CHAPTER 6

FINITE ELEMENT METHODS FOR THE PLATE PROBLEM

Introduction

In this chapter, we study two commonly used finite element approximations of the plate problem.

To begin with, we consider in Section 6.1 various *conforming methods*. Assuming for simplicity that the domain $\bar{\Omega}$ is polygonal, the elaboration of such methods requires the use of *straight finite elements of class* $\mathscr{C}^1$. Although such finite elements cannot be imbedded in affine families in general, we show that they form *almost-affine families*, in the sense that if the P_K-interpolation operator Π_K leaves invariant the space $P_k(K)$, there exists a constant C independent of K such that, for a regular family,

$$\forall v \in H^{k+1}(K), \quad |v - \Pi_K v|_{m,K} \leq C h_K^{k+1-m} |v|_{k+1,K},$$

for all integers $m \leq k+1$ for which $P_K \subset H^m(K)$. This is the case not only of the finite elements of class $\mathscr{C}^1$ introduced in Section 2.2, such as the *Argyris triangle*, but it is also the case of *composite finite elements* such as the *Hsieh–Clough–Tocher triangle*, or of *singular finite elements* such as the *singular Zienkiewicz triangle*.

For finite element spaces made up of such almost-affine families, we obtain (Theorem 6.1.6) error estimates of the form

$$\|u - u_h\|_{2,\Omega} \leq C\|u - \Pi_h u\|_{2,\Omega} = O(h^{k-1}), \quad \text{with} \quad h = \max_{K \in \mathcal{T}_h} h_K,$$

by an application of Céa's lemma. We also show (Theorem 6.1.7) that the minimal assumptions "$u \in H^2(\Omega)$" and "$P_2(K) \subset P_K$, $K \in \mathcal{T}_h$" insure convergence, i.e., $\lim_{h\to 0} \|u - u_h\|_{2,\Omega} = 0$.

The actual implementation of conforming methods offers serious computational difficulties: Either the dimension of the "local" spaces P_K is fairly large (at least 18 for triangular polynomial elements) or the structure of the space P_K is complicated (cf. the Hsieh–Clough–Tocher triangle or the singular Zienkiewicz triangle for example). The basic source of these difficulties is of course the required continuity of the first order partial derivatives across adjacent finite elements.

It is therefore tempting to relax this continuity requirement, and this results in *nonconforming methods*: One looks for a discrete solution in a finite element space V_h which is no longer contained in the space $H^2(\Omega)$ (not even in the space $H^1(\Omega)$ in some cases). The discrete solution then satisfies $a_h(u_h, v_h) = f(v_h)$ for all $v_h \in V_h$, where

$$a_h(.,.) = \sum_{K\in\mathcal{T}_h} \int_K \{\cdots\}\,\mathrm{d}x,$$

the integrand $\{\cdots\}$ being the same as in the bilinear form of the original problem.

The analysis of such nonconforming methods follows exactly the same pattern as in the case of nonconforming methods for second-order problems (cf. Section 4.2). In Section 6.2, we concentrate on one example, where the generic finite element is the *Adini rectangle*. For this finite element, we show that (Theorem 6.2.3)

$$\Big(\sum_{K\in\mathcal{T}_h} |u-u_h|^2_{2,K}\Big)^{1/2} = \mathrm{O}(h),$$

if the solution u is in the space $H^3(\Omega)$.

6.1. Conforming methods

Conforming methods for fourth-order problems

In this section, we study several types of conforming finite element methods which are commonly used for approximating the solution of plate problems. For definiteness, we shall consider the *clamped plate problem*, which corresponds to the following data (cf. Section 1.2):

$$\begin{cases} V = H_0^2(\Omega), \quad \Omega \subset \mathbf{R}^2, \\ a(u,v) = \displaystyle\int_\Omega \{\Delta u \Delta v + \\ \qquad + (1-\sigma)(2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v - \partial_{22}u\partial_{11}v)\}\,\mathrm{d}x, \\ f(v) = \displaystyle\int_\Omega fv\,\mathrm{d}x, \quad f \in L^2(\Omega), \end{cases} \tag{6.1.1}$$

where the constant σ (the Poisson coefficient of the material of which the plate is composed) lies in the interval $]0, \frac{1}{2}[$.

As a matter of fact, the methods which we shall describe apply equally well to *any* fourth-order boundary value problem posed over a space V such as $H_0^2(\Omega)$, $H^2(\Omega) \cap H_0^1(\Omega)$ or $H^2(\Omega)$, whose data $a(.,.)$ and $f(.)$ satisfy the assumptions of the Lax–Milgram lemma. For instance, we could likewise consider the simply supported plate (Exercise 1.2.7) or the biharmonic problem (Section 1.2).

Remark 6.1.1. By contrast, the nonconforming methods studied in the next section are specifically adapted to plate problems (cf. Remark 6.2.1). □

We shall assume that the set Ω is polygonal, so that it may be covered by triangulations composed of straight finite elements. Then in order to develop a conforming method, we face the problem of constructing subspaces of the space $H^2(\Omega)$. Since the functions found in standard finite element spaces are "locally regular" ($P_K \subset H^2(K)$ for all $K \in \mathcal{T}_h$), this construction amounts in practice to finding finite element spaces X_h which satisfy the inclusion $X_h \subset \mathscr{C}^1(\bar{\Omega})$ (Theorem 2.1.2), i.e., whose finite elements are *of class* $\mathscr{C}^1$.

We have already described three finite elements which meet this requirement, the *Argyris triangle*, the *Bell triangle* (cf. Theorem 2.2.13), and the *Bogner–Fox–Schmit rectangle* (cf. Theorem 2.2.15).

Almost-affine families of finite elements

As we pointed out in Section 2.3, Argyris triangles or Bell triangles *cannot* be imbedded in affine families in general, because normal derivatives at some nodes are used either as degrees of freedom (for the Argyris triangle) or in the definition of the space P_K (for the Bell triangle). This is in general the rule for finite elements of class $\mathscr{C}^1$, but there are exceptions. For instance, the Bogner–Fox–Schmit rectangle is a rectangular finite element of class $\mathscr{C}^1$ which *can* be imbedded in an affine family.

Nevertheless, if most finite elements of class $\mathscr{C}^1$ do not form affine families, we shall show that *their interpolation properties are quite similar to those of affine families*, and it is this similarity that motivates the following definition (compare with Theorem 3.1.6).

Consider a family of finite elements (K, P_K, Σ_K) of a given type, for which s denotes the greatest order of partial derivatives occuring in the definition of the set Σ_K. Then such a family is said to be *almost-affine* if, for any integers $k, m \geq 0$ and any numbers $p, q \in [1, \infty]$ compatible with the following inclusions:

$$W^{k+1,p}(K) \hookrightarrow \mathscr{C}^s(K), \tag{6.1.2}$$

$$W^{k+1,p}(K) \hookrightarrow W^{m,q}(K), \tag{6.1.3}$$

$$P_k(K) \subset P_K \subset W^{m,q}(K), \tag{6.1.4}$$

there exists a constant C independent of K such that

$$\forall v \in W^{k+1,p}(K), \quad \|v - \Pi_K v\|_{m,q,K} \leq \\ \leq C(\operatorname{meas}(K))^{1/q - 1/p} h_K^{k+1-m} |v|_{k+1,p,K}, \tag{6.1.5}$$

where $h_K = \operatorname{diam}(K)$.

In order to simplify the exposition, we shall consider in the subsequent examples *only the highest possible value* of the integer k for which the inclusions $W^{k+1,p}(K) \hookrightarrow \mathscr{C}^s(K)$ and $P_k(K) \subset P_K$ are satisfied, but it is implicitly understood that any lower value of k compatible with these two inclusions is also admissible (a related observation was made in Remark 3.1.5).

As expected, *a regular affine family is almost-affine* (cf. Theorem 3.1.6). In particular, this is the case of *a regular family of Bogner–Fox–Schmit rectangles* (cf. Fig. 2.2.20), for which the set K is a rectangle with vertices a_i, $1 \leq i \leq 4$, $P_K = Q_3(K)$, and

$$\Sigma_K = \{p(a_i),\quad \partial_1 p(a_i),\quad \partial_2 p(a_i),\quad \partial_{12} p(a_i),\quad 1 \leq i \leq 4\}.$$

Hence, *for all* $p \in]1, \infty]$ (so as to guarantee the inclusion $W^{4,p}(K) \hookrightarrow \mathscr{C}^2(K) = \operatorname{dom} \Pi_K$) *and all pairs* (m, q) *with* $m \geq 0$ *and* $q \in [1, \infty]$ *compatible with the inclusion*

$$W^{4,p}(K) \hookrightarrow W^{m,q}(K), \tag{6.1.6}$$

there exists a constant C independent of K such that

$$\forall v \in W^{4,p}(K), \quad \|v - \Pi_K v\|_{m,q,K} \leq C(\operatorname{meas}(K))^{1/q-1/p} h_K^{4-m} |v|_{4,p,K}. \tag{6.1.7}$$

A "polynomial" finite element of class $\mathscr{C}^1$: The Argyris triangle

Let us next examine the *Argyris triangle* (the case of Bell's triangle is left as a problem; cf. Exercise 6.1.1). We recall that this finite element is a triple

(K, P_K, Σ_K) where the set K is a triangle with vertices a_i, $1 \leq i \leq 3$, and mid-points $a_{ij} = (a_i + a_j)/2$, $1 \leq i < j \leq 3$, of the sides, the space P_K is the space $P_5(K)$, and the set Σ_K (whose $P_5(K)$-unisolvence has been proved in Theorem 2.2.11) can be chosen in the form (cf. Fig. 2.2.17)

$$\Sigma_K = \{\partial^\alpha p(a_i),\quad 1 \leq i \leq 3,\quad |\alpha| \leq 2,\quad \partial_\nu p(a_{ij}),\quad 1 \leq i < j \leq 3\}.$$

Theorem 6.1.1. *A regular family of Argyris triangles is almost-affine: For all $p \in [1, \infty]$ and all pairs (m, q) with $m \geq 0$ and $q \in [1, \infty]$ compatible with the inclusion*

$$W^{6,p}(K) \hookrightarrow W^{m,q}(K), \tag{6.1.8}$$

there exists a constant C independent of K such that

$$\forall v \in W^{6,p}(K),\quad \|v - \Pi_K v\|_{m,q,K} \leq C(\text{meas}(K))^{1/q-1/p} h_K^{6-m} |v|_{6,p,K}, \tag{6.1.9}$$

where Π_K denotes the associated $P_5(K)$-interpolation operator.

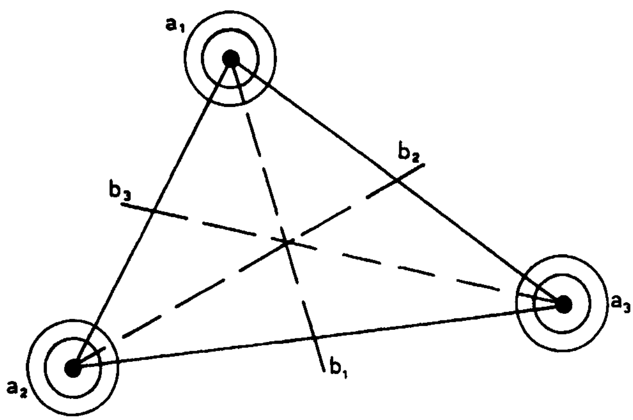
Proof. The key idea is to introduce a finite element similar to the Argyris triangle, but which *can* be imbedded in an affine family, and which will play a crucial intermediary role in obtaining the interpolation error estimate. Inasmuch as it is the presence of the degrees of freedom $\partial_\nu p(a_{ij})$, $1 \leq i < j \leq 3$, which prevents the property of affine-equivalence, we are naturally led to introduce the *Hermite triangle of type* (5), whose associated data are indicated in Fig. 6.1.1. For notational convenience, we shall henceforth denote by b_i the mid-point of the side which does not contain the vertex a_i, $1 \leq i \leq 3$.

It is easily seen that the set Ξ_K is $P_5(K)$-unisolvent and that this is a finite element of class $\mathscr{C}^0$, but not of class $\mathscr{C}^1$ (cf. Exercise 2.3.5). In addition, it is clear that two arbitrary Hermite triangles of type (5) are affine-equivalent. Therefore, if we denote by Λ_K the associated $P_5(K)$-interpolation operator, for all $p \in [1, \infty]$ and all pairs (m, q) with $0 \leq m \leq 6$ and $q \in [1, \infty]$ such that $W^{6,p}(K) \hookrightarrow W^{m,q}(K)$, there exists a constant C independent of K such that, for all functions $v \in W^{6,p}(K)$,

$$|v - \Lambda_K v|_{m,q,K} \leq C(\text{meas}(K))^{1/q-1/p} h_K^{6-m} |v|_{6,p,K}. \tag{6.1.10}$$

It therefore remains to evaluate the semi-norms $|\Pi_K v - \Lambda_K v|_{m,q,K}$. For a given function $v \in W^{6,p}(K)$, the difference

$$\Delta = \Pi_K v - \Lambda_K v \tag{6.1.11}$$



Hermite triangle of type (5)
$P_K = P_5(K)$; dim $P_K = 21$; $\Xi_K = \{\partial^\alpha p(a_i),\quad 1 \leq i \leq 3,\quad \lvert\alpha\rvert \leq 2;\ Dp(b_i)(a_i - b_i),\quad 1 \leq i \leq 3\}$

Fig. 6.1.1

is a polynomial of degree ≤ 5 which satisfies

$$\partial^\alpha \Delta(a_i) = 0, \quad |\alpha| \leq 2, \quad 1 \leq i \leq 3, \tag{6.1.12}$$

since $\partial^\alpha \Pi_K v(a_i) = \partial^\alpha \Lambda_K v(a_i) = \partial^\alpha v(a_i)$, $|\alpha| \leq 2$, $1 \leq i \leq 3$, and

$$\partial_\nu \Delta(b_i) = \partial_\nu (v - \Lambda_K v)(b_i), \quad 1 \leq i \leq 3, \tag{6.1.13}$$

since $\partial_\nu \Pi_K v(b_i) = \partial_\nu v(b_i)$, $1 \leq i \leq 3$. For $1 \leq i \leq 3$, let ν_i and τ_i be the unit outer normal and tangential vectors along the side opposite to the vertex a_i, as indicated in Fig. 6.1.2.

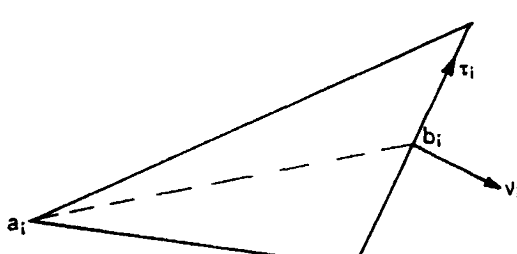


Fig. 6.1.2.

Denoting by $\cdot$ the Euclidean inner product in $\mathbf{R}^2$, we can write, for $1 \leq i \leq 3$,

$$D\Delta(b_i)(a_i - b_i) = \partial_\nu \Delta(b_i)\{(a_i - b_i) \cdot \nu_i\}, \tag{6.1.14}$$

since on the one hand $D\Delta(b_i)\nu_i = \partial_\nu \Delta(b_i)$, and since on the other $D\Delta(b_i)\tau_i = 0$ as a consequence of relations (6.1.12), which imply that the difference Δ vanishes along each side of the triangle. Combining relations (6.1.13) and (6.1.14), we obtain

$$D\Delta(b_i)(a_i - b_i) = \partial_\nu(v - \Lambda_K v)(b_i)\{(a_i - b_i) \cdot \nu_i\}, \quad 1 \leq i \leq 3. \tag{6.1.15}$$

Let q_i denote the basis functions of the Hermite triangle of type (5) which are associated with the degrees of freedom $Dp(b_i)(a_i - b_i)$, $1 \leq i \leq 3$. Then using relations (6.1.12) and (6.1.15), we can write

$$\begin{aligned} \Delta = \Pi_K v - \Lambda_K v &= \sum_{i=1}^{3} \{D\Delta(b_i)(a_i - b_i)\} q_i \\ &= \sum_{i=1}^{3} \partial_\nu(v - \Lambda_K v)(b_i)\{(a_i - b_i) \cdot \nu_i\} q_i. \end{aligned} \tag{6.1.16}$$

Applying Theorem 3.1.5 with $m = 1$, $q = \infty$ and $k = 5$, we obtain

$$\begin{aligned} |\partial_\nu(v - \Lambda_K v)(b_i)| &\leq \sqrt{2}|v - \Lambda_K v|_{1,\infty,K} \\ &\leq C(\operatorname{meas}(K))^{-1/p} \frac{h_K^6}{\rho_K} |v|_{6,p,K}, \quad 1 \leq i \leq 3. \end{aligned} \tag{6.1.17}$$

Next, it is clear that

$$|(a_i - b_i) \cdot \nu_i| \leq h_K, \quad 1 \leq i \leq 3. \tag{6.1.18}$$

Finally, let $\hat{q}_i$ be the basis functions of a reference Hermite triangle of type (5) associated in the usual correspondence with the basis functions q_i. From Theorems 3.1.2 and 3.1.3, we infer that

$$|q_i|_{m,q,K} \leq C \frac{(\operatorname{meas}(K))^{1/q}}{\rho_K^m} |\hat{q}_i|_{m,q,\hat{K}}. \tag{6.1.19}$$

Relations (6.1.16), (6.1.17), (6.1.18) and (6.1.19) then imply that

$$\begin{aligned} |\Pi_K v - \Lambda_K v|_{m,q,K} &\leq C(\operatorname{meas}(K))^{1/q - 1/p} \frac{h_K^7}{\rho_K^{m+1}} |v|_{6,p,K} \\ &\leq C(\operatorname{meas}(K))^{1/q - 1/p} h_K^{6-m} |v|_{6,p,K}, \end{aligned} \tag{6.1.20}$$

since we are considering a regular family. Inequality (6.1.9) is therefore a consequence of inequalities (6.1.10) and (6.1.20). □

A composite finite element of class $\mathscr{C}^1$: The Hsieh–Clough–Tocher triangle

In our next examples, we shall for the first time leave the realm of "purely polynomial" finite elements.

As we already pointed out (cf. Bibliography and Comments of Sections 2.2 and 2.3), Bell's triangle is optimal among triangular polynomial finite elements of class $\mathscr{C}^1$ in the sense that for such finite elements, one has necessarily $\dim P_K \geq 18$, as a consequence of Ženíšek's result. Therefore, *a smaller dimension of the space P_K for triangular finite elements of class $\mathscr{C}^1$ requires that functions other than polynomials be used.*

For example, one can use piecewise polynomials *inside* the set K, a process which results in so-called *composite finite elements*, also named *macroelements*. Or one can add some judiciously selected rational functions to a space of polynomials, a process which results in so-called *singular finite elements* (singular in the sense that some functions in the space P_K or some of their derivatives become infinite and/or are not defined at some points of K). We shall describe and study one example of each type. Other examples of composite and singular finite elements are suggested as problems (cf. Exercises 6.1.3, 6.1.4, 6.1.5, 6.1.6 and 6.1.7).

The *Hsieh–Clough–Tocher triangle*, sometimes abbreviated as the *HCT triangle*, is defined as follows: The set K is a triangle subdivided into three triangles K_i with vertices a, a_{i+1}, a_{i+2}, $1 \leq i \leq 3$ (Fig. 6.1.3), the point a being in the interior of the set K (here and subsequently, the indices are counted modulo 3 when necessary). The space P_K and the set Σ_K are indicated in Fig. 6.1.3. For convenience, we again denote by b_i, $1 \leq i \leq 3$, the mid-point of the side which does not contain the vertex a_i. Our first task is as usual to prove the P_K-unisolvence of the set Σ_K. Since $\dim P_3(K_i) = 10$, it is necessary to find 30 equations to define the three polynomials $P_{|K_i}$, $1 \leq i \leq 3$. First, it is easily seen that the data of the degrees of freedom of the set Σ_K amounts to the data of 21 equations. To see that the condition "$p \in \mathscr{C}^1(K)$" yields 9 additional equations, it suffices to write the continuity of the functions p, $\partial_1 p$ and $\partial_2 p$ at the point a (6 equations) and the continuity of the normal derivatives across the mid-points of the sides $[a, a_i]$ (3 equations).

It therefore remains to show that the 30×30 matrix of the corresponding linear system is invertible, and this is the object of the next theorem (another proof is suggested in Exercise 6.1.2).

Theorem 6.1.2. *With the definitions of Fig.* 6.1.3, *the set Σ_K is P_K-unisolvent.*

The resulting Hsieh–Clough–Tocher triangle is a finite element of class $\mathscr{C}^1$.

Proof. It suffices to show that a function p in the space P_K vanishes if

$$p(a_i) = \partial_1 p(a_i) = \partial_2 p(a_i) = \partial_\nu p(b_i) = 0, \quad 1 \leq i \leq 3. \tag{6.1.21}$$

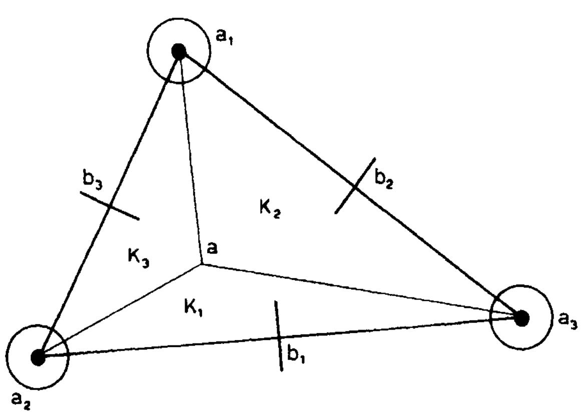
For $1 \leq i \leq 3$, let μ_i denote the unique function which satisfies

$$\mu_i \in P_1(K_i), \quad \mu_i(a) = 1, \quad \mu_i(a_{i+1}) = \mu_i(a_{i+2}) = 0,$$

so that the function $\mu: K \to \mathbf{R}$ defined by

$$\mu_{|K_i} = \mu_i, \quad 1 \leq i \leq 3,$$

is continuous. Since over each triangle K_i, the function $p_{|K_i}$ is a polynomial of degree ≤ 3, assumptions (6.1.21) imply that there exist func-



Hsieh–Clough–Tocher triangle
$P_K = \{p \in \mathscr{C}^1(K);\ p_{\lvert K_i} \in P_3(K_i),\ 1 \leq i \leq 3\}$; $\dim P_K = 12$; $\Sigma_K = \{p(a_i),\ \partial_1 p(a_i),\ \partial_2 p(a_i),\ \partial_\nu p(b_i),\ 1 \leq i \leq 3\}$.

Fig. 6.1.3

tions ν_i such that

$$\nu_i \in P_1(K_i), \quad p_{|K_i} = \nu_i \mu_i^2, \quad 1 \leq i \leq 3.$$

Since the functions $p: K \to \mathbf{R}$ and $\mu: K \to \mathbf{R}$ are continuous, the function $\nu: K \to \mathbf{R}$ defined by

$$\nu_{|K_i} = \nu_i, \quad 1 \leq i \leq 3,$$

is also continuous (the function μ does not vanish in the interior of K).

On each segment $[a, a_{i+2}]$, the gradient ∇p is well-defined since the function p is continuously differentiable, and it is given by either expressions

$$\nabla p\,|_{[a,a_{i+2}]} = \begin{cases} (2\nu_i\mu_i\nabla\mu_i + \mu_i^2\nabla\nu_i)|_{[a,a_{i+2}]}, \\ (2\nu_{i+1}\mu_{i+1}\nabla\mu_{i+1} + \mu_{i+1}^2\nabla\nu_{i+1})|_{[a,a_{i+2}]}, \end{cases}$$

so that we deduce ($\mu \neq 0$ in $\mathring{K}$)

$$2\nu\nabla(\mu_{i+1} - \mu_i) + \mu\nabla(\nu_{i+1} - \nu_i) = 0 \quad \text{along} \quad [a, a_{i+2}].$$

Since $\mu(a_{i+2}) = 0$ and $\nabla(\mu_{i+1} - \mu_i) \neq 0$ (otherwise the lines $\mu_i = 0$ and $\mu_{i+1} = 0$ would be parallel), we conclude that $\nu(a_{i+2}) = \nu_i(a_{i+2}) = 0$. A similar argument would show that $\nu_i(a_{i+1}) = 0$. Consequently each function $\nu_i \in P_1(K_i)$ is of the form

$$\nu_i = C_i\mu_i \quad \text{with} \quad C_i = \text{constant}.$$

The function ν being continuous, we have

$$\nu(a) = \nu_i(a) = C_i, \quad 1 \leq i \leq 3.$$

Denoting by C the common value of the constants C_i, we conclude that

$$\nu_{|K_i} = \nu_i = C\mu_i, \quad 1 \leq i \leq 3,$$

and therefore that

$$p_{|K_i} = C\mu_i^3, \quad \text{whence} \quad \nabla p_{|K_i} = 3C\mu^2\nabla\mu_i, \quad 1 \leq i \leq 3.$$

Then the constant C is necessarily zero for otherwise the function p would not be continuously differentiable along the segment $[a, a_{i+2}]$ since $\nabla\mu_i \neq \nabla\mu_{i+1}$.

That the Hsieh–Clough–Tocher triangle is of class $\mathscr{C}^1$ follows by an argument analogous to the proof of Theorem 2.2.13. □

Remark 6.1.2. The normal derivatives at the mid-point of the sides can be eliminated by requiring that the normal derivative vary linearly along the sides. This elimination results in a finite element of class $\mathscr{C}^1$ for which dim $P_K = 9$ (cf. Exercise 6.1.3). □

There are two reasons that prevent the Hsieh–Clough–Tocher triangle from being imbedded in an affine family. As for the Argyris triangle, one reason is the presence of the normal derivatives $\partial_\nu p(b_i)$ as degrees of freedom. The additional reason is that the point a may be allowed to vary inside the set K. This is why we must adapt to this element the notion of a regular family:

We shall say that *a family of Hsieh–Clough–Tocher triangles K is regular* if the following three conditions are satisfied:

(i) There exists a constant σ such that

$$\forall K, \quad \frac{h_K}{\rho_K} \leq \sigma.$$

(ii) The quantities h_K approach zero.

(iii) Let $\hat{K}$ be any fixed triangle with vertices $\hat{a}_i$, $1 \leq i \leq 3$. For each Hsieh–Clough–Tocher triangle K with vertices $a_{i,K}$, $1 \leq i \leq 3$, let F_K denote the unique affine mapping which satisfies $F_K(\hat{a}_i) = a_{i,K}$, $1 \leq i \leq 3$.

Then (Fig. 6.1.4) *the points* $\hat{a}_K = F_K^{-1}(a_K)$ *all belong to some compact subset* $\hat{B}$ *of the interior of the triangle* $\hat{K}$ (clearly, the compact subset $\hat{B}$

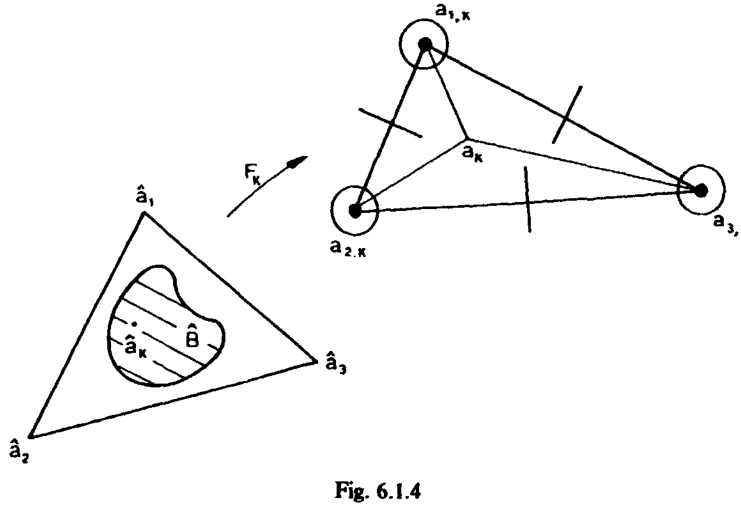


Fig. 6.1.4

may vary from one regular family to another). Notice that the $\hat{K}$ is here simply understood as being a triangle, not a finite element.

Remark 6.1.3. Conditions (i) and (ii) are the familiar ones for a regular family of finite elements. Condition (iii) expresses precisely in which sense the points a_K may vary inside the triangle K, so as to guarantee that the family under consideration is almost-affine, as we next show. □

Theorem 6.1.3. *A regular family of Hsieh–Clough–Tocher triangles is almost affine: For all $p \in [1, \infty]$ and all pairs (m, q) with $m \geq 0$ and $q \in [1, \infty]$ such that*

$$\begin{cases} W^{4,p}(K) \hookrightarrow W^{m,q}(K), \\ P_K \subset W^{m,q}(K), \end{cases} \tag{6.1.22}$$

there exists a constant C independent of K such that

$$\forall v \in W^{4,p}(K), \quad \|v - \Pi_K v\|_{m,q,K} \leq C(\text{meas}(K))^{1/q - 1/p} h_K^{4-m} |v|_{4,p,K}. \tag{6.1.23}$$

Proof. We first observe that the inclusion $W^{4,p}(K) \hookrightarrow \mathscr{C}^1(K) = \text{dom}\, \Pi_K$ holds for all $p \geq 1$. The proof of the theorem consists of three steps.

(i) As expected, we shall introduce a finite element which is similar to the Hsieh–Clough–Tocher triangle but which *can* be imbedded in an affine family. This will be achieved through the replacement of the normal derivatives by appropriate directional derivatives and through a restriction on the position of the points a_K. More precisely, with each Hsieh–Clough–Tocher triangle (K, P_K, Σ_K), we associate the finite element (K, P_K, Ξ_K), where

$$\begin{aligned} \Xi_K = \{&p(a_i), \quad Dp(a_i)(a_{i+1} - a_i), \quad Dp(a_i)(a_{i-1} - a_i), \\ &Dp(b_i)(a - b_i), \quad 1 \leq i \leq 3\} \end{aligned} \tag{6.1.24}$$

(the proof of the P_K-unisolvence of the set Ξ_K is similar to the proof of the P_K-unisolvence of the set Σ_K as given in Theorem 6.1.2), and we denote by Λ_K the P_K-interpolation operator associated with each finite element (K, P_K, Ξ_K).

For each point $\hat{a} \in \hat{B}$, let $\mathscr{K}(\hat{a})$ denote the (possibly empty) subfamily of Hsieh–Clough–Tocher triangles for which $a_K = F_K(\hat{a})$. Then, for each $\hat{a} \in \hat{B}$, the subfamily (K, P_K, Ξ_K), $K \in \mathscr{K}(\hat{a})$ is affine, and consequently, the inclusion

$$P_3(K) \subset P_K \tag{6.1.25}$$

implies that there exists a constant $C(\hat{a}, \hat{K})$ such that

$$\begin{aligned} &\forall v \in W^{4,p}(K), \quad \forall K \in \mathcal{K}(\hat{a}), \\ &|v - \Lambda_K v|_{m,q,K} \leq C(\hat{a}, \hat{K})(\operatorname{meas}(K))^{1/q - 1/p} h_K^{4-m} |v|_{4,p,K}, \end{aligned} \tag{6.1.26}$$

for all pairs (m, q) compatible with the inclusions (6.1.22).

(ii) We next show that, when the points $\hat{a}$ vary in the compact set $\hat{B}$, the constants $C(\hat{a}, \hat{K})$ which appear in the last inequality are bounded. To prove this, we recall that in the proof of Theorem 3.1.4, we found that these constants are of the form (cf. (3.1.33)):

$$C(\hat{a}, \hat{K}) = C(\hat{K}) \| I - \hat{\Lambda}(\hat{a}) \|_{\mathcal{L}(W^{4,p}(\hat{K}); W^{m,q}(\hat{K}))}, \tag{6.1.27}$$

where, for each $\hat{a} \in \hat{B}$, $\hat{\Lambda}(\hat{a})$ denotes the $P_{\hat{K}}$-interpolation operator associated with the corresponding reference finite element $(\hat{K}, \hat{P}(\hat{a}), \hat{\Sigma}(\hat{a}))$.

With self-explanatory notations, we have, for all functions $\hat{v} \in W^{4,p}(\hat{K})$,

$$\begin{aligned} \hat{\Lambda}(\hat{a})\hat{v} = &\sum_{i=1}^{3} \hat{v}(\hat{a}_i)\hat{p}_i(\hat{a}, .) \\ &+ \sum_{\substack{i=1 \\ |j-i|=1}}^{3} \{D\hat{v}(\hat{a}_i)(\hat{a}_j - \hat{a}_i)\}\hat{q}_{ij}(\hat{a}, .) \\ &+ \sum_{i=1}^{3} \{D\hat{v}(\hat{b}_i)(\hat{a} - \hat{b}_i)\}\hat{r}_i(\hat{a}, .), \end{aligned} \tag{6.1.28}$$

and

$$\begin{aligned} &|\hat{v}(\hat{a}_i)| \leq |\hat{v}|_{0,\infty,\hat{K}} \leq C(\hat{K}) \|\hat{v}\|_{4,p,\hat{K}}, \\ &\left.\begin{aligned} &|\{D\hat{v}(\hat{a}_i)(\hat{a}_j - \hat{a}_i)\}| \\ &|\{D\hat{v}(\hat{b}_i)(\hat{a} - \hat{b}_i)\}| \end{aligned}\right\} \leq \sqrt{2}\operatorname{diam}(\hat{K})|\hat{v}|_{1,\infty,\hat{K}} \leq C(\hat{K})\|\hat{v}\|_{4,p,\hat{K}}, \end{aligned} \tag{6.1.29}$$

where the constants $C(\hat{K})$ are independent of $\hat{a}$.

Let us then consider the norm $\|.\|_{m,q,K}$ of any one of the basis functions $\hat{p}_i(\hat{a}, .)$, $\hat{q}_{ij}(\hat{a}, .)$ and $\hat{r}_i(\hat{a}, .)$. On each of the triangles $\hat{K}_i(\hat{a})$, $1 \leq i \leq 3$, which subdivide the triangle $\hat{K}$, the restriction of any one of these basis functions is a polynomial of degree ≤ 3, whose coefficients are obtained through the solution of a linear system with an invertible matrix (the set $\hat{\Sigma}(\hat{a})$ is $\hat{P}(\hat{a})$-unisolvent as long as the point $\hat{a}$ belongs to the interior of the set $\hat{K}$). This matrix depends continuously on the point $\hat{a}$ since its coefficients are polynomial functions of the coordinates of the point $\hat{a}$. Consequently, each coefficient is in turn a continuous function of the point $\hat{a}$ and there exists a constant $\hat{C}$ such that

$$\sup_{\hat{a}\in\hat{B}} \{\|\hat{p}_i(\hat{a},.)\|_{m,q,\hat{K}},\quad \|\hat{q}_{ij}(\hat{a},.)\|_{m,q,\hat{K}},\quad \|\hat{r}_i(\hat{a},.)\|_{m,q,\hat{K}}\} \leq \hat{C}, \tag{6.1.30}$$

since the set $\hat{B}$ is compact. Then it follows from relations (6.1.27) to (6.1.30) that

$$\sup_{\hat{a}\in\hat{B}} C(\hat{a}, \hat{K}) = C(\hat{B}, \hat{K}) < \infty.$$

Combining this result with inequality (6.1.26), we obtain

$$\begin{aligned} &\forall v \in W^{4,p}(K), \quad \forall K,\\ &|v - \Lambda_K v|_{m,q,K} \leq C(\hat{B}, \hat{K})(\mathrm{meas}(K))^{1/q-1/p} h_K^{4-m} |v|_{4,p,K}. \end{aligned} \tag{6.1.31}$$

(iii) By an argument similar to that used in the proof of Theorem 6.1.1 (cf. (6.1.16)), we find that

$$\Pi_K v - \Lambda_K v = \sum_{i=1}^{3} \partial_\nu (v - \Lambda_K v)(b_i)\{(a - b_i)\cdot \nu_i\} r_i, \tag{6.1.32}$$

where the functions r_i, $1 \leq i \leq 3$, are the basis functions associated with the degrees of freedom $\{Dp(b_i)(a - b_i)\}$ in the finite element (K, P_K, Ξ_K). Applying Theorem 3.1.5 with $m = 1$, $q = \infty$ and $k = 3$, we find that

$$|\partial_\nu (v - \Lambda_K v)(b_i)| \leq \sqrt{2} |v - \Lambda_K v|_{1,\infty,K} \leq C(\mathrm{meas}(K))^{-1/p} \frac{h_K^4}{\rho_K} |v|_{4,p,K}. \tag{6.1.33}$$

Next we have

$$|\{(a - b_i)\cdot \nu_i\}| \leq h_K, \tag{6.1.34}$$

$$|r_i|_{m,q,K} \leq C \frac{(\mathrm{meas}(K))^{1/q}}{\rho_K^m} |\hat{r}_i(\hat{a},.)|_{m,q,\hat{K}}, \tag{6.1.35}$$

and we deduce from relations (6.1.30) and (6.1.32) to (6.1.35) that

$$\begin{aligned} |\Pi_K v - \Lambda_K v|_{m,q,K} &\leq C(\mathrm{meas}(K))^{1/q-1/p} \frac{h_K^5}{\rho_K^{m+1}} |v|_{4,p,K}\\ &\leq C(\mathrm{meas}(K))^{1/q-1/p} h_K^{4-m} |v|_{4,p,K}. \end{aligned} \tag{6.1.36}$$

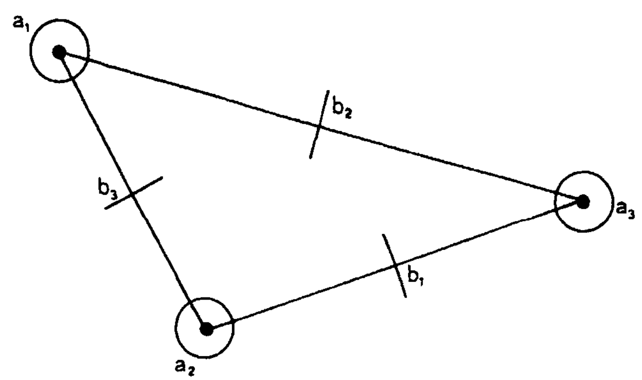
Then the proof is completed by combining the above inequality with inequality (6.1.31). □

Remark 6.1.4. When $q = 2$, it is easily seen that the highest admissible value for the integer m compatible with the inclusion $P_K \subset H^m(K)$ is $m = 2$ (that the integer m is at least 2 follows from an application of Theorem 2.1.2 to the partitionned triangle $K = \bigcup_{i=1}^{3} K_i$; to prove that the integer m cannot exceed 2 requires an argument which shall be used later, cf. Theorem 6.2.1). Notice that this is the first instance of a restriction on the possible inclusions $P_K \subset W^{m,q}(K)$. The next finite element under study will be another instance. Fortunately, the inclusion $P_K \subset H^2(K)$ is precisely that which is needed to insure convergence, as we shall show at the end of this section. □

A singular finite element of class $\mathscr{C}^1$: The singular Zienkiewicz triangle

Let us next turn to an example of a triangular finite element, which is of class $\mathscr{C}^1$ as a result of the addition of appropriate rational functions to a familiar space of polynomials.

The *singular Zienkiewicz triangle* is defined as follows (Fig. 6.1.5): The set K is a triangle with vertices a_i, $1 \leq i \leq 3$, the space P_K is the space $P_3''(K)$ of the Zienkiewicz triangle (cf. (2.2.39)) to which are added three



Singular Zienkiewicz triangle
$P_K = P_3''(K) \oplus \bigvee_{i=1}^{3} \left\{ \dfrac{4\lambda_i \lambda_{i+1}^2 \lambda_{i+2}^2}{(\lambda_i + \lambda_{i+1})(\lambda_i + \lambda_{i+2})} \right\}$ (cf. (2.2.39); dim $P_K = 12$; $\Sigma_K = \{p(a_i), \partial_1 p(a_i), \partial_2 p(a_i), \partial_\nu p(b_i), \quad 1 \leq i \leq 3\}$.

Fig. 6.1.5

functions $q_i: K \to \mathbf{R}$, $1 \leq i \leq 3$, defined by

$$\begin{cases} q_i = \dfrac{4\lambda_i\lambda_{i+1}^2\lambda_{i+2}^2}{(\lambda_i+\lambda_{i+1})(\lambda_i+\lambda_{i+2})} & \text{for } 0 \leq \lambda_i \leq 1,\ 0 \leq \lambda_{i+1}, \lambda_{i+2} < 1, \\ q_i(a_{i+1}) = q_i(a_{i+2}) = 0, & \end{cases} \tag{6.1.37}$$

where the functions λ_i, $1 \leq i \leq 3$, denote the barycentric coordinates in the triangle K (notice that the function given in the first line of definition (6.1.37) is not defined for $\lambda_i + \lambda_{i+1} = 0$ or $\lambda_i + \lambda_{i+2} = 0$, i.e., for $\lambda_{i+2} = 1$ or $\lambda_{i+1} = 1$; this is why we have to assign values to the function q_i at the vertices a_{i+1} and a_{i+2}). Finally the set Σ_K is the same as for the Hsieh–Clough–Tocher triangle.

As usual, we begin by examining the question of unisolvence. Observe that this finite element is an instance where the validity of the inclusions $P_K \subset \mathscr{C}^1(K)$ (which is part of the definition of elements of class $\mathscr{C}^1$) and $P_K \subset H^2(K)$ requires a proof.

Theorem 6.1.4. *With the definitions of Fig.* 6.1.5, *the set* Σ_K *is* P_K-*unisolvent.*

The resulting singular Zienkiewicz triangle is a finite element of class $\mathscr{C}^1$, *and the inclusion* $P_K \subset H^2(K)$ *holds.*

Proof. (i) To begin with, let us verify the inclusions $P_K \subset \mathscr{C}^1(K)$ and $P_K \subset H^2(K)$. Since such properties are invariant through affine transformations, we may consider the case where the set K is the unit triangle $\hat{K}$ with vertices $\hat{a}_1 = (1,0)$, $\hat{a}_2 = (0,1)$, and $\hat{a}_3 = (0,0)$. Then it suffices to study the behavior of the function $\hat{q}_1: \hat{K} \to \mathbf{R}$ in a neighborhood of the origin in $\hat{K}$. We have

$$\begin{cases} \hat{q}_1(x) = \dfrac{x_2x_2^2}{(x_1+x_2)} f(x) & \text{for } x \neq 0, \\ \hat{q}_1(0) = 0, & \end{cases} \tag{6.1.38}$$

where the function $f(x) = 4(1-x_1-x_2)^2/(1-x_2)$ and its derivatives have no singularity at the origin. Since $\lim_{x_1,x_2 \to 0^+} x_1x_2^2/(x_1+x_2) = 0$, we deduce that $\lim_{x_1,x_2\to 0^+} \hat{q}_1(x) = 0$. Therefore the function $\hat{q}_1$ is continuous at the origin. For x_1, $x_2 \geq 0$ and $x \neq 0$, we have:

$$\begin{cases} \partial_1\hat{q}_1(x) = \dfrac{x_2^3}{(x_1+x_2)^2} f(x) + \dfrac{x_1x_2^2}{x_1+x_2} \partial_1 f(x), \\ \partial_2\hat{q}_1(x) = \dfrac{x_1x_2(2x_1+x_2)}{(x_1+x_2)^2} f(x) + \dfrac{x_1x_2^2}{x_1+x_2} \partial_2 f(x), \end{cases} \tag{6.1.39}$$

and thus we conclude that

$$\lim_{x_1,x_2\to 0^+} \partial_j\hat{q}_1(x) = 0 = \partial_j\hat{q}_1(0), \quad j = 1, 2, \tag{6.1.40}$$

which proves that the function $\hat{q}_1$ is continuously differentiable at the origin.

Arguing analogously with the vertex $\hat{a}_2$, and next with the functions $\hat{q}_2$ and $\hat{q}_3$, we conclude that the inclusion

$$P_K \subset \mathscr{C}^1(K)$$

holds. This inclusion implies the inclusion $P_K \subset H^1(K)$ and thus, to obtain the inclusion $P_K \subset H^2(K)$, it remains to show that the second partial derivatives of the function $\hat{q}_1$ are square integrable around the origin. For $x \neq 0$, we find

$$\begin{cases} \partial_{11}\hat{q}_1(x) = -\dfrac{2x_2^3}{(x_1+x_2)^3} f(x) + g_{11}(x), \\ \partial_{12}\hat{q}_1(x) = \dfrac{x_2^2(3x_1+x_2)}{(x_1+x_2)^3} f(x) + g_{12}(x), \\ \partial_{22}\hat{q}_1(x) = \dfrac{2x_1^3}{(x_1+x_2)^3} f(x) + g_{22}(x), \end{cases} \tag{6.1.41}$$

where the functions g_{11}, g_{12} and g_{22} are continuous around the origin. Since the three functions factoring the function $f(x)$ are bounded on the set $\hat{K}$, the inclusion

$$P_K \subset H^2(K)$$

follows.

(ii) The inclusion $P_K \subset \mathscr{C}^1(K)$ proved in (i) guarantees that the degrees of freedom of the set Σ_K are well-defined for the functions in the space P_K. The P_K-unisolvence of the set Σ_K will be an easy consequence of the P_K-unisolvence of the set

$$\Xi_K = \{p(a_i),\quad Dp(a_i)(a_j - a_i),\quad Dp(b_i)(a_i - b_i),\\ 1 \leq i, j \leq 3,\quad |j - i| = 1\}, \tag{6.1.42}$$

which we proceed to show.

Let us denote by p_i, $1 \leq i \leq 3$, and p_{ij}, $1 \leq i, j \leq 3$, $|j - i| = 1$, the basis functions of the space $P_3''(K)$ as given in (2.2.39). By definition, they satisfy

$$p_i(a_k) = \delta_{ik},\quad Dp_i(a_k)(a_l - a_k) = 0, \tag{6.1.43}$$

for $1 \leq i, k, l \leq 3$, $|k - l| = 1$, and

$$p_{ij}(a_k) = 0,\quad Dp_{ij}(a_k)(a_l - a_k) = \delta_{ik}\delta_{jl}, \tag{6.1.44}$$

for $1 \leq i, j, k, l \leq 3$, $|j-i| = |k-l| = 1$. We next show that they satisfy

$$Dp_i(b_k)(a_k - b_k) = -\tfrac{1}{4} + \tfrac{3}{4}\delta_{ik}, \quad 1 \leq i, k \leq 3, \tag{6.1.45}$$

$$Dp_{ij}(b_k)(a_k - b_k) = -\tfrac{1}{4} + \tfrac{3}{8}\delta_{ik} + \tfrac{5}{8}\delta_{jk}, \quad 1 \leq i, j, k \leq 3, \quad |j-i| = 1. \tag{6.1.46}$$

For the purpose of proving these relations, it is convenient to compute the directional derivatives $Dp(b_i)(a_i - b_i)$ for a function $p: K \to \mathbf{R}$ expressed in terms of barycentric coordinates (the computation below is not restricted to $n = 2$). Let then $p(x_1, x_2) = q(\lambda_1, \lambda_2, \lambda_3)$ be such a function. Denoting as usual by $B = (b_{ij})$ the inverse matrix of the matrix A of (2.2.4), we find that

$$\partial_j p = \sum_{k=1}^{3} \partial_k q \partial_j \lambda_k = \sum_{k=1}^{3} b_{kj} \partial_k q, \quad j = 1, 2.$$

Let us compute for example the quantity

$$Dp(b_1)(a_1 - b_1) = \sum_{j=1}^{2} \sum_{k=1}^{3} b_{kj} \partial_k q\left(0, \frac{1}{2}, \frac{1}{2}\right)\left(a_{j1} - \frac{a_{j2} + a_{j3}}{2}\right),$$

where a_{ji}, $j = 1, 2$, denote the coordinates of the vertex a_i. By definition of the matrices B and A,

$$\sum_{j=1}^{2} b_{kj} a_{jl} = \delta_{kl} - b_{k3}, \quad 1 \leq k, l \leq 3,$$

so that

$$Dp(b_1)(a_1 - b_1) = \partial_1 q(0, \tfrac{1}{2}, \tfrac{1}{2}) - \tfrac{1}{2}(\partial_2 q(0, \tfrac{1}{2}, \tfrac{1}{2}) + \partial_3 q(0, \tfrac{1}{2}, \tfrac{1}{2})). \tag{6.1.47}$$

Then relations (6.1.45) and (6.1.46) follow from the above result (and analogous computations for $Dp(b_i)(a_i - b_i)$, $i = 2, 3$) and the following expressions of the basis functions p_i and p_{ij} (which are easily derived from relations (2.2.37) and (2.2.38)):

$$p_i = -2\lambda_i^3 + 3\lambda_i^2 + 2\lambda_1\lambda_2\lambda_3, \tag{6.1.48}$$

$$p_{ij} = \frac{\lambda_i \lambda_j}{2}(\lambda_i - \lambda_j + 1). \tag{6.1.49}$$

On the other hand, the functions q_i as defined in (6.1.37) satisfy

$$\begin{cases} q_i(a_k) = 0, \quad 1 \leq i, k \leq 3, \\ Dq_i(a_k)(a_l - a_k) = 0, \quad 1 \leq i, k, l \leq 3, \quad |k - l| = 1, \\ Dq_i(b_k)(a_k - b_k) = \delta_{ik}, \quad 1 \leq i, k \leq 3. \end{cases} \tag{6.1.50}$$

The second equalities have been obtained in (6.1.40). The last ones are obtained through another application of relations of the form (6.1.47).

Then it follows from relations (6.1.43) to (6.1.50) that the functions (which all belong to the space P_K):

$$\begin{cases} \left\{p_i + \frac{1}{4}\left(-2q_i + \sum_{|j-i|=1} q_j\right)\right\}, & 1 \leqslant i \leqslant 3, \\ \left\{p_{ij} - \frac{1}{8}(q_i + 3q_j - 2q_l)\right\}, & 1 \leqslant i, j, l \leqslant 3, \quad \{i, j, l\} = \{1, 2, 3\}, \\ q_i, \quad 1 \leqslant i \leqslant 3, \end{cases} \tag{6.1.51}$$

form a basis of the space P_K, corresponding to the degrees of freedom of the set Ξ_K of (6.1.42). Thus this set is a P_K-unisolvent set.

It remains to prove that the set Σ_K is also P_K-unisolvent. To prove this, we make the following observation: Along each side K' of the triangle K, the restrictions $p_{|K'}$, $p \in P_K$, are polynomials of degree $\leqslant 3$ in one variable, while the restrictions $Dp(.)\xi_{|K'}$, $p \in P_K$, of any directional derivative are polynomials of degree $\leqslant 2$ in one variable. This is clearly true for the functions in the space $P_3''(K)$, and it is a straightforward consequence of the definition for the functions q_i. Notice in particular that this property implies that the finite element is of class $\mathscr{C}^1$.

Let then $p \in P_K$ be a function which satisfies

$$p(a_i) = \partial_1 p(a_i) = \partial_2 p(a_i) = \partial_\nu p(b_i) = 0, \quad 1 \leqslant i \leqslant 3.$$

The conjunction of these relations and of the above property implies that the normal derivative and the tangential derivative vanish along any side of the triangle K. Consequently, the directional derivatives $Dp(b_i)(a_i - b_i)$, $1 \leqslant i \leqslant 3$, vanish, and therefore the function p is identically zero since the set Ξ_K is P_K-unisolvent. □

Remark 6.1.5. Just as for the Hsieh–Clough–Tocher triangle, the normal derivatives at the mid-point of the sides can be eliminated by requiring that the normal derivatives vary linearly along the sides. Then we obtain in this fashion another finite element of class $\mathscr{C}^1$ for which $\dim(P_K) = 9$ (cf. Exercise 6.1.6). □

Theorem 6.1.5. *A regular family of singular Zienkiewicz triangles is almost affine: For all $p \in]1, \infty]$ and all pairs (m, q) with $m \geqslant 0$ and*

$q \in [1, \infty]$ *such that*

$$\begin{cases} W^{3,p}(K) \hookrightarrow W^{m,q}(K), \\ P_K \subset W^{m,q}(K), \end{cases} \tag{6.1.52}$$

there exists a constant C independent of K such that

$$\forall v \in W^{3,p}(K), \quad \|v - \Pi_K v\|_{m,q,K} \leq C(\text{meas}(K))^{1/q-1/p} h_K^{3-m} |v|_{3,p,K}. \tag{6.1.53}$$

Proof. We shall simply give some indications. The proof of inequality (6.1.53) rests on the inclusion

$$P_2(K) \subset P_K$$

(notice that the inequality $p > 1$ is required so as to guarantce the inclusion $W^{3,p}(K) \hookrightarrow \mathscr{C}^1(K) = \text{dom}\, \Pi_K$). One first argues with the finite element (K, P_K, Ξ_K), with Ξ_K as in (6.1.42), which *can* be imbedded in an affine family. Then one uses the same device as in the proofs of Theorems 6.1.1 and 6.1.3. □

Remark 6.1.6. The second partial derivatives of the basis function $\hat{q}_1$ (as given in (6.1.41)) are not defined at the origin. In fact, for each slope $t > 0$, an easy computation shows that

$$\lim_{\substack{x_2 = tx_1 \\ x_1 \to 0^+}} \partial_{11}\hat{q}_1(x) = \frac{-8t^3}{(1+t)^3}.$$

This phenomenon is observed in ZIENKIEWICZ (1971, p. 199), where it is stated that "second-order derivatives have non-unique values at nodes". Hopefully, this observation carries no consequence since it does not prevent the function q_1 from being in the space $\mathscr{C}^1(K) \cap H^2(K)$. □

Estimate of the error $\|u - u_h\|_{2,\Omega}$

Let us now return to the finite element approximation of the clamped plate problem (6.1.1). We shall consider families of finite element spaces X_h, with the same generic finite element (K, P_K, Σ_K), for which we shall need the following assumptions:

(H1*) *The family* (K, P_K, Σ_K), $K \in \mathscr{T}_h$, *for all h, is an almost-affine family.*

(H2*) *The generic finite element is of class* $\mathscr{C}^1$.

If we assume (as in the subsequent theorems) that the inclusion $P_K \subset H^2(K)$ holds, the inclusion $X_h \subset H^2(\Omega)$ is then a consequence of hypothesis (H2*). This being the case, we let

$$V_h = X_{00h} = \{v_h \in X_h;\quad v_h = \partial_\nu v_h = 0 \quad \text{on} \quad \Gamma\}. \tag{6.1.54}$$

Notice that the X_h-interpolation operator associated with any one of the finite elements of class $\mathscr{C}^1$ considered in this section satisfy the implication

$$v \in \operatorname{dom} \Pi_h \quad \text{and} \quad v = \partial_\nu v = 0 \quad \text{on} \quad \Gamma \Rightarrow \Pi_h v \in X_{00h}, \tag{6.1.55}$$

which will accordingly be an implicit assumption in the remainder of this section.

To begin with, we derive an error estimate in the norm $\|\cdot\|_{2,\Omega}$. As usual, the letter C represents any constant independent of h and of all the functions appearing in a given inequality.

Theorem 6.1.6. *In addition to* (H1*) *and* (H2*), *assume that there exists an integer* $k \geq 2$ *such that the following inclusions are satisfied*:

$$P_k(K) \subset P_K \subset H^2(K), \tag{6.1.56}$$

$$H^{k+1}(K) \hookrightarrow \mathscr{C}^s(K), \tag{6.1.57}$$

where s *is the maximal order of partial derivatives occuring in the definition of the set* Σ_K.

Then if the solution $u \in H_0^2(\Omega)$ *of the clamped plate problem is also in the space* $H^{k+1}(\Omega)$, *there exists a constant* C *independent of* h *such that*

$$\|u - u_h\|_{2,\Omega} \leq C h^{k-1} |u|_{k+1,\Omega}, \tag{6.1.58}$$

where $u_h \in V_h$ *is the discrete solution.*

Proof. Using Céa's lemma, inequality (6.1.7) and relation (6.1.55), we obtain

$$\begin{aligned}
\|u - u_h\|_{2,\Omega} &\leq C \inf_{v_h \in V_h} \|u - v_h\|_{2,\Omega} \\
&\leq C\|u - \Pi_h u\|_{2,\Omega} = C\Big(\sum_{K \in \mathscr{T}_h} \|u - \Pi_K u\|_{2,K}^2\Big)^{1/2} \\
&\leq C h^{k-1}\Big(\sum_{K \in \mathscr{T}_h} |u|_{k+1,K}^2\Big)^{1/2} = C h^{k-1} |u|_{k+1,\Omega}.
\end{aligned}$$

□

Remark 6.1.7. By the previous theorem, the least assumptions which insure an $0(h)$ convergence in the norm $\|.\|_{2,\Omega}$ are the inclusions $P_2(K) \subset P_K$ on the one hand, and the fact that the solution u of the plate problem is in the space $H^3(\Omega)$ on the other. It is remarkable that this last regularity result is precisely obtained if the right-hand side f is in the space $L^2(\Omega)$, and if $\bar{\Omega}$ is a convex polygon, an assumption often satisfied for plates. Therefore, since one cannot expect better regularity *in general*, the choice $P_K = P_2(K)$ appears optimal from the point of view of convergence. However, by Ženíšek's result, this choice is not compatible with the inclusion $X_h \subset \mathscr{C}^1(\bar{\Omega})$. □

Sufficient conditions for $\lim_{h\to 0} \|u - u_h\|_{2,\Omega} = 0$

We next obtain convergence in the norm $\|.\|_{2,\Omega}$ under minimal assumptions (cf. (6.1.59) below).

Theorem 6.1.7. *In addition to* (H1*) *and* (H2*), *assume that the inclusions*

$$P_2(K) \subset P_K \subset H^2(K) \tag{6.1.59}$$

are satisfied, and that the maximal order s of partial derivatives found in the set Σ_K *satisfies* $s \leq 2$.

Then we have

$$\lim_{h\to 0} \|u - u_h\|_{2,\Omega} = 0. \tag{6.1.60}$$

Proof. The argument is the same as in the proof of Theorem 3.2.3 and, for this reason, will be only sketched. Using inequality (6.1.5) with $k = 2$, $p = \infty$, $m = 2$ and $q = 2$, one first shows that the space

$$\mathscr{V} = W^{3,\infty}(\Omega) \cap H_0^2(\Omega)$$

is dense in the space $H_0^2(\Omega)$. Then it suffices to use the inequality

$$\inf_{v_h \in V_h} \|u - v_h\|_{2,\Omega} \leq \|u - v\|_{2,\Omega} + \|v - \Pi_h v\|_{2,\Omega},$$

valid for any function $v \in \mathscr{V}$. □

Conclusions

In the following tableau (Fig. 6.1.6), we have summarized the application of Theorem 6.1.6 to various finite elements of class $\mathscr{C}^1$.

Finite element	dim P_K	$P_k(K) \subset P_K$	$\|u - u_h\|_{2,\Omega}$	Assumed regularity
Argyris triangle	21	$P_5(K) = P_K$	$O(h^4)$	$u \in H^6(\Omega)$
Bell's triangle	18	$P_4(K) \subset P_K$	$O(h^3)$	$u \in H^5(\Omega)$
Bogner-Fox-Schmit rectangle	16	$P_3(K) \subset P_K$	$O(h^2)$	$u \in H^4(\Omega)$
Hsieh-Clough-Tocher triangle	12	$P_3(K) \subset P_K$	$O(h^2)$	$u \in H^4(\Omega)$
Reduced Hsieh-Clough-Tocher triangle (cf. Exercise 6.1.3)	9	$P_2(K) \subset P_K$	$O(h)$	$u \in H^3(\Omega)$
Singular Zienkiewicz triangle	12	$P_2(K) \subset P_K$	$O(h)$	$u \in H^3(\Omega)$
Reduced singular Zienkiewicz triangle (cf. Exercise 6.1.6)	9	$P_2(K) \subset P_K$	$O(h)$	$u \in H^3(\Omega)$

Fig. 6.1.6

One should notice that, if the reduced Hsieh-Clough-Tocher triangle and the reduced singular Zienkiewicz triangle are optimal in that the dimension of the corresponding spaces P_K is the smallest, this reduction in the dimension of the spaces P_K is obtained at the expense of an increased complexity in the *structure* of the functions $p \in P_K$.

Remark 6.1.8. In order to get an $O(h^{k+1})$ convergence in the norm $|\cdot|_{0,\Omega}$, it would be necessary to assume that, for *any* $g \in L^2(\Omega)$, the corresponding solution φ_g of the plate problem belongs to the space $H^4(\Omega) \cap H_0^2(\Omega)$ and that there exists a constant C such that $\|\varphi_g\|_{4,\Omega} \leq C|g|_{0,\Omega}$ for *all* $g \in L^2(\Omega)$. However, *this regularity property is no longer true for convex polygons in general.* It is true only if the boundary Γ is sufficiently smooth: For example, this is the case if the boundary Γ is of class $\mathscr{C}^4$. But then this regularity of the boundary becomes incompatible with our assumption that $\bar{\Omega}$ be a polygonal set. □

Exercises

6.1.1. Show that a regular family of Bell's triangles (cf. Fig. 2.2.18) is almost affine, with the value $k = 4$ in the corresponding inequalities of the form (6.1.5).

6.1.2. The purpose of this problem is to give another proof of unisolvence for the Hsieh–Clough–Tocher triangle (as originally proposed in CIARLET (1974c)). Without loss of generality, it can be assumed that $a = (0, 0)$. Denoting by (x_i, y_i) the coordinates of the vertex a_i, let

$$\alpha_i = \det\begin{pmatrix} x_{i+1} & y_{i+1} \\ x_{i+2} & y_{i+2} \end{pmatrix}, \qquad \alpha = \det\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} = \sum_{i=1}^{3} \alpha_i.$$

For definiteness, it shall be assumed that $\alpha = 1$.

Given a function $p \in P_K$ whose degrees of freedom are all zero, let

$$\delta_0 = p(a), \quad \delta_i = \alpha_i\{Dp(a)(a_i - a)\}, \quad 1 \leq i \leq 3.$$

(i) Show that

$$p_{|K_i} = \mu_i^2\{(-2\mu_i + 3)\delta_0 + (\mu_{i+1} - \mu_i)\delta_{i+1} + (\mu_{i+2} - \mu_i)\delta_{i+2}\}, \quad 1 \leq i \leq 3,$$

where for each i we denote by μ_i the unique function which satisfies

$$\mu_i \in P_1(K_i), \quad \mu_i(a) = 1, \qquad \mu_i(a_{i+1}) = \mu_i(a_{i+2}) = 0.$$

(ii) Show that

$$\sum_{i=1}^{3} \delta_i = 0,$$

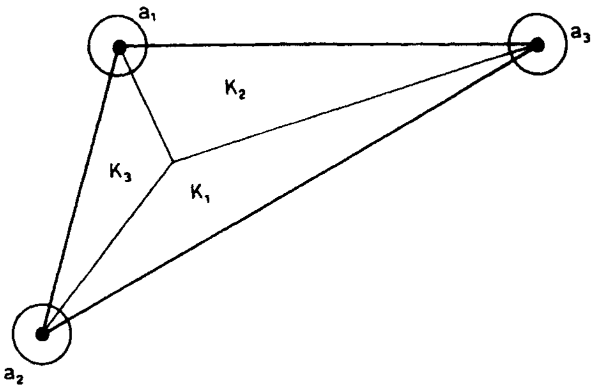
$$\delta_0 - \sum_{k=1}^{3} \delta_k + \frac{2}{\alpha_j}\delta_j = 0, \quad 1 \leq j \leq 3,$$

and conclude that $\delta_i = 0$, $0 \leq i \leq 3$ (the first equality expresses that the function p is differentiable at the point a, while the other relations express the equalities

$$D(p_{i+1} - p_i)\left(\frac{a + a_{i+2}}{2}\right) = 0, \quad 1 \leq i \leq 3).$$

6.1.3. The *reduced Hsieh–Clough–Tocher triangle* is a triangular finite element whose corresponding data P_K and Σ_K are indicated in Fig. 6.1.7.

Show that the set Σ_K is P_K-unisolvent and that a regular family of reduced Hsieh–Clough–Tocher triangles is almost affine, with the value $k = 2$ in the corresponding inequalities of the form (6.1.5).



Reduced Hsieh–Clough–Tocher triangle
$P_K = \{p \in \mathscr{C}^1(K);\ p_{\mid K_i} \in P_3(K_i),\ 1 \le i \le 3,$ $\partial_\nu p \in P_1(K')$ for each side K' of $K\}$; $\dim P_K = 9$; $\Sigma_K = \{p(a_i),\ \partial_1 p(a_i),\ \partial_2 p(a_i),\ 1 \le i \le 3\}$.

Fig. 6.1.7

6.1.4. Following PERCELL (1976), one may define a triangular finite element of class $\mathscr{C}^1$ analogous to the Hsieh–Clough–Tocher triangle, as follows: With an identical subdivision $K = \bigcup_{i=1}^3 K_i$, let

$$P_K = \{P \in \mathscr{C}^1(K);\ p_{\mid K_i} \in P_4(K_i),\quad 1 \le i \le 3\},$$

$$\Sigma_K = \{p(a_i),\quad \partial_1 p(a_i),\quad \partial_2 p(a_i),\quad 1 \le i \le 3;$$

$$p(a_{ij}),\quad 1 \le i < j \le 3;\ \partial_\nu p(a_{iij}),\quad 1 \le i, j \le 3,\quad i \ne j;$$

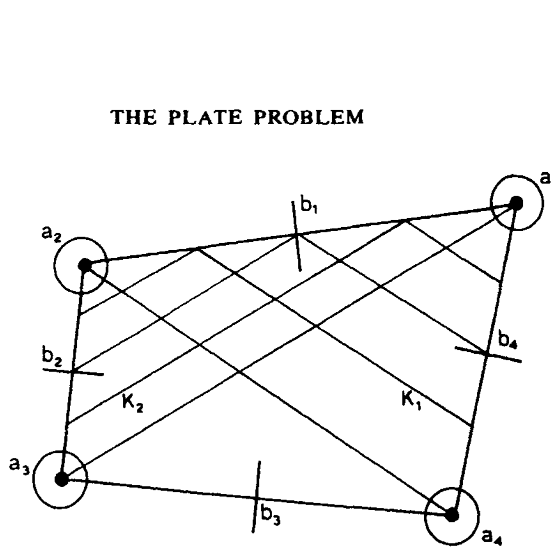
$$p(a),\quad \partial_1 p(a),\quad \partial_2 p(a)\},$$

where

$$a_{ij} = \frac{a_i + a_j}{2},\quad a_{iij} = \frac{2a_i + a_j}{3}.$$

Then show that the set Σ_K is P_K-unisolvent.

6.1.5. *The Fraeijs de Veubeke–Sander quadrilateral* is a finite element (K, P_K, Σ_K) for which the set K is a convex nondegenerate quadrilateral with vertices a_i, $1 \le i \le 4$, and mid-points of the sides b_i, $1 \le i \le 4$. As indicated in Fig. 6.1.8, let K_1 denote the triangle with vertices a_1, a_2 and a_4, and let K_2 denote the triangle with vertices a_1, a_2 and a_3. The space P_K and the set Σ_K are indicated in Fig. 6.1.8.



Fraeijs de Veubeke–Sander quadrilateral
$P_K = R_1(K) + R_2(K)$, where $R_i(K) = \{p \in \mathscr{C}^1(K);\ p_{\mid K_i} \in P_3(K_i),\ p_{\mid K-K_i} \in P_3(K-K_i)\}$, $i = 1, 2$; $\dim P_K = 16$; $\Sigma_K = \{p(a_i),\ \partial_1 p(a_i),\ \partial_2 p(a_i),\ \partial_\nu p(b_i),\ 1 \le i \le 4\}$.

Fig. 6.1.8

(i) Show that the set Σ_K is P_K-unisolvent (CIAVALDINI & NÉDÉLEC (1974)).

(ii) We shall say that *a family of Fraeijs de Veubeke–Sander quadrilaterals is regular* if it is a regular family of finite elements in the usual sense and if, in addition, the following condition is satisfied: For each quadrilateral K in the family, let F_K denote the unique affine mapping which satisfies $F_K(0) = a_K$, $F_K(\hat{a}_1) = a_{1,K}$ and $F_K(\hat{a}_2) = a_{2,K}$, where a_K is the intersection of the two diagonals of the quadrilateral K, and where $\hat{a}_1 = (1, 0)$, $\hat{a}_2 = (0, 1)$ (cf. Fig. 6.1.9). *Then there exist compact intervals $\hat{I}_3$ and $\hat{I}_4$ contained in the half-axes*

$$\{(x_1, x_2) \in \mathbf{R}^2;\ x_1 < 0, x_2 = 0\},\quad \{(x_1, x_2) \in \mathbf{R}^2;\ x_1 = 0, x_2 < 0\},$$

respectively, such that the points $\hat{a}_{j,K} = F_K^{-1}(a_{j,K})$ belong to the intervals $\hat{I}_j$, for $j = 3$ and 4. In other words, the quadrilateral $F_K^{-1}(K)$ is in between the two extremal quadrilaterals $\hat{K}_0$ and $\hat{K}_1$ indicated in Fig. 6.1.9. Then, following CIAVALDINI & NÉDÉLEC (1974), show that a regular family of Fraeijs de Veubeke–Sander quadrilaterals is almost affine, with the value $k = 3$ in the corresponding inequalities of the form (6.1.5).

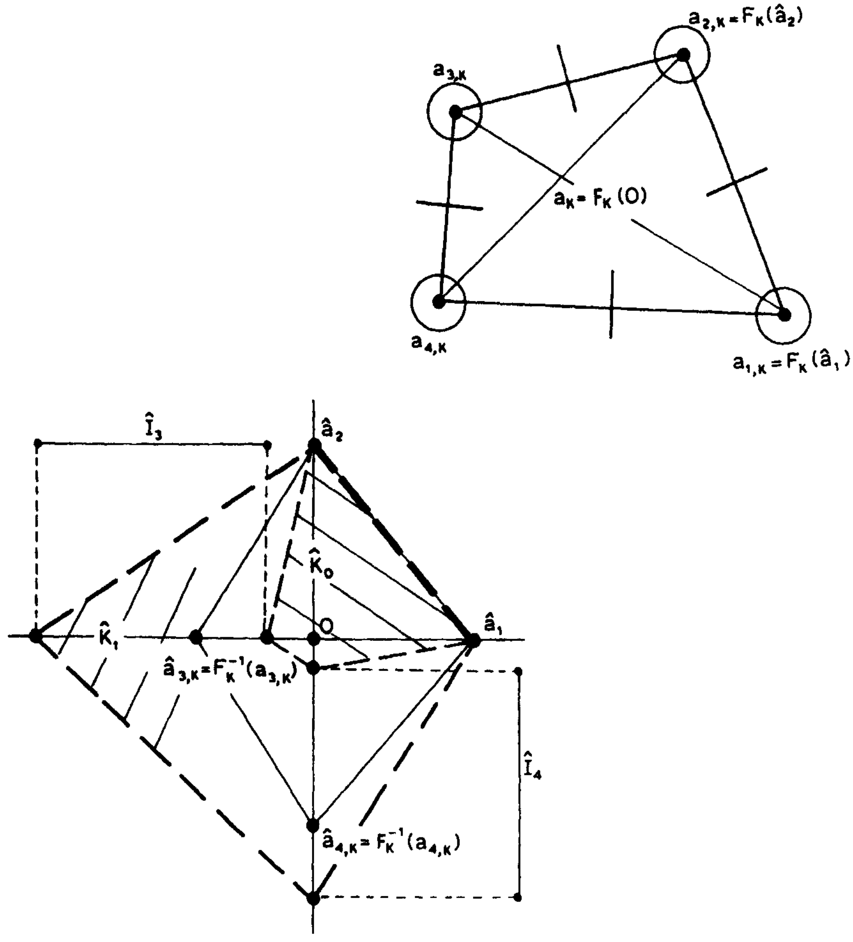
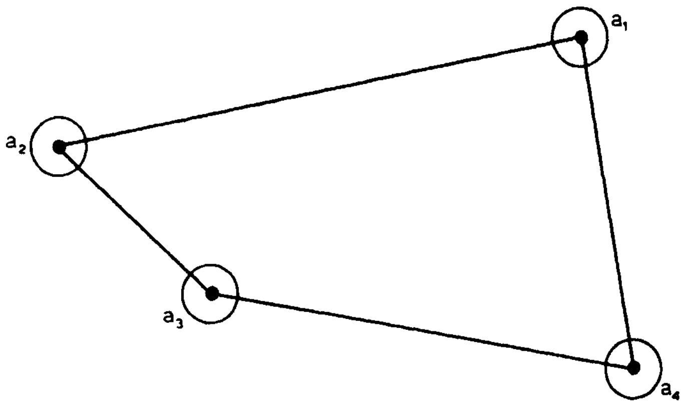


Fig. 6.1.9

(iii) Carry out a similar analysis (unisolvence, interpolation error) for the *reduced Fraeijs de Veubeke–Sander quadrilateral*, whose characteristics are indicated in Fig. 6.1.10 (for the definition of the spaces $R_1(K)$ and $R_2(K)$, see Fig. (6.1.8)).

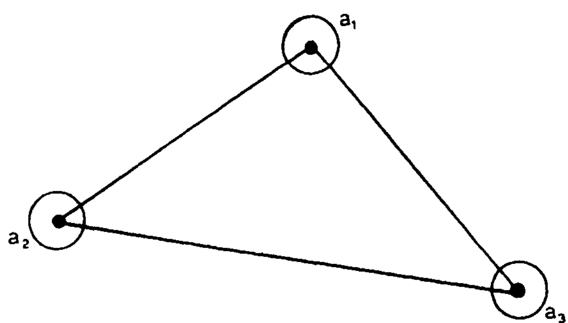
6.1.6. The *reduced singular Zienkiewicz triangle* is a triangular finite element whose corresponding data P_K and Σ_K are indicated in Fig. 6.1.11.

Show that the set Σ_K is P_K-unisolvent and that a regular family of



Reduced Fraeijs de Veubeke–Sander quadrilateral
$P_K = \{p \in R_1(K) + R_2(K);\ \partial_\nu p \in P_1(K')$ along each side K' of $K\}$; $\dim P_K = 12$; $\Sigma_K = \{p(a_i),\ \partial_1 p(a_i),\ \partial_2 p(a_i),\ 1 \leq i \leq 4\}$.

Fig. 6.1.10



Reduced singular Zienkiewicz triangle
$P_K = \{p \in P_3''(K) \oplus \bigvee_{i=1}^{3} \{q_i\};\ \partial_\nu p \in P_1(K')$ for each side K' of $K\}$ (cf. (2.2.39) and (6.1.37)); $\dim P_K = 9$; $\Sigma_K = \{p(a_i),\ \partial_1 p(a_i),\ \partial_2 p(a_i),\ 1 \leq i \leq 3\}$.

Fig. 6.1.11

reduced singular Zienkiewicz triangles is almost affine, with the value $k = 2$ in the corresponding inequalities of the form (6.1.5).

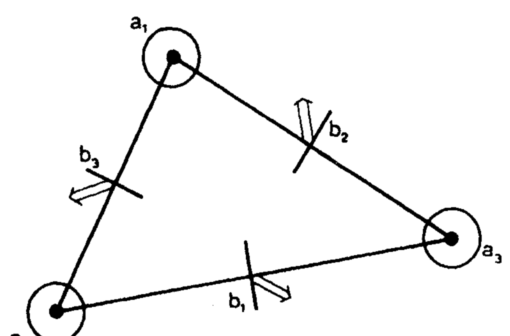
6.1.7. The purpose of this problem is to describe another instance where rational functions are added to a polynomial space so as to obtain a singular finite element of class $\mathscr{C}^1$. An analogous process yielded the singular Zienkiewicz triangle.

(i) Following BIRKHOFF (1971), let $T_3(K)$ denote, for any triangle K, the space of all polynomials whose restrictions along each parallel to any side of K are polynomials of degree ≤ 3 in one variable. Show that the space $T_3(K)$, of so-called *tricubic polynomials*, is the space $P_3(K)$ to which are added linear combinations of the three functions $\lambda_1^2\lambda_2\lambda_3$, $\lambda_1\lambda_2^2\lambda_3$ and $\lambda_1\lambda_2\lambda_3^2$ (which are not linearly independent). Show that $\dim P_3(K) = 12$.

(ii) Following BIRKHOFF & MANSFIELD (1974), we define the *Birkhoff–Mansfield triangle* as indicated in Fig. 6.1.12 (as usual, $\partial_{\nu\tau}p(b_i) = D^2p(b_i)(\nu, \tau)$ where τ is the unit tangential vector at the point b_i).

Show that, along each side of the triangle K, the functions in the space P_K are polynomials of degree ≤ 3 in one variable and that any directional derivative $Dp(.)\xi$, where ξ is any fixed vector in $\mathbf{R}^2$, is also a polynomial of degree ≤ 3 in one variable along each side of the triangle K.

Show that the set Σ_K is P_K-unisolvent.



Birkhoff–Mansfield triangle
$P_K = T_3(K) \oplus \bigvee_{i=1}^{3} \left\{ \dfrac{\lambda_i^2\lambda_{i+1}}{\lambda_i + \lambda_{i+1}} \right\}$; $\dim P_K = 15$; $\Sigma_K = \{p(a_i), \partial_1 p(a_i), \partial_2 p(a_i), \partial_\nu p(b_i), \partial_{\nu\tau} p(b_i), 1 \leq i \leq 3\}$.

Fig. 6.1.12

Show that the resulting finite element is of class $\mathscr{C}^1$ and that the inclusion $P_K \subset H^2(K)$ holds.

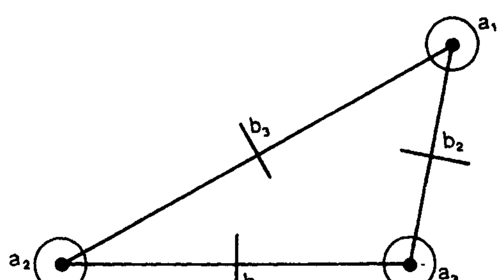
(iii) Show that a regular family of Birkhoff–Mansfield triangles is almost affine, with the value $k = 3$ in the corresponding inequalities of the form (6.1.5).

(iv) Carry out a similar analysis (unisolvence and interpolation error) for the *reduced Birkhoff–Mansfield triangle*, whose characteristics are indicated in Fig. 6.1.13.

6.2. Nonconforming methods

Nonconforming methods for the plate problem

To begin with, we shall give the general definition of a *nonconforming method* for solving the clamped plate problem (corresponding to the data (6.1.1)). Assuming the set $\bar{\Omega}$ polygonal, so that it may be exactly covered with triangulations, we construct *a finite element space X_h whose generic finite element is not of class $\mathscr{C}^1$*. Then the space X_h will not be a subspace of the space $H^2(\Omega)$, as a consequence of the next theorem (which is the converse of Theorem 2.1.2), whose proof is left to the reader (Exercise 6.2.1).



Reduced Birkhoff–Mansfield triangle
$P_K = \left\{p \in T_3(K) \oplus \bigvee_{i=1}^{3} \left\{\frac{\lambda_i^2\lambda_{i+1}}{\lambda_i+\lambda_{i+1}}\right\};\right.$ $\forall \xi \in \mathbf{R}^2$, $Dp(.)\xi \in P_2(K')$ for each side K' of $K\}$; $\dim P_K = 12$; $\Sigma_K = \{p(a_i), \partial_1 p(a_i), \partial_2 p(a_i), \partial_\nu p(b_i), 1 \leq i \leq 3\}$.

Fig. 6.1.13

Theorem 6.2.1. *Assume that the inclusions $P_K \subset \mathscr{C}^1(K)$ for all $K \in \mathscr{T}_h$ and $X_h \subset H^2(\Omega)$ hold. Then the inclusion*

$$X_h \subset \mathscr{C}^1(\bar{\Omega})$$

holds. □

Let us henceforth assume that we have

$$\forall K \in \mathscr{T}_h, \quad P_K \subset H^2(K), \tag{6.2.1}$$

so that, in particular, we have

$$X_h \subset L^2(\Omega). \tag{6.2.2}$$

After defining an appropriate subspace X_{00h} of X_h, so as to take into account the boundary conditions $v = \partial_\nu v = 0$ along Γ as well as possible (this will be illustrated on one example), we define the approximate bilinear form:

$$\begin{aligned} a_h(u_h, v_h) &= \sum_{K \in \mathscr{T}_h} \int_K \{\Delta u_h \Delta v_h + (1-\sigma)(2\partial_{12}u_h\partial_{12}v_h - \partial_{11}u_h\partial_{22}v_h \\ &\qquad - \partial_{22}u_h\partial_{11}v_h)\}\,\mathrm{d}x \\ &= \sum_{K \in \mathscr{T}_h} \int_K \{\sigma\Delta u_h \Delta v_h + (1-\sigma)(\partial_{11}u_h\partial_{11}v_h + \partial_{22}u_h\partial_{22}v_h \\ &\qquad + 2\partial_{12}u_h\partial_{12}v_h)\}\,\mathrm{d}x. \end{aligned} \tag{6.2.3}$$

Observe that this definition is justified by the inclusions (6.2.1). Then the *discrete problem* consists in finding a function $u_h \in V_h = X_{00h}$ such that

$$\forall v_h \in V_h, \quad a_h(u_h, v_h) = f(v_h) \tag{6.2.4}$$

(the linear form need not be approximated in view of the inclusion (6.2.2)). In analogy with the norm $|\cdot|_{2,\Omega}$ of the space $V = H_0^2(\Omega)$, we introduce the *semi-norm*

$$v_h \to \|v_h\|_h = \Big(\sum_{K \in \mathscr{T}_h} |v_h|_{2,K}^2\Big)^{1/2} \tag{6.2.5}$$

over the space V_h. Next we extend the domains of definition of the mappings $a_h(\cdot,\cdot)$ and $\|\cdot\|_h$ to the space $V_h + V$. Thus there exists a constant $\tilde{M}$ independent of the space V_h such that

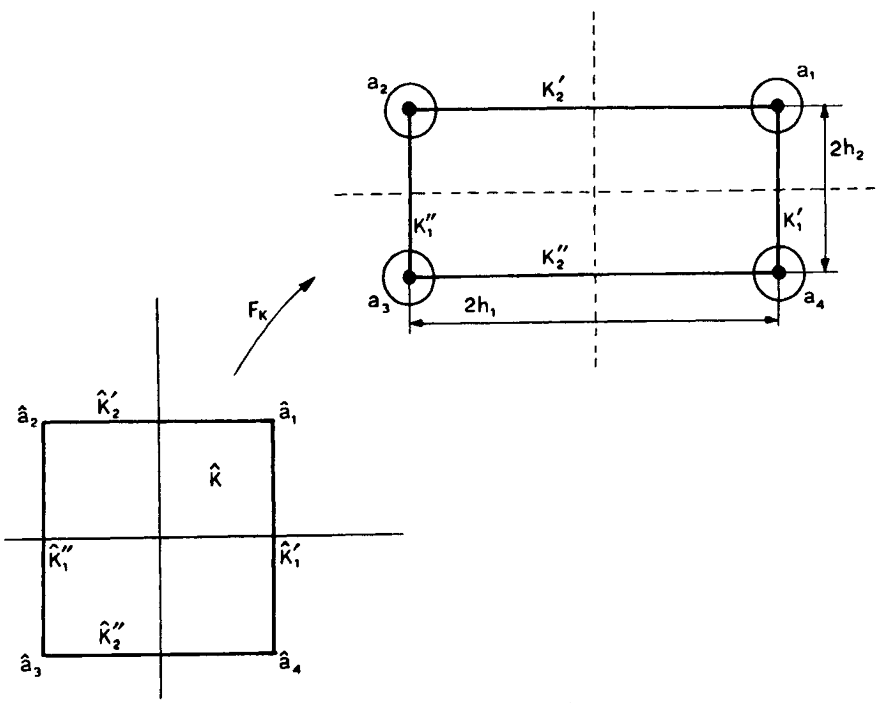
$$\forall u, v \in (V_h + V), \quad |a_h(u, v)| \leq \tilde{M}\|u\|_h\|v\|_h. \tag{6.2.6}$$

An example of a nonconforming finite element: Adini's rectangle

In the remainder of this section, we shall essentially concentrate on one example of a *nonconforming finite element*, in the sense that it yields a nonconforming method when it is used in the approximation of the plate problem. This element, known as *Adini's rectangle*, corresponds to the following data K, P_K and Σ_K: The set K is a rectangle whose vertices a_i, $1 \leq i \leq 4$, are counted as in Fig. 6.2.1.

The space P_K is composed of all polynomials of the form

$$P: x = (x_1, x_2) \to p(x) = \sum_{\alpha_1+\alpha_2 \leq 3} \gamma_{\alpha_1\alpha_2} x_1^{\alpha_1} x_2^{\alpha_2} + \gamma_{13} x_1 x_2^3 + \gamma_{31} x_1^3 x_2,$$



Adini's rectangle
$P_K = P_3(K) \oplus \mathrm{V}\{x_1x_2^3, x_1^3x_2\}$; $\dim P_K = 12$; $\Sigma_K = \{p(a_i), \partial_1 p(a_i), \partial_2 p(a_i), 1 \leq i \leq 4\}$.

Fig. 6.2.1

i.e., we have

$$P_K = P_3(K) \oplus \bigvee \{x_1x_2^3, x_1^3x_2\}. \tag{6.2.7}$$

Notice that the inclusion

$$P_3(K) \subset P_K \tag{6.2.8}$$

holds, and that

$$\dim(P_K) = 12. \tag{6.2.9}$$

To see that *the set*

$$\Sigma_K = \{p(a_i), \partial_1 p(a_i), \partial_2 p(a_i), \quad 1 \leq i \leq 4\} \tag{6.2.10}$$

is a P_K-unisolvent set, let us argue on the square $\hat{K} = [-1, +1]^2$. Then we can write

$$\forall \hat{p} \in P_{\hat{K}}, \quad \hat{p} = \sum_{i=1}^{4} \hat{p}(\hat{a}_i)\hat{p}_i + \sum_{\begin{cases}|j-i|=1\\ (\text{mod. } 4)\end{cases}} D\hat{p}(\hat{a}_i)(\hat{a}_j - \hat{a}_i)\hat{p}_{ij}, \tag{6.2.11}$$

with

$$\begin{cases} \hat{p}_1(x) = \dfrac{(1+x_1)(1+x_2)}{4}\left(1 + \dfrac{x_1+x_2}{2} - \dfrac{x_1^2+x_2^2}{2}\right), \\ \hat{p}_{12}(x) = \dfrac{(1+x_1)(1+x_2)^2(1-x_2)}{8}, \\ \hat{p}_{14}(x) = \dfrac{(1+x_2)(1+x_1)^2(1-x_1)}{8}, \quad \text{etc.}\ldots \end{cases} \tag{6.2.12}$$

Let us assume that the set $\bar{\Omega}$ is rectangular, so that it may be covered by triangulations made up of rectangles. With such a triangulation $\mathcal{T}_h$, we associate a finite element space X_h whose functions v_h are defined as follows:

(i) For each rectangle $K \in \mathcal{T}_h$, the restrictions $v_{h|K}$ span the space P_K of (6.2.7).

(ii) Each function $v_h \in X_h$ is defined by its values and the values of its first derivatives at all the vertices of the triangulation.

Along each side K' of an Adini's rectangle K, the restrictions $p_{|K'}$, $p \in P_K$, are polynomials of degree ≤ 3 in one variable. Since such polynomials are uniquely determined by their values and the values of their first derivative at the end points of K', we conclude that *Adini's rectangle is a finite element of class $\mathscr{C}^0$*. It is *not* of class $\mathscr{C}^1$, however:

Along the side $K_1' = [a_4, a_1]$ for instance (cf. Fig. 6.2.1), the normal derivative is a polynomial of degree ≤ 3 in the variable x_2 on the one hand, and on the other the only degrees of freedom that are available for the normal derivative along the side K_1' are its two values at the end points.

We let $V_h = X_{00h}$, where X_{00h} denotes the space of all functions $v_h \in X_h$ such that $v_h(b) = \partial_1 v_h(b) = \partial_2 v_h(b) = 0$ at all the boundary nodes b. Then the functions $v_h \in V_h$ vanish along the boundary Γ, but their derivatives $\partial_\nu v_h$ do not vanish along the boundary Γ in general, although they vanish at the boundary nodes. To sum up, we have constructed a finite element space V_h whose functions v_h satisfy

$$\begin{cases} v_h \in H_0^1(\Omega) \cap \mathscr{C}^0(\bar{\Omega}), \quad v_{h|K} \in H^2(K) \quad \text{for all} \quad K \in \mathscr{T}_h, \\ \partial_\nu v_h(b) = 0 \quad \text{at the boundary nodes.} \end{cases} \tag{6.2.13}$$

Observe that the associated X_h-interpolation operator Π_h is such that

$$v \in H_0^2(\Omega) \cap \operatorname{dom} \Pi_h \Rightarrow \Pi_h v \in X_{00h} = V_h. \tag{6.2.14}$$

We shall use this implication in particular for functions in the space $H^3(\Omega) \cap H_0^2(\Omega) \subset \mathscr{C}^1(\bar{\Omega}) = \operatorname{dom} \Pi_h$.

Prior to the error analysis, we must examine whether the mapping $\|\cdot\|_h$ of (6.2.5) is indeed a norm.

Theorem 6.2.2. *The mapping*

$$v_h \to \|v_h\|_h = \Big(\sum_{K \in \mathscr{T}_h} |v_h|_{2,K}^2\Big)^{1/2}$$

is a norm over the space V_h.

Proof. Let v_h be a function in the space V_h such that $\|v_h\|_h = 0$. Then the functions $\partial_j(v_{h|K})$, $j = 1, 2$, are constant over each rectangle $K \in \mathscr{T}_h$. Since they are continuous at the vertices, the functions $\partial_j v_h$, $j = 1, 2$, are therefore constant over the set $\bar{\Omega}$, and since they vanish at the boundary nodes, they are identically zero. Thus the function $v_h \in V_h$ is identically zero, as a consequence of the inclusion $V_h \subset H_0^1(\Omega) \cap \mathscr{C}^0(\Omega)$. □

Notice that *the approximate bilinear forms $a_h(\cdot,\cdot)$ are uniformly V_h-elliptic*, since one has (cf. (6.2.3))

$$\forall v_h \in V_h, \quad (1-\sigma)\|v_h\|_h^2 \leq a_h(v_h, v_h), \tag{6.2.15}$$

and the Poisson coefficient σ lies in the interval $]0, \frac{1}{2}[$ (for physical reasons).

Remark 6.2.1. Had we tried to use nonconforming finite element methods for the biharmonic problem (in which case the approximate bilinear form reduces to $\sum_{K\in\mathcal{T}_h}\int_K \Delta u_h \Delta v_h \,dx$), the uniform V_h-ellipticity is no longer automatic, and this is essentially why we restrict ourselves to plate problems. In contrast, conforming methods as described in the previous section apply equally well to any fourth-order elliptic boundary value problem. □

Consistency error estimate. Estimate of the error $(\sum_{K\in\mathcal{T}_h}|u-u_h|_{2,K}^2)^{1/2}$

We are now in a position to apply the abstract error estimate of Theorem 4.2.2, which we recall here for convenience:

$$\|u-u_h\|_h \leq C\left(\inf_{v_h\in V_h}\|u-v_h\|_h + \sup_{w_h\in V_h}\frac{|a_h(u,w_h)-f(w_h)|}{\|w_h\|_h}\right). \tag{6.2.16}$$

In what follows, *the solution u will be assumed to be in the space* $H^3(\Omega)\cap H_0^2(\Omega)$ (this is true for any $f\in L^2(\Omega)$ if $\bar{\Omega}$ is a convex polygon, i.e., a rectangle in the present case). Observing that any family of Adini's rectangles is affine, we obtain for a regular family of triangulations,

$$\inf_{v_h\in V_h}\|u-v_h\|_h \leq \left(\sum_{K\in\mathcal{T}_h}|u-\Pi_K u|_{2,K}^2\right)^{1/2} \leq Ch|u|_{3,\Omega}, \tag{6.2.17}$$

and this estimate takes care of the first term in the right-hand side of inequality (6.2.16). The estimate of the second term, i.e., the *consistency error estimate,* rests on a careful decomposition of the difference

$$D_h(u,w_h) = a_h(u,w_h)-f(w_h), \quad w_h\in V_h. \tag{6.2.18}$$

Let us first show that the term $f(w_h)=\int_\Omega f w_h\,dx$ can be rewritten in the form

$$f(w_h) = -\int_\Omega \nabla(\Delta u)\cdot\nabla w_h\,dx \quad \text{for all} \quad w_h\in V_h \tag{6.2.19}$$

(this equality is obvious if $u\in H^4(\Omega)\cap H_0^2(\Omega)$, in which case $f(w_h)=\int_\Omega \Delta^2 u\, w_h\,dx$, but we only assume here that $u\in H^3(\Omega)\cap H_0^2(\Omega)$). To see this, let $w_h\in V_h$ be given, and let (w_h^k) be a sequence of functions $w_h^k\in\mathcal{D}(\Omega)$ such that $\lim_{k\to\infty}\|w_h^k-w_h\|_{1,\Omega}=0$ (recall that $w_h\in V_h\subset$

$H_0^1(\Omega)$). By making use of Green's formulas (1.2.5) and (1.2.9), we obtain for all integers k,

$$\int_\Omega \Delta u \Delta w_h^k \,\mathrm{d}x = -\int_\Omega \nabla(\Delta u)\cdot\nabla w_h^k \,\mathrm{d}x,$$

$$\int_\Omega \{2\partial_{12}u\partial_{12}w_h^k - \partial_{11}u\partial_{22}w_h^k - \partial_{22}u\partial_{11}w_h^k\}\,\mathrm{d}x = 0,$$

since $\partial_\nu w_h^k = \partial_\tau w_h^k = 0$ along Γ, and thus, by definition of the abstract problem (cf. (6.1.1)),

$$\int_\Omega f w_h^k \,\mathrm{d}x = -\int_\Omega \nabla(\Delta u)\cdot\nabla w_h^k \,\mathrm{d}x.$$

Therefore,

$$\int_\Omega f w_h \,\mathrm{d}x = \lim_{k\to\infty}\int_\Omega f w_h^k \,\mathrm{d}x = \lim_{k\to\infty}\int_\Omega \Delta u\Delta w_h^k\,\mathrm{d}x$$
$$= \lim_{k\to\infty}\left\{-\int_\Omega \nabla(\Delta u)\cdot\nabla w_h^k\,\mathrm{d}x\right\} = -\int_\Omega \nabla(\Delta u)\cdot\nabla w_h\,\mathrm{d}x,$$

and equality (6.2.19) is proved.

Using the same Green's formulas as above, we obtain (cf. (1.2.9) for the notation).

$$\forall K\in\mathcal{T}_h,\quad \forall w_h\in V_h,$$
$$\int_K \{\Delta u\Delta w_h + (1-\sigma)(2\partial_{12}u\partial_{12}w_h - \partial_{11}u\partial_{22}w_h - \partial_{22}u\partial_{11}w_h)\}\,\mathrm{d}x$$
$$= -\int_K \nabla(\Delta u)\cdot\nabla w_h\,\mathrm{d}x + \int_{\partial K}\Delta u\partial_{\nu_K}w_h\,\mathrm{d}\gamma$$
$$+(1-\sigma)\int_{\partial K}\{-\partial_{\tau\tau_K}u\partial_{\nu_K}w_h + \partial_{\nu\tau_K}u\partial_{\tau_K}w_h\}\,\mathrm{d}\gamma.$$

When the above expressions are added up so as to form the approximate bilinear form of (6.2.3), we first find that

$$\sum_{K\in\mathcal{T}_h}\left\{-\int_K \nabla(\Delta u)\cdot\nabla w_h\,\mathrm{d}x\right\} = -\int_\Omega \nabla(\Delta u)\cdot\nabla w_h\,\mathrm{d}x = f(w_h),$$

using the inclusion $V_h\subset H^1(\Omega)$ and equality (6.2.19), and next we shall find that

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}\partial_{\nu\tau_K}u\partial_{\tau_K}(w_{h|K})\,\mathrm{d}\gamma = 0.$$

To prove this last relation, consider separately the case where $K' \subset \partial K$ is a side common to two adjacent rectangles K_1 and K_2, and the case where $K' \subset \partial K$ is a portion of the boundary Γ. In the first case the two corresponding integrals cancel because $u \in H^3(\Omega)$ and $w_h \in \mathscr{C}^0(\bar{\Omega})$, and in the second case the integral vanishes because $w_h = 0$ along Γ.

To sum up, we have found that

$$\forall w_h \in V_h, \quad D_h(u, w_h) = a_h(u, w_h) - f(w_h)$$
$$= \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\Delta u - (1-\sigma)\partial_{\tau\tau_K} u)\partial_{\nu_K}(w_{h|K})\, d\gamma, \tag{6.2.20}$$

i.e., we have obtained *one* decomposition of the expression $D_h(u, w_h)$ as a sum

$$D_h(u, w_h) = \sum_{K \in \mathcal{T}_h} D_K(u_{|K}, w_{h|K}),$$

where each mapping $D_K(.,.)$ appears as a bilinear form over the space $H^3(K) \times P_K$. Just as in the proof of Theorem 4.2.6, the key argument will consist in obtaining another decomposition of the form (6.2.20) (cf. (6.2.23)), which in this case takes into account the "conforming" part of the first order partial derivatives of the functions in the space V_h (for related ideas, cf. Remark 4.2.5). This will in turn allow us to obtain appropriate estimates of the difference $D_h(u, w_h)$, as we shall show in the proof of the next theorem.

Theorem 6.2.3. *Assume that the solution u of the plate problem is in the space $H_0^2(\Omega) \cap H^3(\Omega)$. Then, for any regular family of triangulations, there exists a constant C independent of h such that*

$$\|u - u_h\|_h = \Big(\sum_{K \in \mathcal{T}_h} |u - u_h|_{2,K}^2\Big)^{1/2} \leq Ch|u|_{3,\Omega}. \tag{6.2.21}$$

Proof. In view of the decomposition (6.2.20), we are naturally led to study the bilinear form

$$D_h(.,.): (v, w_h) \in H^3(\Omega) \times V_h \to$$
$$D_h(v, w_h) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\Delta v - (1-\sigma)\partial_{\tau\tau_K} v)\partial_{\nu_K}(w_{h|K})\, d\gamma$$
$$= D_h^1(v, \partial_1 w_h) + D_h^2(v, \partial_2 w_h), \tag{6.2.22}$$

with

$$D_h^j(v, \partial_j w_h) = \sum_{K \in \mathcal{T}_h} \left\{ \int_{K_j'} (\Delta v - (1-\sigma)\partial_{\tau\tau_K} v)\partial_j(w_{h|K})\,\mathrm{d}\gamma \right.$$
$$\left. - \int_{K_j''} (\Delta v - (1-\sigma)\partial_{\tau\tau_K} v)\partial_j(w_{h|K})\,\mathrm{d}\gamma \right\}, \quad j = 1, 2,$$

where, for each $K \in \mathcal{T}_h$, the sides K_j' and K_j'', $j = 1, 2$, are defined as in Fig. 6.2.1.

For each triangulation $\mathcal{T}_h$, we let Y_h denote the finite element space whose generic finite element is the rectangle of type (1) and we let $Z_h = Y_{0h}$ denote the space of all functions $w_h \in Y_h$ which vanish at the boundary nodes. Clearly, the inclusion

$$Z_h \subset \mathscr{C}^0(\bar{\Omega}) \cap H_0^1(\Omega)$$

implies that

$$\forall v \in H^3(\Omega), \quad \forall z_h \in Z_h, \quad D_h^j(v, z_h) = 0, \quad j = 1, 2,$$

with

$$D_h^j(v, z_h) = \sum_{K \in \mathcal{T}_h} \left\{ \int_{K_j'} (\Delta v - (1-\sigma)\partial_{\tau\tau_K} v) z_h\,\mathrm{d}\gamma \right.$$
$$\left. - \int_{K_j''} (\Delta v - (1-\sigma)\partial_{\tau\tau_K} v) z_h\,\mathrm{d}\gamma \right\}, \quad j = 1, 2.$$

Consequently, if, for each $K \in \mathcal{T}_h$, Λ_K denotes the $Q_1(K)$-interpolation operator, we can also write

$$\forall v \in H^3(\Omega), \quad \forall w_h \in V_h, \quad D_h(v, w_h) = \sum_{K \in \mathcal{T}_h} D_K(v, w_h), \tag{6.2.23}$$

where, for each $K \in \mathcal{T}_h$, the bilinear form $D_K(.\,,.)$ is given by

$$\forall v \in H^3(K), \quad \forall p \in P_K, \quad D_K(v, p) = \Delta_{1,K}(v, \partial_1 p) + \Delta_{2,K}(v, \partial_2 p), \tag{6.2.24}$$

with

$$\Delta_{j,K}(v, \partial_j p) = \int_{K_j'} (\Delta v - (1-\sigma)\partial_{\tau\tau_K} v)(\partial_j p - \Lambda_K \partial_j p)\,\mathrm{d}\gamma$$
$$- \int_{K_j''} (\Delta v - (1-\sigma)\partial_{\tau\tau_K} v)(\partial_j p - \Lambda_K \partial_j p)\,\mathrm{d}\gamma, \quad j = 1, 2. \tag{6.2.25}$$

Using the definition of the operator Λ_K, we find a *first polynomial invariance*:

$$\forall v \in H^3(K), \quad \forall q \in Q_1(K), \quad \Delta_{j,K}(v, q) = 0, \quad j = 1, 2, \tag{6.2.26}$$

with

$$\Delta_{j,K}(v, q) = \int_{K'_j} (\Delta v - (1-\sigma)\partial_{\tau\tau_K} v)(q - \Lambda_K q)\,\mathrm{d}\gamma - \int_{K''_j} (\Delta v - (1-\sigma)\partial_{\tau\tau_K} v)(q - \Lambda_K q)\,\mathrm{d}\gamma, \quad j = 1, 2.$$

We next proceed to obtain the *second polynomial invariance*:

$$\forall v \in P_2(K), \quad \forall q \in \partial_j P_K, \quad \Delta_{j,K}(v, q) = 0, \quad j = 1, 2, \tag{6.2.27}$$

where the spaces

$$\partial_j P_K = \{\partial_j p\,;\, p \in P_K\}, \quad j = 1, 2, \tag{6.2.28}$$

both contain the space $Q_1(K)$. To see this, it suffices to show that

$$\forall q \in \partial_j P_K, \quad \int_{K'_j} (q - \Lambda_K q)\,\mathrm{d}\gamma = \int_{K''_j} (q - \Lambda_K q)\,\mathrm{d}\gamma, \quad j = 1, 2. \tag{6.2.29}$$

Let us prove this equality for $j = 1$, for instance. Each function $q \in \partial_1 P_K$ is of the form

$$q = \gamma_0(x_1) + \gamma_1(x_1)x_2 + \gamma_2 x_2^2 + \gamma_3 x_2^3,$$

where γ_0 and γ_1 are polynomials of degree $\leqslant 2$ in the variable x_1. Given any function r defined on a side K', let $\lambda_{K'} r$ denote the linear function along K' which assumes the same values as the function r at the end points of K'. Then we have

$$(q - \Lambda_K q)_{|K'_1}(x_2) = \gamma_2 x_2^2 + \gamma_3 x_2^3 - \lambda_{K'_1}(\gamma_2 x_2^2 + \gamma_3 x_2^3)$$

and therefore

$$(q - \Lambda_K q)_{|K'_1}(x_2) = (q - \Lambda_K q)|_{K''_1}(x_2),$$

which proves (6.2.29). Consequently, the polynomial invariance of (6.2.27) holds.

To estimate the quantities $\Delta_{j,K}(v, \partial_j p)$ of (6.2.25), it suffices to estimate the similar expressions

$$\delta_{j,K}(\varphi, q) = \int_{K'_j} \varphi(q - \Lambda_K q)\,\mathrm{d}\gamma - \int_{K''_j} \varphi(q - \Lambda_K q)\,\mathrm{d}\gamma \tag{6.2.30}$$

for $\varphi \in H^1(K)$, $q \in \partial_j P_K$, $j = 1, 2$. Using the standard correspondences between the functions $\hat{v}: \hat{K} \to \mathbf{R}$ and $v: K \to \mathbf{R}$, we obtain

$$\delta_{1,K}(\varphi, q) = h_2 \delta_{1,\hat{K}}(\hat{\varphi}, \hat{q}), \quad \delta_{2,K}(\varphi, q) = h_1 \delta_{2,\hat{K}}(\hat{\varphi}, \hat{q}), \tag{6.2.31}$$

and we shall also take into account the fact that a function $\hat{q}$ belong to the space $\partial_j P_{\hat{K}}$ when the function q belongs to the space $\partial_j P_K$.

Paralleling the polynomial invariances (6.2.26) and (6.2.27), we now have:

$$\begin{cases} \forall \hat{\varphi} \in H^1(\hat{K}), \quad \forall \hat{q} \in P_0(\hat{K}), \quad \delta_{j,\hat{K}}(\hat{\varphi}, \hat{q}) = 0, \\ \forall \hat{\varphi} \in P_0(\hat{K}), \quad \forall \hat{q} \in \partial_j P_{\hat{K}}, \quad \delta_{j,\hat{K}}(\hat{\varphi}, \hat{q}) = 0, \quad j = 1, 2. \end{cases} \tag{6.2.32}$$

Then if we equip the spaces $\partial_j P_K$ with the norm $\|.\|_{1,K}$, we obtain $\forall \hat{\varphi} \in H^1(\hat{K})$, $\forall \hat{q} \in \partial_j P_{\hat{K}}$,

$$|\delta_{j,\hat{K}}(\hat{\varphi}, \hat{q})| \leq \hat{C} \|\hat{\varphi}\|_{L^2(\partial \hat{K})} \|\hat{q}\|_{L^2(\partial \hat{K})} \leq \hat{C} \|\hat{\varphi}\|_{1,\hat{K}} \|\hat{q}\|_{1,\hat{K}},$$

and thus each bilinear from $\delta_{j,\hat{K}}(.,.)$ is continuous over the space $H^1(\hat{K}) \times \partial_j P_{\hat{K}}$. Using the bilinear lemma (Theorem 4.2.5), there exists another constant $\hat{C}$ such that

$$\begin{aligned} &\forall \hat{\varphi} \in H^1(\hat{K}), \quad \forall \hat{q} \in \partial_j P_{\hat{K}}, \\ &|\delta_{j,\hat{K}}(\hat{\varphi}, \hat{q})| \leq \hat{C} |\hat{\varphi}|_{1,\hat{K}} |\hat{q}|_{1,\hat{K}}. \end{aligned} \tag{6.2.33}$$

By Theorem 3.1.2 and the regularity assumption, there exists a constant C such that

$$|\hat{\varphi}|_{1,\hat{K}} \leq C |\varphi|_{1,K}, \quad |\hat{q}|_{1,\hat{K}} \leq C |q|_{1,K}. \tag{6.2.34}$$

Combining relations (6.2.31), (6.2.33) and (6.2.34), we conclude that

$$\begin{aligned} &\forall \varphi \in H^1(K), \quad \forall q \in \partial_j P_K, \\ &|\delta_{j,K}(\varphi, q)| \leq C h_K |\varphi|_{1,K} |q|_{1,K}, \quad j = 1, 2. \end{aligned} \tag{6.2.35}$$

Let then $v \in H^3(K)$ and $p \in P_K$ be two given functions, so that the functions $\varphi = \Delta v - (1 - \sigma) \partial_{22} v$ and $q = \partial_1 p$ belong to the spaces $H^1(K)$ and $\partial_1 P_K$, respectively. Then we have

$$|\Delta_{1,K}(v, p)| = |\delta_{1,K}(\Delta v - (1 - \sigma) \partial_{22} v, \partial_1 p)| \leq C h_K |v|_{3,K} |p|_{2,K}.$$

Arguing analogously with the term $|\Delta_{2,K}(v, p)|$, we obtain

$$\begin{aligned} &\forall v \in H^3(K), \quad \forall p \in P_K, \\ &|D_K(v, p)| \leq \sum_{j=1}^{2} |\Delta_{j,K}(v, \partial_j p)| \leq C h_K |v|_{3,K} |p|_{2,K}. \end{aligned}$$

Then we are able to estimate the second term in the abstract error estimate (6.2.16): We find that, for all $w_h \in V_h$,

$$|a_h(u, w_h) - f(w_h)| \le \sum_{K \in \mathcal{T}_h} |D_K(u, w_h)| \le Ch|u|_{3,\Omega}\|w_h\|_h$$

and the proof is complete. □

Further results

The error estimate (6.2.21) can be improved *when all the rectangles $K \in \mathcal{T}_h$ are equal.* In this case, LASCAUX & LESAINT (1975) have shown that $\|u - u_h\|_h \le Ch^2|u|_{4,\Omega}$ if the solution u is in the space $H^4(\Omega)$.

For an error estimate in the norm $\|.\|_{1,\Omega}$, see Exercise 6.2.2.

Another nonconforming finite element for solving the plate problem is the *Zienkiewicz triangle* (cf. BAZELEY, CHEUNG, IRONS & ZIENKIEWICZ (1965)) which was described in Section 2.2 (cf. Fig. 2.2.16). Through a refinement of the argument used in the proof of Theorem 6.2.3, LASCAUX & LESAINT (1975) have shown that the necessary polynomial invariances in the difference $D_h(u, w_h)$ (which in turn imply convergence) are obtained if and only if *all sides of all the triangles found in the triangulation are parallel to three directions only.* In this case, one gets $\|u - u_h\|_h \le Ch|u|_{3,\Omega}$ and $\|u - u_h\|_{1,\Omega} \le Ch^2|u|_{3,\Omega}$ assuming the solution u is the space $H^3(\Omega)$. This is therefore an answer to the *Union Jack problem*: As pointed out in ZIENKIEWICZ (1971, p. 188–189), the engineers had empirically discovered that configuration (a) systematically yields poorer results than configuration (b) (Fig. 6.2.2).

The reason why the degree of freedom $p(a_{123})$ (which is normally found in the Hermite triangle of type (3)) should be eliminated is that the presence of the associated basis function $\lambda_1\lambda_2\lambda_3$ (cf. (2.2.37)) would destroy the required polynomial invariances.

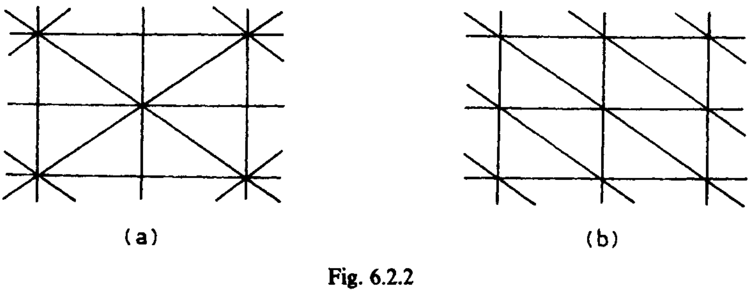


Fig. 6.2.2

Whereas Zienkiewicz triangles yield finite element spaces which satisfy the inclusion $V_h \subset \mathscr{C}^0(\bar{\Omega}) \cap H_0^1(\Omega)$ (just as Adini's rectangles), there exist nonconforming finite elements for the plate problem which are not even of class $\mathscr{C}^0$. Two such finite elements, the *Morley triangle* and the *Fraeijs de Veubeke triangle*, are analyzed in Exercise 6.2.3.

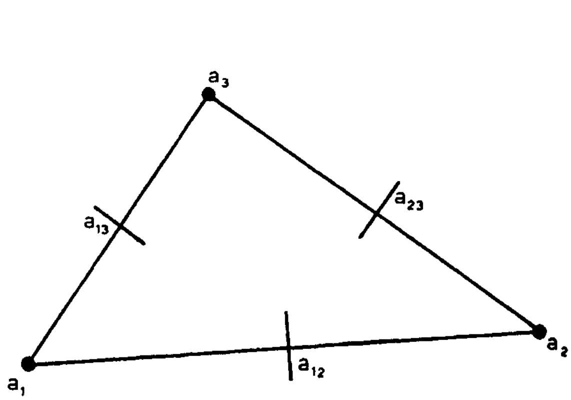
Exercises

6.2.1. Prove Theorem 6.2.1 (cf. Theorem 4.2.1 for a similar argument).
6.2.2. Using the abstract error estimate of Exercise 4.2.3, show that (LASCAUX & LESAINT (1975))

$$\|u - u_h\|_{1,\Omega} \leq Ch^2|u|_{3,\Omega},$$

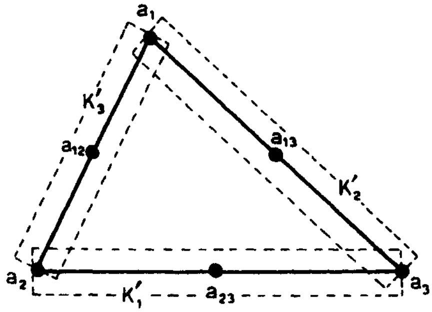
for finite element spaces whose generic element is the Adini rectangle.
6.2.3. Following LASCAUX & LESAINT (1975), the object of this problem is the study of two nonconforming finite elements which are not of class $\mathscr{C}^0$. The first element, known as *Morley's triangle* (cf. MORLEY (1968)) corresponds to the data indicated in Fig. 6.2.3.

The second element, known as *Fraeijs de Veubeke triangle* (cf. FRAEIJS DE VEUBEKE (1974)) is an example of a finite element where some degrees of freedom are *averages* (another related instance is Wilson's



Morley's triangle
$P_K = P_2(K)$; $\dim P_K = 6$; $\Sigma_K = \{p(a_i),\ 1 \leq i \leq 3;\ \partial_\nu p(a_{ij}),\ 1 \leq i < j \leq 3\}$

Fig. 6.2.3.



Fraeijs de Veubeke triangle
$P_K = \{p \in P_3(K);\ \phi(p) = 0\}$; $\dim P_K = 9$; $\phi(p) = 27p(a_{123}) - \sum_{i=1}^{3} p(a_i) - 8 \sum_{1\le i<j\le 3} p(a_{ij})$ $+3\sum_{i=1}^{3} \frac{1}{\lvert K_i'\rvert}\int_{K_i'} Dp(.)(a_{123} - a_i)\,d\gamma$; $\Sigma_K = \left\{p(a_i),\ 1\le i\le 3;\ p(a_{ij}),\ \ 1\le i\le j\le 3;\right.$ $\left.\frac{1}{\lvert K_i'\rvert}\int_{K_i'} \partial_\nu p\,d\gamma,\ 1\le i\le 3\right\}.$

Fig. 6.2.4

brick; cf. Section 4.2). All the relevant data are indicated in Fig. 6.2.4 where, for each $i = 1, 2, 3$, $|K_i'|$ denotes the length of the side K_i'.

(i) In each case, prove the P_K-unisolvence of the given sets Σ_K and that, for regular families, one has

$$\forall v \in H^3(K) \subset \operatorname{dom} \Pi_K, \quad |v - \Pi_K v|_{m,K} \le Ch^{3-m}|v|_{3,K},$$
$$0 \le m \le 3,$$

i.e., *regular families of Morley's triangles or Fraeijs de Veubeke triangles are almost-affine.* Prove in particular that the space P_K corresponding to the Fraeijs de Veubeke triangle contains the space $P_2(K)$.

(ii) For each finite element, describe the associated finite element space X_h, and then let $V_h = X_{00h}$, where X_{00h} is composed of the functions in X_h whose degrees of freedom vanish along the boundary Γ.

Show that neither element is of class $\mathscr{C}^0$. However, show that in each case the averages of the first order partial derivatives are the same across any side common to two adjacent finite elements, while the same averages vanish along a side included in Γ.

(iii) Show that for both elements the semi-norm $\|.\|_h$ of (6.2.5) is a norm over the space V_h.

(iv) Show that if the solution u belongs to the space $H^4(\Omega)$, the error estimates

$$\|u - u_h\|_h \leq C(h|u|_{3,\Omega} + h^2|u|_{4,\Omega})$$

holds. Therefore, contrary to the Zienkiewicz triangle, no restriction need to be imposed on the geometry of the triangulations so as to obtain convergence.

[Hint: The decomposition (6.2.20) is here to be replaced by

$$\begin{aligned}\forall w_h \in V_h,\quad D_h(u, w_h) &= a_h(u, w_h) - f(w_h)\\ &= \sum_{K\in\mathcal{T}_h}\int_{\partial K}(\Delta u - (1-\sigma)\partial_{\tau_K\tau_K}u)\partial_{\nu_K}(w_{h|K})\,\mathrm{d}\gamma\\ &\quad - \sum_{K\in\mathcal{T}_h}\int_{\partial K}\partial_{\nu_K}\Delta u\; w_{h|K} + (1-\sigma)\partial_{\nu_K\tau_K}u\,\partial_{\tau_K}(w_{h|K})\,\mathrm{d}\gamma,\end{aligned}$$

and the key idea is again to subtract off appropriate "conforming" parts in the above expression. Then it is possible to apply the bilinear lemma (one side at a time rather than one element at a time, as in the case of Wilson's brick or Adini's rectangle).]

Bibliography and comments

6.1 and 6.2. The first interpolation error estimates for the Argyris triangle are due to ZLÁMAL (1968), who obtained estimates in the spaces $\mathscr{C}^m(K)$. The results and methods of M. Zlámal were extended by ŽENÍŠEK (1970) to finite elements which yield inclusions of the form $X_h \subset \mathscr{C}^m(\bar{\Omega})$. BRAMBLE & ZLÁMAL (1970) have obtained estimates in Sobolev norms, which are contained in the estimates of Theorem 6.1.1.

The Hsieh–Clough–Tocher triangle appeared in CLOUGH & TOCHER (1965). It is also named after Hsieh who was the first to conceive in 1962 the idea of matching three polynomials so as to get a finite element of class $\mathscr{C}^1$. The interpolation theory given in Theorem 6.1.3 is based on CIARLET (1974c) where a proof of unisolvence was also given along the lines indicated in Exercise 6.1.2. The proof of unisolvence given in

Theorem 6.1.2 is due to PERCELL (1976). See also DOUGLAS, DUPONT, PERCELL & SCOTT (1976). The Fraeijs de Veubeke–Sander quadrilateral (cf. Exercise 6.1.5) is due to SANDER (1964) and FRAEIJS DE VEUBEKE (1965a, 1968), and it has been theoretically studied by CIAVALDINI & NÉDÉLEC (1974).

The singular Zienkiewicz triangle is found in Section 10.10 of ZIENKIEWICZ (1971), where alternate singular finite elements are also described. Since the second derivatives of the functions in the space P_K have singularities at the vertices (Remark 6.1.6), very accurate quadrature schemes are used in practical computations. IRONS & RAZZAQUE (1972b) (see also RAZZAQUE (1973)) obviate this computational difficulty by "smoothing" the second derivatives. Other ways of adding rational functions are mentioned in BIRKHOFF & MANSFIELD (1974) (cf. Exercise 6.1.7), MANSFIELD (1974, 1976b), DUPUIS & GOEL (1970a). Boolean sum interpolation theory can also be used to derive *blending polynomial interpolants*, which interpolate a function $v \in \mathscr{C}^m(K)$ and all its derivatives of order $\leq m$ on the (possibly curved) boundary of a triangle K. In this direction, see BARNHILL (1976a, 1976b), BARNHILL, BIRKHOFF & GORDON (1973), BARNHILL & GREGORY (1975a, 1975b).

For a discussion about the use of finite elements of class $\mathscr{C}^1$ from the engineering viewpoint, see ZIENKIEWICZ (1971, chapter 10). There, finite elements of class $\mathscr{C}^1$ are called "compatible" while finite elements which are not of class $\mathscr{C}^1$ are called "incompatible", and rational functions such as those which are used in the singular Zienkiewicz triangle are called "singular shape functions". The Bogner–Fox–Schmit rectangle is not the only rectangular finite element of class $\mathscr{C}^1$ that may be used in practice. See for example GOPALACHARYULU (1973, 1976).

The general approach followed in Section 6.2 is that of CIARLET (1974a, 1974b). In LASCAUX & LESAINT (1975), a thorough study is made not only of Adini's rectangle, but of other nonconforming finite elements for the plate problem, such as the *Zienkiewicz triangle*, *Morley's triangle* (cf. Exercise 6.2.3) and various instances of *Fraeijs de Veubeke triangles* (an example of which is given in Exercise 6.2.3).

A survey of the use of such nonconforming elements, from an Engineering viewpoint, is found in ZIENKIEWICZ (1971, chapter 10). Adini's rectangle is due to ADINI & CLOUGH (1961) and MELOSH (1963) and, for this reason, it is sometimes called the *ACM rectangle*. The convergence of Adini's rectangle has also been studied by KIKUCHI (1975d, 1976a) and MIYOSHI (1972). KIKUCHI (1975d) considers in addition the use of this element for the approximation of the eigenvalue problem. This last problem is also considered from a numerical standpoint by LINDBERG & OLSON (1970) for conforming and nonconforming finite elements. An extension to the case of *curved nonconforming elements* is considered in BARNHILL & BROWN (1975).

Although some of the references given in Section 4.2 were more specifically concerned with second-order problems, some of them are also relevant in the present situation, notably CÉA (1976), NITSCHE (1974), OLIVEIRA (1976).

There are alternate definitions of nonconforming methods. For example, let us assume that we are given a finite element space V_h which satisfies the inclusion $V_h \subset \mathscr{C}^0(\bar{\Omega}) \cap H_0^1(\Omega)$. Assuming as usual that the functions in the spaces P_K are smooth, the conformity would require the additional conditions that $\partial_\nu(v_{h|K_1}) + \partial_\nu(v_{h|K_2}) = 0$ along any side K' common to two adjacent finite elements K_1 and K_2, and that $\partial_\nu v_h = 0$ along Γ. If these conditions cannot be exactly fulfilled, they may be considered as *constraints*, and accordingly, they may be dealt with either by a *penalty method* or by *duality techniques*.

In the first approach, one minimizes a functional of the form

$$J_h^*(v_h) = \frac{1}{2} a_h(v_h, v_h) - f(v_h) + \frac{1}{\epsilon(h)} \Phi(v_h)$$

where

$$\Phi(v_h) = \sum_{\substack{K_1, K_2 \in \mathscr{T}_h \\ K_1 \neq K_2}} \int_{K_1 \cap K_2} (\partial_\nu(v_{h|K_1}) + \partial_\nu(v_{h|K_2}))^2 \, d\gamma + \int_\Gamma (\partial_\nu v_h)^2 \, d\gamma,$$

and $\epsilon(\cdot)$ is a function of h which approaches zero with h. The function $\epsilon(\cdot)$ is usually of the form $\epsilon(h) = Ch^\sigma$, $C > 0$, where the exponent $\sigma > 0$ is to be chosen so as to maximize the order of convergence. A method of this type has been studied by BABUŠKA & ZLÁMAL (1973) who have shown that the use of the Hermite triangle of type (3) results in the error estimates

$$\|u - u_h\|_h \leq C\sqrt{h}\|u\|_{3,\Omega}, \quad \|u - u_h\|_{1,\Omega} \leq Ch\|u\|_{3,\Omega}$$

if $u \in H^3(\Omega)$, with the optimal choice $\epsilon(h) = Ch^2$, and

$$\|u - u_h\|_h \leq Ch\|u\|_{4,\Omega}, \quad \|u - u_h\|_{1,\Omega} \leq Ch\|u\|_{4,\Omega},$$

if $u \in H^4(\Omega)$, with the optimal choice $\epsilon(h) = Ch^3$ (let us add however that this penalty method is analyzed in the case of the biharmonic problem instead of the plate problem). Such techniques are used in practice: See ZIENKIEWICZ (1974).

The second approach consists in introducing an appropriate Lagrangian. This is done for example by HARVEY & KELSEY (1971) who use the Hermite triangle of type (3) for solving the plate problem.

Let us next review further aspects of the finite element approximation of the plate problem and of more general fourth-order problems. RANNACHER (1976a) has obtained error estimates in the norm $|\cdot|_{0,\infty,\Omega}$. The effect of numerical integration is analyzed in BERNADOU & DUCATEL (1976).

As regards the approximation of fourth-order problems on domains with curved boundaries, we mention MANSFIELD (1976b), who considers in addition the effect of numerical integration. Her approach parallels that given in CIARLET & RAVIART (1972c) for second-order problems. Curved isoparametric finite elements of a new type are suggested by ROBINSON (1973). In the case of the simply supported plate problem (cf. Exercise 1.2.7), we mention the *Babuška paradox* (cf. BABUŠKA (1963); see also BIRKHOFF (1969)): Contrary to second-order problems, no convergent approximation may be found if the curved boundary is replaced by a polygonal domain: This is because the boundary condition $\Delta u - (1-\sigma)\partial_{\tau\tau}u = 0$ on Γ (which is included in the variational formulation) is then replaced by the boundary condition $\partial_{\nu\nu}u = 0$.

Additional references concerning the handling of curved boundaries and/or boundary conditions for the plate problems are NITSCHE (1971, 1972b), CHERNUKA, COWPER, LINDBERG & OLSON (1972), and the survey of SCOTT (1976b).

Finite element approximation of variational inequalities of order four are considered by GLOWINSKI (1975, 1976b). See also GLOWINSKI, LIONS & TRÉMOLIÈRES (1976b, Chapter 4).

When large vertical displacements are considered, the plate problem amounts (cf. LANDAU & LIFSCHITZ (1967, Chapter 2)) to finding a pair $(u_1, u_2) \in (H_0^2(\Omega))^2$, solution of two coupled nonlinear equations, known as *von Karmann's equations*:

$$a_1\Delta^2 u_1 - [u_1, u_2] = f, \quad f \in L^2(\Omega),$$

$$a_2\Delta^2 u_2 + [u_1, u_1] = 0,$$

where

$$[v, w] = \partial_{11}v\partial_{22}w + \partial_{22}v\partial_{11}w - 2\partial_{12}v\partial_{12}w,$$

and a_1, a_2 are two strictly positive constants. The existence of a (possibly non unique) solution is proved in LIONS (1969, p. 53). For an analysis of a finite element approximation by a mixed method, see MIYOSHI (1976a, 1976b, 1976c, 1977). Another finite element method is proposed in BERGAN & CLOUGH (1973) to handle large displacements.

For yet other types of finite element approximation of the plate problem, see ALLMAN (1976), FRIED (1973c), FRIED & YANG (1973), IRONS (1974b), KIKUCHI (1975e), STRICKLIN, HAISLER, TISDALE & GUNDERSON (1969). Plates with cracks have been considered by YAMAMOTO & TOKUDA (1973), and YAMAMOTO & SUMI (1976). Further references are found in the next chapter, specially for the so-called mixed and hybrid methods.