6

The behavior of transformations on intervals and manifolds

This chapter is devoted to a series of examples of transformations on intervals and manifolds whose asymptotic behavior can be explored through the use of the material developed in Chapter 5. Although results are often stated in terms of the asymptotic stability of $\{P^n\}$, where P is a Frobenius–Perron operator corresponding to a transformation S, remember that, according to Proposition 5.6.2 S is exact when $\{P^n\}$ is asymptotically stable and S is measure preserving.

In applying the results of Chapter 5, in several examples we will have occasion to calculate the variation of a function. Thus the first section presents an exposition of the properties of functions of bounded variation.

6.1 Functions of bounded variation

There are a number of descriptors of the "average" behavior of a function f: $[a,b] \to R$. Two of the most common are the **mean value** of f,

$$m(f) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

and its **variance,** $D^2(f) = m((f - m(f))^2)$. However, these are not always satisfactory. Consider, for example, the sequence of functions $\{f_n\}$ with $f_n(x) = \sin 2n\pi x$, $n = 1, 2, \ldots$. They have the same mean value on $[0,1]$, namely $m(f_n) = 0$ and the same variance $D^2(f_n) = \frac{1}{2}$; but they behave quite differently for $n >> 1$ than they do for $n = 1$. To describe these kinds of differences in the behavior of functions, it is useful to introduce the variation of a function (sometimes called the total variation).

Let f be a real-valued function defined on an interval $\Delta \subset R$ and let $[a,b]$ be a subinterval of Δ. Consider a partition of $[a,b]$ given by

$$a = x_0 < x_1 < \cdots < x_n = b \tag{6.1.1}$$

and write

$$s_n(f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| . \tag{6.1.2}$$

If all possible sums $s_n(f)$, corresponding to all subdivisions of $[a, b]$, are bounded by a number that does not depend on the subdivision, f is said to be of **bounded variation** on $[a, b]$. Further, the smallest number c such that $s_n \leqslant c$ for all s_n is called the **variation** of f on $[a, b]$ and is denoted by $\bigvee_a^b f$. Notationally this written as

$$\bigvee_a^b f = \sup s_n(f) , \tag{6.1.3}$$

where the supremum is taken over all possible partitions of the form (6.1.1).

Consider a simple example. Assume that f is a monotonic function, either decreasing or increasing. Then

$$|f(x_i) - f(x_{i-1})| = \theta[f(x_i) - f(x_{i-1})] ,$$

where

$$\theta = \begin{cases} 1 & \text{for } f \text{ increasing} \\ -1 & \text{for } f \text{ decreasing} \end{cases}$$

and, consequently,

$$\begin{aligned} s_n(f) &= \theta \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \\ &= \theta[f(x_n) - f(x_0)] = |f(b) - f(a)| . \end{aligned}$$

Thus any function that is defined and monotonic on a closed interval is of bounded variation. It is interesting (the proof is not difficult) that any function f of bounded variation can be written in the form $f = f_1 + f_2$, where f_1 is increasing and f_2 is decreasing.

Variation of the sum

Let f and g be of bounded variation on $[a, b]$. Then

$$|f(x_i) + g(x_i) - [f(x_{i-1}) + g(x_{i-1})]| \leqslant |f(x_i) - f(x_{i-1})| + |g(x_i) - g(x_{i-1})| ,$$

and, consequently,

$$s_n(f + g) \leqslant s_n(f) + s_n(g) \leqslant \bigvee_a^b f + \bigvee_a^b g .$$

Thus $(f + g)$ is of bounded variation and

$$\bigvee_a^b (f+g) \leqslant \bigvee_a^b f + \bigvee_a^b g.$$

If $f_1, \ldots, f_n$ are of bounded variation on $[a, b]$, then by an induction argument

$$\text{(V1)} \quad \bigvee_a^b (f_1 + \cdots + f_n) \leqslant \bigvee_a^b f_1 + \cdots + \bigvee_a^b f_n \tag{6.1.4}$$

follows immediately.

Variation on the union of intervals

Assume that $a < b < c$ and that the function f is of bounded variation on $[a, b]$ as well as on $[b, c]$. Consider a partition of the intervals $[a, b]$ and $[b, c]$,

$$a = x_0 < x_1 < \cdots < x_n = b = y_0 < y_1 < \cdots < y_m = c \tag{6.1.5}$$

and the corresponding sums

$$\underset{[a,b]}{s_n}(f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

$$\underset{[b,c]}{s_m}(f) = \sum_{i=1}^{m} |f(y_i) - f(y_{i-1})|.$$

It is evident that the partitions (6.1.5) jointly give a partition of $[a, c]$. Therefore,

$$\underset{[a,b]}{s_n}(f) + \underset{[b,c]}{s_m}(f) = \underset{[a,c]}{s_{n+m}}(f) \tag{6.1.6}$$

where the right-hand side of equation (6.1.6) denotes the sum corresponding to the variation of f over $[a, c]$. Observe that (6.1.6) holds only for partitions of $[a, c]$ that contain the point b. However, any additional point in the sum s_n can only increase s_n, but, since we are interested in the supremum, this is irrelevant. From equation (6.1.6) it follows that

$$\bigvee_a^b f + \bigvee_b^c f = \bigvee_a^c f.$$

Again by an induction argument the last formula may be generalized to

$$\text{(V2)} \quad \bigvee_{a_0}^{a_1} f + \cdots + \bigvee_{a_{n-1}}^{a_n} f = \bigvee_{a_0}^{a_n} f, \tag{6.1.7}$$

where $a_0 < a_1 < \cdots < a_n$ and f is of bounded variation on $[a_{i-1}, a_i]$, $i = 1, \ldots, n$.

Variation of the composition of functions

Now let $g\colon [\alpha, \beta] \to [a, b]$ be monotonically increasing or decreasing on the interval $[\alpha, \beta]$ and let $f\colon [a, b] \to R$ be given. Then the composition $f \circ g$ is well defined and, for any partition of $[\alpha, \beta]$,

$$\alpha = \sigma_0 < \sigma_1 < \cdots < \sigma_n = \beta; \tag{6.1.8}$$

the corresponding sum is

$$s_n(f \circ g) = \sum_{i=1}^{n} |f(g(\sigma_i)) - f(g(\sigma_{i-1}))| .$$

Observe that, due to the monotonicity of g, the points $g(\sigma_i)$ define a partition of $[a, b]$. Thus $s_n(f \circ g)$ is a particular sum for the variation of f and, therefore,

$$s_n(f \circ g) \leqslant \bigvee_a^b f$$

for any partition (6.1.8). Consequently,

$$\text{(V3)} \quad \bigvee_\alpha^\beta f \circ g \leqslant \bigvee_a^b f. \tag{6.1.9}$$

Variation of the product

Let f be of bounded variation on $[a, b]$ and let g be C^1 on $[a, b]$. To evaluate the variation of the product $f(x)g(x)$, $x \in [a, b]$, start from the well-known **Abel equality,**

$$\sum_{i=1}^{n} |a_ib_i - a_{i-1}b_{i-1}| = \sum_{i=1}^{n} |b_i(a_i - a_{i-1}) + a_{i-1}(b_i - b_{i-1})| .$$

Applying this equality to the sum [substituting $a_i = f(x_i)$ and $b_i = g(x_i)$]

$$s_n(fg) = \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| ,$$

the immediate result is

$$s_n(fg) \leqslant \sum_{i=1}^{n} \{|g(x_i)|\,|f(x_i) - f(x_{i-1})| + |f(x_{i-1})|\,|g(x_i) - g(x_{i-1})|\} .$$

Now, by applying the mean value theorem, we have

$$s_n(fg) \leqslant (\sup|g|)s_n(f) + \sum_{i=1}^{n} |f(x_{i-1})g'(\tilde{x}_i)|\,(x_i - x_{i-1})$$
$$\leqslant (\sup|g|) \bigvee_a^b f + \sum_{i=1}^{n} |f(x_{i-1})g'(\tilde{x}_i)|\,(x_i - x_{i-1})\,.$$

with $\tilde{x}_i \in (x_{i-1}, x_i)$. Observe that the last term is simply an approximating sum for the Riemann integral of the product $|f(x)g'(x)|$. Thus the function $f(x)g(x)$ is of bounded variation and

$$\text{(V4)} \quad \bigvee_a^b fg \leqslant (\sup|g|) \bigvee_a^b f + \int_a^b |f(x)g'(x)|\,dx\,. \tag{6.1.10}$$

Taking in particular $f \equiv 1$,

$$\text{(V4}'\text{)} \quad \bigvee_a^b g \leqslant \int_a^b |g'(x)|\,dx\,. \tag{6.1.11}$$

However, in this case, the left- and right-hand sides are strictly equal since $s_n(g)$ is a Riemann sum for the integral of g'.

Yorke inequality

Now let f be defined on $[0,1]$ and be of bounded variation on $[a,b] \subset [0,1]$. We want to evaluate the variation of the product of f and the characteristic function $1_{[a,b]}$. Without any loss of generality, assume that the partitions of the interval $[0,1]$ will always contain the points a and b. Then

$$\underset{[0,1]}{s_n}(f1_{[a,b]}) \leqslant \underset{[a,b]}{s_n}(f) + |f(a)| + |f(b)|\,.$$

Let c be an arbitrary point in $[a,b]$. Then, from the preceding inequality,

$$\underset{[0,1]}{s_n}(f1_{[a,b]}) \leqslant \underset{[a,b]}{s_n}(f) + |f(b) - f(c)| + |f(c) - f(a)| + 2|f(c)|$$
$$\leqslant 2\bigvee_a^b f + 2|f(c)|\,.$$

It is always possible to choose the point c such that

$$|f(c)| \leqslant \frac{1}{b-a}\int_a^b |f(x)|\,dx$$

so that

$$\underset{[0,1]}{s_n}(f1_{[a,b]}) \leqslant 2\bigvee_a^b f + \frac{2}{b-a}\int_a^b |f(x)|\,dx\,,$$

which gives

$$\text{(V5)} \qquad \bigvee_0^1 f 1_{[a,b]} \leqslant 2 \bigvee_a^b f + \frac{2}{b-a} \int_a^b |f(x)|\, dx . \tag{6.1.12}$$

6.2 Piecewise monotonic mappings

Two of the most important results responsible for stimulating interest in transformations on intervals of the real line were obtained by Rényi (1957) and by Rochlin (1964). Both were considering two classes of mappings, namely

$$S(x) = \tau(x) \qquad (\mathrm{mod}\ 1),\ 0 \leqslant x \leqslant 1\,, \tag{6.2.1}$$

where τ: $[0,1] \to [0,\infty)$ is a C^2 function such that $\inf_x \tau' > 1$, $\tau(0) = 0$, and $\tau(1)$ is an integer; and the **Rényi transformation**

$$S(x) = rx \qquad (\mathrm{mod}\ 1),\ 0 \leqslant x \leqslant 1\,, \tag{6.2.2}$$

where $r > 1$, is a real constant. (The r-adic transformation considered earlier is clearly a special case of the Rényi transformation.) Using a number-theoretic argument, Rényi was able to prove the existence of a unique invariant measure for such transformations. Rochlin was able to prove that the Rényi transformations on a measure space with the Rényi measure were, in fact, exact.

In this section we unify and generalize the results of Rényi and Rochlin through the use of Theorem 5.6.2.

Consider a mapping S: $[0,1] \to [0,1]$ that satisfies the following four properties:

(2i) There is a partition $0 = a_0 < a_1 < \cdots < a_r = 1$ of $[0,1]$ such that for each integer $i = 1, \ldots, r$ the restriction of S to the interval $[a_{i-1}, a_i)$ is a C^2 function;

(2ii) $S(a_{i-1}) = 0$ for $i = 1, \ldots, r$;

(2iii) There is a $\lambda > 1$ such that $S'(x) \geqslant \lambda$ for $0 \leqslant x < 1$ [$S'(a_i)$ and $S''(a_i)$ denote the right derivatives]; and

(2iv) There is a real finite constant c such that

$$-S''(x)/[S'(x)]^2 \leqslant c, \qquad 0 \leqslant x < 1\,. \tag{6.2.3}$$

An example of a mapping satisfying these conditions is shown in Figure 6.2.1.

Then we may state the following theorem.

Theorem 6.2.1. If S: $[0,1] \to [0,1]$ satisfies the foregoing conditions (2i)–(2iv) and P is the Frobenius–Perron operator associated with S, then $\{P^n\}$ is asymptotically stable.

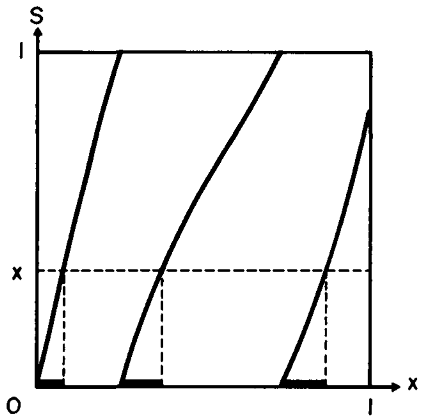

Figure 6.2.1. Function $S(x) = 3x + \frac{1}{4}\sin(7x/4) \pmod 1$ as an example of a transformation on $[0, 1]$ satisfying conditions (2i)–(2iv). In this case $r = 3$, and the counterimage of the set $[0, x]$ consists of the union of the three intervals indicated as heavy lines along the x-axis.

Proof: We first derive an explicit expression for the Frobenius–Perron operator. Note that, for any $x \in [0, 1)$,

$$S^{-1}([0,x]) = \bigcup_{i=1}^{r} [a_{i-1}, g_i(x)],$$

where

$$g_i(x) = \begin{cases} S_{(i)}^{-1}(x) & 0 \leqslant x < b_i \\ a_i & b_i \leqslant x < 1 \end{cases}$$

and $S_{(i)}(x)$ denotes the restriction of S to the interval $[a_{i-1}, a_i)$ whereas

$$b_i = \lim_{x \to a_i} S_{(i)}(x).$$

Thus, by the definition of the Frobenius–Perron operator,

$$\begin{aligned} Pf(x) &= \frac{d}{dx} \int_{S^{-1}([0,x])} f(u)\, du \\ &= \frac{d}{dx} \sum_{i=1}^{r} \int_{a_{i-1}}^{g_i(x)} f(u)\, du \end{aligned}$$

or

$$Pf(x) = \sum_{i=1}^{r} g_i'(x) f(g_i(x)). \tag{6.2.4}$$

If $b_i < 1$, then $g_i'(b_i)$ denotes the right derivative. Thus $g_i'(x) = 0$ for $b_i \leqslant x < 1$ and all of the g_i' are lower left semicontinuous.

Now let D_0 denote the subset of $D([0,1])$ consisting of all functions f that, on the interval $[0,1)$, are bounded, lower left semicontinuous and satisfy the inequality

$$f'_+(x) = \frac{d_+f(x)}{dx} \leqslant k_f f(x), \qquad \text{for } 0 \leqslant x < 1\,, \tag{6.2.5}$$

where k_f is a constant that depends on f. For any $f \in D_0$, the function Pf as calculated from equation (6.2.4) will be bounded and lower left semicontinuous.

For every $f \in D_0$, differentiation of expression (6.2.4) for the Frobenius–Perron operator gives

$$(Pf)'_+ = \sum_{i=1}^{r}(g_i')'_+(f \circ g_i) + \sum_{i=1}^{r}(g_i')^2(f'_+ \circ g_i)\,.$$

By using the inverse function theorem, we have

$$g_i' \leqslant \sup(1/S') \leqslant 1/\lambda$$

and

$$(g_i')'_+/g_i' \leqslant \sup(-S''/[S']^2) \leqslant c$$

so, as a consequence,

$$(Pf)'_+ \leqslant c\sum_{i=1}^{r} g_i'(f \circ g_i) + \frac{1}{\lambda}\sum_{i=1}^{r} g_i'(f'_+ \circ g_i)\,.$$

Using inequality (6.2.5), this expression may be further simplified to

$$(Pf)'_+ \leqslant \left[c + \frac{k_f}{\lambda}\right]Pf.$$

Set $f_n = P^n f$. An induction argument shows that

$$(f_n)'_+ \leqslant \left[\frac{c\lambda}{\lambda - 1} + \frac{k_f}{\lambda^n}\right]f_n\,.$$

Choose a real $k > c\lambda/(\lambda - 1)$. Then

$$(f_n)'_+ \leqslant kf_n \tag{6.2.6}$$

for n sufficiently large, say $n \geqslant n_0(f)$, and thus condition (5.8.3) of Proposition 5.8.1 is satisfied.

We now show that the f_n are bounded and hence satisfy condition (5.8.2) of Proposition 5.8.1. First note that from equation (6.2.4) we may write

$$f_{n+1}(x) = \sum_{i=1}^{r} g_i'(x) f_n(g_i(x)) .$$

Thus, since $g_i' \leqslant 1/\lambda$ and $S(a_{i-1}) = 0$ for $i = 1, \dots, r$,

$$f_{n+1}(0) \leqslant \frac{1}{\lambda} f_n(0) + \frac{1}{\lambda} \sum_{i=2}^{r} f_n(a_{i-1}) . \tag{6.2.7}$$

From (6.2.6) it follows that

$$f_n(a_i) \leqslant f_n(x) e^k \qquad \text{for } x \leqslant a_i ,$$

so that

$$1 \geqslant \int_0^{a_i} f_n(x)\, dx \geqslant e^{-k} f_n(a_i) a_i , \qquad \text{for } i = 1, \dots, r .$$

Thus $f_n(a_i) \leqslant e^k/a_i$, and from (6.2.7) we have

$$f_{n+1}(0) \leqslant (1/\lambda) f_n(0) + L/\lambda \qquad \text{for } n \geqslant n_0(f) ,$$

where

$$L = \sum_{i=2}^{r} e^k/a_{i-1} .$$

Again, using a simple induction argument, it follows that

$$f_n(0) \leqslant (1/\lambda^{n-n_0}) f_{n_0}(0) + L/(\lambda - 1), \qquad \text{for } n \geqslant n_0(f)$$

so

$$f_n(0) \leqslant 1 + [L/(\lambda - 1)]$$

for sufficiently large n, say $n \geqslant n_1(f)$. By using this relation in conjunction with the differential inequality (6.2.6), we therefore obtain

$$f_n(x) \leqslant \{1 + [L/(\lambda - 1)]\} e^k, \qquad \text{for } 0 \leqslant x < 1,\ n \geqslant n_1 . \tag{6.2.8}$$

Thus, by inequalities (6.2.6) and (6.2.8), all the conditions of Proposition 5.8.1 are satisfied and $\{P^n\}$ is asymptotically stable by Theorem 5.6.2. ■

Theorem 6.2.1 is valid only for mappings that are monotonically increasing on each subinterval $[a_{i-1}, a_i)$ of the partition of $[0, 1]$. However, by modification of some of the foregoing properties (2i)–(2iv), we may also prove another theorem valid for transformations that are either monotonically increasing or decreasing on the subintervals of the partition. The disadvantage is that the mapping must be onto for every $[a_{i-1}, a_i)$.

We now consider a mapping S: $[0, 1] \to [0, 1]$ that satisfies a condition slightly different from property (2i):

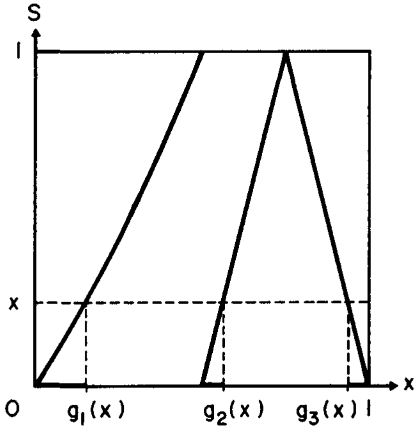

Figure 6.2.2. A piecewise monotonic transformation satisfying the conditions of Theorem 6.2.2.

(2i)′ There is a partition $0 = a_0 < a_1 < \cdots < a_r = 1$ of $[0,1]$ such that for each integer $i = 1, \ldots, r$ the restriction of S to the interval (a_{i-1}, a_i) is a C^2 function; as well as

(2ii)′ $S((a_{i-1}, a_i)) = (0,1)$, that is, S is onto;

(2iii)′ There is a $\lambda > 1$ such that $|S'(x)| \geq \lambda$, for $x \neq a_i$, $i = 0, \ldots, r$; and

(2iv)′ There is a real finite constant c such that

$$|S''(x)|/[S'(x)]^2 \leq c, \qquad \text{for } x \neq a_i,\ i = 0, \ldots, r. \tag{6.2.9}$$

(See Figure 6.2.2 for an example.)

Then we have the following theorem.

Theorem 6.2.2. If S: $[0,1] \to [0,1]$ satisfies the preceding conditions (2i)′–(2iv)′ and P is the Frobenius–Perron operator associated with S, then $\{P^n\}$ is asymptotically stable.

Proof: The proof proceeds much as for Theorem 6.2.1. Using the same notation as before, it is easy to show that for $x \in [0,1]$,

$$S^{-1}((0,x)) = \bigcup_j (a_{j-1}, g_j(x)) + \bigcup_k (g_k(x), a_k),$$

where $g_i = S_{(i)}^{-1}, S_{(i)}$ is as before, and the first union is over all intervals in which $S_{(i)}$ is an increasing function of x whereas the second is over intervals in which $S_{(i)}$ is decreasing. Thus

$$Pf(x) = \sum_j g_j'(x)f(g_j(x)) - \sum_k g_k'(x)f(g_k(x))$$

or, with $\sigma_i(x) = |g_i'(x)|$,

$$Pf(x) = \sum_{i=1}^{r} \sigma_i(x)f(g_i(x)). \tag{6.2.10}$$

Let $D_0 \subset D$ be the set of all bounded continuously differentiable densities such that

$$|f'(x)| \leqslant k_f f(x) \qquad \text{for } 0 < x < 1, \tag{6.2.11}$$

where the constant k_f depends on f. For every $f \in D_0$, differentiating equation (6.2.10) gives

$$(Pf)' = \sum_{i=1}^{r} \sigma_i'(f \circ g_i) + \sum_{i=1}^{r} \sigma_i g_i'(f \circ g_i).$$

Exactly as in the proof of Theorem 6.2.1 we have

$$\sigma_i \leqslant \sup(1/|S'|) \leqslant 1/\lambda$$

and

$$|\sigma_i'|/|g_i'| \leqslant \sup|S''|/[S']^2 \leqslant c.$$

These two inequalities, in combination with (6.2.11), allow us to evaluate $(Pf)'$ as

$$|(Pf)'| \leqslant \left[c + \frac{k_f}{\lambda}\right] Pf.$$

Set $f_n = P^n f$ and use an induction argument to show that

$$|f_n'| \leqslant \left[\frac{c\lambda}{\lambda - 1} + \frac{k_f}{\lambda^n}\right] f_n.$$

Again we may always pick a $k > c\lambda/(\lambda - 1)$ such that

$$|f_n'| \leqslant k f_n \tag{6.2.12}$$

for sufficiently large n [say $n \geqslant n_0(f)$], and thus Proposition 5.8.2 is satisfied. ■

Example 6.2.1. When $\sigma = 10$, $b = 8/3$, and $r = 28$, then all three variables x, y, and z in the Lorenz [1963] equations,

$$\frac{dx}{dt} = yz - bx$$

$$\frac{dy}{dt} = -xz + rz - y \qquad \frac{dz}{dt} = \sigma(y - z),$$

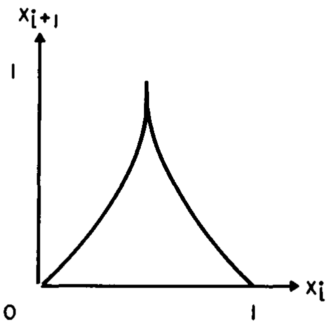

Figure 6.2.3. Successive maxima in the variable $x(t)$ from the Lorenz equations are labeled x_i, and one maximum is plotted against the previous (x_{i+1} vs. x_i) after rescaling so that all $x_i \in [0, 1]$.

show very complicated dynamics. If we label successive maxima in $x(t)$ as x_i ($i = 0, 1, \ldots$), plot each maximum against the previous maximum (i.e., x_{i+1} vs. x_i), and scale the results so that the x_i are contained in the interval $[0, 1]$, then the numerical computations show that the points (x_i, x_{i+1}) are located approximately on the graph of a one-dimensional mapping, as shown in Figure 6.2.3.

As an approximation to this mapping of one maximum to the next, we can consider the transformation

$$S(x) = \begin{cases} \dfrac{(2-a)x}{1-ax} & \text{for } x \in [0, \tfrac{1}{2}] \\[2ex] \dfrac{(2-a)(1-x)}{1-a(1-x)} & \text{for } x \in (\tfrac{1}{2}, 1], \end{cases} \tag{6.2.13}$$

where $a = 1 - \varepsilon$, shown in Figure 6.2.4 for $\varepsilon = 0.01$. Clearly, $S(0) = S(1) = 0$, $S(\frac{1}{2}) = 1$, and, since $S'(x) = (2-a)/(1-ax)^2$, we will always have $|S'(x)| > 1$ for $x \in [0, \frac{1}{2})$ if $\varepsilon > 0$. Finally, since $S''(x) = 2a(2-a)/(1-ax)^3$, $|S''(x)|$ is always bounded above. For $x \in (\frac{1}{2}, 1]$ the calculations are similar. Thus the transformation (6.2.13) satisfies all the requirements of Theorem 6.2.2: $\{P^n\}$ is asymptotically stable and S is exact. □

Remark 6.2.1. The condition that $|S'(x)| > 1$ in Theorem 6.2.2 is essential for S to be exact. We could easily demonstrate this by using (6.2.13) with $\varepsilon = 0$, thus making $|S'(0)| = |S'(1)| = 1$. However, even if $|S'(x)| = 1$ for only one point $x \in [0, 1]$, it is sufficient to destroy the exactness, as can be demonstrated by the transformation

$$S(x) = \begin{cases} x/(1-x) & \text{for } x \in [0, \tfrac{1}{2}] \\ 2x - 1 & \text{for } x \in (\tfrac{1}{2}, 1], \end{cases} \tag{6.2.14}$$

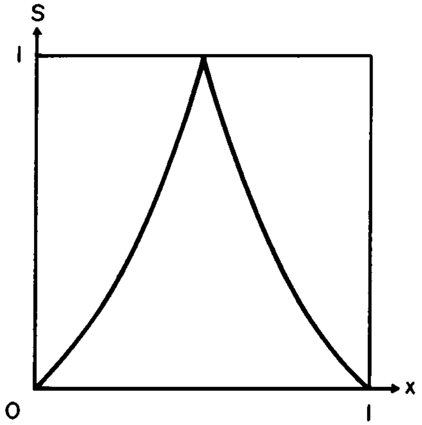

Figure 6.2.4. The transformation $S(x)$ given by equation (6.2.13) with $\varepsilon = 0.01$ as an approximation to the data of Figure 6.2.3.

which we originally considered in Section 1.3 (paradox of the weak repellor). Now, the condition $|S'(x)| > 1$ is violated only at the single point $x = 0$, and, for any $f \in L^1$, the sequence $\{P^n f\}$ converges to zero on $(0, 1]$. Thus, the only solution to the equation $Pf = f$ is the trivial solution $f \equiv 0$, and therefore there is no measure invariant under S.

This is quite difficult to prove. First write the Frobenius–Perron operator corresponding to S as

$$Pf(x) = \frac{1}{(1+x)^2} f\left(\frac{x}{1+x}\right) + \frac{1}{2} f\left(\frac{1}{2} + \frac{x}{2}\right). \tag{6.2.15}$$

Set $q_n(x) = x f_n(x)$, where $f_n = P^n f_0$, and pick the initial density to be $f_0 \equiv 1$. Thus $q_0(x) \equiv x$, and from (6.2.15) we have the recursive formula,

$$q_{n+1}(x) = \frac{1}{1+x} q_n\left(\frac{x}{1+x}\right) + \frac{x}{1+x} q_n\left(\frac{1}{2} + \frac{x}{2}\right). \tag{6.2.16}$$

Proceeding inductively, it is easy to prove that $q_n'(x) \geqslant 0$ for all n, so that the functions $q_n(x)$ are all positive and increasing. From equation (6.2.16) we have

$$q_{n+1}(1) \leqslant \tfrac{1}{2} q_n(\tfrac{1}{2}) + \tfrac{1}{2} q_n(1) \leqslant q_n(1),$$

which shows that

$$\lim_{n\to\infty} q_n(1) = c_0$$

exists. Write $z_0 = 1$ and $z_{k+1} = z_k/(1 + z_k)$. Then from (6.2.16) we have

$$q_{n+1}(z_k) = \frac{1}{1+z_k} q_n(z_{k+1}) + \frac{z_k}{1+z_k} q_n\left(\frac{1}{2} + \frac{z_k}{2}\right).$$

Take k to be fixed and assume that $\lim_{n\to\infty} q_n(x) = c_0$ for $z_k \leqslant x \leqslant 1$ (which is certainly true for $k = 0$). Since $z_k \leqslant \frac{1}{2} + \frac{1}{2}z_k$, taking the limit as $n \to \infty$, we have

$$c_0 = \frac{1}{1+z_k} \lim_{n\to\infty} q_n(z_{k+1}) + \frac{z_k}{1+z_k} c_0,$$

so $\lim_{n\to\infty} q_n(z_{k+1}) = c_0$. Since the functions $q_n(x)$ are increasing, we know that $\lim_{n\to\infty} q_n(x) = c_0$ for all $x \in [z_{k+1}, 1]$. By induction it follows that $\lim_{n\to\infty} q_n(x) = c_0$ in any interval $[z_k, 1]$ and, since $\lim_{k\to\infty} z_k = 0$, we have $\lim_{n\to\infty} q_n(x) = c_0$ for all $x \in (0, 1]$. Thus

$$\lim_{n\to\infty} f_n(x) = c_0/x.$$

Actually, the limit c_0 is zero; to show this, assume $c_0 \neq 0$. Then there must exist some $\varepsilon > 0$ such that

$$\lim_{n\to\infty} \int_\varepsilon^1 f_n(x)\,dx = \int_\varepsilon^1 (c_0/x)\,dx > 1.$$

However, this is impossible since $\|f_n\| = 1$ for every n. By induction, each of the functions $f_n(x)$ is decreasing, so the convergence of $f_n(x)$ to zero is uniform on any interval $[\varepsilon, 1]$ where $\varepsilon > 0$.

Now, let f be an arbitrary function, and write $f = f^+ - f^-$. Given $\delta > 0$, consider a constant h such that

$$\int_0^1 (f^- - h)^+\,dx + \int_0^1 (f^+ - h)^+\,dx \leqslant \delta.$$

Thus, since $|P^n f| \leqslant P^n|f| = P^n f^+ + P^n f^-$, we have

$$\begin{aligned}\int_\varepsilon^1 |P^n f|\,dx &\leqslant \int_\varepsilon^1 P^n f^+\,dx + \int_\varepsilon^1 P^n f^-\,dx \\ &= 2\int_\varepsilon^1 P^n h\,dx + \int_\varepsilon^1 P^n(f^+ - h)\,dx + \int_\varepsilon^1 P^n(f^- - h)\,dx \\ &\leqslant 2h\int_\varepsilon^1 P^n 1\,dx + \delta\end{aligned}$$

and, since $\{P^n 1\}$ converges uniformly to zero on $[\varepsilon, 1]$, we have

$$\lim_{n\to\infty} \int_\varepsilon^1 |P^n f|\,dx = 0 \qquad \text{for } \varepsilon > 0.$$

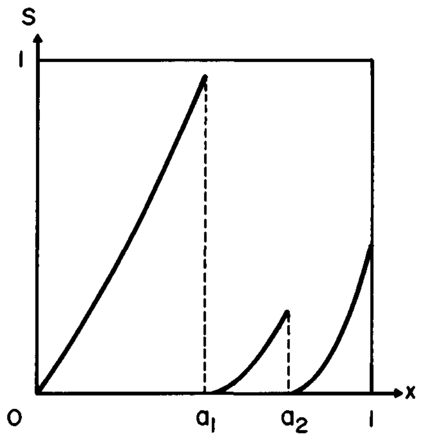

Figure 6.3.1. An example of a piecewise convex transformation satisfying the conditions of Theorem 6.3.1.

Hence the sequence $\{P^n f\}$ converges to zero in $L^1([\varepsilon, 1])$ norm for every $\varepsilon > 0$ and equation $Pf = f$ cannot have a solution $f \in L^1$ except $f \equiv 0$. □

6.3 Piecewise convex transformations with a strong repellor

Although the theorems of the preceding section were moderately easy to prove using the techniques of Chapter 5, the conditions that transformation S must satisfy are highly restrictive. Thus, in specific cases of interest, it may often not be the case that $S'(x) > 1$ or $|S'(x)| > 1$, or that condition (6.2.3) or (6.2.9) is obeyed.

However, for a class of convex transformations, it is known that $\{P^n\}$ is asymptotically stable. Consider S: $[0, 1] \to [0, 1]$ having the following properties:

(3i) There is a partition $0 = a_0 < a_1 < \cdots < a_r = 1$ of $[0, 1]$ such that for each integer $i = 1, \ldots, r$ the restriction of S to $[a_{i-1}, a_i)$ is a C^2 function;

(3ii) $S'(x) > 0$ and $S''(x) \geqslant 0$ for all $x \in [0, 1)$, [$S'(a_i)$ and $S''(a_i)$ are right derivatives];

(3iii) For each integer $i = 1, \ldots, r$, $S(a_{i-1}) = 0$; and

(3iv) $S'(0) > 1$.

An example of a mapping satisfying these criteria is shown in Figure 6.3.1.

Remark 6.3.1. Property (3iv) implies that point $x = 0$ is a **strong repellor** (see also Section 1.3 and Remark 6.2.1), that is, trajectory $\{S(x_0), S^2(x_0), \ldots\}$, starting from a point $x_0 \in (0, a_1)$, will eventually leave $[0, a_1)$. To see this, note that as long as $S^n(x_0) \in [0, a_1)$ there is a $\xi \in (0, a_1)$ such that

$$\begin{aligned} S^n(x_0) &= S(S^{n-1}(x_0)) - S(0) \\ &= S'(\xi)S^{n-1}(x_0) \geqslant \lambda S^{n-1}(x_0), \end{aligned}$$

where $\lambda = S'(0)$. By an induction argument, $S^n(x_0) \geqslant \lambda^n x_0$ and, since $\lambda > 1$, $S^n(x_0)$ must eventually exceed a_1. After leaving the interval $[0, a_1)$ the trajectory will, in general, exhibit very complicated behavior. If at some point it returns to $[0, a_1)$, then it will, again, eventually leave $[0, a_1)$. □

With these comments in mind, we can state the following theorem.

Theorem 6.3.1. Let S: $[0, 1] \to [0, 1]$ be a transformation satisfying the foregoing conditions (3i)–(3iv) and let P be the Frobenius–Perron operator associated with S. Then $\{P^n\}$ is asymptotically stable.

Proof: The complete proof of this theorem, which may be found in Lasota and Yorke [1982], is long and requires some technical details we have not introduced. Rather than give the full proof, here we show only that $\{P^n f\}$ is bounded above, thus implying that there is a measure invariant under S.

We first derive the Frobenius–Perron operator. For any $x \in [0, 1]$ we have

$$S^{-1}([0, x]) = \bigcup_{i=1}^{r} [a_{i-1}, g_i(x)],$$

where

$$g_i(x) = \begin{cases} S_{(i)}^{-1}(x) & \text{for } x \in S([a_{i-1}, a_i)) \\ a_i & \text{for } x \in [0, 1] \backslash S([a_{i-1}, a_i)) \end{cases}$$

and, as before, $S_{(i)}$ denotes the restriction of S to the interval $[a_{i-1}, a_i)$. Thus, as in Section 6.2, we obtain

$$Pf(x) = \sum_{i=1}^{r} g_i'(x) f(g_i(x)). \tag{6.3.1}$$

Even though equations (6.2.4) and (6.3.1) appear to be identical, the functions g_i have different properties. For instance, by using the inverse function theorem, we have

$$g_i' = 1/S' > 0 \quad \text{and} \quad g_i'' = -S''/[S']^2 \leqslant 0.$$

Thus, since $g_i' > 0$ we know that g_i is an increasing function of x, whereas g_i' is a decreasing function of x since $g_i'' \leqslant 0$.

Let $f \in D([0,1])$ be a decreasing density, that is, $x \leqslant y$ implies $f(x) \geqslant f(y)$. Then, by our previous observations, $f(g_i(x))$ is a decreasing function of x as is $g_i'(x)f(g_i(x))$. Since Pf, as given by (6.3.1), is the sum of decreasing functions, Pf is a decreasing function of x and, by induction, so is $P^n f$.

Observe further that, for any decreasing density $f \in D([0,1])$, we have

$$1 \geqslant \int_0^x f(u)\,du \geqslant \int_0^x f(x)\,du = xf(x),$$

so that, for any decreasing density,

$$f(x) \leqslant 1/x \qquad x \in (0,1].$$

Hence, for $i \geqslant 2$, we must have

$$\begin{aligned} g_i'(x)f(g_i(x)) &\leqslant g_i'(0)f(g_i(0)) \\ &\leqslant \frac{g_i'(0)}{g_i(0)} = \frac{g_i'(0)}{a_{i-1}}, \qquad i = 2, \ldots, r. \end{aligned}$$

This formula is not applicable when $i = 1$ since $a_0 = 0$. However, we do have

$$g_1'(x)f(g_1(x)) \leqslant g_1'(0)f(0).$$

Combining these two results with equation (6.3.1) for P, we can write

$$Pf(x) \leqslant g_1'(0)f(0) + \sum_{i=2}^{r} g_i'(0)/a_{i-1}.$$

Set

$$S'(0) = 1/g_1'(0) = \lambda > 1$$

and

$$\sum_{i=2}^{r} g_i'(0)/a_{i-1} = M$$

so

$$Pf(x) \leqslant (1/\lambda)f(0) + M.$$

Proceeding inductively, we therefore have

$$\begin{aligned} P^n f(x) &\leqslant (1/\lambda^n)f(0) + \lambda M/(\lambda - 1) \\ &\leqslant f(0) + \lambda M/(\lambda - 1). \end{aligned}$$

Thus, for decreasing $f \in D([0,1])$, since $f(0) < \infty$ the sequence $\{P^n f\}$ is bounded above by a constant. From Corollary 5.2.1 we, therefore, know that there is a density $f_* \in D$ such that $Pf_* = f_*$, and by Theorem 4.1.1 the measure μ_{f_*} is invariant. ∎

Example 6.3.1. In the experimental study of fluid flow it is commonly observed that for Reynolds numbers R less than a certain value, R_L, strictly laminar flow occurs; for Reynolds numbers greater than another value, R_T, continuously turbulent flow occurs. For Reynolds numbers satisfying $R_L < R < R_T$, a transitional type behavior **(intermittency)** is found. Intermittency is characterized by alternating periods of laminar and turbulent flow, each of a variable and apparently unpredictable length.

Intermittency is also observed in mathematical models of fluid flow, for example, the Lorenz equations [Manneville and Pomeau, 1979]. Manneville [1980] argues that, in the parameter ranges where intermittency occurs in the Lorenz equations, the model behavior can be approximated by the transformation $S: [0,1] \to [0,1]$ given by

$$S(x) = (1 + \varepsilon)x + (1 - \varepsilon)x^2 \qquad (\text{mod } 1) \tag{6.3.2}$$

with $\varepsilon > 0$, where x corresponds to a normalized fluid velocity. This transformation clearly satisfies all of the properties of Theorem 6.2.1 for $0 < \varepsilon < 2$ and is thus exact.

The utility of equation (6.3.2) in the study of intermittency stems from the fact that $x = 0$ is a strong repellor. From Remark 6.3.1 it is clear that any transformation S satisfying conditions (3i)–(3iv) will serve equally well in this approach to the intermittency problem. Exactly this point of view has been adopted by Procaccia and Schuster [1983] in their heuristic treatment of noise spectra in dynamical systems. □

6.4 Asymptotically periodic transformations

In order to prove the asymptotic stability of $\{P^n\}$ in the two preceding sections, we were forced to consider transformations S with very special properties. Thus, for every subinterval of the partition of $[0, 1]$, we used either $S((a_{i-1}, a_i)) = (0, 1)$ or $S(a_{i-1}) = 0$. Eliminating either or both of these requirements may well lead to the loss of asymptotic stability of $\{P^n\}$, as is illustrated in the following example.

Let $S: [0,1] \to [0,1]$ be defined by

$$S(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{4}) \\ 2x - \frac{1}{2} & \text{for } x \in [\frac{1}{4}, \frac{3}{4}) \\ 2x - 1 & \text{for } x \in [\frac{3}{4}, 1], \end{cases}$$

as shown in Figure 6.4.1. Examination of the figure shows that the Borel measure is invariant since $S^{-1}([0,x])$ always consists of two intervals whose union has measure x. However, S is obviously not exact and, indeed, is not even ergodic since $S^{-1}([0,\frac{1}{2}]) = [0,\frac{1}{2}]$ and $S^{-1}([\frac{1}{2},1]) = [\frac{1}{2},1]$. S that is restricted to either $[0,\frac{1}{2}]$ or $[\frac{1}{2},1])$ behaves like the dyadic transformation.

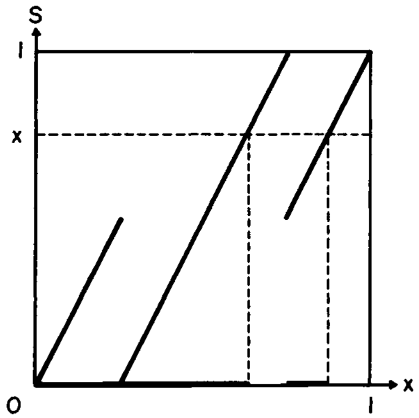

Figure 6.4.1. An example showing that piecewise monotonic transformation that is not onto might not even be ergodic. (See the text for details.)

The loss of asymptotic stability by $\{P^n\}$ may, under certain circumstances, be replaced by the asymptotic periodicity of $\{P^n\}$. To see this, consider a mapping S: $[0,1] \to [0,1]$ satisfying the following three conditions:

(4i) There is a partition $0 = a_0 < a_1 < \cdots < a_r = 1$ of $[0,1]$ such that for each integer $i = 1, \ldots, r$ the restriction of S to (a_{i-1}, a_i) is a C^2 function;

(4ii) $$|S'(x)| \geqslant \lambda > 1, \qquad x \neq a_i,\ i = 0, \ldots, r; \tag{6.4.1}$$

(4iii) There is a real constant c such that

$$\frac{|S''(x)|}{[S'(x)]^2} \leqslant c < \infty, \qquad x \neq a_i,\ i = 0, \ldots, r. \tag{6.4.2}$$

An example of a transformation satisfying these conditions is shown in Figure 6.4.2.

We now state the following theorem.

Theorem 6.4.1. Let S: $[0,1] \to [0,1]$ satisfy conditions (4i)–(4iii) and let P be the Frobenius–Perron operator associated with S. Then, for all $f \in D$, $\{P^n f\}$ is asymptotically periodic.

Proof: We first construct the Frobenius–Perron operator corresponding to S. For any $x \in [0,1]$, we have

$$S^{-1}((0,x)) = \bigcup_{i=1}^{r} \Delta_i(x)$$

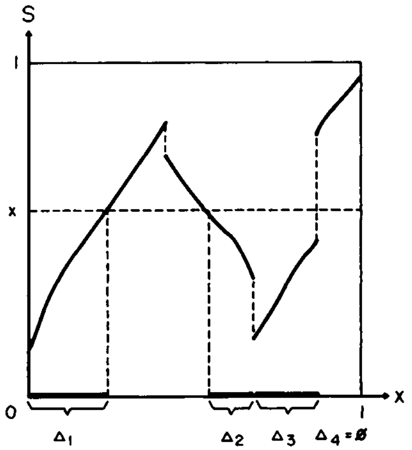

Figure 6.4.2. An example of a transformation on $[0, 1]$ satisfying the conditions of Theorem 6.4.1.

where, for $I_i = S((a_{i-1}, a_i))$,

$$\Delta_i(x) = \begin{cases} (a_{i-1}, g_i(x)) & x \in I_i,\ g_i' > 0 \\ (g_i(x), a_i) & x \in I_i,\ g_i' < 0 \\ \emptyset \text{ or } (a_{i-1}, a_i) & x \notin I_i \end{cases}$$

and, as before, $g_i = S_{(i)}^{-1}$ and $S_{(i)}$ denotes the restriction of S to (a_{i-1}, a_i). Therefore,

$$\begin{aligned} Pf(x) &= \frac{d}{dx} \int_{S^{-1}([0,x])} f(u)\, du \\ &= \sum_{i=1}^{r} \frac{d}{dx} \int_{\Delta_i(x)} f(u)\, du, \end{aligned} \tag{6.4.3}$$

where

$$\frac{d}{dx} \int_{\Delta_i(x)} f(u)\, du = \begin{cases} g_i'(x) f(g_i(x)), & x \in I_i,\ g_i' > 0 \\ -g_i'(x) f(g_i(x)), & x \in I_i,\ g_i' < 0 \\ 0 & x \notin I_i. \end{cases} \tag{6.4.4}$$

The right-hand side of equation (6.4.3) is not defined on the set of end points of the intervals I_i, $S(a_{i-1})$, and $S(a_i)$. However, this set is finite and thus of measure zero. Since a function representing Pf that is an element of L^1 is defined up to a set of measure zero we neglect these end points.

Equation (6.4.4) may be rewritten as

$$\frac{d}{dx}\int_{\Delta_i(x)} f(u)\,du = \sigma_i(x) f(g_i(x)) 1_{I_i}(x),$$

where $\sigma_i(x) = |g_i'(x)|$ and $1_{I_i}(x)$ is the characteristic function of the interval I_i. Thus (6.4.3) may be written as

$$Pf(x) = \sum_{i=1}^{r} \sigma_i(x) f(g_i(x)) 1_{I_i}(x). \tag{6.4.5}$$

Equation (6.4.5) for the Frobenius–Perron operator is made more complicated than those in Sections 6.2 and 6.3 by the presence of the characteristic functions $1_{I_i}(x)$. The effect of these is such that even when a completely smooth initial function $f \in L^1$ is chosen, Pf and all subsequent iterates of f may be discontinuous. As a consequence we do not have simple criteria, such as decreasing functions, to examine the behavior of $P^n f$. Thus we must examine the variation of $P^n f$.

We start by examining the variation of Pf as given by equation (6.4.5). Let a function $f \in D$ be of bounded variation on $[0, 1]$. From property (V1) of Section 6.1, the Yorke inequality (V5), and equation (6.4.5),

$$\begin{aligned}\bigvee_0^1 Pf(x) &\leqslant \sum_{i=1}^{r} \bigvee_0^1 [\sigma_i(x) f(g_i(x)) 1_{I_i}(x)] \\ &\leqslant 2\sum_{i=1}^{r} \bigvee_{I_i} [\sigma_i(x) f(g_i(x))] + \sum_{i=1}^{r} \frac{2}{|I_i|}\int_{I_i} \sigma_i(x) f(g_i(x))\,dx.\end{aligned} \tag{6.4.6}$$

Further, by property (V4),

$$\bigvee_{I_i} [\sigma_i(x) f(g_i(x))] \leqslant \sup \sigma_i(x) \bigvee_{I_i} f(g_i(x)) + \int_{I_i} |\sigma_i'(x)| f(g_i(x))\,dx.$$

Because, from the inverse function theorem, we have $\sigma_i \leqslant 1/\lambda$ and $|\sigma_i'| \leqslant c\sigma_i$, the preceding inequality becomes

$$\bigvee_{I_i} [\sigma_i(x) f(g_i(x))] \leqslant \frac{1}{\lambda} \bigvee_{I_i} f(g_i(x)) + c\int_{I_i} \sigma_i(x) f(g_i(x))\,dx,$$

and, thus, (6.4.6) becomes

$$\begin{aligned}\bigvee_0^1 Pf(x) &\leqslant \frac{2}{\lambda}\sum_{i=1}^{r} \bigvee_{I_i} f(g_i(x)) \\ &\quad + 2\sum_{i=1}^{r}\left[c + \frac{1}{|I_i|}\right]\int_{I_i} \sigma_i(x) f(g_i(x))\,dx.\end{aligned} \tag{6.4.7}$$

Define a new variable $y = g_i(x)$ for the integral in (6.4.7) and use property (V3) for the first term to give

$$\bigvee_0^1 Pf(x) \leqslant \frac{2}{\lambda}\sum_{i=1}^{r}\bigvee_{a_{i-1}}^{a_i} f + 2\sum_{i=1}^{r}\left[c + \frac{1}{|I_i|}\right]\int_{a_{i-1}}^{a_i} f(y)\,dy\,.$$

Set $L = \max_i 2(c + 1/|I_i|)$ and use property (V2) to rewrite this last inequality as

$$\bigvee_0^1 Pf(x) \leqslant \frac{2}{\lambda}\bigvee_0^1 f + L\int_0^1 f(y)\,dy = \frac{2}{\lambda}\bigvee_0^1 f + L \qquad (6.4.8)$$

since $f \in D([0,1])$.

By using an induction argument with inequality (6.4.8), we have

$$\bigvee_0^1 P^n f \leqslant \left(\frac{2}{\lambda}\right)^n \bigvee_0^1 f + L\sum_{j=0}^{n-1}\left(\frac{2}{\lambda}\right)^j. \qquad (6.4.9)$$

Thus, if $\lambda > 2$, then

$$\bigvee_0^1 P^n f \leqslant \left(\frac{2}{\lambda}\right)^n \bigvee_0^1 f + \frac{\lambda L}{\lambda - 2}$$

and, therefore, for every $f \in D$ of bounded variation,

$$\limsup_{n\to\infty} \bigvee_0^1 P^n f < K, \qquad (6.4.10)$$

where $K > \lambda L/(\lambda - 2)$ is independent of f.

Now let the set $\mathcal{F}$ be defined by

$$\mathcal{F} = \{g \in D : \bigvee_0^1 g \leqslant K\}\,.$$

From (6.4.10) it follows that $P^n f \in \mathcal{F}$ for a large enough n and, thus, $\{P^n f\}$ converges to $\mathcal{F}$ in the sense of Definition 5.3.2. We want to show that $\mathcal{F}$ is weakly precompact. From the definition of the variation, it is clear that, for any positive function g defined on $[0,1]$,

$$g(x) - g(y) \leqslant \bigvee_0^1 g$$

for all $x, y \in [0,1]$. Since $g \in D$, there is some $y \in [0,1]$ such that $g(y) \leqslant 1$ and, thus,

$$g(x) \leqslant K + 1\,.$$

Hence, by criterion 1 of Section 5.1, $\mathcal{F}$ is weakly precompact. (Actually, it is strongly precompact, but we will not use this fact.) Since $\mathcal{F}$ is weakly precompact, then P is weakly constrictive by Definition 5.3.3. By Theorem 5.3.1,

P is also strongly constrictive. Finally, by Theorem 5.3.2, $\{P^n f\}$ is asymptotically periodic and the theorem is proved when $\lambda > 2$.

To see that the theorem is also true for $\lambda > 1$, consider another transformation $\tilde{S}: [0,1] \to [0,1]$ defined by

$$\tilde{S}(x) = S \circ \overset{q}{\cdots} \circ S(x) = S^q(x). \tag{6.4.11}$$

Let q be the smallest integer such that $\lambda^q > 2$ and set $\tilde{\lambda} = \lambda^q$. It is easy to see that $\tilde{S}$ satisfies conditions (4i)–(4ii). By the chain rule,

$$|\tilde{S}'(x)| \geq (\inf|S'(x)|)^q \geq \lambda^q = \tilde{\lambda} > 2.$$

Thus, by the preceding part of the proof, $\{\tilde{P}^n\}$ satisfies

$$\limsup_{n\to\infty} \bigvee_0^1 \tilde{P}^n f < \tilde{K}$$

for every $f \in D$ of bounded variation, where the constant $\tilde{K}$ is independent of f. Write an integer n in the form $n = mq + s$, where the remainder s satisfies $0 \leq s < q$. Take m sufficiently large, $m > m_0$, so that

$$\bigvee_0^1 \tilde{P}^m f \leq \tilde{K}, \qquad m > m_0.$$

Now, using inequality (6.4.9), we have

$$\begin{aligned}\bigvee_0^1 P^n f = \bigvee_0^1 P^s(\tilde{P}^m f) &\leq \left(\frac{2}{\lambda}\right)^s \bigvee_0^1 \tilde{P}^m f + L\sum_{j=0}^{q-1}\left(\frac{2}{\lambda}\right)^j \\ &\leq \tilde{K} \sup_{0\leq j\leq q-1}\left(\frac{2}{\lambda}\right)^j + L\sum_{j=0}^{q-1}\left(\frac{2}{\lambda}\right)^j, \qquad n \geq (m_0+1)q.\end{aligned}$$

Thus, for n sufficiently large, the variation of $P^n f$ is bounded by a constant independent of f and the proof proceeds as before. ■

Remark 6.4.1. From the results of Kosjakin and Sandler [1972] or Li and Yorke [1978a], it follows that transformations S satisfying the assumptions of Theorem 6.4.1 are ergodic if $r = 2$. □

Example 6.4.1. In this example we consider one of the simplest heuristic models for the effects of periodic modulation of an autonomous oscillator [Glass and Mackey, 1979].

Consider a system (see Figure 6.4.3) whose activity $x(t)$ increases linearly from a starting time t_i until it reaches a periodic threshold $\theta(t)$ at time $\bar{t}_i$:

$$x(\bar{t}_i) = \theta(\bar{t}_i). \tag{6.4.12}$$

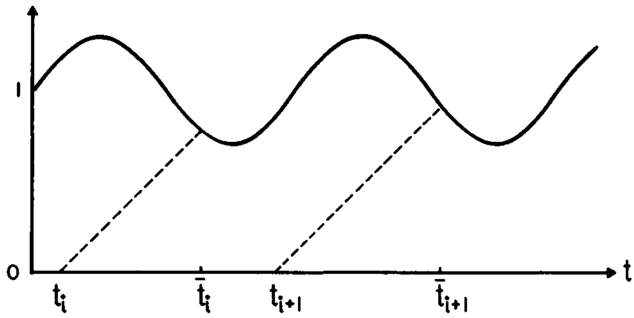

Figure 6.4.3. The periodic threshold $\theta(t)$ is shown as a solid curved line, and the activity $x(t)$ as dashed lines. (See Example 6.4.1 for further details.)

We take

$$x(t) = \lambda(t - t_i) \quad \text{and} \quad \theta(t) = 1 + \phi(t),$$

where ϕ is a continuous periodic function with period 1 whose amplitude satisfies

$$1 \geq \sup \phi(t) = -\inf \phi(t) = K \geq 0.$$

When the activity reaches threshold it instantaneously resets to zero, and the process begins anew at the starting time,

$$t_{i+1} = \bar{t}_i + \gamma^{-1}x(\bar{t}_i). \tag{6.4.13}$$

In (6.4.13), $\bar{t}_i$ is an implicit function of t_i given by (6.4.12) or by

$$\lambda(\bar{t}_i - t_i) = 1 + \phi(\bar{t}_i). \tag{6.4.14}$$

Equation (6.4.14) has exactly one smallest solution $\bar{t}_i \geq t_i$ for every $t_i \in R$.

We wish to examine the behavior of the starting times t_i. Set

$$F(t_i) = \bar{t}_i(t_i) + \gamma^{-1}x(\bar{t}_i(t_i))$$

so that the transformation

$$S(t) = F(t) \qquad (\mathrm{mod}\ 1) \tag{6.4.15}$$

gives the connection between successive starting times.

Many authors have considered the specific case of $\phi(t) = K \sin 2\pi t$, $\gamma^{-1} = 0$, so $\bar{t}_i = t_{i+1}$ and, thus, t_{i+1} is given implicitly by

$$\lambda(t_{i+1} - t_i) = 1 + K \sin 2\pi t_{i+1}.$$

Here, to illustrate the application of the material of this and previous sections, we restrict ourselves to the simpler situation in which $\phi(t)$ is a piecewise linear function of t and θ given by

$$\theta(t) = \begin{cases} 4Kt + 1 - K & t \in [0, \frac{1}{2}] \\ 4K(1-t) + 1 - K & t \in (\frac{1}{2}, 1]. \end{cases}$$

The calculation of $F(t)$ depends on the sign of $\lambda - 4K$. For example, if $\lambda > 4K$, a simple computation shows that

$$F(t) = \begin{cases} \dfrac{1+\alpha}{1-\beta}t + \left(\dfrac{1}{\lambda} + \dfrac{1}{\gamma}\right)\dfrac{1-K}{1-\beta} & t \in [-a, \frac{1}{2}(1-\beta) - a] \\ \dfrac{1-\alpha}{1+\beta}t + \left(\dfrac{1}{\lambda} + \dfrac{1}{\gamma}\right)\dfrac{1+3K}{1+\beta} & t \in (\frac{1}{2}(1-\beta) - a, 1 - a], \end{cases} \tag{6.4.16}$$

where $\alpha = 4K/\gamma$, $\beta = 4K/\lambda$, and $a = (1-K)/\lambda$.

Since $0 \leqslant \beta < 1$, it is clear that $F'(t) > 1$ for all $t \in [-a, \frac{1}{2}(1-\beta) - a]$. However, if $(1-\alpha)/(1+\beta) < -1$, then $|S'(t)| > 1$ for all t and $\{P^n\}$ is asymptotically periodic by Theorem 6.4.1. Should it happen in this case that S is onto for every subinterval of the partition, then $\{P^n\}$ is asymptotically stable by Theorem 6.2.2.

Despite the obvious simplifications in such models they have enjoyed great popularity in neurobiology: the "integrate and fire" model [Knight, 1972a, b]; in respiratory physiology, the "inspiratory off switch" model [Petrillo and Glass, 1984]; in cardiac electrophysiology, the "circle model" [Guevara and Glass, 1982]; and in cell biology, the "mitogen" model [Kauffman, 1974; Tyson and Sachsenmaier, 1978]. □

Example 6.4.2. An interesting problem arises in the rotary drilling of rocks. Usually the drilling tool is in the form of a toothed cone (mass M and radius R) that rotates on the surface of the rock with tangential velocity u. At rest the tool exerts a pressure Q on the rock. In practice it is found that, for sufficiently large tool velocities, after each impact of a tooth with the rock the tool rebounds before the next blow. The energy of each impact, and thus the efficiency of the cutting process, is a function of the angle at which the impact occurs.

Let x be the normalized impact angle that is in the interval $[0,1]$. Lasota and Rusek [1974] have shown that the next impact angle is given by the transformation $S\colon [0,1] \to [0,1]$ defined by

$$S(x) = x + \alpha q(x) - \sqrt{[\alpha q(x)]^2 + 2\alpha x q(x) - \alpha q(x)[1 + q(x)]} \pmod 1, \tag{6.4.17}$$

where

$$q(x) = 1 + \mathrm{int}[(1-2x)/(\alpha - 1)];$$

int(y) denotes the integer part of y, namely, the largest integer smaller than or equal to y, and

$$\alpha = F/(F - 1),$$

where

$$F = Mu^2/QR$$

is Freude's number, the ratio of the kinetic and potential energies.

The Freude number F contains all of the important parameters characterizing this process. It is moderately straightforward to show that with $\tilde{S} = S \circ S$, $|\tilde{S}'(x)| > 1$ if $F > 2$. However, the transformation (6.4.17) is not generally onto, so that by Theorem 6.4.1 the most that we can say is that for $F > 2$, if P is the Frobenius–Perron operator corresponding to S then $\{P^n\}$ is asymptotically periodic. However, it seems natural to expect that $\{P^n\}$ is in fact asymptotically stable. This prediction is supported experimentally, because, once $u > (2QR/M)^{1/2}$, there is a transition from smooth cutting to extremely irregular behavior (chattering) of the tool. □

Example 6.4.3. Kitano, Yabuzaki, and Ogawa [1983] experimentally examined the dynamics of a simple, nonlinear, acoustic feedback system with a time delay. A voltage x, the output of an operational amplifier with response time γ^{-1}, is fed to a speaker. The resulting acoustic signal is picked up by a microphone after a delay τ (due to the finite propogation velocity of sound waves), passed through a full-wave rectifier, and then fed back to the input of the operational amplifier.

Kitano and co-workers have shown that the dynamics of this system are described by the delay-differential equation

$$\gamma^{-1}\dot{x}(t) = -x(t) + \mu F(x(t - \tau)), \tag{6.4.18}$$

where

$$F(x) = -|x + \tfrac{1}{2}| + \tfrac{1}{2} \tag{6.4.19}$$

is the output of the full-wave rectifier with an input x, and μ is the circuit loop gain.

In a series of experiments, Kitano et al. found that increasing the loop gain μ above 1 resulted in very complicated dynamics in x, whose exact nature depends on the value of $\gamma\tau$. To understand these behaviors they considered the one-dimensional difference equation,

$$x_{n+1} = \mu F(x_n),$$

derived from expressions (6.4.18) and (6.4.19) as $\gamma^{-1} \to 0$. In our notation this is equivalent to the map T: $[-\mu/(\mu - 1), \mu/2] \to [-\mu/(\mu - 1), \mu/2]$, defined by

$$T(x) = \begin{cases} \mu(1+x) & \text{for } x \in \left[-\frac{\mu}{\mu-1}, -\frac{1}{2}\right] \\ -\mu x & \text{for } x \in \left(-\frac{1}{2}, \frac{\mu}{2}\right]. \end{cases} \tag{6.4.20}$$

for $1 < \mu \leqslant 2$. Make the change of variables

$$x \to -\frac{\mu}{\mu-1} + x'\frac{\mu}{2}\frac{\mu+1}{\mu-1}$$

so that (6.4.20) is equivalent to the transformation $S\colon [0,1] \to [0,1]$, defined by

$$S(x') = \begin{cases} \mu x' & \text{for } x' \in [0, 1/\mu] \\ 2 - \mu x' & \text{for } x' \in (1/\mu, 1]. \end{cases} \tag{6.4.21}$$

For $1 < \mu \leqslant 2$, the transformation S defined by (6.4.21) satisfies all the conditions of Theorem 6.4.1, and S is thus asymptotically periodic. If $\mu = 2$, then, by Theorem 6.2.2, S is statistically stable. Furthermore, from Remark 6.4.1 it follows that S is ergodic for $1 < \mu < 2$ and will, therefore, exhibit disordered dynamical behavior. This is in agreement with the experimental results. □

Remark 6.4.2. As we have observed in the example of Figure 6.4.1, piecewise monotonic transformations satisfying properties (4i)–(4iii) may not have a unique invariant measure. If the transformation is ergodic, and the invariant measure is thus unique by Theorem 4.2.2, then the invariant measure has many interesting properties. For example, in this case Kowalski [1976] has shown that the invariant measure is continuously dependent on the transformation. □

6.5 Change of variables

In the three preceding sections, we have examined transformations $S\colon [0,1] \to [0,1]$ with very restrictive conditions on the derivatives $S'(x)$ and $S''(x)$. However, most transformations do not satisfy these conditions. A good example is the quadratic transformation,

$$S(x) = 4x(1-x), \qquad \text{for } x \in [0,1].$$

For this transformation, $S'(x) = 4 - 8x$, and $|S'(x)| < 1$ for $x \in (\frac{3}{8}, \frac{5}{8})$. Furthermore, $|S''(x)/[S'(x)]^2| = \frac{1}{2}(1-2x)^2$, which is clearly not bounded at $x = \frac{1}{2}$. However, iteration of any initial density on $[0,1]$ indicates that the iterates rapidly approach the same density (Figure 1.2.2), leading one to suspect that, for the parabolic transformation, $\{P^n\}$ is asymptotically stable.

In this section we show how, by a change of variables, we can sometimes utilize the results of the previous sections to prove asymptotic stability. The idea is originally due to Ruelle [1977] and Pianigiani [1983].

Theorem 6.5.1. Let $S\colon [0,1] \to [0,1]$ be a transformation satisfying properties (2i)′ and (2ii)′ of Section 6.2, and P_S be the Frobenius–Perron operator corresponding to S. If there exists an a.e. positive C^1 function $\phi \in L^1([0,1])$ such that, for some real λ and c,

$$p(x) \equiv \frac{|S'(x)|\phi(S(x))}{\phi(x)} \geqslant \lambda > 1, \qquad 0 < x < 1 \tag{6.5.1}$$

and

$$\left|\frac{1}{\phi(x)}\frac{d}{dx}\left(\frac{1}{p(x)}\right)\right| \leqslant c < \infty, \qquad 0 < x < 1, \tag{6.5.2}$$

then $\{P_S^n\}$ is asymptotically stable.

Proof: Set

$$g(x) = \frac{1}{\|\phi\|}\int_0^x \phi(u)\,du, \qquad \text{for } x \in [0,1] \tag{6.5.3}$$

and consider a new transformation T defined by

$$T(x) = g(S(g^{-1}(x))) \tag{6.5.4}$$

with associated Frobenius–Perron operator P_T. From (6.5.4), $T(g(x)) = g(S(x))$, so

$$\frac{dT}{dg}\frac{dg}{dx} = \frac{dg}{dS}\frac{dS}{dx}$$

or

$$T'(g(x)) = \frac{g'(S(x))S'(x)}{g'(x)}.$$

Using (6.5.3) this may be rewritten as

$$T'(g(x)) = \frac{S'(x)\phi(S(x))}{\phi(x)}.$$

Hence, by (6.5.1), we have $|T'(g)| \geqslant \lambda > 1$. Further, by comparing this equation with (6.5.1), we see that $p(x) = |T'(g(x))|$. It follows that

$$\frac{1}{\phi(x)}\frac{d}{dx}\left(\frac{1}{p(x)}\right) = \frac{1}{\phi(x)}\frac{d}{dx}\left(\frac{1}{|T'(g)|}\right) = \pm\frac{T''(g)}{[T'(g)]^2\|\phi\|},$$

so that, from inequality (6.5.2),

$$\left|\frac{T''(g)}{[T'(g)]^2}\right| \leqslant c\|\phi\| < \infty .$$

Thus the new transformation T satisfies all the conditions of Theorem 6.2.2, and $\{P_T^n\}$ is asymptotically stable as is $\{P_S^n\}$ by (6.5.14). ■

Example 6.5.1. Consider the quadratic transformation $S(x) = 4x(1 - x)$ with $x \in [0, 1]$ and set

$$\phi(x) = \frac{1}{\pi\sqrt{x(1 - x)}} . \tag{6.5.5}$$

Using equations (6.5.3) and (6.5.5), it is easy to verify that all the conditions of Theorem 6.5.1 are satisfied in this case and, thus, for the quadratic transformation, $\{P^n\}$ is asymptotically stable.

Note that with ϕ as given by (6.5.5), the associated function g, as defined by (6.5.3), is given by

$$g(x) = \frac{1}{\pi}\int_0^x \frac{du}{\sqrt{u(1 - u)}} = \frac{1}{2} - \frac{1}{\pi}\sin^{-1}(1 - 2x) , \tag{6.5.6}$$

and thus

$$g^{-1}(x) = \tfrac{1}{2} - \tfrac{1}{2}\cos(\pi x) . \tag{6.5.7}$$

Hence, when $S(x) = 4x(1 - x)$, the transformation T: $[0, 1] \to [0, 1]$, defined by

$$T(x) = g \circ S \circ g^{-1}(x) , \tag{6.5.8}$$

is easily shown to be

$$T(x) = \begin{cases} 2x & \text{for } x \in [0, \tfrac{1}{2}) \\ 2(1 - x) & \text{for } x \in [\tfrac{1}{2}, 1] . \end{cases} \tag{6.5.9}$$

[The transformation defined by (6.5.9) is often referred to as the **tent map** or **hat map.**] The Frobenius–Perron operator, P_T, corresponding to T is given by

$$P_T f(x) = \tfrac{1}{2}f(\tfrac{1}{2}x) + \tfrac{1}{2}f(1 - \tfrac{1}{2}x) ,$$

and, by Theorem 6.2.2, $\{P_T^n\}$ is asymptotically stable. Furthermore, it is clear that $f_* \equiv 1$ is the unique stationary density of P_T, so T is, in fact, exact by Theorem 4.4.1. Reversing the foregoing procedure by constructing a transformation $S = g^{-1} \circ T \circ g$ from T given by (6.5.9) and from g, g^{-1} given by equations (6.5.6) and (6.5.7) yields the transformation $S(x) = 4x(1 - x)$. From this $\{P_S^n\}$ is asymptotically stable, and ϕ, given by (6.5.5), is the stationary density of P_S. □

These comments illustrate the construction of a statistically stable transformation S with a *given* stationary density from an exact transformation T. Clearly, the use of a different exact transformation T_1 will yield a different statistically stable transformation S_1, but one that has the same stationary density as S. Thus we are led to the next theorem.

Theorem 6.5.2. Let $T\colon (0,1) \to (0,1)$ be a measurable, nonsingular transformation and let $\phi \in D((a,b))$, with a and b finite or not, be a given positive density, that is, $\phi > 0$ a.e. Let a second transformation $S\colon (a,b) \to (a,b)$ be given by $S = g^{-1} \circ T \circ g$, where

$$g(x) = \int_a^x \phi(y)\,dy \qquad a < x < b. \tag{6.5.10}$$

Then T is exact if and only if S is statistically stable and ϕ is the density of the measure invariant with respect to S.

Proof: Let P_T and P_S be Frobenius–Perron operators corresponding to the transformations T and S, respectively. We start with the derivation of the relation between P_T and P_S. By the definition of P_S, we have

$$\int_a^y P_S f(x)\,dx = \int_{S^{-1}((a,y))} f(x)\,dx, \qquad \text{for } f \in L^1((a,b)),$$

where $S^{-1}((a,y)) = g^{-1}(T^{-1}(g(a), g(y)))$. Set $x = g^{-1}(z)$ and use equation (6.5.10) to change the variables so that the last integral may be rewritten to give

$$\int_a^y P_S f(x)\,dx = \int_{T^{-1}(g(a),g(y))} f(g^{-1}(z)) \frac{dz}{\phi(g^{-1}(z))}.$$

Defining

$$P_g f(x) = \frac{f(g^{-1}(x))}{\phi(g^{-1}(x))} \qquad \text{for } f \in L^1((a,b)), \tag{6.5.11}$$

we have

$$\int_a^y P_S f(x)\,dx = \int_{T^{-1}(g(a),g(y))} P_g f(z)\,dz = \int_{g(a)}^{g(y)} P_T P_g f(z)\,dz. \tag{6.5.12}$$

Setting

$$P_g^{-1}\tilde{f}(x) = \tilde{f}(g(x))\phi(x) \qquad \text{for } \tilde{f} \in L^1((0,1)) \tag{6.5.13}$$

and substituting $z = g(x)$ in the last integral in (6.5.12) yields

$$\int_a^y P_S f(x)\,dx = \int_a^y P_g^{-1} P_T P_g f(x)\,dx.$$

Thus P_S and P_T are related by

$$P_S f = P_g^{-1} P_T P_g f \qquad \text{for } f \in L^1((a,b)). \tag{6.5.14}$$

By integrating equation (6.5.11) over the entire space, we have

$$\|P_g f\|_{L^1(0,1)} = \|f\|_{L^1(a,b)} \qquad \text{for } f \in L^1((a,b)).$$

Further P_g^{-1}, as given by (6.5.13), is the inverse operator to P_g, and integration of (6.5.13) gives

$$\|P_g^{-1} \tilde{f}\|_{L^1(a,b)} = \|\tilde{f}\|_{L^1(0,1)} \qquad \text{for } \tilde{f} \in L^1((0,1)). \tag{6.5.15}$$

If T is measure preserving, we have $P_T 1 = 1$. Furthermore, from the definition of P_g in (6.5.11), we have $P_g \phi = 1$. As a consequence

$$P_S \phi = P_g^{-1} P_T P_g \phi = P_g^{-1} P_T 1 = P_g^{-1} 1 = \phi,$$

which shows that ϕ is the density of the measure invariant with respect to S. Analogously from $P_S \phi = \phi$, it follows that $P_T 1 = 1$. By using an induction argument with equation (6.5.14), we obtain

$$P_S^n f = P_g^{-1} P_T^n P_g f \qquad \text{for } f \in L^1((a,b)).$$

This, in conjunction with (6.5.15) and the equality $P_g \phi = 1$ gives

$$\begin{aligned} \|P_S^n f - \phi\|_{L^1(a,b)} &= \|P_g^{-1} P_T^n P_g f - P_g^{-1} P_g \phi\|_{L^1(a,b)} \\ &= \|P_T^n P_g f - P_g \phi\|_{L^1(0,1)} = \|P_T^n P_g f - 1\|_{L^1(0,1)} \end{aligned} \tag{6.5.16}$$

By substituting

$$f = P_g^{-1} \tilde{f} \qquad \text{for } \tilde{f} \in L^1((0,1))$$

into (6.5.16), we have

$$\|P_S^n P_g^{-1} \tilde{f} - \phi\|_{L^1(a,b)} = \|P_T^n \tilde{f} - 1\|_{L^1(0,1)}. \tag{6.5.17}$$

Thus, from equations (6.5.16) and (6.5.17), it follows that the strong convergence of $\{P_S^n f\}$ to ϕ for $f \in D((a,b))$ is equivalent to the strong convergence of $\{P_T^n \tilde{f}\}$ to 1 for $\tilde{f} \in D((0,1))$. ■

Example 6.5.2. Let T be the hat transformation of (6.5.9) and pick $\phi(x) = k\exp(-kx)$ for $0 < x < \infty$, which is the density distribution function for the lifetime of an atom with disintegration constant $k > 0$. Then it is straightforward to show that the transformation $S = g^{-1} \circ T \circ g$ is given by

$$S(x) = \ln\left\{\frac{1}{|1 - 2e^{-kx}|^{1/k}}\right\}.$$

The Frobenius–Perron operator associated with S is given by

$$P_S f(x) = \frac{e^{-kx}}{1+e^{-kx}} f\left(\frac{1}{k}\ln\frac{2}{1+e^{-kx}}\right) + \frac{e^{-kx}}{1-e^{-kx}} f\left(\frac{1}{k}\ln\frac{2}{1-e^{-kx}}\right).$$

By Theorem 6.5.2, $\{P_S^n\}$ is asymptotically stable with the stationary density $\phi(x) = k\exp(-kx)$. □

Example 6.5.3. As a second example, consider the Chebyshev polynomials $S_m: (-2,2) \to (-2,2)$,

$$S_m(x) = 2\cos[m\cos^{-1}(x/2)], \qquad m = 0,1,2,\ldots.$$

Define

$$g(x) = \frac{1}{\pi}\int_{-2}^{x} \frac{du}{\sqrt{4-u^2}}$$

corresponding to the density

$$\phi(x) = \frac{1}{\pi\sqrt{4-x^2}}. \tag{6.5.18}$$

The Chebyshev polynomials satisfy $S_{m+1}(x) = xS_m(x) - S_{m-1}(x)$ with $S_0(x) = 2$ and $S_1(x) = x$. It is straightforward, but tedious, to show that the transformation $T_m = g \circ S_m \circ g^{-1}$ is given by

$$T_m(x) = \begin{cases} m\left(x - \dfrac{2n}{m}\right) & \text{for } x \in \left[\dfrac{2n}{m}, \dfrac{2n+1}{m}\right) \\ m\left(\dfrac{2n+2}{m} - x\right) & \text{for } x \in \left[\dfrac{2n+1}{m}, \dfrac{2n+2}{m}\right), \end{cases} \tag{6.5.19}$$

where $n = 0,1,\ldots,[(m-1)/2]$, and $[y]$ denotes the integer part of y. For $m \geqslant 2$, by Theorem 6.2.2, $\{P_{T_m}^n\}$ is asymptotically stable. An explicit computation is easy and shows that $f_* \equiv 1$ is the stationary density of P_{T_m}. Thus T_m is exact. Hence, by Theorem 6.5.2, the Chebyshev polynomials S_m are statistically stable for $m \geqslant 2$ with a stationary density given by equation (6.5.18). This may also be proved more directly as shown by Adler and Rivlin [1964].

This example is of interest from several standpoints. First, it illustrates in a concrete way the nonuniqueness of statistically stable transformations (S_m) with the same stationary density derived from different exact transformations (T_m). Second, it should be noted that the transformation $\tilde{S}_m: (0,1) \to (0,1)$, given by

$$\tilde{S}_m(x) = -\tfrac{1}{4}S_m(4x-2) + \tfrac{1}{2}$$

when $m = 2$, is just the familiar parabola, $\tilde{S}_2(x) = 4x(1-x)$. Finally, we note in passing that cubic maps equivalent to S_3 have arisen in a study of a simple genetic model involving one locus and two alleles [May, 1980] and have also been studied in their own right by Rogers and Whitley [1983]. □

Example 6.5.4. As a further illustration of the power of Theorem 6.5.2, we consider an example drawn from quantum mechanics. Consider a particle of mass m free to move in the x direction and subjected to a restoring force, $-kx$. This is equivalent to the particle being placed in a potential $V(x) = kx^2/2$. The standard solution to this quantized harmonic oscillator problem is [Schiff, 1955]

$$u_n(x) = \left[\frac{\alpha}{\sqrt{\pi}\, 2^n n!}\right]^{1/2} H_n(\alpha x)e^{-(1/2)\alpha^2x^2}, \qquad \text{for } n = 0, 1, \ldots,$$

where

$$\alpha^4 = mk/\hbar^2$$

($\hbar$ is Planck's constant) and $H_n(y)$ denotes the nth-order Hermite polynomial, defined recursively by

$$H_n(y) = (-1)^n e^{y^2}\frac{d^n}{dy^n}(e^{-y^2})$$

[$H_0(y) = 1, H_1(y) = 2y, H_2(y) = 4y^2 - 2, \ldots$]. In accord with the usual interpretation of quantum mechanics, the associated densities are given by $\phi_n(x) = [u_n(x)]^2$, or

$$\phi_n(x) = \frac{\alpha}{\sqrt{\pi}\, 2^n n!} H_n^2(\alpha x)e^{-\alpha^2x^2}, \qquad \text{for } n = 0, 1, \ldots,$$

and the g_n are

$$g_n(x) = \frac{\alpha}{\sqrt{\pi}\, 2^n n!}\int_{-\infty}^{x} H_n^2(\alpha y)e^{-\alpha^2y^2}\, dy, \qquad \text{for } n = 0, 1, \ldots.$$

Then for any exact transformation T, the transformations $S_n(x) = g_n^{-1} \circ T \circ g_n(x)$ have the requisite stationary densities ϕ_n. □

To close this section we note that the following result is a direct extension of Theorem 6.5.2.

Corollary 6.5.1. Let $S\colon (a,b) \to (a,b)$, with a and b finite or not, be a statistically stable transformation with a stationary density $\phi \in D((a,b))$ and let $\tilde{\phi} \in D((\alpha,\beta))$ be given, with α and β also finite or not. Further, set

$$g(x) = \int_a^x \phi(y)\, dy \quad \text{and} \quad \tilde{g}(x) = \int_\alpha^x \tilde{\phi}(y)\, dy.$$

Then the transformation $\tilde{S}: (\alpha, \beta) \to (\alpha, \beta)$, defined by

$$\tilde{S} = \tilde{g}^{-1} \circ g \circ S \circ g^{-1} \circ \tilde{g},$$

is statistically stable with stationary density $\tilde{\phi}$.

Proof: First set $T: (0,1) \to (0,1)$ equal to $T = g \circ S \circ g^{-1}$. This is equivalent to $S = g^{-1} \circ T \circ g$ and, by Theorem 6.5.2, T is exact. Again, using Theorem 6.5.2 with the exactness of T, we have that $\tilde{S} = \tilde{g}^{-1} \circ T \circ \tilde{g}$ is statistically stable. ∎

Remark 6.5.1. Nonlinear transformations with a specified stationary density can be used as pseudorandom number generators. For details see Li and Yorke [1978]. □

6.6 Transformations on the real line

All of the transformations considered in previous sections were defined on the interval $[0,1]$. The particular choice of the interval $[0,1]$ is not restrictive since, given $S: [a,b] \to [a,b]$, we can always consider $T(x) = Q^{-1}(S(Q(x)))$, $T: [0,1] \to [0,1]$, where $Q(x) = a + (b-a)x$. All of the asymptotic properties of S are the same as those of T.

However, if S maps the whole real line (or half-line) into itself, no linear change of variables is available to reduce this problem to an equivalent transformation on a finite interval. Further, transformations on the real line may have some anomalous properties. For example, the requirement that $|S'(x)| \geqslant \lambda > 1$ for $S: R \to R$ is not sufficient for the asymptotic stability of $\{P^n\}$. This is amply illustrated by the specific example $S(x) = 2x$, which was considered in Section 1.3.

There are, however, transformations on the real line for which the asymptotic stability of $\{P^n\}$ can be demonstrated; one example is $S(x) = \beta \tan(\gamma x + \delta)$, $|\beta\gamma| > 1$. This section will treat a class of such transformations.

Assume the transformation $S: R \to R$ satisfies the following conditions:

(6i) There is a partition $\cdots a_{-2} < a_{-1} < a_0 < a_1 < a_2 \cdots$ of the real line such that, for every integer $i = 0, \pm 1, \pm 2, \ldots$, the restriction $S_{(i)}$ of S to the interval (a_{i-1}, a_i) is a C^2 function;

(6ii) $S((a_{i-1}, a_i)) = R$;

(6iii) There is a constant $\lambda > 1$ such that $|S'(x)| \geqslant \lambda$ for $x \neq a_i$, $i = 0, \pm 1, \pm 2, \ldots$;

(6iv) There is a constant $L \geqslant 0$ and a function $q \in L^1(R)$ such that

$$a_i - a_{i-1} \leqslant L, \qquad |g_i'(x)| \leqslant q(x)\,(a_i - a_{i-1}) \tag{6.6.1}$$

where $g_i = S_{(i)}^{-1}$, for $i = 0, \pm 1, \ldots$; and

(6v) There is a real constant c such that

$$\frac{|S''(x)|}{[S'(x)]^2} \leq c \qquad \text{for } x \neq a_i,\ i = 0, \pm 1, \ldots \tag{6.6.2}$$

Then the following theorem summarizes results of Kemperman [1975], Schweiger [1978], Jabłoński and Lasota [1981], and Bugiel [1982].

Theorem 6.6.1. If $S: R \to R$ satisfies conditions (6i)–(6v) and P is the associated Frobenius–Perron operator, then $\{P^n\}$ is asymptotically stable.

Proof: We first calculate the Frobenius–Perron operator. To do this note that

$$S^{-1}((-\infty, x)) = \bigcup_j (a_{j-1}, g_j(x)) + \bigcup_k (g_k(x), a_k),$$

where the first union is over intervals in which g_i is an increasing function of x, and the second is for intervals in which g_i is decreasing. Thus

$$\begin{aligned} Pf(x) &= \frac{d}{dx} \int_{S^{-1}((-\infty,x))} f(u)\, du \\ &= \frac{d}{dx} \sum_j \int_{a_{j-1}}^{g_j(x)} f(u)\, du + \frac{d}{dx} \sum_k \int_{g_k(x)}^{a_k} f(u)\, du \end{aligned}$$

or

$$Pf(x) = \sum_{i=-\infty}^{\infty} \sigma_i(x) f(g_i(x)), \tag{6.6.3}$$

where $\sigma_i(x) = |g_i'(x)|$.

Having an expression for $Pf(x)$ we now caculate the variation of Pf to show that the sequence $f_n = P^n f$ satisfies assumptions (5.8.2) and (5.8.3) of Proposition 5.8.1. Denote by $D_0 \subset D(R)$ the set of all densities of bounded variation on R that are positive, continuously differentiable, and satisfy

$$|f'(x)| \leq k_f f(x), \qquad \text{for } x \in R, \tag{6.6.4}$$

where the constant k_f depends on f. Now

$$\begin{aligned} \bigvee_{-\infty}^{\infty} Pf(x) &\leq \sum_{i=-\infty}^{\infty} \bigvee_{-\infty}^{\infty} \sigma_i(x) f(g_i(x)) \\ &\leq \sum_{i=-\infty}^{\infty} \left\{ \frac{1}{\lambda} \bigvee_{-\infty}^{\infty} f(g_i(x)) + \int_{-\infty}^{\infty} |\sigma_i'(x)| f(g_i(x))\, dx \right\}. \end{aligned}$$

Using $|\sigma_i'(x)| \leq c\sigma_i(x)$, which follows from inequality (6.6.2), and making the change of variables $y = g_i(x)$, we have

$$\bigvee_{-\infty}^{\infty} Pf(x) \leqslant \sum_{i=-\infty}^{\infty} \left\{ \frac{1}{\lambda} \bigvee_{a_i-1}^{a_i} f(y) + c \int_{a_{i-1}}^{a_i} f(y)\,dy \right\}$$

$$\leqslant \frac{1}{\lambda} \bigvee_{-\infty}^{\infty} f(y) + c\,.$$

By an induction argument, we obtain

$$\bigvee_{-\infty}^{\infty} P^n f(x) \leqslant \frac{1}{\lambda^n} \bigvee_{-\infty}^{\infty} f(y) + \frac{\lambda c}{\lambda - 1}\,.$$

Since $\lambda > 1$, then, for real $\alpha > \lambda c/(\lambda - 1)$, there must exist a sufficiently large n, say $n > n_0(f)$, such that

$$\bigvee_{-\infty}^{\infty} P^n f(x) \leqslant \alpha\,. \tag{6.6.5}$$

Now we are in a position to evaluate $P^n f$. From inequalities (6.6.1) and (6.6.3), we have

$$Pf(x) \leqslant q(x) \sum_{i=-\infty}^{\infty} f(g_i(x))\,(a_i - a_{i-1})\,. \tag{6.6.6}$$

For every interval (a_{i-1}, a_i) pick a $z_i \in (a_{i-1}, a_i)$ such that

$$(a_i - a_{i-1}) f(z_i) \leqslant \int_{a_{i-1}}^{a_i} f(x)\,dx, \qquad \text{for } i = 0, \pm 1, \ldots\,.$$

Thus, from (6.6.1) and (6.6.6), we obtain

$$Pf(x) \leqslant q(x) \sum_{i=-\infty}^{\infty} \left\{ L|f(g_i(x)) - f(z_i)| + \int_{a_{i-1}}^{a_i} f(x)\,dx \right\}$$

$$\leqslant Lq(x) \bigvee_{-\infty}^{\infty} f(x) + q(x) \int_{-\infty}^{\infty} f(x)\,dx$$

$$= q(x) \left\{ L \bigvee_{-\infty}^{\infty} f(x) + 1 \right\}.$$

By substituting $P^{n-1}f$ instead of f in this expression and using (6.6.5), we have

$$P^n f(x) \leqslant q(x)\,(\alpha L + 1), \qquad x \in R,\ n > n_0(f) + 1\,. \tag{6.6.7}$$

Thus the sequence of functions $f_n = P^n f$ satisfies condition (5.8.2) of Proposition 5.8.1.

Now, differentiating equation (6.6.3) and using $|\sigma_i'| \leqslant c\sigma_i$, $\sigma_i < 1/\lambda$, and $|f'| \leqslant k_f f$ gives

$$\frac{(Pf)'}{Pf} \leqslant \frac{\Sigma_i |\sigma_i'| (f \circ g_i)}{\Sigma_i \sigma_i (f_i \circ g_i)} + \frac{\Sigma_i (\sigma_i)^2 |f' \circ g_i|}{\Sigma_i \sigma_i (f_i \circ g_i)}$$

$$\leqslant c + \frac{1}{\lambda} \frac{\Sigma_i \sigma_i k_f (f \circ g)}{\Sigma_i \sigma_i (f \circ g_i)} \leqslant c + \frac{1}{\lambda} k_f ,$$

and, by induction,

$$\frac{|(P^n f)'|}{P^n f} \leqslant \frac{c\lambda}{\lambda - 1} + \frac{1}{\lambda^n} k_f .$$

Pick a constant $K > \lambda c/(\lambda - 1)$ so that, since $\lambda > 1$, for n sufficiently large ($n > n_1(f)$), we have

$$|(P^n f)'| \leqslant K P^n f. \tag{6.6.8}$$

Thus the iterates $f_n = P^n f$ satisfy condition (5.8.3) of Proposition 5.8.1. Therefore, by Proposition 5.8.1, $P^n f$ has a nontrivial lower-bound function, and thus, by Theorem 5.6.2, $\{P^n\}$ is asymptotically stable. ∎

Remark 6.6.1. Observe that in the special case where S is periodic (in x) with period $L = a_i - a_{i-1}$, condition (6iv) is automatically satisfied. In fact, in this case $g_i'(x) = g_0'(x)$ so, by setting $q = |g_0'|/L$, we obtain inequality (6.6.1) and, moreover,

$$\|q\| = \frac{1}{L} \int_{-\infty}^{\infty} |g_i'(x)|\, dx = \frac{1}{L} \left| \int_{-\infty}^{\infty} g_i'(x)\, dx \right| = |[g_i(x)]_{-\infty}^{\infty}| = \frac{L}{L} = 1 ,$$

showing that $q \in L^1$. The remaining conditions simply generalize the properties of the transformation $S(x) = \beta \tan(\gamma x + \delta)$ with $|\beta\gamma| > 1$. □

Example 6.6.1. It is easy to show that the Frobenius–Perron operator P associated with $S(x) = \beta \tan(\gamma x + \delta)$, $|\beta\gamma| > 1$, is asymptotically stable. We have

$$S'(x) = \frac{\beta\gamma}{\cos^2(\gamma x + \delta)}$$

hence $|S'(x)| \geqslant \beta\gamma$. Further

$$-\frac{S''(x)}{[S'(x)]^2} = -\frac{1}{\beta} \sin[2(\gamma x + \delta)]$$

so that

$$\left| \frac{S''(x)}{[S'(x)]^2} \right| \leqslant \frac{1}{|\beta|} . \quad \square$$

6.7 Manifolds

The last goal of this chapter is to show how the techniques described in Chapter 5 may be used to study the behavior of transformations in higher-dimensional spaces. The simplest, and probably most striking, use of the Frobenius–Perron operator in d-dimensional spaces is for expanding mappings on manifolds. To illustrate this, the results of Krzyżewski and Szlenk (1969), which may be considered as a generalization of the results of Rényi presented in Section 6.2, are developed in detail in Section 6.8. However, in this section we preface these results by presenting some basic concepts from the theory of manifolds, which will be helpful for understanding the geometrical ideas related to the Krzyżewski–Szlenk results. This elementary description of manifolds is by no means an exhaustive treatment of differential geometry.

First consider the paraboloid $z = x^2 + y^2$. This paraboloid is embedded in three-dimensional space, even though it is a two-dimensional object. If the paraboloid is the state space of a system, then, to study this system, each point on the paraboloid must be described by precisely two numbers. Thus, any point m on the paraboloid with coordinates $(x, y, x^2 + y^2)$ is simply described by its x, y-coordinates. This two-dimensional system of coordinates may be described in a more abstract way as follows. Denote by M the graph of the paraboloid, that is,

$$M = \{(x, y, z): z = x^2 + y^2\},$$

and, as a consequence, there is a one-to-one transformation $\phi: M \to R^2$ described by $\phi(x, y, z) = (x, y)$ for $(x, y, z) \in M$. Of course, other coordinate systems on M are possible, that is, another one-to-one mapping, $\phi^*: M \to R^2$, but ϕ is probably the simplest one.

Now let M be the unit sphere,

$$M = \{(x, y, z): x^2 + y^2 + z^2 = 1\}.$$

In this example it is impossible to find a single smooth invertible function $\phi: M \to R^2$. However, six functions $\phi_i: M \to R^2$ may be defined as follows:

$$\begin{aligned}
\phi_1(x, y, z) &= (x, y) && \text{for } z > 0;\\
\phi_2(x, y, z) &= (x, y) && \text{for } z < 0;\\
\phi_3(x, y, z) &= (x, z) && \text{for } y > 0;\\
\phi_4(x, y, z) &= (x, z) && \text{for } y < 0;\\
\phi_5(x, y, z) &= (y, z) && \text{for } x > 0;\\
\phi_6(x, y, z) &= (y, z) && \text{for } x < 0.
\end{aligned}$$

Each of these functions ϕ_i maps a hemisphere of M onto an open unit disk. This coordinate system has the property that for any $m \in M$ there is an open hemisphere that contains m and on each of these hemispheres one ϕ_i is defined.

In the same spirit, we give a general definition of a smooth manifold.

Definition 6.7.1. A **smooth d-dimensional manifold** consists of a topological Hausdorff space M and a sytem $\{\phi_i\}$ of local coordinates satisfying the following properties:

(a) Each function ϕ_i is defined and continuous on an open subset $W_i \subset M$ and maps it onto an open subset $U_i = \phi_i(W_i)$ of R^d. The inverse functions ϕ_i^{-1} exist and are continuous (i.e., ϕ_i is a homeomorphism of W_i onto U_i);

(b) For each $m \in M$ there is a W_i such that $m \in W_i$, that is, $M = \bigcup_i W_i$;

(c) If the intersection $W_i \cap W_j$ is nonempty, then the mapping $\phi_i \circ \phi_j^{-1}$, which is defined on $\phi_j(W_i \cap W_j) \subset R^d$ and having values in R^d, is a C^∞ mapping.

(Note that a topological space is called a **Hausdorff space** if every two distinct points have nonintersecting neighborhoods.)

Any map ϕ_i gives a coordinate system of a part of M, namely W_i. A local coordinate of a point $m \in W_i$ is $\phi_i(m)$. Having a coordinate system, we may now define what we mean by a C^k function on M. We say that f: $M \to R$ is of **class** C^k if for each ϕ_i: $W_i \to U_i$ the composed mapping $f \circ \phi_i^{-1}$ is of class C^k on U_i.

Next consider the gradient of a function defined on the manifold. For f: $R^d \to R$ the **gradient** of f at a point $x \in R^d$ is simply the vector (sequence of real numbers),

$$\operatorname{grad} f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_d}\right).$$

For f: $M \to R$ of class C^1, the gradient of f at a point $m \in M$ can be calculated in local coordinates as follows:

$$\operatorname{grad} f(m) = (D_{x_1}(m)f, \ldots, D_{x_d}(m)f), \tag{6.7.1a}$$

where

$$D_{x_i}(m)f = \frac{\partial}{\partial x_i}[f(\phi^{-1}(x))]_{x=\phi(m)}. \tag{6.7.1b}$$

Thus the gradient is again a sequence of real numbers that depends on the choice of the local coordinates.

The most important notion from the theory of manifolds is that of tangent vectors and tangent spaces. A continuous mapping γ: $[a, b] \to M$ represents an arc on M with the end points $\gamma(a)$ and $\gamma(b)$. We say that γ starts from $m = \gamma(a)$. The arc γ is C^k if, for any coordinate system ϕ, the composed function $\phi \circ \gamma$ is of class C^k. The **tangent vector** to γ at a point $m = \gamma(a)$ in a coordinate system ϕ is defined by

$$\frac{d}{dt}[\phi(\gamma(t))]_{t=a} = (\xi^1, \ldots, \xi^d), \tag{6.7.2}$$

where, again, the numbers $\xi^1, \ldots, \xi^d$ depend on the choice of the coordinate system ϕ. Of course, γ must be at least of class C^1. Two arcs γ_1 and γ_2 starting from m are called **equivalent** if they produce the same coordinates, that is,

$$\frac{d}{dt}[\phi(\gamma_1(t))]_{t=a_1} = \frac{d}{dt}[\phi(\gamma_2(t))]_{t=a_2}, \tag{6.7.3}$$

where $\gamma_1(a_1) = \gamma_2(a_2) = m$. Observe that, if (6.7.3) holds in a given system of coordinates ϕ, then it holds in any other coordinate system. The class of all equivalent arcs produces the same sequence (6.7.3) for any given system of coordinates. Such a class represents the tangent vector. Tangent vectors are denoted by the Greek letters ξ and η.

Assume that a tangent vector ξ in a coordinate system ϕ has components $\xi^1, \ldots, \xi^d$. What are the components in another coordinate system ψ? Now,

$$\frac{d}{dt}[\psi(\gamma(t))] = \frac{d}{dt}[H(\phi(\gamma(t)))],$$

where $H = \psi \circ \phi^{-1}$ and, therefore, setting $d(\psi \circ \gamma)/dt = (\eta^1, \ldots, \eta^d)$,

$$\eta^i = \sum_{j=1}^{d} \frac{\partial H_i}{\partial x_j} \xi^j. \tag{6.7.4}$$

Equation (6.7.4) shows the transformations of the tangent vector coordinates under the change of coordinate system. Thus from an abstract (tensor analysis) point of view the tangent vector at a point m is nothing but a sequence of numbers in each coordinate system given in such a way that these numbers satisfy condition (6.7.4) when we pass from one coordinate system to another. From this description it is clear that the tangent vectors at m form a linear space, the **tangent space,** which we denote by T_m.

Now consider a transformation F from a d-dimensional manifold M into a d-dimensional manifold $N, F\colon M \to N$. The transformation F is said to be of class C^k if, for any two coordinate systems ϕ on M and ψ on N, the composed function $\psi \circ F \circ \phi^{-1}$ is of class C^k, or its domain is empty. Let ξ be a tangent vector at m, represented by a C^1 arc $\gamma\colon [a, b] \to M$ starting from m. Then $F \circ \gamma$ is an arc starting from $F(m)$, and it is of class C^1 if F is of class C^1. The tangent vector to $F \circ \gamma$ in a coordinate system ψ is given by

$$\frac{d}{dt}[\psi \circ F \circ \gamma]_{t=a} = (\eta^1, \ldots, \eta^d).$$

Setting $\sigma = \psi \circ F \circ \phi^{-1}$, where ϕ is a coordinate system on M,

$$\eta^i = \sum_{j=1}^{d} \frac{\partial \sigma_i}{\partial x_j} \xi^j \tag{6.7.5}$$

results. Equation (6.7.5) gives the linear transformation of a tangent vector ξ at m to a tangent vector η at $F(m)$ without explicit reference to the arc γ. This transformation is called the **differential** of F at a point m and is denoted by $dF(m)$, and thus symbolically

$$\eta = dF(m)\xi.$$

Note that the differential of F is represented in any two coordinate systems, ϕ on M and ψ on N, by the matrix

$$\left(\frac{\partial \sigma_i}{\partial x_j}\right), \qquad i, j = 1, \ldots, d.$$

The same matrix appears in the formula for the gradient of the composed function: If $F: M \to N$ and $f: N \to R$ are C^1 functions, then the differentiation of $(f \circ F) \circ \phi^{-1} = (f \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1})$ gives

$$\operatorname{grad}(f \circ F)(m) = (D_{x_1}(m)(f \circ F), \ldots, D_{x_d}(m)(f \circ F)),$$

where

$$D_{x_i}(m)(f \circ F) = \sum_{j=1}^{d} \frac{\partial}{\partial x_j}[f(\psi^{-1}(x))]_{x=\psi(F(m))} \frac{\partial \sigma_j}{\partial x_i}.$$

This last formula may be written more compactly as

$$\operatorname{grad}((f \circ F)(m)) = (\operatorname{grad} f)(dF(m)).$$

Observe that now $dF(m)$ appears on the right-hand side of the vector.

Finally observe the relationship between tangent vectors and gradients. Let $f: M \to R$ be of class C^1 and let $\gamma: [a, b] \to M$ start from m. Consider the composed function $f \circ \gamma: [a, b] \to R$ that is also of class C^1. Using the local system of coordinates,

$$f \circ \gamma = (f \circ \phi^{-1}) \circ (\phi \circ \gamma),$$

and, consequently,

$$\left.\frac{d(f \circ \gamma)}{dt}\right|_{t=a} = \sum_{i=1}^{d} [D_{x_i}(\gamma(a))f]\xi^i. \tag{6.7.6}$$

Observe that the numbers $D_{x_i} f$ and ξ^i depend on ϕ even though the left-hand side of (6.7.6) does not. Equation (6.7.6) may be more compactly written as

$$\left.\frac{d(f \circ \gamma)}{dt}\right|_{t=a} = [\operatorname{grad} f(\gamma(a))]\gamma'. \tag{6.7.7}$$

In order to construct a calculus on manifolds, concepts such as the length of a tangent vector, the norm of a gradient, and the area of Borel subsets of M are necessary. The most effective way of introducing these is via the Riemannian metric. Generally speaking the Riemannian metric is a scalar product on T_m. This means that, for any two vectors $\xi_1, \xi_2 \in T_m$, there corresponds a real number denoted by $\langle \xi_1, \xi_2 \rangle$. However, the coordinates

$$(\xi_1^1, \ldots, \xi_1^d), (\xi_2^1, \ldots, \xi_2^d)$$

depend on the coordinate system ϕ. Thus the rule that allows $\langle \xi_1, \xi_2 \rangle$ to be calculated given $(\xi_1^i), (\xi_2^i)$ must also depend on ϕ. These facts are summarized in the following definition.

Definition 6.7.2. A **Riemannian metric** on the manifold M is a system of functions

$$g_{ij}^{\phi}(m): M \to R, \qquad i,j = 1, \ldots, d,$$

such that

(a) For any choice of local coordinates $\phi: W \to U$ the functions $g_{ij}^{\phi}(\phi^{-1}(x))$ are defined and C^∞ for $x \in U$.

(b) For each $m \in M$ the quadratic form

$$\sum_{i,j=1}^{d} g_{ij}^{\phi} \lambda^i \lambda^j$$

is symmetric and positive definite (i.e., $g_{ij}^{\phi} = g_{ji}^{\phi}$, and the value of this sum is positive except if all $\lambda^i = 0$).

(c) For every $\xi_1, \xi_2 \in T_m$ the **scalar product**

$$\langle \xi_1, \xi_2 \rangle = \sum_{i,j=1}^{d} g_{ij}^{\phi}(m) \xi_1^i \xi_2^j \tag{6.7.8}$$

does not depend on ϕ.

The last condition (6.7.8) looks somewhat mysterious, but it simply means that

$$\sum_{k,l} g_{kl}^{\psi}(m) \eta_1^k \eta_2^l = \sum_{i,j} g_{ij}^{\phi}(m) \xi_1^i \xi_2^j$$

where η_1, η_2 are calculated from ξ_1, ξ_2 by equation (6.7.4). Thus

$$\sum_{k,l} g^{\psi}_{kl}(m) \sum_{i,j} \frac{\partial H_k}{\partial x_i} \frac{\partial H_l}{\partial x_j} \xi^i_1 \xi^j_2 = \sum_{i,j} g^{\phi}_{ij}(m) \xi^i_1 \xi^j_2 ,$$

which implies that

$$g^{\phi}_{ij}(m) = \sum_{k,l} g^{\psi}_{kl}(m) \frac{\partial H_k}{\partial x_i} \frac{\partial H_l}{\partial x_j} . \tag{6.7.9}$$

Now having introduced the scalar product, the **norm** of $\xi \in T_m$ is defined by $\|\xi\| = (\langle \xi, \xi \rangle)^{1/2}$. If a C^1 arc γ: $[a, b] \to M$ is given, it defines, at each point $m = \gamma(t_0)$, the tangent vector $\gamma'(t_0)$. Thus the length of an arc γ is just

$$l(\gamma) = \int_a^b \|\gamma'(t)\| \, dt . \tag{6.7.10}$$

This equation may be used for any arc γ that is continuous and piecewise C^1. If a manifold M is such that any two points $m_0, m_1 \in M$ can be joined by a continuous piecewise C^1 arc, it is said to be **connected.** On connected manifolds the **distance** between points is given by

$$\rho(m_0, m_1) = \inf l(\gamma) ,$$

where the inf is taken over all possible arcs joining m_0 and m_1. With this distance, M becomes a metric space.

From equation (6.7.7) it is easy to define the **length** of grad f at a point m. It is, by definition,

$$|\mathrm{grad}\, f(m)| = \sup \left| \frac{d}{dt} [f(\gamma(t))]_{t=a} \right| ,$$

where the sup is taken over all possible arcs γ: $[a, b] \to M$ with $\gamma(a) = m$ and $\|\gamma'(a)\| = 1$. From this definition, it follows that, for an arbitrary C^1 arc γ and C^1 function f,

$$\left| \frac{d}{dt} [f(\gamma(t))] \right| \leq |\mathrm{grad}\, f(\gamma(t))| \cdot \|\gamma'(t)\| . \tag{6.7.11}$$

Analogously, for a C^1 mapping F: $M \to N$ we introduce the norm of the differential $dF(m)$ by

$$|dF(m)| = \sup \|dF(m)\xi\| ,$$

where the supremum is taken over all $\xi \in T_m$ such that $\|\xi\| = 1$. Using this notion it can be verified that

$$\|dF(m)\xi\| \leq |dF(m)| \cdot \|\xi\|, \qquad m \in M, \ \xi \in T_m ,$$

and

$$|\mathrm{grad}(f \circ F)(m)| \leqslant |(\mathrm{grad}\, f)(F(m))| \cdot |dF(m)|, \qquad m \in M,$$

where $f: N \to R$ is a C^1 function. Differentiation on manifolds satisfies some other properties analogous to those on R^d. We have, for example,

$$|\mathrm{grad}(fg)| \leqslant |f| \cdot |\mathrm{grad}\, g| + |g| \cdot |\mathrm{grad}\, f|, \qquad (fg)(m) = f(m)g(m)$$

and

$$\left|\mathrm{grad} \sum_i f_i\right| \leqslant \sum_i |\mathrm{grad}\, f_i|,$$

where f, g, and f_i are C^1 functions.

To introduce the measure on M associated with the Riemannian metric, it is first necessary to define the **unit volume function** $V_\phi(m)$. Consider the Riemannian form $g_{ij}^\phi(m)$ corresponding to a coordinate system $\phi: W \to U$. We can find d normalized vectors

$$\begin{pmatrix} \xi_1^1 \\ \vdots \\ \xi_1^d \end{pmatrix} \overset{d}{\cdots} \begin{pmatrix} \xi_d^1 \\ \vdots \\ \xi_d^d \end{pmatrix} \tag{6.7.12}$$

orthogonal with respect to this form, that is,

$$\langle \xi_k, \xi_l \rangle = \sum_{i,j=1}^{d} g_{ij}^\phi(m) \xi_k^i \xi_l^j = \delta_{kl},$$

where $\delta_{kl} = 1$ if $k = l$, and $\delta_{kl} = 0$, $k \neq l$, is the Kronecker delta. Write

$$V_\phi(m) = \left| \det \begin{vmatrix} \xi_1^1 & \cdots & \xi_d^1 \\ \cdots & \cdots & \cdots \\ \xi_1^d & \cdots & \xi_d^d \end{vmatrix} \right|.$$

This same procedure can be carried out algebraically by setting $V_\phi(m) = |\det(g_{ij}^\phi(m))|^{-1/2}$.

Function $V_\phi(m)$ has a simple heuristic interpretation. Vectors $\xi_1, \ldots, \xi_d$, which correspond to the components (6.7.12), are orthogonal and normalized. Thus the volume spanned by them should be equal to 1. The volume spanned by the representation of $\xi_1, \ldots, \xi_d$ in a local coordinate system is $V_\phi(m)$. Thus $V_\phi(m)$ gives the volume in local coordinates corresponding to a unit volume in M. By using this interpretation we define the **measure μ of a Borel set** $B \subset W$ as

$$\mu(B) = \int_{\phi(B)} \frac{dx}{V_\phi(\phi^{-1}(x))}. \tag{6.7.13}$$

The idea leading to this definition is obvious, as the elementary volume $dx = dx_1 \cdots dx_d$ in R^d corresponds to the volume $V_\phi(\phi^{-1}(x))\,dx_1 \cdots dx_d$ in M. Thus, in order to reproduce the "original" volume in M, we must divide dx over V_ϕ. It can be shown that $\mu(B)$ defined by (6.7.13) does not depend on the choice of ϕ, which is quite obvious from the heuristic interpretation of $V_\phi(m)$.

Analogous considerations lead to the definition of the **determinant of the differential** of a C^1 transformation F from a d-dimensional manifold M into a d-dimensional manifold N. Take a point $m \in M$ and define

$$|\det dF(m)| = \left|\frac{d\sigma}{dx}\right| \frac{V_\phi(m)}{V_\psi(F(m))},$$

where $|d\sigma/dx|$ denotes the absolute value of the determinant of the $d \times d$ matrix

$$\left(\frac{\partial \sigma_i}{\partial x_j}\right), \qquad i,j = 1,\ldots,d.$$

It can be shown that this definition does not depend on the choice of coordinate systems ϕ and ψ in M and N, respectively. Note also that the determinant per se is not defined, but only its absolute value. This is because our manifolds M, N are not assumed to be oriented.

The following calculation will justify our definition of $|\det dF(m)|$. Let B be a small set on M, and $F(B)$ its image on N. What is the ratio $\mu(F(B))/\mu(B)$? From equation (6.7.13),

$$\frac{\mu(F(B))}{\mu(B)} = \int_{\psi(F(B))} \frac{dy}{V(\psi^{-1}(y))} \Big/ \int_{\phi(B)} \frac{dx}{V_\phi(\phi^{-1}(x))}.$$

Setting $\sigma = \psi \circ F \circ \phi^{-1}$ and substituting $y = \sigma(x)$,

$$\frac{\mu(F(B))}{\mu(B)} = \int_{\phi(B)} \left|\frac{d\sigma}{dx}\right| \frac{dx}{V_\psi(F(\phi^{-1}(x)))} \Big/ \int_{\phi(B)} \frac{dx}{V_\phi(\phi^{-1}(x))}$$

results. Thus, for small B containing a point m, we have approximately

$$\frac{\mu(F(B))}{\mu(B)} \simeq \left|\frac{d\sigma}{dx}\right| \frac{V_\phi(m)}{V_\psi F(m))} = |\det(dF(m))|. \tag{6.7.14}$$

6.8 Expanding mappings on manifolds

With the background material of the preceding section, we now turn to an examination of the asymptotic behavior of expanding mappings on manifolds.

We assume that M is a finite dimensional compact connected smooth (C^∞) manifold with a Riemannian metric. As we have seen in Section 6.7, this metric induces the natural (Borel) measure μ and distance ρ on M. We use $|f'(m)|$ to denote the length of the gradient of f at point $m \in M$.

Before stating and proving our main result, we give a sufficient condition for the existence of a lower-bound function in the same spirit as contained in Propositions 5.8.1 and 5.8.2. We use the notation of Section 5.8.

Proposition 6.8.1. Let $P: L^1(M) \to L^1(M)$ be a Markov operator and if we assume that there is a set D_0, dense in D, so that for every $f \in D_0$ the trajectory

$$P^n f = f_n, \qquad \text{for } n \geqslant n_0(f) \tag{6.8.1}$$

is such that the functions f_n are C^1 and satisfy

$$|f_n'(m)| \leqslant k f_n(m) \qquad \text{for } m \in M, \tag{6.8.2}$$

where $k \geqslant 0$ is a constant independent of f, then there exists $\varepsilon > 0$ such that $h = \varepsilon 1_M$ is a lower-bound function for P.

Proof: The proof of this proposition proceeds much as for Proposition 5.8.2. As before, $\|f_n\| = 1$. Set

$$\varepsilon = [1/2\mu(M)]e^{-kr},$$

where

$$r = \sup_{m_0, m_1 \in M} \rho(m_0, m_1).$$

Let $\gamma(t)$, $a \leqslant t \leqslant b$, be a piecewise smooth arc joining points $m_0 = \gamma(a)$ and $m_1 = \gamma(b)$. Differentiation of $f_n \circ \gamma$ gives [see inequality (6.7.11)]

$$\begin{aligned} \left|\frac{d[f_n(\gamma(t))]}{dt}\right| &\leqslant |f_n'(\gamma(t))| \cdot \|\gamma'(t)\| \\ &\leqslant k\|\gamma'(t)\| f_n(\gamma(t)) \end{aligned}$$

so that

$$f_n(m_1) \leqslant f_n(m_0) \exp\left\{k \int_a^b \|\gamma'(s)\|\, ds\right\}.$$

Since γ was an arbitrary arc, this gives

$$f_n(m_1) \leqslant f_n(m_0)e^{k\rho(m_0, m_1)} \leqslant f_n(m_0)e^{kr}.$$

Now suppose that $h = \varepsilon 1_M$ is not a lower-bound function for P. This means that there must be some $n' > n_0$ and $m_0 \in M$ such that $f_{n'}(m_0) < \varepsilon$. Therefore,

$$f_{n'}(m_1) \leqslant \varepsilon e^{kr} = (1/2\mu(M)) \qquad \text{for } m_1 \in M,$$

which contradicts $\|f_n\| = 1$ for all $n > n_0(f)$. Thus we must have $f_n \geqslant h = \varepsilon 1_M$ for $n > n_0$. ■

Next we turn to a definition of an expanding mapping on a manifold.

Definition 6.8.1. Let M be a finite dimensional compact connected smooth (C^∞) manifold with Riemannian metric and let μ be the corresponding Borel measure. A C^1 mapping $S: M \to M$ is called **expanding** if there exists a constant $\lambda > 1$ such that the differential $dS(m)$ satisfies

$$\|dS(m)\xi\| \geqslant \lambda\|\xi\| \tag{6.8.3}$$

at each $m \in M$ for each tangent vector $\xi \in T_m$.

With this definition, Krzyżewski and Szlenk [1969] and Krzyżewski [1977] demonstrate the existence of a unique absolutely continuous normalized measure invariant under S and establish many of its properties. Most of these results are contained in the next theorem.

Theorem 6.8.1. Let $S: M \to M$ be an expanding mapping of class C^2, and P the Frobenius–Perron operator corresponding to S. Then $\{P^n\}$ is asymptotically stable.

Proof: From equation (6.7.5) with $F = S$, since S is expanding, $\eta \neq 0$ for any $\xi \neq 0$, and, thus, the matrix $(\partial\sigma_i/\partial x_j)$ must be nonsingular for every $m \in M$.

In local coordinates the transformation S has the form

$$x \to \phi(S(\phi^{-1}(x))) = \sigma(x)$$

and consequently is locally invertible. Therefore, for any point $m \in M$ the counterimage $S^{-1}(m)$ consists of isolated points, and, since M is compact, the number of these points is finite. Denote the counterimages of m by $m_1, \ldots, m_k$. Because S is locally invertible there exists a neighborhood W of m and neighborhoods W_i of m_i such that S restricted to W_i is a one to one mapping from W_i onto W. Denote the inverse mapping of S on W_i by g_i. We have $S \circ g_i = I_{W_i}$, where I_{W_i} is the identity mapping on W_i and, consequently, $(dS) \circ (dg_i)$ is the identity mapping on the tangent vector space. From this, in conjunction with (6.8.3), it immediately follows that

$$\|(dg_i)\xi\| \leqslant (1/\lambda)\,\|\xi\|. \tag{6.8.4}$$

Now take a set $B \subset W$, so

$$S^{-1}(B) = \bigcup_{i=1}^{k} g_i(B),$$

and, by the definition of the Frobenius–Perron operator,

$$\int_B Pf(m)\mu(dm) = \int_{S^{-1}(B)} f(m)\mu(dm) = \sum_{i=1}^{k} \int_{g_i(B)} f(m)\mu(dm).$$

This may be rewritten as

$$\frac{1}{\mu(B)}\int_B Pf(m)\mu(dm) = \sum_{i=1}^{k} \frac{\mu(g_i(B))}{\mu(B)} \cdot \frac{1}{\mu(g_i(B))}\int_{g_i(B)} f(m)\mu(dm)\,.$$

If B shrinks to m, then $g_i(B)$ shrinks to $g_i(m)$,

$$\frac{1}{\mu(B)}\int_B Pf(m)\mu(dm) \to Pf(m) \text{ a.e.}$$

and

$$\frac{1}{\mu(g_i(B))}\int_{g_i(B)} f(m)\mu(dm) \to f(g_i(m)) \text{ a.e.,} \qquad i = 1,\ldots,k.$$

Moreover, by (6.7.14),

$$\frac{\mu(g_i(B))}{\mu(B)} \to |\det(dg_i(m))|\,.$$

Thus, by combining all the preceding expressions, we have

$$Pf(m) = \sum_{i=1}^{k} |\det(dg_i(m))| f(g_i(m))\,, \tag{6.8.5}$$

which is quite similar to the result in equation (6.2.10).

Now let $D_0 \subset D(M)$ be the set of all strictly positive C^1 densities. For $f \in D_0$, differentiation of $Pf(m)$ as given by (6.8.5) yields

$$\begin{aligned}\frac{|(Pf)'|}{Pf} &= \frac{\sum_{i=1}^k |(J_i(f\circ g_i))'|}{\sum_{i=1}^k J_i(f\circ g_i)} \\ &\leqslant \frac{\sum_{i=1}^k |J_i'|\,(f\circ g_i)}{\sum_{i=1}^k J_i(f\circ g_i)} + \frac{\sum_{i=1}^k J_i|f'\circ g_i|\,|dg_i|}{\sum_{i=1}^k J_i(f\circ g_i)} \\ &\leqslant \max_i \frac{|J_i'|}{J_i} + \max_i \frac{|f'\circ g_i|\,|dg_i|}{(f\circ g_i)}\,,\end{aligned}$$

where $J_i = |\det dg_i(m)|$. From equation (6.8.4), it follows that $|dg_i| \leqslant 1/\lambda$, so that

$$\sup \frac{|(Pf)'|}{Pf} \leqslant c + \frac{1}{\lambda} \sup \frac{|f'|}{f}\,,$$

where

$$c = \sup_{i,m} \frac{|J_i'(m)|}{J_i(m)}\,.$$

Thus, by induction, for $n = 1, 2, \ldots,$ we have

$$\sup \frac{|(P^n f)'|}{P^n f} \leqslant \frac{c\lambda}{\lambda - 1} + \frac{1}{\lambda^n} \sup \frac{|f'|}{f}.$$

Choose a real $K > \lambda c/(\lambda - 1)$, then

$$\sup \frac{|(P^n f)'|}{P^n f} \leqslant K \tag{6.8.6}$$

for n sufficiently large, say $n > n_0(f)$. A straightforward application of Proposition 6.8.1 and Theorem 5.6.2 finishes the proof. ∎

Example 6.8.1. Let M be the two-dimensional torus, namely, the Cartesian product of two unit circles:

$$M = \{(m_1, m_2)\colon m_1 = e^{ix_1},\ m_2 = e^{ix_2},\ x_1 x_2 \in R\}.$$

M is evidently a Riemannian manifold, and the inverse functions to

$$m_1 = e^{ix_1} \quad \text{and} \quad m_2 = e^{ix_2} \tag{6.8.7}$$

define the local coordinate system. In these local coordinates the Riemannian metric is given by $g_{jk} = \delta_{jk}$, the Kronecker delta, and defines a Borel measure μ identical with that obtained from the product of the Borel measures on the circle.

We define a mapping $S\colon M \to M$ that, in local coordinates, has the form

$$S(x_1, x_2) = (3x_1 + x_2, x_1 + 3x_2) \qquad (\text{mod } 2\pi). \tag{6.8.8}$$

Thus S maps each point (m_1, m_2) given by (6.8.7) to the point $(\tilde{m}_1, \tilde{m}_2)$, where

$$\tilde{m}_1 = \exp[i(3x_1 + x_2)] \quad \text{and} \quad \tilde{m}_2 = \exp[i(x_1 + 3x_2)].$$

We want to show that S is an expanding mapping.

From (6.8.8) we see that $dS(m)$ maps the vector $\xi = (\xi_1, \xi_2)$ into the vector $(3\xi_1 + \xi_2, \xi_1 + 3\xi_2)$. Also, since $g_{jk} = \delta_{jk}$, hence $\langle \xi_1, \xi_2 \rangle = \xi_1^2 + \xi_2^2$ from (6.7.8). Thus

$$\begin{aligned} \|dS(m)\xi\|^2 &= (3\xi_1 + \xi_2)^2 + (\xi_1 + 3\xi_2)^2 \\ &= 4(\xi_1^2 + \xi_2^2) + 6(\xi_1 + \xi_2)^2 \\ &\geqslant 4\|\xi\|^2, \end{aligned}$$

and we see that inequality (6.8.3) is satisfied with $\lambda = 2$, therefore S is an expanding mapping. Further, if P is the Frobenius–Perron operator corresponding to S, then, by Theorem 6.8.1, $\{P^n\}$ is asymptotically stable. It is also possible to show that S is measure preserving, so by Proposition 5.6.2 this transformation is exact. This proves our earlier assertion in Section 4.3. □