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ON THE STRONG LAW OF LARGE NUMBERS FOR MULTIVARIATE MARTINGALES WITH CONTINUOUS TIME

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We prove the strong law of large numbers for vector martingales with arbitrary operator normalizations. From the theorem proved, we deduce several known results on the strong law of large numbers for martingales with continuous time.

In the present paper, we investigate the strong law of large numbers with matrix normalizations for multivariate martingales with continuous time. Earlier, this problem was considered in [1–5].

We introduce the following notation: R^d is the Euclidean space of column vectors $x = (x_1, x_2, \ldots, x_d)^T$, where T denotes transposition ($R^1 = R$), $R^{q\times d}$ is the space of $q \times d$ matrices, $\operatorname{tr} A$ is the trace of a matrix A, $\|\cdot\|$ is the Euclidean norm of a vector or a matrix, which is clear from the context, a.s. means "almost surely," E denotes mathematical expectation, $\mathfrak{R}(R)$ is the set of all sequences of positive numbers monotonically increasing to infinity, and $\sup\varnothing = 0$.

First, we prove a general theorem. Let $(X_t,\ t \ge 0)$ be a separable random process in R^d defined on a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and let $(A_t,\ t \ge 0)$ be a nonrandom function with values in $R^{q\times d}$.

Theorem 1. *Suppose that a random process* $(A_t X_t,\ t \ge 0)$ *is separable and* $\|A_t\| \to 0,\ t \to \infty$. *If, for any sequence* $(t_n,\ n \ge 1) \in \mathfrak{R}(R)$, *the condition*

$$\sum_{n=1}^{\infty} \mathrm{P}\left(\sup_{t_n < t \le t_{n+1}} \left\| A_{t_{n+1}}\left(X_t - X_{t_n}\right)\right\| > \varepsilon \right) < \infty \tag{1}$$

is satisfied for all $\varepsilon > 0$, *then the strong law of large numbers*

$$\|A_t X_t\| \to 0 \quad a.s., \qquad t \to \infty, \tag{2}$$

is true.

The proof of Theorem 1 is based on the lemma presented below, which follows from Lemma 4.4.2 in [6].

Lemma 1. *Let* $(Y_t,\ t \ge 0)$ *be a separable random process in* R^q. *In order that*

$$\lim_{t\to\infty} \|Y_t\| = 0 \quad a.s.,$$

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it is necessary and sufficient that the following relation be true for any sequence $(t_n,\ n \ge 1) \in \mathfrak{R}(R)$:

$$\mathrm{P}\left(\lim_{n\to\infty} \| Y_{t_n} \| = 0\right) = 1.$$

Proof of Theorem 1. By virtue of Lemma 1, relation (2) holds if and only if, for any $(t_n,\ n \ge 1) \in \mathfrak{R}(R)$, we have

$$\| A_{t_n} X_{t_n} \| \to 0 \quad \text{a.s.}, \qquad n \to \infty. \tag{3}$$

It follows from Corollary 2.1 in [7] that relation (3) is true for a fixed sequence $(t_n,\ n \ge 1)$ if, for any sequence of natural numbers $(n(j),\ j \ge 1)$ monotonically increasing to infinity, the following condition is satisfied for all $\varepsilon > 0$:

$$\sum_{j=1}^{\infty} \mathrm{P}\left(\max_{n(j) < n \le n(j+1)} \left\| A_{t_{n(j+1)}} \left(X_{t_n} - X_{t_{n(j)}} \right) \right\| > \varepsilon \right) < \infty.$$

This condition is obviously satisfied by virtue of (1). Theorem 1 is proved.

In what follows, we assume that all processes (functions) under consideration have trajectories from the space D a.s., i.e., the trajectories are right-continuous and there exist finite limits from the left.

Denote by $(M_t,\ t \ge 0)$ a locally square-integrable martingale in R^d $(M_0 = 0)$ defined on $(\Omega, \mathcal{F}, \mathrm{P})$ with filtration $(\mathcal{F}_t,\ t \ge 0)$ (see, e.g., [8]). The matrix quadratic characteristic of the martingale is denoted by $(\langle M \rangle_t,\ t \ge 0)$. As above, $(A_t,\ t \ge 0)$ is a nonrandom function with values in $R^{q\times d}$ such that $\| A_t \| \to 0$ as $t \to \infty$.

Theorem 2. *If, for any sequence* $(t_n,\ n \ge 1) \in \mathfrak{R}(R)$, *the condition*

$$\sum_{j=1}^{\infty} \operatorname{tr}\left[A_{t_{n+1}} \left(\mathrm{E}\langle M \rangle_{t_{n+1}} - \mathrm{E}\langle M \rangle_{t_n} \right) A_{t_{n+1}}^T \right] < \infty \tag{4}$$

is satisfied, then

$$\| A_t M_t \| \to 0 \quad a.s., \qquad t \to \infty. \tag{5}$$

Proof. For fixed t_n and t_{n+1}, the random process $\left(\hat{M}_t = A_{t_{n+1}}\left(M_t - M_{t_n}\right),\ \mathcal{F}_t,\ t \ge t_n\right)$ is a locally square-integrable martingale in R^q with quadratic characteristic $\left(\langle \hat{M} \rangle_t = A_{t_{n+1}}\left(\langle M \rangle_t - \langle M \rangle_{t_n}\right) A_{t_{n+1}}^T,\ \ t \ge t_n\right)$. Then, for any $\varepsilon > 0$, by using the Doob inequality (see, e.g., [8]), we obtain

$$\mathrm{P}\left(\sup_{t_n < t \le t_{n+1}} \| \hat{M}_t \| > \varepsilon\right) \le \sum_{k=1}^{q} \mathrm{P}\left(\sup_{t_n < t \le t_{n+1}} \left| \hat{M}_t^{(k)} \right| > \varepsilon q^{-1/2}\right) \le q\varepsilon^{-2} \sum_{k=1}^{q} \mathrm{E}\left\langle \hat{M}^{(k)} \right\rangle_{t_{n+1}} = q\varepsilon^{-2}\, \mathrm{E}\operatorname{tr}\left\langle \hat{M} \right\rangle_{t_{n+1}},$$

where $\hat{M}_t^{(k)}$, $k = \overline{1, q}$, are the coordinates of the vector $\hat{M}_t$. This implies that, by virtue of (4), condition (1) is satisfied. Theorem 2 is proved.

Below, we present several corollaries of Theorem 2.
The following result was obtained in [4]:

Corollary 1. *Suppose that the condition* $\langle M \rangle_t = \mathrm{E}\left(M_t M_t^T\right) = B_t$ *is satisfied. Assume that* $B_{t_0} > 0$, *i.e., the matrix* B_{t_0} *is positive definite for certain* $t_0 \geq 0$ *and* $\left\| B_t^{-1} \right\| \to 0$ *as* $t \to \infty$. *Then*

$$\left\| B_t^{-1} M_t \right\| \to 0 \quad a.s., \qquad t \to \infty.$$

Proof. We use the following matrix inequality (see [9]): For arbitrary symmetric positive-definite matrices A and B such that $A \geq B$, the following inequality is true:

$$A^{-1}(A - B)A^{-1} \leq B^{-1} - A^{-1}. \tag{6}$$

By virtue of this inequality, we have

$$\mathrm{tr}\left[B_{t_{n+1}}^{-1}\left(B_{t_{n+1}} - B_{t_n}\right)B_{t_{n+1}}^{-1} \right] \leq \mathrm{tr}\left(B_{t_n}^{-1} - B_{t_{n+1}}^{-1}\right).$$

Since

$$\sum_{n=1}^{\infty} \mathrm{tr}\left(B_{t_n}^{-1} - B_{t_{n+1}}^{-1}\right) = \mathrm{tr} B_{t_1}^{-1} < \infty,$$

condition (4) is satisfied, which proves Corollary 1.

Corollary 2. *If*

$$\int_0^{+\infty} \sup_{s \geq t} \mathrm{tr}\left[A_s d\left(\mathrm{E}\langle M \rangle_t\right) A_s^T \right] < \infty, \tag{7}$$

then relation (5) is true.

Proof. We show that condition (4) follows from condition (7). Indeed,

$$\sum_{n=1}^{\infty} \mathrm{tr}\left[A_{t_{n+1}}\left(\mathrm{E}\langle M\rangle_{t_{n+1}} - \mathrm{E}\langle M\rangle_{t_n}\right)A_{t_{n+1}}^T \right] = \sum_{n=1}^{\infty} \int_{t_n}^{t_{n+1}} \mathrm{tr}\left[A_{t_{n+1}} d\left(\mathrm{E}\langle M\rangle_t\right) A_{t_{n+1}}^T \right]$$

$$\leq \sum_{n=1}^{\infty} \int_{t_n}^{t_{n+1}} \sup_{s \geq t} \mathrm{tr}\left[A_s d\left(\mathrm{E}\langle M\rangle_t\right) A_s^T \right] \leq \int_0^{+\infty} \sup_{s \geq t} \mathrm{tr}\left[A_s d\left(\mathrm{E}\langle M\rangle_t\right) A_s^T \right] < \infty.$$

Corollary 2 is proved.

The following result was obtained in [5]:

Corollary 3. *Suppose that the function* $(A_t,\ t \ge 0)$ *satisfies the condition* $A_s^T A_s \le A_t^T A_t$ *for all* $s \ge t$. *If*

$$\int_0^{+\infty} \operatorname{tr}\left[A_t d(\mathrm{E}\langle M \rangle_t) A_t^T \right] < \infty,$$

then relation (5) is true.

Corollary 3 follows directly from Corollary 2 and the following lemma:

Lemma 2. *If* $A_s^T A_s \le A_t^T A_t$, *then, for any symmetric matrix* $D \ge 0$, *the following inequality is true:*

$$\operatorname{tr}\left(A_s D A_s^T\right) \le \operatorname{tr}\left(A_t D A_t^T\right).$$

Proof. Denote by $D^{1/2}$ the square root of a matrix D. By virtue of the conditions of Lemma 2, we have

$$\operatorname{tr}\left(D^{1/2} A_s^T A_s D^{1/2}\right) \le \operatorname{tr}\left(D^{1/2} A_t^T A_t D^{1/2}\right),$$

or

$$\operatorname{tr}\left[D^{1/2} A_s^T \left(D^{1/2} A_s^T\right)^T \right] \le \operatorname{tr}\left[D^{1/2} A_t^T \left(D^{1/2} A_t^T\right)^T \right].$$

This yields

$$\operatorname{tr}\left[\left(D^{1/2} A_s^T\right)^T D^{1/2} A_s^T \right] \le \operatorname{tr}\left[\left(D^{1/2} A_t^T\right)^T D^{1/2} A_t^T \right].$$

The statement of Lemma 2 follows from the last inequality.

Remark 1. The condition $A_s^T A_s \le A_t^T A_t$ is equivalent to the condition $\| A_s x \| \le \| A_t x \| \quad \forall x \in R^d$, which was introduced in [10] in the course of investigation of the strong law of large numbers for multivariate martingales with discrete time.

The following result was obtained in [1]:

Corollary 4. *Suppose that a function* $(A_t,\ t \ge 0)$ *with values in* $R^{d \times d}$ *satisfies the following conditions:*

(i) *for all* $t \ge 0$, *the matrices* A_t *are symmetric and* $A_t > 0$;

(ii) $A_s \le A_t$ *for all* $s \ge t$;

(iii) $c = \sup\limits_{t \ge 0} [\lambda_{\max}(A_t) / \lambda_{\min}(A_t)] < \infty$, *where* $\lambda_{\max}(\cdot)$ *is the maximum eigenvalue of a matrix and* $\lambda_{\min}(\cdot)$ *is its minimum eigenvalue.*

If

$$\int_0^{+\infty} \operatorname{tr}[A_t d(\mathrm{E}\langle M \rangle_t) A_t] < \infty,$$

then relation (5) is true.

Corollary 4 follows directly from Corollary 2 and the following lemma:

Lemma 3. *Suppose that conditions (i)–(iii) of Corollary 4 are satisfied. Then, for all* $s \ge t$ *and any symmetric matrix* $D \ge 0$, *the following inequality is true:*

$$\operatorname{tr}(A_s D A_s) \le d c^2 \operatorname{tr}(A_t D A_t).$$

Proof. We have

$$\begin{aligned}\operatorname{tr}(A_s D A_s) &\le d\lambda_{\max}\left(A_s A_t^{-1} (A_t D A_t) A_t^{-1} A_s\right) \\ &\le d\lambda_{\max}^2 (A_s A_t^{-1}) \lambda_{\max}(A_t D A_t) \le d\lambda_{\max}^2 (A_s) \lambda_{\min}^{-2}(A_t) \operatorname{tr}(A_t D A_t) \\ &\le d\lambda_{\max}^2 (A_t) \lambda_{\min}^{-2}(A_t) \operatorname{tr}(A_t D A_t) \le d c^2 \operatorname{tr}(A_t D A_t).\end{aligned}$$

Lemma 3 is proved.

Finally, consider the application of Theorem 2 to the investigation of the strong consistency of the least-squares estimators of unknown parameters in a multivariate linear regression. Assume that the following random process is observed in R^q (see Sec. 4.3 in [2]):

$$Y_t = \int_0^t D_s^T da_s \theta + X_t, \qquad t \ge 0,$$

where $(X_t,\ t \ge 0)$ is a locally square-integrable martingale in R^q whose quadratic characteristic $\langle X \rangle_t$ is a deterministic function, $(D_t,\ t \ge 0)$ is a nonrandom function with values in $R^{d \times q}$, $(a_t,\ t \ge 0)$ is a monotonically nondecreasing function with values in R, and θ is an unknown parameter. Assume that the density

$$Q_t = \frac{d\langle X \rangle_t}{da_t}$$

exists and, moreover, $Q_t > 0$. We set $B_t = \int_0^t D_s Q_s^{-1} D_s^T \, da_s$ and assume that $B_t > 0$, $t > 0$. As an estimator of the parameter θ, we consider the following statistic:

$$\hat{\theta}_t = B_t^{-1} \int_0^t D_s Q_s^{-1} dY_s.$$

The following result was obtained in [2]:

Corollary 5. *If* $\lambda_{\min}(B_t) \to \infty$ *as* $t \to \infty$, *then*

$$\left\| \hat{\theta}_t - \theta \right\| \to 0 \quad a.s., \qquad t \to \infty. \tag{8}$$

Proof. Since

$$\hat{\theta}_t = \theta + B_t^{-1} \int_0^t D_s Q_s^{-1} dX_s,$$

to prove relation (8) it is necessary to show that

$$\left\| B_t^{-1} \int_0^t D_s Q_s^{-1} dX_s \right\| \to 0 \quad \text{a.s.}, \qquad t \to \infty.$$

It follows from the condition $\lambda_{\min}(B_t) \to \infty$ that $\| B_t^{-1} \| \to 0$, $t \to \infty$. Furthermore, the random process $\left(M_t = \int_0^t D_s Q_s^{-1} dX_s, \ t \geq 0 \right)$ is a locally square-integrable martingale with the deterministic quadratic characteristic

$$\langle M \rangle_t = \int_0^t D_s Q_s^{-1} d\langle X \rangle_s Q_s^{-1} D_s^T = \int_0^t D_s Q_s^{-1} D_s^T da_s = B_t.$$

By using inequality (6), we establish that condition (4) is satisfied.

Corollary 5 is proved.

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