold type. The end of the proof of a theorem, corollary, or proposition is marked with a ■; the end of a remark or example is denoted by a □.

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A. L.
M. C. M.