CHAPTER 7

A MIXED FINITE ELEMENT METHOD

Introduction

In this chapter, we consider the problem of approximating the solution of the *biharmonic problem*: Find $u \in H_0^2(\Omega)$, $\Omega \subset \mathbf{R}^2$, such that

$$J(u) = \inf_{v \in H_0^2(\Omega)} J(v), \quad \text{with} \quad J(v) = \frac{1}{2}\int_\Omega |\Delta v|^2 \,\mathrm{d}x - \int_\Omega fv \,\mathrm{d}x.$$

Our objective is to study a method *based on a different variational formulation* of the biharmonic problem (it being implicitly understood that the above variational formulation is the standard one). Such methods fall themselves into several categories (cf. the discussion in the section "Additional Bibliography and Comments" at the end of this chapter), and it is the purpose of this chapter to study one of these, of the so-called *mixed* type. Basically, it corresponds to a variational formulation where the function u is the first argument of the minimum (u, φ) of a new functional. In this fashion, we shall directly get approximations not only of the solution u, but also of the second argument φ. Since this function φ turns out to be $-\Delta u$ in the present case, this approach is particularly appropriate for the study of two-dimensional steady-state flows, where $-\Delta u$ represents the *vorticity*.

Thus our first task in Section 7.1 is to construct a functional $\mathcal{J}$ and a space $\mathcal{V}$ such that (Theorem 7.1.2)

$$\mathcal{J}(u, -\Delta u) = \inf_{(v,\psi)\in\mathcal{V}} \mathcal{J}(v, \psi).$$

The space $\mathcal{V}$ consists of pairs $(v, \psi) \in H_0^1(\Omega) \times L^2(\Omega)$ which satisfy specific linear relations of the form $\beta((v, \psi), \mu) = 0$ for all functions $\mu \in H^1(\Omega)$.

Next, this problem is discretized in a natural way: Given a finite element space X_h contained in the space $H^1(\Omega)$, one looks for a pair

$(u_h, \varphi_h) \in \mathcal{V}_h$ such that

$$\mathcal{J}(u_h, \varphi_h) = \inf_{(v_h, \psi_h) \in \mathcal{V}_h} \mathcal{J}(v_h, \psi_h),$$

where the space $\mathcal{V}_h$ consists of those pairs $(v_h, \psi_h) \in X_{0h} \times X_h$ which satisfy linear relations of the form $\beta((v_h, \psi_h), \mu_h) = 0$ for all functions $\mu_h \in X_h$.

The major portion of Section 7.1 is then devoted to the study of convergence (Theorems 7.1.5 and 7.1.6): Our main conclusion is that, if the inclusions $P_k(K) \subset P_K$, $K \in \mathcal{T}_h$, hold, the error estimate

$$\|u - u_h\|_{1,\Omega} + |\Delta u + \varphi_h|_{0,\Omega} = O(h^{k-1})$$

holds. The main difficulty in this error analysis is that, in general, the space $\mathcal{V}_h$ is *not* a subspace of the space $\mathcal{V}$ (if this were the case, it would suffice to use the convergence analysis valid for conforming methods).

The advantages and drawbacks inherent in this method are easily understood: The main advantage is that *it suffices to use finite elements of class* $\mathscr{C}^0$, whereas finite elements of class $\mathscr{C}^1$ would be required for conforming methods. Another advantage (from the point of view of fluid mechanics) is that the present method not only yields a continuous approximation of the function u, but also of the vorticity $-\Delta u$, whereas a standard approximation using finite elements of class $\mathscr{C}^1$ would result in a discontinuous approximation $-\Delta u_h$ of the vorticity (which, in addition, needs to be computed).

The major drawback is that the computation of the discrete solution (u_h, φ_h) requires the solution of a *constrained* minimization problem, since the functions $v_h \in X_{0h}$ and $\psi_h \in X_h$ do not vary independently from one another. It is the object of Section 7.2 to show how such a problem may be solved, using *duality techniques.*

The basic idea consists in introducing an appropriate space $\mathcal{M}_h \subset X_h$ of "multipliers" and then in applying *Uzawa's method* for solving the saddle-point equations (cf. Theorem 7.2.2) of the Lagrangian associated with the present variational formulation. The convergence of Uzawa's method is established in Theorem 7.2.5.

In the process, we find an answer to a problem which has been often considered for the biharmonic problem and its various possible discretizations: We show (Theorem 7.2.4) that, *in this particular case, Uzawa's method amounts to solving a sequence of discrete Dirichlet problems for the operator* $-\Delta$.

Therefore we have at our disposal a method for approximating the solution of a fourth-order problem *which uses the same finite element programs as those needed for second-order problems.*

7.1. A mixed finite element method for the biharmonic problem

Another variational formulation of the biharmonic problem

Consider the variational problem which corresponds to the following data:

$$\begin{cases} V = H_0^2(\Omega), \\ a(u, v) = \int_\Omega \Delta u \Delta v \, \mathrm{d}x, \\ f(v) = \int_\Omega f v \, \mathrm{d}x, \end{cases} \tag{7.1.1}$$

where the set $\bar{\Omega}$ is a convex polygonal subset of $\mathbf{R}^2$ and the function f belongs to the space $L^2(\Omega)$. We recognize here the *biharmonic problem,* whose solution $u \in H_0^2(\Omega)$ also satisfies

$$J(u) = \inf_{v \in H_0^2(\Omega)} J(v), \tag{7.1.2}$$

with

$$J(v) = \frac{1}{2}\int_\Omega |\Delta v|^2 \, \mathrm{d}x - \int_\Omega f v \, \mathrm{d}x. \tag{7.1.3}$$

Thus we may equivalently consider that we are minimizing the functional

$$\mathcal{J}(v, \psi) = \frac{1}{2}\int_\Omega |\psi|^2 \, \mathrm{d}x - \int_\Omega f v \, \mathrm{d}x, \tag{7.1.4}$$

over those pairs $(v, \psi) \in H_0^2(\Omega) \times L^2(\Omega)$ whose elements v and ψ are related through the equality $-\Delta v = \psi$. This observation is the basis for another variational formulation of the biharmonic problem (Theorem 7.1.2), which depends on the fact that *the space*

$$\{(v, \psi) \in H_0^2(\Omega) \times L^2(\Omega);\ -\Delta v = \psi\}$$

can be described in an alternate way, as we now show.

Theorem 7.1.1. *Define the space*

$$\mathcal{V} = \{(v, \psi) \in H_0^1(\Omega) \times L^2(\Omega);\ \forall \mu \in H^1(\Omega),\quad \beta((v, \psi), \mu) = 0\}, \tag{7.1.5}$$

where

$$\beta((v, \psi), \mu) = \int_\Omega \nabla v \cdot \nabla \mu \, \mathrm{d}x - \int_\Omega \psi\mu \, \mathrm{d}x. \tag{7.1.6}$$

Then the mapping

$$(v, \psi) \in \mathcal{V} \to |\psi|_{0,\Omega}$$

is a norm over the space $\mathcal{V}$, *which is equivalent to the product norm* $(v, \psi) \in \mathcal{V} \to (|v|_{1,\Omega}^2 + |\psi|_{0,\Omega}^2)^{1/2}$, *and which makes* $\mathcal{V}$ *a Hilbert space. In addition, we have*

$$\mathcal{V} = \{(v, \psi) \in H_0^2(\Omega) \times L^2(\Omega);\ -\Delta v = \psi\}.$$

Proof. Equipped with the product norm, the space $\mathcal{V}$ is a Hilbert space since it is a closed subspace of the space $H_0^1(\Omega) \times L^2(\Omega)$.

Let (v, ψ) be any element of the space $\mathcal{V}$. The particular choice $\mu = v$ in the definition (7.1.5) of this space gives

$$|v|_{1,\Omega}^2 = \int_\Omega \psi v \, \mathrm{d}x \leq C(\Omega)|\psi|_{0,\Omega}|v|_{1,\Omega},$$

where $C(\Omega)$ is the constant appearing in the Poincaré–Friedrichs inequality (cf. (1.2.2)). Therefore,

$$(|v|_{1,\Omega}^2 + |\psi|_{0,\Omega}^2)^{1/2} \leq ((C(\Omega))^2 + 1)^{1/2}|\psi|_{0,\Omega},$$

and the first assertion is proved.

Since the set Ω has a Lipschitz-continuous boundary Γ, the Green formula

$$\forall v \in H^2(\Omega), \quad \forall \mu \in H^1(\Omega),$$

$$\int_\Omega \nabla v \cdot \nabla \mu \, \mathrm{d}x = -\int_\Omega \Delta v \, \mu \, \mathrm{d}x + \int_\Gamma \partial_\nu v \, \mu \, \mathrm{d}\gamma \tag{7.1.7}$$

holds. Let then the functions $v \in H_0^2(\Omega)$ and $\psi \in L^2(\Omega)$ be related through $-\Delta v = \psi$. For any function $\mu \in H^1(\Omega)$, an application of Green's formula (7.1.7) shows that $\beta((v, \psi), \mu) = 0$, since $\partial_\nu v = 0$ on Γ.

Conversely, let the functions $v \in H_0^1(\Omega)$ and $\psi \in L^2(\Omega)$ satisfy

$\beta((v, \psi), \mu) = 0$ for all $\mu \in H^1(\Omega)$. In particular then, we have

$$\forall \mu \in H_0^1(\Omega), \quad \int_\Omega \nabla v \cdot \nabla \mu \, \mathrm{d}x = \int_\Omega \psi \mu \, \mathrm{d}x,$$

so that v appears as the solution of a homogeneous Dirichlet problem for the operator $-\Delta$ on the set Ω. Since the set $\bar{\Omega}$ is convex, such a second-order boundary value problem is regular, i.e., the function v is in the space $H^2(\Omega)$. Using Green's formula (7.1.7) with functions μ in the space $H_0^1(\Omega)$, we first deduce that $-\Delta v = \psi$, and using the same Green formula with functions μ in the space $H^1(\Omega)$, we next deduce that $\partial_\nu v = 0$ along Γ. □

Theorem 7.1.2. *Let $u \in H_0^2(\Omega)$ denote the solution of the minimization problem* (7.1.2). *Then we also have*

$$\mathcal{J}(u, -\Delta u) = \inf_{(v,\psi) \in \mathcal{V}} \mathcal{J}(v, \psi), \tag{7.1.8}$$

where the functional $\mathcal{J}$ and the space $\mathcal{V}$ are defined as in (7.1.4) *and* (7.1.5), *respectively. In addition, the pair $(u, -\Delta u) \in \mathcal{V}$ is the unique solution of the minimization problem* (7.1.8).

Proof. The symmetric bilinear form

$$((u, \varphi), (v, \psi)) \in \mathcal{V} \times \mathcal{V} \to \int_\Omega \varphi \psi \, \mathrm{d}x$$

is continuous and $\mathcal{V}$-elliptic (by Theorem 7.1.1), and the linear form

$$(v, \psi) \in \mathcal{V} \to \int_\Omega f v \, \mathrm{d}x$$

is continuous. Therefore the minimization problem: Find an element $(u^*, \varphi) \in \mathcal{V}$ such that

$$\mathcal{J}(u^*, \varphi) = \inf_{(v,\psi) \in \mathcal{V}} \mathcal{J}(v, \psi), \tag{7.1.9}$$

has one and only one solution, also solution of the variational equations

$$\forall (v, \psi) \in \mathcal{V}, \quad \int_\Omega \varphi \psi \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x. \tag{7.1.10}$$

Let us establish the relationship between this solution (u^*, φ) and the solution of problem (7.1.2). Since the pair (u^*, φ) is an element of the space $\mathcal{V}$, we deduce from Theorem 7.1.1 that the function u^* belongs to the space $H_0^2(\Omega)$ and that $-\Delta u^* = \varphi$. Applying again the same theorem in conjunction with relations (7.1.10), we find that

$$\forall v \in H_0^2(\Omega), \quad \int_\Omega \Delta u^* \Delta v \, dx = \int_\Omega f v \, dx,$$

and thus the function u^* coincides with the solution u of problem (7.1.2). □

The corresponding discrete problem. Abstract error estimate

We are now in a position to describe a discrete problem associated with this new variational formulation of the biharmonic problem.

Let there be given a finite element space X_h which satisfies the inclusion

$$X_h \subset H^1(\Omega). \tag{7.1.11}$$

We define as usual the finite element space

$$X_{0h} = \{v_h \in X_h;\ v_h = 0 \quad \text{on} \quad \Gamma\}, \tag{7.1.12}$$

and we let (compare with (7.1.5))

$$\mathcal{V}_h = \{(v_h, \psi_h) \in X_{0h} \times X_h;\ \forall \mu_h \in X_h, \quad \beta((v_h, \psi_h), \mu_h) = 0\}, \tag{7.1.13}$$

where the mapping $\beta((.,.),.)$ is defined as in (7.1.6).

Then, in analogy with (7.1.8), we define the *discrete problem* as follows: *Find an element* $(u_h, \varphi_h) \in \mathcal{V}_h$ *such that*

$$\mathcal{J}(u_h, \varphi_h) = \inf_{(v_h, \psi_h) \in \mathcal{V}_h} \mathcal{J}(v_h, \psi_h), \tag{7.1.14}$$

where $\mathcal{J}$ *is the functional defined in* (7.1.4).

Remark 7.1.1. It is thus realized that the *same* space X_h is used for the approximation of both spaces $H^1(\Omega)$ and $L^2(\Omega)$. It is indeed possible to develop a seemingly more general theory where another space, say Y_h, is used for approximating the space $L^2(\Omega)$, but eventually the advantage is nil: As shown in CIARLET & RAVIART (1974), one is naturally led, in the process of getting error estimates, to assume that the inclusion $Y_h \subset X_h$ holds, and this is precisely contrary to what one would have naturally expected. Besides, the assumption $X_h = Y_h$ yields significant simplifications in the developments to come. □

Theorem 7.1.3. *The discrete problem* (7.1.14) *has one and only one solution.*

Proof. Arguing as in the proof of Theorem 7.1.1, we deduce that the mapping

$$(v_h, \psi_h) \in \mathcal{V}_h \to |\psi_h|_{0,\Omega}$$

is a norm over the space $\mathcal{V}_h$. Thus, the existence and uniqueness of the solution of the discrete problem follows by an argument similar to that of Theorem 7.1.2. □

As a consequence of this result, the element $(u_h, \varphi_h) \in \mathcal{V}_h$ is also solution of the variational equations

$$\forall (v_h, \psi_h) \in \mathcal{V}_h, \quad \int_\Omega \varphi_h \psi_h \,\mathrm{d}x = \int_\Omega f v_h \,\mathrm{d}x. \tag{7.1.15}$$

We next begin our study of the convergence of this approximation process. As usual, we shall first establish an *abstract error estimate* (in two steps; cf. Theorems 7.1.4 and 7.1.5) and we shall then apply this to some typical finite element spaces (Theorem 7.1.6).

The abstract error estimate consists in getting an upper bound for the expression

$$|u - u_h|_{1,\Omega} + |\Delta u + \varphi_h|_{0,\Omega}$$

(recall that φ_h is an approximation of $-\Delta u$, whence the unusual sign in the second term). Notice that the above expression is a natural analogue in the present situation of the error in the norm $\|.\|_{2,\Omega}$ that arises in conforming methods.

As a first step towards getting the error estimate, we prove:

Theorem 7.1.4. *There exists a constant C independent of the space X_h such that*

$$|u - u_h|_{1,\Omega} + |\Delta u + \varphi_h|_{0,\Omega} \leq$$
$$\leq C\Big(\inf_{(v_h, \psi_h) \in \mathcal{V}_h} (|u - v_h|_{1,\Omega} + |\Delta u + \psi_h|_{0,\Omega}) + \inf_{\mu_h \in X_h} \|\Delta u + \mu_h\|_{1,\Omega}\Big). \tag{7.1.16}$$

Proof. Since the set Ω is convex, the solution u of problem (7.1.2) belongs to the space $H^3(\Omega)$. Thus we can write

$$\forall v \in \mathcal{D}(\Omega), \quad -\int_\Omega \nabla v \cdot \nabla(\Delta u)\,\mathrm{d}x = \int_\Omega \Delta v \Delta u\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x,$$

and consequently

$$\forall v \in H_0^1(\Omega), \quad -\int_\Omega \nabla v \cdot \nabla(\Delta u)\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x.$$

Using the definition (7.1.6) of the mapping $\beta((.\,,.),.)$, we have therefore shown that, given any function $v \in H_0^1(\Omega)$ and any function $\psi \in L^2(\Omega)$, we have

$$\beta((v, \psi), -\Delta u) = \int_\Omega fv\,\mathrm{d}x + \int_\Omega \psi\Delta u\,\mathrm{d}x. \tag{7.1.17}$$

Let then (v_h, ψ_h) be an arbitrary element of the space $\mathcal{V}_h$ and let μ_h be an arbitrary element of the space X_h. Using the definition (7.1.5) of the space $\mathcal{V}$, the variational equations (7.1.15) and relation (7.1.17), we obtain

$$\int_\Omega (\Delta u + \varphi_h)(\varphi_h - \psi_h)\,\mathrm{d}x = -\beta((u_h - v_h, \varphi_h - \psi_h), \Delta u + \mu_h).$$

From this equality, we deduce that

$$\left|\int_\Omega (\Delta u + \varphi_h)(\varphi_h - \psi_h)\,\mathrm{d}x\right| \leqslant D(\Omega)|\varphi_h - \psi_h|_{0,\Omega}\|\Delta u + \mu_h\|_{1,\Omega},$$

where the constant $D(\Omega)$ depends solely on the constant $C(\Omega)$ of the Poincaré–Friedrichs inequality (argue as in the beginning of the proof of Theorem 7.1.1). Using this inequality, we get

$$\begin{aligned}
|\varphi_h - \psi_h|_{0,\Omega}^2 &= \int_\Omega (\varphi_h - \psi_h)(\Delta u + \varphi_h)\,\mathrm{d}x \\
&\quad - \int_\Omega (\varphi_h - \psi_h)(\Delta u + \psi_h)\,\mathrm{d}x \\
&\leqslant D(\Omega)|\varphi_h - \psi_h|_{0,\Omega}\|\Delta u + \mu_h\|_{1,\Omega} \\
&\quad + |\varphi_h - \psi_h|_{0,\Omega}|\Delta u + \psi_h|_{0,\Omega},
\end{aligned}$$

and hence,

$$|\varphi_h - \psi_h|_{0,\Omega} \leqslant D(\Omega)\|\Delta u + \mu_h\|_{1,\Omega} + |\Delta u + \psi_h|_{0,\Omega}. \tag{7.1.18}$$

On the other hand, we have

$$\begin{aligned}|u-u_h|_{1,\Omega}+|\Delta u+\varphi_h|_{0,\Omega} &\le |u-v_h|_{1,\Omega}+|v_h-u_h|_{1,\Omega} \\ &\quad +|\Delta u+\psi_h|_{0,\Omega}+|\psi_h-\varphi_h|_{0,\Omega} \\ &\le |u-v_h|_{1,\Omega}+|\Delta u+\psi_h|_{0,\Omega} \\ &\quad +(1+C(\Omega))|\psi_h-\varphi_h|_{0,\Omega}. \end{aligned} \tag{7.1.19}$$

Upon combining inequalities (7.1.18) and (7.1.19), we obtain

$$\begin{aligned}|u-u_h|_{1,\Omega}+|\Delta u+\varphi_h|_{0,\Omega} &\le (|u-v_h|_{1,\Omega}+(2+C(\Omega))|\Delta u+\psi_h|_{0,\Omega}) \\ &\quad +(1+C(\Omega))D(\Omega)\|\Delta u+\mu_h\|_{1,\Omega},\end{aligned}$$

and the inequality of (7.1.16) follows. □

To apply Theorem 7.1.4, we have to estimate on the one hand the expression $\inf_{\mu_h\in X_h}\|\Delta u+\mu_h\|_{1,\Omega}$, which is a standard problem. On the other hand, we also have to estimate the expression

$$\inf_{(v_h,\psi_h)\in\mathscr{V}_h}(|u-v_h|_{1,\Omega}+|\Delta u+\psi_h|_{0,\Omega}),$$

and this is no longer a standard problem, because the functions v_h and ψ_h do not vary independently in their respective spaces X_{0h} and X_h. Nevertheless, it is possible to estimate the above expression by means of the "unconstrained" terms $\inf_{v_h\in X_{0h}}|u-v_h|_{1,\Omega}$ and $\inf_{\mu_h\in X_h}|\Delta u-\mu_h|_{0,\Omega}$ (cf. (7.1.22)), provided we make use of an appropriate *inverse inequality*, as our next result shows.

Theorem 7.1.5. *Let $\alpha(h)$ be a strictly positive constant such that*

$$\forall\mu_h\in X_h,\quad |\mu_h|_{1,\Omega}\le\alpha(h)|\mu_h|_{0,\Omega}. \tag{7.1.20}$$

Then there exists a constant C independent of the space X_h such that

$$|u-u_h|_{1,\Omega}+|\Delta u+\varphi_h|_{0,\Omega}\le$$

$$\le C((1+\alpha(h))\inf_{v_h\in X_{0h}}|u-v_h|_{1,\Omega}+\inf_{\mu_h\in X_h}\|\Delta u-\mu_h\|_{1,\Omega}). \tag{7.1.21}$$

Proof. Let (v_h,ψ_h) be an arbitrary element in the space $\mathscr{V}_h$, and let μ_h be an arbitrary element in the space X_h. The function $\nu_h=\mu_h+\psi_h$ belongs to the space X_h and thus,

$$\beta((v_h,\psi_h),\nu_h)=0.$$

Next, using the fact that $\partial_\nu u = 0$ on Γ, we get

$$\int_\Omega \Delta u\, \nu_h\, dx = -\int_\Omega \nabla u \cdot \nabla \nu_h\, dx,$$

so that, combining the two above equalities, we obtain

$$\int_\Omega (\Delta u + \psi_h)\nu_h\, dx = \int_\Omega \nabla(v_h - u)\cdot \nabla \nu_h\, dx.$$

Consequently, we get the inequality

$$\left|\int_\Omega (\Delta u + \psi_h)\nu_h\, dx\right| \leq |u - v_h|_{1,\Omega}|\nu_h|_{1,\Omega}$$
$$\leq \alpha(h)|u - v_h|_{1,\Omega}|\nu_h|_{0,\Omega},$$

which in turn implies that

$$|\nu_h|_{0,\Omega}^2 = \int_\Omega (\mu_h - \Delta u)\nu_h\, dx + \int_\Omega (\Delta u + \psi_h)\nu_h\, dx$$
$$\leq |\Delta u - \mu_h|_{0,\Omega}|\nu_h|_{0,\Omega} + \alpha(h)|u - v_h|_{1,\Omega}|\nu_h|_{0,\Omega}.$$

From this inequality, we deduce that

$$|\Delta u + \psi_h|_{0,\Omega} \leq |\Delta u - \mu_h|_{0,\Omega} + |\nu_h|_{0,\Omega}$$
$$\leq 2|\Delta u - \mu_h|_{0,\Omega} + \alpha(h)|u - v_h|_{1,\Omega},$$

and thus,

$$\inf_{(v_h,\psi_h)\in \mathcal{V}_h} (|u - v_h|_{1,\Omega} + |\Delta u + \psi_h|_{0,\Omega}) \leq$$
$$\leq (1 + \alpha(h)) \inf_{v_h\in X_{0h}} |u - v_h|_{1,\Omega} + 2 \inf_{\mu_h \in X_h} |\Delta u - \mu_h|_{0,\Omega}. \tag{7.1.22}$$

To finish the proof, it suffices to combine inequalities (7.1.16) and (7.1.22). □

Estimate of the error ($|u - u_h|_{1,\Omega} + |\Delta u + \phi_h|_{0,\Omega}$)

To apply the abstract error estimate proved in the previous theorem, we shall need the following standard assumptions on the family of finite element spaces X_h:

(H1) The associated family of triangulations $\mathcal{T}_h$ is regular.

(H2) All the finite elements (K, P_K, Σ_K), $K \in \bigcup_h \mathcal{T}_h$, are affine-equivalent to a single reference finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$.

(H3) All the finite elements (K, P_K, Σ_K), $K \in \bigcup_h \mathcal{T}_h$, are of class $\mathscr{C}^0$.

(H4) The family of triangulations satisfies an inverse assumption (cf. 3.2.28)).

Theorem 7.1.6. *In addition to* (H1), (H2), (H3) *and* (H4), *assume that there exists an integer* $k \geq 2$ *such that the following inclusions are satisfied*:

$$P_k(\hat{K}) \subset \hat{P} \subset H^1(\hat{K}). \tag{7.1.23}$$

Then if the solution $u \in H_0^2(\Omega)$ *of the minimization problem* (7.1.2) *belongs to the space* $H^{k+2}(\Omega)$, *there exists a constant* C *independent of* h *such that*

$$|u - u_h|_{1,\Omega} + |\Delta u + \varphi_h|_{0,\Omega} \leq Ch^{k-1}(|u|_{k+1,\Omega} + |\Delta u|_{k,\Omega}). \tag{7.1.24}$$

Proof. In view of the inclusions (7.1.23), there exist constants C independent of h such that

$$\inf_{v_h \in X_{0h}} |u - v_h|_{1,\Omega} \leq Ch^k |u|_{k+1,\Omega}, \tag{7.1.25}$$

$$\inf_{\mu_h \in X_h} \|\Delta u - \mu_h\|_{1,\Omega} \leq Ch^{k-1} |\Delta u|_{k,\Omega}. \tag{7.1.26}$$

Next, the inverse assumption allows us to conclude (cf. (3.2.35)) that the constants $\alpha(h)$ in inequalities (7.1.20) may be taken of the form

$$\alpha(h) = \frac{C}{h}, \tag{7.1.27}$$

for another constant C independent of h. Then the conclusion follows by using relations (7.1.25), (7.1.26) and (7.1.27) in the error estimate (7.1.21). □

Remark 7.1.2. In principle, an inclusion such as $H^{k+1}(\hat{K}) \hookrightarrow \mathscr{C}^s(\hat{K})$ (where s is the maximal order of partial derivatives occurring in the definition of the set $\hat{\Sigma}$) should have been added, but it is always satisfied *in practice*: Since $s = 0$ or 1 for finite elements of class $\mathscr{C}^0$ and since $n = 2$, the inclusion $H^3(\hat{K}) \subset \mathscr{C}^s(\hat{K})$ holds. □

Concluding remarks

Let us briefly discuss the application of this theorem: The major conclusion is that *one can solve the biharmonic problem with the same* *finite element spaces that are normally used for solving second-order problems, provided the inclusions $P_2(K) \subset P_K$, $K \in \mathcal{T}_h$, hold.* If we are using in particular triangles of type (k), $k \geqslant 2$, we get

$$\|u - u_h\|_{1,\Omega} = O(h^{k-1}) \quad \text{and} \quad |\Delta u + \varphi_h|_{0,\Omega} = O(h^{k-1}).$$

We shall therefore retain two basic advantages of this method: First, we get a convergent approximation to the solution u (albeit in the norm $\|\cdot\|_{1,\Omega}$ instead of the norm $\|\cdot\|_{2,\Omega}$) with much less sophisticated finite element spaces than would be required in conforming methods. The second advantage is that we obtain a convergent approximation φ_h of the vorticity $-\Delta u$, a physical quantity of interest in steady-state flows.

Nevertheless, one should keep in mind that, in spite of the simplicity of the spaces X_h, there remains the practical problem of actually computing the pair (u_h, φ_h). This is the object of the next section.

Exercise

7.1.1. Following CIARLET & GLOWINSKI (1975), the object of this problem is to show that *the solution of the biharmonic problem can be reduced to the solution of a sequence of Dirichlet problems for the operator $-\Delta$* (indeed, the analysis which shall be developed in the next section is nothing but the discrete analogue of what follows).

We recall that (cf. Theorem 7.1.2)

$$\mathcal{J}(u, -\Delta u) = \inf_{(v,\psi)\in\mathcal{V}} \mathcal{J}(v, \psi),$$

where the functional $\mathcal{J}$ and the space $\mathcal{V}$ are defined as in (7.1.4) and (7.1.5), respectively. Let there be given a subspace $\mathcal{M}$ of the space $H^1(\Omega)$ such that we may write the direct sum

$$H^1(\Omega) = H_0^1(\Omega) \oplus \mathcal{M}.$$

We next introduce the space

$$\mathcal{W} = \{(v, \psi) \in H_0^1(\Omega) \times L^2(\Omega);\ \forall \mu \in H_0^1(\Omega),\ \beta((v, \psi), \mu) = 0\},$$

where the mapping β is defined as in (7.1.6), and we define the *Lagrangian*

$$\mathcal{L}((v, \psi), \mu) = \mathcal{J}(v, \psi) + \beta((v, \psi), \mu).$$

(i) Show that, given a function $\lambda \in \mathcal{M}$, the problem: Find an element

$(u_\lambda, \varphi_\lambda) \in \mathscr{W}$ such that

$$\mathscr{L}((u_\lambda, \varphi_\lambda), \lambda) = \inf_{(v,\psi)\in\mathscr{W}} \mathscr{L}((v, \psi), \lambda),$$

has one and only one solution, which may also be obtained by solving the following Dirichlet problems for the operator $-\Delta$:

(*) Find a function $\varphi_\lambda \in H^1(\Omega)$ such that

$$(\varphi_\lambda - \lambda) \in H_0^1(\Omega),$$
$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \nabla\varphi_\lambda \cdot \nabla v \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x.$$

(**) Find a function $u_\lambda \in H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \nabla u_\lambda \cdot \nabla v \,\mathrm{d}x = \int_\Omega \varphi_\lambda v \,\mathrm{d}x$$

(notice that since $\bar{\Omega}$ is a convex polygon, the function u_λ is in fact in the space $H^2(\Omega) \cap H_0^1(\Omega)$).

(ii) Let u denote the solution of problem (7.1.2), and let λ^* be that function in the space $\mathscr{M}$ which is such that the function $(\Delta u + \lambda^*)$ belongs to the space $H_0^1(\Omega)$. Show that $((u, -\Delta u), \lambda^*)$ is the unique *saddle-point* of the Lagrangian $\mathscr{L}$ over the space $\mathscr{W} \times \mathscr{M}$, in the sense that

$$\forall (v, \psi) \in \mathscr{W}, \quad \forall \mu \in \mathscr{M},$$
$$\mathscr{L}((u, -\Delta u), \mu) \leq \mathscr{L}((u, -\Delta u), \lambda^*) \leq \mathscr{L}((v, \psi), \lambda^*).$$

(iii) As a consequence of (ii), show that

$$\mathscr{L}((u, -\Delta u), \lambda^*) = \max_{\lambda\in\mathscr{M}} g(\lambda),$$

where the function $g: \mathscr{M} \to \mathbf{R}$ is defined by

$$g: \lambda \in \mathscr{M} \to g(\lambda) = \min_{(v,\psi)\in\mathscr{W}} \mathscr{L}((v, \psi), \lambda)$$
$$= \mathscr{L}((u_\lambda, \varphi_\lambda), \lambda) = \frac{1}{2}\int_\Omega |\varphi_\lambda|^2 \,\mathrm{d}x$$

(this is a standard device in duality theory; see for example EKELAND & TEMAM (1974, chapter VI).

(iv) We next apply the *gradient method* to the maximization problem of question (iii) (a technique known as *Uzawa's method* for the original minimization problem, then called the *primal problem*): Given any function $\lambda_0 \in \mathcal{M}$ and a parameter $\rho > 0$ (to be specified in (v)), we define a sequence of functions $\lambda^n \in \mathcal{M}$ by the recurrence relation:

$$\forall \mu \in \mathcal{M}, \quad (\lambda^{n+1} - \lambda^n, \mu)_{\mathcal{M}} = -\rho \langle Dg(\lambda^n), \mu \rangle,$$

where $(.,.)_{\mathcal{M}}$ is an inner product in the space $\mathcal{M}$, whose associated norm is assumed to be equivalent to the norm $\|.\|_{1,\Omega}$, and $\langle .,. \rangle$ denotes the pairing between the spaces $\mathcal{M}'$ (= dual space of $\mathcal{M}$) and $\mathcal{M}$. Show that the function g is indeed everywhere differentiable over the space $\mathcal{M}$ and that one iteration of Uzawa's method consists of the following steps:

(*) Given a function $\lambda^n \in \mathcal{M}$, find the function $\varphi^n \in H^1(\Omega)$ which satisfies:

$$(\varphi^n - \lambda^n) \in H_0^1(\Omega),$$

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \nabla \varphi^n \cdot \nabla v \, dx = \int_\Omega f v \, dx.$$

(**) Find the function $u^n \in H_0^1(\Omega)$ which satisfies

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \nabla u^n \cdot \nabla v \, dx = \int_\Omega \varphi^n v \, dx.$$

(***) Find the function $\lambda^{n+1} \in \mathcal{M}$ which satisfies

$$\forall \mu \in \mathcal{M}, \quad (\lambda^{n+1} - \lambda^n, \mu)_{\mathcal{M}} = \rho \beta((u^n, \varphi^n), \mu).$$

(v) Show that the method described in question (iv) is convergent, in the sense that

$$\lim_{n \to \infty} \|u^n - u\|_{1,\Omega} = 0,$$

$$\lim_{n \to \infty} |\varphi^n + \Delta u|_{0,\Omega} = 0,$$

provided that

$$0 < \rho < 2c^2\sigma^2,$$

where the quantity σ is defined by

$$\sigma = \inf_{v \in H^2(\Omega) \cap H_0^1(\Omega)} \frac{|\Delta v|_{0,\Omega}}{\|\partial_\nu v\|_{L^2(\Gamma)}},$$

and c is any constant such that

$$\forall \mu \in H^1(\Omega), \quad c\|\mu\|_{L^2(\Gamma)} \leq \sqrt{(\mu, \mu)_{\mathcal{M}}}.$$

7.2. Solution of the discrete problem by duality techniques

Replacement of the constrained minimization problem by a saddle-point problem

Let us briefly review the definition of the discrete problem: We must find the unique element $(u_h, \varphi_h) \in \mathcal{V}_h$ which satisfies

$$\mathcal{J}(u_h, \varphi_h) = \inf_{(v_h, \psi_h) \in \mathcal{V}_h} \mathcal{J}(v_h, \psi_h), \tag{7.2.1}$$

with

$$\mathcal{V}_h = \{(v_h, \psi_h) \in X_{0h} \times X_h;\ \forall \mu_h \in X_h,\ \beta((v_h, \psi_h), \mu_h) = 0\}, \tag{7.2.2}$$

$$\beta((v_h, \psi_h), \mu_h) = \int_\Omega \nabla v_h \cdot \nabla \mu_h \, \mathrm{d}x - \int_\Omega \psi_h \mu_h \, \mathrm{d}x, \tag{7.2.3}$$

$$\mathcal{J}(v_h, \psi_h) = \frac{1}{2} \int_\Omega |\psi_h|^2 \, \mathrm{d}x - \int_\Omega f v_h \, \mathrm{d}x. \tag{7.2.4}$$

In the sequel, we assume that $\mathcal{M}_h$ is any supplementary subspace of the space X_{0h} in the space X_h, i.e., one has

$$X_h = X_{0h} \oplus \mathcal{M}_h \tag{7.2.5}$$

(practical choices of such subspaces $\mathcal{M}_h$ will be given later on). We also define the space

$$\mathcal{W}_h = \{(v_h, \psi_h) \in X_{0h} \times X_h;\ \forall \mu_h \in X_{0h},\ \beta((v_h, \psi_h), \mu_h) = 0\}, \tag{7.2.6}$$

and the *Lagrangian*

$$\mathcal{L}: (X_{0h} \times X_h) \times X_h \to \mathbf{R} \tag{7.2.7}$$

defined for all functions $v_h \in X_{0h}$, $\psi_h \in X_h$, $\mu_h \in X_h$ by

$$\mathcal{L}((v_h, \psi_h), \mu_h) = \mathcal{J}(v_h, \psi_h) + \beta((v_h, \psi_h), \mu_h), \tag{7.2.8}$$

the mappings β and $\mathcal{J}$ being given as in (7.2.3) and (7.2.4), respectively.

The next result is basic to the subsequent analysis.

Theorem 7.2.1. *Given a function $\lambda_h \in \mathcal{M}_h$, the minimization problem: Find an element $(u_{\lambda_h}, \varphi_{\lambda_h}) \in \mathcal{W}_h$ such that*

$$\mathcal{L}((u_{\lambda_h}, \varphi_{\lambda_h}), \lambda_h) = \inf_{(v_h, \psi_h) \in \mathcal{W}_h} \mathcal{L}((v_h, \psi_h), \lambda_h), \tag{7.2.9}$$

has one and only one solution, which may also be obtained through the consecutive solutions of the following problems:

(i) *Find a function* $\varphi_{\lambda_h} \in X_h$ *such that*

$$(\varphi_{\lambda_h} - \lambda_h) \in X_{0h}, \tag{7.2.10}$$

$$\forall v_h \in X_{0h}, \quad \int_\Omega \nabla \varphi_{\lambda_h} \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx. \tag{7.2.11}$$

(ii) *Find a function* $u_{\lambda_h} \in X_{0h}$ *such that*

$$\forall v_h \in X_{0h}, \quad \int_\Omega \nabla u_{\lambda_h} \cdot \nabla v_h \, dx = \int_\Omega \varphi_{\lambda_h} v_h \, dx. \tag{7.2.12}$$

Proof. Since the mapping

$$(v_h, \psi_h) \in \mathcal{W}_h \to |\psi_h|_{0,\Omega}$$

is a norm over the space $\mathcal{W}_h$ (argue as in the proof of Theorem 7.1.1), the minimization problem (7.2.9) has a unique solution.

Let us define a mapping

$$A_h : X_h \to X_{0h} \tag{7.2.13}$$

as follows: Given a function $\psi_h \in X_h$, the function $A_h\psi_h \in X_{0h}$ is the unique solution of the equations

$$\forall \mu_h \in X_{0h}, \quad \int_\Omega \nabla(A_h\psi_h) \cdot \nabla \mu_h \, dx = \int_\Omega \psi_h \mu_h \, dx.$$

Then the space $\mathcal{W}_h$ can also be written as

$$\mathcal{W}_h = \{(A_h\psi_h, \psi_h) \in X_{0h} \times X_h;\ \psi_h \in X_h\}.$$

Given a function $\lambda_h \in \mathcal{M}_h$, problem (7.2.9) consists in *minimizing the function* $\mathcal{L}((.,.), \lambda_h)$ *of the two variables* $v_h \in X_{0h}$ *and* $\psi_h \in X_h$ *when these two variables satisfy a relation of the form*

$$\Phi(v_h, \psi_h) = 0.$$

In the present case, the mapping ϕ: $X_{0h} \times X_h \to X_{0h}$ is given by

$$\Phi : (v_h, \psi_h) \in X_{0h} \times X_h \to \Phi(v_h, \psi_h) = (A_h\psi_h - v_h) \in X_{0h}.$$

The functions $\mathcal{L}((.,.), \lambda_h)$ and Φ are both differentiable. Thus there necessarily exists a unique *Lagrange multiplier* $\Xi_{\lambda_h} \in X'_{0h}$ (X'_{0h} = dual space of X_{0h}) such that

$$D\mathscr{L}((u_{\lambda_h}, \varphi_{\lambda_h}), \lambda_h) = \Xi_{\lambda_h} \cdot D\Phi(u_{\lambda_h}, \varphi_{\lambda_h}).$$

By taking the partial derivatives with respect to each argument, the above equality is seen to be equivalent to the two relations

$$\forall v_h \in X_{0h}, \quad \langle \Xi_{\lambda_h}, v_h \rangle + \int_\Omega \nabla\lambda_h \cdot \nabla v_h \,\mathrm{d}x = \int_\Omega f v_h \,\mathrm{d}x, \tag{7.2.14}$$

$$\forall \psi_h \in X_h, \quad \langle \Xi_{\lambda_h}, A_h\psi_h \rangle + \int_\Omega \psi_h\lambda_h \,\mathrm{d}x = \int_\Omega \varphi_{\lambda_h}\psi_h \,\mathrm{d}x, \tag{7.2.15}$$

where $\langle .\,,.\rangle$ stands for the duality pairing between the spaces X'_{0h} and X_{0h}. Consequently, the two relations (7.2.14) and (7.2.15) and the equation

$$u_{\lambda_h} = A_h\varphi_{\lambda_h} \tag{7.2.16}$$

(which expresses that $(u_{\lambda_h}, \varphi_{\lambda_h})$ is an element of the space $\mathscr{W}_h$) allow for the determination of the functions u_{λ_h} and φ_{λ_h}.

In order to put relations (7.2.14) and (7.2.15) in a more convenient form, let us introduce the (unique) function $\xi_{\lambda_h} \in X_{0h}$ which satisfies

$$\forall v_h \in X_{0h}, \quad \langle \Xi_{\lambda_h}, v_h \rangle = \int_\Omega \nabla\xi_{\lambda_h} \cdot \nabla v_h \,\mathrm{d}x.$$

Then relations (7.2.14) become

$$\forall v_h \in X_{0h}, \quad \int_\Omega \nabla(\xi_{\lambda_h} + \lambda_h) \cdot \nabla v_h \,\mathrm{d}x = \int_\Omega f v_h \,\mathrm{d}x, \tag{7.2.17}$$

while relations (7.2.15) become

$$\forall \psi_h \in X_h, \quad \int_\Omega (\xi_{\lambda_h} + \lambda_h - \varphi_{\lambda_h})\psi_h \,\mathrm{d}x = 0,$$

and thus we deduce that

$$\varphi_{\lambda_h} - \lambda_h = \xi_{\lambda_h}. \tag{7.2.18}$$

Consequently, the proof is complete: The assertions (7.2.10), (7.2.11) and (7.2.12) have been proved in (7.2.18), (7.2.17) and (7.2.16), respectively. □

In the next theorem, we show that the Lagrangian $\mathscr{L}$ defined in (7.2.7) possesses a (unique) *saddle-point* (cf. (7.2.19) below) over the product space $\mathcal{W}_h \times \mathcal{M}_h$, *whose first argument is precisely the solution* (u_h, φ_h) *of the original minimization problem* (7.2.1).

Theorem 7.2.2. *Let* φ_{0h} *be the (unique) function in the space* X_{0h} *such that the function* $(\varphi_h - \varphi_{0h})$ *belongs to the space* $\mathcal{M}_h$. *Then the element* $((u_h, \varphi_h), \varphi_h - \varphi_{0h}) \in \mathcal{W}_h \times \mathcal{M}_h$ *is the unique saddle-point of the Lagrangian* $\mathcal{L}$ *over the space* $\mathcal{W}_h \times \mathcal{M}_h$, i.e., *one has*

$$\forall (v_h, \psi_h) \in \mathcal{W}_h, \quad \forall \mu_h \in \mathcal{M}_h,$$

$$\mathcal{L}((u_h, \varphi_h), \mu_h) \leq \mathcal{L}((u_h, \varphi_h), \varphi_h - \varphi_{0h}) \leq \mathcal{L}((v_h, \psi_h), \varphi_h - \varphi_{0h}). \tag{7.2.19}$$

Proof. Since the pair (u_h, φ_h) belongs to the space $\mathcal{V}_h$, we deduce that

$$\forall \mu_h \in \mathcal{M}_h, \quad \mathcal{L}((u_h, \varphi_h), \mu_h) = \mathcal{J}(u_h, \varphi_h),$$

and thus the first inequality of (7.2.19) is proved. The second inequality amounts to showing that

$$\mathcal{L}((u_h, \varphi_h), \varphi_h - \varphi_{0h}) = \inf_{(v_h, \psi_h) \in \mathcal{W}_h} \mathcal{L}((v_h, \psi_h), \varphi_h - \varphi_{0h}).$$

Thus, by Theorem 7.2.1, it suffices to verify that (cf. (7.2.11))

$$\forall v_h \in X_{0h}, \quad \int_\Omega \nabla \varphi_h \cdot \nabla v_h \,\mathrm{d}x = \int_\Omega f v_h \,\mathrm{d}x, \tag{7.2.20}$$

since the relations corresponding to (7.2.10) and (7.2.12) are clearly satisfied. Given any function $v_h \in X_{0h}$, let ψ_h denote the (unique) function in the space X_h such that $(v_h, \psi_h) \in \mathcal{V}_h$, i.e., which satisfies

$$\forall \mu_h \in X_h, \quad \int_\Omega \nabla v_h \cdot \nabla \mu_h \,\mathrm{d}x = \int_\Omega \psi_h \mu_h \,\mathrm{d}x. \tag{7.2.21}$$

Since, by (7.1.15),

$$\int_\Omega f v_h \,\mathrm{d}x = \int_\Omega \varphi_h \psi_h \,\mathrm{d}x,$$

an application of (7.2.21) with $\mu_h = \varphi_h$ yields (7.2.20).

Next, let $((u_h^*, \varphi_h^*), \lambda_h^*) \in \mathcal{W}_h \times \mathcal{M}_h$ be a saddle-point of the Lagrangian

$\mathscr{L}$ over the space $\mathscr{W}_h \times \mathscr{M}_h$. From Theorem 7.2.1, we deduce that

$$(\varphi_h^* - \lambda_h^*) \in X_{0h}, \tag{7.2.22}$$

$$\forall v_h \in X_{0h}, \quad \int_\Omega \nabla\varphi_h^* \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx, \tag{7.2.23}$$

$$\forall v_h \in X_{0h}, \quad \int_\Omega \nabla u_h^* \cdot \nabla v_h \, dx = \int_\Omega \varphi_h^* v_h \, dx, \tag{7.2.24}$$

since, by definition,

$$\mathscr{L}((u_h^*, \varphi_h^*), \lambda_h^*) = \inf_{(v_h, \psi_h) \in \mathscr{W}_h} \mathscr{L}((v_h, \psi_h), \lambda_h^*).$$

On the other hand, we have

$$\mathscr{L}((u_h^*, \varphi_h^*), \lambda_h^*) = \sup_{\mu_h \in \mathscr{M}_h} \mathscr{L}((u_h^*, \varphi_h^*), \mu_h),$$

so that

$$\forall \mu_h \in \mathscr{M}_h, \quad \int_\Omega \nabla u_h^* \cdot \nabla \mu_h \, dx = \int_\Omega \varphi_h^* \mu_h \, dx. \tag{7.2.25}$$

Since the space X_h is the direct sum of the subspaces X_{0h} and $\mathscr{M}_h$, we deduce from (7.2.24) and (7.2.25) that the pair (u_h^*, φ_h^*) is an element of the space $\mathscr{V}_h$.

Let (v_h, ψ_h) be an arbitrary element of the space $\mathscr{V}_h$. Using the definition of this space and relations (7.2.23), we have

$$\int_\Omega \varphi_h^* \psi_h \, dx = \int_\Omega \nabla\varphi_h^* \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx.$$

Therefore, we have shown that (cf. (7.1.15))

$$u_h^* = u_h \quad \text{and} \quad \varphi_h^* = \varphi_h.$$

Finally, it results from (7.2.22) that the function $\lambda_h^* \in \mathscr{M}_h$ is equal to the function $\varphi_h - \varphi_{0h}$. ☐

Use of Uzawa's method. Reduction to a sequence of Dirichlet problems for the operator $-\Delta$

Using a well-known result in duality theory (cf. Exercise 7.2.1), the property for $((u_h, \varphi_h), \varphi_h - \varphi_{0h})$ to be a saddle-point of the Lagrangian $\mathscr{L}$ implies that we also have

$$\mathscr{L}((u_h, \varphi_h), \varphi_h - \varphi_{0h}) = \max_{\lambda_h \in \mathscr{M}_h} g(\lambda_h), \tag{7.2.26}$$

where the function $g: \mathcal{M}_h \to \mathbf{R}$ is defined by

$$g: \lambda_h \in \mathcal{M}_h \to g(\lambda_h) = \min_{(v_h, \psi_h) \in \mathcal{W}_h} \mathcal{L}((v_h, \psi_h), \lambda_h) = \mathcal{L}((u_{\lambda_h}, \varphi_{\lambda_h}), \lambda_h)$$
$$= -\frac{1}{2}\int_\Omega |\varphi_{\lambda_h}|^2 \, dx, \tag{7.2.27}$$

with the notations of Theorem 7.2.1.

The basic idea is then to apply the *gradient method* to the maximization problem (7.2.26), this technique for solving the so-called "*primal*" *problem* (7.2.1) being known in optimization theory as *Uzawa's method.* Thus we need to show that the function g is differentiable, and we need to compute its derivative: This is the object of the next theorem (as usual, $\mathcal{M}'_h$ denotes the dual space of the space $\mathcal{M}_h$ and $\langle .\,,.\rangle$ denotes the duality pairing between the spaces $\mathcal{M}'_h$ and $\mathcal{M}_h$).

Theorem 7.2.3. *At any point $\lambda_h \in \mathcal{M}_h$, the function g defined in* (7.2.27) *is differentiable, and its derivative $Dg(\lambda_h) \in \mathcal{M}'_h$ is defined by the relations*

$$\forall \mu_h \in \mathcal{M}_h, \quad \langle Dg(\lambda_h), \mu_h\rangle = \beta((u_{\lambda_h}, \varphi_{\lambda_h}), \mu_h). \tag{7.2.28}$$

Proof. The function $g: \mathcal{M}_h \to \mathbf{R}$ can be written as

$$g = g_1 \cdot g_0,$$

where the functions $g_1: X_h \to \mathbf{R}$ and $g_0: \mathcal{M}_h \to X_h$ are respectively given by

$$g_1: v_h \in X_h \to -\frac{1}{2}\int_\Omega |v_h|^2 \, dx,$$
$$g_0: \lambda_h \in \mathcal{M}_h \to \varphi_{\lambda_h} \in X_h.$$

The mapping g_0 is affine (cf. (7.2.10) and (7.2.11)) and thus we may assume that $f = 0$ for computing its derivative, in which case we find that

$$\forall \mu_h \in \mathcal{M}_h, \quad Dg_0(\lambda_h)\mu_h = \varphi^\circ_{\mu_h}, \tag{7.2.29}$$

where the function $\varphi^\circ_{\mu_h}$ is such that

$$\begin{cases} (\varphi^\circ_{\mu_h} - \mu_h) \in X_{0h}, \\ \forall v_h \in X_{0h}, \quad \displaystyle\int_\Omega \nabla\varphi^\circ_{\mu_h} \cdot \nabla v_h \, dx = 0. \end{cases} \tag{7.2.30}$$

On the other hand, we have

$$\forall v_h, w_h \in \mathcal{M}_h, \quad Dg_1(v_h)w_h = -\int_\Omega v_h w_h \, dx, \tag{7.2.31}$$

so that, by (7.2.29) and (7.2.31),

$$\langle Dg(\lambda_h), \mu_h\rangle = -\int_\Omega \varphi_{\lambda_h}\varphi^\circ_{\mu_h}\,\mathrm{d}x.$$

Using (7.2.12) and (7.2.30), this last expression can be transformed into

$$-\int_\Omega \varphi_{\lambda_h}\varphi^\circ_{\mu_h}\,\mathrm{d}x = \int_\Omega \varphi_{\lambda_h}(\mu_h - \varphi^\circ_{\mu_h})\,\mathrm{d}x - \int_\Omega \varphi_{\lambda_h}\mu_h\,\mathrm{d}x \tag{7.2.32}$$

$$= \beta((u_{\lambda_h}, \varphi_{\lambda_h}), \mu_h). \qquad \square$$

We recall that the gradient method as applied to the maximization problem (7.2.26) consists in defining a sequence $(\lambda_h^n)_{n=0}^\infty$ of functions $\lambda_h^n \in \mathcal{M}_h$ by the iterative scheme:

$$\forall \mu_h \in \mathcal{M}_h, \quad (\lambda_h^{n+1} - \lambda_h^n, \mu_h)_{\mathcal{M}_h} = -\rho\langle Dg(\lambda_h^n), \mu_h\rangle, \quad n \geqslant 0, \tag{7.2.33}$$

where:

$(.\,,.)_{\mathcal{M}_h}$ = an arbitrary inner product in the space $\mathcal{M}_h$,

ρ = a strictly positive parameter, the admissible range of which will be determined later (Theorem 7.2.5),

λ_h^0 = an arbitrary function of the space $\mathcal{M}_h$.

Using Theorems 7.2.1 and 7.2.3, we can immediately convert one iteration (7.2.33) in a more explicit form:

Theorem 7.2.4. *One iteration of Uzawa's method amounts to consecutively solving the following three problems:*

(i) *Given the function $\lambda_h^n \in \mathcal{M}_h$, find the (unique) function $\varphi_h^n \in X_h$ which satisfies*

$$(\varphi_h^n - \lambda_h^n) \in X_{0h}, \tag{7.2.34}$$

$$\forall v_h \in X_{0h}, \quad \int_\Omega \nabla\varphi_h^n \cdot \nabla v_h\,\mathrm{d}x = \int_\Omega f v_h\,\mathrm{d}x. \tag{7.2.35}$$

(ii) *Find the function $u_h^n \in X_{0h}$ which satisfies*

$$\forall v_h \in X_{0h}, \quad \int_\Omega \nabla u_h^n \cdot \nabla v_h\,\mathrm{d}x = \int_\Omega \varphi_h^n v_h\,\mathrm{d}x. \tag{7.2.36}$$

(iii) *Find the function $\lambda_h^{n+1} \in \mathcal{M}_h$ which satisfies*

$$\forall \mu_h \in \mathcal{M}_h, \quad (\lambda_h^{n+1} - \lambda_h^n, \mu_h)_{\mathcal{M}_h} = \rho\beta((u_h^n, \varphi_h^n), \mu_h). \quad \square \tag{7.2.37}$$

In other words, *the problem of approximating the solution of a fourth-order problem* (the biharmonic problem) *is reduced here to a sequence of "discrete second-order problems"*, namely problems (7.2.34)–(7.2.35) and (7.2.36), which correspond to the discretization of a nonhomogeneous and a homogeneous Dirichlet problems for the operator $-\Delta$, respectively. As will be explained later, the solution of problem (7.2.37) requires in principle a comparatively much smaller amount of work.

Convergence of Uzawa's method

Of course, these considerations implicitly assume that for some choices of the parameter ρ, Uzawa's method is convergent: This is what we shall prove in the next theorem.

First, we need to define a mapping

$$B_h: X_h \to \mathcal{M}_h, \tag{7.2.38}$$

as follows: For each function $\psi_h \in X_h$, the function $B_h\psi_h$ is the unique function in the space $\mathcal{M}_h$ which satisfies

$$\forall \mu_h \in \mathcal{M}_h, \quad (B_h\psi_h, \mu_h)_{\mathcal{M}_h} = \beta((A_h\psi_h, \psi_h), \mu_h), \tag{7.2.39}$$

where $A_h: X_h \to X_{0h}$ is the mapping of (7.2.13). Then we let

$$\|B_h\| = \sup_{v_h \in X_h} \frac{|B_h v_h|_{\mathcal{M}_h}}{|v_h|_{0,\Omega}}, \tag{7.2.40}$$

where $|.|_{\mathcal{M}_h}$ is the norm associated with the inner product $(.,.)_{\mathcal{M}_h}$.

Theorem 7.2.5. *If the parameter ρ satisfies*

$$0 < \rho < 2\sigma_h^2, \tag{7.2.41}$$

with (cf. (7.2.40))

$$\sigma_h = \frac{1}{\|B_h\|}, \tag{7.2.42}$$

Uzawa's method is convergent, in the sense that

$$\lim_{n\to\infty} u_h^n = u_h \quad \text{in} \quad X_{0h}, \tag{7.2.43}$$

$$\lim_{n\to\infty} \varphi_h^n = \varphi_h \quad \text{in} \quad X_h. \tag{7.2.44}$$

Proof. It suffices to show that $\lim_{n\to\infty} u_h^n = 0$ in X_{0h} and $\lim_{n\to\infty} \varphi_h^n = 0$ in X_h in the special case where $f = 0$. Using the definition (7.2.39) of the mapping B_h, the recurrence relation (7.2.37) takes the form

$$\lambda_h^{n+1} = \lambda_h^n + \rho B_h \varphi_h^n,$$

which, in conjunction with (7.2.32), yields

$$|\lambda_h^{n+1}|^2_{\mathcal{M}_h} = |\lambda_h^n|^2_{\mathcal{M}_h} - 2\rho|\varphi_h^n|^2_{0,\Omega} + \rho^2|B_h\varphi_h^n|^2_{\mathcal{M}_h},$$

since $f = 0$. Therefore, we get the inequality

$$|\lambda_h^n|^2_{\mathcal{M}_h} - |\lambda_h^{n+1}|^2_{\mathcal{M}_h} \geq (2\rho - \rho^2\|B_h\|^2)|\varphi_h^n|^2_{0,\Omega},$$

which in turn shows that

$$\lim_{n\to\infty} \varphi_h^n = 0,$$

provided the parameter ρ satisfies inequality (7.2.41). In addition, we deduce that

$$\lim_{n\to\infty} u_h^n = \lim_{n\to\infty} A_h\varphi_h^n = 0,$$

and the proof is complete. □

Concluding remarks

It is worth pointing out that the convergence of the present method is thus guaranteed for *any* choice of subspace $\mathcal{M}_h$ satisfying relation (7.2.5) and *any* choice of inner product $(.,.)_{\mathcal{M}_h}$ over the space $\mathcal{M}_h$. What is *not* independent of these data, however, is the quantity σ_h of (7.2.41) and it is of course desirable to get a concrete estimate of this quantity: This is the object of Exercise 7.2.2.

Although the space $\mathcal{M}_h$ is not uniquely determined by the sole equation $X_h = X_{0h} \oplus \mathcal{M}_h$, there is a "canonical" choice: Let us assume for definiteness that we are using Lagrange finite elements. Then the space $\mathcal{M}_h$ consists of those functions in the space X_h which are zero at the *interior nodes*, i.e., those nodes which are situated in the set Ω.

With the above choice for the space $\mathcal{M}_h$, assume that the inner product $(.,.)_{\mathcal{M}_h}$ is the inner product of the space $L^2(\Gamma)$. Then if we denote by M the dimension of the space X_{0h}, the solution of either problem (i) or (ii) (cf. (7.2.34)–(7.2.35) and (7.2.36)) requires the solution of a system of M linear equations, while the solution of problem (iii) (cf. (7.2.37)) amounts to solving a system of $0(\sqrt{M})$ linear equations. As a consequence, the amount of work required for solving problem (iii) is negligible compared with the total amount of work required in one iteration of Uzawa's method, at least asymptotically.

There remains in addition the possibility of reducing the computations involved in step (iii), simply by using *numerical integration* for computing the integrals over Γ, and this is precisely why Theorem 7.2.5 was proved with an arbitrary inner product over the space $\mathcal{M}_h$. In this direction, see Exercise 7.2.3.

Exercises

7.2.1. Let V and M be two arbitrary sets and let $L\colon V \times M \to \mathbf{R}$ be a given mapping. A pair $(v^*, \mu^*) \in V \times M$ is a *saddle-point* of the function L if

$$\sup_{\mu \in M} L(v^*, \mu) = L(v^*, \mu^*) = \inf_{v \in V} L(v, \mu^*).$$

Show that

$$\sup_{\mu \in M} \inf_{v \in V} L(v, \mu) = L(v^*, \mu^*) = \inf_{v \in V} \sup_{\mu \in M} L(v, \mu).$$

7.2.2. Let us assume that the finite element space X_h is made up of Lagrange finite elements and that $\mathcal{M}_h$ consists of those functions in the space X_h whose values are zero at all the nodes which belong to the set Ω. Assume in addition that the inner product $(\cdot\,,\cdot)_{\mathcal{M}_h}$ is the inner product of the space $L^2(\Gamma)$. The purpose of this problem is to show that (cf. Ciarlet & Glowinski (1975)) the quantity σ_h defined in (7.2.42) satisfies

$$\lim_{h \to 0} \sigma_h = \sigma,$$

where

$$\sigma = \inf_{v \in H^2(\Omega) \cap H_0^1(\Omega)} \frac{|\Delta v|_{0,\Omega}}{\|\partial_\nu v\|_{L^2(\Gamma)}}.$$

Such a quantity can be estimated for simple domains: See the section "Bibliography and Comments".

(i) Let ψ and μ be arbitrary functions in the space $H^1(\Omega)$. Show that

$$\limsup_{h \to 0} \sigma_h \leqslant \frac{|\psi|_{0,\Omega} \|\mu\|_{L^2(\Gamma)}}{\left| \int_\Gamma \partial_\nu v \, \mu \, \mathrm{d}\gamma \right|},$$

and deduce from this inequality that

$$\lim_{h\to 0} \sup \sigma_h \leqslant \sigma.$$

[Hint: Let, for each h, ψ_h and μ_h be two functions in the space X_h such that

$$\lim_{h\to 0} \|\psi_h - \psi\|_{1,\Omega} = 0, \quad \lim_{h\to 0} \|\mu_h - \mu\|_{1,\Omega} = 0.$$

Then prove and use the inequality

$$\sigma_h \leqslant \frac{|\psi_h|_{0,\Omega}\|\mu_h\|_{L^2(\Gamma)}}{\int_\Gamma B_h\psi_h\,\mu_h\,\mathrm{d}\gamma}\Bigg].$$

(ii) For each h, let ψ_h be an arbitrary function in the space X_h and let μ_h be an arbitrary function in the space $\mathcal{M}_h$. Show that there exists a constant C independent of h such that

$$\frac{1}{\sigma} \geqslant \frac{\left|\int_\Gamma B_h\psi_h\,\mu_h\,\mathrm{d}\gamma\right|}{|\psi_h|_{0,\Omega}\|\mu_h\|_{L^2(\Gamma)}} - Ch^{1/2},$$

and deduce from this inequality that

$$\lim_{h\to 0} \inf \sigma_h \geqslant \sigma.$$

[Hint: For each h, let $u_h = A_h\psi_h$ and let $\tilde{u}_h \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfy $-\Delta\tilde{u}_h = \psi_h$. Prove and use the inequality

$$\frac{1}{\sigma} \geqslant \frac{\left|\int_\Gamma B_h\psi_h\,\mu_h\,\mathrm{d}\gamma + \int_\Omega \nabla(\tilde{u}_h - u)\cdot\nabla\mu_h\,\mathrm{d}x\right|}{|\psi_h|_{0,\Omega}\|\mu_h\|_{L^2(\Gamma)}}\Bigg].$$

7.2.3. Assume that the space X_h is made up of triangles of type (2), and let a quadrature scheme be given by

$$\int_\Gamma \varphi(x)\,\mathrm{d}x \sim \sum_{K'\subset\Gamma} \frac{\text{length}\,(K')}{6}\{\varphi(a_{K'}) + 4\varphi(b_{K'}) + \varphi(c_{K'})\},$$

where the symbol $\Sigma_{K'\subset\Gamma}$ means that the summation is extended over all sides K' (of the triangles) contained in the boundary Γ, and where, for each such side K', $a_{K'}$ and $c_{K'}$ denote the end-points of the segment K' while $b_{K'}$ stands for its mid-point.

Show that this quadrature scheme induces an inner product over the space $\mathcal{M}_h$. What is the corresponding structure of the matrix of the linear system found in the solution of problem (7.2.37)?

Bibliography and comments

7.1. As mentioned in the introduction of this chapter, a general discussion of analogous and related methods (equilibrium, mixed, hybrid methods) for second-order and fourth-order problems is postponed until the next section "Additional Bibliography and Comments". We shall discuss here only the particular mixed finite element approximation of the biharmonic problem considered in this chapter.

The content of this section is based on CIARLET & RAVIART (1974). The starting point was the work of GLOWINSKI (1973), who studied a related method. R. Glowinski obtained convergence, without orders of convergence, however, for piecewise polynomials of degree ≤ 1 or ≤ 2. In the first case (which is not covered by the present analysis), convergence holds provided a certain "patch test" is satisfied, which amounts to saying that there are only three admissible directions for the sides of the triangles (this condition is reminiscent of the analogous condition found when Zienkiewicz triangles are used for solving the plate problem). In addition, R. Glowinski made the interesting observation that for specific choices of subspaces, the method is identical to the usual 13-point difference approximation of the operator Δ^2. Likewise, MERCIER (1974) has also studied a similar method, again proving convergence without orders of convergence. For recent developments, see FALK (1976d).

7.2. The results contained in this section are proved in CIARLET & GLOWINSKI (1975). A further, and significant, step has been recently taken by GLOWINSKI & PIRONNEAU (1976a, 1976b, 1976c, 1976d) who reduced the approximation of the biharmonic problem to (i) a *finite* number of approximate Dirichlet problems for the operator $-\Delta$ and (ii) the solution of a linear system with a symmetric and positive definite matrix. The key idea consists in transforming the biharmonic problem into a variational problem posed over the boundary Γ, in which the unknown is $-\Delta u_{|\Gamma}$.

KESAVAN & VANNINATHAN (1977) have analyzed mathematically the effect of numerical integration, combined with the use of isoparametric finite elements, in the discrete second-order problems found in the method described in Section 7.2. We also mention that BOURGAT (1976) has implemented this method (with numerical integration and isoparametric finite elements). It turns out that the results compare favorably with those obtained with more familiar finite element methods. From a practical standpoint, it is clear that this is much less complex than a direct application of numerical integration and isoparametric finite elements to the more standard discretization of the biharmonic problem.

Several authors have considered either the problem of reducing the biharmonic problem to a sequence of Dirichlet problems for the Laplacian Δ, or its discrete counterpart. In this direction, we quote BOSSAVIT (1971), EHRLICH (1971), MCLAURIN (1974), SMITH (1968, 1973). In particular, it is shown in SMITH (1968) that the quantity σ (cf. Exercises 7.1.1 and 7.2.2) is equal to $2/R$ if $\bar{\Omega}$ is a disk of radius R and satisfies the inequalities $(2\sqrt{\pi}/\sqrt{ab}) \leqslant \sigma \leqslant (\pi^2(a^2+b^2)^2/4ab(a^3+b^3))$ if $\bar{\Omega}$ is a rectangle with sides a and b. For other estimates, see PAYNE (1970).

Although the exposition is self-contained, the reader who wishes to get a better acquaintance with optimization theory, in particular with the methods and techniques of duality theory referred to here (Lagrangian, saddle-point, gradient's method, Uzawa's method, etc...) may consult the books of AUSLENDER (1976), CÉA (1971), EKELAND & TÉMAM (1974), LAURENT (1972).

Additional bibliography and comments

Primal, dual and primal-dual formulations

As a preliminary step towards a better understanding of the various finite element methods which shall be described later, we must shed a new light on our familiar minimization problem: Find $u \in V$ such that $J(u) = \inf_{v \in V} J(v)$. We begin our discussion with the most illuminating example, the *elasticity problem*: With the notations of Section 1.2 (cf. Fig. 1.2.3 in particular), this problem consists in minimizing *the energy* (cf. (1.2.36))

$$\begin{aligned} J(\boldsymbol{v}) &= \frac{1}{2}\int_{\Omega}\left\{\lambda(\operatorname{div}\boldsymbol{v})^2 + 2\mu\sum_{i,j=1}^{3}(\epsilon_{ij}(\boldsymbol{v}))^2\right\}\mathrm{d}x \\ &\quad - \left(\int_{\Omega}\boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_1}\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}\gamma\right) \\ &= \frac{1}{2}a(\boldsymbol{v},\boldsymbol{v}) - f(\boldsymbol{v}), \end{aligned}$$

over the space (cf. (1.2.30))

$$V = \{v \in (H^1(\Omega))^3;\ v = 0 \quad \text{on} \quad \Gamma_0\}$$

of *admissible displacements*, and the associated boundary value problem is

$$\begin{cases} -\mu\Delta u - (\lambda+\mu)\,\mathrm{grad\,div}\, u = f & \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \Gamma_0, \\ \sum_{j=1}^{3} \sigma_{ij}(u)\nu_j = g_i \quad \text{on} \quad \Gamma_1, & 1 \leq i \leq 3. \end{cases}$$

We shall call this formulation of the elasticity problem the *displacement model*, or *formulation*, and we shall call *primal problem* the corresponding minimization, or variational, problem.

It turns out that in the analysis of actual structures, the knowledge of the *stress tensor* $(\sigma_{ij})_{i,j=1}^3$ is often of greater interest than the knowledge of the displacement u. To make this tensor appear as the unknown of a new variational problem, we note from (1.2.33) that (with the convention that σ_{ij} denotes an unknown component of the stress tensor, while τ_{ij} is the corresponding "generic" component, $1 \leq i, j \leq 3$)

$$a(v, v) = \int_\Omega \sum_{i,j=1}^{3} \tau_{ij}(v)\epsilon_{ij}(v)\,\mathrm{d}x$$
$$= \int_\Omega \left\{\left(\frac{1+\sigma}{E}\right)\sum_{i,j=1}^{3}\tau_{ij}^2 - \frac{\sigma}{E}\left(\sum_{l=1}^{3}\tau_{ll}\right)^2\right\}\mathrm{d}x,$$

since relations (1.2.32) can be inverted into

$$\epsilon_{ij} = \left(\frac{1+\sigma}{E}\right)\tau_{ij} - \frac{\sigma}{E}\left(\sum_{k=1}^{3}\tau_{kk}\right)\delta_{ij}, \quad 1 \leq i, j \leq 3,$$

where $E = \mu(3\lambda + 2\mu)/(\lambda+\mu)$ and $\sigma = (\lambda/2(\lambda+\mu))$ are respectively the Young modulus and the Poisson coefficient.

Using relations (cf. (1.2.39))

$$-\mu\Delta u_i - (\lambda+\mu)\partial_i\,\mathrm{div}\, u = -\sum_{j=1}^{3}\partial_j\sigma_{ij}, \quad 1 \leq i \leq 3,$$

and the boundary conditions $\Sigma_{j=1}^3 \sigma_{ij}(u)\nu_j = g_i$ on Γ_1, $1 \leq i \leq 3$, one finds (cf. DUVAUT & LIONS (1972, Chapter 3, Section 3.5)) that the tensor $\sigma = (\sigma_{ij})_{i,j=1}^3$ minimizes the *complementary energy*

$$I(\tau) = \frac{1}{2}\int_\Omega \left\{\left(\frac{1+\sigma}{E}\right)\sum_{i,j=1}^{3}\tau_{ij}^2 - \frac{\sigma}{E}\left(\sum_{l=1}^{3}\tau_{ll}\right)^2\right\}\mathrm{d}x$$

over the set

$$\mathcal{V}(f, g) = \Big\{\tau \in \mathcal{W};\ -\sum_{j=1}^{3} \partial_j \tau_{ij} = f_i \quad \text{in} \quad \Omega,$$
$$\sum_{j=1}^{3} \tau_{ij}\nu_j = g_i \quad \text{on} \quad \Gamma_1, \quad 1 \leqslant i \leqslant 3\Big\}$$

of *admissible stresses*, where

$$\mathcal{W} = \{\tau = (\tau_{ij})_{i,j=1}^{3} \in (L^2(\Omega))^9;\ \tau_{ij} = \tau_{ji}, \quad 1 \leqslant i, j \leqslant 3\}.$$

In the definition of the set $\mathcal{V}(f, g)$, the relations $-\Sigma_{j=1}^{3}\, \partial_j\tau_{ij} = f_i$ in Ω are to be understood in the sense of distributions, while the interpretation of the relations $\Sigma_{j=1}^{3}\, \tau_{ij}\nu_j = g_i$ on Γ_1 will be hinted at on a simpler problem. Then it is easily shown that the above minimization problem has one and only one solution σ, which is precisely related to the displacement u by relations (1.2.31) and (1.2.32). The reason the functional I is called the "complementary" energy is that $J(u) + I(\sigma) = 0$.

We shall call this formulation of the elasticity problem the *equilibrium model*, or *formulation* (it is called "equilibrium" model because the relations which define the set $\mathcal{V}(f, g)$ express the equilibrium of internal and boundary forces, respectively) and we shall call *dual problem* the corresponding minimization, or variational, problem.

It is therefore natural to conceive finite element approximations of this dual problem, but then the *major difficulty lies in the constraints used in the definition of the set* $\mathcal{V}(f, g)$. To obviate this difficulty, one key idea is to use techniques from *duality theory*. This is precisely the basis of the *mixed* and *hybrid* finite element methods, which we shall describe later.

For the sake of simplicity in the exposition, we shall often consider the model problem

$$\begin{cases} -\Delta u = f \quad \text{in} \quad \Omega \subset \mathbf{R}^n, \quad f \in L^2(\Omega), \\ u = 0 \quad \text{on} \quad \Gamma, \end{cases}$$

for which the *primal problem* consists in finding a function $u \in V = H_0^1(\Omega)$ such that

$$J(u) = \inf_{v \in V} J(v),$$

with

$$J(v) = \frac{1}{2}\int_\Omega \|\nabla v\|^2\, \mathrm{d}x - \int_\Omega fv\, \mathrm{d}x,$$

or equivalently, such that

$$\forall v \in V, \quad \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx.$$

In this case, we shall consider that the unknowns which play the role of the "stresses" are the components of the gradient ∇u of the solution u (the "displacement") of the primal problem, and therefore, the objective is that these components be obtained by solving a minimization problem. In this direction, we introduce the space

$$H(\mathrm{div}; \Omega) = \{q \in (L^2(\Omega))^n;\ \mathrm{div}\, q \in L^2(\Omega)\},$$

a Hilbert space when equipped with the norm

$$\|q\|_{H(\mathrm{div};\Omega)} = (|q|_{0,\Omega}^2 + |\mathrm{div}\, q|_{0,\Omega}^2)^{1/2}.$$

Then for any function $f \in L^2(\Omega)$, we define the affine hyperplane

$$\mathcal{V}(f) = \{q \in H(\mathrm{div}; \Omega);\ \mathrm{div}\, q + f = 0 \quad \text{in} \quad \Omega\}.$$

Using the $\mathcal{V}(0)$-ellipticity of the bilinear form $(p, q) \to \int_\Omega p \cdot q \, dx$, it is easily proved that there exists a unique function $p \in \mathcal{V}(f)$ such that

$$I(p) = \inf_{q \in \mathcal{V}(f)} I(q),$$

with

$$I(q) = \frac{1}{2} \int_\Omega \|q\|^2 \, dx,$$

or equivalently, such that

$$\forall q \in \mathcal{V}(0), \quad \int_\Omega p \cdot q \, dx = 0.$$

Since in addition one has precisely

$$p = \nabla u,$$

we have therefore constructed an adequate dual problem.

In order to get rid of the constraint $\mathrm{div}\, q + f = 0$ which appears in the definition of the set $\mathcal{V}(f)$, we use a device standard in duality theory (cf. for example CÉA (1971), EKELAND & TÉMAM (1974)), which makes it possible to construct a problem in which the unknown p is no longer subjected to a constraint (i.e., it shall be simply required that $p \in H(\mathrm{div}; \Omega)$). To achieve this goal, we have to find an appropriate space $\mathcal{M}$ of *Lagrange multipliers* and a *Lagrangian* $\mathcal{L}: H(\mathrm{div};\Omega)\times\mathcal{M}\to\mathbf{R}$, in such a way that *the unknown* $\boldsymbol{p}$ *is obtained as the first argument of the saddle-point* $(\boldsymbol{p},\lambda)$ *of the Lagrangian* $\mathcal{L}$ over the space $H(\mathrm{div};\Omega)\times\mathcal{M}$ (this is the process that was followed in Section 7.2 for the solution of the discrete problem; see also Exercise 7.1.1 for the biharmonic problem itself).

In the present case, it turns out that we may choose

$$\mathcal{M}=L^2(\Omega)\quad\text{and}\quad\mathcal{L}(\boldsymbol{q},\mu)=I(\boldsymbol{q})+\int_\Omega\mu(\mathrm{div}\,\boldsymbol{q}+f)\,\mathrm{d}x$$

(the particular form of the above Lagrangian is no coincidence; it is based on the fact that the functions $\boldsymbol{q}$ in the set $\mathcal{V}(f)$ may be equally characterized as those functions $\boldsymbol{q}\in H(\mathrm{div};\Omega)$ for which $\int_\Omega\mu(\mathrm{div}\,\boldsymbol{q}+f)\,\mathrm{d}x=0$ for all $\mu\in L^2(\Omega)$). Then one can show that the above Lagrangian has a unique saddle-point $(\boldsymbol{p},\lambda)$ over the space $H(\mathrm{div};\Omega)\times L^2(\Omega)$, and that one has precisely

$$\boldsymbol{p}=\nabla u\quad\text{and}\quad\lambda=u$$

(that the second argument of the saddle-point is related to the solution of either the primal or dual problem is no coincidence either; cf. Section 7.2 for a similar circumstance). In other words, one has

$$\forall\boldsymbol{q}\in H(\mathrm{div};\Omega),\quad\forall\mu\in L^2(\Omega),\quad\mathcal{L}(\boldsymbol{p},\mu)\leqslant\mathcal{L}(\boldsymbol{p},\lambda)\leqslant\mathcal{L}(\boldsymbol{q},\lambda),$$

or equivalently,

$$\forall\boldsymbol{q}\in H(\mathrm{div};\Omega),\quad\int_\Omega\boldsymbol{p}\cdot\boldsymbol{q}\,\mathrm{d}x+\int_\Omega\lambda\,\mathrm{div}\,\boldsymbol{q}\,\mathrm{d}x=0,$$

$$\forall\mu\in L^2(\Omega),\quad\int_\Omega\mu(\mathrm{div}\,\boldsymbol{p}+f)\,\mathrm{d}x=0$$

(these variational equations simply express the necessary, and sufficient in this case, conditions that the two partial derivatives of the Lagrangian vanish at the saddle-point). Notice in passing that the above variational equations mean that, at least formally, the given second-order problem has been replaced by a *first-order system*, namely,

$$\begin{cases}\boldsymbol{p}-\nabla\lambda=0 & \text{in}\quad\Omega,\\ -\mathrm{div}\,\boldsymbol{p}=f & \text{in}\quad\Omega,\\ \lambda=0\quad\text{on}\quad\Gamma.\end{cases}$$

For the finite element approximation of general first-order systems, see LESAINT (1973, 1975).

The verification of the above statements offers no difficulties. It relies in particular on the following result (LIONS & MAGENES (1968)). Given a function $q \in H(\mathrm{div}; \Omega)$, one may define its "outer normal component" (denoted by definition) $q \cdot \nu$ along Γ as an element of the space $H^{-1/2}(\Gamma)$, in such a way that the Green formula

$$\forall v \in H^1(\Omega), \quad \int_\Omega \{\nabla v \cdot q + v \operatorname{div} q\} \, \mathrm{d}x = \langle q \cdot \nu, v \rangle_\Gamma$$

holds, where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality pairing between the spaces $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ (recall that $\langle q \cdot \nu, v \rangle_\Gamma = \int_\Gamma (q \cdot \nu) v \, \mathrm{d}\gamma$ if it so happens that $q \cdot \nu \in L^2(\Gamma) \subset H^{-1/2}(\Gamma)$; the spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ have been defined in connection with problems on unbounded domains; cf. the section "Additional Bibliography and Comments" in Chapter 4).

We shall call this formulation (of the model problem) the *primal-dual formulation*, and *primal-dual problem* the corresponding saddle-point, or variational, problem.

In the case of the elasticity problem, the corresponding Lagrangian is called the *Hellinger–Reissner energy*. It takes the form

$$\mathscr{L}(v, \tau) = -I(\tau) + \int_\Omega \sum_{i,j=1}^3 \epsilon_{ij}(v) \tau_{ij} \, \mathrm{d}x - \left(\int_\Omega f \cdot v \, \mathrm{d}x + \int_{\Gamma_1} g \cdot v \, \mathrm{d}\gamma \right),$$

and it can be shown that the pair (u, σ) is a saddle-point of the Lagrangian $\mathscr{L}$ over the space $V \times \mathscr{W}$.

The proper framework for justifying the previous considerations is of course that of *duality theory* (as was implicitly indicated by the use of the adjectives "primal" and "dual", for instance). For a thorough reference concerning duality theory in general, see EKELAND & TÉMAM (1974). For the application of duality theory to problems in elasticity, see FRÉMOND (1971a, 1971b, 1972, 1973), WASHIZU (1968), ODEN & REDDY (1974, 1976b), TONTI (1970). For applications to variational inequalities, see GLOWINSKI (1976a), GLOWINSKI, LIONS & TRÉMOLIÈRES (1976a, 1976b).

Displacement and equilibrium methods

Except in Chapter 7, the finite element methods described in this book are based on the *primal formulation* of a given problem. This explains why, by reference to the elasticity problem, such methods are sometimes known as *displacement methods.*

As we pointed out, it is also desirable to develop methods in which the "stresses" are directly computed. In this direction, the engineers have devised various ways of computing the stresses directly from the knowledge of the displacements, as in BARLOW (1976), HINTON & CAMPBELL (1974), STEIN & AHMAD (1974). Our primary interest, however, concerns methods directly based on a *dual formulation.*

To be more specific, consider the model problem $-\Delta u = f$ in Ω, $u = 0$ on Γ, whose dual problem consists in finding an element $\boldsymbol{p} \in \mathcal{V}(f)$ such that

$$\forall \boldsymbol{q} \in \mathcal{V}(0), \quad \int_\Omega \boldsymbol{p} \cdot \boldsymbol{q} \,\mathrm{d}x = 0,$$

where

$$\mathcal{V}(f) = \{\boldsymbol{q} \in H(\mathrm{div}; \Omega); \mathrm{div}\, \boldsymbol{q} + f = 0 \quad \text{in} \quad \Omega\}.$$

Let us assume that an element $\boldsymbol{p}_0 \in \mathcal{V}(f)$ is known, so that this problem is reduced to finding the (unique) element $\boldsymbol{p}^* = (\boldsymbol{p} - \boldsymbol{p}_0) \in \mathcal{V}(0)$ which satisfies

$$\forall \boldsymbol{q} \in \mathcal{V}(0), \quad \int_\Omega \boldsymbol{p}^* \cdot \boldsymbol{q} \,\mathrm{d}x = -\int_\Omega \boldsymbol{p}_0 \cdot \boldsymbol{q} \,\mathrm{d}x.$$

Since it is possible to construct subspaces $\mathcal{V}_h(0)$ of $\mathcal{V}(0)$ (as in the approximation of the Stokes problem; cf. the section "Additional Bibliography and Comments" of Chapter 4), the discrete problem consists in finding the (unique) element $\boldsymbol{p}_h^* \in \mathcal{V}_h(0)$ which satisfies

$$\forall \boldsymbol{q}_h \in \mathcal{V}_h(0), \quad \int_\Omega \boldsymbol{p}_h^* \cdot \boldsymbol{q}_h \,\mathrm{d}x = -\int_\Omega \boldsymbol{p}_0 \cdot \boldsymbol{q}_h \,\mathrm{d}x,$$

and one gets in this fashion an approximation $\boldsymbol{p}_h = \boldsymbol{p}_0 + \boldsymbol{p}_h^*$ of the solution $\boldsymbol{p}$ of the dual problem.

It is however exceptional that an element be known in the set $\mathcal{V}(f)$, so that the major difficulty is to take appropriately into account the constraint $\mathrm{div}\, \boldsymbol{p} + f = 0$ in Ω. There are essentially three ways to circumvent this difficulty.

First, *the constraint is approximated*, in such a way that the discrete solution $\boldsymbol{p}_h$ satisfies a relation of the form $\mathrm{div}\, \boldsymbol{p}_h + f_h = 0$ in Ω, where f_h is a typical finite element approximation of the function f (e.g. one has

$f_{h|K} \in P_k(K)$ for all $K \in \mathcal{T}_h$). Again by reference to the elasticity problem, such methods, which are directly based on the dual formulation, are known as *equilibrium methods* (of course, they include those where the constraint may be exactly satisfied). They have been first advocated by FRAEIJS DE VEUBEKE (1965b, 1973). Their numerical analysis is thoroughly made in THOMAS (1975, 1976, 1977). See also FALK (1976b) for a related method, and RAVIART (1975).

While the equilibrium methods are, by definition, based on a formulation where there is *only one unknown* (the gradient ∇u for the model problem, the stress tensor $\boldsymbol{\sigma}$ for the elasticity problem, etc...), one may use the *techniques of duality theory* to get rid of the constraint, a process which results in the addition of a *second unknown*, the Lagrange multiplier. This is in particular the basis of the *mixed* methods and the *dual hybrid* methods, which are other alternatives for handling the constraint.

Mixed methods

It is customary to call *mixed method* any finite element method based on the primal-dual formulation (notice that we shall later extend this definition; then we shall return to the mixed method described in Chapter 7, in the light of the present definitions).

Especially for second-order problems, the study of such methods may be based on a general approach of BREZZI (1974b) (notice that it does not directly apply to the method described in Section 7.2, however, even though several features are common to both analyses).

F. Brezzi considers the following variational problem (irrespective of whether it is obtained from a saddle-point problem): Find a pair $(p, \lambda) \in \mathcal{W} \times \mathcal{M}$ such that

$$\forall q \in \mathcal{W}, \quad a(p, q) + b(q, \lambda) = f(q),$$

$$\forall \mu \in \mathcal{M}, \quad b(p, \mu) = g(v),$$

where $\mathcal{W}$ and $\mathcal{M}$ are Hilbert spaces, $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are continuous bilinear forms on $\mathcal{W} \times \mathcal{W}$ and $\mathcal{W} \times \mathcal{M}$ respectively, f and g are given elements in the dual spaces $\mathcal{W}'$ and $\mathcal{M}'$ respectively. First, F. Brezzi gives necessary and sufficient conditions on the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$, which insure that the above problem has one and only one solution for all $f \in \mathcal{W}'$, $g \in \mathcal{M}'$. Secondly, he considers the associated discrete problem: Find a pair $(p_h, \lambda_h) \in \mathcal{W}_h \times \mathcal{M}_h$ such that

$$\forall q_h \in \mathcal{W}_h, \quad a(p_h, q_h) + b(q_h, \lambda_h) = f(q_h),$$
$$\forall \mu_h \in \mathcal{M}_h, \quad b(p_h, \mu_h) = g(v_h),$$

where $\mathcal{W}_h$ and $\mathcal{M}_h$ are closed subspaces (finite-dimensional in practice) of the spaces $\mathcal{W}$ and $\mathcal{M}$, respectively. Then under suitable assumptions, F. Brezzi obtains an abstract error estimate for the quantity $\|p - p_h\|_{\mathcal{W}} + \|\lambda - \lambda_h\|_{\mathcal{M}}$.

To indicate the flavor of his results, let us return to our model problem. As we have seen, the primal-dual formulation consists in finding a pair $(p, \lambda) \in H(\mathrm{div}; \Omega) \times L^2(\Omega)$ such that

$$\forall q \in H(\mathrm{div}; \Omega), \int_\Omega p \cdot q \, \mathrm{d}x + \int_\Omega \lambda \, \mathrm{div}\, q \, \mathrm{d}x = 0,$$
$$\forall \mu \in L^2(\Omega), \quad \int_\Omega \mu(\mathrm{div}\, p + f) \, \mathrm{d}x = 0$$

(recall that $p = \nabla u$ and $\lambda = u$). Accordingly, a *mixed finite element method* for solving this problem is defined as follows: Given two finite-dimensional spaces $\mathcal{W}_h$ and $\mathcal{M}_h$ which satisfy the inclusions

$$\mathcal{W}_h \subset H(\mathrm{div}; \Omega), \quad \mathcal{M}_h \subset L^2(\Omega),$$

find a pair $(p_h, \lambda_h) \in \mathcal{W}_h \times \mathcal{M}_h$ such that

$$\forall q_h \in \mathcal{W}_h, \quad \int_\Omega p_h \cdot q_h \, \mathrm{d}x + \int_\Omega \lambda_h \, \mathrm{div}\, q_h \, \mathrm{d}x = 0,$$
$$\forall \mu_h \in \mathcal{M}_h, \quad \int_\Omega \mu_h(\mathrm{div}\, p_h + f) \, \mathrm{d}x = 0.$$

In this particular case, the spaces $\mathcal{W}_h$ and $\mathcal{M}_h$ should be related as follows: First, the implication

$$q_h \in \mathcal{W}_h \quad \text{and} \quad \forall \mu_h \in \mathcal{M}_h, \quad \int_\Omega \mu_h \, \mathrm{div}\, q_h \, \mathrm{d}x = 0 \Rightarrow \mathrm{div}\, q_h = 0,$$

should hold. Secondly, *Brezzi's condition*:

$$0 < \beta = \inf_{\mu_h \in \mathcal{M}_h} \sup_{q_h \in \mathcal{W}_h} \frac{\int_\Omega \mu_h \, \mathrm{div}\, q_h \, \mathrm{d}x}{|\mu_h|_{0,\Omega} \|q_h\|_{H(\mathrm{div};\Omega)}},$$

should hold. Under these assumptions, the discrete problem has a unique solution and there exists a constant C independent on the subspaces $\mathscr{W}_h$ and $\mathscr{M}_h$ such that

$$\|p - p_h\|_{H(\mathrm{div};\Omega)} + |\lambda - \lambda_h|_{0,\Omega} \leq$$
$$\leq C\left(\inf_{q_h\in\mathscr{W}_h} \|p - q_h\|_{H(\mathrm{div};\Omega)} + \inf_{\mu_h\in\mathscr{M}_h} |\lambda - \mu_h|_{0,\Omega}\right).$$

Raviart & Thomas (1977a) have constructed various finite element spaces $\mathscr{W}_h$ and $\mathscr{M}_h$ which satisfy Brezzi's condition and they have obtained the corresponding orders of convergence. See also Mansfield (1976a) for further results. Scholz (1976, 1977), has obtained estimates of the error $(|\lambda - \lambda_h|_{0,\infty,\Omega} + h|p - p_h|_{0,\infty,\Omega})$, by adapting the method of weighted norms of J.A. Nitsche.

Let us also briefly review mixed finite element methods for the plate problem, as first proposed by Hermann (1967) (for another mixed method for plates, see Poceski (1975)). In this case, the dual formulation is defined as follows: Let

$$\mathscr{V}(f) = \{q = (q_i)_{i=1}^3 \in (L^2(\Omega))^3;$$
$$\partial_{11}q_1 + 2\partial_{12}q_2 + \partial_{22}q_3 = f \quad \text{in} \quad \Omega\},$$

$$I(q) = \frac{1}{2}\int_\Omega (q_1^2 + 2q_2^2 + q_3^2)\,\mathrm{d}x.$$

Then there exists a unique element $p \in \mathscr{V}(f)$ such that $I(p) = \inf_{q\in\mathscr{V}(f)} I(q)$, and one has precisely

$$p = (\partial_{11}u, \partial_{12}u, \partial_{22}u).$$

This is of particular interest for plates, where the second partial derivatives $\partial_{ij}u$ yield in turn the moments. In the primal-dual formulation, the triple $(\partial_{11}u, \partial_{12}u, \partial_{22}u)$ is the first argument of the saddle-point, while the second argument, i.e., the Lagrange multiplier, turns out to be the displacement u itself. For an analysis of such methods, see Johnson (1972, 1973), Kikuchi & Ando (1972a, 1973a), Miyoshi (1973a), Samuelsson (1973).

Recently, Brezzi & Raviart (1976) have developed a general theory of mixed methods for fourth-order problems, which contains in particular the analysis of Section 7.1, as well as the analyses of C. Johnson and T. Miyoshi quoted above. F. Brezzi and P.-A. Raviart also obtain optimal error estimates in the norm $|\cdot|_{0,\Omega}$.

As advocated in particular by Taylor & Hood (1973), finite element methods of mixed type seem more and more popular for approximating the solutions of Stokes and Navier–Stokes problems (cf. the sections

"Additional Bibliography and Comments" at the end of Chapters 4 and 5). Such methods have been studied by BERCOVIER (1976), BERCOVIER & LIVNE (1976), FORTIN (1976), GIRAULT (1976c), RAVIART (1976).

Mixed methods are also increasingly used for solving nonlinear problems such as the von Karmann equations (cf. MIYOSHI (1976a, 1976c, 1977)), elastoplastic plates (cf. BREZZI, JOHNSON & MERCIER (1977)), nonlinear problems of monotone type as considered in Section 5.3 (cf. BERCOVIER (1976), SCHEURER (1977)).

To sum up, mixed methods yield simultaneous approximations of the solutions of both the primal and dual problems. Since the solution of the dual problem consists in practice of derivatives of the solution of the primal problem (e.g. ∇u for the model second-order problem), the terminology *mixed method* can also be used more generally for any approximation procedure in which an unknown *and* some of its derivatives are simultaneously approximated, irrespective of whether this is achieved through duality techniques. This is in particular the definition of J.T. Oden, who has done a thorough study of such methods. See ODEN (1972b, 1973c), ODEN & LEE (1975), ODEN & REDDY (1975, 1976a, Section 8.10, 1976c), REDDY (1973), REDDY & ODEN (1973), BABUŠKA, ODEN & LEE (1977).

In the light of the two above possible definitions of mixed methods, let us return to the method studied in the present chapter. The method described in Section 7.1 is mixed in the general sense only: The pair $(u, -\Delta u)$ is obtained through a minimization problem which, although not the standard one, is regarded as a primal problem. By contrast, the method described in Section 7.2 is mixed in the restricted sense, in that it is a natural discretization of a primal-dual problem (as described in Exercise 7.1.1).

For further references concerning the mathematical analysis of mixed methods, see KIKUCHI (1976a), HASLINGER & HLAVÁČEK (1975, 1976a, 1976b). The first abstract analyses of such methods are due to AUBIN & BURCHARD (1971) (see also AUBIN (1972)), and BABUŠKA (1971b). In particular, I. Babuška developed an abstract theory which resembles that of F. Brezzi and which is the basis of the paper of BABUŠKA, ODEN & LEE (1977).

Hybrid methods

A problem to be approximated by a mixed method is in practice formulated in such a way that the unknown u is a function together with some derivatives of this function *over the set* $\bar{\Omega}$, e.g. $(u, \nabla u)$ for the model problem, $(u, (\partial_{11}u, \partial_{12}u, \partial_{22}u))$ for the plate problem, etc....

In another class of finite element methods, the unknown is a function together with some derivatives of this function *along the boundaries of appropriate subdomains of the set* $\bar{\Omega}$. Accordingly, an appropriate variational formulation of the given problem needs to be developed. To show how this is achieved, consider again the model problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \quad f \in L^2(\Omega), \\ u = 0 & \text{on} \quad \Gamma. \end{cases}$$

Assume that a triangulation $\mathcal{T}_h$ is established over the set $\bar{\Omega}$, in the sense that relations $(\mathcal{T}_h i)$, $1 \leq i \leq 4$ (cf. Section 2.1), are satisfied (at this stage, the sets K occurring in the decomposition $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$ need not be related to actual finite elements), and assume that the objective is to compute not only the "displacement" u, but also the "stresses" along the boundaries ∂K of the sets K, understood here as the normal derivatives $\partial_{\nu_K} u$, $K \in \mathcal{T}_h$. We introduce the spaces

$$V(\mathcal{T}_h) = \prod_{K \in \mathcal{T}_h} H^1(K) \subset L^2(\Omega),$$

$$M(\mathcal{T}_h) = \Big\{ \mu \in \prod_{K \in \mathcal{T}_h} H^{-1/2}(\partial K);\ \exists q \in H(\mathrm{div}; \Omega), \\ \forall K \in \mathcal{T}_h,\ q \cdot \nu_K = \mu|_{\partial K} \Big\},$$

provided with the norms

$$\|v\|_{V(\mathcal{T}_h)} = \Big(\sum_{K \in \mathcal{T}_h} \|v\|_{1,K}^2 \Big)^{1/2},$$

$$\|\mu\|_{M(\mathcal{T}_h)} = \inf \Big\{ \|q\|_{H(\mathrm{div};\Omega)};\ \exists q \in H(\mathrm{div}; \Omega), \\ \forall K \in \mathcal{T}_h,\ q \cdot \nu_K = \mu|_{\partial K} \Big\},$$

respectively (recall that if a function q belongs to the space $H(\mathrm{div}; K)$, its outer normal component $q \cdot \nu_K$ is well-defined as an element of the space $H^{-1/2}(\partial K)$). It can be verified that there exists a unique pair $(\bar{u}, \lambda) \in V(\mathcal{T}_h) \times M(\mathcal{T}_h)$ such that

$$\forall v \in V(\mathcal{T}_h), \quad \sum_{K \in \mathcal{T}_h} \int_K \nabla \bar{u} \cdot \nabla v \, dx - \sum_{K \in \mathcal{T}_h} \langle \lambda, v \rangle_{\partial K} = \int_\Omega f v \, dx,$$

$$\forall \mu \in M(\mathcal{T}_h), \qquad - \sum_{K \in \mathcal{T}_h} \langle \mu, v \rangle_{\partial K} = 0,$$

where $\langle\cdot,\cdot\rangle_{\partial K}$ denotes the duality between the spaces $H^{-1/2}(\partial K)$ and $H^{1/2}(\partial K)$. Moreover one has

$$\bar{u} = u, \quad \text{and} \quad \forall K \in \mathcal{T}_h, \quad \lambda|_{\partial K} = \partial_{\nu_K} u.$$

Notice that even though the unknown u lies in the space $\Pi_{K\in\mathcal{T}_h} H^1(K)$, where no continuity is *a priori* required, the function u is nevertheless automatically in the space $H_0^1(\Omega)$ (this is so because of the particular form of the constraint which appear in the definition of the space $M(\mathcal{T}_h)$).

Because the first unknown in such a formulation is the solution of the *primal* problem, we shall refer to this formulation as a *primal hybrid model*, or *formulation*, and the corresponding saddle-point, or variational, problem will be called *primal hybrid problem*. Accordingly, finite element methods based on this formulation shall be called *primal hybrid methods*.

Let us briefly describe such a method for the model problem. Assume that $n = 2$ and that the sets K are triangles. Then for some integer $k \geqslant 1$, we let $\bar{V}_h = \Pi_{K\in\mathcal{T}_h} P_k(K)$, so that the space $\bar{V}_h$ is contained in the space $V(\mathcal{T}_h)$. Next, for some integer $m \geqslant 0$ and for each triangle $K \in \mathcal{T}_h$, we let $S_m(\partial K) = \Pi_{i=1}^3 P_m(K_i')$ where K_i', $1 \leqslant i \leqslant 3$, denote the three sides of K. Then the other space M_h consists of those functions μ in the space $\Pi_{K\in\mathcal{T}_h} S_m(\partial K)$ which satisfy $\mu|_{K_1} + \mu|_{K_2} = 0$ along the side $K_1 \cap K_2$ whenever it happens that the triangles K_1 and K_2 are adjacent. In this fashion, we have constructed a subspace M_h of the space $M(\mathcal{T}_h)$. As in the case of mixed methods for second-order problems, the error analysis can be again based on the abstract approach of Brezzi (1974b). Once Brezzi's condition is verified (in the case of the above spaces $\bar{V}_h$ and M_h, it is satisfied if $k \geqslant m + 1$, m even, or $k \geqslant m + 2$ if m is odd), estimates are obtained for the error

$$\Big(\sum_{K\in\mathcal{T}_h} \|\bar{u} - \bar{u}_h\|_{1,K}^2\Big)^{1/2} + \Big(\sum_{K\in\mathcal{T}_h} h_K \|\lambda - \lambda_h\|_{L^2(\partial K)}^2\Big)^{1/2}.$$

These results are found in Raviart & Thomas (1977b), Thomas (1977). See also the survey of Raviart (1975).

Notice that since the first unknown is in a finite element space not contained in the space $H^1(\Omega)$, this method may be regarded as "nonconforming for the unknown u". In fact, the connection with nonconforming methods is deeper, as shown in the above references in particular. For related ideas, see Babuška, Oden & Lee (1977), Crouzeix & Raviart (1973), Irons & Razzaque (1972a).

Additional references for second-order problems are BREZZI (1974a) for the nonhomogeneous Neumann problem for the operator $-\Delta$, KIKUCHI (1973) for plane stress problems.

A complementary approach to the theory of mixed methods has been contributed by BABUŠKA, ODEN & LEE (1977), who have developed a general theory of *mixed-hybrid methods*, which includes results on both mixed and hybrid methods as special cases. In this direction, see also ODEN (1976a), ODEN & REDDY (1976a, Section 8.10), ODEN & LEE (1975, 1977).

One may analogously construct another formulation in which the first unknown is the solution of the dual problem while the second unknown is the trace along the boundaries ∂K of the solution of the primal problem. In the case of the model problem, we define the spaces

$$\mathcal{V}(f, \mathcal{T}_h) = \{q \in (L^2(\Omega))^n;\ \forall K \in \mathcal{T}_h,\ \operatorname{div} q + f = 0 \quad \text{in} \quad K\},$$

$$M'(\mathcal{T}_h) = \Big\{\mu \in \prod_{K\in\mathcal{T}_h} H^{1/2}(\partial K),\ \exists v \in H_0^1(\Omega),$$
$$\forall K \in \mathcal{T}_h,\ v|_{\partial K} = \mu|_{\partial K}\Big\},$$

provided with the norms $|\cdot|_{0,\Omega}$ and

$$\|\mu\|_{M'(\mathcal{T}_h)} = \inf\{\|v\|_{1,\Omega};\ \exists v \in H_0^1(\Omega),\ \forall K \in \mathcal{T}_h,\ v|_{\partial K} = \mu|_{\partial K}\},$$

respectively. Then there exists a unique pair $(p, \lambda) \in \mathcal{V}(f, \mathcal{T}_h) \times M'(\mathcal{T}_h)$ such that

$$\forall q \in \mathcal{V}(0, \mathcal{T}_h),\quad \int_\Omega p \cdot q \,\mathrm{d}x - \sum_{K\in\mathcal{T}_h} \langle q \cdot \nu_K, \lambda\rangle_{\partial K} = 0,$$

$$\forall \mu \in M'(\mathcal{T}_h),\quad -\sum_{K\in\mathcal{T}_h} \langle p \cdot \nu_K, \mu\rangle_{\partial K} = 0,$$

and one has in addition

$$p = \nabla u \quad \text{and} \quad \forall K \in \mathcal{T}_h,\quad \lambda|_{\partial K} = u|_{\partial K}.$$

Such a formulation is called *dual hybrid model*, or *formulation*, and the corresponding saddle-point, or variational, problem, is called *dual hybrid problem*. The finite element approximation of such problems yields to *dual hybrid methods*, for an extensive study of which we refer to THOMAS (1976, 1977), in the case of second-order problems. Dual hybrid methods for the plate problem have been thoroughly studied by BREZZI (1975) and BREZZI & MARINI (1975). In this case, the first unknown is the triple $(\partial_{11}u, \partial_{12}u, \partial_{22}u)$ (i.e., the solution of the corresponding dual problem), while the second unknown are the triples $(u|_{\partial K}, \partial_1 u|_{\partial K}, \partial_2 u|_{\partial K})$, $K \in \mathcal{T}_h$. Hybrid methods for plates have been also studied by KIKUCHI (1973), KIKUCHI & ANDO (1972b, 1972c, 1972d, 1973b). REDDY (1976) has extended to fourth-order problems the mixed-hybrid approach of BABUŠKA, ODEN & LEE (1977).

Hybrid methods have been proposed and advocated by ALLMAN (1976), FRAEIJS DE VEUBEKE (1965b, 1973), HENSHELL (1973), JONES (1964), PIAN (1971, 1972), PIAN & TONG (1969a, 1969b), TORBE & CHURCH (1975), WOLF (1975).

In the same fashion as we extended the definition of mixed methods, we may define more generally as a *hybrid method* any finite element method based on a formulation where one unknown is a function, or some of its derivatives, on the set Ω, and the other unknown is the trace of some of the derivatives of the same function, or the trace of the function itself, along the boundaries of the set K. In other words, we ignore in this new acceptation that in practice, such methods are based on appropriate primal hybrid, or dual hybrid, formulations.

Even more general acceptations exist. For example FIX (1976) states that a finite element method is *hybrid* as soon as (any kind of) duality techniques are used for treating troublesome constraints. The use of Lagrange multipliers for handling boundary conditions, as proposed by BABUŠKA (1973a), is an example of such methods.

An attempt of general classification of finite element methods

Table 1 summarizes the previous considerations. For definiteness, we have formulated the problems as minimization or saddle-point problems, but they could have been equally expressed in the more general form of variational equations.

The reader will notice that notable omissions among the definitions of this table are those of *conforming* and *nonconforming methods*, amply illustrated in this book for displacement methods. The reason behind these omissions is that these make up *another* classification on their own. We shall simply illustrate by two examples the possible connections that may be established between the two classifications: First, mixed methods may be subdivided into conforming and nonconforming methods. For instance, the mixed method studied by JOHNSON (1972, 1973) for plates is "nonconforming with respect to the argument u",

Table 1.

Variational problem	Particular nomenclature for the elasticity problem	Special case of the model problem $-\Delta u = f$ in Ω, $u = 0$ on Γ	Name of the finite element methods based on the same formulations
$\boxed{\text{Primal problem}}$: Find $u \in V$ such that $J(u) = \inf_{v \in V} J(v)$	Displacement model u: displacement J: potential energy V: space of admissible displacements	$J(v) = \frac{1}{2}\int_\Omega \lVert\nabla v\rVert^2 \,dx - \int_\Omega fv \,dx$ $V = H_0^1(\Omega)$	$\boxed{\text{Displacement methods}}$
$\boxed{\text{Dual problem}}$: Find $p \in \mathscr{V}(f)$ such that $I(p) = \inf_{q \in \mathscr{V}(f)} I(q)$	Equilibrium model p: stress tensor I: complementary energy $\mathscr{V}(f)$: set of admissible stresses	$I(q) = \frac{1}{2}\int_\Omega \lVert q\rVert^2 \,dx$ $\mathscr{V}(f) = \{q \in H(\mathrm{div};\Omega);\ \mathrm{div}\, q + f = 0 \text{ in } \Omega\}$ $\boxed{p = \nabla u}$	$\boxed{\text{Equilibrium methods}}$
$\boxed{\text{Primal-dual problem}}$: Find: $(p, \lambda) \in \mathscr{W} \times \mathscr{M}$ such that $\forall q \in \mathscr{W}, \forall \mu \in \mathscr{M}$, $\mathscr{L}(p, \mu) \leq \mathscr{L}(p, \lambda) \leq \mathscr{L}(q, \lambda)$	$\mathscr{L}$: Hellinger–Reissner's energy	$\mathscr{L}(q, \mu) = I(q) + \int_\Omega \mu(\mathrm{div}\, q + f)\,dx$ $\mathscr{W} = H(\mathrm{div};\Omega)$, $\mathscr{M} = L^2(\Omega)$ $\boxed{p = \nabla u, \lambda = u}$	$\boxed{\text{Mixed methods}}$

Primal hybrid problem: Find $(\bar{u}, \lambda) \in V(\mathcal{T}_h) \times M(\mathcal{T}_h)$ such that $\forall v \in V(\mathcal{T}_h)$, $\forall \mu \in M(\mathcal{T}_h)$, $L_h(\bar{u}, \mu) \leq L_h(\bar{u}, \lambda) \leq L_h(v, \lambda)$	$L_h(v, \mu) = \frac{1}{2} \sum_K \int_K \lVert \nabla v \rVert^2 \,dx - \int_\Omega f v \,dx - \sum_K \langle \mu, v \rangle_{\partial K}$ $V(\mathcal{T}_h) = \prod_K H^1(K)$ $M(\mathcal{T}_h) = \left\{ \mu \in \prod_K H^{-1/2}(\partial K);\ \exists \boldsymbol{q} \in H(\mathrm{div}; \Omega),\ \forall K \in \mathcal{T}_h,\ \boldsymbol{q} \cdot \boldsymbol{\nu}_K = \mu\vert_{\partial K} \right\}$ $\boxed{\bar{u} = u,\ \lambda\vert_{\partial K} = \partial_{\nu_K} \text{ for all } K \in \mathcal{T}_h}$	**Primal hybrid methods**
Dual hybrid problem: Find $(\boldsymbol{p}, \lambda) \in \mathcal{V}(f, \mathcal{T}_h) \times M'(\mathcal{T}_h)$ such that $\forall \boldsymbol{q} \in \mathcal{V}(f, \mathcal{T}_h)$, $\forall \mu \in M'(\mathcal{T}_h)$, $L'_h(\boldsymbol{p}, \mu) \leq L'_h(\boldsymbol{p}, \lambda) \leq L'_h(\boldsymbol{q}, \lambda)$	$L'_h(\boldsymbol{q}, \mu) = \frac{1}{2} \int_\Omega \lVert \boldsymbol{q} \rVert^2 \,dx - \sum_K \langle \boldsymbol{q} \cdot \boldsymbol{\nu}_K, \mu \rangle_{\partial K}$ $\mathcal{V}(f, \mathcal{T}_h) = \left\{ \boldsymbol{q} \in \prod_K H(\mathrm{div}; K);\ \forall K \in \mathcal{T}_h,\ \mathrm{div}\, \boldsymbol{q} + f = 0 \text{ in } K \right\}$ $M'(\mathcal{T}_h) = \left\{ \mu \in \prod_K H^{1/2}(\partial K);\ \exists v \in H_0^1(\Omega),\ \forall K \in \mathcal{T}_h,\ v\vert_{\partial K} = \mu\vert_{\partial K} \right\}$ $\boxed{\boldsymbol{p} = \nabla u,\ \lambda\vert_{\partial K} = u\vert_{\partial K} \text{ for all } K \in \mathcal{T}_h}$	**Dual hybrid methods**

which is not required to belong to a subspace of the space $H^2(\Omega)$. Secondly, primal hybrid methods as described in the case of the model problem are automatically "nonconforming for the argument u", which lies "only" in a subspace of the space $\Pi_{K\in\mathcal{T}_h} H^1(K)$.

We could likewise take into consideration the effect of *numerical integration* and/or the effect of the *approximation of the boundary* in the case of curved domains. We suggest that a classification according to such *variational crimes* make up a *secondary classification of finite element methods*, while the classification of the above table, i.e., based on the *formulation* of the problem, make up the *primary classification of finite element methods.*