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### THE USE OF POINTWISE INTERPOLATION IN DOMAIN DECOMPOSITION METHODS WITH NONNESTED MESHES\*

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**Abstract.** In this paper, the author develops a new technique and a corresponding theory for Schwarz-type overlapping domain decomposition methods for solving large sparse linear systems which arise from finite element discretization of elliptic partial differential equations. The theory provides an optimal convergence of an additive Schwarz algorithm that is constructed with a nonnested coarse space and a not necessarily shape regular subdomain partitioning. The theory is also applicable to the graph partitioning algorithms recently developed by Cai and Saad [*Overlapping domain decomposition algorithms for general sparse matrices*, Preprint 93-027, Univ. of Minnesota] and Farhat and Lesoine [*Internat. J. Numer. Methods Engrg.*, 36 (1993), pp. 745–764] for problems defined on unstructured meshes.

**Key words.** unstructured grids, nonnested meshes, domain decomposition, iterative method

**AMS subject classifications.** 65F10, 65N30

**1. Introduction.** Considerable interest has developed regarding Schwarz-type overlapping domain decomposition methods for the numerical solution of partial differential equations; see, for example, [2]–[4], [6], [7], [12], [13], [18], and the references therein. This class of methods offers a great deal of parallelism and is very promising for modern parallel computers. The success of the methods depends heavily on the existence of a uniformly, or nearly uniformly, bounded decomposition of the function space in which the problem is defined. In this paper, we further enrich the Schwarz theory by providing a new technique of constructing a uniformly bounded decomposition of the problem space, which is more flexible and convenient for large, geometrically complicated, practical problems. It has a convergence rate that is similar to that of the regular Dryja–Widlund-type decomposition [13], and does not require the coarse space to be a subspace of the original finite element space, in which the partial differential equation is discretized. Nor does it require that the collection of the unextended subdomains forms a regular finite element subdivision.

We shall only discuss a two-level additive Schwarz algorithm with a coarse and a fine grid. It is well known that the fine grid determines the accuracy of the discrete problem and that the only role of the coarse grid is to accelerate the convergence of the iterative method. In this paper, we try to minimize the interconnection between the two grids by using a not necessarily nested coarse grid. As a result, the same coarse grid can be used even if the fine grid is locally refined, or remeshed, to deal with the local singularity of the underlining problem. There are a number of ways to handle the communication between the two grids. In this paper, we insist on the computationally simplest one, i.e., pointwise interpolation. We show that this, sometimes

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troublesome, interpolation operator behaves well in both two- and three-dimensional space in our applications. For technical reasons, we assume that the coarse mesh is quasi-uniform; however, no such assumption is needed for the fine mesh. Other recent developments along this line can be found in [9]–[11], [19], [20]. Some theory and experiments with the pointwise interpolation in the context of nonnested multigrid methods can be found in [1], [8], [16], [17], and the references therein.

In [14], Dryja and Widlund developed a general theory for Schwarz-type algorithms which has a convergence rate characterized by the quantity  $(1 + H/\delta)$ , where  $H$  measures the diameters of the subdomains (as well as the coarse mesh size) and  $\delta$  the overlap between neighboring subdomains. This quantity indicates that subdomains with uniform aspect ratio are desired. In this paper, we develop a result involving  $\min\{1 + H_c^2/\delta^2, 1 + H/\delta + H_c/\delta \cdot H_c/H\}$ , where  $H_c$  is the size of the coarse grid, which generally has nothing to do with the subdomain diameter  $H$ . The first quantity is independent of  $H$ . This allows us to use subdomains of arbitrary shape. As a consequence, our theory applies to the type of unstructured mesh problems decomposed by some graph-based partitioning techniques discussed by Cai and Saad in [5]. The second quantity reduces to that of Dryja and Widlund when  $H \approx H_c$  and comes into play in the case of small overlap.

When solving a system of equations arising from the discretization of nonselfadjoint, or indefinite, or nonlinear elliptic problems by a Schwarz-type algorithm, a fine enough coarse mesh space is usually used in order to make the convergence rate optimal; see e.g., [6], [7]. In such a case, using the Dryja–Widlund construction [13] would normally result in a large number of subdomains that must be combined in order for the number of subproblems to fit the number of processors of a parallel computer. With our new construction, the size of the coarse mesh is totally independent of the number of subdomains.

In this paper, we shall focus only on a simple selfadjoint model problem, namely, the homogeneous Dirichlet boundary value problem: Find  $u_h^* \in V_h \subset H_0^1(\Omega)$  such that

$$(1) \quad a(u_h^*, v) = (f, v), \quad \forall v \in V_h,$$

where the bilinear form  $a(u, v)$  is defined by  $a(u, v) = \int_{\Omega} \nabla u \nabla v dx$ ,  $f(x) \in L^2(\Omega)$  is given, and  $\Omega$  is an open bounded polygon in  $R^d$  ( $d = 2$  or  $3$ ) with boundary  $\partial\Omega$ . We shall use  $a(\cdot, \cdot)$  and  $\|\cdot\|_a$  to denote the inner product and norm of  $H_0^1(\Omega)$ . To introduce the finite element discretization and the finite element space  $V_h$ , we let  $\Omega_h = \{k_i\}$  be a standard finite element triangulation of  $\Omega$  that satisfies the minimal angle condition; i.e., in the two-dimensional case  $\gamma_k \geq \gamma_0 > 0$  for any  $k \in \Omega_h$ . Here  $\gamma_k$  is the minimal interior angle of  $k \in \Omega_h$  and  $\gamma_0$  a constant. We do not assume that the triangulation is quasi-uniform. We allow the use of highly refined unstructured meshes. We define the corresponding finite element space  $V_h \subset H_0^1(\Omega)$  as the regular piecewise linear continuous triangular finite element space on  $\Omega$ . Let us denote by  $\bar{h}$  the maximum diameter of this finite element mesh which will be used later to restrict the size of the coarse grid.

Throughout this paper,  $c$  and  $C$ , with or without subscripts, denote generic, strictly positive constants which are independent of any of the mesh parameters.

**2. A nonnested coarse mesh space.** We begin by introducing several notations. Let  $\Omega_{H_c} = \{\tau_i\}$  be a quasi-uniform triangulation of  $\Omega$  and  $\tau_i$  one of the triangles whose diameter is of order  $H_c$ .  $\Omega_{H_c}$  will be referred to as the coarse grid. Here  $H_c$  is the maximum diameter of this coarse triangulation. We assume, throughout this paper, that each fine triangle intersects with a finite number of coarse triangles,

$$(2) \quad \bar{h} \leq C H_c, \quad \text{and}$$

$$(3) \quad |k_i| \leq C |\tau_j| \quad \text{if } k_i \cap \tau_j \neq \emptyset.$$

Here and in the rest of the paper,  $|\cdot|$  means the area in  $R^2$  and the volume in  $R^3$ . Let  $V_{H_c} \subset H_0^1(\Omega)$  be a shape-regular finite element space over  $\Omega$  consisting of piecewise linear continuous functions. Note that, in general,  $V_{H_c} \not\subset V_h$ , and it is not necessary for  $V_{H_c}$  to have the same type of elements as  $V_h$ . Let  $\Pi_h : C^0(\bar{\Omega}) \rightarrow V_h$  be the usual piecewise linear continuous interpolation operator, which uses values only at the nodal points of the fine mesh triangulation. This operator has the following properties.

LEMMA 2.1. *There exists a constant  $C > 0$ , independent of  $\bar{h}$ ,  $H_c$ , such that*

- (i)  $\|\Pi_h v\|_a \leq C \|v\|_a \quad \forall v \in V_{H_c}$ ;
- (ii)  $\|v - \Pi_h v\|_{L^2(\Omega)} \leq C \bar{h} \|v\|_a \quad \forall v \in V_{H_c}$ .

We note that estimates (i) and (ii) do not hold if  $v$  is an arbitrary function in  $H^1(\Omega)$ . However, we need the bounds only for functions in the subspace  $V_{H_c}$ . A proof of Lemma 2.1 will be given in §4. Let

$$V_0 \equiv \Pi_h V_{H_c} \equiv \{v \in V_h, \text{ there exists } w \in V_{H_c}, \text{ such that } v = \Pi_h w\},$$

which is a subspace of  $V_h$ . We shall use the  $L^2$  projection operator  $Q_{H_c} : H_0^1(\Omega) \rightarrow V_{H_c}$  defined by  $(Q_{H_c} u, v) = (u, v)$ , for any  $u \in H_0^1(\Omega)$ , and  $v \in V_{H_c}$ .

We now partition  $\Omega$  into nonoverlapping subdomains  $\{\Omega_i\}$ , such that no  $\partial\Omega_i$  cuts through any elements  $k_i$ , and  $\bar{\Omega} = \cup_{i=1}^N \bar{\Omega}_i$ . Note that we do not assume that  $\{\Omega_i\}$  forms a regular finite element subdivision of  $\Omega$ , nor that the diameters of  $\Omega_i$  are of the same order. In practice, a graph-based partitioning technique, such as those introduced in [5], [15], can often be used to obtain  $\Omega_i$ , especially if  $\Omega_h$  is an unstructured grid. To obtain an overlapping decomposition of  $\Omega$ , we extend each  $\Omega_i$  to a larger subdomain  $\Omega'_i \supset \Omega_i$ , which is also assumed not to cut any fine mesh triangles, such that  $\text{distance}(\partial\Omega'_i \cap \Omega, \partial\Omega_i \cap \Omega) \geq c\delta$ ,  $\forall i$ , for a constant  $c > 0$ . Here  $\delta > 0$  will be referred to as the overlapping size. For each  $\Omega'_i$ , we define a finite element space  $V_i \equiv V_h \cap H_0^1(\Omega'_i)$  and extended by zero outside  $\Omega'_i$ . In the next lemma, we prove that the decomposition  $V_h = V_0 + V_1 + \cdots + V_N$  exists and is uniformly bounded.

LEMMA 2.2. *For any  $v \in V_h$ , there exist  $v_i \in V_i$ ,  $i = 1, \dots, N$ , and  $v'_0 \in V_{H_c}$  such that  $v = \Pi_h v'_0 + v_1 + \cdots + v_N$ , and, in addition, there exists a constant  $C_0 > 0$  independent of the mesh parameters such that*

$$(4) \quad \|v'_0\|_a^2 + \sum_{i=1}^N \|v_i\|_a^2 \leq C_0 \left(1 + \frac{H_c^2}{\delta^2}\right) \|v\|_a^2 \quad \forall v \in V_h.$$

*Proof.* For any  $v \in V_h$ , let  $v'_0 = Q_{H_c} v \in V_{H_c}$ ,  $v_0 = \Pi_h v'_0 \in V_0$ , and  $w = v - v_0 \in V_h$ . Because of the boundedness of  $Q_{H_c}$  in the  $H_0^1$  norm, we clearly have  $\|v'_0\|_a \leq C \|v\|_a$ . Let  $\{\theta_i(x)\}$  be a partition of unity of  $\Omega$  corresponding to  $\{\Omega'_i\}$  such that  $|\nabla \theta_i|_2 \leq C/\delta$  and  $\sum_{i=1}^N \theta_i(x) \equiv 1$  ( $|\cdot|_2$  is the usual Euclidean norm in  $R^2$  or  $R^3$ ). Of course,  $\theta_i$  are smooth and  $0 \leq \theta_i \leq 1$ . We define  $v_i = \Pi_h(\theta_i w) \in V_i$ . It is easy to see that  $v_0 + \sum_{i=1}^N v_i = v$ , therefore the existence of such a decomposition is proved.

Let  $k$  be a single triangular element in  $\Omega'_i$  with diameter  $h$ . We assume that the average of  $\theta_i$  over  $k$  is  $\bar{\theta}_{i,k}$ . It can be seen that

$$(5) \quad |v_i|_{H^1(k)}^2 \leq 2|\bar{\theta}_{i,k} \Pi_h w|_{H^1(k)}^2 + 2|\Pi_h((\theta_i - \bar{\theta}_{i,k})w)|_{H^1(k)}^2.$$

Because  $\Pi_h w = w$ , the first term on the right-hand side of this inequality presents no problems. We next estimate the second term. With the help of the elementwise inverse inequality, we have

$$\begin{aligned}
 |\Pi_h((\theta_i - \bar{\theta}_{i,k})w)|_{H^1(k)} &\leq C \frac{1}{h} \|\Pi_h((\theta_i - \bar{\theta}_{i,k})w)\|_{L^2(k)} \\
 &\leq C \frac{1}{h} \frac{h}{\delta} \|w\|_{L^2(k)} = C \frac{1}{\delta} \|w\|_{L^2(k)},
 \end{aligned}
 \tag{6}$$

where  $C$  depends only on the finite element subdivision of  $\Omega$ . The fact  $|\theta_i - \bar{\theta}_{i,k}| \leq C h/\delta$  is also used. By taking the sum over all elements  $k \in \Omega'_i$ , we arrive at the estimate

$$|v_i|_{H^1(\Omega'_i)}^2 \leq C \left( |w|_{H^1(\Omega'_i)}^2 + \frac{1}{\delta^2} \|w\|_{L^2(\Omega'_i)}^2 \right), \tag{7}$$

which implies that

$$\sum_{i=1}^N |v_i|_{H^1(\Omega'_i)}^2 \leq C \left( \|w\|_a^2 + \frac{1}{\delta^2} \|w\|_{L^2(\Omega)}^2 \right). \tag{8}$$

Here the fact that each point in  $\Omega$  is covered by only a finite number of overlapping subdomains is assumed. To bound the first term on the right-hand side of (8) in terms of  $\|v\|_a$ , we use the boundedness of the operators  $\Pi_h$  and  $\mathcal{Q}_{H_c}$  in the  $H_0^1$  norm; i.e.,

$$\|w\|_a = \|v - \Pi_h \mathcal{Q}_{H_c} v\|_a \leq C (\|v\|_a + \|\mathcal{Q}_{H_c} v\|_a) \leq C \|v\|_a. \tag{9}$$

To estimate the second term on the right-hand side of (8) in terms of  $\|v\|_a$ , we need the  $L^2$  regularity estimates of  $\Pi_h$  (Lemma 2.1 (ii)) and  $\mathcal{Q}_{H_c}$ , which give us

$$\|w\|_{L^2(\Omega)} \leq \|v - \mathcal{Q}_{H_c} v\|_{L^2(\Omega)} + \|\mathcal{Q}_{H_c} v - \Pi_h \mathcal{Q}_{H_c} v\|_{L^2(\Omega)} \leq C H_c \|v\|_a. \tag{10}$$

Recall that  $\bar{h} \leq C H_c$  by assumption. The proof of the lemma thus follows immediately by combining the estimates (8), (9), and (10).  $\square$

**3. An estimate for the small overlap case.** The decomposition bound (4), provided in the previous section, grows at a rate proportional to  $1/\delta^2$ , which is rather large when a small overlap is used. In this section, we discuss an alternative estimate, for the same decomposition described in Lemma 2.2, and prove that it is in fact proportional only to  $1/\delta$ . However, for the small overlap case, we do need to assume that the  $\Omega'_i$ 's have approximately the same size; i.e., if  $H_i$  is the diameter of  $\Omega'_i$ , then there exists a constant  $\beta$  such that  $1 \geq \min\{H_i\}/\max\{H_i\} \geq \beta$ .

LEMMA 3.1 (Dryja and Widlund [14]). *Let  $\Gamma_{\delta,i} \subset \Omega'_i$  be the set of points that is within a distance of  $\delta$  of  $\partial\Omega_i \cap \Omega$ . Then*

$$\|w\|_{L^2(\Gamma_{\delta,i})}^2 \leq C \delta^2 \left( \left(1 + \frac{H}{\delta}\right) |w|_{H^1(\Omega'_i)}^2 + \frac{1}{\delta H} \|w\|_{L^2(\Omega'_i)}^2 \right), \quad \forall w \in H^1(\Omega'_i), \quad \forall i. \tag{11}$$

Here  $H$  is the maximum diameter of these  $\Omega'_i$ 's. The constant  $C$  may depend on  $\beta$ .

LEMMA 3.2. *The same decomposition described in Lemma 2.2 exists and is bounded in the sense that there exists a constant  $C_0 > 0$  independent of the mesh parameters such that*

$$\|v'_0\|_a^2 + \sum_{i=1}^N \|v_i\|_a^2 \leq C_0 \left( 1 + \frac{H}{\delta} + \frac{H_c^2}{\delta H} \right) \|v\|_a^2 \quad \forall v \in V_h. \tag{12}$$

*Proof.* The proof is nearly the same as that for Lemma 2.2, except that we make use of the fact that  $\theta_i - \bar{\theta}_{i,k} = 0$  if  $k \subset \Omega'_i \setminus \Gamma_{\delta,i}$ .  $\square$

We note that the aspect ratio, or diameter, of the subdomains  $H$  appears in the estimate in the case when small overlap is being used. The factor  $H$  is introduced into the estimate by Lemma 3.1. We do not know whether it can be removed or replaced by a quantity that is independent of the subdomain aspect ratio. We also comment that if  $H_c \approx H$ , then the result of Lemma 3.2 coincides with that of [14].

**4. The boundedness of  $\Pi_h$ .** If the interpolation operator  $\Pi_h$  is considered as a map from the space  $H_0^1(\Omega)$  to  $V_h$ , then it generally does not satisfy the bounds stated in Lemma 2.1, because the values of a  $H^1(\Omega)$ -function are not necessarily well defined at the mesh points. In this section, we prove, however, that  $\Pi_h$ , restricted to the subspace  $V_{H_c} \subset H_0^1(\Omega)$ , is indeed bounded. In a trivial case, when  $V_{H_c} \subset V_h$  then Lemma 2.1(i) holds as an equality with  $C = 1$ , and (ii) holds with  $C = 0$ . The elementary proof provided in this section applies to both  $d = 2$  and 3. Several different proofs have been obtained recently; see [9], [11], [19], [20].

We begin with Lemma 2.1(i). The essential step is to establish the estimate

$$(13) \quad |\Pi_h u|_{H^1(k)}^2 \leq C \left( \sum |\nabla u|_\tau|_2^2 \right) |k|, \quad \forall u \in V_{H_c},$$

where  $k \in \Omega_h$ , and the summation is taken over all  $\tau \in \Omega_{H_c}$  that have nonempty intersection with  $k$ . We note that  $\nabla u|_\tau$  is a constant vector (since  $u$  is linear in  $\tau$ ) and  $|\cdot|_2$  is the usual Euclidean norm in  $R^d$ .

If  $k$  belongs completely to a single  $\tau$ , then (13) is obviously true. Otherwise, we denote by  $A_i$ ,  $i = 1, \dots, d+1$ , the vertices of  $k$ , and it is known that

$$(14) \quad |\Pi_h u|_{H^1(k)}^2 \leq C \sum_{i,j=1, i < j}^{d+1} (u(A_i) - u(A_j))^2 h_k^{d-2}.$$

Here  $h_k$  is the diameter of  $k$ . Let  $A_i A_j$  be the line segment connecting points  $A_i$  and  $A_j$ . We assume that  $A_i A_j$  is cut into  $l$  pieces by the coarse tetrahedra  $\tau_1^{ij}, \dots, \tau_l^{ij}$ , and  $u(\cdot)$  is linear on each piece. By the assumption made at the beginning of §2,  $l$  is finite. Therefore, by using the triangle inequality and the mean value theorem, we have

$$(15) \quad (u(A_i) - u(A_j))^2 \leq 2 \sum_{m=1}^l \left| \nabla u|_{\tau_m^{ij}} \right|_2^2 h_k^2.$$

(13) can thus be proved by combining the estimates (15) and (14). For  $\tau \in \Omega_{H_c}$ , we denote by  $\tau_j$ ,  $j = 1, \dots, l_1$ , all the coarse tetrahedra that share at least one of the fine tetrahedra with  $\tau$  (i.e., this fine tetrahedron intersects with both  $\tau$  and  $\tau_j$ ).  $l_1$  is a finite number. By summing (13) over all  $k_i \in \Omega_h$ ,  $i = 1, \dots, m$ , whose intersection with  $\tau$  is nonempty, we obtain

$$(16) \quad |\Pi_h u|_{H^1(\tau)}^2 \leq \sum_{i=1}^m |\Pi_h u|_{H^1(k_i)}^2 \leq C \sum_{j=1}^{l_1} |\nabla u|_{\tau_j}|_2^2 |\tau_j|.$$

Here we used the fact that, for each  $\tau_j$ , the sum of the areas of the fine tetrahedra that intersect with  $\tau_j$  is less than  $C |\tau_j|$  because of the assumption  $\bar{h} \leq C H_c$ . The proof for Lemma 2.1(i) follows immediately by summing (16) over all  $\tau$  in  $\Omega_{H_c}$  (the number of repetitions, for each  $\tau$ , in the summation is finite).

We now turn to the proof of Lemma 2.1(ii). Let  $k$  be a fine tetrahedron and  $A$  one of its vertices, which implies that  $w(A) = 0$ . Here  $w = u - \Pi_h u$ . We estimate the integral

$$(17) \quad \|w\|_{L^2(k)}^2 = \int_k w^2(X) d\Omega = \int_k (w(X) - w(A))^2 d\Omega.$$

Let  $XA$  be the line segment connecting points  $X$  and  $A$ . Using the same argument as before, we assume  $XA$  is cut into  $l_2$  pieces by coarse tetrahedra  $\tau_1^k, \dots, \tau_{l_2}^k$ . Since  $w$  is linear on each piece, we have

$$(18) \quad (w(X) - w(A))^2 \leq 2 \sum_{m=1}^{l_2} |\nabla w|_{\tau_m^k}|_2^2 h_k^2.$$

By combining the results of (17), (18), and assumptions made at the beginning of §2, we arrive at

$$(19) \quad \|w\|_{L^2(k)}^2 \leq Ch_k^2 \sum |\nabla w|_{L^2(\tau)}^2,$$

where the sum is taken over all coarse tetrahedra which intersect with  $k$ . Summing (19) over all  $k$  and noting that the number of coarse tetrahedra that overlap with each  $k$  is finite, we obtain

$$\|w\|_{L^2(\Omega)}^2 \leq C\bar{h}^2 |\nabla w|_{L^2(\Omega)}^2.$$

Thus, the proof of Lemma 2.1(ii) follows immediately by using the result of part (i).

**5. An additive Schwarz method based on a nonnested coarse space.** In this section, we define and analyze an additive Schwarz algorithm for solving the finite element problem (1). For  $1 \leq i \leq N$ , we define the operator  $P_i : V_h \rightarrow V_i$  by  $a(P_i u, \phi) = a(u, \phi)$ , for any  $u \in V_h$  and  $\phi \in V_i$ , and define  $P'_0 : V_h \rightarrow V_{H_c}$  by  $a(P'_0 u, \phi) = a(u, \Pi_h \phi)$  for any  $u \in V_h$  and  $\phi \in V_{H_c}$ . Let  $P_0 = \Pi_h P'_0$ . Similar operators were used in the context of nonnested subspaces based multigrid methods; see, e.g., [1], [17]. Let  $P$  be defined by  $P = \sum_{i=0}^N P_i$ . It can be seen easily that  $P$  is symmetric in the inner product  $a(\cdot, \cdot)$ . Let  $g = \Pi_h g_0 + \sum_{i=1}^N g_i$  and  $g_i = P_i u_h^*$ . Following the Schwarz theory of Dryja and Widlund [13], it can be shown that if the operator  $P$  is nonsingular, then the linear operator equation

$$(20) \quad Pu_h^* = g$$

has the same solution as that of (1). We show in the next theorem that  $P$  is not only nonsingular but also uniformly bounded from both above and below. A proof can be obtained by using the standard techniques in [13].

**THEOREM 5.1.** *The following estimate holds:*

$$ca(u, u) \leq a(Pu, u) \leq Ca(u, u), \quad \forall u \in V_h,$$

where  $c = \max\{1/[C_0(1 + H_c^2/\delta^2)], 1/[C_0(1 + H/\delta + H_c^2/\delta H)]\}$ .

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