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Lars B. Wahlbin

Superconvergence in Galerkin Finite Element Methods

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Preface.

These notes are from a graduate seminar at Cornell in Spring 1994. They are devoted mainly to basic concepts of superconvergence in second–order time–independent elliptic problems.

A brief chapter–by–chapter description is as follows: Chapter 1 considers one–dimensional problems and is intended to get us moving quickly into the subject matter of superconvergence. (The results in Sections 1.8 and 1.10 are new.) Some standard results and techniques used there are then expounded on in Chapters 2 and 3. Chapter 4 gives a few selected results about superconvergence in L_2–projections in any number of space dimensions. In Chapter 5 we elucidate local maximum–norm error estimates in second order elliptic partial differential equations and the techniques used in proving them, without aiming for complete detail. Theorems 5.5.1 and 5.5.2 are basic technical results; they will be used over and over again in the rest of the notes.

In Chapters 6 through 12 we treat a variety of topics in superconvergence for second order elliptic problems. Some are old and established, some are very recent and not yet published. Some of the earlier contributions have benefitted from later sharpening of tools, in particular with respect to local maximum–norm estimates.

In Chapter 6 we consider tensor–product elements. Using ideas of [Douglas, Dupont and Wheeler 1974b] we show that in some situations one–dimensional superconvergence results automatically translate to several dimensions. Chapter 7, "Superconvergence by local symmetry", presents recent fundamental results from [Schatz, Sloan and Wahlbin 1994]. Chapter 8 treats difference quotients for approximating derivatives of any order on translation invariant meshes. Here we follow the basic ideas of [Nitsche and Schatz 1974, Section 6]. In Chapter 9 we briefly comment on how, in many cases, results about superconvergence in linear problems automatically carry over to nonlinear problems. The essential idea is from [Douglas and Dupont 1975], which in turn is essentially the quadratic convergence of Newton's method. Chapter 10 is concerned with superconvergence on curved meshes which come about via isoparametric mappings of straight–lined meshes; it is based on [Cayco, Schatz and Wahlbin 1994]. Chapter 11 reverts back to the seventies. It is mainly concerned with the K–operator of [Bramble and Schatz 1974]; the presentation follows [Thomée 1977]. We give an application to boundary integral equations, [Tran 1993]. Also, we briefly mention a method for obtaining higher order accuracy in outflow derivatives, [Douglas, Dupont and Wheeler 1974a], and an averaging method of [Louis 1979]. Finally, in Chapter 12, we review the computational investigation of [Babuška, Strouboulis, Upadhyay and Gangaraj 1993] and comment on it in light of the theories of Chapters 6 and 7.

Previous treatises of superconvergence are [Chen 1982a] and [Zhu and Lin 1989]. Both are in Chinese and, although there is a description of Zhu and Lin's book in Mathematical Reviews, it is hard for me to judge how they compare with the present account. The use of local maximum–norm estimates seems common to all three. *For what appears to be a major difference of approach, see Remark 7.4.i.*

Surveys of superconvergence with a more limited scope have appeared in [Křížek and Neittaanmäki 1987a] and [Wahlbin 1991, Chapter VII].

Let me next list some topics that are *not* included in these notes. The first is superconvergence in collocation finite element methods for differential equations. For this topic I have not even included references; the reader is referred to [Křížek and Neittaanmäki 1987a]. The second omitted topic is superconvergence in boundary integral methods or in integral equations (except for Section 11.5). Third, extrapolation methods, involving computation on two or more meshes, and fourth, the use of superconvergence in construction of smooth stress fields and the related use of superconvergence for a posteriori error estimation and adaptive refinement. For the second through fourth topics I have included a fair number of references so that the interested reader may, by glancing through the list of references, easily gain an inroad to the literature on the subjects. Finally, I have not considered superconvergence in Galerkin finite element methods for time-dependent problems. Here, though, I have included all references that I know of.

The references are likewise "complete" with respect to the mathematical literature for the main topics treated. Of course, many of these references touch only briefly on superconvergence. In preparing the references I have used Mathsci ($\simeq$ Mathematical Reviews since 1972 on line) which does not systematically cover the vast engineering literature. The number of references given is more than it makes sense to actually refer to in the text, unless one resorts to plain listing, which I have not done. I hope that nevertheless some readers will find the list of references useful. (As an example, if a reader interested in early history wants to find references to papers on superconvergence in finite element methods before 1970, our list gives: [Stricklin 1966], [Filho 1968], [Stricklin 1968], [Oganesyan and Rukhovetz 1969] and [Tong 1969].)

Basic discoveries continue to be made at present. Furthermore, the theory of superconvergence is very immature in carrying results up close to boundaries (or, internal lines of discontinuity). Today, such investigations are carried out on a case–by–case basis, and, of course, not all results hold all the way up to boundaries (see Section 1.7). Most results that are proven up to boundaries in several dimension pertain to axes–parallel parallelepipedes, or, locally, to straight boundaries. For these and other reasons I have decided to offer these notes essentially as they were written week–by–week during the seminar, rather than rework them into "text–book" form; such a textbook would most likely be out–of–date when it appears. The reader may be warned that, reflecting my lecturing style, proofs often appear before theorems; indeed, the "theorem" may be only an informal statement. Also, true to the principle that repetition is the mother of studies, there is a fair amount of such. E.g., symmetry considerations are first met with in two–point boundary value problems, then in L_2–projections and finally in multidimensional elliptic problems; difference quotients first ocur in two–point boundary value problems and later in many dimensions; and, tensor product elements are considered as well for L_2–projections as for elliptic problems.

Let me penultimately remark that there are three concepts which may be confused with superconvergence. "Supraconvergence" is a concept in finite–difference theory for irregular meshes, cf. [Kreiss, Manteuffel, Swartz, Wendroff and White 1986], [Manteuffel and White 1986], [Heinrich 1987, p. 107], and cf. also [Bramble 1970]. "Superapproximation" will be explained in these notes and "superlinear convergence" occurs in the theory of iterative methods, cf. e.g. [Ortega and Rheinboldt 1979, pp. 285 and 291].

Finally, I thank Arletta Havlik for her superior typing, Al Schatz for many stimulating discussions, Stig Larsson for timely help and the National Science Foundation for financial support.

Ithaca, January 1995
Lars B. Wahlbin

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Chapter 1. Some one–dimensional superconvergence results.

We shall move briskly into the subject of superconvergence. In Chapters 2 and 3 we shall then backtrack and elucidate some of the techniques and results used in this chapter.

1.1. Introduction.

Consider the following two–point boundary value problem: Find $u = u(x)$ such that

$$\text{(1.1.1)} \qquad \begin{cases} -(a_2(x)u')' - (a_1(x)u)' + a_0(x)u = f \ \text{ in } \ I = (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

We shall assume throughout this chapter that the coefficients a_i and the right hand side f are as smooth as necessary on $\bar{I}$ for our analyses to carry through. Furthermore, we demand that

$$\text{(1.1.2)} \qquad a_2(x) \geq \underline{a}_2 > 0, \ \text{for all } x \in \bar{I}.$$

The weak formulation of (1.1.1) is to find $u \in \overset{\circ}{H}{}^1(I)$ such that

$$\text{(1.1.3)} \qquad A(u,\chi) \equiv \int_I (a_2 u'\chi' + a_1 u\chi' + a_0 u\chi)dx = \int_I f\chi dx, \ \text{for all } \chi \in \overset{\circ}{H}{}^1(I).$$

Let T_h, $0 < h < 1/2$, be a sequence of subdivisions of $\bar{I}$,

$$\text{(1.1.4)} \qquad T_h = \{x_i\}_{i=0}^{N(h)}, \ 0 = x_0 < x_1 < \cdots < x_N = 1,$$
$$I_i = (x_i, x_{i+1}), \ \bar{I} = \bigcup_{i=0}^{N(h)-1} \bar{I}_i.$$

Let $h_i = x_{i+1} - x_i$. With, perhaps, a slight abuse of notation, we shall let $h = \max_i h_i$. With integers $0 \leq \mu < r - 1$, we set

$$\text{(1.1.5)} \qquad \overset{\circ}{S}_h = \overset{\circ}{S}{}_h^{\mu,r} = \{\chi(x) : \chi \in C^\mu(I) \cap C^0(\bar{I}), \chi(0) = \chi(1) = 0,$$
$$\chi\big|_{I_i} \in \Pi_{r-1}(I_i)\}$$

where $\Pi_{r-1}(I_i)$ denotes the polynomials of degree $\leq r-1$ on I_i. When no confusion can arise we use the shorter notation $\overset{\circ}{S}_h$. Note that $\overset{\circ}{S}_h \subseteq \overset{\circ}{H}{}^1(I)$ since $\mu \geq 0$.

Well known examples are:

$\mu = 0, \ r = 2:$ piecewise linears

$\mu = 1, \ r = 4:$ Hermite cubics

$\mu = r - 2:$ smoothest splines.

The finite element solution is sought as follows: Find $u_h \in \overset{\circ}{S}_h$ such that

$$\text{(1.1.6)} \qquad A(u_h, \chi) = (f, \chi), \ \text{for all } \chi \in \overset{\circ}{S}_h.$$

(We shall not take into account numerical integration in (1.1.6).)

In our analyses below we shall need to use various results from the literature. These can be found with varying degrees of generality, in particular with respect to dependence on the distribution of mesh sizes h_i in a subdivision. To keep things simple in this chapter we make the following explicit assumption of quasi-uniformity of meshes, except in the case of continuous elements ($\mu = 0$) when no such restriction is made.

(1.1.7) If $\mu \geq 1$, then there exists a positive constant C_{QU} independent of h such that in any subdivision τ_h, we have $h \leq C_{QU} \min_i h_i$.

We shall assume that (1.1.3) has a unique solution u for any f in $L_2(I)$, say. Following [Schatz 1974], cf. also [Hildebrandt and Weinholz 1964] and [Schatz and Wang 1994], we then know that there exists $h_0 > 0$ such that for $h \leq h_0$, given any $f \in L_2(I)$ there is a unique solution u_h to (1.1.6). Furthermore, under our general conditions on smooth enough data, we have the following estimates for the error $e = u - u_h$:

$$\|e\|_{L_2(I)} + h\|e\|_{H^1(I)} \leq Ch^r \|u\|_{H^r(I)} \tag{1.1.8}$$

and

$$\|e\|_{H^{-s}(I)} \leq Ch^{r+s}\|u\|_{H^r(I)}, \text{ for } s \leq r-2. \tag{1.1.9}$$

Here

$$\|v\|_{H^{-s}(I)} = \sup_{\substack{w \in H^s(I) \\ \|w\|_{H^s(I)}=1}} (v, w). \tag{1.1.10}$$

Correspondingly we have in maximum–norm,

$$\|e\|_{L_\infty(I)} + h\|e\|_{W^1_\infty(I)} \leq Ch^r \|u\|_{W^r_\infty(I)}, \tag{1.1.11}$$

and

$$\|e\|_{W^{-s}_\infty(I)} \leq Ch^{r+s}\|u\|_{W^r_\infty(I)}, \text{ for } s \leq r-2, \tag{1.1.12}$$

where now

$$\|v\|_{W^{-s}_\infty(I)} = \sup_{\substack{w \in W^s_1(I) \\ \|w\|_{W^s_1(I)}=1}} (v, w). \tag{1.1.13}$$

The constants C occurring are independent of h and u. They do depend on the coefficients a_i and on μ and r; in the case of $\mu \geq 1$ they may also depend on C_{QU} in (1.1.7).

The $\overset{\circ}{H}{}^1$–estimate in (1.1.8) can be found in this generality in [Schatz 1974], after use of standard approximation theory. The L_2–estimate and the negative norm estimates (1.1.9) follow by standard duality arguments, cf. (2.2.4) below. The L_∞–estimate (1.1.11) for $\mu = 0$ is in [Wheeler, M. F. 1973]. The case of general μ, with the quasi–uniformity condition (1.1.7), is in [Douglas, Dupont and Wahlbin 1975]. From this the W^1_∞–estimate follows in the quasi–uniform case. The W^1_∞–estimate in the case $\mu = 0$, although "well–known folklore", we have not been able to locate without restrictions on the meshes. We will therefore give a proof, in Remark 1.3.2 below. The negative norm estimates in (1.1.12) follow in a standard duality fashion from (1.1.11).

It is not hard to convince one–self that the powers of h occurring in (1.1.11) are the best possible in general. E.g.,

$$
\begin{aligned}
(1.1.14)\qquad &\min_{\chi\in\Pi_{r-1}(0,h)} \|x^r-\chi\|_{L_\infty(0,h)}\\
&= h^r \min_{\chi\in\Pi_{r-1}(0,h)} \left\|\left(\frac{x}{h}\right)^r - \frac{1}{h^r}\chi(x)\right\|_{L_\infty(0,h)}\\
&= h^r \min_{\chi\in\Pi_{r-1}(0,h)} \left\|\xi^r - \frac{1}{h^r}\chi(h\xi)\right\|_{L_\infty(0,1)}\\
&= h^r \min_{\widetilde{\chi}\in\Pi_{r-1}(0,1)} \|\xi^r - \widetilde{\chi}(\xi)\|_{L_\infty(0,1)} = Ch^r.
\end{aligned}
$$

A superconvergent "point" for function values of order σ is now a family of points $\xi=\xi(h)$ such that

$$(1.1.15)\qquad |e(\xi)| \le Ch^{r+\sigma},$$

where $\sigma>0$ and $C=C(u,a_2,a_1,a_0)$ (and possibly also depending on G_{QU} in (1.1.7)). Similarly, $\eta=\eta(h)$ is superconvergent of order σ for first derivatives if

$$(1.1.16)\qquad |e'(\eta)| \le Ch^{r-1+\sigma},$$

with $\sigma>0$.

In principle, we could talk about superconvergence points for a particular problem or for a particular solution. Generally what we have in mind, though, is some class of problems with (locally) smooth coefficients and solutions, and we then wish to determine points which are superconvergent for the whole class.

Furthermore, we point out that what is described in (1.1.15) and (1.1.16) is only so–called "natural" superconvergence. That is, u_h (or u_h') is simply evaluated at the point ξ (or η) and then compared with $u(\xi)$ (or $u'(\eta)$). In these notes we shall also see many examples of superconvergence involving postprocessing of u_h (trivial, or not so trivial). In fact, in terms of implementation on a computer, many postprocessing methods are simpler to implement than is evaluating u_h at a (non–node) point.

1.2. Nodal superconvergence for function values in continuous elements ($\mu=0$).

We shall give the argument of [Douglas and Dupont 1974]. Let $G(\xi;\cdot)$ be the Green's function for (1.1.1), cf. e.g. [Birkhoff and Rota 1969, Theorem 10, p.52–53], so that

$$(1.2.1)\qquad e(\xi) = A(e, G(\xi;\cdot)).$$

From (1.1.3) and (1.1.6), $A(e,\chi)=0$, for $\chi\in \overset{\circ}{S}_h$ and thus

$$(1.2.2)\qquad e(\xi) = A(e, G(\xi;\cdot)-\chi),\ \text{ for any } \chi\in \overset{\circ}{S}_h.$$

Let now $\xi=x_i$, a meshpoint. Since $G(\xi;\cdot)$ is continuous and (uniformly in ξ) smooth on both sides of $x=\xi$, and since $\mu=0$, we have from standard approximation theory that, for a suitable χ,

$$(1.2.3)\qquad \|G(\xi;\cdot)-\chi\|_{H^1(I)} \le Ch^{r-1}.$$

Since by (1.1.8) also $\|e\|_{H^1(I)} \le Ch^{r-1}\|u\|_{H^r(I)}$ we thus obtain from (1.2.2),

$$|e(x_i)| \le Ch^{2r-2}\|u\|_{H^r(I)}, \tag{1.2.4}$$

for x_i a meshpoint. Thus, provided $r \ge 3$ (i.e., continuous piecewise quadratics or higher elements are used), we have superconvergence at the knots. There is no restriction on the geometries of the meshes allowed.

We state the above as a theorem.

Theorem 1.2.1. *Under the assumptions of Section 1.1, for $\mu = 0$ (with no mesh restrictions),*

$$|(u - u_h)(x_i)| \le Ch^{2r-2}\|u\|_{H^r(I)}. \tag{1.2.5}$$

Let us mention that [Douglas and Dupont 1974] gives an explicit example (with $a_2 \in C^{r-1}(I)$ only, though) showing that the power h^{2r-2} is sharp.

1.3. Reduction to a model problem.

In Sections 1.5–1.11 we shall investigate superconvergence of, typically, order one (i.e., $\sigma = 1$ in (1.1.15) or (1.1.16)). It turns out to be convenient to reduce the investigations to the case $a_2 \equiv 1$, $a_1 = a_0 \equiv 0$. The argument follows [Wahlbin 1992] which in turn was based on [Douglas, Dupont and Wahlbin 1975]. Let thus

$$A(u - u_h, \chi) = 0, \text{ for } \chi \in \overset{\circ}{S}_h, \tag{1.3.1}$$

(cf., (1.1.3) and (1.1.6)) and let $\widetilde{u}_h \in \overset{\circ}{S}_h$ be another approximation to u given by

$$(u' - \widetilde{u}_h', \chi') = 0, \text{ for } \chi \in \overset{\circ}{S}_h. \tag{1.3.2}$$

Then, with $\theta = \widetilde{u}_h - u_h \in \overset{\circ}{S}_h$,

$$\begin{aligned}(\theta', \chi') &= ((\widetilde{u}_h - u)', \chi') - ((u_h - u)', \chi') \\ &= -\Big(a_2(u_h - u)', \frac{1}{a_2}\chi'\Big) \\ &= -\Big(a_2(u_h - u)', \Big(\frac{1}{a_2}\chi\Big)'\Big) + \Big(u_h - u, \Big(\frac{a_2'}{a_2}\chi\Big)'\Big) \\ &= -\Big(a_2(u_h - u)', \Big(\frac{1}{a_2}\chi\Big)' - \psi'\Big) + \Big(a_1(u_h - u), \psi'\Big) \\ &\quad + \Big(a_0(u_h - u), \psi\Big) + \Big(u_h - u, \Big(\frac{a_2'}{a_2}\chi\Big)'\Big), \text{ for } \chi, \psi \in \overset{\circ}{S}_h.\end{aligned} \tag{1.3.3}$$

Let us now consider $(\frac{1}{a_2}\chi)' - \psi'$, and let us treat in detail the continuous case ($\mu = 0$). Let then ψ be the natural (Lagrange) interpolant (at x_i, x_{i+1}; and the appropriate number of equispaced points in the interior of I_i) to $\frac{1}{a_2}\chi$. By standard approximation theory,

$$\Big\|\Big(\frac{1}{a_2}\chi\Big)' - \psi'\Big\|_{L_1(I_i)} \le Ch_i^{r-1}\Big\|\Big(\frac{1}{a_2}\chi\Big)^{(r)}\Big\|_{L_1(I_i)}, \tag{1.3.4}$$

and since $\chi^{(r)} \equiv 0$ we have by Leibniz' formula and inverse estimates (cf. Section 2.1),

$$\left\| \left(\frac{1}{a_2} \chi \right)' - \psi' \right\|_{L_1(I_i)} \leq C h_i \|\chi\|_{W_1^1(I_i)}, \tag{1.3.5}$$

where C depends on a_2. Thus, summing over all intervals,

$$\left\| \left(\frac{1}{a_2} \chi \right)' - \psi' \right\|_{L_1(I)} \leq C h \|\chi\|_{W_1^1(I)}. \tag{1.3.6}$$

This result is a special case of "superapproximation", cf. [Nitsche and Schatz 1974]. Employing a quasi–interpolant à la [deBoor and Fix 1973] (see e.g. [Wahlbin 1991, Lemma 3.2, p.367]), (1.3.6) is true for any $\overset{\circ}{S}_h$ considered by us, cf. (1.1.7). We shall give some detail on this in Section 2.3 below. From (1.3.3) we now have using also (1.1.11),

$$|(\theta', \chi')| \leq C h^r \|u\|_{W_\infty^r(I)} \|\chi\|_{W_1^1(I)}. \tag{1.3.7}$$

We proceed to estimate $\|\theta'\|_{L_\infty(I)}$. We have

$$\|\theta'\|_{L_\infty(I)} = \sup_{\|\psi\|_{L_1(I)}=1} (\theta', \psi). \tag{1.3.8}$$

Let

$$\$_h = \$_h^{\mu-1,r-1} = \left\{ \chi(x) : \chi \in C^{\mu-1}(I); \chi\big|_{I_i} \in \Pi_{r-2} \right\}, \tag{1.3.9}$$

and let P denote the L_2–projection into $\$_h$. For each ψ occurring in (1.3.8) we have

$$(\theta', \psi) = (\theta', P\psi) \tag{1.3.10}$$

since $\theta' \in \$_h$. Now set

$$\phi(x) = \int_0^x P\psi(y)dy - xMV(P\psi) \tag{1.3.11}$$

where $MV(P\psi) = \int_0^1 P\psi dy$. Then $\phi \in \overset{\circ}{S}_h$, and $\phi' = P\psi - MV(P\psi)$. Since $(\theta', 1) = 0$ we have from (1.3.10) and (1.3.7)

$$|(\theta', \psi)| = |(\theta', \phi')| \leq C h^r \|u\|_{W_\infty^r(I)} \|\phi\|_{W_1^1(I)}. \tag{1.3.12}$$

Since $\|\phi\|_{W_1^1(I)} \leq C \|P\psi\|_{L_1(I)}$ and the L_2–projection into $\$_h$ is bounded in L_1 (easy if $\mu = 0$ so that P is completely local; for the cases $\mu \geq 1$ see [Douglas, Dupont and Wahlbin 1975] or [Wahlbin 1991, Lemma 3.5] or, Theorem 3.2.3 below) we thus have $\|\phi\|_{W_1^1(I)} \leq C \|\psi\|_{L_1(I)} = C$. Hence

$$\|\theta'\|_{L_\infty(I)} \leq C h^r \|u\|_{W_\infty^r(I)}. \tag{1.3.13}$$

We state this result as a theorem.

Theorem 1.3.1. *Let u_h be the projection into $\overset{\circ}{S}_h$ of u based on the form A, which satisfies the assumptions of Section 1.1. Let $\widetilde{u}_h \in \overset{\circ}{S}_h$ be given by $(\widetilde{u}_h' - u', \chi') = 0$, for $\chi \in \overset{\circ}{S}_h$. For $\mu \geq 1$ assume quasiuniformity as in (1.1.7). Then*

$$\|(u_h - \widetilde{u}_h)'\|_{L_\infty(I)} \leq C h^r \|u\|_{W_\infty^r(I)}. \tag{1.3.14}$$

We shall next prove an analogue of this for function values.

Theorem 1.3.2. *With assumptions as in Theorem 1.3.1 and, in addition, $r \geq 3$, we have*

$$\|u_h - \widetilde{u}_h\|_{L_\infty(I)} \leq Ch^{r+1}\|u\|_{W^r_\infty(I)}. \tag{1.3.15}$$

Proof: In the case $\mu = 0$ we have from Theorem 1.2.1 that $\theta(x_i) \equiv (u_h - \widetilde{u}_h)(x_i) = 0(h^{2r-2}) \leq Ch^{r+1}$. For $x \in I_i$, $\theta(x) = \theta(x_i) + \int_{x_i}^{x} \theta'(y)dy$ and (1.3.15) follows in this case from Theorem 1.3.1.

For the general case of $\mu \geq 1$ we write

$$\|\theta\|_{L_\infty(I)} = \sup_{\|v\|_{L_1(I)}=1} (\theta, v). \tag{1.3.16}$$

For each such v, let $-w'' = v$ in I, $w(0) = w(1) = 0$. Then

$$(\theta, v) = (\theta', w') = (\theta', P(w')) \tag{1.3.17}$$

where P is the L_2–projection into $\$_h = \$_h^{\mu-1,r-1}$. Since $MV(P(w')) = MV(w') = 0$, $Q = \int_0^x P(w') \in \overset{\circ}{S}_h$ with $Q' = P(w')$ so that from (1.3.3), with $e = u_h - u$,

$$\begin{aligned}(\theta, v) &= (\theta', Q') \\ &= -\left(a_2 e', \left(\frac{1}{a_2}Q\right)' - \psi'\right) + (a_1 e, \psi') + (a_0 e, \psi) \\ &\qquad + \left(e, \left(\frac{a_2'}{a_2}Q\right)'\right) \\ &\equiv I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{1.3.18}$$

Integrating by parts, and using (1.1.11) and superapproximation,

$$\begin{aligned}|I_1| &= \left|\left(e, \left(a_2\left(\frac{1}{a_2}Q\right)' - \psi'\right)'\right)\right| \\ &\leq C\|e\|_{L_\infty(I)}\left\|\left(\frac{1}{a_2}Q\right)' - \psi'\right\|_{W_1^1(I)} \\ &\leq Ch^{r+1}\|Q\|_{W_1^2(I)}.\end{aligned} \tag{1.3.19}$$

Similarly, using the W_∞^{-1} error estimate of (1.1.12),

$$|I_2 + I_3 + I_4| \leq Ch^{r+1}\|Q\|_{W_1^2(I)}. \tag{1.3.20}$$

Under our quasi–uniformly assumption (1.1.7) it is easy to see that, since P is stable in L_1, it is also stable in $W_1^1(I)$. Thus $\|Q\|_{W_1^2(I)} \leq C\|w\|_{W_1^2(I)} \leq C\|v\|_{L_1} = C$. It follows from this and (1.3.16)–(1.3.18) that

$$\|\theta\|_{L_\infty(I)} \leq Ch^{r+1}. \tag{1.3.21}$$

This completes the proof of the theorem. □

Remark 1.3.1. In various special cases, better results can be had. E.g., if $r \geq 3$, if $a_2 \equiv 1$ and $a_1 \equiv 0$, then (1.3.14) in Theorem 1.3.1 may be replaced by

$$\|(u_h - \widetilde{u}_h)'\|_{L_\infty(I)} \leq Ch^{r+1}\|u\|_{W^r_\infty(I)}.$$

This is easily seen from the proof. Similarly we then have Ch^{r+2} in Theorem 1.3.2 for $r \geq 4$. $\square$

Remark 1.3.2. In the case of $\mu = 0$ we will now briefly point out how the W^1_∞ error estimate in (1.1.11) follows without any mesh restrictions. With $e = u_h - u$ we have as in (1.3.3)

$$\begin{aligned}(\theta', \chi') &= -\Big(a_2 e', \frac{1}{a_2}\chi'\Big) = -\Big(a_2 e', \Big(\frac{1}{a_2}\chi\Big)'\Big) + \Big(e', \frac{a_2'}{a_2}\chi\Big) \\ &= -\Big(a_2 e', \Big(\frac{1}{a_2}\chi\Big)' - \psi'\Big) + (a_1 e, \psi') + (a_0 e, \psi) + \Big(e', \frac{a_2'}{a_2}\chi\Big) \\ &\equiv I_1 + I_2 + I_3 + I_4, \ \text{ for } \chi, \psi \in \overset{\circ}{S}_h.\end{aligned}$$

Using the H^1 estimate (1.1.10),

$$|I_1| \leq Ch^{r-1}\|u\|_{H^r(I)} \Big\| \Big(\frac{1}{a_2}\chi\Big)' - \psi' \Big\|_{L_2(I)}.$$

By the L_2 analogue of the superapproximation result (1.3.6), and using inverse estimates, cf. Section 2.1 below (recall that $\mu = 0$ now),

$$\Big\| \Big(\frac{1}{a_2}\chi\Big)' \Big\|^2_{L_2(I_i)} \leq C^2 h_i^2 \|\chi\|^2_{W_2^1(I_i)} \leq C^2 h_i \|\chi\|^2_{W_1^1(I_i)} \leq C^2 h_i \|\chi\|^2_{W_1^1(I)}.$$

Thus,

$$\Big\| \Big(\frac{1}{a_2}\chi\Big)' - \psi' \Big\|^2_{L_2(I)} \leq C^2 \Big(\sum_i h_i\Big) \|\chi\|^2_{W_1^1(I)} \leq C^2 \|\chi\|^2_{W_1^1(I)},$$

and hence,

$$|I_1| \leq Ch^{r-1}\|u\|_{H^r(I)} \|\chi\|_{W_1^1(I)}.$$

Similarly, integrating by parts and using also a Sobolev inequality in one space dimension,

$$\begin{aligned}|I_2| &= |-((a_1 e)', \psi)| \leq C\|e\|_{H^1(I)}\|\psi\|_{L_2(I)} \leq Ch^{r-1}\|u\|_{H^r(I)}\|\psi\|_{L_2(I)} \\ &\leq Ch^{r-1}\|u\|_{H^r(I)}\|\psi\|_{W_1^1(I)} \leq Ch^{r-1}\|u\|_{H^r(I)}\|\chi\|_{W_1^1(I)}.\end{aligned}$$

One treats the two remaining terms I_3 and I_4 in a similar manner and arrives at

$$|(\theta', \chi')| \leq Ch^{r-1}\|u\|_{H^r(I)}\|\chi\|_{W_1^1(I)}, \ \text{ for } \chi \in \overset{\circ}{S}_h.$$

Note that we have only used the H^1-error estimates for $u_h - u$ in obtaining this result. These are well-known to hold for $\mu = 0$ without mesh restrictions by use of Lagrange interpolation on each mesh interval (cf. [Schatz 1974]). As in (1.3.8)–(1.3.13) this leads to

$$\|\theta'\|_{L_\infty} \leq Ch^{r-1}\|u\|_{H^r(I)} \leq Ch^{r-1}\|u\|_{W^r_\infty(I)}.$$

Thus, to conclude the W^1_∞-error estimate, it now suffices to consider the error in $(u - \tilde{u}_h)'$. But $\tilde{u}_h'$ is the L_2-projection of u' into the now discontinuous space $\$_h = \$_h^{-1,r-1}$ and as such $\tilde{u}_h'$ is completely determined on each I_i by $u'\big|_{I_i}$. It follows easily that this L_2-projection is bounded in L_∞ without any mesh restrictions. So, $\|(u - \tilde{u}_h)'\|_{L_\infty(I)} \leq C \min_{\chi \in \$_h} \|u' - \chi\|_{L_\infty(I)} \leq Ch^{r-1}\|u\|_{W^r_\infty(I)}$. Thus the W^1_∞ error estimate of (1.1.11) obtains without any mesh restriction in the case $\mu = 0$. $\square$

1.4. Existence of superconvergence points in general.

In this section we shall consider only $\widetilde{u}_h \in \overset{\circ}{S}_h$ given by

$$(1.4.1) \qquad \big((u-\widetilde{u}_h)', \chi'\big) = 0, \text{ for } \chi \in \overset{\circ}{S}_h.$$

No mesh restrictions at all will be involved in this section, even for $\mu \geq 1$. The results may be combined with those of Section 1.3 which, however, do involve mesh restrictions for $\mu \geq 1$.

Our first result concerns derivatives. Let $[\sigma]^+$ denote the smallest integer $\geq \sigma$.

Theorem 1.4.1. *Let $k_d = \left[\frac{r-1}{r-1-\mu}\right]^+$, and let $\mathcal{J}_i = (x_i, x_{i+k_d})$, for any $i = 0, 1, \ldots, N-k_d$. There exists a point $\eta_i \in \mathcal{J}_i$ such that $(u-\widetilde{u}_h)'(\eta_i) = 0$.*

Proof: By [deBoor 1978, pp.108–] we may construct $B_i \in \$^{\mu-1,r-1}$, cf. (1.3.9), with support in $\overline{\mathcal{J}}_i$ and $B_i(x) > 0$ on $\mathcal{J}_i$ (a B–spline basis function). Let now

$$(1.4.1) \qquad \psi_i(x) = \int_0^x B_i(y)ds - xMV(B_i)$$

which belongs to $\overset{\circ}{S}_h$. Then

$$(1.4.2) \qquad \int_{\mathcal{J}_i} (u-\widetilde{u}_h)'B_i = \big((u-\widetilde{u}_h)', \psi_i' + MV(B_i)\big) = \big((u-\widetilde{u}_h)', \psi_i'\big) = 0.$$

The theorem follows since $B_i > 0$ on $\mathcal{J}_i$ (note that if $\mu = 0$, $\mathcal{J}_i = I_i$). □

A corresponding result for function values is as follows.

Theorem 1.4.2. *a) For $\mu = 0$, $(u-\widetilde{u}_h)(x_i) = 0$, for all meshpoints x_i.*
b) For $r \geq 3$, let $k_f = \left[\frac{r-2}{r-3-\max(-1,\mu-2)}\right]^+$ and $\mathcal{J}_i = (x_i, x_{i+k_f})$.
There exists a point $\xi_i \in \mathcal{J}_i$ such that $(u-\widetilde{u}_h)(\xi_i) = 0$.

Proof: a) follows as in Section 1.2 by noting that now $G(x_i;\cdot) \in \overset{\circ}{S}_h$. For b), as in the proof of Theorem 1.4.1, we may find $B_i \in \$^{\mu-2,r-2}$ such that B_i is supported in $\overline{\mathcal{J}}_i$ and positive in the interior. Letting $-\psi'' = B_i$ on I, $\psi(0) = \psi(1) = 0$ we have $\psi \in \overset{\circ}{S}_h$ and so

$$(1.4.3) \qquad \int_{\mathcal{J}_i} (u-\widetilde{u}_h)B_i = \big((u-\widetilde{u}_h)', \psi'\big) = 0. \qquad \square$$

Of course, Theorems 1.4.1 and 1.4.2 give no information as how to nail down what these points may be. They do, however, give us a hunting license.

1.5. Superconvergence for interior points of mesh–intervals for continuous elements ($\mu = 0$).

For the results in this section we refer to [Chen 1979], [Lesaint and Zlamal 1979] and also to [Bakker 1982, 1984], where many results were rediscovered but also extended to $2m^{th}$ order problems. We point out that no mesh restrictions will be imposed.

We shall first show superconvergence for derivatives at certain points. For a mesh interval I_i, $L_{i,k}(x)$ shall denote the k^{th} Legendre polynomial normalized to I_i, i.e., if $I_i = (x_i, x_{i+1})$, then $L_{i,k}(x) = L_k(2(x - (x_i + x_{i+1})/2)/h_i)$ so that x_i corresponds to -1, the midpoint to 0 and x_{i+1} to 1.

Theorem 1.5.1. *For $\mu = 0$,*

$$|(u - u_h)'(\eta_i)| \le Ch^r\left(\|u\|_{W^r_\infty(I)} + \|u\|_{W^{r+1}_\infty(I_i)}\right) \tag{1.5.1}$$

where η_i is any zero of $L_{i,r-1}$.

We note that $L_{i,r-1}$ has $r-1$ zeroes inside I_i and that u_h' is a polynomial of degree $r-2$. The number of superconvergence points found is thus "maximal". For, if we had r stable points η_j on each I_i where $(u-u_h)'(\eta_j) = 0(h^r)$, then we would have $\|(u-u_h)'\|_{L_\infty(I)} = 0(h^r)$ which is not true in general.

Proof: By Theorem 1.3.1 it suffices to consider $\widetilde{u}_h$ instead of u_h. Now with P the L_2–projection into $\$_h = \$_h^{-1,r-1}$, i.e., discontinuous piecewise polynomials of degree $\le r-2$, we have $\widetilde{u}_h' = Pu'$, or, that $\widetilde{e}' \equiv \widetilde{u}_h' - u'$ satisfies

$$(\widetilde{e}', \psi) = 0, \text{ for all } \psi \in \$_h, \tag{1.5.2}$$

or,

$$\int_{I_i} \widetilde{e}'\psi = 0, \text{ for all } \psi \in \Pi_{r-2}(I_i). \tag{1.5.3}$$

Taylor–expanding $\widetilde{e}'$ around the midpoint $(x_i + x_{i+1})/2$ and writing the expansion in terms of the Legendre polynomials, we have

$$\widetilde{e}'(x) = c_0 L_{i,0}(x) + c_1 L_{i,1}(x) + \cdots + c_{r-1}L_{i,r-1}(x) + 0(h^r), \tag{1.5.4}$$

where the $0(h^r)$ term now involves $(r+1)^{\text{th}}$ derivatives of u on I_i. By (1.5.3), $c_0, c_1, \ldots, c_{r-2}$ are all $0(h^r)$, so that

$$\widetilde{e}'(x) = c_{r-1}L_{i,r-1}(x) + 0(h^r). \tag{1.5.5}$$

The desired result follows. □

The corresponding superconvergence result for function values is as follows.

Theorem 1.5.2. *For $\mu = 0$, $r \ge 3$,*

$$(u - u_h)(\xi_i) \le Ch^{r+1}\left(\|u\|_{W^r_\infty(I)} + \|u\|_{W^{r+1}_\infty(I_i)}\right) \tag{1.5.6}$$

where ξ_i is any zero of $L'_{i,r-1}$ (or, a mesh point, where we have $0(h^{2r-2})$)).

Again, the number of superconvergence points found is maximal.

Proof: From Theorem 1.3.2 it is again enough to consider $\widetilde{u}_h$. By (1.5.5) and Theorem 1.2.1,

$$\begin{aligned}\widetilde{e}(x) &= \widetilde{e}(x_i) + \int_{x_i}^{x} \widetilde{e}'(y)dy \\ &= 0(h^{2r-2}) + c_{r-1}\int_{x_i}^{x} L_{i,r-1}(y)dy + 0(h^{r+1}).\end{aligned} \tag{1.5.7}$$

Since $(r-1)rL_{r-1} = -((1-x^2)L'_{r-1})'$ by Legendre's differential equation, the result follows. □

1.6. Superconvergence in derivatives at points about which the meshes are locally symmetric (r even).

In this section we shall consider general μ and since Theorem 1.5.1 gives a maximally possible number of superconvergence points when $\mu = 0$, identifiable a priori to boot, we have in mind here $\mu \geq 1$. Our arguments in this section follow [Wahlbin 1992] and are restricted to one–dimensional problems. In return, we then obtain more sharply delineated results than in several dimensions, in particular with respect to mesh–geometry outside an immediate neighborhood of the point of interest and with respect to how close to the boundary of I the results are valid. By Theorem 1.3.1 it turns out to be sufficient to treat only $\widetilde{u}_h \in \overset{\circ}{S}_h$ given by $((u - \widetilde{u}_h)', \chi') = 0$, for $\chi \in \overset{\circ}{S}_h$; this is because only superconvergence of order 1 will be at issue. The quasi–uniformity condition (1.1.7) is assumed throughout.

Let $w = u'$ and $w_h = \widetilde{u}_h' \in \$_h = \$^{\mu-1,r-1}$; then as we have seen before,

$$w_h = P(w) \tag{1.6.1}$$

where P is the L_2–projection into $\$_h$. Fixing a point η, there exists $\delta = \delta_h^\eta \in \$_h$ such that

$$\chi(\eta) = (\chi, \delta), \text{ for all } \chi \in \$_h. \tag{1.6.2}$$

We shall refer to δ as a "discrete delta–function" centered at η. In case $\mu = 0$, δ is of course a function with support on $I_i \ni \eta$. For $\mu \geq 1$ this is no longer true; δ_h^η suffers some influence from outside I_i. However such influences are small. In fact we have:

Lemma 1.6.1. *Assume (1.1.7). There exist constants C and $c > 0$, independent of h and η, such that*

$$|\delta_h^\eta(x)| \leq \frac{C}{h} e^{-c|x-\eta|/h}. \tag{1.6.3}$$

Such an exponential decay property was noted in [Descloux 1972] and [Douglas, Dupont and Wahlbin 1975]. Lemma 1.6.1 is taken straight from [Wahlbin 1991, Lemma 3.4]. It will also be proven below in Theorem 3.2.2.

Let now $\widehat{w}_s(\cdot;\eta)$ denote the Taylor polynomial of degree s for w centered at η and consider in particular

$$\widehat{w}_{r-1}(x;\eta) = \widehat{w}_{r-2}(x;\eta) + w^{(r-1)}(\eta)(x-\eta)^{r-1}/(r-1)!\ . \tag{1.6.4}$$

Then since $\widehat{w}_{r-2}(\cdot;\eta) \in \$_h$ and $w_h = P(w)$,

$$\begin{aligned} w(\eta) - w_h(\eta) &= \widehat{w}_{r-2}(\eta;\eta) - w_h(\eta) \\ &= (\widehat{w}_{r-2} - w_h, \delta) = (\widehat{w}_{r-2} - w, \delta) \\ &= (\widehat{w}_{r-1} - w^{(r-1)}(\eta)(x-\eta)^{r-1}/(r-1)! - w, \delta) \\ &= (\widehat{w}_{r-1} - w, \delta) - \frac{w^{(r-1)}(\eta)}{(r-1)!}((x-\eta)^{r-1}, \delta). \end{aligned} \tag{1.6.5}$$

Here

$$|(\widehat{w}_{r-1} - w)(x)| \leq \frac{1}{r!}|x-\eta|^r \|w\|_{W_\infty^r(I)} \leq \frac{1}{r!}|x-\eta|^r \|u\|_{W_\infty^{r+1}(I)} \tag{1.6.6}$$

and using (1.6.3) we thus find that

$$(1.6.7)\qquad \begin{aligned} &|(\widehat{w}_{r-1} - w, \delta)| \\ &\le C\Big(\int_0^1 |x-\eta|^r h^{-1}\exp(-c|x-\eta|/h)dx\Big)\|u\|_{W_\infty^{r+2}(I)} \\ &\le Ch^r\|u\|_{W_\infty^{r+1}(I)}. \end{aligned}$$

Thus, in total, from (1.6.5)–(1.6.7),

$$(1.6.8)\qquad |(u' - \widetilde{u}_h')(\eta)| \le Ch^r\|u\|_{W_\infty^{r+1}(I)} + \frac{|u^{(r)}(\eta)|}{(r-1)!}|((x-\eta)^{r-1}, \delta_h^\eta(x))|.$$

In general there is no reason to expect the last term on the right in (1.6.8) to be less in order than $0(h^{r-1})$ (which it clearly is from (1.6.3), cf. the calculation in (1.6.7)). However, if the mesh happened to be symmetric around η we might expect δ_h^η to be evenly symmetric around η, at least "locally" (Occam's razor). If then r is even so that $(\chi - \eta)^{r-1}$ is oddly symmetric around η, maybe the term will be of higher order due to cancellation.

Let us formalize "mesh–symmetry around η" as follows: For a sufficiently large constant C_1, with $\mathcal{J} = \mathcal{J}(h, \eta, C_1) = \{x \in R : |x-\eta| \le C_1 h \ln 1/h\}$, we assume that $\mathcal{J} \subseteq I$ and also, if $\chi(x) \in \$_h(\mathcal{J}) = \$_h\big|_{\mathcal{J}}$, then

$$(1.6.9)\qquad \overline{\chi}(x) \equiv \chi(\eta - (x-\eta)) \in \$_h(\mathcal{J}).$$

In words, if we reflect (locally) through η we stay in $\$_h$.

Sharpening the razor, we shall prove the following:

Theorem 1.6.1. *Assume the quasi–uniformity condition (1.1.7), that r is even and that $\mu \ge 1$. There exist constants C_1 and C_2, independent of u, h and η, such that if η $(= \eta(h))$ is such that the mesh is symmetric around η with constant C_1 (i.e., in a symmetric $C_1 h \ln 1/h$ neighborhood inside I), then for u_h the solution of (1.1.6),*

$$(1.6.10)\qquad |(u-u_h)'(\eta)| \le C_2 h^r \|u\|_{W_\infty^{r+1}(I)}.$$

Remark 1.6.1. The reader may wonder about the restriction $\mu \ge 1$. Note that if $\mu = 0$ and η a mesh point (which is symmetric) then u_h' is not well defined. This can be gotten around by taking limits of averages of left and right derivatives. However, the case $\mu = 0$ is so thoroughly analyzed in Section 1.5 that we shall wait to give the argument until the multi–dimensional case, where it is more urgent. □

Proof of Theorem 1.6.1: As already noted, by Theorem 1.3.1 it suffices to consider $u - \widetilde{u}_h$, and by (1.6.8) we need only consider the last term there. Introducing the notation $(f, g)_{I'} = \int_{I'} fg$ for I' a set in I we first have using (1.6.3),

$$(1.6.11)\qquad \begin{aligned} &|((x-\eta)^{r-1}, \delta)_{I\setminus\mathcal{J}}| \\ &\le C\int_{|x-\eta|\ge C_1 h \ln 1/h} |x-\eta|^{r-1}h^{-1}\exp(-c|x-\eta|/h)dx \\ &\le Ch^{r-1}\int_{C_1 \ln 1/h}^\infty y^{r-1}e^{-cy}dy \\ &\le Ch^{r-1}e^{-(c/2)C_1 \ln 1/h} \le Ch^r, \end{aligned}$$

provided $C_1c/2 \geq 1$. Furthermore, since $(x-\eta)^{r-1}$ is odd around η,

$$((x-\eta)^{r-1},\delta)_{\mathcal{J}} = ((x-\eta)^{r-1},\delta_{odd})_{\mathcal{J}} \tag{1.6.12}$$

where

$$f_{odd} = (f(x) - \overline{f}(x))/2 = (f(x) - f(2\eta - x))/2, \text{ for } x \in \mathcal{J}, \tag{1.6.13}$$

and $f = f_{odd} + f_{even}$. We shall need an extension operator E from $\$_h(\mathcal{J})$ to $\$_h = S_h(I)$. Assuming (as we may possibly after a slight enlargement) that $\mathcal{J}$ is a union of mesh intervals, a simple such operator E is constructed in [Douglas, Dupont and Wahlbin 1975, Lemma 2.1] in such a fashion that

$$\|E\chi\|_{L_p(I\setminus\mathcal{J})} \leq C\|\chi\|_{L_p(\mathcal{J}')}, \text{ for } \chi \in \$_h(\mathcal{J}), 1 \leq p \leq \infty. \tag{1.6.14}$$

Here $\mathcal{J}'$ denotes the union of the two mesh intervals at the very ends of $\mathcal{J}$. Now since $(\delta_{odd}, \delta_{even})_{\mathcal{J}} = 0$,

$$\begin{aligned}\|\delta_{odd}\|^2_{L_2(\mathcal{J})} &= (\delta_{odd}, \delta_{odd})_{\mathcal{J}} = (\delta_{odd}, \delta)_{\mathcal{J}} \\ &= (E\delta_{odd}, \delta) - (E\delta_{odd}, \delta)_{I\setminus\mathcal{J}}.\end{aligned} \tag{1.6.15}$$

Since $(E\delta_{odd}, \delta) = \delta_{odd}(\eta) = 0$ (here we used that $E\delta_{odd} \in \$_h(I)$, i.e., that $\delta_{odd} \in \$_h(\mathcal{J})$, i.e., the symmetry of the mesh for the one and only time!) and using (1.6.3) twice,

$$|(E\delta_{odd}, \delta)_{I\setminus\mathcal{J}}| \leq Ch^{2k}\|\delta_{odd}\|_{L_1} \leq Ch^{2k}, \tag{1.6.16}$$

for any given k, provided $C_1 = C_1(k)$ is taken large enough. It follows from (1.6.15) that

$$\|\delta_{odd}\|_{L_2(\mathcal{J})} \leq Ch^k \tag{1.6.17}$$

and hence from (1.6.12) that

$$|((x-\eta)^{r-1},\delta)_{\mathcal{J}}| \leq Ch^r \tag{1.6.18}$$

(with C_1 further enlarged if necessary). This and (1.6.11) shows that

$$|((x-\eta)^{r-1},\delta)| \leq Ch^r \tag{1.6.19}$$

and, as noted, the theorem now follows from (1.6.8). □

Remark 1.6.2. It is easily seen from (1.6.7) that the full $\|u\|_{W^{r+1}_\infty(I)}$ norm may be replaced by $\|u\|_{W^{r+1}_\infty(\mathcal{J})} + \|u\|_{W^r_\infty(I)}$. In fact, further localization with respect to smoothness demands is possible but we shall wait with that argument until the multi–dimensional case so as not to overburden the present exposition. □

A particular example of symmetry points η which satisfy (1.6.9) occurs in a uniform mesh, $h_i \equiv h$ for all i. Then both meshpoints and midpoints of mesh–intervals are symmetric, *provided* they stay away from the boundary by a distance $C_1 h \ln 1/h$.

Corollary 1.6.2. *Let r be even and the meshes uniform. Then for η a meshpoint or midpoint with*

$$dist(\eta, \{0,1\}) \geq C_1 h \ln 1/h, \tag{1.6.20}$$

we have

$$|(u-u_h)'(\eta)| \leq C_2 h^r \|u\|_{W_\infty^{r+1}(I)}. \tag{1.6.21}$$

Perhaps the most interesting aspect of this result is that the condition (1.6.20) is necessary in general, as we shall see in the next section. Intuitively, the discrete delta–function, which now suffers from outside influences, looses its symmetry when the point η at which it is centered approaches the boundary.

1.7. Necessity of staying $C_1 h \ln 1/h$ away from the boundary for superconvergence in the uniform mesh case ($\mu = 1$).

We place ourselves in the situation of Corollary 1.6.2, i.e., uniform meshes, and we take the particular "lowest" case of Hermite cubics, i.e., $r = 4$, $\mu = 1$. If our original problem is $-u'' = -3x^2$, $u(0) = u(1) = 0$, with solution $u(x) = (x^4 - x)/4$, we thus look at the L_2–projection $P(x^3 - 1/4)$ of u' into $\$_h = \$_h^{0,3}(I)$, the continuous quadratics. Since constants are clearly reproduced by P, we consider the error in

$$x^3 - P(x^3) \tag{1.7.1}$$

at mesh and midpoints. Notice that now $u^{(r+1)} = u^{(5)} \equiv 0$ which perhaps makes the counterexample even more convincing. Again, our arguments are taken from [Wahlbin 1992, Section 3]; indeed, here we shall merely describe them and refer to this reference for further details.

There, the error of meshpoints and midpoints was explicitly calculated for $x^3 - P(x^3)$, modulo terms which are of order e^{-cN}, $c > 0$ constant, $N = 1/h$. This was done by using different basis functions for $\$_h$ when regarded as a trial or test–space, respectively; the system of equations to be solved was then reduced to easily analyzable three–term recurrence relations.

Let, in this section, $e = x^3 - P(x^3)$ and

$$e_{2i} = e(x_i),\ e_{2i+1} = e(x_{i+1/2}) \equiv e((x_i + x_{i+1})/2),\ i = 0, N-1. \tag{1.7.2}$$

From [Wahlbin 1992], by explicit calculation and a simple algebraic manipulation,

$$-e_{2i-1} + 6e_{2i+1} - e_{2i+3} = 0,\ i = 1, 2, \dots, N-2 \tag{1.7.3}$$

and

$$-e_{2i-2} + 6e_{2i} - e_{2i+2} = 0,\ i = 1, 2, \dots, N-1, \tag{1.7.4}$$

as well as

$$e_0 + 2e_1 = -\frac{h^3}{20}. \tag{1.7.5}$$

From (1.7.3)–(1.7.5), with $\rho = 3 - \sqrt{8}$ (considering points in the left half of I),

$$(u - \widetilde{u}_h)'(\eta_i) = \begin{cases} A_1 h^3 \rho^i + E_{i,1}, & \text{if } \eta_i = x_i, \\ A_2 h^3 \rho^i + E_{i,2}, & \text{if } \eta_i = x_{i+1/2} \end{cases} \tag{1.7.6}$$

where $A_1 \simeq -.071$, $A_2 \simeq .010$ (and $\rho \simeq .172$). Here, for $i \le [N/2]$, $|E_{i,j}| \le e^{-cN}$, with $c > 0$.

Thus, if we want convergence as $Ch^{3+\sigma}$ for some $\sigma > 0$ we need (for $i \le [N/2]$) $\rho^i \le Ch^\sigma$ or, $i \ge (\ln C + \sigma \ln 1/h)/(\ln 1/\rho)$, i.e.,

$$dist(\eta_i, 0) \simeq ih \ge C\sigma h(\ln 1/h)/(\ln 1/\rho).$$

(Of course, in general the presence of a non–zero fifth derivative of u precludes superconvergence to extra order > 1 in the case of Hermite cubics.)

1.8. Finding all superconvergent points for function values and derivatives in the case of a locally uniform mesh for $\mu = 1$ and r even (with a remark about smoothest cubics).

We place ourselves in a situation with r even and locally uniform meshes. Then for $\mu \ge 1$, meshpoints and midpoints are superconvergent for derivatives to order 1, provided the patches of uniform meshes, of extent $C_1 h \ln 1/h$ themselves, stay $Ch \ln 1/h$ away from the boundary of I. In this section we shall investigate the consequences which this has for superconvergence in both function–values and derivatives, mainly in the case of $\mu = 1$. In fact, we shall find all such superconvergence points.

We take the case of $\widetilde{u}_h$, i.e., $a_2 \equiv 1$, $a_1 \equiv a_0 \equiv 0$, cf. Section 1.3. Let us work on an interval $I_i = [x_i, x_{i+1}]$ where we now know that $e \equiv u - \widetilde{u}_h$ satisfies (with $x_{i+1/2} = (x_i + x_{i+1})/2$)

$$e'(x_i),\ e'(x_{i+1/2}) \text{ and } e'(x_{i+1}) = 0(h^r). \tag{1.8.1}$$

Let

$$\varphi_k = \begin{cases} x^k \text{ on } I_i, \\ 0 \text{ on } I \setminus \overline{I}_i, \end{cases} \qquad k = 0, 1, \dots, r-3, \tag{1.8.2}$$

(note that now $r \ge 4$ since $\mu \ge 1$ and r is even). With w_k given by

$$\begin{aligned} -w_k'' &= \varphi_k,\ x \in I, \\ w_k(0) &= w_k(1) = 0, \end{aligned} \tag{1.8.3}$$

we have the relations

$$\int_{I_i} e\varphi_k = \int_I e' w_k'. \tag{1.8.4}$$

Since $\mu = 1$ and $k \le r - 3$, we find that $w_k \in \overset{\circ}{S}_h$ and so

$$\int_{I_i} ex^k = 0,\ k = 0, 1, \dots, r-3. \tag{1.8.5}$$

If we now Taylor–expand e on I_i around $x_{i+1/2}$ to order r and write the expansion in terms of (normalized) Legendre polynomials $L_{i,k} = L_k(2(x - x_{i+1/2})/h_i)$ where $L_k(1) = 1$, we have

$$e(x) = c_0 L_{i,0} + c_1 L_{i,1} + \cdots + c_{r-3} L_{i,r-3} + c_{r-2} L_{i,r-2} + c_{r-1} L_{i,r-1} + c_r L_{i,r} + R, \tag{1.8.6}$$

where

$$R = R(x, x_{i+1/2}) = \int_{x_{i+1/2}}^{x} \frac{(x-t)^r}{r!} u^{(r+1)}(t)dt = 0(h^{r+1})\|u\|_{W_\infty^{r+1}(I_i)}. \tag{1.8.7}$$

Due to (1.8.5), c_0, $c_1, \ldots, c_{r-3}$ are all $0(h^{r+1})$. Hence, since $R' = 0(h^r)$ and $L'_{i,k} = \frac{1}{h_i}(L'_k)$,

$$e'(x) = c_{r-2}L'_{i,r-2} + c_{r-1}L'_{i,r-1} + c_r L'_{i,r} + 0(h^r). \tag{1.8.8}$$

Thus, from (1.8.1), since $L'_{i,r-2}$ and $L'_{i,r}$ are odd about $x_{i+1/2}$, and $L'_{i,r-1}$ is even,

$$e'(x_{i+1/2}) = c_{r-1}L'_{i,r-1}(x_{i+1/2}) + 0(h^r) = 0(h^r) \tag{1.8.9}$$

so that since $L'_{i,r-1}(x_{i+/2}) = \frac{1}{h_i}L'_{r-1}(0) \neq 0$, we have

$$c_{r-1} = 0(h^{r+1}). \tag{1.8.10}$$

Therefore

$$e'(x) = c_{r-2}L'_{i,r-2} + c_r L'_{i,r} + 0(h^r). \tag{1.8.11}$$

We next plug in the fact that $e'(x_i)$, $e'(x_{i+1}) = 0(h^r)$ from (1.8.1):

$$\begin{aligned} e'(x_i) &= c_{r-2}\frac{1}{h_i}L'_{r-2}(-1) + \frac{c_r}{h_i}L'_r(-1) = 0(h^r), \\ e'(x_{i+1}) &= c_{r-2}\frac{1}{h_i}L'_{r-2}(-1) + \frac{c_r}{h_i}L'_r(1) = 0(h^r), \end{aligned} \tag{1.8.12}$$

from which we obtain

$$c_r = -c_{r-2}\frac{L'_{r-2}(1)}{L'_r(1)} + 0(h^{r+1}). \tag{1.8.13}$$

Hence, from (1.8.6) and the above

$$\begin{aligned} e(x) &= c_{r-2}\left[L_{i,r-2}(x) - \frac{L'_{r-2}(1)}{L'_r(1)}L_{i,r}(x)\right] + 0(h^{r+1}) \\ &= c_{r-2}Q(x) + 0(h^{r+1}) \end{aligned} \tag{1.8.14}$$

where

$$Q(x) = L_{i,r-2}(x) - \frac{(r-2)(r-1)}{r(r+1)}L_{i,r}(x) \tag{1.8.15}$$

if we use the common ($L_k(1) = 1$) normalization of the Legendre polynomials. One may use Rodrigues' formula and elementary manipulations to see that (normalized to $(-1, 1)$)

$$Q(x) = const.\frac{d^{r-2}}{dx^{r-2}}\left[(x^2-1)^{r-2}\left(x^2 - \frac{r+2}{r-2}\right)\right] \tag{1.8.16}$$

so that repeated application of Rolle's theorem shows that $Q(x)$ has exactly $r-2$ roots inside I_i (and two roots outside).

Using now also Theorem 1.3.2 (note that $r \geq 4$ since r is even and $\mu \geq 1$) we thus have the following:

Theorem 1.8.1. *Let r be even, $\mu = 1$ and, with $e = u - u_h$, u_h the solution to (1.1.6), $e'(x_i)$, $e'(x_{i+1/2})$ and $e'(x_{i+1}) = 0(h^r)$ on I_i. Then*

$$|e(\xi)| = 0(h^{r+1}) \tag{1.8.17}$$

for ξ any of the $r-2$ roots in $\overline{I}_i$ of $Q(x)$ given in (1.8.15), and only then in general. In particular, this is so if the mesh is uniform in a symmetric $C_1 h \ln 1/h$ neighborhood of I_i which is inside I, with C_1 large enough.

In contradistinction to the case $\mu = 0$ (see Theorem 1.5.2) we now don't have the "maximum" possible number of superconvergence points on $\overline{I}_i$, namely r, but merely $r-2$ of them. (The two extra roots occur outside the interval $\overline{I}_i$; in principle there would be no problem in using them too.)

Example 1.8.1. In the case of Hermite cubics ($r = 4$, $\mu = 1$), the two superconvergent points for function values are given by

$$\xi = x_{i+1/2} \pm \left(\frac{1}{4} - \frac{1}{\sqrt{30}}\right)^{1/2} h \simeq x_{i+1/2} \pm (.2596648112\ldots)h. \qquad \square \tag{1.8.18}$$

One may now go back and differentiate (1.8.14) to deduce that $e'(x)$ is superconvergent to order 1 for points $\eta \in \overline{I}_i$ such that

$$Q'(\eta) = L'_{i,r-2}(\eta) - \frac{L'_{r-2}(1)}{L'_r(1)} L'_{i,r}(\eta) = 0. \tag{1.8.19}$$

There are of course always such roots at x_i, $x_{i+1/2}$ and x_{i+1}; the corresponding superconvergence points are now "rediscovered". However, apart from these, Q' has $r-4$ additional roots inside I_i which we didn't know about before. Indeed, this gives a "full" complement of $r-1$ superconvergence points on $\overline{I}_i$ for first derivatives.

Theorem 1.8.2. *Let r be even, $\mu = 1$, and with $e = u - u_h$, u_h the solution to (1.1.6), $e'(x_i)$, $e'(x_{i+1/2})$ and $e'(x_{i+1}) = 0(h^r)$ on I_i. Then*

$$|e'(\eta)| = 0(h^r) \tag{1.8.20}$$

where η is any of the roots of Q' in $\overline{I}_i$. In particular, this is so if the mesh is uniform in a symmetric $C_1 h \ln 1/h$ neighborhood of I_i which is inside I, with C_1 large enough.

Example 1.8.2. One may treat cases of other μ in ad hoc manners. For example, in the case of smoothest cubics ($\mu = 2$, $r = 4$) one may consider two adjoining equal length intervals $\overline{I}_i \cup \overline{I}_{i+1}$ and use that $e' = 0(h^4)$ at five points: the meshpoints and the midpoints; that e is C^2 across x_{i+1}; *and also* the fact that (if $a_2 \equiv 1$, $a_1 = a_0 \equiv 0$) $\int_{\overline{I}_i \cup \overline{I}_{i+1}} eB_i dx = 0$ where B_i is the piecewise linear "tent" function; $B_i \equiv 0$ outside $I \setminus (\overline{I}_i \cup \overline{I}_{i+1})$, $B_i(x_{i+1}) = 1$ and $B_i \in C^0(I)$. This follows since, if $-w'' = B_i$, $w(0) = w(1) = 0$, then $w \in \overset{\circ}{S}_h$; it corresponds to the relations (1.8.5). The two superconvergent points for function values turn out to be the very same as for Hermite cubics (Example 1.8.1, (1.8.18)). $\square$

1.9. Superconvergence in function values at points about which the meshes are locally symmetric (r odd).

This section is analogous to the investigation for even r in Section 1.6.

Theorem 1.9.1. *Assume the quasi-uniformity condition (1.1.7), that r is odd and that $\mu \geq 1$. There exist constants C_1 and C_2, independent of u, h and ξ such that if ξ $(= \xi(h))$ is such that the mesh is symmetric around ξ with constant C_1 (i.e., symmetric in a $C_1 h \ln 1/h$ neighborhood, cf. definition (1.6.9)), then for u_h the solution of (1.1.6),*

$$|(u - u_h)(\xi)| \leq C_2 h^{r+1} \|u\|_{W_\infty^{r+1}(I)}. \tag{1.9.1}$$

Proof: Since $\mu \geq 1$ and hence $r \geq 3$, by Theorem 1.3.2 it suffices to treat $\widetilde{u}_h$ given by $((u - \widetilde{u}_h)', \chi') = 0$, for $\chi \in \overset{\circ}{S}_h$. Let ξ be a point in I and

$$\varphi = \varphi(x; \xi) = \frac{1}{2}(\chi_{[0,\xi]} - \chi_{[\xi,1]}), \tag{1.9.2}$$

half the difference of the characteristic functions of $[0, \xi]$ and $[\xi, 1]$, respectively. Then

$$(u - \widetilde{u}_h)(\xi) = ((u - \widetilde{u}_h)', \varphi), \tag{1.9.3}$$

or, with $w = u'$ since then $\widetilde{u}_h' = P(u')$, the L_2–projection into $\$_h = \$_h^{\mu-1,r-1}$,

$$\begin{aligned} (u - \widetilde{u}_h)(\xi) &= (w - P(w), \varphi) = (w - P(w), \varphi - P(\varphi)) \\ &= (w - \chi, \varphi - P(\varphi)), \text{ for any } \chi \in \$_h. \end{aligned} \tag{1.9.4}$$

Letting $\widehat{w}_s = \widehat{w}_s(x; \xi)$ be the Taylor polynomial of degree s centered at ξ we have taking $\chi = \widehat{w}_{r-2} \in \$_h$ in (1.9.4),

$$(u - u_h)(\xi) = (w - \widehat{w}_{r-1}, \varphi - P(\varphi)) - \frac{u^{(r)}(\xi)}{(r-1)!}((x - \xi)^{r-1}, \varphi - P(\varphi)). \tag{1.9.5}$$

We now need a suitable analogue of the exponential decay properties for the discrete delta–function, given in Lemma 1.6.1. In fact, based on this, we have the following:

Lemma 1.9.1. *There exist C and $c > 0$ such that with φ as above,*

$$|(\varphi - P(\varphi))(x)| \leq Ce^{-c|x-\xi|/h}. \tag{1.9.6}$$

Proof: From [Wahlbin 1991, Theorem 3.3] or Theorem 3.2.4 below we have with $A_d = A_d(x) = \{y : |x - y| \leq d\} \cap I$, for $d \geq c_0 h$ (c_0 large enough),

$$\begin{aligned} &|(\varphi - P(\varphi))(x)| \\ &\quad \leq C \min_{\chi \in \$_h} \|\varphi - \chi\|_{L_\infty(A_d)} + Cd^{-1/2} e^{-cd/h} \|\varphi - P(\varphi)\|_{L_2(A_d)}, \end{aligned} \tag{1.9.7}$$

with $c > 0$. If A_d doesn't include ξ, then $\varphi \equiv 1/2$ or $-1/2$ so that the first term on the right of (1.9.7) is zero. Furthermore, we have already remarked that P is stable in any L_p–space, cf. Theorem 3.2.3 below. Since then

$$\|\varphi - P(\varphi)\|_{L_2(A_d)} \leq Cd^{1/2} \|\varphi - P(\varphi)\|_{L_\infty(A_d)} \leq Cd^{1/2} \tag{1.9.8}$$

we obtain the lemma. □

Continuing now with the proof of Theorem 1.9.1, from (1.9.6) we have for the first term in (1.9.5),

$$\begin{aligned}(1.9.9)\qquad |(w-\widehat{w}_{r-1},\varphi-P(\varphi))| &\le C\|u\|_{W_\infty^{r+1}(I)}\int_0^1 |x-\xi|^r e^{-c|x-\xi|/h}dx\\ &\le Ch^{r+1}\|u\|_{W_\infty^{r+1}(I)},\end{aligned}$$

so that (corresponding to (1.6.8)),

$$(1.9.10)\qquad |(u-u_h)(\xi)|\le Ch^{r+1}\|u\|_{W_\infty^{r+1}(I)}+\frac{|u^{(r)}(\xi)|}{(r-1)!}|((x-\xi)^{r-1},\varphi-P(\varphi))|.$$

Let now $\mathcal{J}=\mathcal{J}(h,\xi,C_1)$ be a symmetric $C_1 h\ln 1/h$ neighborhood about ξ, assumed inside I. Again by (1.9.6), and with the notation $(f,g)_{\mathcal{J}}=\int_{\mathcal{J}} fg$,

$$(1.9.11)\qquad |((x-\xi)^{r-1},\varphi-P(\varphi))_{I\setminus\mathcal{J}}|\le Ch^k,\ \text{for any } k=k(C_1).$$

It remains to estimate

$$\begin{aligned}(1.9.12)\qquad ((x-\xi)^{r-1},\varphi-P(\varphi))_{\mathcal{J}} &= ((x-\xi)^{r-1},\varphi_{even}-(P(\varphi))_{even})_{\mathcal{J}}\\ &= -((x-\xi)^{r-1},(P(\varphi))_{even})_{\mathcal{J}}\end{aligned}$$

where we used that $(r-1)$ is now even and φ odd about ξ, and where f_{even} denotes the even part of f (on $\mathcal{J}$). Further, again by trivial odd–even considerations,

$$\begin{aligned}(1.9.13)\qquad \|(P(\varphi))_{even}\|^2_{L_2(\mathcal{J})} &= ((P(\varphi))_{even},(P(\varphi))_{even})_{\mathcal{J}}\\ &= ((P(\varphi))_{even},P(\varphi))_{\mathcal{J}}=((P(\varphi))_{even},P(\varphi)-\varphi)_{\mathcal{J}}.\end{aligned}$$

Using then for the first and only time in this proof the assumption that the mesh is symmetric about ξ (on $\mathcal{J}$), cf. (1.6.9), we have $(P(\varphi))_{even}\in \$_h(\mathcal{J})$. Employing the extension operator E from $\$_h(\mathcal{J})$ to $\$_h(I)$, cf. (1.6.14),

$$\begin{aligned}(1.9.14)\qquad \|(P(\varphi))_{even}\|^2_{L_2(\mathcal{J})} &= (E((P(\varphi))_{even}),P(\varphi)-\varphi)_{\mathcal{J}}\\ &= (E((P(\varphi)_{even}),P(\varphi)-\varphi)-(E((P(\varphi))_{even}),P(\varphi)-\varphi)_{I\setminus\mathcal{J}}\\ &= -(E((P(\varphi))_{even}),P(\varphi)-\varphi)_{I\setminus\mathcal{J}}\\ &\le Ch^{2k},\end{aligned}$$

for any k, for C_1 large enough, where we again used (1.9.6). The theorem obtains from this, reported into (1.9.12), and then used together with (1.9.11) in (1.9.10). □

Remark 1.9.1. From (1.9.9) it is easy to replace $\|u\|_{W_\infty^{r+1}(I)}$ in (1.9.1) by $\|u\|_{W_\infty^{r+1}(\mathcal{J})}+\|u\|_{W_\infty^{r}(I)}$. □

Corresponding to Corollary 1.6.2 we now have the following.

Corollary 1.9.2. *Let r be odd and the meshes uniform. Then for ξ a meshpoint or midpoint with*

$$dist(\xi, \{0,1\}) \geq C_1 h \ln 1/h, \tag{1.9.15}$$

we have

$$|(u - u_h)(\xi)| \leq C_2 h^{r+1} \|u\|_{W_\infty^{r+1}(I)}. \tag{1.9.16}$$

1.10. Finding all superconvergent points for function values and derivatives in the case of a locally uniform mesh for $\mu = 1$ and r odd.

Corresponding to Section 1.8 we consider locally uniform meshes so that for r odd, meshpoints and midpoints are superconvergent for function values, locally. Let $e = u - \widetilde{u}_h$, $\widetilde{u}_h$ the projection with respect to the simple form (e', χ'). Taking $\mu = 1$, as in (1.8.5) we have for any I_i,

$$\int_{I_i} e x^k = 0, \text{ for } k = 0, 1, \ldots, r-3. \tag{1.10.1}$$

Thus, using Taylor expansion about the midpoint $x_{i+1/2}$ to polynomial degree r, and writing this in terms of the (normalized) Legendre polynomials.

$$e(x) = c_{r-2} L_{i,r-2}(x) + c_{r-1} L_{i,r-1}(x) + c_r L_{i,r}(x) + 0(h^{r+1}). \tag{1.10.2}$$

Since r is odd, $e(x_{i+1/2}) = 0(h^{r+1})$ means that $c_{r-1} = 0(h^{r+1})$ and so, (if L_k normalized to $L_k(1) = 1$)

$$e(x_{i+1}) = (c_{r-2} + c_r) + 0(h^{r+1}); \tag{1.10.3}$$

the assumption $e(x_{i+1}) = 0(h^{r+1})$ then gives

$$e(x) = c_{r-2}(L_{i,r-2}(x) - L_{i,r}(x)) + 0(h^{r+1}). \tag{1.10.4}$$

(Use of $e(x_i) = 0(h^{r+1})$ gives nothing new.)

Now, [Gradshteyn and Ryzhik 1979, 8.914, p.1026],

$$L_{r-2}(x) - L_r(x) = const(r)(1 - x^2) L'_{r-1}(x) \tag{1.10.5}$$

and we thus have the following (which corresponds in a sense rather to Theorem 1.8.2 than Theorem 1.8.1).

Theorem 1.10.1. *Let r be odd, $\mu = 1$ and, with $e = u - u_h$, u_h the solution to (1.1.6), and $e(x_i)$, $e(x_{i+1/2})$ and $e(x_{i+1}) = 0(h^{r+1})$ on I_i. Then*

$$|e(\xi)| = 0(h^{r+1}) \tag{1.10.6}$$

where ξ is any root of $L'_{i,r-1}$ on I_i (or, the given meshpoints and midpoints). In particular, this is so if the mesh is uniform in a symmetric $C_1 h \ln 1/h$ neighborhood of I_i which is inside I, with C_1 large enough.

Now, L_{r-1} has $r-1$ roots on $(-1,1)$ and hence L'_{r-1} has $r-2$ roots. One of those is the midpoint. Thus, the number of superconvergence points on $\overline{I}_i$ for function values is now r, a full complement; this is in contrast to the situation for r even for function values. We also note that the points in Theorem 1.10.1 are exactly those occurring for $\mu = 0$ (without mesh restrictions or staying away from the boundary), see Theorem 1.5.2.

Finally, differentiating (1.10.4), (1.10.5) and using Legendre's differential equation, $((1-x^2)L'_{r-1})' = const. L_{r-1}$, we have:

Theorem 1.10.2. *Let r be odd, $\mu = 1$ and $e(x_i)$, $e(x_{i+1/2})$ and $e(x_{i+1}) = 0(h^{r+1})$ on I_i. Then*

$$|e'(\eta)| = 0(h^r) \tag{1.10.7}$$

where η is any root of $L_{i,r-1}$ on I_i. In particular, this is so if the mesh is uniform in a symmetric $C_1 h \ln 1/h$ neighborhood of I_i which is inside I, with C_1 large enough.

Thus we have exactly the same superconvergence points for derivatives as for $\mu = 0$, see Theorem 1.5.1.

The following table summarizes our superconvergence results in Sections 1.2 and 1.5–1.10.

$\mu : r$	Function Values	First Derivative
$\mu = 0$: Any r Completely General Meshes.	a) $O(h^{2r-2})$ at Meshpoints. b) $O(h^{r+1})$ at Zeroes of $L'_{r-1}(x)$. (For $r \geq 3$, Total of r Points, "Maximal #".)	$O(h^r)$ at Zeroes of $L_{r-1}(x)$. (Total of $r - 1$ Points, "Maximal #".)
$\mu = 1$: r Even Meshes Uniform in a $C_1 h \ln 1/h$ neighborhood of the Point (and similarly, away from the boundary ∂I).	$O(h^{r+1})$ at the $r - 2$ Zeroes of $Q(x) := L_{r-2}(x) - \frac{L'_{r-2}(1)}{L'_r(1)} L_r(x)$. (Total of $r - 2$ Points, **NOT** "Maximal #".)	$O(h^r)$ at Mesh- and Midpoints, and at the Additional $r - 4$ Zeroes of $Q'(x)$. (Total of $r - 1$ Points, "Maximal #".)
$\mu = 1$: r Odd Meshes as for $\mu = 1$, r Even, above.	$O(h^{r+1})$ at Mesh- and Midpoints, and at Zeroes of $L'_{r-1}(x)$. (Cf. $\mu = 0$, any r (≥ 3) above.) ("Maximal #".)	$O(h^r)$ at Zeroes of $L_{r-1}(x)$. (Cf. $\mu = 0$, any r above.) ("Maximal #".)
Smoothest Cubics $\mu = 2 : r = 4$ Meshes as immediately above.	$O(h^{r+1})$ at Two Points, same as for Hermite Cubics ($\mu = 1$, $r = 4$) above. (**NOT** "Maximal #".)	$O(h^r)$ at Mesh- and Midpoints. ("Maximal #".)
$\mu \geq 1$: r Even General Meshes Symmetric About the Point in a $C_1 h \ln 1/h$–neighborhood (and similarly away from the boundary ∂I).		$O(h^r)$ at the Point.
$\mu \geq 1$: r Odd General Meshes: Same as for $\mu \geq 1$, r Even, immediately above.	$O(h^{r+1})$ at the Point.	

1.11. First order difference quotients of u_h as superconvergent approximations to u' on locally uniform meshes.

This section is an exposé of some of the ideas of [Nitsche and Schatz 1974, Section 6]; that work was in turn inspired by [Thomée and Westergren 1968] in finite difference approximations. In the present situation we shall rely on the results of Section 1.3 and thus treat only the case of first derivatives. Similar results for higher order derivatives will be treated in the multidimensional situation; they would not follow from the techniques of this section for variable coefficients a_2, a_1 and a_0.

On the other hand, the present restricted technique, viz., utilizing that $u_h' \simeq P(u')$ to the hilt, gives somewhat sharper results with respect to how close to the boundary of I the result is valid and the size of domains on which the mesh is assumed uniform. Let us also remark that if approximations to u and u' of a certain order are available, then in our present case, a same order approximation to u'' obtains readily from the differential equation (1.1.1) itself.

We shall thus consider $\widetilde{u}_h \in \overset{\circ}{S}_h$ given by

$$((u-\widetilde{u}_h)', \chi') = 0, \text{ for } \chi \in \overset{\circ}{S}_h = \overset{\circ}{S}{}_h^{\mu,r}. \tag{1.11.1}$$

With P the L_2–projection into $\$_h = \$_h^{\mu-1,r-1}$ and $w = u'$, we then have $w_h = \widetilde{u}_h' = P(w) \in \$_h$, i.e.,

$$(w - w_h, \chi) = 0, \text{ for } \chi \in \$_h. \tag{1.11.2}$$

For $\mathcal{J} \subseteq I$ an interval, let $\$_h^{comp}(\mathcal{J})$ denote the functions in $\$_h(\mathcal{J})$ which have compact support in $\mathcal{J}$. For technical reasons we shall need to operate with four intervals inside each other, all centered about the same point. $\mathcal{J}_1$ is the basic interval on which we wish to prove our final estimate. $\mathcal{J}_2 = \{x \in R : dist(x, \mathcal{J}_1) \le Kh\}$ where K is to be given below, (1.11.5) (and, if necessary, slightly enlarged later). $\mathcal{J}_3 = \{x \in R : dist(x, \mathcal{J}_2) \le C_1 h \ln 1/h\}$ where C_1 will be taken large enough. Finally, $\mathcal{J}_4 = \{x \in R : dist(x, \mathcal{J}_3) \le Kh\}$. Note that $\mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \mathcal{J}_3 \subseteq \mathcal{J}_4$. It is assumed that

$$\mathcal{J}_4 \subseteq I \text{ and that the mesh is uniform on } \mathcal{J}_4;\ h_i \equiv h \text{ there.} \tag{1.11.3}$$

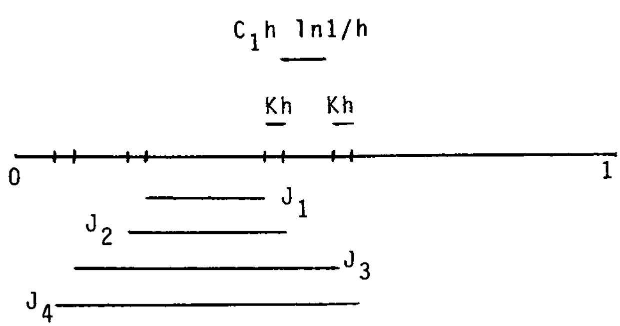


Figure 1.11.1

With $T_h f(x) = f(x+h)$, translation by h, we have

$$(T_h(w - w_h), \chi) = (w - w_h, T_{-h}\chi) = 0, \text{ for } \chi \in \$_h^{comp}(\mathcal{J}_3), \tag{1.11.4}$$

since then $T_{-h}\chi \in \$_h$.

Let now

$$\partial_h = \sum_{i=-K}^{K} c_i(h)T_{ih} \tag{1.11.5}$$

denote a difference quotient operator which is supposed to be an approximation to $\partial/\partial x$. Specifically we assume that

$$|\partial_h f(x)| \le C\|f\|_{W^1_\infty(A_{Kh}(x))} \tag{1.11.6}$$

where $A_{Kh}(x) = \{y : |x-y| \le Kh\}$, and also that

$$|(\partial_h f - f')(x)| \le Ch^s\|f\|_{W^{s+1}_\infty(A_{Kh}(x))}, \text{ for } s = r-1 \text{ and } r. \tag{1.11.7}$$

Now, as in (1.11.4),

$$(\partial_h(w - w_h), \chi) = 0, \text{ for } \chi \in \$_h^{comp}(\mathcal{J}_3). \tag{1.11.8}$$

Since $\partial_h w_h \in \$_h(\mathcal{J}_3)$ we may utilize the following result on local behavior in "local L_2–projections", see [Wahlbin 1991, Theorem 3.3] or Theorem 3.2.4 below. In fact, we have already used it in proving Lemma 1.9.1, see (1.9.7).

Lemma 1.11.1. *There exist three constants C, c_0, and $c > 0$ such that the following holds. For any v and $v_h \in \$_h(\mathcal{J})$ such that*

$$(v - v_h, \chi) = 0, \text{ for all } \chi \in \$_h^{comp}(\mathcal{J}), \tag{1.11.9}$$

if $d \ge c_0 h$ with $A_d = A_d(x) = \{y : |x-y| \le d\} \subseteq \mathcal{J}$,

$$|(v - v_h)(x)| \le C \min_{\chi\in\$_h(\mathcal{J})} \|v - \chi\|_{L_\infty(A_d)} + Cd^{-1/2}e^{-cd/h}\|v - v_h\|_{L_2(A_d)}. \tag{1.11.10}$$

We clearly have

$$d^{-1/2}\|v - v_h\|_{L_2(A_d)} \le C\|v - v_h\|_{L_\infty(A_d)}. \tag{1.11.11}$$

We apply Lemma 1.11.1 to $v = \partial_h w = \partial_h u'$ and $v_h = \partial_h w_h = \partial_h P(w)$ for $x \in \mathcal{J}_2$, and we may then take $d = C_1 h \ln 1/h$ with $A_d(x) \subseteq \mathcal{J}_3$. Since P is L_p–stable and hence W^1_p–stable on our quasi-uniform mesh, by (1.11.6) we find that

$$\begin{aligned} \|\partial_h(w - w_h)\|_{L_\infty(A_d)} &\le \|\partial_h(w - w_h)\|_{L_\infty(\mathcal{J}_3)} \\ &\le C\|w - w_h\|_{W^1_\infty(I)} \le C\|u\|_{W^2_\infty(I)}. \end{aligned} \tag{1.11.12}$$

Then for $x \in \mathcal{J}_2$, using (1.11.10) and (1.11.11),

$$\begin{aligned} &|(\partial_h w - \partial_h w_h)(x)| \\ &\quad \le C \min_{\chi\in\$_h(\mathcal{J}_3)} \|\partial_h w - \chi\|_{L_\infty(\mathcal{J}_3)} + Ch^k\|u\|_{W^2_\infty(I)}, \\ &\qquad \text{for } x \in \mathcal{J}_2, \end{aligned} \tag{1.11.13}$$

for any $k = k(C_1)$ provided C_1 is large enough. Finally, using (1.11.7) with $s = r-1$ and standard approximation theory for a suitable χ,

$$\begin{aligned} \|\partial_h w - \chi\|_{L_\infty(\mathcal{J}_3)} &\le \|\partial_h w - w'\|_{L_\infty(\mathcal{J}_3)} + \|w' - \chi\|_{L_\infty(\mathcal{J}_3)} \\ &\le Ch^{r-1}\|w\|_{W^r_\infty(I)} \le Ch^{r-1}\|u\|_{W^{r+1}_\infty(I)}. \end{aligned} \tag{1.1.14}$$

Thus

$$\|\partial_h(u' - \tilde{u}'_h)\|_{L_\infty(\mathcal{J}_2)} \leq Ch^{r-1}\|u\|_{W^{r+1}_\infty(I)}. \tag{1.11.15}$$

To convert this to a result about $\partial_h(u - \tilde{u}_h)$, we first consider the case $r \geq 3$. Let $x \in \mathcal{J}_1$. Following the technique of Section 1.4, we introduce $B \in \$_h^{\mu-2,r-2}$, a nonnegative function with small compact support containing x and of length $M(\mu, r)h$. With $\partial_h^* = \sum_{i=-K}^{K} c_i(h)T_{-ih}$ we have $(\partial_h\chi, \psi) = (x, \partial_h^*\psi)$ for ψ with compact support in $\mathcal{J}_3$. After enlarging K (in the definition of $\mathcal{J}_2$) if necessary, we have $\partial_h^* B \in \$_h^{comp}(\mathcal{J}_2)$. Defining φ by $-\varphi'' = \partial_h^* B$ on I, $\varphi(0) = \varphi(1) = 0$, we see $\varphi \in \overset{\circ}{S}_h(I)$ so that

$$\int_{supp\, B} \partial_h(u - \tilde{u}_h)B = (u - \tilde{u}_h, \partial_h^* B) = ((u - \tilde{u}_h)', \varphi') = 0. \tag{1.11.16}$$

Thus if we consider a point $x \in \mathcal{J}_1$, $\partial_h(u - \tilde{u}_h)$ vanishes at a point $\xi = \xi(x)$ within a distance $hM(\mu, r)$ of x. Again enlarging K in the definition of $\mathcal{J}_2$ if necessary, we may assume that $\xi(x) \in \mathcal{J}_2$. Hence

$$\partial_h(u - \tilde{u}_h)(x) = \int_\xi^x (\partial_h(u - \tilde{u}_h))' = \int_\xi^x \partial_h(u' - \tilde{u}'_h) \tag{1.11.17}$$

so that by (1.11.15),

$$|\partial_h(u - \tilde{u}_h)(x)| \leq Ch^r\|u\|_{W^{r+1}_\infty(I)}, \text{ for } r \geq 3,\ x \in \mathcal{J}_1. \tag{1.11.18}$$

Using (1.11.7) with $s = r$, finally, for $r \geq 3$,

$$\|u' - \partial_h\tilde{u}_h\|_{L_\infty(\mathcal{J}_1)} \leq Ch^r\|u\|_{W^{r+1}_\infty(I)}. \tag{1.11.19}$$

For the case $r = 2$, i.e., the piecewise linear case, we know in our present setting $(a_2 \equiv 1, a_1 \equiv a_0 \equiv 0)$ that $(u - \tilde{u}_h)(x_i) = 0$ at meshpoints. Consequently, also $\partial_h(u - \tilde{u}_h)(x_i) = 0$ (for $x_i \in \mathcal{J}_1$). The result (1.11.19) follows as before.

Finally, we now combine (1.11.19), derived for the case $a_2 \equiv 1$, $a_1 \equiv a_0 \equiv 0$, with Theorem 1.3.1. By (1.11.6) and that theorem, with θ the difference between our present simple projection and that based on the general form $A(\cdot, \cdot)$, we find that

$$\|\partial_h\theta\|_{L_\infty(\mathcal{J}_1)} \leq C\|\theta\|_{W^1_\infty(I)} \leq Ch^r. \tag{1.11.20}$$

Noting that $C_1 h \ln 1/h >> Kh$ for h small we thus have upon using (1.11.20) in (1.11.19):

Theorem 1.11.1. *There exist constants C_1 and C_2 such that the following holds. Let $\mathcal{J}_1$ be any interval in I such that $\mathcal{J}_4 = \{x \in R : dist(x, \mathcal{J}_1) \leq C_1 h \ln 1/h\} \subseteq I$ and the mesh is uniform on $\mathcal{J}_4$ with meshsize h. Let ∂_h be a difference operator, based on translations by h, approximating $\frac{\partial}{\partial x}$ in the sense of (1.11.6) and (1.11.7). Let u_h be the solution to (1.1.6). Then*

$$\|u' - \partial_h u_h\|_{L_\infty(\mathcal{J}_1)} \leq C_2 h^r\|u\|_{W^{r+1}_\infty(I)}. \tag{1.11.21}$$

In other words, unless $x \in \mathcal{J}_1$ happens to be a superconvergent point for derivatives, $\partial_h u_h(x)$ is an asymptotically better approximation to $u'(x)$ than $u'_h(x)$.

1.12. Two examples of superconvergence by "iteration".

The general structure is as follows: Q is a quantity of interest, $Q = Q(u)$, which may be given by two different "expressions", $Q = Q_1(u)$ as well as $Q = Q_2(u)$. Using $Q_{1,h} := Q_1(u_h)$ or $Q_{2,h} := Q_2(u_h)$ as an approximation may give approximations of different order.

1.12.a. This example is due to [Wheeler, J. A. 1973], cf. also [Douglas, Dupont and Wheeler 1974a]. The quantity of interest is the "outflow" derivative $u'(1)$, say. Using $Q_{1,h} = u_h'(1)$ as an approximation we have no reason to expect better rate of convergence than

$$|u'(1) - u_h'(1)| \le Ch^{r-1}. \tag{1.12.1}$$

Now, multiplying the equation (1.1.1) by x and integrating by parts over I a few times, we have using that $u(0) = u(1) = 0$,

$$u'(1) = Q_2(u) = \frac{1}{a_2(1)} \int_0^1 (-fx - ua_2' + a_1 u + a_0 ux)dx. \tag{1.12.2}$$

Define then

$$Q_{2,h} := \frac{1}{a_2(1)} \int_0^1 (-fx - u_h a_2' + a_1 u_h + a_0 u_h x)dx. \tag{1.12.3}$$

Leaving numerical quadrature out of the picture (easily handled for smooth data) we have with $e = u - u_h$,

$$u'(1) - Q_{2,h} = \frac{1}{a_2(1)} \int_0^1 (-ea_2' + a_1 e + a_0 ex)dx. \tag{1.12.4}$$

Thus,

$$|u'(1) - Q_{2,h}| \le C\|e\|_{W_1^{-s}(I)} \equiv \sup_{\substack{\varphi \in W_\infty^s \\ \|\varphi\|_{W_\infty^{-s}(I)}=1}} (e, \varphi). \tag{1.12.5}$$

A standard duality argument then establishes that, for smooth data, taking $s = r - 2$,

$$|u'(1) - Q_{2,h}| \le Ch^{2r-2}, \tag{1.12.6}$$

an asymptotically considerable improvement over (1.12.1).

1.12.b. This iteration method was first used and analyzed in connection with Fredholm integral equations of the second kind, cf. [Nyström 1930] (where the motivation for its use was somewhat different than here), [Sloan, Burn and Datyner 1975] and [Sloan 1976]. The analogue for two-point boundary value problems presented here is in [Volk 1986], where also numerical illustrations can be found.

First, if necessary by fiddling around in (1.1.1), let us assume that the highest order coefficient $a_2 \equiv 1$. then

$$-u'' = f + (a_1 u)' - a_0 u, \;\; u(0) = u(1) = 0. \tag{1.12.7}$$

At issue is to approximate $u(x)$ itself at any point; of course we could let $Q_{1,h}(x) = u_h(x)$ do it. However, if somewhat laboriously, the operator on the left of (1.12.7),

$-d^2/dx^2$ with boundary conditions, can be inverted by integration: I.e., if v such that

$$-v'' = g \text{ on } I,\ v(0) = v(1) = 0 \tag{1.12.8}$$

is desired, then

$$v(x) = \int_0^x \int_0^y g(z)dzdy - x\int_0^1 \int_0^y g(z)dzdy. \tag{1.12.9}$$

(Again, numerical quadrature will not be taken into account here.) Thus we replace u by u_h on the right hand side of (1.12.7) and consequently define

$$u_*(x) = Q_{2,h}(x) := v(x) \tag{1.12.10}$$

where $v(x)$ is as in (1.12.9) with

$$g = f + (a_1 u_h)' - a_0 u_h. \tag{1.12.11}$$

Then

$$\|u - u_*\|_{L_\infty(I)} = \sup_{\|v\|_{L_1}=1} (u - u_*, v). \tag{1.12.12}$$

Letting $-\psi'' = v$ on I, $\psi(0) = \psi(1) = 0$, and letting $e = u - u_h$,

$$\begin{aligned}(u-u_*, v) &= ((u - u_*)', \psi') = -((u - u_*)'', \psi)\\ &= ((a_1 e)', \psi) - (a_0 e, \psi) = -(a_1 e, \psi') - (a_0 e, \psi)\\ &\le C(a_1)\|e\|_{W_\infty^{-1}(I)} + C(a_0)\|e\|_{W_\infty^{-2}(I)}.\end{aligned} \tag{1.12.13}$$

Thus,

$$\|u - u_*\|_{L_\infty(I)} \le \begin{cases} Ch^{r+1}, & \text{if } \ge 3,\\ Ch^{r+2}, & \text{if } a_1 \equiv 0 \text{ and } r \ge 4.\end{cases} \tag{1.12.14}$$

(The reader may wish to compare this to Remark 1.3.1.)

1.13. A graphical illustration of superconvergence.

We consider the equation

$$-u'' + u = f \text{ in } I = (0, 1), \tag{1.13.1}$$

with the boundary conditions

$$u'(0) = u'(1) = 0. \tag{1.13.2}$$

We shall give one example from [Dunlap, Li, Taylor, Warren and Wahlbin 1995].

Let us first describe the general approach. Working with a known function and having in mind $\mu \ge 1$, let the meshes be uniform of size h and $\widetilde{I} \subset\subset I$ an interior subinterval of I, cf. Section 1.7. Introduce *testpoints* $t_{h,i,j} \in I_i$, $j = 0, \dots, J$, by taking $J+1$ uniformly spaced points on each mesh interval $\overline{I}_i$. The error for the j^{th} set of testpoints with respect to $\widetilde{I}$ (and the given function and problem) is then

$$\widetilde{e}_{h,j} = \max_{t_{h,i,j} \in \widetilde{I}} |(u - u_h)(t_{h,i,j})|. \tag{1.13.3}$$

The observed rate of convergence $\widetilde{r}_{h,j}$ at the j^{th} set of testpoints is now naturally defined as

$$\widetilde{r}_{h,j} = \log_2\left(\frac{\widetilde{e}_{2h,j}}{\widetilde{e}_{h,j}}\right). \tag{1.13.4}$$

(A similar computation may of course be done for the error in derivatives.)

For the graph below we have $u(x) = \sin(5x) - \frac{(5\cos 5-5)}{2}x^2 - 5x$, $r = 4$, $\mu = 1$, $\widetilde{I} = [0.2, 0.8]$, $h = 1/50$ and $J = 1000$. The graph gives $\widetilde{r}_{h,j}$ versus j, cf. (1.8.18).

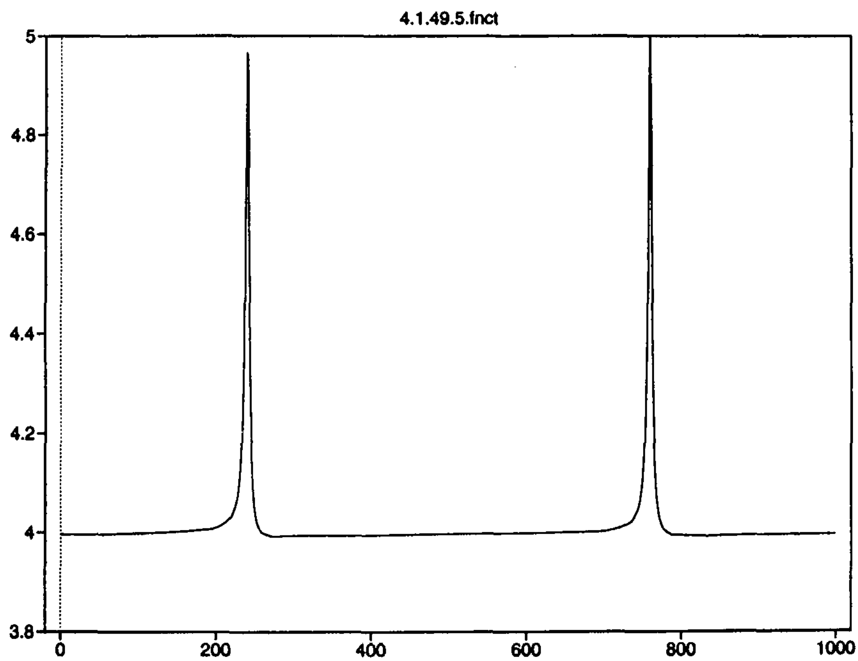


Figure 1.13.1

Chapter 2. Remarks about some of the tools used in Chapter 1.

A seasoned Finite Elementer need not bother with this chapter.

We shall give sketchy generalities about certain standard tools of the trade such as inverse estimates, duality estimates and superapproximation. We shall not, however, give much detail concerning basic approximation theory; doing so would swell this chapter to unmanageable proportions. As for the local properties of L_2–projections, so heavily used in Chapter 1, we shall derive them in Chapter 3. In fact, they will be given there in any number of space dimensions.

2.1. Inverse estimates.

Let $\mathcal{J}$ be an interval (a,b) of length $|\mathcal{J}| = (b-a)$ and consider $\Pi_s(\mathcal{J})$, the polynomials of degree $\le s$ on $\mathcal{J}$. We have the following:

Theorem 2.1.1. *There exists a constant $C(s)$ such that for any p and q with $1 \le p \le \infty$, $1 \le q \le \infty$,*

$$\|\chi\|_{L_p(\mathcal{J})} \le C(s)|\mathcal{J}|^{-(\frac{1}{q}-\frac{1}{p})}\|\chi\|_{L_q(\mathcal{J})}, \text{ for } \chi \in \Pi_s(\mathcal{J}). \tag{2.1.1}$$

Proof: We treat the case $p < \infty$. By Hölder's inequality we have

$$\|\chi\|_{L_p(\mathcal{J})} \le |\mathcal{J}|^{1/p}\|\chi\|_{L_\infty(\mathcal{J})}. \tag{2.1.2}$$

With the transformation $\widetilde{x} = (x-a)/|\mathcal{J}|$ from $\mathcal{J}$ to $I = [0,1]$, and with $\widetilde{\chi}(\widetilde{x}) = \chi(x)$,

$$\|\chi(x)\|_{L_\infty(\mathcal{J})} = \|\widetilde{\chi}(\widetilde{x})\|_{L_\infty(I)}. \tag{2.1.3}$$

Now, $\widetilde{\chi} \in \Pi_s(I)$ and hence by the fact that any norms on a finite dimensional space are equivalent,

$$\|\widetilde{\chi}\|_{L_\infty(I)} \le C(s)\|\widetilde{\chi}\|_{L_1(I)}. \tag{2.1.4}$$

By transformation of variables and Hölder's inequality with $1/q + 1/q' = 1$,

$$\begin{aligned}\|\widetilde{\chi}\|_{L_1(I)} &= \int_{\mathcal{J}} |\chi(x)|\frac{dx}{|\mathcal{J}|} \le \frac{1}{|\mathcal{J}|}\|\chi\|_{L_q(\mathcal{J})}|\mathcal{J}|^{1/q'} \\ &= |\mathcal{J}|^{-1/q}\|\chi\|_{L_q(\mathcal{J})}.\end{aligned} \tag{2.1.5}$$

The desired result follows from (2.1.2)–(2.1.5). □

Let next τ be a quasi–uniform partition of $I = [0,1]$ with $h = \max_i h_i$, and $h \le C_{QU} \min_i h_i$, cf. (1.1.7). Let $S_h = S_h^{\mu,s+1}(I)$.

Theorem 2.1.2. *There exists a constant $C(s)$ such that for $p > q$,*

$$\|\chi\|_{L_p(I)} \le C(s)\Big(\frac{h}{C_{QU}}\Big)^{-(\frac{1}{q}-\frac{1}{p})}\|\chi\|_{L_q(I)}, \text{ for } \chi \in S_h. \tag{2.1.6}$$

Proof: We have, for $q < p < \infty$, using Theorem 2.1.1,

$$\|\chi\|^p_{L_p(I)} = \sum_{i=0}^{N-1} \|\chi\|^p_{L_p(I_i)} \tag{2.1.7}$$
$$\le C(s)^p \sum_{i=0}^{N-1} |h_i|^{(1-p/q)} \|\chi\|^p_{L_q(I_i)}$$
$$\le C(s)^p (\min_i h_i)^{1-p/q} \sum_{i=0}^{N-1} \|\chi\|^p_{L_q(I_i)}$$
$$\le C(s)^p \Big(\frac{h}{C_{QU}}\Big)^{1-p/q} \sum_{i=0}^{N-1} \|\chi\|^p_{L_q(I_i)}.$$

For an array $z = \{z_i\}_{i=1}^{N-1}$ we have $\|z\|_{\ell_p} \le \|z\|_{\ell_q}$, for $p > q$. Thus, from (2.1.7),

$$\|\chi\|^p_{L_p(I)} \le C(s)^p \Big(\frac{h}{C_{QU}}\Big)^{1-p/q} \Big(\sum_{i=0}^{N-1} \|\chi\|^q_{L_q(I_i)}\Big)^{p/q}. \tag{2.1.8}$$

Taking p^{th} roots concludes the argument. The case $p = \infty$ is similar. □

The estimate (2.1.6) is a typical case of an "inverse estimate". It is a property that does not hold for "general" functions, but only for some restricted class. Inverse estimates do not only come into play when going from a high L_p–norm to a lower one, but also in bounding higher derivatives by lower ones.

Theorem 2.1.3. *There exists a constant $C(s)$ such that for any $1 \le p \le \infty$ and any integers $k > \ell$,*

$$\|\chi^{(k)}\|_{L_p(\mathcal{J})} \le C(s)|\mathcal{J}|^{-(k-\ell)} \|\chi^{(\ell)}\|_{L_p(\mathcal{J})}, \quad \text{for } \chi \in \Pi_s(\mathcal{J}). \tag{2.1.9}$$

Proof: If $k > s$ there is nothing to prove. Since $\chi^{(k)} = \frac{\partial^{k-\ell}}{\partial x^{k-\ell}}(\chi^{(\ell)})$ it is enough to consider $\ell = 0$. Now, by Hölder's inequality,

$$\|\chi^{(k)}\|_{L_p(\mathcal{J})} \le |\mathcal{J}|^{1/p} \|\chi^{(k)}\|_{L_\infty(\mathcal{J})}. \tag{2.1.10}$$

Letting $\tilde{x} = \frac{(x-a)}{|\mathcal{J}|}$, $\tilde{\chi}(\tilde{x}) = \chi(x)$, we have $\frac{d^k}{dx^k}\chi(x) = \frac{1}{|\mathcal{J}|^k} \frac{d^k}{d\tilde{\chi}^k}\tilde{\chi}(\tilde{x})$, so that

$$\|\chi^{(k)}\|_{L_p(\mathcal{J})} \le |\mathcal{J}|^{1/p-k} \Big\| \frac{d^k}{d\tilde{x}^k}\tilde{\chi}\Big\|_{L_\infty(I)}. \tag{2.1.11}$$

and again appealing to the equivalence of any two norms on a finite dimensional space,

$$\|\chi^{(k)}\|_{L_p(\mathcal{J})} \le C(s)|\mathcal{J}|^{1/p-k} C_1(s) \|\tilde{\chi}\|_{L_\infty(I)} \tag{2.1.12}$$
$$= C(s)C_1(s)|\mathcal{J}|^{1/p-k} \|\chi\|_{L_\infty(\mathcal{J})}$$
$$\le C(s)C_1(s)|\mathcal{J}|^{-k} \|\chi\|_{L_p(\mathcal{J})}$$

where we used Theorem 2.1.1 in the last step. This proves the theorem. □

Corresponding to Theorem 2.1.2 we now have in the quasi–uniform situation:

Theorem 2.1.4. *There exists a constant $C(s)$ s.t. for $k > \ell$, $1 \le p \le \infty$,*

$$\|\chi^{(k)}\|_{L_p(I)} \le C(s)\Big(\frac{h}{C_{QU}}\Big)^{-(k-\ell)}\|\chi^{(\ell)}\|_{L_p(I)}, \text{ for } \chi \in S_h^{\mu,s+1}(I). \tag{2.1.13}$$

As a final comment to this section, the reader will have noticed that care has been taken to avoid constants depending on p and q in Theorems 2.1.1 and 2.1.2. It is important to, at least, keep uniform control of such constants. As we shall see in the multi–dimensional situation, one may want to apply the results with p depending on h.

2.2. On approximation theory, and duality.

We shall merely give some scattered remarks. Of course, of central importance (for us) is the approximation of smooth functions. Thus, with $\$_h = \$_h^{\mu,r}(I)$ we want to ascertain that, e.g.,

$$\min_{\chi\in\$_h} \|v - \chi\|_{L_\infty(I)} \le Ch^r\|v\|_{W^r_\infty(I)} \tag{2.2.1}$$

where C is a constant which does not depend on h or v. The case of $\mu = 0$ (or, $\mu = -1$, of course) is simple: On each I_i, we choose the two endpoints $x_{i,1} = x_i$ and $x_{i,r} = x_{i+1}$ and then $r-2$ additional points $x_{i,j}$, $j = 2, \ldots, r-1$ forming an equispaced patition of $\bar{I}_i$. Letting $Int(v)$ be the Lagrange interpolant of polynomial degree $r-1$ to v at these points, almost any elementary text in Numerical Analysis tells us that there is a constant $C = C(r)$ such that

$$\|v - Int(v)\|_{L_\infty(I_i)} \le Ch_i^r\|v^{(r)}\|_{L_\infty(I_i)}. \tag{2.2.2}$$

Since the mesh points x_i are interpolation points, $Int(v)$ will belong to $\$_h^{\mu,r}(i)$ when $\mu = 0$ (or $\mu = -1$). Also, if $\$_h$ is replaced by $\overset{\circ}{S}_h$ there is no additional problem if $v(0) = v(1) = 0$; the piecewise Lagrange interpolant will be in $\overset{\circ}{S}_h$.

A similar classical procedure which takes into account only what happens on I_i for the approximation occurs in the Hermite case, (or, rather "a" Hermite case), when $r = 2(\mu+1)$. Then we find $Int(v) \in \Pi_{r-1}(I_i)$ to match $v^{(j)}(x_i)$ and $v^{(j)}(x_{i+1})$, for $j = 0, \ldots, \mu$. The combined result will lie in $\$_h^{\mu,r}$.

If $2(\mu + 1) < r$, throw in a few equispaced interpolation points in between … .

However, if $2(\mu + 1) > r$ we run out of free parameters in this game. The construction now becomes more elaborate, but the result (2.2.1) is still valid, also if $\$_h$ is replaced by $\overset{\circ}{S}_h$, i.e., boundary conditions are taken into account. We refer the reader to, say, [Powell 1981, Chapters 19–20]. The interpolant used can be "global" in the sense that the value used at one point influences the interpolant all over I (but with exponentially diminishing influence away from the point) or it can be "local", i.e., the value of the interpolant at a point depends only on the values of the function over a few mesh–intervals around the point. For such local interpolants, "quasi–interpolants", see in particular [deBoor and Fix 1973] and also [Powell 1981, Section 20.3].

However, there is a need not only for high–order approximation as in (2.2.1) but also for low–order approximation. Let us motivate this need via a duality argument. Let us then say we know, for $e = u - u_h$ where u_h is the solution to (1.1.6), that

$$\|e\|_{W^1_\infty(I)} \le Ch^{r-1}. \tag{2.2.3}$$

Now we wish to show that, for $r \geq 3$,

$$\|e\|_{W_\infty^{-1}(I)} \leq Ch^{r+1} \tag{2.2.4}$$

where

$$\|v\|_{W_\infty^{-1}(I)} = \sup_{\|w\|_{W_1^1(I)}=1} (v, w) \tag{2.2.5}$$

(cf. Remark 1.3.2 and Section 1.12 where (2.2.4) = (1.1.12) was used.)

A typical duality argument now proceeds as follows:

$$\|e\|_{W_\infty^{-1}(I)} = \sup_{\|w\|_{W_1^1(I)}=1} (e, w). \tag{2.2.6}$$

For each such w, let $\psi \in \overset{\circ}{H}{}^1(I)$ be the solution of the adjoint problem to (1.1.1), in weak form,

$$A(\chi, \psi) = (\chi, w), \text{ for any } \chi \in \overset{\circ}{H}{}^1(I). \tag{2.2.7}$$

By Fredholm's alternative, if (1.1.1) has unique solutions, so does (2.2.7). Furthermore, in this one-dimensional case, it is not hard to show the a priori estimate that

$$\|\psi\|_{W_1^3(I)} \leq C\|w\|_{W_1^1(I)} = C. \tag{2.2.8}$$

Then

$$(e, w) = A(e, \psi) = A(e, \psi - \chi), \text{ for any } \chi \in \overset{\circ}{S}_h, \tag{2.2.9}$$

and so by Hölder's inequality, and (2.2.3),

$$|(e, w)| \leq Ch^{r-1} \min_{\chi \in \overset{\circ}{S}_h} \|\psi - \chi\|_{W_1^1(I)}. \tag{2.2.10}$$

Thus, to complete the program of deriving (2.2.4) it would suffice that, cf. (2.2.8),

$$\min_{\chi \in \overset{\circ}{S}_h} \|\psi - \chi\|_{W_1^1(I)} \leq Ch^2 \|\psi\|_{W_1^3(I)}, \tag{2.2.11}$$

where $\psi(0) = \psi(1) = 0$. We note that the Hermite interpolant given above would not work in general: it requires point-values at mesh-points for certain derivatives of ψ and such would in general not be available if ψ is merely in $W_1^3(I)$.

A solution lies in first smoothing ψ before applying the "interpolant". This is easy if one disregards the boundary conditions $\chi(0) = \chi(1) = 0$. Let K be a "smoothing kernel", a C_0^∞ function with certain properties to be given below. Assume that ψ is extended over the boundary ∂I in some suitable way, stable in W_p^m for m as large as needed, cf. [Nečas 1967, p. 75] or [Friedman 1969, p. 10] where the rather well-known and simple Lions-Nikolskii-Peetre extension operator is given. Setting then

$$Sm_h(\psi)(x) = \frac{1}{h} \int \psi(y) K\Big(\frac{x-y}{h}\Big) dy \tag{2.2.12}$$

it is not hard to see that a "high–order" interpolant applied to $Sm_h(\psi)(x)$ does the trick, provided "moment conditions" such as

$$\int K(x-y)y^\ell dy = x^\ell,\ \ell = 0,1,\dots,L \tag{2.2.13}$$

for L sufficiently high are fulfilled. (I.e., the smoothing operator reproduces polynomials up to a certain order.) We refer to [Hilbert 1973] or [Strang 1973] for details.

Finally, if one must respect boundary conditions (as in (2.2.11) as stated!), one uses instead of $Int(Sm_h(\psi))$ as an approximation a perturbed one, $\widetilde{Int}(Sm_h(\psi))$, which equals $\psi(0)$ at $x=0$ (or $\psi(1)$ at $x=1$). In the case of (2.2.11), e.g., one checks that (for Int being the local quasi–interpolant, cf. [deBoor and Fix 1973], e.g.)

$$|\widetilde{Int}(v)(x) - Int(v)(x)| \le \begin{cases} C|Int(v)(0)|, & \text{for} \quad 0 \le x \le Ch, \\ 0, & \text{for} \quad x > Ch. \end{cases} \tag{2.2.14}$$

Since, if $\psi(0)=0$,

$$|Int(Sm_h(\psi))(0)| \le Ch^2\|\psi\|_{W^2_\infty(I)} \le Ch^2\|\psi\|_{W^3_1(I)}, \tag{2.2.15}$$

we obtain upon using inverse estimates (there is no problem if $\mu = 0$ or -1), that

$$\begin{aligned} \|\psi - &\widetilde{Int}(Sm_h(\psi))\|_{W^1_1(I)} \\ &\le \|\psi - Int(Sm_h(\psi))\|_{W^1_1(I)} \\ &\qquad + \|(Int - \widetilde{Int})(Sm_h(\psi))\|_{W^1_1(0,Ch)} \\ &\le Ch^2\|\psi\|_{W^3_1(I)} + \frac{C}{h}\|(Int - \widetilde{Int})(Sm_h(\psi))\|_{L_1(0,Ch)} \\ &\le Ch^2\|\psi\|_{W^3_1(I)} + C\|(Int - \widetilde{Int})(Sm_h(\psi)\|_{L_\infty(0,Ch)} \\ &\le Ch^2\|\psi\|_{W^3_1(I)} + C|Int(Sm_h(\psi))(0)| \\ &\le Ch^2\|\psi\|_{W^3_1(I)} + Ch^2\|\psi\|_{W^2_\infty(I)} \quad (\text{since } \psi(0)=0) \\ &\le Ch^2\|\psi\|_{W^3_1(I)} \quad (\text{Sobolev in one dimension}). \end{aligned} \tag{2.2.16}$$

Applying a cut–off argument at $x = 1/2$ and similar techniques at $x=1$, this then shows (2.2.11) in general even when respecting the boundary conditions $\chi(0) = \chi(1) = 0$.

2.3. Superapproximation.

Recall that this played a major role in Section 1.3. We shall assume that we have at our disposal a high order local approximation operator I_h into $\$_h = S_h^{\mu,s}(I)$ (or, into $\overset{\circ}{S}_h$ if boundary conditions are to be respected). This operator is a linear operator such that

$$I_h\chi \equiv \chi \text{ for } \chi \in \$_h. \tag{2.3.1}$$

It is a high-order local approximation in the sense that there exist two constants C_1 and C_2 such that

(2.3.2)
$$\|v - I_h(v)\|_{W^k_\infty(I_i)} \leq C_1 h^{s-k} \|v^{(s)}\|_{L_\infty(I'_i)}, \quad \text{for } k = 0, 1, \ldots, s-1, \text{ any } v \in W^s_\infty(I'_i),$$

where

(2.3.3)
$$I'_i = \{x \in I : dist(x, I_i) \leq C_2 h\} \cap I.$$

Furthermore,

(2.3.4)
$$I_h(v) \text{ has support in } (Supp(v) + C_2 h) \cap I,$$

where for any set B and real $d > 0$, $B + d = \{x \in R : dist(x, B) \leq d\}$.

We shall assume that, cf. (1.1.7),

the meshes are quasi-uniform.

The above approximation operator I_h is in general the quasi-interpolant of [de Boor and Fix 1973].

Remark 2.3.1. The case of $\mu = 0$ (or, $\mu = -1$) is much simpler and is left to the reader. The operator I_h can then be taken completely local on each I_i, namely, as the Lagrange interpolation operator on equidistanced points including x_i and x_{i+1}. The quasi-uniform condition is then not necessary, as the reader should check. (Actually, the quasi-uniformity is not necessary in any case in which the approximation operator is completely local on each I_i; this hapens e.g. in Hermite cases.) □

We shall use piecewise norms with respect to the partitions τ,

(2.3.5)
$$\|v\|_{W^{k,h}_p(I)} = \Big(\sum_{i=0}^{N-1} \|v\|^p_{W^k_p(I_i)} \Big)^{1/p},$$

with the usual modification if $p = \infty$. We have the following as the major (very complete!) result on superapproximation in one space dimension.

Theorem 2.3.1. *There exist constants C_3 and C_4 depending on C_1, C_2 above and on s and C_{QU} (in (1.1.7)) such that the following holds.*

Let ω be a smooth function such that

(2.3.6)
$$\|\omega\|_{W^k_\infty(I)} \leq \Lambda d^{-k}, \text{ for } k = 0, 1, \ldots, s,$$

where $d \geq h$. Then for any $\chi \in S^{\mu,s}_h(I)$ there exists $\psi \in \$^{\mu,s}_h(I)$ such that for any $k \leq s-1$ and any $1 \leq p \leq \infty$,

(2.3.7)
$$\|\omega\chi - \psi\|_{W^{k,h}_p(I)} \leq C_3 \Lambda \Big(\frac{h}{d}\Big) \|\chi\|_{W^{k,h}_p((Supp(\omega)+C_4 h)\cap I)}.$$

Furthermore,

(2.3.8)
$$Supp(\psi) \subseteq (Supp(\omega) + C_4 h) \cap I.$$

Proof: The natural choice is $\psi = I_h(\omega\chi)$. We are now faced with the problem that, in general, $\omega\chi \notin W^s_\infty(I'_i)$ so that the error estimate (2.3.2) is not immediately applicable. We employ the following trick taken from [Douglas, Dupont and Wahlbin 1975]: With each interior mesh point x_i we associate the functions

$$v_{ik}(x) = (x - x_i)_+^k, \; k = \mu + 1, \dots, s - 1. \tag{2.3.9}$$

Starting from the left we then determine a linear combination w of the v_{ik} so that $\omega\chi + w \in \mathcal{C}^{s-1}(I)$; i.e., we have soaked up the jumps in certain derivatives. Then $\omega\chi + w \in W_\infty^s(I)$. Now $w \in \$_h(I)$ so that $I_h w = w$ and so,

$$I_h(\omega\chi) = I_h(\omega\chi + w) - w. \tag{2.3.10}$$

We now use the approximation hypothesis (2.3.2) so that

$$\begin{aligned} \|\omega\chi - I_h(\omega\chi)\|_{W_\infty^k(I_i)} &= \|\omega\chi + w - I_h(\omega\chi + w)\|_{W_\infty^k(I_i)} \\ &\le C_1 h^{s-k} \|(\omega\chi + w)^{(s)}\|_{L_\infty(I_i')} \\ &= C_1 h^{s-k} \|(\omega\chi)^{(s)}\|_{L_\infty^h(I_i')} \\ &= C_1 h^{s-k} \|(\omega\chi)^{(s)}\|_{L_\infty^h(I_i' \cap Supp(\omega))}, \end{aligned} \tag{2.3.11}$$

where the notation L_∞^h means that the s^{th} derivative is taken piecewise over the partition. By Leibniz' formula, and (2.3.6), since $\chi^{(s)} \equiv 0$ (in the piecewise sense),

$$\begin{aligned} &\|(\omega\chi)^{(s)}\|_{L_\infty^h(I_i' \cap Supp(\omega))} \\ &\quad \le C(s) \sum_{\ell=0}^{s-1} \Lambda d^{-(s-\ell)} \|\chi\|_{W_\infty^{\ell,h}(I_i' \cap Supp(\omega))}. \end{aligned} \tag{2.3.12}$$

Hence, using the inverse estimates of Theorem 2.1.3,

$$\begin{aligned} &\|(\omega\chi)^{(s)}\|_{L_\infty^h(I_i' \cap Supp(\omega))} \\ &\quad \le C\Big(\sum_{\ell=0}^{s-1} \Lambda d^{-(s-\ell)} h^{-(\ell-k)}\Big) \|\chi\|_{W_\infty^{k,h}(I_i' \cap Supp(\omega)+h)} \\ &\quad \le C\Lambda d^{-s} h^k \Big(\frac{d}{h}\Big)^{s-1} \|\chi\|_{W_\infty^{k,h}(I_i' \cap Supp(\omega)+h)}. \end{aligned} \tag{2.3.13}$$

Hence, from (2.3.11),

$$\begin{aligned} &\|\omega\chi - I_h(\omega\chi)\|_{W_\infty^k(I_i)} \\ &\quad \le C\Lambda d^{-s} h^s \Big(\frac{d}{h}\Big)^{s-1} \|\chi\|_{W_\infty^{k,h}(I_i' \cap Supp(\omega)+k)} \\ &\quad \le C\Lambda \Big(\frac{h}{d}\Big) \|\chi\|_{W_\infty^{k,h}(I_i' \cap Supp(\Omega)+h)}. \end{aligned} \tag{2.3.14}$$

By use of Hölder's inequality and the inverse estimates of Theorem 2.1.1, we then obtain

$$(2.3.15)\qquad \begin{aligned}\|\omega\chi - I_h(\omega\chi)\|_{W_p^k(I_i)} &\le h^{1/p}\|\omega\chi - I_h(\omega\chi)\|_{W_\infty^k(I_i)}\\ &\le Ch^{1/p}\Lambda\Big(\frac{h}{d}\Big)\|\chi\|_{W_\infty^{k,h}(I_i'\cap Supp(\omega)+h)}\\ &\le C\Lambda\Big(\frac{h}{d}\Big)\|\chi\|_{W_p^{k,h}(I_i'\cap Supp(\omega)+2h)}.\end{aligned}$$

Raising to the p^{th} power and summing, the estimate (2.3.7) obtains. The statement about the support of ψ is easily verified. □

2.4. A typical combination of inverse estimates and approximation theory used in Chapter 1.

Consider the operator P, the L_2–projection into $\$_h$. At various points in Chapter 1 we used the following: Assume that the meshes are quasi–uniform, and that the operator P is stable in $L_p(I)$, $1 \le p \le \infty$, i.e.,

$$(2.4.1)\qquad \|Pv\|_{L_p(I)} \le C_0\|v\|_{L_p(I)}$$

where C_0 does not depend on h or v. (This will be shown for $\mu \ge 0$ and quasi–uniform meshes in Chapter 3; of course, the case $\mu = -1$ is trivial.)

We then asserted, and used, that

$$(2.4.2)\qquad \|Pv\|_{W_p^1(I)} \le C\|v\|_{W_p^1(I)}.$$

(Again, this is trivial if $\mu = -1$.) In general for quasi–uniform meshes, one would proceed as follows to deduce (2.4.2) from (2.4.1).

Now,

$$(2.4.3)\qquad \|v - Pv\|_{W_p^1(I)} \le \|v - \chi\|_{W_p^1(I)} + \|\chi - Pv\|_{W_p^1(I)}.$$

Now since the mesh is quasi–uniform, by inverse estimates we have, since $\chi - Pv \in \$_h$,

$$(2.4.4)\qquad \begin{aligned}\|\chi - Pv\|_{W_p^1(I)} &\le C_{inv}h^{-1}\|\chi - Pv\|_{L_p(I)}\\ &\le C_{inv}h^{-1}\|\chi - v\|_{L_p(I)} + C_{inv}h^{-1}\|v - Pv\|_{L_p(I)}.\end{aligned}$$

But since $\chi \in \$_h$, $P\chi = \chi$, and we have $v - Pv = (v-\chi) - P(v-\chi)$ so that

$$(2.4.5)\qquad \|v - Pv\|_{L_p(I)} \le \|(v-\chi) - P(v-\chi)\|_{L_p(I)} \le (1 + C_0)\|v - \chi\|_{L_p(I)}.$$

From the above, for any $\chi \in \$_h$,

$$(2.4.6)\qquad \|v - Pv\|_{W_p^1(I)} \le \|v - \chi\|_{W_p^1(I)} + C_{inv}(1 + C_0)h^{-1}\|v - \chi\|_{L_p(I)}.$$

The argument to show (2.4.2) is now capped off by "low order" approximation theory: we may find $\chi \in \$_h$ such that

$$(2.4.7)\qquad \|v - \chi\|_{W_p^1(I)} + h^{-1}\|v - \chi\|_{L_p(I)} \le C_{app}\|v\|_{W_p^1(I)}.$$

Chapter 3. Local and global properties of L_2–projections.

We shall prove the relevant estimates used in Chapter 1 in any number of space dimensions. We then need appropriate inverse estimates and superapproximation results which we proceed to state and discuss in the first section.

Let us remark that the L_2–projection occurs in its own right in Numerical Analysis and that hence the results of this chapter, and those of Chapter 4, are of independent interest.

3.1. Assumptions.

We shall treat in detail only the case when no imposed boundary conditions are present (this is the case used in Chapter 1 in the one–dimensional situation). Let thus T_h, $0 < h < 1/2$, be a sequence of partitions of domains $\mathcal{D}_h \subset\subset R^n$ into disjoint open "finite elements" τ_i^h, $\mathcal{D}_h = \bigcup_{i=0}^{N(h)} \bar{\tau}_i^h$.

The parameter h may be thought of as $\max_i(diam(\tau_i^h))$, and, to fix thoughts, $\mathcal{D}_h$ as approximations to a fixed domain $\mathcal{D}$.

Let now $S_h = S_h(\mathcal{D}_h)$ be a family of finite dimensional function spaces on $\mathcal{D}_h$. A very typical case is that of τ_i^h being n–simplices in R^n and S_h a space of continuous functions on $\mathcal{D}_h$ such that for $\chi \in S_h$, χ restricted to each τ_i^h is a polynomial of total degree $\leq r-1$. These are the Lagrange finite element spaces. We refer to [Ciarlet 1991] for generalities about finite element spaces.

We shall use some special notation relative to the domains $\mathcal{D}_h$. If $\Omega_0 \subseteq \Omega_1 \subseteq \mathcal{D}_h$, then

$$\partial_<(\Omega_0, \Omega_1) = dist(\partial\Omega_0 \setminus \partial\mathcal{D}_h, \partial\Omega_1 \setminus \partial\mathcal{D}_h). \tag{3.1.1}$$

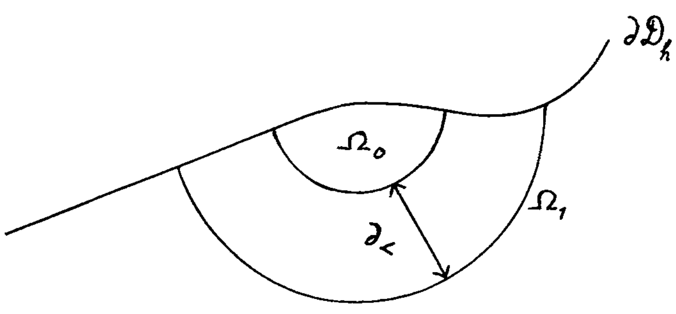


Figure 3.1.1.

We let $C_\infty^<(\overline{\Omega}) = \{v \in C_\infty(\overline{\Omega}) : \partial_<(supp(v), \Omega) > 0\}$ and $S_h^<(\Omega) = \{v \in S_h : \partial_<(supp(v), \Omega) > 0\}$.

We first state our superapproximation assumption.

Assumption 3.1.1. (Superapproximation) There exist constants c and C and a number L, independent of h, such that the following holds: Let $\Omega_0 \subseteq \Omega_1 \subseteq \mathcal{D}_h$ with $d = \partial_<(\Omega_0, \Omega_1) > ch$. Let further $\omega \in C_\infty^<(\overline{\Omega}_0)$ with

$$\|\omega\|_{W_\infty^\ell(\Omega_0)} \leq \Lambda d^{-\ell}, \ \ell = 0, \ldots, L. \tag{3.1.2}$$

Then for any $\chi \in S_h$ there exists $\psi \in S_h^{<}(\Omega_1)$ such that

$$\|\omega\chi - \psi\|_{L_2(\Omega_1)} \le C\Lambda\left(\frac{h}{d}\right)\|\chi\|_{L_2(\Omega_1)}. \tag{3.1.3}$$ □

We shall also need an inverse estimate:

Assumption 3.1.2. (Inverse estimate) There exists a constant C independent of h such that for any element τ_i^h, for any $1 \le p \le \infty$, $1 \le q \le \infty$,

$$\|\chi\|_{L_p(\tau_i^h)} \le Ch^{-n(\frac{1}{q}-\frac{1}{p})}\|\chi\|_{L_q(\tau_1^h)}, \text{ for } \chi \in S_h. \tag{3.1.4}$$ □

We note that for $q < p$, the estimate (3.1.4) is valid with $L_p(\tau_i^h)$ and $L_q(\tau_i^h)$ replaced by $L_p(\mathcal{D}_h)$, respectively $L_q(\mathcal{D}_h)$, cf. the proof of Theorem 2.1.2.

An inverse estimate in general is proven essentially as in Section 2.1 by mapping to a standard (reference) element of unit size and appealing to finite–dimensionality. In the form we have stated (3.1.4), one has to avoid degeneracies in meshes such as triangles getting thin. We refer to [Ciarlet 1991, Theorem 17.2, p. 135] for more details.

Our superapproximation Assumption 3.1.1 can often be proven by following the lines of Section 2.3. E.g., in the Lagrange case described above, there is an interpolation operator I_h locally determined on each τ_i^h which preserves polynomials of total degree $\le r-1$ such that

$$\|v - I_h v\|_{L_\infty(\tau_i^h)} \le Ch^r|v|_{W_\infty^r(\tau_i^h)}, \tag{3.1.5}$$

in case certain geometry constraints are not violated, see [Ciarlet 1991, Theorem 16.2, p. 128]. Here $|\cdot|_{W_\infty^r}$ denotes the r^{th} semi-norm involving only r^{th} derivatives. Applying this to $v = \omega\chi$ and noting that r^{th} derivatives of χ vanish, and using inverse estimates, one easily obtains our Assumption 3.1.1.

The systematic derivation of error estimates such as (3.1.5) using polynomial invariance of I_h goes back to [Bramble and Hilbert 1970], the "Bramble–Hilbert Lemma". (In this respect, cf. also (2.2.13).) For the rather laborious derivation of error estimates before the Bramble–Hilbert Lemma, the reader may ponder the following quote from [Prenter 1975, p. 127] (who did not employ the Bramble–Hilbert Lemma): "Waning sadism forbids us to go further."

A key idea in the Bramble–Hilbert Lemma, namely that $\inf_{\pi\in\Pi_{s-1}}\|v-\pi\|_{W_p^s(\Omega)}$ is equivalent to $|v|_{W_p^s(\Omega)}$, can be traced at least as far back as [Deny and Lions 1953–4, Théorème 5.1]. Of course, there it had nothing to do with finite element appoximation theory!

In other cases, Assumption 3.1.1 is less trivial. E.g., if (in some piece over $\mathcal{D}_h \subset\subset R^2$) the mesh is rectangular and S_h then consists of continuous bilinear functions $a_0 + a_1x + a_2y + a_3xy$ on the rectangle τ_i^h, then the counterpart of (3.1.5) would not be enough since $\frac{\partial^2}{\partial x\partial y}\chi \ne 0$ in general, for $\chi \in S_h$. What is needed then is the so–called "sharp" form of the Bramble-Hilbert Lemma [Bramble and Hilbert 1971], cf. also [Nitsche and Schatz 1974, Appendix] and [Bramble, Nitsche and Schatz 1975, Appendix]. In our present case of bilinears this means that

$$\|v - I_h(v)\|_{L_\infty(\tau_i^h)} \le Ch^2|v|'_{W_\infty^2(\tau_i^h)} \tag{3.1.6}$$

where $|v|'_{W^2_\infty(\tau_i^h)} = \left|\frac{\partial^2 v}{\partial x^2}\right|_{L_\infty(\tau_i^h)} + \left|\frac{\partial^2 v}{\partial y^2}\right|_{L_\infty(\tau_i^h)}$, i.e., *only those derivatives which annihilate $\chi \in S_h$ are involved.* Then superapproximation is easily proved assuming suitable inverse properties.

3.2. Estimates for L_2–projections.

Assuming that the superapproximation Assumption 3.1.1 holds, we have first the following:

Lemma 3.2.1. *There exist positive constants c_0, c_1 and C such that the following holds. Let $\Omega_0 \subseteq \Omega_1 \subseteq \mathcal{D}_h$ with $d = \partial_<(\Omega_0, \Omega_1) \geq c_0 h$. Assume further that $v_h \in S_h$ is such that*

$$(v_h, \chi) = 0, \text{ for all } \chi \in S_h^<(\Omega_1). \tag{3.2.1}$$

Then

$$\|v_h\|_{L_2(\Omega_0)} \leq Ce^{-c_1 d/h}\|v_h\|_{L_2(\Omega_1)}. \tag{3.2.2}$$

Of course, if S_h consists of discontinuous elements this is trivial!

Proof: Let $\delta = ch$ with c as in Assumption 3.1.1, i.e., the "separation" between domains required there. Let further $\Omega_0 \subseteq \Omega_\delta \subseteq \Omega_{2\delta}$ with $\delta = \partial_<(\Omega_0, \Omega_\delta) = \partial_<(\Omega_\delta, \Omega_\delta)$. Let $\omega \in C_\infty^{\leq}(\Omega_\delta)$ be nonnegative with

$$\omega \equiv 1 \text{ on } \Omega_0, \tag{3.2.3}$$

and

$$\|\omega\|_{W^\ell_\infty} \leq \Lambda\delta^{-\ell}, \; \ell = 0, \dots, L. \tag{3.2.4}$$

(It is well known how to construct such a function.) Then

$$\|v_h\|^2_{L_2(\Omega_0)} \leq (v_h, \omega v_h). \tag{3.2.5}$$

For any $\chi \in S_h^<(\Omega_1)$, we have

$$(v_h, \omega v_h) = (v_h, \omega v_h - \chi). \tag{3.2.6}$$

Assuming that $\Omega_{2\delta} \subseteq \Omega_1$, we choose χ according to superapproximation, in particular with $\chi \in S_h^<(\Omega_{2\delta})$ and obtain from (3.1.3),

$$\begin{aligned} \|v_h\|^2_{L_2(\Omega_0)} &\leq C\|v_h\|_{L_2(\Omega_{2\delta})}\Lambda\left(\frac{h}{\delta}\right)\|v_h\|_{L_2(\Omega_{2\delta})} \\ &\leq \left(C\Lambda\frac{h}{\delta}\right)\|v_h\|^2_{L_2(\Omega_{2\delta})}. \end{aligned} \tag{3.2.7}$$

Repeat the argument with domains $\Omega_{2\delta} \subseteq \cdots \subseteq \Omega_{2N\delta}$ as long as they remain inside Ω_1, i.e., since $\delta = ch$, for $N \geq const.\, d/h$. Increasing δ in Assumption 3.1.1 if necessary, we may assume $C\Lambda h/\delta < 1$ and set $C\Lambda h/\delta = e^{-c_1'}$; then

$$\|v_h\|^2_{L_2(\Omega_0)} \leq e^{-const. c_1' d/h}\|v_h\|^2_{L_2(\Omega_1)}. \tag{3.2.8}$$

The desired result follows. □

We next investigate the discrete δ–function centered at $x_0 \in \mathcal{D}_h$, i.e., the function $\delta = \delta_h^{x_0} \in S_h$ such that

$$(\chi, \delta) = \chi(x_0), \text{ for all } \chi \in S_h. \tag{3.2.9}$$

(Here we assume that $S_h \subseteq C^0(\mathcal{D}_h)$; the case of totally discontinous elements is much simpler and left to the reader.) From the Fundamental Theorem of Linear Algebra, it is easily seen that there exists a unique such discrete δ–function centered at x_0.

Assuming now both Assumption 3.1.1 and 3.1.2 from Section 3.1, we have the following:

Theorem 3.2.2. *There exist positive constants C and c_2 such that*

$$|\delta_h^{x_0}(x)| \le Ch^{-n}e^{-c_2|x-x_0|/h}. \tag{3.2.10}$$

Let us remark that this estimate was heavily used in Section 1.5.

Proof: From Lemma 3.2.1 we have for Ω_0 not containing x_0, and $d = dist(\Omega_0, x_0) \ge ch$ with c sufficiently large,

$$\|\delta_h\|_{L_2(\Omega_0)} \le Ce^{-c_1 d/h}\|\delta_h\|_{L_2(\mathcal{D}_h)}. \tag{3.2.11}$$

This is true by default if Ω_0 contains x_0 (then $d = 0$). Thus, by use of the inverse Assumption 3.1.2,

$$|\delta_h(x_0)| \le Ch^{-n/2}e^{-c_1 d/h}\|\delta_h\|_{L_2(\mathcal{D}_h)}. \tag{3.2.12}$$

From (3.2.9) and again using inverse assumptions,

$$\|\delta_h\|^2_{L_2(\mathcal{D}_h)} = \delta_h(x_0) \le Ch^{-n/2}\|\delta_h\|_{L_2(\mathcal{D}_h)}. \tag{3.2.13}$$

The desired result follows. □

From Theorem 3.2.2 we obtain the L_p–stability of the L_2–projection P into S_h.

Theorem 3.2.3. *There exists a constant C such that for any $1 \le p \le \infty$,*

$$\|Pv\|_{L_p(\mathcal{D}_h)} \le C\|v\|_{L_p(\mathcal{D}_h)}, \text{ for all } v \in L_p(\mathcal{D}_h). \tag{3.2.14}$$

Proof: We have

$$Pv(x_0) = (Pv, \delta_h^{x_0}) = (v, \delta_h^{x_0}). \tag{3.2.15}$$

Thus, from (3.2.10),

$$|Pv(x_0)| \le C\|v\|_{L_\infty(\mathcal{D}_h)} \int_{\mathcal{D}_h} h^{-n}e^{-c_1|x-x_0|/h}dx \le C\|v\|_{L_\infty(\mathcal{D}_h)}, \tag{3.2.16}$$

and so

$$\|Pv\|_{L_\infty(\mathcal{D}_h)} \le C\|v\|_{L_\infty(\mathcal{D}_h)}. \tag{3.2.17}$$

By duality (note that Pv is perfectly well defined for $v \in L_1(\mathcal{D}_h)$, for "typical" finite element spaces),

$$\begin{aligned} \|Pv\|_{L_1(\mathcal{D}_h)} &= \sup_{\|w\|_{L_\infty(\mathcal{D}_h)}=1} (Pv, w) \\ &= \sup(v, Pw) \le C\|v\|_{L_1(\mathcal{D}_h)}\|w\|_{L_\infty(\mathcal{D}_h)} \\ &\le C\|v\|_{L_1(\mathcal{D}_h)}. \end{aligned} \tag{3.2.18}$$

The Riesz–Thorin interpolation theorem (cf. [Bennett and Sharpley 1988, p. 192]) now gives the stability of the L_2–projection with respect to any L_p–norm. □

Finally we derive a general result used in Section 1.9 (cf. Lemma 1.9.1) and in Section 1.11 (Lemma 1.11.1).

Theorem 3.2.4. *Under the Assumptions 3.1.1 and 3.1.2, there exist positive constants c_3, c_4 and C such that the following holds. Assume that v is a continuous function and $v_h \in S_h$ is such that*

$$(v - v_h, \chi) = 0, \text{ for all } \chi \in S_h^<(\Omega_d(x_0)) \tag{3.2.19}$$

where $\Omega_d(x_0) = \{x : |x - x_0| \leq d\} \cap I$ with $d \geq c_3 h$. Then

$$\begin{aligned} |(v - v_h)(x_0)| \leq C \min_{\chi \in S_h} \|v - \chi\|_{L_\infty(\Omega_d(x_0))} \\ + C d^{-n/2} e^{-c_4 d/h} \|v - v_h\|_{L_2(\Omega_d(x_0))}. \end{aligned} \tag{3.2.20}$$

Proof: Let $\widetilde{P}$ denote the L_2–projection into $S_h^<(\Omega_d(x_0))$. We may assume that $\Omega_d(x_0)$ is a union of elements. By Theorem 3.2.3, this is stable in any L_p–space. (It is easy to see that the size of $\mathcal{D}_h$ doesn't enter in any essential way in the proof of Theorem 3.2.3, cf. in particular (3.2.16).) Then

$$(v - v_h)(x_0) = (v - \widetilde{P}v)(x_0) + (\widetilde{P}v - v_h)(x_0). \tag{3.2.21}$$

Here, by the stability of $\widetilde{P}$,

$$|(v - \widetilde{P}v)(x_0)| \leq C \|v\|_{L_\infty(\Omega_d(x_0))}. \tag{3.2.22}$$

Since

$$(\widetilde{P}v - v_h, \chi) = (v - v_h, \chi) = 0, \text{ for } \chi \in S_h^<(\Omega_d(x_0)), \tag{3.2.23}$$

and $\widetilde{P}v - v_h \in S_h(\Omega_d(x_0))$, our inverse estimate and Lemma 3.2.1 lead to, for $x_0 \in \tau_{i_0}^h$,

$$\begin{aligned} |(\widetilde{P}v - v_h)(x_0)| &\leq C h^{-n/2} \|\widetilde{P}v - v_h\|_{L_2(\tau_{i_0}^h)} \\ &\leq C h^{-n/2} e^{-c_1 d/h} \|\widetilde{P}v - v_h\|_{L_2(\Omega_d(x_0))}. \end{aligned} \tag{3.2.24}$$

Now,

$$\begin{aligned} &\|\widetilde{P}v - v_h\|_{L_2(\Omega_d(x_0))} \\ &\quad\leq \|\widetilde{P}v - v\|_{L_2(\Omega_d(x_0))} + \|v - v_h\|_{L_2(\Omega_d(x_0))} \\ &\quad\leq \|v\|_{L_2(\Omega_d(x_0))} + \|v - v_h\|_{L_2(\Omega_d(x_0))} \\ &\quad\leq C d^{n/2} \|v\|_{L_\infty(\Omega_d(x_0))} + \|v - v_h\|_{L_2(\Omega_d(x_0))}. \end{aligned} \tag{3.2.25}$$

Thus, using also elementary calculus (with a change of constants),

$$\begin{aligned} |(\widetilde{P}v - v_h)(x_0)| &\leq C h^{-n/2} d^{n/2} e^{-c_1 d/h} \|v\|_{L_\infty(\Omega_d(x_0))} \\ &\quad + C h^{-n/2} e^{-c_1 d/h} \|v - v_h\|_{L_2(\Omega_d(x_0))} \\ &\leq C \|v\|_{L_\infty(\Omega_d(x_0))} + C d^{-n/2} e^{-c_4 d/h} \|v - v_h\|_{L_2(\Omega_d(x_0))}. \end{aligned} \tag{3.2.26}$$

Combining (3.2.26) and (3.2.22) into (3.2.21),

$$|(v - v_h)(x_0)| \leq C \|v\|_{L_\infty(\Omega_d(x_0))} + C d^{-n/2} e^{-c_4 d/h} \|v - v_h\|_{L_2(\Omega_d(x_0))}. \tag{3.2.27}$$

Writing now $v - v_h = (v - \chi) + (\chi - v_h)$ for any $\chi \in S_h$, we obtain the theorem. □

With respect to the local results for the L_2–projection in several space dimensions given here, original work can be found in [Descloux 1972], [Nitsche and Schatz 1972] and [Douglas, Dupont and Wahlbin 1975b]. Our treatment above has followed [Nitsche and Schatz 1972] with some ideas from [Schatz and Wahlbin 1983] thrown in to show exponential decay.

In our development above we have kept the assumption of global quasi–uniformity of meshes, as implicit in Section 3.1, for simplicity. It is apparent from the proofs that there is room for extensions of the results to less regular meshes. For an investigation of the technique of [Descloux 1972] in this regard, see [Crouzeix and Thomée 1985].

(There was at one time an on–going battle to show general L_p–stability of the L_2–projection in one space dimension under no (or "minimal") mesh–conditions. The interested reader will find references to some work in [deBoor 1981].)

Chapter 4. Introduction to several space dimensions: some results about superconvergence in L_2–projections.

From Chapter 3 we now have recourse to rather sharp results concerning the L_2–projection. In this brief chapter we shall give examples of superconvergence results for the L_2–projection. The chapter is intended as an introductory one to multi–dimensional superconvergence results for solutions to elliptic boundary value problems: the technicalities there become more formidable. However, as already remarked, the L_2–projection is of independent interest in Numerical Analysis.

In our first section we shall investigate the existence of local superconvergence points for function values in general.

4.1. Negative norm estimates and existence of general superconvergence points for function values.

This section is somewhat analogous to Section 1.4. If the finite element space S_h allows us to construct a nonnegative function $B \in S_h$, then

$$(v - Pv, B) = 0 \tag{4.1.1}$$

so that $v - Pv$ vanishes at some point in the support of B. (If S_h consists of discontinuous functions, then $B \equiv 1$ on an element τ_i^h, $B \equiv 0$ outside, may be taken.) Very often, this may be accomplished with $Supp(B)$, and thus a zero of $v - Pv$, in a ball of radius $\leq Ch$.

We shall next look at negative norm estimates for the L_2–projection.

$$\|v - Pv\|_{W_\infty^{-r}(\mathcal{D}_h)} = \sup_{\|w\|_{W_1^r(\mathcal{D}_h)}=1} (v - Pv, w). \tag{4.1.2}$$

For any $\chi \in S_h$, we have upon using various approximation assumptions (which we leave to the reader to make explicit)

$$\begin{aligned}(v - Pv, w) &= (v - Pv, w - \chi)\\ &\leq \|v - Pv\|_{L_\infty(\mathcal{D}_h)}\|w - \chi\|_{L_1(\mathcal{D}_h)}\\ &\leq Ch^{2r}\|v\|_{W_\infty^r(\mathcal{D}_h)}\|w\|_{W_1^r(\mathcal{D}_h)}\end{aligned} \tag{4.1.3}$$

Thus, for v smooth,

$$\|v - Pv\|_{W_\infty^{-r}(\mathcal{D}_h)} \leq Ch^{2r}. \tag{4.1.4}$$

We shall now show that this negative norm estimate leads to a rather "dense" occurrence of superconvergence points.

Now, if say B_d is an open ball of radius d in R^n, then we may construct w such that $w > 0$ in B_d, $supp\ w = B_d$,

$$|w|_{W_\infty^\ell(B_d)} \leq Cd^{-\ell},\ \ell = 0, \dots, r \tag{4.1.5}$$

and, with $c > 0$,

$$\|w\|_{L_1(B_d)} \geq cd^n. \tag{4.1.6}$$

Thus, since by (4.1.5) $\|w\|_{W_1^r(\mathcal{D}_h)} \leq \|w\|_{W_1^r(B_d)} \leq Cd^{n-r}$,

$$|(v - Pv, w)| \leq Ch^{2r}d^{n-r}. \tag{4.1.7}$$

Assuming now that $v - Pv$ does not vanish on B_d we have since $w > 0$ (taking $v - Pv > 0$, say),

$$(4.1.8)\qquad \begin{aligned}\min_{x\in B_d}(v - Pv)(x) &\le \int_{B_d}(v - Pv)(y)w(y)dy/\|w\|_{L_1(B_d)}\\ &\le Ch^{2r}d^{-r} = Ch^r\left(\frac{h}{d}\right)^r.\end{aligned}$$

Thus, in any ball of radius $d = h^{1-\sigma/r}$, there is a supercovergent point of order σ (at least). (Here we have taken h^r to be the general best order of approximation in L_p, the so-called "optimal order".) Of course, the above analysis gives no clue as to how to locate such superconvergence points a priori. They do, again, give us a hunting license.

4.2. Superconvergence in L_2–projections on n–dimensional tensor product spaces.

Our second theme will be to show that any superconvergence result we know for one-dimensional L_2-projections will "automatically" carry over to n-dimensional L_2-projections on finite element spaces which are locally tensor-products.

We shall give the argument in detail for $n = 2$.

Let $\mathcal{D}_h$ be plane domains and $\Omega_0 = \Omega_0(h) \subseteq \Omega_1 = \Omega_1(h)$ two rectangles in $\mathcal{D}_h$ such that in Ω_1, the finite element space $S_h(\Omega_1)$ is given as linear combinations of $\chi_1(x)\chi_2(y)$ where χ_i, $i = 1, 2$, belong to one-dimensional finite element spaces (with partitions matching the boundaries of Ω_1). In "obvious" notation we may then write

$$(4.2.1)\qquad S_h(\Omega_1) = S_h^x(I_x) \otimes S_h^y(I_y).$$

Let now v be a given function and v_h a function in $S_h(\Omega_1)$ such that

$$(4.2.2)\qquad (v - v_h, \chi) = 0, \text{ for } \chi \in S_h^<(\Omega_1).$$

Let P_x denote the one-dimensional L_2-projection into $S_h^x(I_x)$ and similarly P_y for $S_h^y(I_y)$. Writing $P_x \otimes P_y = (P_x \otimes I)(I \otimes P_y)$ it is easy to see that, in fact, $P_x \otimes P_y = P$, the L_2-projection into $S_h(\Omega_1)$. Thus,

$$(4.2.3)\qquad (v_h - Pv, \chi) = 0, \text{ for } \chi \in S_h^<(\Omega_1)$$

so that assuming that

$$(4.2.4)\qquad \partial_<(\Omega_0, \Omega_1) \ge C_1 h \ln 1/h,$$

and also that

$$(4.2.5)\qquad \|v - v_h\|_{L_1(\mathcal{D}_h)} \le C,$$

we have from Theorem 3.2.4 (making the implicit assumptions that it is applicable!), that

$$(4.2.6)\qquad \|v_h - Pv\|_{L_\infty(\Omega_0)} \le Ch^k$$

for any k provided $C_1 = C_1(k)$ in (4.2.4) is large enough. To estimate $v - v_h$ it now remains to look at $v - P_x \otimes P_y v = (v - (P_x \otimes I)v) + (P_x \otimes I)(v - I \otimes P_y v)$. We obtain:

Theorem 4.2.1. *With assumptions as given above, if $\overline{x} = \overline{x}(h)$ is a superconvergence point for function values for P_x and $\overline{y} = \overline{y}(h)$ a superconvergence point for function values for P_y, then, if $(\overline{x}, \overline{y}) \in \Omega_0$, it is a superconvergent point for $v - v_h$.*

The reader will have no trouble in extending this result to n–dimensional tensor products.

Also, assuming inverse estimates for derivatives, the result is true for superconvergence points in derivatives. (It is easy to "descend" in the results of Chapter 1 to give superconvergence results for derivatives in one–dimensional L_2–projections. We leave this to the interested reader.) Likewise, it is not hard to consider superconvergence for derivatives of suitable difference quotients on translation invariant meshes, cf. Sections 1.11 and Chapter 8.

4.3. Superconvergence by symmetry in L_2–projections.

The results in this section are due to [Schatz, Sloan and Wahlbin 1994].

We continue to assume the results of Chapter 3. Assume now further that the finite element spaces are symmetric about points $x_0 = x_0(h)$ in a $C_1 h \ln 1/h$ neighborhood with respect to the anti–podal map (cf. (1.6.9)): If $N_1 = \{x : |x - x_0| \leq C_1 h \ln 1/h\}$ we assume that $N_1 \subseteq \mathcal{D}_h$ and that with $\overline{\chi}(x) := \chi(x_0 - (x - x_0))$,

(4.3.1) $$\overline{\chi}(x) \in S_h(N_1) \text{ for any } \chi \in S_h(N_1).$$

We shall give some examples taken from [Schatz, Sloan and Wahlbin 1994] at the end of this section. Note that x_0 may vary with h. We also assume the approximation property that

(4.3.2) $$\min_{\chi \in S_h} \|v - \chi\|_{L_\infty(N_1)} \leq C h^r |v|_{W^r_\infty(N_2)}$$

where $N_2 = N_1 + Ch$ and $|\cdot|_{W^r_\infty(N_2)}$ *denotes the seminorm involving only* r^{th} *order derivatives* (r is again the "optimal" order in L_p). This dependence on the *seminorm* only is crucial for our arguments. We further assume that

(4.3.3) $$\|v - v_h\|_{L_2(\mathcal{D}_h)} \leq C.$$

With C_1 sufficiently large we then have:

Theorem 4.3.1. *Assume that S_h consists of continuous functions. Assume further that r is odd and that the assumptions described above hold. Then for v and $v_h \in S_h$ such that*

(4.3.4) $$(v - v_h, \chi) = 0, \ \textit{for } \chi \in S_h^<(N_1),$$

we have for v sufficiently smooth and x_0 a symmetry point,

(4.3.5) $$|(v - v_h)(x_0)| \leq C h^{r+1} \ln 1/h.$$

<u>Proof</u>: The key is to consider the even part of v and v_h,

(4.3.6) $$v_{even}(x) := (v(x) + \overline{v}(x))/2$$

and similarly for v_h. Then $v_{h,even} \in S_h(N_1)$ by (4.3.1). It also follows from (4.3.1) by a simple transformation of variables argument that

(4.3.7) $$(v_{even} - v_{h,even}, \chi) = 0, \text{ for } \chi \in S_h^<(N_1).$$

Since $v(x_0) - v_h(x_0) = (v_{even} - v_{h,even})(x_0)$, Theorem 3.2.4 with (4.3.3) tells us that

$$|(v - v_h)(x_0)| \leq \min_{\chi \in S_h} \|v_{even} - \chi\|_{L_\infty(N_1)} + Ch^{r+1} \tag{4.3.8}$$

provided C_1 is large enough.

Using now (4.3.2) and the fact that r is odd,

$$\begin{aligned} \min_{\chi \in S_h} \|v_{even} - \chi\|_{L_\infty(N_1)} &\leq Ch^r |v_{even}|_{W^r_\infty(N_2)} \\ &\leq Ch^{r+1} \ln 1/h |v|_{W^{r+1}_\infty(N_2)}. \end{aligned} \tag{4.3.9}$$

This proves the theorem. □

For r even, one similarly obtains $0(h^r \ln 1/h)$ superconvergent estimates for derivatives at symmetry points x_0. The key is now to consider the odd part, $v_{odd} - v_{h,odd}$. The details are easy for quasi–uniform meshes. (In Chapter 7 we shall show that for C^0 elements, if a symmetry point is such that the derivative is not continuous there, then averaging works.)

Finally, we shall elucidate locally symmetric meshes, following [Schatz, Sloan and Wahlbin 1994]. Here, of course, one assumes that the piecewise polynomial spaces or, more general isoparametric ones, cf. [Ciarlet 1991, Chapter VI], are "the same" on each element. If one takes a mesh locally symmetric about a point x_0 on the line, then extends that mesh to the upper half–plane in any manner whatsoever, and then reflects that mesh via the antipodal map about x_0 to create a mesh in the plane, one obtains a mesh locally symmetric about x_0. (• denotes the symmetry point(s).)

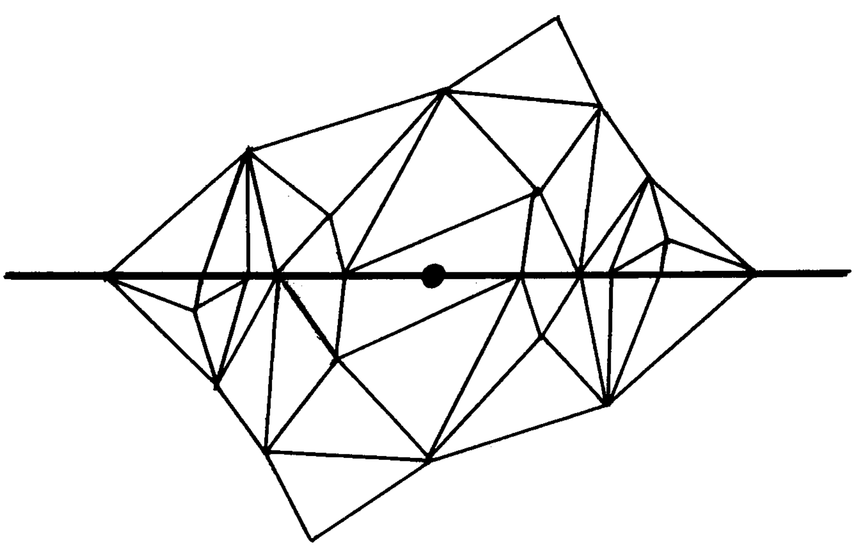


Figure 4.3.1.

Another example, not constructed by the above method, is as follows:

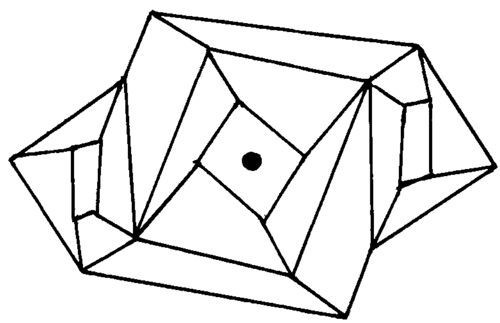


Figure 4.3.2.

Perhaps more traditional examples are as follows:
Rectangular grids:

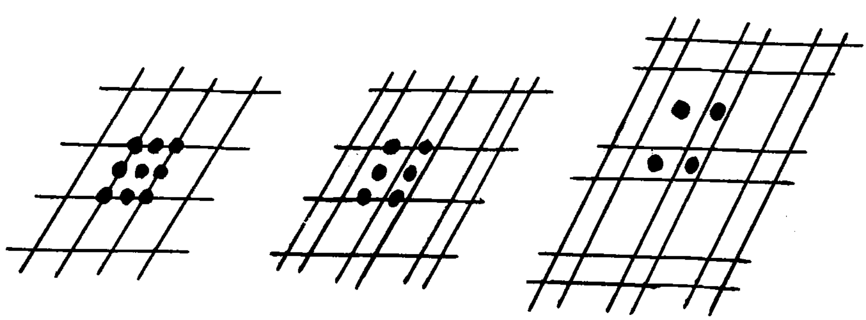


Figure 4.3.3.

A triangular grid of the criss–cross type:

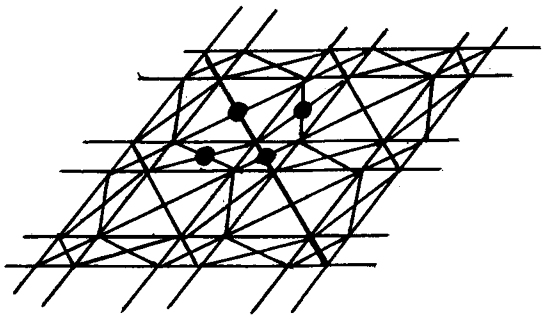


Figure 4.3.4.

If the mesh is locally composed of congruent axes–parallel parallelepipedes in R^3, each parallelepiped has 27 symmetry points:

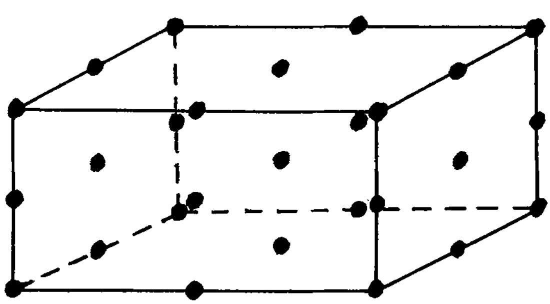


Figure 4.3.5.

Chapter 5. Second order elliptic boundary value problems in any number of space dimensions: preliminary considerations on local and global estimates and presentation of the main technical tools for showing superconvergence.

5.1. Introduction.

Let $\mathcal{D}$ be a bounded domain in R^n. The basic boundary value problem to be approximated is the following: Find u such that

$$(5.1.1)\qquad -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial u}{\partial x_i}\Big) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a_i(x)u) + a(x)u = f \text{ in } \mathcal{D},$$

some boundary condition on $\partial\mathcal{D}$.

We shall assume that the coefficients a_{ij}, a_i and a are sufficiently smooth (on $\mathcal{D}$, or the relevant domain under consideration) and that the a_{ij} satisfy the uniform ellipticity condition

$$(5.1.2)\qquad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c_{ell}|\xi|^2, \text{ for } \xi \in R^n,$$

with $c_{ell} > 0$ independent of x (for $x \in \mathcal{D}$, or x in the domain under consideration). The formal bilinear form corresponding to the elliptic operator in (5.1.1) is

$$(5.1.3)\qquad A(u,w) = \int \Big(\sum_{i,j=1}^{n} a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j} + \sum_{i=1}^{n} a_i v \frac{\partial w}{\partial x_i} + avw\Big)dx.$$

In examples later on, we shall be specific about the boundary condition on $\partial\mathcal{D}$.

Let $S_h = S_h(\mathcal{D}) \subseteq W^1_\infty(\mathcal{D})$, $0 < h < 1/2$, be a one–parameter family of "finite–element" spaces (the "h–method"). For $\Omega \subseteq \mathcal{D}$ we let S_h denote the restrictions of functions in S_h to Ω and we let $S_h^{comp}(\Omega)$ denote the set of those functions in $S^h(\mathcal{D})$ with compact support in the interior of Ω. Various assumptions concerning S_h will be introduced as we go along.

In our general treatment we shall only consider domains which are compactly contained in $\mathcal{D}$. In the main technical development we then do not have to consider the actual boundary conditions on $\mathcal{D}$; of course, in various examples, we shall be specific.

Our *general starting point* is then that we have a function u and a function $u_h \in S_h$ such that for domains $\Omega \subset\subset \mathcal{D}$ (*which may depend on* h),

$$(5.1.4)\qquad A(u - u_h, \chi) = F(\chi), \text{ for } \chi \in S_h^{comp}(\Omega).$$

Here $F(\chi)$ is a bounded linear functional on $\overset{\circ}{W}{}^1_1(\Omega)$.

If u_h is a "usual" finite element approximation to the solution of (5.1.1), then most often $F \equiv 0$ (if we disregard numerical integration, as we shall unless explicitly stated to the contrary). As we shall see, however, it will be important to allow a general right hand side in (5.1.4).

In general, we shall assume that there is an integer $r \geq 2$ such that for smooth functions v on $\mathcal{D}$,

$$(5.1.5)\qquad \min_{\chi \in S_h} \|v - \chi\|_{L_\infty(\Omega)} \leq C(u)h^r.$$

If $\Omega \subset\subset \mathcal{D}$ with enough separation between Ω and $\mathcal{D}$, this does not take into consideration essential boundary conditions (such as Dirichlet boundary conditions). We shall call this integer r the *optimal order*, assuming that no higher order of approximation is possible for "general" u, cf. (1.1.14). Similarly, we shall assume that

$$\min_{\chi \in S_h} \|v - \chi\|_{W^1_\infty(\Omega)} \leq C(u)h^{r-1} \tag{5.1.6}$$

and no better in general. We refer to [Ciarlet 1991] for basics.

It is now rather obvious what we shall call *superconvergence of order* $\sigma > 0$ cf. (1.1.15) and (1.1.16). Again, we shall also consider postprocessing.

The main technical tools for proving superconvergence will be presented in Section 5.5. The intervening sections are devoted to elucidating these tools.

5.2. Existence of superconvergence points in general: an example (also an example of a multi–dimensional duality argument).

Let $\mathcal{D}$ be a bounded domain in R^n with sufficiently smooth boundary. Consider the Neumann problem of finding u such that

$$\begin{cases} -\Delta u + u = f & \text{in} \quad \mathcal{D}, \\ \frac{\partial u}{\partial n} = 0 \quad \text{on} & \partial\mathcal{D}. \end{cases} \tag{5.2.1}$$

The weak formulation of this problem is to find $u \in H^1(\mathcal{D})$ such that

$$A(u, \chi) \equiv \int_{\mathcal{D}} (\nabla u \cdot \nabla \chi + u\chi)dx = (f, \chi) \equiv \int_{\mathcal{D}} f\chi, \text{ for all } \chi \in H^1(\mathcal{D}). \tag{5.2.2}$$

(Note that the "natural" boundary conditions $\frac{\partial u}{\partial n} = 0$ on $\partial\mathcal{D}$ are not seen in the weak formulation, cf. e.g., [Nečas 1967, Example 2.8, p. 33].)

In our case the form A is coercive over $H^1(\mathcal{D})$, indeed,

$$A(v, v) = \|v\|^2_{H^1(\mathcal{D})}, \text{ for } v \in H^1(\mathcal{D}) \tag{5.2.3}$$

so that the Riesz representation theorem guarantees a unique solution for any f in $L_2(\mathcal{D})$ (say).

One may now introduce finite elements partitions of $\mathcal{D}$; if numerical integration is not taken into account we may assume that the elements that meet $\partial\mathcal{D}$ are curved to fit $\partial\mathcal{D}$ exactly. Under conditions on quasi–uniformity (and other general standard conditions that we shall not give here) it may be shown (see [Schatz and Wahlbin 1994, Theorem 4.1], cf. also [Scott 1976] for an original, somewhat weaker, result when $n = 2$), that with $u_h \in S_h$ the finite element approximation given by

$$A(u_h, \chi) = (f, \chi), \text{ for all } \chi \in S_h, \tag{5.2.4}$$

we have for $e = u - u_h$,

$$\|e\|_{W^1_\infty(\mathcal{D})} \leq Ch^{r-1}, \tag{5.2.5}$$

where C depends on the smoothness of u (i.e., f), the coefficients and various parameters describing S_h; however, it can be taken independent of h.

We now make explicit the approximation hypothesis that

$$\min_{\chi \in S_h} \|w - \chi\|_{W^1_p(\mathcal{D})} \leq Ch^{r-1}\|w\|_{W^r_p(\mathcal{D})}, \text{ for } 1 \leq p \leq \infty, \tag{5.2.6}$$

where C is independent of w, h and p (see [Ciarlet 1991]; in certain cases smoothing à la [Hilbert 1973] or [Strang 1973], cf. also Section 2.2, may be required).

We shall estimate, via a typical duality argument, the following negative norm,

$$\|e\|_{W_p^{-(r-2)}(\mathcal{D})} \equiv \sup_{\|v\|_{W_{p'}^{r-2}(\mathcal{D})}=1} (e, v) \tag{5.2.7}$$

where p is close to ∞ and $\frac{1}{p} + \frac{1}{p'} = 1$, and thus p' is close to 1. For any v as above, let

$$\begin{cases} -\Delta w + w = v & \text{in} \quad \mathcal{D}, \\ \frac{\partial w}{\partial n} = 0 & \text{on} \quad \partial\mathcal{D}. \end{cases} \tag{5.2.8}$$

We then have the following a priori estimate, for $\partial\mathcal{D}$ smooth enough,

$$\|w\|_{W_{p'}^{r}(\mathcal{D})} \leq C \frac{1}{p'-1} \|v\|_{W_{p'}^{r-2}(\mathcal{D})}. \tag{5.2.9}$$

We refer the reader to [Gilbarg and Trudinger 1983, Chapter 9] for tracing constants in a priori estimates to see that (5.2.9) is true. (The argument there is for a Dirichlet problem but easily adapted to our present Neumann problem. The reader who pursues this will connect with the constant in the Calderon–Zygmund Theorem and, ultimately, with that in L_p–estimates for the Hilbert transform.) The point is that, in contrast to the one–dimensional situation, we cannot take $p' = 1$. We now have

$$(e, v) = A(e, w) = A(e, w - \chi), \text{ for any } \chi \in S_h(\mathcal{D}). \tag{5.2.10}$$

Thus, by Hölder's inequality and (5.2.5), (5.2.6), (5.2.9), for a suitable choice of $\chi \in S_h$,

$$\begin{aligned} |(e, v)| &\leq Ch^{r-1} \|w - \chi\|_{W_{p'}^{1}(\mathcal{D})} \\ &\leq Ch^{2r-2} \|w\|_{W_{p'}^{r}(\mathcal{D})} \leq \frac{Ch^{2r-2}}{(p'-1)} \|v\|_{W_{p'}^{r-2}(\mathcal{D})} \\ &\equiv \frac{Ch^{2r-2}}{(p'-1)}. \end{aligned} \tag{5.2.11}$$

Thus

$$\|e\|_{W_p^{-r(-2)}(\mathcal{D})} \leq Ch^{2r-2} p, \tag{5.2.12}$$

for $p < \infty$.

Analogously to our investigation in Section 4.1, cf. (4.1.4) we shall find that, now for $r \geq 3$, the negative norm estimate (5.2.12) leads to an abundance of (nonidentifiable) superconvergence points.

Let thus B_d be a ball of radius d, inside $\mathcal{D}$, and v a smooth nonnegative function with support in B_d such that

$$v = 1 \quad \text{on} \quad B_{d/2} \tag{5.2.13}$$

and

$$\|v\|_{W_\infty^{\ell}(B_d)} \leq \Lambda d^{-\ell}, \ \ell = 0, 1, \ldots, r-2. \tag{5.2.14}$$

Then assuming that e does not vanish on B_d, say $e > 0$ there, with $\frac{1}{p} + \frac{1}{p'} = 1$,

(5.2.15) $$\min_{x \in B_d} e(x) \le \int_{B_d} e(y)v(y)/\|v\|_{L_1(B_d)}.$$

By (5.2.13),

(5.2.16) $$\|v\|_{L_1(B_d)} \ge cd^n.$$

From (5.2.12) we then obtain

(5.2.17) $$\min_{\chi \in B_d} e(x) \le Cd^{-n}h^{2r-2}p\|v\|_{W_{p'}^{r-2}(B_d)}$$

and using (5.2.14) and Hölder's inequality, this is bounded as

(5.2.18) $$\begin{aligned}\min_{\chi \in B_d} e(x) &\le Cd^{-n}h^{2r-2}pd^{-(r-2)}d^{n/p'} \\ &= Ch^r\left(\frac{h}{d}\right)^{r-2} pd^{-n/p} \\ &\le Ch^r\left(\frac{h}{d}\right)^{r-2} \ln(1/d)\end{aligned}$$

where we have chosen $p = \ln(1/d)$.

We conclude the following upon choosing $d = h^{1-\sigma/(r-2)}$:

Proposition 5.2.1. *For $r \ge 3$, under the various assumptions above, any ball of radius $d = h^{1-\sigma/(r-2)}$ contains a point ξ where*

(5.2.19) $$|e(\xi)| \le Ch^{r+\sigma}\ln(1/h),$$

i.e., a point of superconvergence of (almost) order σ.

A similar investigation considering any first derivative $\frac{\partial e}{\partial x_i}$ and $\|\frac{\partial e}{\partial x_i}\|_{W_p^{-(r-1)}}$, for p close to ∞ (using now a priori estimates for the problem $-\Delta w + w = \partial v/\partial x$ in $\mathcal{D}$, with homogeneous Neumann conditions on $\partial\mathcal{D}$), gives that in any ball of size $d = h^{1-\sigma/(r-1)}$, there is a point where $\partial e/\partial x_i$ changes sign or, a point superconvergent (almost) to order σ (i.e., $\partial e/\partial x_i = 0(h^{r-1+\sigma}\ln(1/h))$. Note that $\partial e/\partial x_i$ may not be continuous in general.

The arguments given above in a simple case of "natural" boundary conditions may rather easily be extended to variable coefficient operators. They may also be extended to some essential boundary conditions such as homogeneous Dirichlet conditions on smooth domains, provided boundary data approximations are performed to sufficiently high accuracy, cf. [Schatz, Sloan and Wahlbin, Example 3.2]. (One notes that it is *not* in general enough to employ isoparametric elements at the boundary; these only approximate the boundary to order $0(h^r)$. Super–parametric approximations are required to obtain (5.2.12).) Due to the use of duality arguments, the arguments need to be carefully reconsidered when considering nonsmooth boundaries (such as polygonal domains) and nonsmooth coefficients; the a priori estimates used above in situations with sufficient smoothness do not necessarily hold in such cases.

5.3. General comments on local a priori error estimates.

The main technical tool for investigating superconvergence in the situation introduced in Section 5.1 will be given in Section 5.5. This section will be devoted to generalities and loose discussion, and the aim is to "appreciate" the rather formidable general tool to come ...

We note that our main tool in the one–dimensional situation, viz. that $u'_h \simeq P(u')$ where P is the L_2–projection into $\$_h = \$_h^{\mu-1,r-1}$, is now lost. To our knowledge, no one has been able to recover a suitable general analogue of it in several dimensions. (Only the case of tensor–product elements will build on one–dimensional results.)

Let us thus consider first the following simple situation, corresponding to the one in Lemma 3.2.1. Let $u_h \in S_h$ be a function such that

$$A(u_h, \chi) = 0, \text{for } \chi \in S_h^{comp}(\Omega), \tag{5.3.1}$$

where $\Omega \subset\subset \mathcal{D}$. For further simplicity here, we shall take the "simplest" case,

$$A(v, w) \equiv D(v, w) \equiv \int \nabla v \cdot \nabla w dx, \tag{5.3.2}$$

corresponding to Poisson's equation in (5.1.1).

Now, a rather well–known result about harmonic functions is the following: Let B_0 be any ball of radius r and B_d a concentric ball of radius $r + d$. (The base radius r is rather immaterial here; what matters is the difference d.) There is then a universal constant C such that if u is harmonic in B_d, then

$$\|\nabla u\|_{L_2(B_0)} \leq C d^{-1} \|u\|_{L_2(B_d)}. \tag{5.2.3}$$

We would now wish to prove a corresponding result for $u_h \in S_h$ a "discrete harmonic function", i.e., satisfying

$$D(u_h, \chi) = 0, \text{ for } \chi \in S_h^{comp}(B_d). \tag{5.3.4}$$

A first step is to try to prove the continuous case (5.3.3) by use of the similar property,

$$D(u, \chi) = 0, \text{ for } \chi \in \overset{\circ}{H}_1(B_d). \tag{5.3.5}$$

(I.e., we eschew, at least for the moment, any use of estimates for fundamental solutions or of "high–powered" Harmonic Analysis.)

Here is one "elementary" argument: Let ω be a "cut–off" function,

$$\omega \equiv 1 \quad \text{on} \quad B_0, \tag{5.3.6}$$

$$Supp(\omega) \subset\subset B_d, \tag{5.3.7}$$

and

$$\|\omega\|_{W_\infty^\ell(B_d)} \leq C d^{-\ell}, \; \ell = 0, 1. \tag{5.3.8}$$

It is well–known how to construct such a function (by use of $\exp(-1/x^2)$, e.g.). Then

$$\|\nabla u\|^2_{L_2(B_0)} = \int_{B_0} \nabla u \cdot \nabla u \leq \int_{B_d} \omega^2 \nabla u \cdot \nabla u = \|\omega \nabla u\|^2_{L_2(B_d)}. \tag{5.3.9}$$

Now,

$$\|\omega\nabla u\|^2_{L_2(B_d)} = \int_{B_d} \nabla u \nabla(\omega^2 u) - \int_{B_d} \nabla u \cdot 2\omega(\nabla\omega)u \tag{5.3.10}$$

and from (5.3.5), the first term on the right vanishes!

Thus, by Cauchy–Schwarz' inequality, and (5.3.8),

$$\begin{aligned} \|\omega\nabla u\|^2_{L_2(B_d)} &\le \Big| \int_{B_d} (\omega\nabla u)\cdot(\nabla\omega)u \Big| \\ &\le \|\omega\nabla u\|_{L_2(B_d)} C d^{-1} \|u\|_{L_2(B_d)} \end{aligned} \tag{5.3.11}$$

and we have proven that

$$\|\omega\nabla u\|_{L_2(B_d)} \le C d^{-1} \|w\|_{L_2(B_d)} \tag{5.3.12}$$

which of course implies (5.3.3) by use of (5.3.9).

Well, let us now see how this travels to the discrete situation, (5.3.4)! Equations (5.3.9) and (5.3.10) are fine, with u replaced by u_h. However, now we can't scrap the first term on the right of (5.3.10) since $\omega^2 u_h$ does not in general belong to S_h^{comp}. Instead of (5.3.10) we have for any $\chi \in S_h^{comp}(B_d)$,

$$\|\omega\nabla u_h\|^2_{L_2(B_d)} = \int_{B_d} \nabla u_h \nabla(\omega^2 u_h - \chi) - 2\int_{B_d} \omega\nabla u_h(\nabla\omega)u_h. \tag{5.3.13}$$

This is where superapproximation enters; this point is one of the essential discoveries in [Nitsche and Schatz 1974]. So, we now take a little bit more care and, say, restrict the cut–off function ω to have support in $B_{d/2}$. If we then let $d \ge c_1 h$ for c_1 large enough (d sufficiently large compared to element diameters), we make the appropriate superapproximation assumption: There exists $\chi \in S_h^{comp}(B_d)$ such that

$$\|\omega^2 u_h - \chi\|_{H^1(B_d)} \le C(h/d)\|u_h\|_{H^1(B_d)}, \tag{5.3.14}$$

cf. Theorem 2.3.1. From (5.3.13) we then have

$$\begin{aligned} \|\omega\nabla u_h\|^2_{L_2(B_d)} &\le \|\nabla u_h\|_{L_2(B_d)} C(h/d)\|u_h\|_{H^1(B_d)} \\ &\quad + C d^{-1}\|\omega\nabla u_h\|_{L_2(B_d)}\|u_h\|_{L_2(B_d)}. \end{aligned} \tag{5.3.15}$$

It follows from this and (5.3.9) (with u replaced by u_h), that

$$\|u_h\|_{H^1(B_0)} \le C_1\Big(\frac{h}{d}\Big)^{1/2}\|u_h\|_{H^1(B_d)} + C_1 d^{-1}\|u_h\|_{L_2(B_d)}. \tag{5.3.16}$$

Now, taking a little bit more care so that the original cut–off has support in $B_{d/4}$, say, we have upon repeating the argument with the obvious changes (B_0 replaced by $B_{d/2}$),

$$\|u_h\|_{H^1(B_0)} \le C_2\Big(\frac{h}{d}\Big)\|u_h\|_{H^1(B_d)} + C_2 d^{-1}\|u_h\|_{L_2(B_d)}. \tag{5.3.17}$$

To cap off the argument, we now assume an inverse estimate for derivatives: Shrinking B_d a little bit if necessary to coincide with a mesh–domain,

$$\|u_h\|_{H^1(B_d)} \le C_3 h^{-1}\|u_h\|_{L_2(B_d)}, \tag{5.3.18}$$

cf. Theorem 1.2.4. We then obtain the desired estimate

$$\|u_h\|_{H^1(B_0)} \le C_4 d^{-1} \|u_h\|_{L_2(B_d)}, \tag{5.3.19}$$

for $d \ge c_1 h$.

[Nitsche and Schatz 1974] now go on to show, e.g., that if

$$A(u - u_h, \chi) = 0, \text{ for } \chi \in S_h^{comp}(B_d), \tag{5.3.20}$$

then

$$\begin{aligned} \|u - u_h\|_{H^1(B_0)} \\ &\le C \min_{\chi \in S_h} \Big(\|u - \chi\|_{H^1(B_d)} + d^{-1}\|u - \chi\|_{L_2(B_d)}\Big) \\ &\quad + C d^{-1}\|u - u_h\|_{L_2(B_d)}. \end{aligned} \tag{5.3.21}$$

The argument for showing (5.3.21) goes briefly as follows: We first asume that B_0 is a ball of radius d and B_d one of radius $2d$. Let $\omega \in C_0^\infty(B_d)$ be a cutoff function roughly as before, but now with

$$\omega \equiv 1 \quad \text{on} \quad B_{d/2}. \tag{5.3.22}$$

Set $\widetilde{u} := \omega u$ and let $\widetilde{u}_h \in S_h^{comp}(B_d)$ be the Ritz–Galerkin projection into $S_h^{comp}(B_d)$, i.e., given by

$$D(\widetilde{u} - \widetilde{u}_h, \chi) = 0, \text{ for } \chi \in S_h^{comp}(B_d). \tag{5.3.23}$$

As is well known and trivial to prove, with the semi–norm $|u|_{\overset{\circ}{H}{}^1} = \|\nabla u\|_{L_2}$,

$$|\widetilde{u}_h|_{\overset{\circ}{H}{}^1(B_d)} \le |\widetilde{u}|_{\overset{\circ}{H}{}^1(B_d)} \le C\Big(|u|_{\overset{\circ}{H}{}^1(B_d)} + d^{-1}\|u\|_{L_2(B_d)}\Big). \tag{5.3.24}$$

We now write

$$u - u_h = (\widetilde{u} - \widetilde{u}_h) + (\widetilde{u}_h - u_h) \text{ on } B_0. \tag{5.3.25}$$

Since $\widetilde{u} - \widetilde{u}_h \in \overset{\circ}{H}{}^1(B_d)$ we have by Poincaré–Friedrichs' inequality and then using (5.3.24),

$$\begin{aligned} \|\widetilde{u} - \widetilde{u}_h\|_{H^1(B_0)} &\le C(1 + d)|\widetilde{u} - \widetilde{u}_h|_{\overset{\circ}{H}{}^1(B_d)} \\ &\le C(1 + d)|\widetilde{u}|_{\overset{\circ}{H}{}^1(B_d)} \\ &\le C\Big(|u|_{\overset{\circ}{H}{}^1(B_d)} + d^{-1}\|u\|_{L_2(B_d)}\Big). \end{aligned} \tag{5.3.26}$$

It remains to estimate the second term on the right of (5.3.25).

Now, by (5.3.22), for $\chi \in S_h^{comp}(B_{d/2})$,

$$D(\widetilde{u}_h - u_h, \chi) = D(\widetilde{u} - u, \chi) = 0, \tag{5.3.27}$$

i.e., $\widetilde{u}_h - u_h \in S_h$ is discrete harmonic in $B_{d/2}$. By (5.3.19) then, and again using that $\omega \equiv 1$ on $B_{d/2}$,

$$\begin{aligned} \|\widetilde{u}_h - u_h\|_{H^1(B_0)} &\le \frac{C}{d}\|\widetilde{u}_h - u_h\|_{L_2(B_{d/2})} \\ &\le \frac{C}{d}\Big(\|\widetilde{u}_h - \widetilde{u}\|_{L_2(B_{d/2})} + \|u - u_h\|_{L_2(B_{d/2})}\Big). \end{aligned} \tag{5.3.28}$$

By the same argument as in (5.3.26) we have

$$(5.3.29)\quad \frac{C}{d}\|\widetilde{u}_h - \widetilde{u}\|_{L_2(B_{d/2})} \le C|\widetilde{u}_h - \widetilde{u}|_{\overset{\circ}{H}{}^1(B_d)} \le C\Big(|u|_{\overset{\circ}{H}{}^1(B_d)} + d^{-1}\|u\|_{L_2(B_d)}\Big).$$

From (5.3.25), (5.3.26) and (5.3.28), (5.3.29) we have

$$(5.3.30)\quad \|u - u_h\|_{H^1(B_1)} \le C\Big(|u|_{\overset{\circ}{H}{}^1(B_d)} + d^{-1}\|u\|_{L_2(B_d)}\Big) + Cd^{-1}\|u - u_h\|_{L_2(B_d)}.$$

Recall that this was proven under the assumption that B_0, B_d were concentric balls of radii d, $2d$, respectively. (We did use Poincaré–Friedrichs' inequality.) By squaring and summing, (5.3.30) holds with the original setup of B_0 any ball and B_d the concentric one separated by a distance d. The desired estimate (5.3.21) now follows by writing $u - u_h = (u - \chi) + (\chi - u_h)$, for any $\chi \in S_h$.

We shall informally refer to the first term on the right of (an estimate such as) (5.3.21) as "the best local approximation", and the weaker norm last term as the "slush term". Note that if we try to take local approximability into account in a very sharp fashion, i.e., taking d smaller (while still $\ge c_1 h$), we pay for this in the slush term. Also note that the slush term in the similar result for the L_2–projection, (3.2.14), enjoys exponential decay $\exp(-d/h)$. The "decay" in (5.3.21) is much weaker, and [Nitsche and Schatz 1974] thus felt obliged to further weaken the slush–term into one involving negative norms:

$$(5.3.31)\quad \begin{aligned} &\|u - u_h\|_{H^1(B_0)} \\ &\le C\min_{\chi\in S_h}\big(\|u-\chi\|_{H^1(B_d)} + d^{-1}\|u-\chi\|_{L_2(B_d)}\big) \\ &+ Cd^{-1-s}\|u - u_h\|_{H^{-s}(B_d)}, \end{aligned}$$

cf. (1.1.10).

Correspondingly in the L_2–norm, again of course under suitable assumptions,

$$(5.3.32)\quad \begin{aligned} \|u - u_h\|_{L_2(B_0)} \le C\min_{\chi\in S_h}\big(\|u-\chi\|_{L_2(B_d)} + h\|u-\chi\|_{H^1(B_d)}\big) \\ + Cd^{-s}\|u - u_h\|_{H^{-s}(B_d)}. \end{aligned}$$

The inclusion of the term $h\|u - \chi\|_{H^1(B_d)}$ on the right reflects the fact that in general the H^1–projection is not a stable operator in L_2. E.g., in the one–dimensional situation for the problem $-u'' = f$ on I, $u(0) = u(1) = 0$, with piecewise linears, u_h is the interpolant of u. Pondering then the following figure

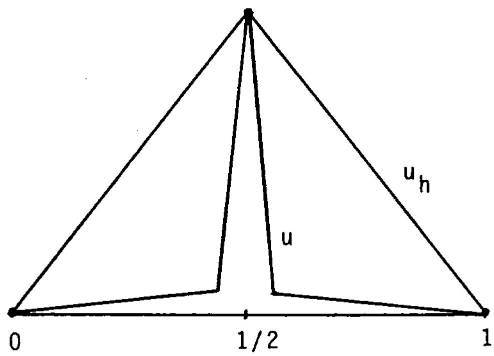


Figure 5.3.1

for $h = 1/2$, it is obvious that the $\overset{\circ}{H}{}^1$-projection cannot be a stable operator in *any* L_p-norm except for $p = \infty$. (This counterexample was noted in [Babuška and Osborn 1980, page 58]; if interelement continuity is C^1 or better, L_2-stability holds, cf. [Schatz and Wahlbin 1977, Appendix 2] for the local estimate, and [Babuška and Aziz 1972, Theorem 6.3.8] for the easier global case.)

[Nitsche and Schatz 1974] also gave corresponding L_2-based estimates in the case of $u - u_h$ satisfying (5.1.4), i.e.,

$$A(u - u_h, \chi) = F(\chi), \text{ for } \chi \in S_h^{comp}(\Omega). \tag{5.3.33}$$

The argument is, in brief, as follows. Again, first let B_0 be a ball of radius d and B_d a ball of radius $2d$. Now $F(\cdot)$ is a functional on $\overset{\circ}{H}{}^1(B_d)$, and we set

$$|||F|||_{H^{-1}(B_d)} = \sup_{\substack{v \in \overset{\circ}{H}{}^1(B_d) \\ |v|_{\overset{\circ}{H}{}^1(B_d)} = 1}} F(v). \tag{5.3.34}$$

Again, let us assume that $A(v, w) = D(v, w) \equiv \int \nabla v \cdot \nabla w$, for simplicity. Solve now the problem $-\Delta w = f$ in B_d, $w = 0$ on ∂B_d, i.e., find $w \in \overset{\circ}{H}{}^1(B_d)$ such that

$$D(w, v) = F(v), \text{ for all } v \in \overset{\circ}{H}{}^1(B_d). \tag{5.3.35}$$

This problem has a unique solution by Riesz representation theorem.

Then

$$D(u + w - u_h, \chi) = 0, \text{ for } \chi \in S_h^{comp}(B_d). \tag{5.3.36}$$

Thus by the triangle inequality and (5.3.31), taking $\chi = 0$ there for the moment,

$$(5.3.37)\qquad \begin{aligned} \|u - u_h\|_{H^1(B_0)} &\le \|(u+w) - u_h\|_{H^1(B_0)} + \|w\|_{H^1(B_0)} \\ &\le C\Big(|u+w|_{\overset{\circ}{H}{}^1(B_d)} + d^{-1}\|u+w\|_{L_2(B_d)}\Big) \\ &\quad + Cd^{-1}\|u+w-u_h\|_{L_2(B_d)} + \|w\|_{H^1(B_d)} \\ &\le C\Big(|u|_{\overset{\circ}{H}{}^1(B_d)} + d^{-1}\|u\|_{L_2(B_d)}\Big) \\ &\quad + Cd^{-1}\|u-u_h\|_{L_2(B_d)} \\ &\quad + C\Big(|w|_{\overset{\circ}{H}{}^1(B_d)} + d^{-1}\|w\|_{L_2(B_d)}\Big). \end{aligned}$$

Setting $v = w$ in (5.3.35) we find that

$$(5.3.38)\qquad |w|_{\overset{\circ}{H}{}^1(B_d)} \le |||F|||_{H^{-1}(B_d)}$$

and by Poincaré–Friedrichs' inequality, thus

$$(5.3.39)\qquad d^{-1}\|w\|_{L_2(B_d)} \le C|w|_{\overset{\circ}{H}{}^1(B_d)} \le C|||F|||_{H^{-1}(B_d)}.$$

So, from (5.3.37),

$$(5.3.40)\qquad \begin{aligned} \|u-u_h\|_{H^1(B_0)} &\le C\big(|u|_{\overset{\circ}{H}{}^1(B_d)} + d^{-1}\|u\|_{L_2(B_d)}\big) \\ &\quad + Cd^{-1}\|u-u_h\|_{L_2(B_d)} + C|||F|||_{H^{-1}(B_d)}. \end{aligned}$$

This was proven for B_0 and B_d balls of radii d and $2d$. To get it for a general B_0 of radius r, and B_d of radius $r+d$, one again squares and sums. One then needs the following lemma, which is not completely obvious.

Lemma 5.3.1. *If Ω_j, $j = 1, \dots, J$ are such that $\overline{\Omega}_j$ are disjoint sets, we have*

$$(5.3.41)\qquad |||F|||^2_{H^{-1}(\bigcup_1^J \Omega_j)} \ge \sum_{j=1}^{J} |||F|||^2_{H^{-1}(\Omega_j)}.$$

Assuming this lemma for the moment, we cover B_0 by (finite) families of disjoint balls of radius d (how many families? — solve the sphere packing problem!). The result (5.3.40) is then extended to general B_0 and B_d as before. (Indeed, to any domain Ω_0 and $\Omega_d = \{x : dist(x, \Omega_0) \le d\}$.)

Writing $u - u_h = (u-\chi) - (u_h - \chi)$ for $\chi \in S_h$ then gives for u and u_h satisfying (5.3.33) (i.e., (5.1.4)),

$$(5.3.42)\qquad \begin{aligned} &\|u-u_h\|_{H^1(B_0)} \\ &\le C \min_{\chi \in S_h} \Big(|u-\chi|_{\overset{\circ}{H}{}^1(B_d)} + d^{-1}\|u-\chi\|_{L_2(B_d)}\Big) \\ &\quad + Cd^{-1}\|u-u_h\|_{L_2(B_d)} + C|||F|||_{H^{-1}(B_d)}. \end{aligned}$$

(Again, the term $\|u-u_h\|_{L_2(B_d)}$ can be replaced by a negative norm at the expense of additional inverse powers of d.)

Remark 5.3.1. In (5.3.33), $F(\cdot)$ is seen only by how it acts on $\chi \in S_h^{comp}(\Omega)$. Since $S_h^{comp}(\Omega) \subseteq \overset{\circ}{H}{}^1(\Omega)$, an argument involving the Hahn–Banach extension theorem then shows that $|||F|||_{H^{-1}(B_d)}$ in (5.3.43) may be replaced by

$$\max_{\substack{\chi \in S_h^{comp}(B_d) \\ |\chi|_{\overset{\circ}{H}{}^1(B_d)}=1}} F(\chi). \tag{5.3.43}$$ □

It remains to prove Lemma 5.3.1.

Proof of Lemma 5.3.1. We shall prove it for $J = 2$; it will be obvious that the proof generalizes to any J.

If Ω_1 and Ω_2 are two balls such that $\overline{\Omega}_1$ and $\overline{\Omega}_2$ are disjoint, we let $v_j \in \overset{\circ}{H}{}^1(\Omega_j)$, with $|v_j|_{\overset{\circ}{H}{}^1(\Omega_j)} = 1$, for $j = 1, 2$. Setting (for simplicity in notation),

$$\theta_j = |||F|||_{H^{-1}(\Omega_j)}, \ j = 1, 2$$

we have with $\Omega = \Omega_1 \cup \Omega_2$ and

$$v := \theta_1 v_1 + \theta_2 v_2 \in \overset{\circ}{H}{}^1(\Omega).$$

Then

$$|||F|||^2_{H^{-1}(\Omega)} \geq \left(\frac{F(v)}{|v|_{\overset{\circ}{H}{}^1(\Omega)}}\right)^2 = \frac{(\theta_1 F(v_1) + \theta_2 F(v_2))^2}{\theta_1^2 + \theta_2^2}.$$

We take the supremum over such v_1 and v_2 as above and obtain

$$|||F|||^2_{H^{-1}(\Omega)} \geq \frac{(\theta_1^2 + \theta_2^2)^2}{\theta_1^2 + \theta_2^2} = |||F|||^2_{H^{-1}(\Omega_1)} + |||F|||^2_{H^{-1}(\Omega_2)}.$$ □

5.4. General comments on L_∞ estimates.

In the years 1975–1976 the following basic papers on L_∞-estimates in finite element methods in multi-dimensions on general meshes were published: [Natterer 1975], [Nitsche 1975], [Frehse and Rannacher 1976] and [Scott 1976].

To this was quickly added an investigation into the harder problem of *localized* maximum norm estimates, [Schatz and Wahlbin 1977]. (There had been previous papers on local maximum norm estimates in several space dimensions, on uniform or translation invariant meshes, cf. [Bramble and Thomée 1974], [Bramble, Nitsche and Schatz 1975] and [Bramble and Schatz 1976].)

In the next section we shall give — finally! — our basic tool to be used in our superconvergence investigations. We therefore feel no need to review the papers mentioned above. However, some readers may appreciate seeing how the local H^1 estimates of [Nitsche–Schatz 1974] actually lead to global maximum norm error estimates. So, as a longish aside, we shall give the argument in the plane case of two space dimensions and the particular case of piecewise linear functions on triangular finite elements. (In this case much technical detail is simplified; cf. [Schatz and Wahlbin 1982, Remark 5.3].)

Let, for simplicity, $\mathcal{D}$ be a convex plane domain with a smooth boundary $\partial\mathcal{D}$. Let $\tau_h = \{\tau_i^h\}_{i=1}^{N(h)}$ be disjoint open triangles such that

$$\overline{\mathcal{D}}_h = \bigcup_{i=1}^{N(h)} \overline{\tau_i^h} \subseteq \overline{\mathcal{D}} \tag{5.4.1}$$

and such that

$$dist(\mathcal{D}_h, \partial\mathcal{D}) \leq Ch^2. \tag{5.4.2}$$

We assume that the triangles are edge–to–edge, i.e., no vertex of one triangle falls in the interior of the edge of another. We also assume that with $h = \max_i diam(\tau_i^h)$, the family of triangulations is quasi–uniform in the sense that with ρ_i^h the radius of the maximal inscribed disc in τ_i^h,

$$\rho_i^h \geq c_0 h \tag{5.4.3}$$

where $c_0 > 0$ is independent of i and h.

The problem we shall consider is, again for simplicity, that of finding u such that

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \mathcal{D}, \\ u &= 0 \quad \text{on} \quad \partial\mathcal{D}. \end{aligned} \tag{5.4.4}$$

The weak form of (5.4.3) namely, to find $u \in \overset{\circ}{H}{}^1(\mathcal{D})$ such that

$$D(u,\chi) \equiv \int_{\mathcal{D}} \nabla u \cdot \nabla \chi = (f,\chi), \text{ for } \chi \in \overset{\circ}{H}{}^1(\mathcal{D}), \tag{5.4.5}$$

incorporates the essential Dirichlet boundary conditions. Our piece–wise linear finite element family is now

$$\overset{\circ}{S}_h(\mathcal{D}) = \{\chi \in C^0(\mathcal{D}_h) : \chi = 0 \text{ on } \partial\mathcal{D}_h,\ \chi|_{\tau_i^h} = c_0 + c_1 x + c_2 y\}. \tag{5.4.6}$$

We regard the functions in $\overset{\circ}{S}_h(\mathcal{D})$ as $\equiv 0$ in the skin–layer $\mathcal{D}\backslash\mathcal{D}_h$. The approximation $u_h \in \overset{\circ}{S}_h$ is sought as

$$D(u_h,\chi) = (f,\chi), \text{ for } \chi \in \overset{\circ}{S}_h,$$

i.e.,

$$D(u - u_h,\chi) = 0, \text{ for } \chi \in \overset{\circ}{S}_h. \tag{5.4.7}$$

Let us now fix $x_0 \in \mathcal{D}$. If $x_0 \in \mathcal{D}\backslash\mathcal{D}_h$, obviously by (5.4.2), $|(u-u_h)(x_0)| \leq Ch^2$ if u is smooth. We assume now that $x_0 \in \mathcal{D}_h$. Using a suitable $\chi \in \overset{\circ}{S}_h$, we have if $x_0 \in \tau_{i_0}^h \equiv \tau_0$,

$$\begin{aligned} |(u-u_h)(x_0)| &\leq |(u-\chi)(x_0)| + |(\chi - u_h)(x_0)| \\ &\leq |(u-\chi)(x_0)| + Ch^{-1}\|\chi - u_h\|_{L_2(\tau_0)} \end{aligned} \tag{5.4.8}$$

where we used an easily verifiable inverse estimate, cf. [Ciarlet 1991]. Thus, again by the triangle inequality and also Hölder's inequality,

$$|(u-u_h)(x_0)| \leq C\|u-\chi\|_{L_\infty(\tau_0)} + Ch^{-1}\|u-u_h\|_{L_2(\tau_0)}. \tag{5.4.9}$$

Hence,

$$(5.4.10) \qquad |(u-u_h)(x_0)| \le C \min_{\chi \in \overset{\circ}{S}_h} \|u-\chi\|_{L_\infty(\mathcal{D})} + Ch^{-1}\|u-u_h\|_{L_2(\tau_0)}.$$

Now for a duality argument:

$$(5.4.11) \qquad \|u-u_h\|_{L_2(\tau_0)} = \sup_{\|v\|_{L_2(\tau_0)}=1} (u-u_h, v).$$

For each such v, let $-\Delta w = v$ in $\mathcal{D}$, $w = 0$ on $\partial\mathcal{D}$. Then

$$(5.4.12) \qquad (u-u_h, v) = D(u-u_h, w).$$

With $w_h \in \overset{\circ}{S}_h$ defined by

$$(5.4.13) \qquad D(\chi, w-w_h) = 0, \text{ for } \chi \in \overset{\circ}{S}_h$$

we then have, for any $\chi \in \overset{\circ}{S}_h$,

$$(5.4.14) \qquad (u-u_h, v) = D(u-\chi, w-w_h).$$

So,

$$(5.4.15) \qquad |(u-u_h, v)| \le C \min_{\chi \in \overset{\circ}{S}_h} \|u-\chi\|_{W^1_\infty(\mathcal{D})} \, \|w-w_h\|_{W^1_1(\mathcal{D})}.$$

Next, let

$$(5.4.16) \qquad \Omega_j = \{x : 2^{-j} \le |x-x_0| \le 2^{-j+1}\} \cap \mathcal{D},$$

and let $\mathcal{J}_0$ be such that $2^{-\mathcal{J}_0} = c_0 h$ with c_0 large enough (or, the nearest integer to accomplish this). Also let

$$(5.4.17) \qquad d_j = 2^{-j},$$

and

$$(5.4.18) \qquad \Omega_{\mathcal{J}_0} = \{x : |x-x_0| \le 2^{-\mathcal{J}_0}\} \cap \mathcal{D}.$$

Then

$$(5.4.19) \qquad \|w-w_h\|_{W^1_1(\mathcal{D})} = \|w-w_h\|_{W^1_1(\Omega_{\mathcal{J}_0})} + \sum_{j=I}^{\mathcal{J}_0-1} \|w-w_h\|_{W^1_1(\Omega_j)}$$

where I is small enough so that $\cup\, \Omega_j$ covers $\mathcal{D}$. Here,

$$(5.4.20) \qquad \begin{aligned} \|w-w_h\|_{W^1_1(\Omega_{\mathcal{J}_0})} &\le Ch\|w-w_h\|_{H^1(\mathcal{J}_0)} \\ &\le Ch\|w-w_h\|_{H^1(\mathcal{D})} \le Ch^2\|w\|_{H^2(\mathcal{D})} \le Ch^2, \end{aligned}$$

where we used that w_h is the $\overset{\circ}{H}{}^1$ projection of w so that $\|w-w_h\|_{H^1(\mathcal{D})} = \min_{\chi \in \overset{\circ}{S}_h} \|w-\chi\|_{H^1(\mathcal{D})}$ and then approximation theory: take χ as the interpolant at vertices of w; note that $w \in H^2(\mathcal{D})$ since $\partial\mathcal{D}$ is smooth. (See any standard textbook on Partial Differential Equations.) Then $w \in \mathcal{C}^0(\mathcal{D})$ by Sobolev's theorem and the piecewise linear interpolant χ of w makes sense and is well–known to satisfy

$\|w-\chi\|_{H^1(\mathcal{D})} \le Ch\|w\|_{H^2(\mathcal{D})}$. The reader is encouraged to check this with respect to the "skin–layer" $\mathcal{D} \setminus \mathcal{D}_h$, where $\chi \equiv 0$.

For $j < \mathcal{J}_0$ we have by local H^1–error estimates, see (5.3.20) (cf. also [Schatz and Wahlbin 1982, Theorem 4.1] for the extension needed in case the domains involved abutt on the boundary $\partial\mathcal{D}$),

$$
\begin{aligned}
(5.4.21) \qquad \|w-w_h\|_{W_1^1(\Omega_j)} &\le d_j\|w-w_h\|_{H^1(\Omega_j)} \\
&\le Cd_j\Big\{\min_{\chi\in\mathring{S}_h}\big(\|w-\chi\|_{H^1(\Omega_j')}+d_j^{-1}\|w-\chi\|_{L_2(\Omega_j')}\big) \\
&\qquad\qquad + d_j^{-1}\|w-w_h\|_{L_2(\Omega_j')}\Big\} \\
&\le Cd_j\min_{\chi\in\mathring{S}_h}\big(\|w-\chi\|_{H^1(\Omega_j')}+d_j^{-1}\|w-\chi\|_{L_2(\Omega_j')}\big) \\
&\qquad\qquad + C\|w-w_h\|_{L_2(\mathcal{D})}.
\end{aligned}
$$

Here, $\Omega_j' = \Omega_{j-1}\cup\overline{\Omega}_j\cup\Omega_{j+1}$ (recall that $d_j \ge c_0 h$ with c_0 large enough). Again by standard duality argument over L_2, and standard approximation theory (and checking the skin layer $\mathcal{D}\setminus\mathcal{D}_h$!),

$$
(5.4.22) \qquad \|w-w_h\|_{L_2(\mathcal{D})} \le Ch^2\|w\|_{H^2(\mathcal{D})} \le Ch^2
$$

where we again used a standard a priori estimate in the last step. Furthermore, again by approximation theory, taking χ to be the piece–wise linear interpolant,

$$
(5.4.23) \qquad \min_{\chi\in\mathring{S}_h}\big(\|w-\chi\|_{H^1(\Omega_j')}+d_j^{-1}\|w-\chi\|_{L_2(\Omega_j')}\big) \le Ch\|w\|_{H^2(\Omega_j'')},
$$

with $\Omega_j'' = \Omega_{j-2}\cup\overline{\Omega}_{j-1}\cup\overline{\Omega}_j\cup\overline{\Omega}_{j+1}\cup\Omega_{j+2}$. Since c_0 is large enough is assumed, the support of v $(=\tau_0)$ may be assumed $\simeq d_j$ away from Ω_j''. Now,

$$
(5.4.24) \qquad w(x) = \int_{\tau_0} v(y)G^{(x)}(y)dy
$$

where $G^{(x)}(y)$ is the Green's function for the problem $-\Delta w = v$ in $\mathcal{D}$, $w=0$ on $\partial\mathcal{D}$. It is well known that for any k^{th} derivative with respect to x, $k\ge 1$, D_x^k, we have (remember, $n=2$)

$$
(5.4.25) \qquad |D_x^k G^{(x)}(y)| \le C|x-y|^{-k}
$$

since $\partial\mathcal{D}$ is smooth (cf. [Krasovskiǐ 1967]). Hence, using also Hölder's inequality,

$$
\begin{aligned}
(5.4.26) \qquad \|w\|_{H^2(\Omega_j'')} &\le Cd_j\|w\|_{W_\infty^2(\Omega_j'')} \le Cd_j\int_{\tau_0}|v(y)|d_j^{-2}dy \\
&\le Cd_jh\|v\|_{L_2(\tau_0)}d_j^{-2} = Chd_j^{-1}.
\end{aligned}
$$

Then, from (5.4.23),

$$
(5.4.27) \qquad \min_{\chi\in\mathring{S}_h}\big(\|w-\chi\|_{H^1(\Omega_j')}+d_j^{-1}\|w-\chi\|_{L_2(\Omega_j')}\big) \le Ch^2d_j^{-1},
$$

and so from (5.4.21) and (5.4.22),

$$
(5.4.28) \qquad \|w-w_h\|_{W_1^1(\Omega_j)} \le Ch^2.
$$

Combining this with (5.4.20), we have from (5.4.19), and since $2^{-\mathcal{J}_0} = c_0 h$, i.e., $\mathcal{J}_0 \simeq \ln(1/h)$,

$$\|w - w_h\|_{W_1^1(\mathcal{D})} \le Ch^2 + C \sum_{j=I}^{\mathcal{J}_0 - 1} h^2 \le Ch^2 \ln(1/h). \tag{5.4.29}$$

Hence from (5.4.15),

$$|(u - u_h, v)| \le C \min_{\chi \in \overset{\circ}{S}_h} \|u - \chi\|_{W_\infty^1(\mathcal{D})} h^2 \ln(1/h) \tag{5.4.30}$$

so that, finally, from (5.4.11) and (5.4.10),

$$|(u - u_h)(x_0)| \le C \ln(1/h) \min_{\chi \in \overset{\circ}{S}_h} \left\{ \|u - \chi\|_{L_\infty(\mathcal{D})} + h\|u - \chi\|_{W_\infty^1(\mathcal{D})} \right\}, \tag{5.4.31}$$

or, if u is smooth enough, with $C = C(u)$,

$$\|u - u_h\|_{L_\infty(\mathcal{D})} \le Ch^2 \ln 1/h. \tag{5.4.32}$$

Actually, by an argument involving integration by parts over each element in (5.4.14), and for general r, cf. [Schatz and Wahlbin 1982], we have

$$\|u - u_h\|_{L_\infty(\mathcal{D}_h)} \le C(\ln(1/h))^{\overline{r}} \min_{\chi \in S_h} \|u - \chi\|_{L_\infty(\mathcal{D}_h)} \tag{5.4.33}$$

where

$$\overline{r} = \begin{cases} 1, & \text{if} \quad r = 2, \\ 0, & \text{if} \quad r \ge 3. \end{cases}$$

We are now ready to describe:

5.5. The main technical tools for proving superconvergence in second order elliptic problems in several space dimensions.

As an introduction here, let us recall (1.3.7),

$$|(\theta', \chi')| \le Ch^r \|\chi\|_{W_1^1(I)}, \text{ for } \chi \in \overset{\circ}{S}_h.$$

From this, in the one–dimensional situation, we were able to deduce that, (1.3.13),

$$\|\theta'\|_{L_\infty(I)} \le Ch^r.$$

What we shall describe are localized generalizations of this to any number of space dimensions. Or, we can view them as generalizations of the local H^1–estimate (5.3.43) to W_∞^1 and L_∞. They are proven "essentially" following the scheme in Section 5.3: First, consider "discrete harmonic" functions (or, more generally functions satisfying the discrete analogue of $Lu = 0$, L a second order elliptic operator). Show that one may bound the $W_\infty^1(B_0)$–norm in terms of $L_\infty(B_d)$, or $L_2(B_d)$, or negative norms, at the expense of inverse powers of d. (Decidedly nontrivial, now!) Then let $A(u - u_h, \chi) = 0$, for $\chi \in S_h^{comp}(B_d)$. Use a cut–off function ω and consider $\widetilde{u} = \omega u$ and $\widetilde{u}_h$, a local Ritz–Galerkin projection, this time actually w.r.t. a local Neumann problem (cf. (5.3.22) of seq.). To treat the difference $\widetilde{u} - \widetilde{u}_h$ requires so to speak *global* L_∞–based estimates for this local problem, cf. (5.4.32) and (5.4.33). Then $u_h - \widetilde{u}_h \in S_h$ is locally "discrete harmonic" and already taken care of. The right hand side functional $F(\cdot)$ has to be worked in, of course. The details are longish and given in [Schatz and Wahlbin 1994].

Let thus Ω_0 be a basic domain compactly contained in $\mathcal{D}$, and Ω_d with $\Omega_0 \subset\subset \Omega_d \subseteq \mathcal{D}$ where $d = dist(\partial\Omega_0, \partial\Omega_d)$, and u and u_h given functions with $u_h \in S_h$ such that

$$A(u - u_h, \chi) = F(\chi), \text{ for } \chi \in S_h^{comp}(\Omega_d). \tag{5.5.1}$$

Let with $\frac{1}{q} + \frac{1}{q'} = 1$,

$$\|v\|_{W_q^{-s}(\Omega)} = \sup_{\substack{\varphi \in C_0^\infty(\Omega) \\ |\varphi|_{W_{q'}^s(\Omega)}=1}} (v, \varphi) \tag{5.5.2}$$

and

$$|||F|||_{-1,\infty,\Omega_0} = \sup_{\substack{\varphi \in C_0^\infty(\Omega) \\ |\varphi|_{W_1^1(\Omega)}=1}} F(\varphi). \tag{5.5.3}$$

Under various conditions given in [Schatz and Wahlbin 1994] ("typical finite element spaces", quasi-uniform meshes on Ω_d) we then have for $d \geq c_0 h$, with c_0 sufficiently large.

Theorem 5.5.1. *There exist constants c_0 and C such that for $d \geq c_0 h$,*

$$\begin{aligned} |u - u_h|_{W_\infty^1(\Omega_0)} &+ d^{-1}\|u - u_h\|_{L_\infty(\Omega_0)} \\ &\leq C \min_{\chi \in S_h} \left(|u - \chi|_{W_\infty^1(\Omega_d)} + d^{-1}\|u - \chi\|_{L_\infty(\Omega_d)}\right) \\ &+ C d^{-1-s-n/q}\|u - u_h\|_{W_q^{-s}(\Omega_d)} \\ &+ C \ln(d/h)|||F|||_{-1,\infty,\Omega_d}. \end{aligned} \tag{5.5.4}$$

Remark 5.5.1. Corresponding to Remark 5.3.1, we may replace $|||F|||_{-1,\infty,\Omega_d}$ in (5.5.4) by

$$\max_{\substack{\chi \in S_h^{comp}(B_d) \\ |\chi|_{\overset{\circ}{W}{}_1^1(B_d)}=1}} F(\chi). \qquad \square$$

The constants C and c_0 depend on the finite element spaces and its local properties, such as constants occurring in quasi-uniformity conditions, approximation hypotheses and superapproximation conditions. They also depend on the smoothness properties of the coefficients a_{ij}, a_i and a (cf. (5.1.1)) and on the ellipticity constant in (5.1.2). Of course, they also depend on n, s and q. However, they are independent of h, u, u_h and F; and, no conditions on coercivity, local or global, of the form $A(v, w)$ are involved (beyond what has already been said about ellipticity for the highest order coefficients a_{ij}.)

The corresponding result for the error in function values for $u - u_h$ is as follows. Here we have, if r is the optimal order of approximation in L_p–spaces,

$$\bar{r} = \begin{cases} 1, & \text{if } \quad r = 2, \\ 0, & \text{if } \quad r \geq 3. \end{cases} \tag{5.5.5}$$

Theorem 5.5.2. *There exist constants c_0 and C such that for $d \geq c_0 h$,*

$$\begin{aligned}(5.5.6) \qquad \|u - u_h\|_{L_\infty(\Omega_0)} \leq C(\ln d/h)^{\bar{r}} \min_{\chi \in S_h} \|u - \chi\|_{L_\infty(\Omega_d)} \\ + Cd^{-s-n/q}\|u - u_h\|_{W_q^{-s}(\Omega_d)} \\ + Ch(\ln d/h)^{\bar{r}} |||F|||_{-1,\infty,\Omega_d} \\ + C(\ln d/h)|||F|||_{-2,\infty,\Omega_d}.\end{aligned}$$

Here, in addition to the definitions in (5.5.2) and (5.5.3),

$$(5.5.7) \qquad |||F|||_{-2,\infty,\Omega} = \sup_{\substack{\varphi \in C_0^\infty(\Omega) \\ |\varphi|_{W_1^2(\Omega)}=1}} F(\varphi).$$

For the necessity of the logarithmic factor $(\ln d/h)^{\bar{r}}$ in the case of $r = 2$ in (5.5.6), we refer to [Haverkamp 1984]. And, for the history of Theorems 5.5.1 and 5.5.2, see the discussion in [Schatz and Wahlbin 1994].

Chapter 6. Superconvergence in tensor–product elements.

6.1. Introduction.

We give some results of the general nature that superconvergence points which are present in the case of one–dimensional two–point boundary value problems automatically give rise to superconvergence points in a multi–dimensional tensor product situation. We shall give details only in the two–dimesional case; generalizations to arbitrary n will be fairly obvious. The general principle was applied to L_2–projections in Section 4.2.

Our investigation is limited in that we shall not treat the general case of operators of the form (5.1.1). For superconvergence in derivatives we restrict ourselves to the case of

$$-div(\alpha(x,y)\nabla u) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a_i(x,y)u) + a(x,y)u = f, \tag{6.1.1}$$

with α a scalar smooth uniformly positive function, $\alpha \geq c_{ell} > 0$, and with corresponding bilinear form

$$A(v,w) = \int \Big(\alpha\nabla v \cdot \nabla w + \sum_{i=1}^{n} a_i v \,\frac{\partial w}{\partial x_i} + avw\Big)dxdy. \tag{6.1.2}$$

For superconvergence in function values we shall only treat the case of the Laplacian, $\alpha \equiv 1$, $a_i \equiv a \equiv 0$.

Our general starting point is that we have u and $u_h \in S_h(\Omega_1)$ such that

$$A(u - u_h, \chi) = 0, \text{ for } \chi \in S_h^{comp}(\Omega_1). \tag{6.1.3}$$

We shall then investigate superconvergence points for $u - u_h$ in $\Omega_0 \subset\subset \Omega_1$. For simplicity, we shall only consider the case when $dist(\Omega_0, \partial\Omega_1)$ is of unit magnitude. The essential ideas in this chapter are those of [Douglas, Dupont and Wheeler 1974b], combined with those of Theorems 5.5.1 and 5.5.2.

6.2. Superconvergence in derivatives for the case of the Laplacian.

Let $\Omega_0 = I_0^x \times I_0^y$ be a rectangle and $\Omega_1 = I_1^x \times I_1^y$ another rectangle containing Ω_0 with $dist(\partial\Omega_0, \partial\Omega_1) = 1$. We assume that on Ω_1, $S_h(\Omega_1)$ consists of linear combinations of $\chi^x(x)\chi^y(y)$ where χ^x and χ^y belong to one–dimensional finite element spaces $S_h^x(I_1^x)$ and $S_h^y(I_1^y)$. We also assume that Ω_1 is adjusted so that the endpoints of I_1^x and I_1^y coincide with meshpoints. We write in common notation in the field (it differs from the "tensor–product" ideas of Algebra),

$$S_h(\Omega_1) = S_h^x(I_1^x) \otimes S_h^y(I_1^y). \tag{6.2.1}$$

We shall follow here the ideas of [Douglas, Dupont and Wheeler 1974b]. Our starting point is a function u and $u_h \in S_h$ such that

$$D(u - u_h, \chi) \equiv \iint \Big(\frac{\partial(u-u_h)}{\partial x}\,\frac{\partial\chi}{\partial x} + \frac{\partial(u-u_h)}{\partial y}\,\frac{\partial\chi}{\partial y}\Big)dxdy = 0, \quad \text{for } \chi \in S_h^{comp}(\Omega_1). \tag{6.2.2}$$

Let now ω be a smooth cut–off function with $supp(\omega) \subset \Omega_1$ and $\omega \equiv 1$ on $\Omega_{1/2}$ where $\Omega_0 \subset\subset \Omega_{1/2} \subset\subset \Omega_1$ with fixed separation. Let

$$\widetilde{u} := \omega u. \tag{6.2.3}$$

We let $\overset{\circ}{S}{}^x_h(I^x_1)$ and $\overset{\circ}{S}{}^y_h(I^y_1)$ be the subspaces vanishing at the endpoints and let R^x and R^y denote the one dimensional projections into them: E.g., $R^x v$ is given by

$$\int_{I^x_1} (R^x v)' \chi' = \int_{I^x_1} v' \chi', \text{ for } \chi \in \overset{\circ}{S}{}^x_h(I^x_1). \tag{6.2.4}$$

We then define

$$W_h := R^x \otimes R^y(\widetilde{u}), \tag{6.2.5}$$

i.e., we first take the y–projection to obtain $R^y(\widetilde{u}(x,\cdot))(y)$ for each fixed x, and then the x–projection of that for each fixed y (or, vice versa). Clearly, due to the tensor–product structure, $W_h \in \overset{\circ}{S}_h(\Omega_1)$. We shall now consider $u_h - W_h$. For $\chi \in S_h^{comp}(\Omega_{1/2})$, since $\omega \equiv 1$ on $\Omega_{1/2}$, we have using (6.2.4) and its counterpart for y,

$$\begin{aligned}
D(u_h - W_h, \chi) &= D(\widetilde{u} - W_h, \chi) \\
&= \iint \frac{\partial}{\partial x}(\widetilde{u} - W_h)\frac{\partial}{\partial x}\chi + \frac{\partial}{\partial y}(\widetilde{u} - W_h)\frac{\partial}{\partial y}\chi \\
&= \iint \frac{\partial}{\partial x}(I - R^x \otimes R^y)(\widetilde{u})\frac{\partial \chi}{\partial x} + \frac{\partial}{\partial y}(I - R^x \otimes R^y)(\widetilde{u})\frac{\partial \chi}{\partial y} \\
&= \int_{I^y_1} dy\Big(\int_{I^x_1} \frac{\partial}{\partial x}(I - R^x \otimes R^y)(\widetilde{u})\frac{\partial \chi}{\partial x}dx\Big) \\
&\quad + \int_{I^x_1} dx\Big(\int_{I^y_1} \frac{\partial}{\partial y}(I - R^x \otimes R^y)(\widetilde{u})\frac{\partial \chi}{\partial y}dy\Big) \\
&= \int_{I^y_1} dy\Big(\int_{I^x_1} \frac{\partial}{\partial x}(I - I^x \otimes R^y)(\widetilde{u})\frac{\partial \chi}{\partial x}dx\Big) \\
&\quad + \int_{I^x_1} dx\Big(\int_{I^y_1} \frac{\partial}{\partial y}(I - R^x \otimes I^y)(\widetilde{u})\frac{\partial \chi}{\partial y}dy\Big) \\
&= \iint (I - I^x \otimes R^y)\Big(\frac{\partial \widetilde{u}}{\partial x}\Big)\frac{\partial \chi}{\partial x} + \iint (I - R^x \otimes I^y)\Big(\frac{\partial \widetilde{u}}{\partial y}\Big)\frac{\partial \chi}{\partial y} \\
&\equiv F(\chi), \quad \text{for} \quad \chi \in S_h^{comp}(\Omega_{1/2}).
\end{aligned} \tag{6.2.6}$$

Now, for $\varphi \in C_0^\infty(\Omega_{1/2})$,

$$\begin{aligned}
&\Big|\iint (I - I^x \otimes R^y)\Big(\frac{\partial \widetilde{u}}{\partial x}\Big)\frac{\partial \varphi}{\partial x}dxdy\Big| \\
&= \Big|\int_{I^x_1} dx\Big(\int_{I^y_1} I^x \otimes (I^y - R^y)\Big(\frac{\partial \widetilde{u}}{\partial x}\Big)(x,y)\frac{\partial \varphi}{\partial x}(x,y)dy\Big)\Big| \\
&\le \max_x \Big\|(I^y - R^y)\Big(\frac{\partial \widetilde{u}}{\partial x}\Big)(x,\cdot)\Big\|_{L_\infty(I^y)} \int_{I^x_1}\int_{I^y_1} \Big|\frac{\partial \varphi}{\partial x}(x,y)\Big|dxdy.
\end{aligned} \tag{6.2.7}$$

We next make the assumption, if you like, that

$$\|(I^y - R^y)v\|_{L_\infty(I_1^y)} \le Ch^r \|v\|_{W_\infty^r(I_1^y)}, \tag{6.2.8}$$

so that from (6.1.9) and the symmetric argument for the second term in $F(\cdot)$,

$$|||F|||_{-1,\infty,\Omega_{1/2}} \le Ch^r \|u\|_{W_\infty^{r+1}(\Omega_{1/2})}. \tag{6.2.9}$$

Thus, $D(u_h - W_h, \chi) = F(\chi)$, for $\chi \in S_h^{comp}(\Omega_{1/2})$ with $F(\cdot)$ satisfying (6.2.9). We may then use Theorem 5.5.1 (with "$u = 0$") and find that

$$\|u_h - W_h\|_{W_\infty^1(\Omega_0)} \le C\|u_h - W_h\|_{W_q^{-s}(\Omega_{1/2})} + C(\ln 1/h)h^r. \tag{6.2.10}$$

Hence, by use of the triangle inequality and that $\omega \equiv 1$ on $\Omega_{1/2}$, we have for any point $\overline{x}, \overline{y} \in \Omega_0$, with $\partial = \partial/\partial x$ or $\partial/\partial y$,

$$\begin{aligned} |\partial(u - u_h)(\overline{x}, \overline{y})| &\le |\partial(\widetilde{u} - W_h)(\overline{x}, \overline{y})| + \|W_h - u_h\|_{W_\infty^1(\Omega_0)} \\ &\le |\partial(\widetilde{u} - W_h)(\overline{x}, \overline{y})| + C\|W_h - u_h\|_{W_q^{-s}(\Omega_{1/2})} + C(\ln 1/h)h^r \\ &\le |\partial(\widetilde{u} - W_h)(\overline{x}, \overline{y})| + C\|u - u_h\|_{W_q^{-s}(\Omega_{1/2})} \\ &\qquad + C\|\widetilde{u} - W_h\|_{W_q^{-s}(\Omega_{1/2})} + C(\ln 1/h)h^r \\ &\le |\partial(\widetilde{u} - W_h)(\overline{x}, \overline{y})| + C\|\widetilde{u} - W_h\|_{L_\infty(\Omega_{1/2})} + C(\ln 1/h)h^r \\ &\qquad + C\|u - u_h\|_{W_q^{-s}(\Omega_{1/2})}. \end{aligned} \tag{6.2.11}$$

We now observe that

$$\begin{aligned} \widetilde{u} - W_h &= (I^x \otimes I^y - R^x \otimes R^y)(\widetilde{u}) \\ &= (I^x - R^x) \otimes I^y(\widetilde{u}) + I^x \otimes (I^y - R^y)(\widetilde{u}) \\ &\qquad - (I^x - R^x) \otimes (I^y - R^y)(\widetilde{u}). \end{aligned} \tag{6.2.12}$$

Under appropriate conditions on the one–dimensional projections (cf. Chapter 1) we have for u smooth,

$$\|\widetilde{u} - W_h\|_{L_\infty(\Omega_{1/2})} \le Ch^r \|u\|_{W_\infty^r(\Omega_1)}. \tag{6.2.13}$$

Thus from (6.2.11),

$$\begin{aligned} &|\partial(u - u_h)(\overline{x}, \overline{y})| \\ &\qquad \le |\partial(\widetilde{u} - W_h)(\overline{x}, \overline{y})| + C(\ln 1/h)h^r + C\|u - u_h\|_{W_q^{-s}(\Omega_{1/2})}. \end{aligned} \tag{6.2.14}$$

We need two more ingredients to deduce that $(\overline{x}, \overline{y})$ is a superconvergence point for ∂. First, we need an estimate for $u - u_h$ in any negative norm. We will discuss this in the next section and for the moment we simply assume that for some s, q,

$$\|u - u_h\|_{W_q^{-s}(\Omega_{1/2})} \le C(\ln 1/h)h^r. \tag{6.2.15}$$

Then

$$|\partial(u - u_h)(\overline{x}, \overline{y})| \le |\partial(\widetilde{u} - W_h)(\overline{x}, \overline{y})| + C(\ln 1/h)h^r. \tag{6.2.16}$$

We now come to the heart of the matter! Going back to (6.1.14) and taking for definiteness $\partial = \frac{\partial}{\partial x}$, we have

$$\begin{aligned}\frac{\partial}{\partial x}(\widetilde{u} - W_h) &= \frac{\partial}{\partial x}(I^x - R^x) \otimes I^y(\widetilde{u}) + I^x \otimes (I^y - R^y)\Big(\frac{\partial \widetilde{u}}{\partial x}\Big) \\ &\qquad - \Big(\frac{\partial}{\partial x}(I^x - R^x)\Big) \otimes (I^y - R^y)(\widetilde{u}) \\ &\equiv I_1 + I_2 + I_3.\end{aligned} \tag{6.2.17}$$

Here, again under assumptions on appropriate behavior of the one–dimensional projections,

$$|I_2| + |I_3| \le Ch^r \|u\|_{W_\infty^{r+1}(\Omega_1)}. \tag{6.2.18}$$

We conclude that if $\overline{x}$ is a superconvergence point for the first derivative in the one–dimensional projection R^x of order σ, then $(\overline{x}, y)$, for any y, if it belongs to Ω_0, is a superconvergence point for $\frac{\partial}{\partial x}(u - u_h)$ of order $\overline{\sigma} = \min(\sigma, 1)$ (modulo a $\ln 1/h$–factor).

We collect the above and its obvious consequence in a somewhat informally stated theorem:

Theorem 6.2.1. *Assume that the one–dimensional projections R^x and R^y are stable in L_∞ and W_∞^1 and that $\overset{\circ}{S}{}_h^x$ and $\overset{\circ}{S}{}_h^y$ have approximation to orders r and $r-1$, for function values and derivatives, respectively. Assume also that $S_h(\Omega_1)$ is the tensor product $S_h(I^x) \otimes S_h(I^y)$. Assume further that for u and u_h satisfying*

$$D(u - u_h, \chi) = 0, \ \text{for } \chi \in S_h^{comp}(\Omega_1), \tag{6.2.19}$$

we have for some s and q,

$$\|u - u_h\|_{W_q^{-s}(\Omega_1)} \le C(\ln 1/h)h^r. \tag{6.2.20}$$

Then:

a) if $\overline{x} \in I_0^x$ is a superconvergence point of order σ for the first derivative for R^x, then for any $(\overline{x}, y) \in \Omega_0$,

$$\Big|\frac{\partial}{\partial x}(u - u_h)(\overline{x}, y)\Big| \le C(\ln 1/h)h^{r-1+\overline{\sigma}}, \ \overline{\sigma} = \min(1, \sigma). \tag{6.2.21}$$

b) if $\overline{y} \in I_0^y$ is a superconvergence point of order σ for the first derivative of R^y, then for any $(x, \overline{y}) \in \Omega_0$,

$$\Big|\frac{\partial}{\partial y}(u - u_h)(x, \overline{y})\Big| \le C(\ln 1/h)h^{r-1+\overline{\sigma}}, \ \overline{\sigma} = \min(1, \sigma). \tag{6.2.22}$$

c) If $\overline{x} \in I_0^x$ and $\overline{y} \in I_0^y$ are similarly superconvergence points for first derivatives of order σ_x, σ_y, respectively, then

$$|\nabla(u - u_h)(\overline{x}, \overline{y})| \le C(\ln 1/h)h^{r-1+\overline{\sigma}}, \ \overline{\sigma} = \min(1, \sigma_x, \sigma_y). \tag{6.2.23}$$

<u>Remark 6.2.1</u>. Clearly, if

$$\|u - u_h\|_{W_q^{-s}(\Omega_1)} \le Ch^{r-1+\tau} \tag{6.2.24}$$

with $\tau > 0$, corresponding superconvergence results hold but with $\overline{\sigma}$ replaced by $\min(\overline{\sigma}, \tau)$. □

6.3. Negative norm estimates for $u - u_h$: Examples.

The term $\|u - u_h\|_{W_q^{-s}(\Omega_1)}$ in the previous section is the only one that takes into account what happens outside of Ω_1: all other considerations are local to Ω_1. We shall give four brief examples of how to estimate it. For future purposes we shall also give estimates of higher order than immediately necessary in the previous section. Note that with our definition, $\|v\|_{W_q^{-s}(\Omega_1)} \le \|v\|_{W_q^{-s}(\Omega_2)}$ for $\Omega_1 \subseteq \Omega_2$.

Example 6.3.1. Homogeneous Neumann problems in smooth domains in R^n.

In Section 5.2 we considered a model Neumann problem, $-\Delta u + u = f$ in $\mathcal{D}$, $\partial u/\partial n = 0$ on $\partial\mathcal{D}$, with f and $\partial\mathcal{D}$ smooth. Assuming a perfect match for elements meeting $\partial\mathcal{D}$, we showed that

$$\|u - u_h\|_{W_q^{-(r-2)}(\mathcal{D})} \le Ch^{2r-2}q, \text{ for } q \ge 2 \text{ (say)}. \tag{6.3.1}$$

The argument easily extends to general operators of the form (5.1.1) with conormal derivative conditions on $\partial\mathcal{D}$, provided the problem has uniqueness (and smooth coefficients). □

Example 6.3.2. Dirichlet problems in smooth domains in R^n.

Here we consider (5.1.1) with homogeneous Dirichlet boundary conditions, $u = 0$ on $\partial\mathcal{D}$, and smooth data. We impose these essential boundary conditions so that $\mathring{S}_h(\mathcal{D}_h) \subseteq \mathring{H}^1(\mathcal{D})$, say. Using isoparametric elements at the boundary which approximate it to order h^r, it was shown in [Schatz and Wahlbin 1982] that

$$\|u - u_h\|_{L_\infty(\mathcal{D})} \le C(\ln 1/h)^{\bar{r}} h^r. \tag{6.3.2}$$

This estimate suffices for the purposes of Section 6.2.

For the Laplacian, an easy argument involving the maximum principle will convince the reader that it is hopeless in general to try for higher order accuracy in any negative norm for isoparametric elements (if $\mathcal{D}_h \subseteq \mathcal{D}$, say). Note in this connection that in general for isoparametrics it is not true that

$$\min_{\chi \in \mathring{S}_h(\mathcal{D})} \|v - \chi\|_{H^1(\mathcal{D})} \le Ch^{r-1}\|v\|_{H^r(\mathcal{D})}, \tag{6.3.3}$$

except in the case of piecewise linears, $r = 2$. □

If one approximates the smooth boundary to order $0(h^{2r-2})$, e.g., by use of "superparametric" modifications at the boundary, one obtains essentially (6.3.1) also in this case. The argument is the same. □

The final two problems are more special.

Example 6.3.3. Dirichlet problems in smooth plane domains.

In [Scott 1975] a special way of treating Dirichlet problems in plane domains without use of isoparametric or superparametric modification at the boundary was given and shown to satisfy

$$\|u - u_h\|_{W_2^{-(r-2)}(\mathcal{D})} \le Ch^{2r-2}. \tag{6.3.4}$$ □

Example 6.3.4. A Dirichlet problem in a plane polygonal domain.

Consider the problem

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \mathcal{D}, \\ u &= 0 \quad \text{on} \quad \partial\mathcal{D}, \end{aligned} \tag{6.3.5}$$

where $\mathcal{D}$ is a plane *polygonal* domain. As is well known, even if f is smooth ($f \equiv 1$ is the classical torsion engineering problem), the solution u has singularities at corners of the domain. E.g., even if $f \equiv 0$ near a corner of interior angle α, the solution behaves like

$$u(r,\theta) = const.\ r^{\pi/\alpha} \sin(\theta\pi/\alpha) + \cdots, \tag{6.3.6}$$

expressed in polar coordinates at the corner. Thus, although we have no trouble fitting the boundary exactly here, the relevant a priori estimates for a duality argument may be lacking: In fact, if α is the maximum angle, then $0(h^{2\pi/\alpha})$ is the best we can get in *any negative norm* if we use globally quasi-uniform elements and the constant in (6.3.6) is nonzero, see [Wahlbin 1984]. However, the exact behavior indicated in (6.3.6) is known, see [Grisvard 1985, Theorem 5.1.3.5], and if one uses suitable mesh refinements towards the corners, cf. e.g. [Babuška 1970], one obtains

$$\min_{\chi \in \overset{\circ}{S}_h(\mathcal{D})} \|u-\chi\|_{H^1(\mathcal{D})} \le Ch^{r-1}\|f\|_{H^{r-2}(\mathcal{D})}. \tag{6.3.7}$$

A standard duality argument *then* gives

$$\|u-u_h\|_{W_2^{-(r-2)}(\mathcal{D})} \le Ch^{2r-2}. \tag{6.3.8}$$

If one thus assumes that the patch of tensor-product elements considered in Section 6.2 sits away from corners and has quasi-uniform partitions, the theory there (and that to come) will apply. □

6.4. Superconvergence in derivatives for the case of (6.1.1).

Recall that in (6.1.1) we imposed that the highest order term in the elliptic operator is of the form $-div(\alpha(x,y)\nabla u)$, with α a scalar function. Again following essentially [Douglas, Dupont and Wheeler 1974b] and Section 5.5, or, viewed alternatively, extending our one-dimensional results of Section 1.3 to several dimensions, we shall show that the results of Section 6.2 hold for a general form of the type (6.1.2). We shall introduce the necessary superapproximation hypothesis as we go along.

As in Section 6.2 we let $\Omega_0 \subset\subset \Omega_{1/2} \subset\subset \Omega_1$ be rectangles with $S_h(\Omega_1)$ having a tensor-product structure. Again, $\omega \equiv 1$ on $\Omega_{1/2}$ and has compact support in Ω_1. This time, we define a comparison function $Z_h \in \overset{\circ}{S}_h(\Omega_1)$ by

$$D(\widetilde{u} - Z_h, \chi) = 0, \text{ for } \chi \in \overset{\circ}{S}_h(\Omega_1). \tag{6.4.1}$$

Now, for $\chi \in S_h^{comp}(\Omega_{1/4})$ (what $\Omega_{1/4}$ should be is obvious) and $\psi \in S_h^{comp}(\Omega_{1/2})$,

$$
\begin{aligned}
(6.4.2) \qquad D(Z_h - u_h, \chi) &= D(u - u_h, \chi) \\
&= \int \nabla(u - u_h) \cdot \nabla\chi \\
&= \int \alpha\nabla(u - u_h) \cdot \Big(\frac{1}{\alpha}\nabla\chi\Big) \\
&= \int \alpha\nabla(u - u_h) \cdot \nabla\Big(\frac{1}{\alpha}\chi\Big) + \int \nabla(u - u_h) \cdot \frac{1}{\alpha}(\nabla\alpha)\chi \\
&= \int \alpha\nabla(u - u_h)\nabla\Big(\frac{1}{\alpha}\chi - \psi\Big) - \int \sum_{i=1}^{n} a_i(u - u_h)\frac{\partial\psi}{\partial x_i} \\
&\qquad - \int a(u - u_h)\psi - \int (u - u_h)div\Big(\frac{1}{\alpha}(\nabla\alpha)\chi\Big),
\end{aligned}
$$

since $A(u - u_h, \psi) = 0$.

We now make the superapproximation hypothesis that we can find $\psi \in S_h^{comp}(\Omega_{1/2})$ such that

$$(6.4.3) \qquad \Big\|\frac{1}{\alpha}\chi - \psi\Big\|_{W_1^1(\Omega_{1/2})} \leq Ch\|\chi\|_{W_1^1(\Omega_{1/2})}, \text{ for } \chi \in S_h^{comp}(\Omega_{1/4}).$$

We then have

$$(6.4.4) \qquad D(Z_h - u_h, \chi) = F(\chi), \text{ for } \chi \in S_h^{comp}(\Omega_{1/4})$$

where

$$(6.4.5) \qquad \sup_{\substack{\chi \in S_h^{comp}(\Omega_{1/4}) \\ |\chi|_{W_1^1(\Omega_{1/4})}=1}} F(\chi) \leq C\Big(h\|u - u_h\|_{W_\infty^1(\Omega_{1/2})} + \|u - u_h\|_{L_\infty(\Omega_{1/2})}\Big).$$

Thus, from Theorem 5.5.1, cf. Remark 5.5.1,

$$
\begin{aligned}
(6.4.6) \qquad &\|Z_h - u_h\|_{W_\infty^1(\Omega_0)} \\
&\leq C(\ln 1/h)\Big(h\|u - u_h\|_{W_\infty^1(\Omega_{1/2})} + \|u - u_h\|_{L_\infty(\Omega_{1/2})}\Big) \\
&\qquad + C\|Z_h - u_h\|_{W_q^{-s}(\Omega_{1/2})} \\
&\leq C(\ln 1/h)\Big(h\|u - u_h\|_{W_\infty^1(\Omega_{1/2})} + \|u - u_h\|_{L_\infty(\Omega_{1/2})}\Big) \\
&\qquad + C\|Z_h - \widetilde{u}\|_{W_q^{-s}(\Omega_{1/2})} + C\|u - u_h\|_{W_q^{-s}(\Omega_{1/2})}.
\end{aligned}
$$

We next attack the negative norm term involving $Z_h - \widetilde{u}$. We have H^2–regularity for the Laplace operator on a rectangle, i.e., if

$$
(6.4.7) \qquad \begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega_1, \\ u &= 0 \quad \text{on} \quad \partial\Omega_1 \end{aligned}
$$

then

$$(6.4.8) \qquad \|u\|_{H^2(\Omega_1)} \leq C\|f\|_{L_2(\Omega_1)}.$$

A standard duality argument then gives (under natural assumptions)

$$\|Z_h - \tilde{u}\|_{L_2(\Omega_1)} \le Ch^r, \tag{6.4.9}$$

for u smooth enough.

Further, by Theorems 5.5.1 and 5.5.2, with $F \equiv 0$, we have since $A(u - u_h, \chi) = 0$, for $\chi \in S_h^{comp}(\Omega_1)$,

$$\begin{aligned} h\|u - u_h\|_{W^1_\infty(\Omega_{1/2})} &+ \|u - u_h\|_{L_\infty(\Omega_{1/2})} \\ &\le C(\ln 1/h) \min_{\chi \in S_h} \Big(h\|u - \chi\|_{W^1_\infty(\Omega_1)} + \|u - \chi\|_{L_\infty(\Omega_1)} \Big) \\ &\quad + C\|u - u_h\|_{W_q^{-s}(\Omega_1)} \\ &\le Ch^r(\ln 1/h) + C\|u - u_h\|_{W_q^{-s}(\Omega_1)}, \end{aligned} \tag{6.4.10}$$

again under natural approximation hypotheses.

From (6.4.6), (6.4.9) and (6.4.10) then,

$$\|Z_h - u_h\|_{W^1_\infty(\Omega_0)} \le Ch^r(\ln 1/h)^2 + C\|u - u_h\|_{W_q^{-s}(\Omega_1)}. \tag{6.4.11}$$

Thus, with $\partial = \partial/\partial x$ or $\partial/\partial y$,

$$\begin{aligned} |\partial(u - u_h)(\bar{x}, \bar{y})| &\le |\partial(\tilde{u} - Z_h)(\bar{x}, \bar{y})| + \|Z_h - u_h\|_{W^1_\infty(\Omega_0)} \\ &\le |\partial(\tilde{u} - Z_h)(\bar{x}, \bar{y})| + Ch^r(\ln 1/h)^2 + C\|u - u_h\|_{W_q^{-s}(\Omega_1)}. \end{aligned} \tag{6.4.12}$$

Hence, if we have a suitable negative norm estimate for $u - u_h$, apart from an additional logarithmic factor, *the conclusions of Theorem 6.2.1 given there for the Laplacian remain unchanged for the more general bilinear form (6.1.2) $A(\cdot,\cdot)$ in (6.1.3).*

6.5. Superconvergence in function values for the Laplacian and $r \ge 3$.

Provided $r \ge 3$ one similarly obtains that superconvergence for function values on tensor product meshes is inherited from the one-dimensional case, if the underlying partial differential operator is the Laplacian. We shall merely point out where modifications are necessary in Section 6.2.

For $F(\cdot)$ in (6.2.6), one now also needs an estimate for $|||F|||_{-2,\infty,\Omega_{1/2}}$. This will be inherited from W_∞^{-1}-error estimates for the one-dimensional cases, and this is why $r \ge 3$ is required to get

$$|||F|||_{-2,\infty,\Omega_{1/2}} \le Ch^{r+1}. \tag{6.5.1}$$

Of course one then applies Theorem 5.5.2 to obtain, corresponding to (6.2.10),

$$\|u_h - W_h\|_{L_\infty(\Omega_0)} \le C\|u_h - W_h\|_{W_q^{-s}(\Omega_{1/2})} + C(\ln 1/h)h^{r+1}. \tag{6.5.2}$$

Corresponding to (6.2.11) one then has

$$\begin{aligned} |(u - u_h)(\bar{x}, \bar{y})| \le C|(\tilde{u} - W_h)(\bar{x}, \bar{y})| &+ C\|\tilde{u} - W_h\|_{W_q^{-s}(\Omega_{1/2})} \\ &+ C(\ln 1/h)h^{r+1} + C\|u - u_h\|_{W_q^{-s}(\Omega_{1/2})}. \end{aligned} \tag{6.5.3}$$

Again, from (6.2.12), one easily shows for $r \ge 3$, that $\tilde{u} - W_h$ is bounded by Ch^{r+1} in negative norms. Thus, if also $u - u_h$ is so bounded in some negative norm,

$$|(u - u_h)(\bar{x}, \bar{y})| \le C|(\tilde{u} - W_h)(\bar{x}, \bar{y})| + C(\ln 1/h)h^{r+1}. \tag{6.5.4}$$

Finally, as in (6.2.17) et seq. we now see that $u - u_h$ inherits the one–dimensional superconvergence points from $W_h = R^x \otimes R^y(\widetilde{u})$.

Chapter 7. Superconvergence by local symmetry.

The present chapter basically reviews [Schatz, Sloan and Wahlbin 1994].

7.1. Introduction.

As with tensor–product elements, we have already met the main idea in the simpler case of L_2–projections in Chapter 4 (Section 4.3, in which the reader may find illuminating pictures). We have also met the idea that symmetry implies superconvergence in the one–dimensional case of two–point boundary value problems in Sections 1.6 and 1.9. We now place ourselves in the situation of the general bilinear form in (5.1.3) and assume that

$$A(u-u_h,\chi)=0, \text{ for } \chi\in S_h^{comp}(B_d), \tag{7.1.1}$$

where $B_d=B_d(x_0)\subseteq\mathcal{D}$ is a ball of radius $d=d(h)$ around the point(s) $x_0=x_0(h)$, which in general will vary with h.

Our symmetry assumption is that if $\chi\in S_h(B_d)$, then

$$\overline{\chi}(x)\equiv\chi(x_0-(x-x_0))\in S_h(B_d). \tag{7.1.2}$$

We shall also require that

$$\min_{\chi\in S_h}\left(\|v-\chi\|_{L_\infty(B_{d/2})}+h|v-\chi|_{W^1_\infty(B_{d/2})}\right)\le Ch^r|v|_{W^r_\infty(B_d)}, \tag{7.1.3}$$

in which now a crucial point is that the W^r_∞–seminorm on the right involves only the pure r^{th} derivatives of v.

Let us point out that the symmetry assumption (7.1.2) refers only to the finite element spaces. Symmetry is not required of the coefficients of the underlying second order partial differential operator in (5.1.3).

Remark 7.1.1. Anticipating the results below, we shall show that superconvergence at symmetry points occurs as follows:

for r odd, in function values,

for r even, in gradients.

Going back to Chapter 1 and, say, Hermite cubics in the uniform mesh case ($r=4$), we there have superconvergence for first derivatives (on interior subintervals) at the mesh and midpoints, i.e., the local symmetry points. However, for function values, the superconvergence points are not symmetry points (see Example 1.8.1, eq. (1.8.16)). Thus, the dichotomy between odd and even r is real. □

7.2. The case of a symmetric form with constant coefficients.

To bring out the central *idea* in this chapter, we first consider the case of the Laplacian, i.e.,

$$D(u-u_h,\chi)\equiv\int\sum_{i=1}^{n}\frac{\partial(u-u_h)}{\partial x_i}\,\frac{\partial\chi}{\partial x_i}=0, \text{ for } \chi\in S_h^{comp}(B_d), \tag{7.2.1}$$

where the underlying partial differential operator is also "symmetric" with respect to the antipodal map about x_0. A change of variables via the antipodal map then establishes that

$$D(\overline{u}-\overline{u}_h,\chi)=D(u-u_h,\overline{\chi})=0, \text{ for } \chi\in S_h^{comp}(B_d), \tag{7.2.2}$$

since $\overline{\chi} \in S_h^{comp}(B_d)$. (The same conclusion clearly applies if the a_{ij} are constant, $a_i \equiv 0$ and a is constant.)

We now separate the argument according to whether r is odd or even.

7.2.a. Odd r.

Let $e = u - u_h$ and

$$e_{even}(x) = (e(x) + \overline{e}(x))/2; \tag{7.2.3}$$

note that $e_{even} = u_{even} + u_{h,even}$ where $u_{h,even} \in S_h(B_d)$ by the symmetry assumption (7.1.2). Thus,

$$D(e_{even}, \chi) = 0, \text{ for } \chi \in S_h^{comp}(B_d). \tag{7.2.4}$$

We then apply Theorem 5.5.2 and the approximation assumption (7.1.3) to obtain that, provided $d \geq c_0 h$,

$$\begin{aligned} |e_{even}(x_0)| &\leq C \min_{\chi \in S^h} \|u_{even} - \chi\|_{L_\infty(B_{d/2})} \\ &\qquad + C d^{-s-n/q} \|e_{even}\|_{W_q^{-s}(B_{d/2})} \\ &\leq C h^r |u_{even}|_{W_\infty^r(B_d)} + C d^{-s-n/q} \|u - u_h\|_{W_q^{-s}(\mathcal{D})}. \end{aligned} \tag{7.2.5}$$

We now assume that we have a suitable negative norm estimate,

$$\|u - u_h\|_{W_q^{-s}(\mathcal{D})} \leq C h^{r+\tau}, \tag{7.2.6}$$

for some $\tau > 0$. Such estimates were discussed in Section 6.3. Using then that $e_{even}(x_0) = (u - u_h)(x_0)$ and Taylor's theorem, we have if $u \in W_\infty^{r+1}(B_d)$,

$$|(u - u_h)(x_0)| \leq C h^r d + C d^{-s-n/q} h^{r+\tau}. \tag{7.2.7}$$

So far we haven't nailed down how d may depend on h. The most favorable case (from the above analysis!) is clearly to take (asymptotically),

$$h^r d = d^{-s-n/q} h^{r+\tau}, \tag{7.2.8}$$

i.e.,

$$d = h^\sigma, \text{ with } \sigma = \tau/(s + n/q + 1). \tag{7.2.9}$$

We then have:

Theorem 7.2.1. *Assuming that r is odd, (7.2.1), the negative norm estimate (7.2.6), the symmetry assumption (7.1.2) with (7.2.9) where $\sigma \leq 1$, that Theorem 5.5.2 applies, the approximation assumption (7.1.3), and that $u \in W_\infty^{r+1}(B_d)$, we have*

$$|(u - u_h)(x_0)| \leq C h^{r+\sigma}. \tag{7.2.10}$$

Let us remark that superconvergence of lower orders than σ above results if the symmetry assumption (7.1.2) is satisfied on balls B_d of radii smaller than those required in (7.2.9).

Example 7.2.1. The most "favorable" negative norm estimates we can reasonably expect occur in Example 6.3.1, or, Example 6.3.2 with superparametric modifications near the smooth boundary. Then

$$\|u - u_h\|_{W_q^{-(r-2)}(\mathcal{D})} \le Ch^{2r-2}q.$$

We take $q = \ln 1/h$, $s = r - 2$, $(\tau \simeq r - 2)$, or, more precisely,

$$d = h^\sigma,\ \sigma = (r-2)/(r-1) = 1 - 1/(r-1),$$

and obtain from (7.2.5) and (7.2.7) that

$$|(u - u_h)(x_0)| \le Ch^{r+1-1/(r-1)} \ln 1/h.$$

For r large we thus have superconvergence of close to a full extra order, while requiring symmetry only in a ball of very small radius, $h^{1-1/(r-1)}$, around x_0. □

Remark 7.2.1. In the case that the mesh is symmetric around x_0 in an $O(1)$ neighborhood, a recent argument, [Schatz 1995], shows that we have (7.2.10) with $\sigma = 1$. This remark also applies to Sections 7.2.b and 7.3 below. □

7.2.b. Even r.

We shall assert under conditions similar to the ones above for odd r, that, now for even r, ∇u_h is a superconvergent approximation to ∇u at the symmetry point x_0. In general, if the finite element spaces are merely continuous (which is our minimal assumption, but also very frequent in practice), if x_0 is a point on the boundary of two elements, ∇u_h is not continuous here. To include this case in our analysis we proceed as follows:

Let ℓ denote a direction, i.e., unit vector, through x_0 such that $\nabla u_h(x_0 + \varepsilon\ell)$ is unique for $\varepsilon \neq 0$ and small in magnitude and the right and left hand limits

$$\lim_{\varepsilon\to 0\pm} \nabla u_h(x_0 + \varepsilon\ell) \quad \text{exist.} \tag{7.2.11}$$

We then set

$$\widehat{\nabla} u_h(x_0) = \frac{1}{2}\Big(\lim_{\varepsilon\to 0+} \nabla u_h + \lim_{\varepsilon\to 0-} \nabla u_h\Big). \tag{7.2.12}$$

We have as before,

$$D(u_{odd} - u_{h,odd}, \chi) = 0, \text{ for } \chi \in S_h^{comp}(B_d). \tag{7.2.13}$$

By Theorem 5.5.1 we then have $(e = u - u_h)$,

$$\begin{aligned} &\|\nabla(u_{odd} - u_{h,odd})\|_{L_\infty(B_{d/4})} \\ &\le C \min_{\chi\in S_h} \Big(|u_{odd} - \chi|_{W^1_\infty(B_{d/2})} + d^{-1}\|u_{odd} - \chi\|_{L_\infty(B_{d/2})}\Big) \\ &\qquad + Cd^{-1-s-n/q}\|e_{odd}\|_{W_q^{-s}(B_{d/2})} \\ &\le Ch^{r-1}d + Cd^{-1-s-n/q}h^{r-1+\tau'}, \end{aligned} \tag{7.2.14}$$

where we have assumed that the approximation hypothesis (7.1.3) holds and that

$$\|e\|_{W_q^{-s}(\mathcal{D})} \le Ch^{r-1+\tau'}, \tag{7.2.15}$$

cf. again Section 6.2 for such negative norm estimates.

Again performing a balancing act à la (7.2.8), we have

$$\|\nabla(u_{odd} - u_{h,odd})\|_{L_\infty(B_{d/4})} \leq Ch^{r-1-\sigma'}, \tag{7.2.16}$$

if

$$d = h^{\sigma'},\ \sigma' = \tau'/(s + n/q + 2) \leq 1. \tag{7.2.17}$$

If the symmetry point x_0 is a point of continuity for ∇u_h, (7.2.16) gives immediately that

$$|\nabla u(x_0) - \nabla u_h(x_0)| \leq Ch^{r-1+\sigma'}. \tag{7.2.18}$$

If ∇u_h is not continuous at x_0, we have from (7.2.16) for $|\varepsilon| \neq 0$ small (with $x_0 \pm \varepsilon\ell$ continuity points for ∇u_h),

$$\begin{aligned}\Big|\frac{1}{2}\{(\nabla u)(x_0 + \varepsilon\ell) + (\nabla u)(x_0 - \varepsilon\ell)\} \\ - \frac{1}{2}\{(\nabla u_h)(x_0 + \varepsilon\ell) + (\nabla u_h)(x_0 - \varepsilon\ell)\}\Big| \\ \leq Ch^{r-1-\sigma'}.\end{aligned} \tag{7.2.19}$$

Letting $\varepsilon \to 0+$ we obtain

$$|\nabla u(x_0) - \widehat{\nabla} u_h(x_0)| \leq Ch^{r-1+\sigma'}. \tag{7.2.20}$$

Hence,

Theorem 7.2.2. *Assuming that r is even, the negative norm estimate (7.2.15), the symmetry assumption (7.1.2) with (7.2.17) where $\sigma' \leq 1$, that Theorem 5.5.1 applies, the approximation assumption (7.1.3) and that $u \in W_\infty^{r+1}(B_d)$, we have*

$$|\nabla u(x_0) - \widehat{\nabla} u_h(x_0)| \leq Ch^{r-1+\sigma'}. \tag{7.2.21}$$

Here $\widehat{\nabla} u_h$ is given in (7.2.12); it coincides with ∇u_h in case x_0 is a continuity point for ∇u_h.

Again, if symmetry holds only in a smaller domain than dictated now by (7.2.17), superconvergence of lower order obtains.

Example 7.2.2. In the most favorable cases considered in Example 7.2.1, we now have

$$|(\nabla u - \widehat{\nabla} u_h)(x_0)| \leq Ch^{r-1+(1-1/r)} \ln 1/h,$$

while imposing symmetry only in a ball of radius $h^{1-1/r}$ around x_0. □

Example 7.2.3. If we consider a Dirichlet problem over a smooth domain but use only isoparametric modifications at the boundary, then we have (Example 6.3.2),

$$\|u - u_h\|_{L_\infty(\mathcal{D})} \leq C(\ln 1/h)^{\bar{r}} h^r.$$

Taking $\tau' \simeq 1$, $s = 0$, $\sigma' = 1/2$ we may then assert that

$$|(\nabla u - \widehat{\nabla} u_h)(x_0)| \leq Ch^{(r-1)+1/2}(\ln 1/h)^{\bar{r}},$$

for meshes symmetric in an $0(h^{1/2})$ neighborhood of x_0. □

7.3. The general case (5.1.3) with variable smooth coefficients.

7.3.a. Odd r.

Now

$$A(e,\chi)=0,\ \text{ for } \chi\in S_h^{comp}(B_d). \tag{7.3.1}$$

Thus, by the antipodal change of variables about x_0, $y=x_0-(x-x_0)$,

$$\begin{aligned} A(\bar{e},\chi) &= \int\Big(\sum_{i,j=1}^{n}\bar{a}_{ij}\frac{\partial e}{\partial y_i}\,\frac{\partial\overline{\chi}}{\partial y_j}-\sum_{i=1}^{n}\bar{a}_i e\frac{\partial\overline{\chi}}{\partial y_i}+\bar{a}e\overline{\chi}\Big)dy \\ &\equiv B(e,\overline{\chi}),\quad \text{for}\quad \chi\in S_h^{comp}(B_d). \end{aligned} \tag{7.3.2}$$

Thus, using now that $A(e,\chi)=0$, and by symmetry, $A(e,\overline{\chi})=0$, for $\chi\in S_h^{comp}(B_d)$,

$$\begin{aligned} A(e_{even},\chi) &= \frac{1}{2}A(e,\chi)+\frac{1}{2}B(e,\overline{\chi}) \\ &= \frac{1}{2}B(e,\overline{\chi})-\frac{1}{2}A(e,\overline{\chi}) \\ &= \frac{1}{2}\int\sum_{i,j=1}^{n}(\bar{a}_{ij}-a_{ij})\frac{\partial e}{\partial y_i}\,\frac{\partial\overline{\chi}}{\partial y_j}dy \\ &\quad -\frac{1}{2}\int\sum_{i=1}^{n}(\bar{a}_i+a_i)e\frac{\partial\overline{\chi}}{\partial y_j}dy+\frac{1}{2}\int(\bar{a}-a)e\overline{\chi}dy \\ &\equiv F_1(\overline{\chi})+F_2(\overline{\chi})+F_3(\overline{\chi})\equiv F(\overline{\chi}),\ \text{ for } \chi\in S_h^{comp}(B_d). \end{aligned} \tag{7.3.4}$$

Here, for $\varphi\in C_0^\infty(B_{d/4})$, by the mean–value theorem applied to $\bar{a}_{ij}-a_{ij}$,

$$|F_1(\varphi)|\le Cd\|e\|_{W^1_\infty(B_{d/4})}|\varphi|_{W^1_1(B_{d/4})}. \tag{7.3.5}$$

By integration by parts and Poincaré–Friedrichs' inequality, again for $\varphi\in C_0^\infty(B_{d/4})$,

$$\begin{aligned} |F_2(\varphi)| &= \Big|\sum_{i=1}^{n}\Big(\frac{\partial}{\partial y_i}(\bar{a}_i+a_i)e\Big)\varphi dy\Big| \\ &\le C\|e\|_{W^1_\infty(B_{d/4})}\|\varphi\|_{L_1(B_{d/4})} \\ &\le Cd\|e\|_{W^1_\infty(B_{d/4})}|\varphi|_{W^1_1(B_{d/4})} \end{aligned} \tag{7.3.6}$$

and the same trivially goes for $F_3(\varphi)$. So,

$$|||F|||_{-1,\infty,d/4}\le Cd\|e\|_{W^1_\infty(B_{d/4})}. \tag{7.3.7}$$

Similarly, for $\varphi\in C_0^\infty(B_{d/4})$, after integrating by parts and then once more using Poincaré–Friedrichs' inequality and the mean–value theorem,

$$\begin{aligned} |F_1(\varphi)| &= \frac{1}{2}\Big|\sum_{i,j=1}^{n}e\Big(\frac{\partial}{\partial y_i}(\bar{a}_{ij}-a_{ij})\frac{\partial\varphi}{\partial y_i}+(\bar{a}_{ij}-a_{ij})\frac{\partial^2\varphi}{\partial y_i\partial y_j}\Big)dy\Big| \\ &\le C\|e\|_{L_\infty(B_{d/4})}d|\varphi|_{W^2_1(B_{d/4})}. \end{aligned} \tag{7.3.8}$$

Treating F_2 and F_3 in obvious analogous ways, we find that

$$|||F|||_{-2,\infty,d/4}\le Cd\|e\|_{L_\infty(B_{d/4})}. \tag{7.3.9}$$

So, from Theorem 5.5.2, by (7.3.7) and (7.3.9),

$$
\begin{aligned}
(7.3.10) \qquad |e(x_0)| \le C \min_{\chi \in S_h} \|u_{even} - \chi\|_{L_\infty(E_{d/2})} \\
+ Cd^{-s-n/q}\|e\|_{W_q^{-s}(\mathcal{D})} \\
+ C(\ln 1/h)\Big(hd\|e\|_{W^1_\infty(B_{d/4})} + d\|e\|_{L_\infty(B_{d/4})}\Big).
\end{aligned}
$$

By use of an inverse estimate and approximation theory, using a suitable $\chi \in S_h$,

$$
\begin{aligned}
(7.3.11) \qquad h\|e\|_{W^1_\infty(B_{d/4})} &\le h\|u-\chi\|_{W^1_\infty(B_{d/4})} + h\|\chi - u_h\|_{W^1_\infty(B_{d/4})} \\
&\le Ch\|u-\chi\|_{W^1_\infty(B_{d/4})} + C\|\chi - u_h\|_{L_\infty(B_{d/2})} \\
&\le C\Big(h\|u-\chi\|_{W^1_\infty(B_{d/4})} + \|u-\chi\|_{L_\infty(B_{d/2})}\Big) \\
&\qquad + C\|e\|_{L_\infty(B_{d/2})} \\
&\le Ch^r + C\|e\|_{L_\infty(B_{d/2})}.
\end{aligned}
$$

Thus from (7.3.10), using approximation theory (7.1.3) and Taylor's theorem for the first term there (assuming $u \in W^{r+1}_\infty(B_d)$),

$$
\begin{aligned}
(7.3.12) \qquad |e(x_0)| \le Ch^r d(\ln 1/h) + Cd^{-s-n/q}\|e\|_{W_q^{-s}(\mathcal{D})} \\
+ C(\ln 1/h)d\|e\|_{L_\infty(B_{d/2})}.
\end{aligned}
$$

Now again falling back to Theorem 5.5.2 (with $F \equiv 0$) for the last term on the right of (7.3.12),

$$
\begin{aligned}
(7.3.13) \qquad |e(x_0)| \le Ch^r d(\ln 1/h) + Cd^{-s-n/q}\|e\|_{W_q^{-s}(\mathcal{D})} \\
+ C(\ln 1/h)d^{1-s-n/q}\|e\|_{W_q^{-s}(\mathcal{D})}.
\end{aligned}
$$

We conclude, somewhat informally stated:

Theorem 7.3.1. *Under the "the same" assumptions, the conclusion of Theorem 7.2.1 remains valid for the case of the general bilinear form (5.1.3), modulo a multiplicative factor* $\ln 1/h$.

7.3.b. Even r.

Similarly (we leave the argument to the reader, cf. [Schatz, Sloan and Wahlbin 1994]), we assert *that the conclusions of Theorem 7.2.2 remain valid for the more general bilinear form (5.1.3), again modulo a multiplicative factor* $\ln 1/h$.

Of course, now we also used Theorem 5.5.1.

Finally we remark that Examples 7.2.1–7.2.3, which were based on the negative norm estimates of Examples 6.3.1 and 6.3.2, apply essentially as before. We leave it to the reader to consider the negative norm estimates in Examples 6.3.3 and 6.3.4 in the present context.

7.4. Historical remarks.

We shall rediscover a few known results through our present symmetry techniques. We shall also explain why an earlier attempt at a general principle fails. Details will be given only in the plane, although the results are relevant in any number of space dimensions.

7.4.i. Piecewise linear and quadratic triangles, and beyond.

Assume that the mesh is locally, in an 0(1) neighborhood, say, generated by three linearly independent directions.

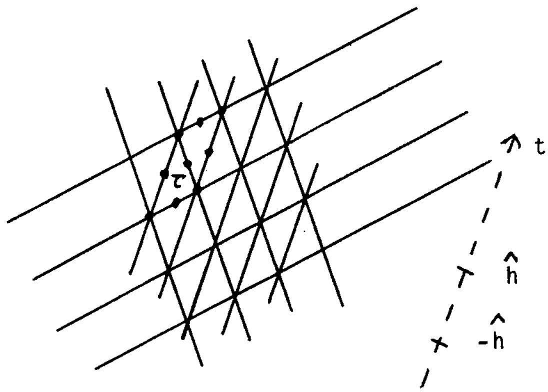


Figure 7.4.1

Then meshpoints and midpoints on common diagonals between two triangles are symmetry points. In particular then, if globally $\|u - u_h\|_{W_q^{-s}(\mathcal{D})} \leq Ch^r$ for some negative norm, the average of the constant vectors ∇u_h over two adjoining triangles is a superconvergent approximation to ∇u at the midpoint of the shared diagonal (cf. Remark 7.2.1).

The history of this particular result can be traced back at least as far as [Oganesyan and Rukhovetz 1969, eq. (3.18)] in the L_2–based case for piecewise linear elements. Phrased in our present language,

$$|A(u - u^I, \chi)| \leq Ch^r \|\chi\|_{W_1^1}, \text{ for } \chi \in S_h^{comp}(\Omega_1), \tag{7.4.1}$$

where $r = 2$ and u^I is the natural piecewise linear interpolant at nodes. Of course, having this we automatically have

$$|A(u_h - u^I, \chi)| \leq Ch^r \|\chi\|_{W_1^1}, \text{ for } \chi \in S_h^{comp}(\Omega_1) \tag{7.4.2}$$

and our present techniques, viz., Theorem 5.5.1, gives that u_h is "superclose" to u^I in $W_\infty^1(\Omega_0)$. To then find out about superconvergence for $\nabla(u - u_h)$, it simply remains to consider the locally defined error in $\nabla(u - u^I)$. This is frequently elementary.

To verify (7.4.1) is often a slightly laborious task involving taking into account local cancellation effects due to symmetry. In the piecewise linear situation, typical examples can be found in [Chen and Liu 1987] and [Lin and Xu 1985]. In other words, to state (7.4.1) as a hypothesis leaves a major part of the work to the reader.

The hypothesis (7.4.1) and its counterpart for function values were formalized e.g. in [Zhu 1983, eqs. (2.11) and (2.12)]. He then proceeded "essentially" as in our Theorems 5.5.1 and 5.5.2. Earlier, [Zlámal 1977, 1978a,b] and [Lesaint and Zlámal 1979] had used the same idea in a discrete ℓ_1–setting for the error in higher order quadrilateral elements.

For our additional results about averaging from two sides about a symmetric nodal point in the piecewise linear situation, cf. also [Levine 1985].

However, an objection against basing a "general" theory of superconvergence on (7.4.1) is that it is not a general enough principle. For uniform triangular meshes we shall "see" via computational investigations in Chapter 12 that, beyond quadratics, the standard interpolant is not superclose to the finite element approximation. Furthermore, a little known but in my opinion important paper, [Li 1990], proves that on uniform triangular meshes, for $\mathcal{C}^0$ Lagrange cubics or Hermite cubics, the Galerkin finite element solution (for the Laplacian on a rectangle) and the standard interpolants are not superclose in H^1 or L_2. Thus, basing a general theory of superconvergence on the hope that the Galerkin finite element approximation is close to a standard interpolant is doomed to fail.

7.4.ii. More on $\mathcal{C}^0$ quadratics on triangles.

Again we have a mesh consisting of triangles involving three linearly independent directions. Now $r = 3$ and from our theory in this chapter, mesh and midpoints are superconvergent in function values.

Letting then u^I denote the piecewise quadratic interpolant of u in one variable on any side of τ, by stability $u_h - u^I$ is superconvergent over that side in function values, and thus, from inverse assumptions, $\frac{\partial}{\partial t}(u_h - u^I)$ is superconvergent for the tangential derivative $\frac{\partial}{\partial t}$ along that side. Hence superconvergence for $\frac{\partial}{\partial t}(u - u_h)$ occurs at the same points as for $\frac{\partial}{\partial t}(u - u^I)$. These are trivially found to be the Gauss points: If $-\widehat{h} \leq t \leq \widehat{h}$ describe a side of τ, they are the zeroes of the normalized second degree Legendre polynomial, i.e.,

$$t = \pm\widehat{h}/\sqrt{3}. \tag{7.4.3}$$

We have rediscovered a result which can be traced at least as far back as [Andreev 1984b].

We note that our "fast" derivation above was based on the fact that there were enough symmetry points present on an element to determine the "natural" local interpolant. This will be the key also in the following two examples.

7.4.iii. Incomplete cubics.

We are now on rectangles, say, axis–parallel for simplicity. The incomplete cubics are given as $A + Bx + Cy + Dx^2 + Exy + Fy^2 + Gx^2y + Hyx^2$ on each rectangle (and are, of course, globally continuous). Again, $r = 3$. They are determined by interpolation at the following eight points:

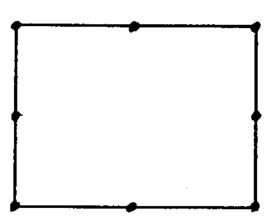


Figure 7.4.2

These now happen to be among the (nine) symmetry points and thus superconvergent for function values. If u^I denotes the interpolant indicated, then $u_h - u^I$ is superconvergent there for function values, and so, by stability, "superclose" on the whole of τ. By an inverse inequality, $\nabla(u_h - u^I)$ is "superclose" on τ. To detect superconvergence points for $\nabla(u-u_h)$ it thus remains to consider such for $\nabla(u-u^I)$. Since x^2y and yx^2 are reproduced by u^I it is enough to look at what happens for $u = x^3$ and $u = y^3$. This is elementary and we find that the four interior Gauss points (h_x and h_y are the meshsizes in the x and y directions, respectively), are superconvergent for $\nabla(u - u_h)$.

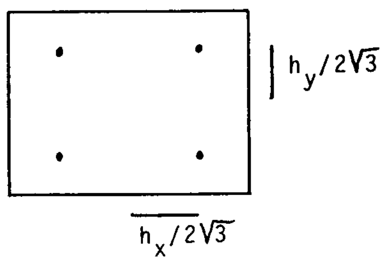


Figure 7.4.3

We have rediscovered a result which may be traced back at least as far as to work by Zlámal and Lesaint in the late 1970's, cf. the references. Cf. also [Barlow 1976].

7.4.iv. C^0 or C^1 biquadratic tensor–products.

We are again on a rectangle and we now have nine symmetry points for function values,

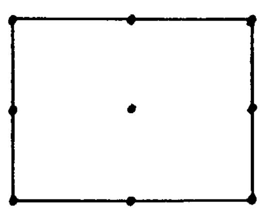


Figure 7.4.4

which is enough to determine a local interpolant u^I of the form $A + Bx + Cy + Dx^2 + Exy + Fy^2 + Gx^2y + Hxy^2 + Ix^2y^2$. (For the C^1–case, this u^I will not be globally in S_h, but that doesn't matter for our argument.)

(As for superconvergence at the four nodes, in the C^1–case we have rediscovered [Bramble and Schatz 1976, Theorem 10].) It follows as in 7.4.iii for the incomplete cubics, that the interior Gauss points are again superconvergent for gradients. (The now simple details are left to the reader.)

Again, work by Zlámal and Lesaint in the late 1970's has been rediscovered.

We may also remark that although the present situation is that of tensor products, we have *not* used the theory of Chapter 6, with its restrictions on the form of the bilinear form $A(\cdot,\cdot)$. We have relied instead on superconvergence by symmetry which we have verified for a general form $A(\cdot,\cdot)$.

Chapter 8. Superconvergence for difference quotients on translation invariant meshes.

8.1. Introduction.

We have met a simple case in Section 1.11. In this chapter we shall consider derivatives of any order, but we shall not manage to be as precise with respect to local influences as we were in the one-dimensional situation. The whole chapter is by and large based on [Nitsche and Schatz 1974, Section 6] where the theory was presented in L_2-based error estimates. Using later technology, here we work with pointwise error estimates (but cf. Remark 8.4.1 below!).

The work of Nitsche and Schatz was inspired by the finite difference theory in [Thomée and Westergren 1968]. An earlier investigation into using difference quotients to approximate derivatives in a multidimensional finite element setting is given for parabolic problems in [Fix and Nassif 1972]. Another discovery of the fact that it is better to difference than to differentiate in finite elements is in [Long and Morton 1977].

Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \mathcal{D}$ be separated by d and let ℓ be a direction, i.e., a unit vector in R^n. Let $H > 0$ be a parameter, which will typically be a constant times h. By *translation invariance by H in the direction ℓ* we shall mean the following.

Let T_H^ℓ denote translation by H in the direction ℓ, i.e.,

$$T_H^\ell v(x) = v(x + H\ell), \tag{8.1.1}$$

and for ν an integer,

$$T_{\nu H}^\ell v(x) = v(x + \nu H\ell). \tag{8.1.2}$$

Then the finite element spaces are called translation invariant by H in the direction ℓ if

$$T_{\nu H}^\ell \chi \in S_h^{comp}(\Omega_2), \text{ for } \chi \in S_h^{comp}(\Omega_1). \tag{8.1.3}$$

This will be required for $|\nu| \leq M$, some fixed M; the dependence on M will be suppressed in our notation.

Let us give some pictorial examples, where we display what we think of as h.

Example 8.1.1.

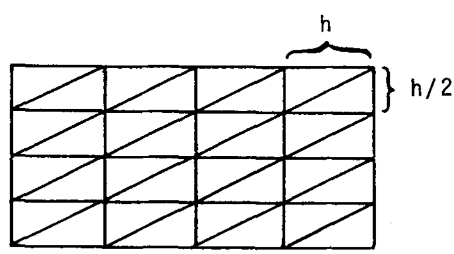


Figure 8.1.1

Here we may take the following combinations:

$$\ell = (1,0),\ \ H = h,$$
$$\ell = (0,1),\ \ H = h/2,$$
$$\ell = \frac{2}{\sqrt{5}}(1, 1/2),\ \ H = h\sqrt{5}/2.$$

(Also, the direction $\frac{2}{\sqrt{5}}(1,-1/2)$.)

Of course, the same example works if the triangular elements are replaced by rectangular ones. □

Example 8.1.2.

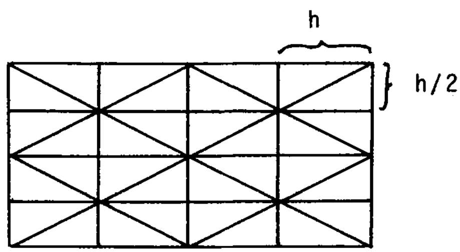


Figure 8.1.2

Here

$$\ell = (1,0),\ \ H = 2h,$$
$$\ell = (0,1),\ \ H = h,$$

and the "diagonal" directions. □

Example 8.1.3.

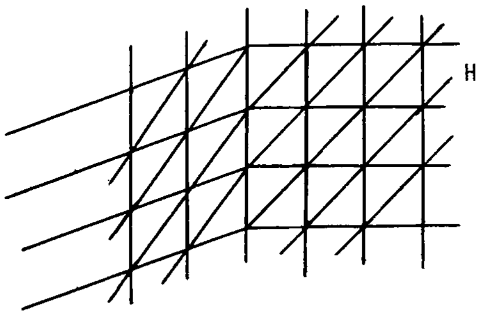


Figure 8.1.3

Here, $\ell = (0,1)$ and H as indicated. No other direction works. □

We now consider a function u which satisfies, with $u_h \in S_h(\Omega_1)$,

$$A(u - u_h, \chi) = 0, \text{ for } \chi \in S_h^{comp}(\Omega_2). \tag{8.1.4}$$

The problem is to approximate $D_\ell^k u$, the k^{th} derivative in the locally translation invariant direction ℓ. We shall utilize a suitable difference quotient based on translations in the direction ℓ,

$$\partial_{\ell,H}^k v(x) = \sum_{|\nu| \le M} C_{\nu,H} v(x + \nu H \ell). \tag{8.1.5}$$

The dependence of $C_{\nu,H}$ on k and ℓ will be suppressed in the notation. We shall assume that $\partial_{\ell,H}^k$ is an r^{th} order approximation to D_ℓ^k in the sense that

$$\|D_\ell^k \widetilde{T}_h v - \partial_{\ell,H}^k v\|_{L_\infty(\Omega_1)} \le CH^s \|v\|_{W_\infty^{k+s}(\Omega_2)}, \text{ for } 0 \le s \le r. \tag{8.1.6}$$

Since $H \simeq h$, we may replace H by h on the right. Here, $\widetilde{T}_h$ is a "suitable" shift operator, sometimes $\widetilde{T}_h = I$. See Remark 8.2.1 for another, where $\widetilde{T}_h$ is translation by $h/2$ in the ℓ–direction. The meshes do not have to be translation invariant with respect to $\widetilde{T}_h$.

8.2. Constant coefficient operators and unit separation, $d \simeq 1$.

So, let a_{ij}, a_i and a be constant in A. Then

$$A(T_{\nu H}^\ell v, w) = A(v, T_{-\nu H}^\ell w) \equiv A(v, (T_{\nu H}^\ell)^* w), \tag{8.2.1}$$

and thus, from (8.1.3) and (8.1.4),

$$A(\partial_{\ell,H}^k (u - u_h), \chi) = A(u - u_h, (\partial_{\ell,h}^k)^* \chi) = 0, \text{ for } \chi \in S_h^{comp}(\Omega_1). \tag{8.2.2}$$

From (8.1.3) again, $\partial_{\ell,H}^k u_h \in S_h$ and Theorem 5.5.2 (with $F \equiv 0$) thus implies that

$$\begin{aligned} \|\partial_{\ell,H}^k (u - u_h)\|_{L_\infty(\Omega_0)} \le\ & C(\ln d/h)^{\bar{r}} \min_{\chi \in S_h} \|\partial_{\ell,H}^k u - \chi\|_{L_\infty(\Omega_1)} \\ & + Cd^{-s-n/q} \|\partial_{\ell,H}^k (u - u_h)\|_{W_q^{-s}(\Omega_1)}. \end{aligned} \tag{8.2.3}$$

Here $d \geq c_0 h$ is the separation between Ω_0 and Ω_1 (and the separation between Ω_1 and Ω_2 is assumed $\geq$ cte h, too). By (8.1.6), with $s = r$, and standard approximation assumptions,

$$\text{(8.2.4)} \qquad \begin{aligned} &\min_{\chi \in S_h} \|\partial^k_{\ell,H} u - \chi\|_{L_\infty(\Omega_1)} \\ &\leq \|\partial^k_{\ell,H} u - D^k_\ell \widetilde{T}_h u\|_{L_\infty(\Omega_1)} + \min_{\chi \in S_h} \|D^k_\ell \widetilde{T}_h u - \chi\|_{L_\infty(\Omega_1)} \\ &\leq C h^r \|u\|_{W^{r+k}_\infty(\Omega_2)}. \end{aligned}$$

Thus, from (8.2.3) using again (8.1.6), if u is assumed smooth enough on Ω_2,

$$\text{(8.2.5)} \qquad \begin{aligned} &\|D^k_\ell \widetilde{T}_h u - \partial^k_{\ell,H} u_h\|_{L_\infty(\Omega_0)} \\ &\qquad \leq C(\ln 1/h)^{\bar{r}} h^r + C d^{-s-n/q} \|\partial^k_{\ell,H}(u - u_h)\|_{W^{-s}_q(\Omega_1)}. \end{aligned}$$

It remains to attack the negative norm on the right of (8.2.5). We have

$$\text{(8.2.6)} \qquad \|\partial^k_{\ell,H}(u - u_h)\|_{W^{-s}_q(\Omega_1)} = \sup_{\substack{\varphi \in C^\infty_0(\Omega_1) \\ \|\varphi\|_{W^s_{q'}(\Omega_1)} = 1}} (\partial^k_{\ell,H}(u - u_h), \varphi).$$

Here,

$$\text{(8.2.7)} \qquad \begin{aligned} (\partial^k_{\ell,H}(u - u_h), \varphi) &= (u - u_h, (\partial^k_{\ell,H})^* \varphi) \\ &\leq C\|u - u_h\|_{L_\infty(\Omega_2)} \|(\partial^k_{\ell,H})^* \varphi\|_{L_1(\Omega_2)}. \end{aligned}$$

From the dual version of (8.1.6) with $s = 0$ (which we leave to the audience to make precise), if $s \geq k$,

$$\text{(8.2.8)} \qquad \|(\partial^k_{\ell,H})^* \varphi\|_{L_1(\Omega_2)} \leq C.$$

From (8.2.5)–(8.2.8), since in our present case $d \simeq 1$,

$$\text{(8.2.9)} \qquad \|D^k_\ell \widetilde{T}_h u - \partial^k_{\ell,H} u_h\|_{L_\infty(\Omega_0)} \leq C(\ln 1/h)^{\bar{r}} h^r + C\|u - u_h\|_{L_\infty(\Omega_2)}.$$

We now once again apply Theorem 5.5.1 to the last term to obtain with $\Omega_2 \subset\subset \mathcal{D}$ separated by a unit distance, and u smooth enough,

$$\text{(8.2.10)} \qquad \|D^k_\ell \widetilde{T}_h u - \partial^k_{\ell,H} u_h\|_{L_\infty(\Omega_0)} \leq C(\ln 1/h)^{\bar{r}} h^r + C\|u - u_h\|_{W^{-s}_q(\mathcal{D})}.$$

We summarize informally:

Theorem 8.2.1. *Let the differential operator have constant coefficients; let $\Omega_0 \subset\subset \Omega_2 \subset\subset \mathcal{D}$ be separated by unit distances; let the finite element spaces be translation invariant in the direction ℓ on Ω_2 with parameter $H \simeq h$ (cf. (8.1.3)); let $\partial^k_{\ell,H}$ be a finite difference operator based on translations in the ℓ-direction which approximates the directional derivative D^k_ℓ to order h^r (cf. (8.1.6)); let the finite element space have approximation order h^r for function values; let $A(u - u_h, \chi) = 0$ for $\chi \in S^{comp}_h(\Omega_2)$; let u be locally smooth enough; and assume that for some q and s,*

$$\text{(8.2.11)} \qquad \|u - u_h\|_{W^{-s}_q(\mathcal{D})} \leq C h^r.$$

Furthermore assume that Theorem 5.5.2 is applicable. Then

$$\text{(8.2.12)} \qquad \|D^k_\ell \widetilde{T}_h u - \partial^k_{\ell,H} u_h\|_{L_\infty(\Omega_0)} \leq C(\ln 1/h)^{\bar{r}} h^r,$$

where $\bar{r} = 1$ *if* $r = 2$, $\bar{r} = 0$ *for* $r \geq 3$.

So, suitable difference quotients provide optimal order h^r (modulo a logarithmic term if $r = 2$) approximation to D_ℓ^k, *for any* k, some negative norm permitting, of course. Let us remark that above we have given the details for pure k^{th} derivatives in the direction ℓ. There are no essential changes for mixed derivatives, but now one needs of course translation invariance in all directions involved.

Remark 8.2.1. Here we present again a rediscovery from Section 7.4.i, now rediscovered from the results of this section. Consider piecewise linear plane triangular elements on a locally "uniform" mesh such as depicted in Example 8.1.1. Let us consider e.g. the x–direction, $\ell = (1,0)$ with $H = h$ as there, and first derivatives $\partial u/\partial x$. Let ∂ be the difference operator

(8.2.13) $$\partial v(x,y) = (v(x+h,y) - v(x,y))/h,$$

and let

(8.2.14) $$\widetilde{T}_h v(x) = v\Big(x + \frac{h}{2}(1,0)\Big).$$

Then (8.1.6) is satisfied with $r = 2$. Thus, some negative norm permitting,

(8.2.15) $$\Big\|\frac{\partial u}{\partial x}\Big(x + \frac{h}{2}, y\Big) - \partial u_h(x,y)\Big\|_{L_\infty(\Omega_0)} \leq Ch^2 \ln 1/h.$$

Let now x be a "mesh–point"; while y is a "meshpoint" shifted by $\widetilde{h}/2$, where $\widetilde{h}$ is the mesh length in the y–direction (i.e., y is a "midpoint"):

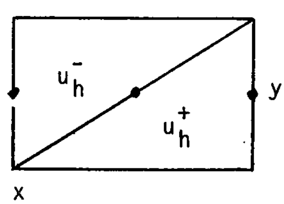


Figure 8.2.1

Denote u_h^- and u_h^+ as indicated. Then

(8.2.16) $$\begin{aligned}\partial u_h(x,y) &= (u_h(x+h,y) - u_h(x,y))/h \\ &= \big[(u_h(x+h,y) - u_h(x+h/2,y)) \\ &\qquad + (u_h(x+h/2,y) - u_h(x,y))\big]/h \\ &= \frac{1}{2}\big[(u_h^+(x+h,y) - u_h^+(x+h/2,y))/(h/2) \\ &\qquad + (u_h^-(x+h/2,y) - u_h^-(x,y))/(h/2)\big].\end{aligned}$$

Combining this with (8.2.15) we deduce that the average of $\partial u_h/\partial x$ (which is constant in each element) over two adjoining elements as above is an $h^2 \ln 1/h$ order approximation to $\frac{\partial u}{\partial x}(x,y)$ if x and y are both "midpoints".

Doing the same in the y–direction, and after an easy generalization to any mesh generated locally by three fixed directions, we have rediscovered the result about superconvergence of averages of ∇u_h at midpoints of shared sides. $\square$

8.3. Constant coefficient operators and general separation d.

Of course, we assume that $d \geq c_0 h$ with c_0 large enough. We shall sketch the argument; the reader can easily supply missing details. Repeating (8.2.5),

$$\begin{aligned}(8.3.1)\qquad &\|D_\ell^k \widetilde{T}_h u - \partial_{\ell,H}^k u_h\|_{L_\infty(\Omega_0)} \\ &\leq C(\ln 1/h)^{\bar{r}} h^r + C d^{-s-n/q} \|\partial_{\ell,H}^k (u - u_h)\|_{W_q^{-s}(\Omega_1)}.\end{aligned}$$

As in (8.2.7), now using also Poincaré–Friedrichs' inequality, and, Hölder's inequality,

$$\begin{aligned}(8.3.2)\qquad |(\partial_{\ell,H}^k(u - u_h), \varphi)| &\leq \|u - u_h\|_{L_\infty(\Omega_2)} \|(\partial_{\ell,H}^k)^* \varphi\|_{L_1(\Omega_2)} \\ &\leq C\|u - u_h\|_{L_\infty(\Omega_2)} \|\varphi\|_{W_1^k(\Omega_1)} \\ &\leq C\|u - u_h\|_{L_\infty(\Omega_2)} d^{s-k} \|\varphi\|_{W_1^s(\Omega_1)} \\ &\leq C\|u - u_h\|_{L_\infty(\Omega_2)} d^{s-k+n/q} \|\varphi\|_{W_{q'}^s(\Omega_1)}, \text{ for } s \geq k.\end{aligned}$$

So,

$$(8.3.3)\qquad \|D_\ell^k \widetilde{T}_h u - \partial_{\ell,H}^k u_h\|_{L_\infty(\Omega_0)} \leq C(\ln 1/h)^{\bar{r}} h^r + d^{-k} \|u - u_h\|_{L_\infty(\Omega_2)}.$$

If one now assumes that

$$(8.3.4)\qquad \|u - u_h\|_{L_\infty(\mathcal{D})} \leq C h^r (\ln 1/h)^{\bar{r}},$$

cf. Example 6.2.2, one obtains

$$(8.3.5)\qquad \|D_\ell^k \widetilde{T}_h u - \partial_{\ell,H}^k u_h\|_{L_\infty(\Omega_0)} \leq C(\ln 1/h)^{\bar{r}} d^{-k} h^r.$$

E.g., for first derivatives, $k = 1$, if the mesh is translation invariant in an $0(h^{1/2})$ neighborhood one obtains superconvergence of almost a half extra order.

If (8.3.4) does not hold one may of course further localize the term $\|u - u_h\|_{L_\infty(\Omega_2)}$ in (8.3.3) by use of Theorem 5.5.2 (with $F \equiv 0$). We leave this to the reader.

8.4. Variable coefficients.

We shall consider a fairly simple case which we hope will convince the reader that he or she can handle the general case. Let thus

$$(8.4.1)\qquad A(v, w) = \int \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} dx,$$

i.e., we consider only the highest order terms, but these are of a general form. Let further $d \simeq 1$ and the mesh locally translation invariant in the x_1–direction with parameter $H \equiv h$. For $r = 2$, i.e., typically piecewise linears or bilinears, consider approximations to $\partial^2 u/\partial x^2$ based on the traditional second order accurate operator

$$(8.4.2)\qquad \partial^2 v(x) = (v(x + he_1) - 2v(x) + v(x - he_1))/h^2,$$

where $e_1 = (1, 0, \ldots, 0)$. (Here, $\widetilde{T}_h \equiv I$.)

Our aim is to show that if $A(u-u_h,\chi)=0$, for $\chi \in S_h^{comp}(\Omega_2)$, then

$$\left\|\frac{\partial^2 u}{\partial x^2}-\partial^2 u_h\right\|_{L_\infty(\Omega_0)} \le Ch^2(\ln 1/h)^3. \tag{8.4.3}$$

(Higher order elements are admitted but the final result is still only $0(h^2)$ as in (8.4.3), for obvious reasons.) For this, we shall assume that in some suitable negative norm,

$$\|u-u_h\|_{W_q^{-s}(\mathcal{D})} \le Ch^2. \tag{8.4.4}$$

We start by noting that

$$\partial^2 = \partial_-\partial_+ \tag{8.4.5}$$

where $\partial_+ v(x) = (v(x+he_1)-v(x))/h$ and $\partial_- v(x) = (v(x)-v(x-he_1))/h$.

We shall need two preliminary results. The first is that

$$\|\partial_\pm e\|_{W_\infty^1(\Omega_1)} \le Ch\ln 1/h. \tag{8.4.6}$$

Here, and below, $e=u-u_h$. The second preliminary result is that

$$\|\partial_\pm e\|_{L_\infty(\Omega_1)} \le Ch^2(\ln 1/h)^2. \tag{8.4.7}$$

We proceed to prove (8.4.6). For technical reasons we introduce a further domain $\Omega_{1.5}$ between Ω_1 and Ω_2. Consider then, for $\chi \in S_h^{comp}(\Omega_{1.5})$,

$$\begin{aligned} A(\partial_+ e,\chi) &= \sum_{i,j=1}^n \int a_{ij}(x)\Big(\partial_+\frac{\partial e}{\partial x_i}\Big)\frac{\partial \chi}{\partial x_j} \\ &= \frac{1}{h}\sum_{i,j=1}^n \int a_{ij}(x)\frac{\partial \chi}{\partial x_j}(x)\Big(\frac{\partial e}{\partial x_i}(x+he_1)-\frac{\partial e}{\partial x_i}(x)\Big) \\ &= \frac{1}{h}\sum_{i,j=1}^n \int \Big(a_{ij}(x-he_1)\frac{\partial \chi}{\partial x_j}(x-he_1)-a_{ij}(x)\frac{\partial \chi}{\partial x_j}(x)\Big)\frac{\partial e}{\partial x_i}(x) \\ &= -\frac{1}{h}\sum_{i,j=1}^n \int a_{ij}(x)\Big(\frac{\partial \chi}{\partial x_j}(x)-\frac{\partial \chi}{\partial x_j}(x-he_1)\Big)\frac{\partial e}{\partial x_i}(x) \\ &\quad +\frac{1}{h}\sum_{i,j=1}^n \int (a_{ij}(x-he_1)-a_{ij}(x))\frac{\partial \chi}{\partial x_j}(x-he_1)\frac{\partial e}{\partial x_i}(x). \end{aligned} \tag{8.4.8}$$

Here, the first term on the right vanishes since $\chi(x)-\chi(x-he_1) \in S_h^{comp}(\Omega_2)$ by translation invariance. Thus,

$$\begin{aligned} |A(\partial_+ e,\chi)| &= \Big|-\sum_{i,j=1}^n \int (\partial_- a(x))\frac{\partial \chi}{\partial x_j}(x-he_1)\frac{\partial e}{\partial x_i}(x)\Big| \\ &\le C\|e\|_{W_\infty^1(\Omega_2)}\|\chi\|_{W_1^1(\Omega_{1.5})}, \ \text{for } \chi \in S_h^{comp}(\Omega_{1.5}). \end{aligned} \tag{8.4.9}$$

In other words,

$$A(\partial_+ e,\chi) = F(\chi), \ \text{for } \chi \in S_h^{comp}(\Omega_{1.5}), \tag{8.4.10}$$

where

$$\sup_{\substack{\chi\in S_h^{comp}(\Omega_{1.5})\\ \|\chi\|_{W_1^1(\Omega_{1.5})}=1}} |F(\chi)| \le C\|e\|_{W_\infty^1(\Omega_2)} \le Ch, \tag{8.4.11}$$

by use of Theorem 5.5.1 (with $F \equiv 0$), local approximation and the negative norm estimate (8.4.4). Another application of Theorem 5.5.1 then gives (8.4.6).

As for (8.4.7), we have declared that

$$F(\varphi) \equiv -\sum_{i,j=1}^{n} \int \Big(\partial_- a(x)\Big)\frac{\partial\varphi}{\partial x_j}(x - he_1)\frac{\partial e}{\partial x_i}(x)\Big). \tag{8.4.12}$$

Integration by parts, freeing e from its derivatives, using Theorem 5.2.2 twice (once for e, with $F \equiv 0$, and once for $A(\partial_+ e, \chi) = F(\chi)$), then gives (8.4.7).

We next consider $A(\partial^2 e, \chi)$. We have for $\chi \in S_h^{comp}(\Omega_1)$,

$$\begin{aligned} A(\partial^2 e, \chi) &= \sum_{i,j=1}^{n} \int a_{ij}(x)\Big((\partial_-\partial_+)\frac{\partial e}{\partial x_i}\Big)\frac{\partial \chi}{\partial x_j} \\ &= -\sum_{i,j=1}^{n} \int \partial_+\Big(a_{ij}\frac{\partial\chi}{\partial x_j}\Big)\partial_+\frac{\partial e}{\partial x_i} \\ &= \sum_{i,j=1}^{n} \int (\partial^2)\Big(a_{ij}\frac{\partial\chi}{\partial x_j}\Big)\frac{\partial e}{\partial x_i}. \end{aligned} \tag{8.4.13}$$

Now, using the discrete analogue of Leibniz' rule, $\partial^2(fg) = (\partial^2 f)g + (\partial_+ f)(\partial_+ g) + (\partial_- f)(\partial_- g) + f(\partial^2 g)$, when second differences fall on $\frac{\partial\chi}{\partial x_j}$, the result is zero, by translation invariance and (8.1.4). When second differences fall on a_{ij}, that difference quotient is bounded, since the a_{ij} are smooth. The typical middle term is

$$\sum_{i,j=1}^{n} \int \partial_\pm a_{ij}\Big(\partial_\pm\frac{\partial\chi}{\partial x_j}\Big)\frac{\partial e}{\partial x_i}, \tag{8.4.14}$$

and we transfer the difference quotient falling on $\frac{\partial\chi}{\partial x_j}$ to $\partial_- a_{ij}$ or $\frac{\partial e}{\partial x_i}$. Using then (8.4.6) we have

$$A(\partial^2 e, \chi) = F(\chi), \text{ for } \chi \in S_h^{comp}(\Omega_1), \tag{8.4.15}$$

with

$$\|F\|_{-1,\infty,\Omega_1} \le Ch\ln 1/h. \tag{8.4.16}$$

Similarly, using now (8.4.7) after integration by parts, cf. (8.4.12) et. seq.,

$$\|F\|_{-2,\infty,\Omega_1} \le Ch^2(\ln 1/h)^2. \tag{8.4.17}$$

We may thus apply Theorem 5.5.2 to deduce that

$$\|\partial^2 e\|_{L_\infty(\Omega_0)} \le Ch^2(\ln 1/h)^3. \tag{8.4.18}$$

We conclude that

$$\left\| \frac{\partial^2 u}{\partial x^2} - \partial^2 u_h \right\|_{L_\infty(\Omega_0)} \le Ch^2(\ln 1/h)^3. \tag{8.4.19}$$

The reader has no doubt found an induction argument hidden above, cf. (8.4.6) and (8.4.7) . For the general case, see [Nitsche and Schatz 1974, Section 6] where this induction is brought out into the open in the case of L_2–based estimates. Details are not particularly hard to translate to pointwise estimates.

Remark 8.4.1. The reader will also have noticed a proliferation of $\ln 1/h$ factors. Actually, one may avoid this by going back to an earlier technique for getting L_∞–estimates, in the case of meshes which are now required to be translation invariant in all of n linearly independent directions. This technique involves estimating L_∞–norms by Sobolev's inequality in L_2, but now with *difference quotients* rather than derivatives of order $> n/2$ involved. Somewhat more regularity of u is required than in our arguments above in this chapter. We refer to [Bramble, Nitsche and Schatz 1975, Lemma 4.2 in particular] for details. Cf. also Section 11.4 below.

In fact, as remarked by [Liu 1994b] and [Schatz 1994] if one is interested in interior maximu–norm esitmates only on translation invariant meshes but for a more complicated problem than the scalar one treated in Section 5.5 (e.g., a Stokes' problem or a linear elasticity problem), this technique may give a much quicker derivation than could be had by doing the analogues of the proofs of Theorems 5.5.1 and 5.5.2. (After all, these two theorems are valid for general quasi–uniform meshes.) □

Chapter 9. On superconvergence in nonlinear problems.

For simplicity in notation, we shall consider only an equation involving highest order terms, and, for additional brevity, we shall only give a "global" argument. The reader will have no problem in furnishing details with respect to these omissions.

Let $\mathbf{q}(w)$ denote the gradient vector $\mathbf{q}(w) = \nabla w$ and let $a_{ij}(w, \mathbf{q})$, $i, j = 1, \ldots, n$, be smooth functions for $(w, \mathbf{q}) \in R^{1+n}$. The basic problem is now

$$(9.1) \qquad \begin{cases} L(u) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(u, \mathbf{q}(u)) \frac{\partial u}{\partial x_j}\right) = f \text{ on } \mathcal{D}, \\ \text{some linear, homogeneous boundary condition on } \partial\mathcal{D}. \end{cases}$$

We assume that there is a unique solution u to (9.1) which is as smooth as necessary on $\overline{\mathcal{D}}$. It is not assumed that the $a_{ij}(w, \mathbf{q})$ are bounded on the whole of R^{1+n}, nor is it assumed that the operator based on $\{a_{ij}(w, \mathbf{q})\}$ is uniformly elliptic on the whole of R^{1+n}.

We let $A(u, \mathbf{q}(u))(v, w)$ denote the corresponding form which is bilinear in v and w. The finite element problem is now to find $u_h \in S_h$ such that

$$(9.2) \qquad A(u_h, \mathbf{q}(u_h))(u_h, \chi) = (f, \chi), \text{ for } \chi \in S_h$$

where, of course, S_h would respect essential boundary conditions (at least in some sense). Again we assume that there is a unique solution u_h to (9.2), at least for h sufficiently small.

Let us pause to remark that the essential ideas of this chapter are due to [Douglas and Dupont 1975], in an L_2–based case. Conditions which ensure that (9.2) has a unique solution may be found there, see also the treatise [Ženíšek 1990]. For L_∞–estimates, cf. also [Nitsche 1977] and [Frehse and Rannacher 1978]. For superconvergence, cf. [Chen 1983a].

The main theme in this chapter is that if certain points (or postprocessors) are superconvergent for a certain linearized problem associated with (9.1), then they are automatically superconvergent for the error in the nonlinear problem. We proceed to introduce this linearized problem, which involves the derivative of A at u in the direction v.

Consider $u + v$ where v is to be thought of as "small". Then

$$(9.3) \qquad \begin{aligned} &a_{ij}(u + v, \mathbf{q}(u) + \mathbf{q}(v)) \frac{\partial(u + v)}{\partial x_i} \\ &\quad = a_{ij}(u, \mathbf{q}(u)) \frac{\partial u}{\partial x_i} + a_{ij}(u, \mathbf{q}(u)) \frac{\partial v}{\partial x_i} \\ &\qquad + [a_{ij}(u + v, \mathbf{q}(u) + \mathbf{q}(v)) - a_{ij}(u, \mathbf{q}(u))] \left[\frac{\partial u}{\partial x_i} + \frac{\partial v}{\partial x_i}\right]. \end{aligned}$$

By Taylor's theorem,

$$(9.4) \qquad \begin{aligned} &a_{ij}(u + v, \mathbf{q}(u) + \mathbf{q}(v)) - a_{ij}(u, \mathbf{q}(u)) \\ &\quad = \frac{\partial a_{ij}}{\partial u}(u, \mathbf{q}(u)) v + \sum_{k=1}^{n} \frac{\partial a_{ij}}{\partial q_k}(u, \mathbf{q}(u)) \frac{\partial v}{\partial x_k} + R, \end{aligned}$$

where $R = R(x) = R(u(x), \mathbf{q}(u(x)), v(x), \mathbf{q}(v(x)))$ may be bounded by

$$(9.5) \qquad |R| \leq C_1(u,v)\|v\|^2_{W^1_\infty(\mathcal{D})},$$

with $C_1(u,v)$ depending on $\|u\|_{W^1_\infty(\mathcal{D})}$ and $\|u+v\|_{W^1_\infty(\mathcal{D})}$ (through the a_{ij} and their derivatives).

Since also

$$(9.6) \qquad |a_{ij}(u+v, \mathbf{q}(u)+\mathbf{q}(v)) - a_{ij}(u, \mathbf{q}(u))| \leq C_2(u,v)\|v\|_{W^1_\infty(\mathcal{D})},$$

we find from the above that

$$(9.7) \qquad A(u+v, \mathbf{q}(u+v))(u+v, \chi) = A(u, \mathbf{q}(u))(u, \chi) + \widetilde{A}(v, \chi) + F(\chi)$$

where $\widetilde{A}$, which depends on u, is the bilinear form

$$(9.8) \qquad \widetilde{A}(v,\chi) = A(u, \mathbf{q}(u))(v, \chi) + \int_{\mathcal{D}} \sum_{i,j=1}^{n} \Big(\frac{\partial a_{ij}}{\partial u}(u, \mathbf{q}(u))v + \sum_{k=1}^{n} \frac{\partial a_{ij}}{\partial q_k}(u, \mathbf{q}(u)) \frac{\partial v}{\partial x_k} \Big) \frac{\partial u}{\partial x_i} \frac{\partial \chi}{\partial x_j},$$

and where

$$(9.9) \qquad |||F|||_{-1,\infty,\mathcal{D}} \leq C_3(u,v)\|v\|^2_{W^1_\infty(\mathcal{D})},$$

with $C_3(u,v)$ depending (through the a_{ij}) only on $\|u\|_{W^1_\infty(\mathcal{D})}$ and $\|u+v\|_{W^1_\infty(\mathcal{D})}$.

We note that $\widetilde{A}$ is *not* gotten by simply freezing the coefficients of A at u.

Under our general assumptions, the form $\widetilde{A}(v,\chi)$ has smooth coefficients.

We now introduce a major assumption:

(9.10) The bilinear form $\widetilde{A}(v,\chi)$ is such that the "global" variant of Theorems 5.5.1 holds.

(This "global" variant is obtained by setting $d = 1$, $\Omega_0 = \Omega_d = \mathcal{D}$, and scrapping the negative norm term, in (5.5.4), cf. [Schatz and Wahlbin 1994, Theorem 4.1] for a proof in a simple case.) That (9.10) holds must be checked in each application, cf. [Douglas and Dupont 1975]. Under various simplifying assumptions, the maximum–principle may apply to show that the corresponding linearized boundary value problem has a unique solution.

Example 9.1. Let $n = 2$ and the partial differential operator in (9.1) equal

$$-div(a(q^2)\nabla u)$$

where $q^2 = q^2(u) = |\nabla u|^2 = \sum_{i=1}^{2} \left|\frac{\partial u}{\partial x_i}\right|^2$. A short calculation establishes that the bilinear form $\widetilde{A}$ in (9.8) has only terms involving $\frac{\partial v}{\partial x_k} \frac{\partial \chi}{\partial x_j}$ and that the corresponding matrix of coefficients $\widetilde{\mathbb{A}}$ is

$$a(q^2)I + 2a'(q^2)\{q_k q_i\} = \begin{bmatrix} a(q^2) + 2a'(q^2)q_1^2 & 2a'(q^2)q_1q_2 \\ 2a'(q^2)q_1q_2 & a(q^2) + 2a'(q^2)q_2^2 \end{bmatrix}.$$

In the special case of

$$a(q^2) = (1+q^2)^t, \ t \text{ real},$$

we have the matrix

$$\widetilde{\mathbb{A}} = (1+q^2)^{t-1} \begin{bmatrix} 1+q^2+2tq_1^2 & 2tq_1q_2 \\ 2tq_1q_2 & 1+q^2+2tq_2^2 \end{bmatrix}.$$

A simple calculation shows that if q is bounded, $\widetilde{\mathbb{A}}_{11}$ and $\det \widetilde{\mathbb{A}}$ are both uniformly positive for $t \geq -\frac{1}{2}$. Hence, cf. e.g. [Shilov 1961, Section 46, Theorem 27] the form $\widetilde{A}$ is uniformly elliptic in these cases. If the original boundary conditions were $u = 0$ on $\partial\mathcal{D}$, the maximum principle gives that the linearized problem has unique solutions. The case $t = -1/2$ is the minimal surface equation, cf. [Johnson and Thomée 1975], and, for the reader interested in the history of finite elements, [Schellbach 1851]. For the case of the p–Laplacian, cf. [Barrett and Liu 1993]. □

Remark 9.1. If instead we consider the Neumann problem

$$\begin{aligned} &-div(a(q^2(u))\nabla u) + u = f \quad \text{in} \quad \mathcal{D}, \\ &\frac{\partial u}{\partial u} = 0 \quad \text{on} \quad \partial\mathcal{D}, \end{aligned}$$

with $a(q^2) = (1+q^2)^t$, $t \geq -\frac{1}{2}$, then [Schatz and Wahlbin 1994, Theorem 4.1] immediately applies to show that the assumption (9.10) holds. □

Remark 9.2. With $L(u)$ as in (9.1), perhaps with lower order terms, L is called *monotone* if, in a formal sense,

$$(L(u+v) - L(u), v) \geq 0,$$

for u and v satisfying the boundary conditions under consideration. This can serve to show existence of solutions to (9.1) and (9.2) (for (9.1), in some exact function class). Under further conditions, including so–called strict monotonicity, one may assert that the linearized operator $\widetilde{L}$ corresponding to our form $\widetilde{A}$ in (9.8) is coercive and so has unique solutions. We shall not pursue the details but refer to e.g. [Lions 1969, Chapter 2, Section 2] and [Ženíšek 1990, Chapter 4]. □

Let now $\widetilde{u}_h$ denote the projection of u into S_h with respect to the derivative form $\widetilde{A}$;

$$\widetilde{A}(u - \widetilde{u}_h, \chi) = 0, \text{ for } \chi \in S_h. \tag{9.11}$$

We assume that (cf. (9.10))

$$\|u - \widetilde{u}_h\|_{W^1_\infty(\mathcal{D}_h)} \leq Ch^{r-1}. \tag{9.12}$$

(E.g., with essential boundary conditions, $\mathcal{D}_h \subseteq \mathcal{D}$ denote mesh domains.)

The idea is now to use the above to compare $\widetilde{u}_h$ to u_h; under certain assumptions, the difference $\widetilde{u}_h - u_h$ will be shown to be of higher order. If so, known "linear" superconvergence results about $u - \widetilde{u}_h$ automatically translate to the nonlinear case.

Taking u to be the solution of (9.1) and $u + v = u_h$ the solution to (9.2), (9.7) gives

$$\widetilde{A}(\widetilde{u}_h - u_h, \chi) = \widetilde{A}(u - u_h, \chi) = -F(\chi), \text{ for } \chi \in S_h. \tag{9.13}$$

Yet another assumption will now be introduced! Namely, that

$$\varepsilon \equiv \|u - u_h\|_{W^1_\infty(\mathcal{D}_h)} \equiv \|v\|_{W^1_\infty(\mathcal{D}_h)} = o\Big(\frac{1}{\ln 1/h}\Big), \text{ as } h \to 0. \tag{9.14}$$

E.g., cf. [Douglas and Dupont 1975], if it is known that $\|u - u_h\|_{H^1(\mathcal{D}_h)} \le Ch^{r-1}$, (9.14) would easily follow from inverse assumptions if $r > n/2 + 1$. Of course, (9.14) controls the constant $C_3(u, v)$ in (9.9).

Using the "global" form of Theorem 5.5.1, we now have via (9.9),

$$\|\widetilde{u}_h - u_h\|_{W^1_\infty(\mathcal{D}_h)} \le C\varepsilon^2(\ln 1/h). \tag{9.15}$$

Since

$$\begin{aligned} \varepsilon = \|u - u_h\|_{W^1_\infty(\mathcal{D}_h)} &\le \|\widetilde{u}_h - u_h\|_{W^1_\infty(\mathcal{D}_h)} + \|u - \widetilde{u}_h\|_{W^1_\infty(\mathcal{D}_h)} \\ &\le C\|\widetilde{u}_h - u_h\|_{W^1_\infty(\mathcal{D}_h)} + Ch^{r-1}, \end{aligned} \tag{9.16}$$

where we used (9.12), setting $\eta = \|\widetilde{u}_h - u_h\|_{W^1_\infty(\mathcal{D}_h)}$, we then have

$$\eta \le C\eta^2(\ln 1/h) + Ch^{2(r-1)}(\ln 1/h). \tag{9.17}$$

From (9.12) and (9.14) we know that $\eta = o(1/(\ln 1/h))$, as $h \to 0$. Then (9.17) implies that

$$\|\widetilde{u}_h - u_h\|_{W^1_\infty(\mathcal{D}_h)} \le Ch^{2(r-1)}(\ln 1/h), \tag{9.18}$$

which is our final estimate in this brief chapter.

In conclusion, if superconvergence points for derivatives or function values (or, postprocessors) are known for the linearized problem $\widetilde{A}(\widetilde{u}_h - u, \chi) = 0$, for $\chi \in S_h$, then under the various assumptions introduced in this chapter, the same points are superconvergent if (9.18) permits.

Remark 9.3. (On tensor products) If $L(u)$ in (9.1) is of the form

$$Lu = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\alpha(u)\frac{\partial u}{\partial x_i}\Big)$$

then the linearized bilinear form is

$$\widetilde{A}(v, \chi) = \int_{\mathcal{D}} \sum_{i=1}^{n} \Big[\alpha(u)\frac{\partial v}{\partial x_i}\frac{\partial \chi}{\partial x_i} + \alpha'(u)\frac{\partial u}{\partial x_i}v\frac{\partial \chi}{\partial x_i}\Big].$$

I.e., it is of the type so that the tensor–product theory in Section 6.3 for superconvergence in gradients applies to it.

In a homogeneous Dirichlet problem, e.g., we may also allow a lower order term of the type $a(u, \mathbf{q}(u))u$ with the theory of Section 6.3 still applying. □

Remark 9.4. (On Newton's method) Let u^1 be any approximation to u, not necessarily in S_h, and then perform a Newton step in S_h. I.e., with $\widetilde{A}^1(v, \chi)$ denoting the bilinear form in (9.8) but with u and $\mathbf{q}(u)$ replaced by u^1 and $\mathbf{q}(u^1)$, i.e., the derivative of A at u^1 in the direction v, define $u_h^2 \in S_h$ by

$$\widetilde{A}^1(u_h^2 - u^1, \chi) = -A(u^1, \mathbf{q}(u^1))(u^1, \chi) + (f, \chi), \text{ for } \chi \in S_h. \tag{9.20}$$

We then have, using (9.11) and (9.20),

$$
\begin{aligned}
(9.21)\qquad \widetilde{A}(u_h^2 - \widetilde{u}_h, \chi) &= \widetilde{A}(u_h^2 - u^1, \chi) + \widetilde{A}(u^1 - u, \chi) \\
&= (\widetilde{A} - \widetilde{A}^1)(u_h^2 - u^1, \chi) + \widetilde{A}^1(u_h^2 - u^1, \chi) + \widetilde{A}(u^1 - u, \chi) \\
&= [(\widetilde{A} - \widetilde{A}^1)(u_h^2 - u^1, \chi)] + [-A(u^1, \mathbf{q}(u^1))(u^1, \chi) \\
&\qquad + A(u, \mathbf{q}(u))(u, \chi) + \widetilde{A}(u^1 - u, \chi)] \\
&\equiv F_1(\chi) + F_2(\chi).
\end{aligned}
$$

Here, $F_2(\chi)$ is as in (9.7), with $v = u^1 - u$. Setting

$$(9.22)\qquad \delta = \|u^1 - u\|_{W^1_\infty(\mathcal{D}_h)}$$

(which we assume is ≤ 1) we then have from (9.9),

$$(9.23)\qquad |||F_2|||_{-1,\infty,\mathcal{D}_h} \leq C\delta^2.$$

For F_1, we string it out as

$$(9.24)\quad F_1(\chi) = (\widetilde{A} - \widetilde{A}^1)(u_h^2 - \widetilde{u}_h, \chi) + (\widetilde{A} - \widetilde{A}^1)(\widetilde{u}_h - u, \chi) + (\widetilde{A} - \widetilde{A}^1)(u - u^1, \chi).$$

Setting

$$(9.25)\qquad \eta = \|u_h^2 - \widetilde{u}_h\|_{W^1_\infty(\mathcal{D}_h)}$$

we then clearly have

$$(9.26)\qquad |||F_1|||_{-1,\infty,\mathcal{D}_h} \leq C\delta\Big(\eta + \|\widetilde{u}_h - u\|_{W^1_\infty(\mathcal{D}_h)} + \delta\Big).$$

Assume for definiteness also the linear result that

$$(9.27)\qquad \|\widetilde{u}_h - u\|_{W^1_\infty(\mathcal{D}_h)} \leq Ch^{r-1},$$

cf. (9.10), (9.11) and (9.12). Then

$$(9.28)\qquad |||F_1|||_{-1,\infty,\mathcal{D}_h} \leq C\delta\eta + C\delta h^{r-1} + C\delta^2.$$

It follows from (9.21), (9.23) and (9.28), under the assumption (9.10), that

$$(9.29)\qquad \eta \leq C(\ln 1/h)[\delta\eta + \delta h^{r-1} + \delta^2].$$

If we now finally assume that $\delta \ln 1/h \to 0$ as $h \to 0$, we have

$$(9.30)\qquad \|u_h^2 - \widetilde{u}_h\|_{W^1_\infty(\mathcal{D}_h)} \leq C(\ln 1/h)\Big[h^{2r-2} + \|u^1 - u\|^2_{W^1_\infty(\mathcal{D}_h)}\Big].$$

E.g., if $\|u^1 - u\|_{W^1_\infty(\mathcal{D}_h)} \leq Kh^{r-1}$, and if $K = K(h)$ is not too badly growing with $1/h$, we have asserted that one Newton–step in S_h gives us an approximation u_h^2 in the nonlinear problem which exhibits "the same superconvergence phenomena as in the linearized problem".

Of course, the reader will have noted that this remark, indeed, this whole chapter, reflects the quadratic convergence of Newton's method. One may, e.g., start with u^1 a rough approximation, say on a "coarse" grid and iterate this remark, leading to a "multi–grid" approach. In this respect, cf. [Xu, J. 1994]. □

Chapter 10. Superconvergence in isoparametric mappings of translation invariant meshes: an example.

The example given in this chapter is a special case of the theory in [Cayco, Schatz and Wahlbin 1994].

10.1. Introduction.

Let $\widehat{R} = \{a \le \widehat{x} \le b, c \le \widehat{y} \le d\}$ be a rectangle, and $\varphi = \varphi(\widehat{x}, \widehat{y}) = (\varphi^1, \varphi^2)$ a smooth mapping $\widehat{R} \to R \subset\subset \mathcal{D} \subset\subset R^2$, which is one–to–one and onto, and has a smooth inverse.

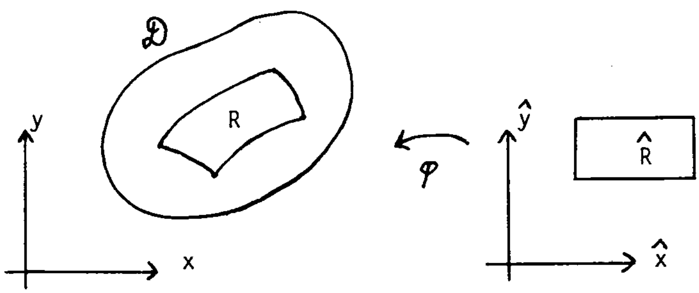


Figure 10.1.1

Here, $\mathcal{D}$ is our "physical domain" in the $x - y$ plane and on the piece R of it, we have, for whatever reason, decided to install a mesh associated with the map φ of $\widehat{R}$.

Let us say that we are interested in using quadratic isoparametric triangles. We would then first subdivide $\widehat{R}$ into triangles, say, translation invariant in the $\widehat{x}$ direction with parameter h, and quasi–uniform in the $\widehat{y}$ direction.

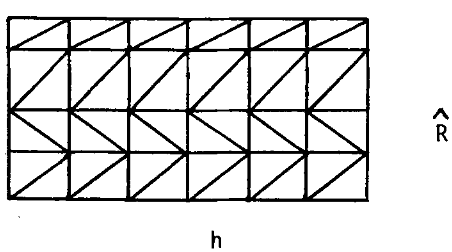


Figure 10.1.2

Let $\widehat{\tau}_i$, $i = 1, \dots, I$ denote these triangles. For each $\widehat{\tau}_i$, let

$$\text{(10.1.1)} \qquad \widehat{a}^i_j = \text{ vertices, } j = 1,2,3; \ \widehat{a}^i_{jk} = \text{ midpoints of sides, } 1 \le j < k = 3.$$

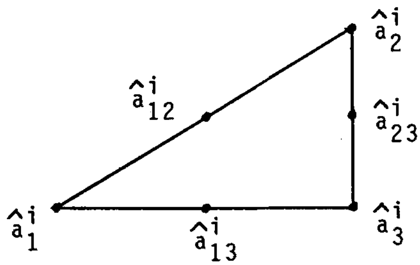


Figure 10.1.3

By the mapping φ, these points (let us collectively call them $\{\widehat{a}\}$) map to points $\{a\}$ in R:

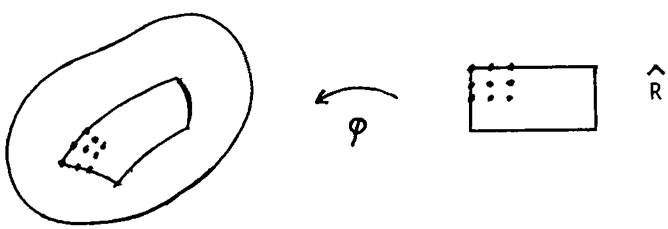


Figure 10.1.4

On each $\widehat{\tau}_i$ we now construct the map

$$\varphi_{i,h} = (\varphi^1_{i,h}(\widehat{x}, \widehat{y}), \varphi^2_{i,h}(\widehat{x}, \widehat{y})) \tag{10.1.2}$$

where each $\varphi^j_{i,h}$, $j = 1, 2$, is a quadratic in $\widehat{x}$ and $\widehat{y}$,

$$\varphi^j_{i,h}(\widehat{x}, \widehat{y}) = A_j + B_j\widehat{x} + C_j\widehat{y} + D_j\widehat{x}^2 + E_j\widehat{x}\widehat{y} + F_j\widehat{y}^2,\ j = 1, 2, \tag{10.1.3}$$

such that the vertices $\widehat{a}^i_j$ go to $a^i_j = \varphi(\widehat{a}^i_j)$ and the midpoints $\widehat{a}^i_{jk}$ go to $a^i_{jk} = \varphi(\widehat{a}^i_{jk})$.
If φ is fixed smooth and h is small enough, each such $\varphi_{i,h}$ exists and is invertible, see e.g. [Ciarlet 1991, Chapter VI, Theorem 37.2]. Also their Jacobians do not vanish. The main point in applying that theorem is the following: If $\widetilde{\tau}_i$ denotes the result of an *affine* map of $\widehat{\tau}_i$ respecting the vertices only, then with $\widetilde{a}_{jk}$ the corresponding midpoints of $\widetilde{\tau}_i$, we have for the points $a_{jk} = \varphi(\widehat{a}_{jk})$,

$$|\widetilde{a}_{jk} - a_{jk}| = 0(h^2). \tag{10.1.4}$$

In practice, for h not "small enough", one has to watch out for the map being one–to–one and onto, cf. e.g. [Frey, Hall and Porsching 1978]. We denote by φ_h the total map of $\widehat{R}$ to R_h (an approximation to φ) thus constructed, and set $\tau_i = \varphi_h(\widehat{\tau}_i)$. In pictures,

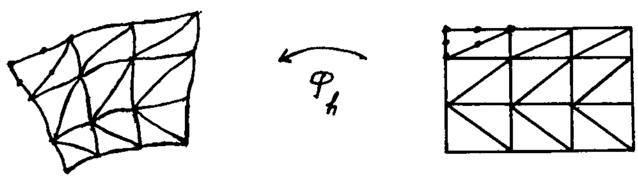


Figure 10.1.5

Each τ_i is thus a curved "triangle" and under our assumptions on φ, its "length dimensions" are $0(h)$ in a quasi–uniform fashion for h small enough.

So far we have constructed a "mesh" on R, or, on R_h to be precise. As for the finite element function spaces in this part of the physical domain, they will be given as follows: On each $\widehat{\tau}_i$, consider quadratics $\widehat{\chi}(\widehat{x},\widehat{y}) = A + B\widehat{x} + C\widehat{y} + D\widehat{x}^2 + E\widehat{x}\widehat{y} + F\widehat{y}^2$. Under the map φ_h, they naturally give rise to a function $\chi(x,y)$ on τ_i. Or, put otherwise, $\chi(x,y)$ is such that when mapped to $\widehat{R}$ by the inverse map φ_h^{-1}, it becomes a quadratic in $\widehat{x}$, $\widehat{y}$.

We shall refer to $\widehat{R}$ as the "computational domain".

If then $\widehat{\chi}(\widehat{x},\widehat{y})$ is a C^0 quadratic on $\widehat{R}$, then $\chi(x,y)$ is still a continuous function on R_h; we refer to them as "isoparametric quadratic". (The word *iso*parametric means that the same polynomial functions have been used in approximating φ as in giving $\widehat{\chi}$. With a slight abuse of terminology, $\widehat{\tau}_i$ may be referred to as a "reference element" for τ_i; although, $\widehat{\tau}_i$ would then commonly be blown up to unit size, which we shall not do. We refer to [Ciarlet 1991, Chapter VI] for details.)

We denote by S_h the isoparametric quadratic finite elements thus constructed, at least on R_h. Let then as usual u be given and $u_h \in S_h$ such that

$$A(u - u_h, \chi) = 0, \text{ for } \chi \in S_h^{comp}(R_h). \tag{10.1.5}$$

Remark 10.1.1. As for the actual computation of a typical term like $\int_\tau a_{12}(x,y)\frac{\partial u_h}{\partial x}\frac{\partial \chi}{\partial y}dxdy$, with the correspondence $(x,y) = \varphi_h(\widehat{x},\widehat{y})$, one has

$$\int_{\widehat{\tau}} \widehat{a}_{12}\left[\frac{\partial}{\partial \widehat{x}}\,\widehat{u}_h\,\frac{\partial \widehat{x}}{\partial x} + \frac{\partial}{\partial \widehat{y}}\,\widehat{u}_h\,\frac{\partial \widehat{y}}{\partial x}\right]\left[\frac{\partial \widehat{\chi}}{\partial \widehat{x}}\,\frac{\partial \widehat{x}}{\partial y} + \frac{\partial \widehat{\chi}}{\partial \widehat{y}}\,\frac{\partial \widehat{y}}{\partial y}\right] J(\varphi_h)d\widehat{x}d\widehat{y}, \tag{10.1.6}$$

where $\partial\widehat{x}/\partial x$ involves derivatives of φ_h^{-1}. Computations are, in practice, done on the "reference elements" $\widehat{\tau}$ in the "computational" domain and clearly numerical integration is called for, even if the coefficients of the bilinear form were very simple. We shall not consider this numerical integration; if of high enough order, our final superconvergence results still hold. □

10.2. Superconvergence in difference quotients for first derivatives.

We shall first assume that "standard" theory applied to (10.1.5) gives that

$$h\|u - u_h\|_{W^1_\infty(R^0_h)} + \|u - u_h\|_{L_\infty(R^0_h)} \le Ch^3, \tag{10.2.1}$$

where $R^0_h \subset\subset R_h$ with 0(1) separation, for simplicity.

Letting

$$\widehat{e} = (u - u_h)(\varphi_h^{-1}) \equiv \widehat{u} - \widehat{u}_h, \tag{10.2.2}$$

we then have

$$h\|\widehat{e}\|_{W^1_\infty(\widehat{R}^0)} + \|\widehat{e}\|_{L_\infty(\widehat{R}^0)} \le Ch^3. \tag{10.2.3}$$

(It is elementary to verify that under our assumptions that φ and φ_h are one to one, with the Jacobians and their inverses bounded, φ^{-1} and φ_h^{-1} behave "similarly" in L_∞ and W^1_∞.)

Using now the map φ_h^{-1} we translate (10.1.5) to the computational domain $\widehat{R}$ so that

$$\widehat{A}(\widehat{e}, \widehat{\chi}) = 0, \text{ for } \widehat{\chi} \in \widehat{S}_h^{comp}(\widehat{R}). \tag{10.2.4}$$

The finite element functions $\widehat{u}_h$ and $\widehat{\chi}$ are here ordinary $\mathcal{C}^0$ quadratics on the translation invariant triangular mesh on $\widehat{R}$, but the form $\widehat{A}$ has only piecewise smooth coefficients on each triangle $\widehat{\tau}$; the coefficients are in general discontinuous across element boundaries, cf. (10.1.6). Likewise, the function $\widehat{u}(\widehat{x}, \widehat{y}) = u(\varphi_h(\widehat{x}, \widehat{y}))$ is now only piecewise smooth. For clarity we shall write

$$\widehat{A}(\cdot\,,\cdot) = \widehat{A}_{\varphi_h}(\cdot\,,\cdot). \tag{10.2.5}$$

Consider now the forward difference operator ∂_+ in the translation invariant $\widehat{x}$–direction,

$$\partial_+ v(\widehat{x}, \widehat{y}) = \frac{1}{h}(v(\widehat{x} + h, \widehat{y}) - v(\widehat{x}, \widehat{y})). \tag{10.2.6}$$

We let $\widehat{A}_\varphi$ denote the form obtained by the mapping φ^{-1} from R to $\widehat{R}$. Note that this form has smooth coefficients and satisfies our ellipticity conditions on the highest order terms, by standard arguments.

Let us look $\widehat{A}_\varphi(\partial v, w)$! Consider e.g. one of the highest order terms, where $\widehat{A}_{12}$

denotes the coefficient associated with $\frac{\partial v}{\partial \widehat{x}}\frac{\partial w}{\partial \widehat{y}}$ in the form $\widehat{A}_\varphi$.

$$\begin{aligned}(10.2.7)\qquad &\int_{\widehat{R}} \widehat{A}_{12}(\widehat{x},\widehat{y})\frac{\partial}{\partial \widehat{x}}(\partial_+ v)\frac{\partial w}{\partial \widehat{y}} d\widehat{x}d\widehat{y}\\
&= \frac{1}{h}\int_{\widehat{R}} \widehat{A}_{12}(\widehat{x},\widehat{y})\Big[\frac{\partial v}{\partial \widehat{x}}(\widehat{x}+h,\widehat{y}) - \frac{\partial v}{\partial \widehat{x}}(\widehat{x},\widehat{y})\Big]\frac{\partial w}{\partial \widehat{y}}(\widehat{x},\widehat{y})d\widehat{x}d\widehat{y}\\
&= \frac{1}{h}\int_{\widehat{R}} \frac{\partial v}{\partial \widehat{x}}(\widehat{x},\widehat{y})\Big[\widehat{A}_{12}(\widehat{x}-h,\widehat{y})\frac{\partial w}{\partial \widehat{y}}(\widehat{x}-h,\widehat{y})\\
&\qquad - \widehat{A}_{12}(\widehat{x},\widehat{y})\frac{\partial w}{\partial \widehat{y}}(\widehat{x},\widehat{y})\Big]d\widehat{x}d\widehat{y}\\
&= -\int_{\widehat{R}} \widehat{A}_{12}(\widehat{x},\widehat{y})\frac{\partial v}{\partial \widehat{x}}(\widehat{x},\widehat{y})\frac{\partial}{\partial \widehat{y}}(\partial_- w)(\widehat{x},\widehat{y})d\widehat{x}d\widehat{y}\\
&\qquad - \int_{\widehat{R}} (\partial_-\widehat{A}_{12})(\widehat{x},\widehat{y})\frac{\partial v}{\partial \widehat{x}}(\widehat{x},\widehat{y})\frac{\partial w}{\partial \widehat{y}}(\widehat{x}-h,\widehat{y})d\widehat{x}d\widehat{y},\end{aligned}$$

where ∂_- denotes the backward difference operator.

We conclude that for w with compact support,

$$(10.2.8)\qquad \widehat{A}_\varphi(\partial_+ v, w) = -\widehat{A}_\varphi(v,\partial_- w) + F_1(v,w)$$

where, over the relevant domains,

$$(10.2.9)\qquad |F_1(v,\psi)| \le C\|v\|_{W^1_\infty}\|\psi\|_{W^1_1}$$

and, after integration by parts, also,

$$(10.2.10)\qquad |F_1(v,\psi)| \le C\|v\|_{L_\infty}\|\psi\|_{W^2_1}.$$

We apply (10.2.8) to $v = \widehat{e}$ and $w = \widehat{\chi} \in \widehat{S}_h^{comp}(\widehat{R}_1)$ where $\widehat{R}_1 \subset\subset \widehat{R}$, with 0(1) separation. From (10.2.4) we have since $\partial_-\widehat{\chi} \in S_h^{comp}(\widehat{R})$ by translation invariance,

$$(10.2.11)\qquad -\widehat{A}_\varphi(\widehat{e},\partial_-\widehat{\chi}) = (\widehat{A}_{\varphi_h} - \widehat{A}_\varphi)(\widehat{e},\partial_-\widehat{\chi})$$

and by the properties of $\varphi - \varphi_h$, cf. (10.1.6) which shows how the forms $\widehat{A}_{\varphi_h}$ and $\widehat{A}_\varphi$ involves derivatives of φ_h and φ, and their inverses, and by (10.2.3),

$$\begin{aligned}(10.2.12)\qquad |(\widehat{A}_{\varphi_h} - \widehat{A}_\varphi)(\widehat{e},\partial_-\widehat{\chi})| &\le Ch^2\|e\|_{W^1_\infty(\widehat{R})}\frac{1}{h}\|\widehat{\chi}\|_{W^1_1(\widehat{R}_1)}\\
&\le Ch^3\|\widehat{\chi}\|_{W^1_1(\widehat{R}_1)}.\end{aligned}$$

In total from (10.2.8) et. seq., for the smooth and uniformly elliptic form $\widehat{A}_\varphi$,

$$(10.2.13)\qquad \widehat{A}_\varphi(\partial_+\widehat{e},\widehat{\chi}) = F(\widehat{\chi}),\ \text{for } \widehat{\chi} \in \widehat{S}_h^{comp}(\widehat{R}_1)$$

where, using again (10.2.3) in (10.2.9) and (10.2.10),

$$(10.2.14)\qquad |||F|||_{-1,\infty,\widehat{R}_1} \le Ch^2$$

and

$$(10.2.15)\qquad |||F|||_{-2,\infty,\widehat{R}_1} \le Ch^3.$$

It follows then from Theorem 5.5.2 that, for $\widehat{R}_0 \subset\subset \widehat{R}_1$, again with unit separation,

$$\|\partial_+\widehat{e}\|_{L_\infty(\widehat{R}_0)} \le C \min_{\chi\in\widehat{S}_h} \|\partial_+\widehat{u} - \widehat{\chi}\|_{L_\infty(\widehat{R}_1)} + Ch^3 \ln 1/h + C\|\partial_+\widehat{e}\|_{W_q^{-s}(\widehat{R}_1)}. \tag{10.2.16}$$

The general negative norm term on the right of (10.2.16) can easily be converted into $\|\widehat{e}\|_{W_q^{-s+1}(\widehat{R}_{1.5})}$ and thus, from (10.2.3),

$$\|\partial_+\widehat{e}\|_{L_\infty(R_0)} \le C \min_{\widehat{\chi}\in\widehat{S}_h} \|\partial_+\widehat{u} - \widehat{\chi}\|_{L_\infty(\widehat{R}_1)} + Ch^3 \ln 1/h. \tag{10.2.17}$$

It now becomes necessary to introduce some further notation in order to be perfectly clear about the differences incurred by the mappings φ or φ_h.

The "fixed" points are $\widehat{x}$, $\widehat{y}$ in the computational domain.

To treat the first term on the right of (10.2.17), we note that there $\widehat{u}(\widehat{x},\widehat{y}) \equiv \widehat{u}_{\varphi_h}(\widehat{x},\widehat{y}) \equiv u(\varphi_h(\widehat{x},\widehat{y}))$. With $\widehat{u}_\varphi(\widehat{x},\widehat{y}) \equiv u(\varphi(\widehat{x},\widehat{y}))$, a smooth function, we have

$$\partial_+\widehat{u}_{\varphi_h} - \widehat{\chi} = \partial_+(\widehat{u}_{\varphi_h} - \widehat{u}_\varphi) + (\partial_+\widehat{u}_\varphi - \widehat{\chi}) \equiv \widehat{I}_1 + \widehat{I}_2. \tag{10.2.18}$$

We clearly have for, say, $\widehat{\chi}$ the interpolant of $\partial_+\widehat{u}_\varphi$,

$$\|\widehat{I}_2\|_{L_\infty(\widehat{R}_1)} \le Ch^3 \|\partial_+\widehat{u}_\varphi\|_{W_\infty^3(\widehat{R}_1+Ch)} \le Ch^3, \tag{10.2.19}$$

if $u \in C^4$ locally.

For $\widehat{I}_1$, we have by Taylor expansion

$$\begin{aligned}
&\partial_+(\widehat{u}_{\varphi_h} - \widehat{u}_\varphi)\\
&\quad= \frac{1}{h}\big(u(\varphi_h(\widehat{x}+h,\widehat{y})) - u(\varphi(\widehat{x}+h,\widehat{y}))\\
&\qquad - (u(\varphi_h(\widehat{x},\widehat{y})) - u(\varphi(\widehat{x},\widehat{y}))\big)\\
&\quad= \frac{1}{h}\Big(\frac{\partial u}{\partial x}(\varphi(\widehat{x}+h,\widehat{y}))\big[\varphi_h^1(\widehat{x}+h,\widehat{y}) - \varphi^1(\widehat{x}+h,\widehat{y})\big]\\
&\qquad + \frac{\partial}{\partial y}u(\varphi(\widehat{x}+h,\widehat{y})\big[\varphi_h^2(\widehat{x}+h,\widehat{y}) - \varphi^2(\widehat{x}+h,\widehat{y})\big]\\
&\qquad - \Big(\frac{\partial u}{\partial x}(\varphi(\widehat{x},\widehat{y})\big[\varphi_h^1(\widehat{x},\widehat{y}) - \varphi^1(\widehat{x},\widehat{y})\big]\\
&\qquad + \frac{\partial u}{\partial y}(\varphi(\widehat{x},\widehat{y}))\big[\varphi_h^2(\widehat{x},\widehat{y}) - \varphi^2(\widehat{x},\widehat{y})\big]\Big)\\
&\qquad + \text{HOTS},
\end{aligned} \tag{10.2.20}$$

where HOTS denotes terms of the form $\frac{1}{h}0(Q)$, with Q quadratic in $\varphi_h - \varphi$. Such HOTS are $0(h^5)$.

Shifting the derivative $\frac{\partial u}{\partial x}(\varphi(\widehat{x}+h,\widehat{y}))$ to $\frac{\partial u}{\partial x}\varphi(\widehat{x},\widehat{y})$ (and the same for $\frac{\partial u}{\partial y}$) introduces an error which is bounded by $0(h^3)$. So, it remains to consider the error in,

$$\partial_+(\varphi_h^i - \varphi^i),\ i = 1,2. \tag{10.2.21}$$

But, clearly, a difference quotient of the interpolant φ_h^i is the interpolant to the same difference quotient of φ^i in our present translation invariant situation. It follows that

$$\|\widehat{I}_1\|_{L_\infty(\widehat{R}_1)} = \|\partial_+(\widehat{u}_{\varphi_h} - \widehat{u}_\varphi)\|_{L_\infty(\widehat{R}_1)} \le Ch^3. \tag{10.2.22}$$

From (10.2.17) and the above we then have

$$\|\partial_+\widehat{e}\|_{L_\infty(R_0)} \equiv \|\partial_+(\widehat{u}_{\varphi_h}(\widehat{x},\widehat{y}) - \widehat{u}_h(\widehat{x},\widehat{y}))\|_{L_\infty(\widehat{R}_0)} \le Ch^3 \ln 1/h. \tag{10.2.23}$$

Let us now consider a difference operator which is an $0(h^3)$ approximation to $\partial v/\partial\widehat{x}$, e.g.,

$$\begin{aligned}\partial v(\widehat{x},\widehat{y}) = \frac{1}{h}\Big[&-\frac{1}{6}(v(\widehat{x}+h,\widehat{y}) - v(\widehat{x}-h,\widehat{y})) \\ &+\frac{4}{3}\Big(v\Big(\widehat{x}+\frac{h}{2},\widehat{y}\Big) - v\Big(\widehat{x}-\frac{h}{2},\widehat{y}\Big)\Big)\Big].\end{aligned} \tag{10.2.24}$$

This operator is built up from linear combinations of ∂_+ (and translations by h or $h/2$) and is, in fact, an $0(h^4)$ approximation to $\partial v/\partial\widehat{x}$, as is elementary to check by Taylor expansion.

Then from (10.2.23),

$$\|\partial(\widehat{u}_{\varphi_h}(\widehat{x},\widehat{y}) - \widehat{u}_h(\widehat{x},\widehat{y}))\|_{L_\infty(R_0)} \le C\Big(\ln\frac{1}{h}\Big)h^3. \tag{10.2.25}$$

So far we haven't used that ∂ is an accurate approximation to $\partial/\partial\widehat{x}$. We shall, shortly.

Now, $\widehat{u}_{\varphi_h}$ is merely a piecewise smooth function. It remains to determine what, if anything, is approximated by $\partial\widehat{u}_h$ down in the physical domain.

Let us consider

$$\begin{aligned}\frac{\partial}{\partial\widehat{x}}\widehat{u}_\varphi(\widehat{x},\widehat{y}) &= \frac{\partial}{\partial\widehat{x}}u(\varphi(\widehat{x},\widehat{y})) \\ &= \frac{\partial u}{\partial x}(\varphi(\widehat{x},\widehat{y}))\frac{\partial\varphi^1}{\partial\widehat{x}} + \frac{\partial u}{\partial y}(\varphi(\widehat{x},\widehat{y}))\frac{\partial\varphi^2}{\partial\widehat{x}} \\ &\equiv \frac{\partial u}{\partial\ell}(\varphi(\widehat{x},\widehat{y})),\end{aligned} \tag{10.2.26}$$

i.e., the directional derivative of u in a certain direction determined by φ (not φ_h!) at the corresponding point in the physical domain.

Then

$$\begin{aligned}&\frac{\partial u}{\partial\ell}(\varphi(\widehat{x},\widehat{y})) - \partial\widehat{u}_h(\widehat{x},\widehat{y}) \\ &\quad= \Big(\frac{\partial}{\partial\widehat{x}} - \partial\Big)\widehat{u}_\varphi(\widehat{x},\widehat{y}) + \partial(\widehat{u}_\varphi - \widehat{u}_{\varphi_h})(\widehat{x},\widehat{y}) + \partial\widehat{e} \\ &\quad\equiv \widehat{J}_1 + \widehat{J}_2 + \partial\widehat{e}.\end{aligned} \tag{10.2.27}$$

From (10.2.25) and now using the approximation of properties of the difference quotient ∂,

$$\|\widehat{J}_1 + \partial\widehat{e}\|_{L_\infty(\widehat{R}_0)} \le Ch^3 \ln 1/h. \tag{10.2.28}$$

The result (10.2.22) shows that also

$$\|\widehat{J}_2\|_{L_\infty(\widehat{R}_0)} \le Ch^3. \tag{10.2.29}$$

We conclude that

$$\|\frac{\partial u}{\partial \ell}(\varphi(\widehat{x},\widehat{y})) - \partial\widehat{u}_h(\widehat{x},\widehat{y})\|_{L_\infty(\widehat{R}_0)} \le Ch^3 \ln 1/h. \tag{10.2.30}$$

We may, of course, since $\varphi - \varphi_h = 0(h^3)$, conclude also that

$$\|\frac{\partial u}{\partial \ell}(\varphi_h(\widehat{x},\widehat{y})) - \partial\widehat{u}_h(\widehat{x},\widehat{y})\|_{L_\infty(\widehat{R}_0)} \le Ch^3 \ln 1/h \tag{10.2.31}$$

so that we are now comparing things at the *same* point $\varphi_h(\widehat{x},\widehat{y})$ in the physical domain. However, the directional derivative $\frac{\partial u}{\partial \ell}$ is based on the original smooth map φ, cf. (10.2.26), i.e.,

$$\frac{\partial u}{\partial \ell}(\varphi_h(\widehat{x},\widehat{y})) = \frac{\partial u}{\partial x}(\varphi_h(\widehat{x},\widehat{y}))\frac{\partial \varphi^1(\widehat{x},\widehat{y})}{\partial \widehat{x}} + \frac{\partial u}{\partial y}(\varphi_h(\widehat{x},\widehat{y}))\frac{\partial \varphi^2(\widehat{x},\widehat{y})}{\partial \widehat{x}}. \tag{10.2.32}$$

Remark 10.2.1. In our present special case of quadratics we may proceed further as follows. If we assume in our situation of approximating $\frac{\partial}{\partial x}$ that $\widehat{y}$ is a mesh point $\widehat{y}_M$ and that $\widehat{x}$ is a Gauss-point $\widehat{x}_G$, cf. (7.5.3), so that $\partial\varphi_h^1/\partial\widehat{x}$ and $\partial\varphi_h^2/\partial\widehat{x}$ are superconvergent approximations to $\partial\varphi^1/\partial\widehat{x}$ and $\partial\varphi^2/\partial\widehat{x}$ at those points (if the mesh is translation invariant in the $\widehat{y}$ direction as well), then

$$|\frac{\partial u}{\partial \ell_h}(\varphi_h(\widehat{x}_G,\widehat{y}_M)) - \partial\widehat{u}_h(\widehat{x}_G,\widehat{y}_M)| \le Ch^3 \ln 1/h \tag{10.2.33}$$

where

$$\frac{\partial u}{\partial \ell_h}(\cdot) = \frac{\partial u}{\partial x}(\cdot)\frac{\partial \varphi_h^1(\cdot)}{\partial \widehat{x}} + \frac{\partial u}{\partial y}(\cdot)\frac{\partial \varphi_h^2(\cdot)}{\partial \widehat{x}}. \tag{10.2.34}$$

Note that such a point $(\widehat{x}_G,\widehat{y}_M)$, when mapped by φ_h, is on the boundary of a physical element and that $\frac{\partial u}{\partial \ell_h}$ is in the tangential direction along the mesh boundary at that point. □

There are no particular difficulties in treating higher order difference quotients, except, it becomes progressively less clear to identify what they "naturally" correspond to in the physical domain. Actually, it is often more appropriate to work "the other way": If one desires an approximation to a certain higher order derivative in certain directions at a certain point in the physical domain, one should express that (in, perhaps, a complicated fashion) in the computational domain, and then use difference quotients for derivatives of $\widehat{u}$ and also use difference quotients to approximate derivatives involved of the mapping function.

Remark 10.2.2. (On mesh-perturbations) In [Zlamal 1977, 1978a, 1978b] and in [Lesaint and Zlamal 1979] (and in a multitude of later papers), meshes were considered which are "slight" perturbations of translation invariant ones (or, slight perturbations of tensor product meshes). Basically, if one desires superconvergence of order $0(h^s)$, $s = r + \ell$, for function values, (or, $s = (r-1) + \ell$ for derivatives), it is assumed that the mesh is on each element an $0(h^s)$ perturbation of a regular (translation invariant, or, tensor-product) mesh.

(The perturbation argument is not hard although sometimes longish; at this point in our lectures it may safely be left to the reader.)

Our present curved mesh in the physical domain *cannot* for general φ be viewed as such a perturbation, except for $s = 2$, which would not suffice to prove superconvergence by such perturbation techniques for quadratics or higher elements. In the piecewise linear case, a mapped mesh can be viewed on each element as an $0(h^2)$ perturbation of a regular mesh. Thus, the piecewise linear results on superconvergence of averages of derivatives at symmetry points carry over without much fuss to that situation. $\square$

Chapter 11. Superconvergence by averaging: mainly, the K–operator.

11.1. Introduction.

In certain cases for second order elliptic partial differential equations we know that

$$\|u - u_h\|_{W_2^{-\ell}(\mathcal{D})} \le Ch^{r+\ell}, \tag{11.1.1}$$

for some $\ell > 0$, $\ell = r - 2$ being the best we can reasonably expect in the most favorable situation, see Section 6.3. Such higher rate of convergence in a negative norm suggests oscillatory behavior of the error, cf. Section 4.1 and 5.2. For meshes which are locally translation invariant in all of the axes directions, [Bramble and Schatz 1974] found a local "filter–device" to average out such local oscillations in a systematic fashion, the K–operator, which when applied to u_h gives $K * u_h(x) = \int K(x-y)u_h(y)dy$ satisfying

$$\|u - K * u_h\|_{L_\infty(\Omega_0)} \le Ch^{r+\ell}. \tag{11.1.2}$$

The function K depends on h in a simple scaling manner, and it also depends on r and ℓ. The convolution $K * u_h(x)$ involves the values of u_h only in an $0(h)$ neighborhood of x. Since the function K is a piecewise polynomial, the convolution is easy to evaluate, involving only integrals of piecewise polynomials if S_h consists of piecewise polynomials. Aside from its dependence on r, the function K is independent of the actual form of S_h, and it is also independent of the underlying partial differential operator. It is, of course, enough to calculate $K * \varphi$ for basis functions φ of S_h.

If superconvergent approximations are desired, say, at nodal points, evaluation of $K * u_h$ can be expressed in terms of fixed values of $K * \varphi$ and the particular coefficients of u_h in that basis. We refer to [Bramble and Schatz 1977, (5.10) and Appendix] for some examples.

In [Thomée 1977] the agument was extended to include $0(h^{r+\ell})$ order approximation to any derivative of u.

For applications of the K–operator, cf. e.g. [Douglas 1984 and 1985].

In Section 11.3 we shall give Thomée's argument; in fact, we shall follow his presentation very closely. It involves use of Fourier transforms and we therefore start by introducing a few concepts and results needed in this connection.

11.2. Preliminaries on Fourier transforms and multipliers.

For functions $u \in L_1(R^n)$ we set

$$\mathcal{F}u(\xi) \equiv \widehat{u}(\xi) = \int e^{-i\langle x,\xi\rangle} u(x)dx, \tag{11.2.1}$$

where $\langle x, \xi\rangle = \sum_{j=1}^n x_j\xi_j$, with the formal inverse

$$\mathcal{F}^{-1}v(x) \equiv \check{v}(x) = (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} v(\xi)d\xi. \tag{11.2.2}$$

Parseval's relation is that

$$\int u\bar{v}dx = (2\pi)^{-n} \int \widehat{u}\bar{\widehat{v}}d\xi, \tag{11.2.3}$$

which also holds if u and v are in $L_2(R^n)$ (Plancherel). Also recall that with $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$,

$$\mathcal{F}(D^\alpha u)(\xi) = (i\xi)^\alpha \widehat{u}(\xi), \tag{11.2.4}$$

that

$$\mathcal{F}(u(\cdot + y))(\xi) = e^{i\langle y,\xi\rangle} \widehat{u}(\xi), \tag{11.2.5}$$

that

$$\mathcal{F}(h^{-n} u(h^{-1} \cdot))(\xi) = \widehat{u}(h\xi), \tag{11.2.6}$$

and that

$$\mathcal{F}(u * v)(\xi) = \mathcal{F}\Big(\int u(\cdot - y) v(y) dy\Big)(\xi) = \widehat{u}(\xi)\widehat{v}(\xi). \tag{11.2.7}$$

We have from Parseval's relation that

$$\begin{aligned} \|u\|_{W_2^s(R^n)} &= \Big(\sum_{|\alpha| \le s} \|D^\alpha u\|^2_{L_2(R^n)}\Big)^{1/2} \\ &\simeq \|(1 + |\xi|^2)^{s/2} \widehat{u}\|_{L_2(R^n)}, \text{ for } s \ge 0, \end{aligned} \tag{11.2.8}$$

where $\simeq$ denotes equivalence of norms, and, by duality, that

$$\|u\|_{W_2^{-s}(R^n)} = \sup_{\substack{\varphi \in C_0^\infty(R^n) \\ \|\varphi\|_{W_2^s(R^n)} = 1}} (u, \varphi) \simeq \|(1 + |\xi|^2)^{-s/2} \widehat{u}\|_{L_2(R^n)}. \tag{11.2.9}$$

We shall consider operators where $\widehat{u}(\xi) \to a(\xi)\widehat{u}(\xi)$ for $a(\xi)$ a smooth function on R^n; we call $a(\xi)$ the symbol of the operator. E.g., $a(\xi) = e^{i\langle y,\xi\rangle}$ corresponds to translation by y for u. We shall say that $a(\xi)$ is a Fourier multiplier on L_p, with $1 \le p \le \infty$, or, $a \in M_p$, if

$$M_p(a) = \sup\{\|\mathcal{F}^{-1}(a\widehat{v})\|_{L_p} : v \in C_0^\infty(R^n),\ \|v\|_{L_p} \le 1\} < \infty. \tag{11.2.10}$$

Clearly, from Parseval's relation,

$$M_2(a) \le \sup_\xi |a(\xi)|; \tag{11.2.11}$$

in fact equality holds. Also, it is easy to see that the product of two functions in M_p is again in M_p, and an elementary change of variables argument establishes that if $a(\xi) \in M_p$, then for $h > 0$, $a(h\xi) \in M_p$ with

$$M_p(a(\cdot)) = M_p(a(h\,\cdot)). \tag{11.2.12}$$

(A fact not unconnected with (11.2.6)!) Further, if $a(\xi_1, \dots, \xi_m) \in M_p(R^m)$ and $b(\xi_{m+1}, \dots, \xi_n) \in M_p(R^{n-m})$, then

$$a(\xi_1, \dots, \xi_m) b(\xi_{m+1}, \dots, \xi_n) \in M_p(R^n). \tag{11.2.13}$$

Here is a simple and convenient criterion which gives that a symbol is in M_p for any $1 \le p \le \infty$. The criterion is taken from [Bramble and Schatz 1976, Lemma 2.3].

Lemma 11.2.1. *Suppose that*

$$\Big(\prod_{j=1}^{n}(1+\partial/\partial\xi_j)\Big)a(\xi) \in L_q(R^n) \tag{11.2.14}$$

for some $1 < q \le 2$. *Then* $a \in M_p$, *for* $1 \le p \le \infty$.

Proof: We shall show that (11.2.14) implies that $\overset{\vee}{a}(x) \in L_1(R^n)$. Then

$$\mathcal{F}^{-1}(a\widehat{u})(x) = (\overset{\vee}{a} * u)(x) \tag{11.2.15}$$

and so (Young's inequality, which is essentially Fubini's theorem),

$$\|\mathcal{F}^{-1}(a\widehat{u})\|_{L_p} \le \|\overset{\vee}{a}\|_{L_1}\|u\|_{L_p}, \tag{11.2.16}$$

i.e., $a \in M_p$, with $M_p(a) = \|\overset{\vee}{a}\|_{L_1}$.
We have, cf. (11.2.4),

$$\overset{\vee}{a}(x) = \Big(\prod_{j=1}^{n}\frac{1}{(1-ix_j)}\Big)\Big(\prod_{j=1}^{n}\Big(1+\frac{\partial}{\partial\xi_j}\Big)a(\xi)\Big)^{\vee}(x) \tag{11.2.17}$$

so that by Hölder's inequality, with q and q' conjugate indices,

$$\int|\overset{\vee}{a}(x)|dx \le \Big(\int\frac{1}{\prod_{j=1}^{n}(1+x_j^2)^{q/2}}dx\Big)^{1/q} \cdot\Big(\int\Big|\Big(\prod_{j=1}^{n}\Big(1+\frac{\partial}{\partial\xi_j}\Big)a\Big)^{\vee}\Big|^{q'}dx\Big)^{1/q'}. \tag{11.2.18}$$

Since $q > 1$ and since by Hausdorff–Young's inequality, $\|\overset{\vee}{f}\|_{q'} \le C\|f\|_q$, for $1 \le q \le 2$, we obtain that $\overset{\vee}{a} \in L_1$. □

Remark 11.2.1. Hausdorff–Young's inequality is elementary for $q = 1$: $\|\overset{\vee}{f}\|_{L_\infty} \le (2\pi)^{-n}\|f\|_{L_1}$. For $q = 2$, it is Parseval's relation. It follows in between by the Riesz–Thorin interpolation theorem, cf. [Bennet and Sharpley 1988, Theorem 4.1.7.]. □

The construction of the K–operator is based on the (smoothest) B–splines. Let χ be the characteristic function of $[-1/2, 1/2]$ and, for $\ell \ge 1$,

$$\psi_{(\ell)} = \chi * \chi * \cdots * \chi, \tag{11.2.19}$$

convolution with ℓ factors. $\psi_{(\ell)}$ is thus the one–dimensional (smoothest) B–spline of "order" ℓ; e.g., $\psi_{(2)}$ is the piecewise linear basis function on $[-1, 1]$. We set, for $h > 0$,

$$\psi_{(\ell),h}(x) = h^{-1}\psi_{(\ell)}(h^{-1}x). \tag{11.2.20}$$

We have by a simple calculation that $\widehat{\chi}(\sigma) = \sin(\sigma/2)/(\sigma/2)$ and so,

$$\widehat{\psi}_{(\ell),h}(\xi) = \widehat{\psi}_{(\ell)}(h\xi) = \Big(\frac{\sin(\frac{1}{2}h\xi)}{\frac{1}{2}h\xi}\Big)^{\ell}, \tag{11.2.21}$$

where we used (11.2.6) and (11.2.7). Since $\|\psi_\ell\|_{L_1} = 1$,

$$M_p(\widehat{\psi}_{(\ell),h}) \le 1, \text{ independent of } p,\ \ell \text{ and } h, \tag{11.2.22}$$

cf. (11.2.16) and (11.2.12). For $\alpha = (\alpha_1, \dots, \alpha_N)$ a multi-index, we set

$$\psi_{(\alpha)}(x) = \prod_{j=1}^{n} \psi_{(\alpha_j)}(x_j) \tag{11.2.23}$$

and correspondingly, $\psi_{(\alpha),h}(x) = h^{-n}\psi_{(\alpha)}(h^{-1}x)$. (For simplicity, we assume that "h" is the same in each direction.) The analogue of (11.2.22) holds, cf. (11.2.13).

We shall have occasion to use the special multi-index

$$\mathbf{m} = (1, 1, \dots, 1) \in \mathbb{Z}^n. \tag{11.2.24}$$

Hence $\psi_{(\ell\mathbf{m})}$ denotes the tensor-product B-spline of the same order ℓ in each variable.

Finally, we shall use the difference operators in the axes directions,

$$\partial_h^\alpha = \partial_{h,1}^{\alpha_1} \cdots \partial_{h,n}^{\alpha_n}, \text{ where } \partial_{h,j} v(x) = \Big(v\Big(x + \frac{1}{2}he_j\Big) - v\Big(x - \frac{1}{2}he_j\Big)\Big)/h. \tag{11.2.25}$$

We have from an elementary calculation, using (11.2.5),

$$\mathcal{F}(\partial_h^\alpha v)(\xi) = (2i)^{|\alpha|} \prod_{j=1}^{n} \Big(\frac{\sin(\frac{1}{2}h\xi_j)}{h}\Big)^{\alpha_j} \widehat{v}(\xi). \tag{11.2.26}$$

From (11.2.4), (11.2.21) and (11.2.26) we see that

$$\begin{aligned} \mathcal{F}(D^\alpha \psi_{(\alpha+\ell\mathbf{m}),h}) &= (i\xi)^\alpha \prod_{j=1}^{n} \Big(\frac{\sin(\frac{1}{2}h\xi_j)}{\frac{1}{2}h\xi_j}\Big)^{\alpha_j+\ell} \\ &= (2i)^\alpha \prod_{j=1}^{n} \Big(\frac{\sin(\frac{1}{2}h\xi_j)}{h}\Big)^{\alpha_j} \prod_{j=1}^{n} \Big(\frac{\sin(\frac{1}{2}h\xi_j)}{\frac{1}{2}h\xi_j}\Big)^{\ell} \\ &= \mathcal{F}(\partial_h^\alpha \xi_{(\ell\mathbf{m}),h}), \end{aligned} \tag{11.2.27}$$

or,

$$D^\alpha \psi_{(\alpha+\ell\mathbf{m}),h} = \partial_h^\alpha \psi_{(\ell\mathbf{m}),h}. \tag{11.2.28}$$

In words, the derivative of a certain spline is the difference quotient of another spline. *This simple result is fundamental to our development.*

For more details about Fourier multipliers we refer to [Hörmander 1960] or [Brenner, Thomée and Wahlbin 1975, Chapter 1].

11.3. The K–operator in general.

In the present section we have a smooth function u and another function v that is continuous. For the purposes of this section it will play no role that v is in fact equal to u_h, a finite element approximation to u, and it will consequently also be immaterial that the finite element space is translation invariant. Our main result is the following, where $K_h^{(\alpha,\ell,p)}$ is a "simple" local function built up from tensor–product smoothest B–splines. (It is defined in (11.3.15) below.)

Theorem 11.3.1. *Let α be a given multi-index, let $\ell \geq 1$ and $p \geq 1$. Let $u \in C^{2p+|\alpha|}(\Omega_1)$, $v \in C(\Omega_2)$, where $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2$ with unit size separations. Further, set $N_0 = [n/2]+1$. Then for h sufficiently small,*

$$\begin{aligned}(11.3.1)\qquad &\|D^\alpha u - K_h^{(\alpha,\ell,p)} * (\partial_h^\alpha v)\|_{L_\infty(\Omega_0)} \\ &\leq C\Big(h^{2p}\|u\|_{W_\infty^{2p+|\alpha|}(\Omega_1)} + \sum_{|\beta|\leq \ell+N_0} \|\partial_h^{\alpha+\beta}(u-v)\|_{W_2^{-\ell}(\Omega_1)} \\ &\qquad + h^\ell \sum_{|\beta|\leq\ell} \|\partial_h^{\alpha+\beta}(u-v)\|_{L_\infty(\Omega_1)}\Big).\end{aligned}$$

Note that ∂_h^α is not a high–order approximation to D^α, but that in turn the K–operator depends on α.

A basic step in the construction of the K–operator is the following in the one–dimensional case. The construction is given in the proof.

Lemma 11.3.1. *Let $q \geq 1$, $p \geq 1$. There exists a unique trigonometric polynomial $k_{q,p}(\sigma)$ of order $p-1$ such that*

$$(11.3.2)\qquad k_{q,p}(\sigma)(\widehat{\chi}(\sigma))^q = 1 + 0(|\sigma|^{2p}),\ \ \textit{as } |\sigma| \to 0.$$

Proof: We have, cf. (11.2.21), that $\widehat{\chi}(\sigma)^q = \left(\frac{\sin(\frac{1}{2}\sigma)}{\frac{1}{2}\sigma}\right)^q$, or, setting $\tau = \sin(\frac{1}{2}\sigma)$, for small τ or σ, Maclaurin expanding we have

$$(11.3.3)\qquad \widehat{\chi}(\sigma)^{-q} = \Big(\frac{\arcsin\tau}{\tau}\Big)^q = \sum_{j=0}^{\infty} \gamma_{q,j}\tau^{2j}.$$

We may therefore choose

$$(11.3.4)\qquad k_{q,p}(\sigma) = \sum_{j=0}^{p-1} \gamma_{q,j}\Big(\sin\Big(\frac{1}{2}\sigma\Big)\Big)^{2j} = \sum_{j=0}^{p-1} \gamma_{q,j}2^{-j}(1-\cos\sigma)^j.$$

Uniqueness of $k_{q,p}$ is immediate. For, if $k_{q,p}^1$ and $k_{q,p}^2$ both satisfy (11.3.2), then $k_{q,p}^1 - k_{q,p}^2$ is a trigonometric polynomial of degree $\leq p-1$ which vanishes to order $2p$ at the origin. It is hence identically zero. □

For $\beta = (\beta_1, \dots, \beta_n)$ with all $\beta_j > 0$ and $p \geq 1$, we now define the n–dimensional trigonometric polynomial

$$(11.3.5)\qquad k^{(\beta,p)}(\xi) = \prod_{j=1}^{n} k_{\beta_j,p}(\xi_j) = \sum_\gamma k_\gamma^{(\beta,p)} e^{-i\langle\xi,\gamma\rangle}.$$

Note that the sum on the right is finite.

Using the coefficient of this polynomial and the B–splines introduced in Section 11.2, we set

$$\mathcal{K}^{(\beta,p)}(x) = \sum_{\gamma} k_{\gamma}^{(\beta,p)} \psi_{(\beta)}(x-\gamma). \tag{11.3.6}$$

Then, from (11.2.5) and the above,

$$\begin{aligned}\widehat{\mathcal{K}}^{(\beta,p)}(\xi) &= k^{(\beta,p)}(\xi)\widehat{\psi}_{(\beta)}(\xi) = \prod_{j=1}^{n} k_{\beta_j,p}(\xi_j)\widehat{\chi}(\xi_j)^{\beta_j} \\ &= 1 + 0(|\xi|^{2p}), \quad \text{as} \quad |\xi| \to 0.\end{aligned} \tag{11.3.7}$$

We also set

$$\mathcal{K}_h^{(\beta,p)}(x) = h^{-n}\mathcal{K}^{(\beta,p)}(h^{-1}x). \tag{11.3.8}$$

The relation (11.3.7) says that, in some sense, convolution with $\mathcal{K}_h^{(\beta,p)}$ is an approximate identity of order h^{2p}. More precisely we have the following.

Lemma 11.3.2. *Let $\beta = (\beta_1, \dots, \beta_n)$ with all $\beta_j > 0$, and let $p \geq 1$. Then for Ω_1, with $\Omega_2 = \Omega_1 + Mh$, $M = M(\beta, p)$,*

$$\|v - \mathcal{K}_h^{(\beta,p)} * v\|_{L_\infty(\Omega_1)} \leq Ch^{2p}\|v\|_{W_\infty^{2p}(\Omega_2)}. \tag{11.3.9}$$

Proof: Set

$$\widetilde{k}_j(\sigma) = k_{\beta_j,p}(\sigma)\widehat{\chi}(\sigma)^{\beta_j}. \tag{11.3.10}$$

From (11.3.2) and since $k_{\beta_j,p}(\sigma)$ and $\widehat{\chi}(\sigma)$ are bounded together with their first derivatives, we find using Lemma 11.2.1 that the analytic function $(1-\widetilde{k}_j(\xi_j))\xi_j^{-2p} \in M_\infty(R^1)$. Thus, by (11.2.4), using also (11.2.12) in one dimension, and (11.2.13),

$$\begin{aligned}&\|\mathcal{F}^{-1}(1-\widetilde{k}_j(h\xi_j))\widehat{v}\|_{L_\infty(R^n)} \\ &\quad = h^{2p}\|\mathcal{F}^{-1}((1-\widetilde{k}_j(h\xi_j))(h\xi_j)^{-2p}(\widehat{D_j^{2p}v}))\|_{L_\infty(R^n)} \\ &\quad \leq Ch^{2p}\|v\|_{W_\infty^{2p}(R^n)}.\end{aligned} \tag{11.3.11}$$

Now write by elementary algebra

$$1 - \widehat{\mathcal{K}}_h^{(\beta,p)}(h\xi) = 1 - \prod_{j=1}^{n} \widetilde{k}_j(h\xi_j) = \sum_{j=1}^{n} \prod_{\ell<j} \widetilde{k}_\ell(h\xi_\ell)(1-\widetilde{k}_j(h\xi_j)). \tag{11.3.12}$$

It is easily seen (e.g., from (11.2.5), any trigonometric polynomial is in M_∞, and $\widehat{\chi}$ is in M_∞ by (11.2.22)) that $\widetilde{k}_j \in M_\infty$, and hence we have via (11.3.11) that

$$\|v - \mathcal{K}_h^{(\beta,p)} * v\|_{L_\infty(R^n)} \leq Ch^{2p}\|v\|_{W_\infty^{2p}(R^n)}. \tag{11.3.13}$$

If, as in (11.3.9), we are only interested in the norm over Ω_1, since $\mathcal{K}_h^{(\beta,p)} * v$ only involves the values of v over Ω_2, we may extend v from Ω_2 in any manner whatsoever to R^n and obtain (11.3.9) from (11.3.13). □

We are now ready to define $K_h^{(\alpha,\ell,p)}$. With $\mathbf{m} = (1, 1, \ldots, 1)$ and $\ell \geq 1$, recall from (11.2.28) that $D^\alpha \psi_{(\alpha+\ell\mathbf{m}),h} = \partial_h^\alpha \psi_{(\ell\mathbf{m}),h}$. We hence have, cf. (11.3.6),

$$
\begin{aligned}
D^\alpha K_h^{(\alpha+\ell\mathbf{m},p)} &= D^\alpha \Big(\sum_\gamma k_\gamma^{(\alpha+\ell\mathbf{m},p)} \psi_{(\alpha+\ell\mathbf{m}),h}(x - h\gamma) \Big) \\
&= \sum_\gamma k_\gamma^{(\alpha+\ell\mathbf{m},p)} \partial_h^\alpha \psi_{(\ell\mathbf{m}),h}(x - h\gamma) \\
&\equiv \partial_h^\alpha K_h^{(\alpha,\ell,p)}
\end{aligned}
\tag{11.3.14}
$$

where thus

$$
K_h^{(\alpha,\ell,p)}(x) = \sum_\gamma k_\gamma^{(\alpha+\ell\mathbf{m})} \psi_{(\ell\mathbf{m}),h}(x - \gamma h).
\tag{11.3.15}
$$

Having finally defined the operator occurring in Theorem 11.3.1, we shall now prove that theorem.

Proof of Theorem 11.3.1. We have

$$
\begin{aligned}
&D^\alpha u - K_h^{(\alpha,\ell,p)} * (\partial_h^\alpha v) \\
&\quad = [D^\alpha u - K_h^{(\alpha,\ell,p)} * (\partial_h^\alpha u)] + K_h^{(\alpha,\ell,p)} * \partial_h^\alpha (u - v) \\
&\quad \equiv I_1 + I_2.
\end{aligned}
\tag{11.3.16}
$$

Here, from (11.3.14), and by the nature of a convolution,

$$
\begin{aligned}
I_1 &= D^\alpha u - \partial_h^\alpha (K_h^{(\alpha,\ell,p)} * u) \\
&= D^\alpha u - (\partial_h^\alpha K_h^{(\alpha,\ell,p)}) * u \\
&= D^\alpha u - (D^\alpha K_h^{(\alpha+\ell\mathbf{m},p)}) * u \\
&= D^\alpha u - K_h^{(\alpha+\ell\mathbf{m},p)} * D^\alpha u.
\end{aligned}
\tag{11.3.17}
$$

Thus from Lemma 11.3.2,

$$
\|I_1\|_{L_\infty(\Omega_0)} \leq C h^{2p} \|u\|_{W_\infty^{|\alpha|+2p}(\Omega_1)}.
\tag{11.3.18}
$$

We see that the correct estimate for I_2, and hence the theorem, would come from the following, cf. (11.3.15):

For $\ell \geq 1$, w continuous,

$$
\begin{aligned}
&\|\psi_{(\ell\mathbf{m}),h} * w\|_{L_\infty(\Omega_0)} \\
&\quad \leq C \Big(\sum_{|\beta| \leq \ell + N_0} \|\partial_h^\beta w\|_{W_2^{-\ell}(\Omega_1)} + h^\ell \sum_{|\beta| \leq \ell} \|\partial_h^\beta w\|_{L_\infty(\Omega_1)} \Big).
\end{aligned}
\tag{11.3.19}
$$

To show (11.3.19), let $\varphi \in C_0^\infty(-1, 1)$ with $\varphi \equiv 1$ near the origin, and set $\Phi(\xi) = \prod_{i=1}^{n} \varphi(\xi_j)$. We then have (for w with compact support — we may change the function at will outside of Ω_2),

$$\begin{aligned}(11.3.20)\qquad \psi_{(\ell\mathbf{m}),h} * w &= \mathcal{F}^{-1}(\widehat{\psi}_{(\ell\mathbf{m})}(h\xi)\widehat{w}(\xi)) \\ &= \mathcal{F}^{-1}(\widehat{\psi}_{(\ell\mathbf{m})}(h\xi)\Phi(h\xi)\widehat{w}(\xi)) \\ &\quad + \mathcal{F}^{-1}(\widehat{\psi}_{(\ell\mathbf{m})}(h\xi)(1-\Phi(h\xi))\widehat{w}(\xi)) \\ &\equiv \mathcal{J}_1 + \mathcal{J}_2.\end{aligned}$$

For $\mathcal{J}_1 = \mathcal{J}_1(x)$ we have from Cauchy–Schwarz's inequality,

$$\begin{aligned}(11.3.21)\qquad |\mathcal{J}_1| &= (2\pi)^{-n}\Big|\int (1+|\xi|^2)^{-N_0/2} e^{i\langle x,\xi\rangle}(1+|\xi|^2)^{N_0/2} \\ &\qquad \cdot \widehat{\psi}_{(\ell\mathbf{m})}(h\xi)\Phi(h\xi)\widehat{w}(\xi)d\xi\Big| \\ &\le C\|(1+|\xi|^2)^{N_0/2}\widehat{\psi}_{(\ell\mathbf{m})}(h\xi)\Phi(h\xi)\widehat{w}(\xi)\|_{L_2(R^n)}.\end{aligned}$$

For $h\xi$ in the support of $\Phi(h\xi)$,

$$(11.3.22)\qquad 1+|\xi|^2 \le C\Big(1+\sum_{j=1}^{n}\Big(\sin\Big(\frac{1}{2}h\xi_j\Big)/h\Big)^2\Big).$$

Next note that the function $\frac{a}{1+a} = 1 - \frac{1}{1+a}$ is increasing. Since $\left(\frac{\sin(hy)}{h}\right)^2 \le y^2$ it follows that

$$(11.3.23)\qquad \Big(\frac{\sin(hy)}{hy}\Big)^2 = \Big(\frac{\sin(hy)}{h}\Big)^2\frac{1}{y^2} \le \frac{1+\left(\frac{\sin(hy)}{h}\right)^2}{1+y^2}.$$

Thus, since $\frac{\sin(\sigma)}{\sigma}$ is decreasing for $0 \le \sigma \le 1$, and bounded by 1, we have with ξ_0 denoting the largest component of ξ (for $|h\xi_0| < 1$),

$$\begin{aligned}(11.3.24)\qquad \widehat{\psi}_{(\ell\mathbf{m})}(h\xi)^2 &\le \Big(\frac{\sin(\frac{1}{2}h\xi_0)}{\frac{1}{2}h\xi_0}\Big)^{2\ell} \le \Big(\frac{1+(\sin(\frac{1}{2}h\xi_0)/h)^2}{1+\frac{1}{2}\xi_0^2}\Big)^{\ell} \\ &\le C\frac{\left(1+\sum_{j=1}^{n}(\sin(\frac{1}{2}h\xi_j)/h)^{2\ell}\right)}{(1+|\xi|^2)^{\ell}}.\end{aligned}$$

It follows from (11.3.22) and (11.3.24) that

$$\begin{aligned}(11.3.25)\qquad &|(1+|\xi|^2)^{N_0/2}\widehat{\psi}_{(\ell\mathbf{m})}(h\xi)\Phi(h\xi)| \\ &\le C\Big(1+\sum_{j=1}^{n}\Big(\sin\Big(\frac{1}{2}h\xi_j\Big)/h\Big)^{2(N_0+\ell)}\Big)^{1/2}(1+|\xi|^2)^{-\ell/2}.\end{aligned}$$

By (11.2.9), $\|w\|_{W_2^{-\ell}(R^n)} \simeq \|(1+|\xi|^2)^{-\ell/2}\widehat{w}\|_{L_2(R^n)}$ and from (11.2.26), we find via

Parseval's relation that

$$(11.3.26)\qquad |\mathcal{J}_1| \le C\Big(\|w\|_{W_2^{-\ell}(R^n)} + \sum_{j=1}^{n} \|\partial_{h,j}^{N_0+\ell} w\|_{W_2^{-\ell}(R^n)}\Big)$$
$$\le C \sum_{|\beta|\le \ell+N_0} \|\partial_h^\beta w\|_{W_2^{-\ell}(R^n)}$$
$$\le C \sum_{|\beta|\le \ell+N_0} \|\partial_h^\beta w\|_{W_2^{-\ell}(\Omega_1)}.$$

(In the last step we again used the freedom to change w at will outside our basic domain.)

For $\mathcal{J}_2$ (defined in (11.3.20)), using elementary algebra we write it as (cf. 11.3.12),

$$(11.3.27)\qquad \mathcal{J}_2 = \mathcal{F}^{-1}\Big(\Big(\prod_j \widehat{\chi}(h\xi_j)^\ell\Big)\Big(\sum_j \prod_{i<j} \varphi(h\xi_i)(1-\varphi(h\xi_j))\widehat{w}\Big)\Big).$$

Now $\widehat{\chi}$ and φ are in M_∞, and likewise is $(1-\varphi(\xi_j))/\xi_j^\ell$, by Lemma 11.2.1 and (11.2.13). Thus, cf. (11.2.26),

$$(11.3.28)\qquad |\mathcal{J}_2| \le C\sum_{j=1}^{n} \|\mathcal{F}^{-1}((1-\varphi(h\xi_j))\widehat{\chi}(h\xi_j)^\ell \widehat{w})\|_{L_\infty(R^n)}$$
$$= C\sum_{j=1}^{n} \Big\|\mathcal{F}^{-1}\Big(\Big(\frac{1-\varphi(h\xi_j)}{(\frac{1}{2}h\xi_j)^\ell}\Big)\Big(\sin\Big(\frac{1}{2}h\xi_j\Big)\Big)^\ell \widehat{w}\Big)\Big\|_{L_\infty(R^n)}$$
$$= Ch^\ell \sum_{j=1}^{n} \Big\|\mathcal{F}^{-1}\Big(\Big(\frac{1-\varphi(h\xi_j)}{(h\xi_j)^\ell}\Big)(\widehat{\partial_{h,j}^\ell w})\Big)\Big\|_{L_\infty(R^n)}$$
$$\le Ch^\ell \sum_{j=1}^{n} \|\partial_{h,j}^\ell w\|_{L_\infty(R^n)} \le Ch^\ell \sum_{j=1}^{n} \|\partial_{h,j}^\ell w\|_{L_\infty(\Omega_1)}.$$

Together, (11.3.26) and (11.3.28) show (11.3.19). As we have already noticed, this proves Theorem 11.3.1. □

11.4. The K–operator applied to finite element approximations in second order elliptic problems.

We apply Theorem 11.3.1 with $v = u_h$, where $u_h \in S_h$ is given with $A(u-u_h,\chi) = 0$ for $\chi \in S_h^{comp}(\Omega_2)$, $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2$. Here S_h is translation invariant on Ω_2 in all the axes–directions x_j, $j = 1,\dots,n$. (For simplicity we assume that the translation parameter is the same, h, in all directions.) From our investigations in Chapter 8 we have that

$$(11.4.1)\qquad \|\partial_h^{\alpha+\beta}(u-u_h)\|_{L_\infty(\Omega_1)} \le Ch^r$$

under various assumptions. (The logarithmic factor occurring in Chapter 8 may be avoided, cf. Remark 8.4.1.) Choosing $2p \ge r+\ell$, in order to obtain superconvergence of order $h^{r+\ell}$, it thus remains to estimate $\|\partial_h^{\alpha+\beta}(u-u_h)\|_{W_2^{-\ell}(\Omega_1)}$.

In the case of a basic operator with constant coefficients, we have

$$(11.4.2)\qquad A(\partial_h^{\alpha+\beta}(u-u_h),\chi) = A(u-u_h,(\partial_h^{\alpha+\beta})\chi) = 0,\ \text{for } \chi \in S_h^{comp}(\Omega_2).$$

We now need a local error estimate in a negative norm. We have not treated such estimates in these lectures, and we shall not give much detail here. Referring to [Nitsche and Schatz 1974, Lemma 4.2] we have

$$\begin{aligned}(11.4.3)\qquad &\|\partial_h^{\alpha+\beta}(u-u_h)\|_{W_2^{-\ell}(\Omega_1)}\\ &\le Ch^{\gamma}\|\partial_h^{\alpha+\beta}(u-u_h)\|_{W_2^1(\Omega_2)}+C\|\partial_h^{\alpha+\beta}(u-u_h)\|_{W_2^{-\ell-L}(\Omega_2)}\end{aligned}$$

for any L, with $\gamma=\min(\ell+1,r-1)$. The H^1-estimate for $\partial_n^{\alpha+\beta}(u-u_h)$ is localized essentially as in Chapter 8 (again, details can be found in [Nitsche and Schatz 1974]), and the term involving $W_2^{-\ell-L}$ on the right may easily be converted to $\|u-u_h\|_{W_2^{-\ell}}$, by taking $L\ge|\alpha|+|\beta|$. The upshot is that, if u is smooth enough,

$$\begin{aligned}(11.4.4)\qquad &\|\partial_h^{\alpha+\beta}(u-u_h)\|_{W_2^{-\ell}(\Omega_1)}\\ &\le Ch^{\gamma}\min_{\chi\in S_h}\|\partial_h^{\alpha+\beta}u-\chi\|_{W_2^1(\Omega_2)}+C\|u-u_h\|_{W_2^{-\ell}(\Omega_2)}\\ &\le Ch^{r+\ell}\|u\|_{W_2^{r+|\alpha|+|\beta|}(\Omega_2)}+C\|u-u_h\|_{W_2^{-\ell}(\Omega_2)},\end{aligned}$$

for $\ell\le r-2$.

Thus, provided $1\le\ell\le r-2$ *and* $\|u-u_h\|_{W_2^{-\ell}(\Omega_2)}\le Ch^{r+\ell}$, *we have*

$$(11.4.5)\qquad \|D^{\alpha}u-K_h^{(\alpha,\ell,p)}*(\partial_h^{\alpha}u_h)\|_{L_\infty(\Omega_0)}\le Ch^{r+\ell}.$$

The same estimate obtains in the case of variable coefficients, again using the ideas of [Nitsche and Schatz 1974] to prove (11.4.4). (There will be a lot of garbage terms when commuting the difference operator on to χ in (11.4.2). These are handled by induction.) The reader will find details in [Bramble and Schatz 1977, Lemma 4.1].

11.5. Boundary integral equations and the K–operator: an example.

A particularly dramatic use of the K-operator is given in [Tran 1993] in the context of boundary integral equations. We shall need to give a certain amount of background material in order to appreciate it. For deeper background, cf. e.g. [Sloan 1992].

We consider the Dirichlet problem on a disc in the plane,

$$(11.5.1)\qquad \Delta u=0 \quad\text{in}\quad B_R=\{x:|x|<R\}\subset\subset R^2,$$

$$(11.5.2)\qquad u=g\quad\text{on}\quad \Gamma_R=\partial B_R.$$

The so-called "direct" method of formulating this problem as an integral equation over Γ_R goes as follows.

Let $\widetilde{p}(y)=\frac{\partial u}{\partial n}(y)$, the outward normal derivative for $y\in\Gamma_R$. By Green's theorem, for x in $\overset{\circ}{B}_R$, the interior of B_R (and log being the natural, e-based, logarithmic function, as is traditional in the present setting),

$$\begin{aligned}(11.5.3)\qquad u(x)&=\frac{1}{2\pi}\int_{\Gamma_R}\Big[\frac{\partial}{\partial n_y}(\log|x-y|)u(y)-\log|x-y|\widetilde{p}(y)\Big]d\ell_y\\ &=\frac{1}{2\pi}\int_{\Gamma_R}\Big[\frac{\partial}{\partial n_y}(\log|x-y|)g(y)-\log|x-y|\widetilde{p}(y)\Big]d\ell_y,\, x\in\overset{\circ}{B}_R.\end{aligned}$$

Letting x tend to Γ_R we have, in a standard way, the following integral equation over Γ_R,

(11.5.4)
$$-\frac{1}{\pi}\int_{\Gamma_R} \log|x-y|\widetilde{p}(y)d\ell_y = g(x) - \frac{1}{\pi}\int_{\Gamma_R} \frac{\partial}{\partial n_y}(\log|x-y|)g(y)d\ell_y, \text{ for } x \in \Gamma_R.$$

(The method is called "direct" because we haven't used reformulation in terms of double or single layer potentials, or, some such thing; in fact our unknown, the outflow derivative $\widetilde{p}(y)$, is frequently of major interest in applications. Note that we have to know a fundamental solution to (11.5.1) in order to derive the boundary integral equation (11.5.4). This is the principal limitation of the method.)

It can be shown that (11.5.4) is equivalent with (11.5.1)–(11.5.2), provided $R \neq 1$. (If $R = 1$, since then $\int_{\Gamma_1} \log|x-y|d\ell_y = 0$, we are in trouble, at least we loose uniqueness in (11.5.4).)

Now parametrize the circle by

(11.5.5)
$$x(t) = R(\cos 2\pi t, \sin 2\pi t),\ 0 \le t \le 1.$$

By use of elementary trigonometry, we then have

(11.5.6)
$$\begin{aligned} -\frac{1}{\pi}\int_{\Gamma_R} & \log(|x-y|)\widetilde{p}(y)d\ell_y \\ &= \frac{-1}{2\pi}\int_0^1 \log(R^2(\cos 2\pi t - \cos 2\pi s)^2 \\ &\qquad + (\sin 2\pi t - \sin 2\pi t)^2)\widetilde{p}(y(s))|y'(s)|ds \\ &= -2\int_0^1 \log(|2R\sin(\pi(t-s))|)\widetilde{p}(y(s))\frac{R}{2\pi}ds. \end{aligned}$$

We also note that

(11.5.7)
$$\frac{\partial}{\partial n_y}\log|x-y| = \frac{n_y\cdot(y-x)}{|x-y|^2} = \frac{n_{y(s)}\cdot(y(s)-x(t))}{|x(t)-y(s)|^2}$$

is a smooth function of t and s if the boundary is smooth. Indeed, it equals the constant $(2R)^{-1}$ for the circle. Letting then $G(t)$ denote the right hand side of (11.5.4), and setting $p(s) = \frac{R}{2\pi}\widetilde{p}(y(s))$, we thus obtain the equation

(11.5.8)
$$Ap(t) \equiv -2\int_0^1 \log|2R\sin(\pi(t-s))|p(s)ds = G(t),\ 0 \le t \le 1,$$

where $G(t)$ is a smooth periodic function (provided g is smooth).

From [Gradshteyn and Ryzhik 1979, 1.441.2, p. 38] we have

(11.5.9)
$$-\frac{1}{2}\log(1-\cos(2\pi t)) = \sum_{k=1}^{\infty}\frac{\cos(k2\pi t)}{k}$$

and so,

(11.5.10)
$$-2\log(2R|\sin\pi t|) = -2\log R + \sum_{k\neq 0}\frac{1}{|k|}e^{2\pi i k t}.$$

From this it follows that the effect of the operator A on $p(s) = \sum c_j e^{2\pi ijs}$ is, essentially, to multiply the j^{th} Fourier coefficient c_j, $j \neq 0$, by $\frac{1}{|j|}$. (We also smell a rat if $R = 1$!) Introducing the natural periodic Sobolev spaces H^m by the norms

$$\|p\|_{H^m} = \Big(c_0^2 + \sum_{j\neq 0} |j|^{2m}|c_j|^2\Big)^{1/2}, \quad \text{for } -\infty < m < \infty, \tag{11.5.11}$$

we thus have

$$\|Ap\|_{H^{m+1}} \leq C\|p\|_{H^m}, \tag{11.5.12}$$

i.e., the operator is "smoothing".

Let us now further, for simplicity, assume that

$$R < 1. \tag{11.5.13}$$

Then clearly the operator A is coercive over $H^{-1/2}$ in the sense that with a positive constant c, and the L_2 inner product

$$(p, q) = \int_0^1 p(s)\overline{q(s)}ds, \tag{11.5.14}$$

we have

$$(Ap, p) \geq c\|p\|^2_{H^{-1/2}}. \tag{11.5.15}$$

It follows from Riesz' representation theorem that the equation $Ap = G$ has a unique solution p in $H^{-1/2}$, for every G in $H^{1/2}$. Furthermore, it is easy to derive by Fourier analysis the regularity result that

$$\|p\|_{H^m} \simeq \|G\|_{H^{m+1}}. \tag{11.5.16}$$

(The particular case $R = e^{-1/2}$ leads to an isometry in (11.5.16), cf. (11.5.10).)

Now for finite element solutions to $Ap = G$! Let T_h be uniform subdivisions of length h of $[0, 1]$, $T_h = \{I_j\}_{j=1}^N$, and set

$$S_h = \{\chi : \chi\big|_{I_j} \text{ is constant}\}, \tag{11.5.17}$$

i.e., S_h consists of discontinuous piecewise constants. The standard Galerkin finite element method is then to find $p_h \in S_h$ by

$$(Ap_h, \chi) = (G, \chi), \quad \text{for } \chi \in S_h. \tag{11.5.18}$$

(Here the inner product is the L_2–inner product (11.5.14).) We assume that we can evaluate A exactly for $q \in S_h$, and that we can likewise evaluate exactly the inner product.

By the above, $(Ap, p) \simeq \|p\|^2_{H^{-1/2}}$, and so by standard techniques, and approximation theory,

$$\|p - p_h\|_{H^{-1/2}} \leq C \min_{\chi\in S_h} \|p - \chi\|_{H^{-1/2}} \leq Ch^{3/2}, \tag{11.5.19}$$

if $p \in H^1$ (which it does if $G \in H^2$). An obvious duality argument then gives

$$\|p - p_h\|_{H^{-2}} \leq Ch^3. \tag{11.5.20}$$

Let us pause here to remark that negative norm error estimates are of interest for boundary integral equations for the following reason (among others): If we wish to evaluate $u(x)$ at a point x inside B_R, we would naturally do it via (11.5.3), using $\widetilde{p}_h$. Since $\log|x-y|$ is smooth for $x \in \overset{\circ}{B}_R$, $y \in \Gamma_R$, it is easy to see that the error in $|u(x) - u_h(x)|$ would then be bounded by the error in any negative norm for $p - p_h$. Of course, constants deteriorate as x comes close to the boundary.

The so–called qualocation method (for "quadrature–modified collocation method"[1)]) was introduced to obtain even higher order negative norm estimates than in (11.5.20). (We continue to treat only the piecewise constant case.) Our presentation is based on [Sloan 1992a, Chapter 7] and [Sloan 1992b]; references to original work by Sloan and others such as [Sloan and Wendland 1989] and [Chandler and Sloan 1990] can be found in these expositions.

In the variant of the qualocation method that we shall give, it is still assumed that $(Aq)(t)$ can be exactly evaluated for $q \in S_h$. Also, it will now be essential that the mesh is uniform (above, we could have considered quasi–uniform meshes, e.g.).

We now find $p_h^Q \in S_h$ via

$$(Ap_h^Q, \chi)_h = (G, \chi)_h, \text{ for } \chi \in S_h, \tag{11.5.21}$$

where $(p,q)_h$ is based on a composite quadrature rule; for $I_j = (jh, (j+1)h)$,

$$\int_{I_j} f ds \cong \omega f(jh) + (1-\omega) f\Big(\Big(j + \frac{1}{2}\Big)h\Big). \tag{11.5.22}$$

(This form of the quadrature rule anticipates what is to come.) Since the functions in our space S_h are discontinuous at the nodes jh, we shall have to assign them a definite value there. We take that to be the mean from the left and right,

$$\chi(jh) = (\chi(jh-\varepsilon) + \chi(jh+\varepsilon))/2, \text{ for } \varepsilon < h, \ \chi \in S_h. \tag{11.5.23}$$

(We may note that $A\chi \in H^1$, for $\chi \in S_h$, and hence $A\chi$ is continuous.)

It turns out to be possible to introduce a particular basis ψ_μ for S_h which diagonalizes the matrix $(A\psi_\mu, \psi_\nu)_h$ in (11.5.21). Thereafter, it turns out to be possible to derive a very precise expression for the error, in terms of the Fourier coefficients for p_h and p. Considering the leading term in that error expansion and choosing

$$\omega = \frac{3}{7}, \ (1-\omega) = \frac{4}{7}, \tag{11.5.24}$$

in the integration rule (11.5.22) to knock out that leading term, one then finds that

$$\|p - p_h^Q\|_{H^{-4}} \leq Ch^5, \tag{11.5.25}$$

for $p \in H^4$. The details are lengthy and will not be further commented on here.

The operator A is of convolution type, hence, translation invariant. So is our piecewise constant finite element space S_h, which is periodic so that we need not worry about boundaries. The analysis in [Tran 1993] now combines this with the K–operator, much in the spirit of Section 11.4. It also extends the analysis to the case when Γ is a general smooth curve in the plane and A a more general operator,

[1)] The method was introduced and, by and large, developed in Australia. It is sometimes referred to as the koalacation method.

not necessarily coming from the problem (11.5.1–2). (Both these extensions involve compact perturbations of our present type operators, not unlike going to variable coefficients in the second order partial differential equation case, cf. the remark in 11.6.iii below.)

Tran illustrates his analysis with a numerical example which I think is rather stunning, so, I'll reproduce it here.

The domain is the ellipse $\frac{x_1^2}{4} + \frac{x_2^2}{9} \le 1$ and the boundary function g is $g(x_1, x_2) = \sin(x_1 - 0.1)\cosh(x_2 - 0.2)$. The exact solution p is then known. He uses the qualocation method as above, and the particular K–operator is based on cubic (smoothest) splines, as Section 11.3 suggests ($\ell = 4$). The function K is supported in $[-4h, 4h]$. The following is a sketch of it:

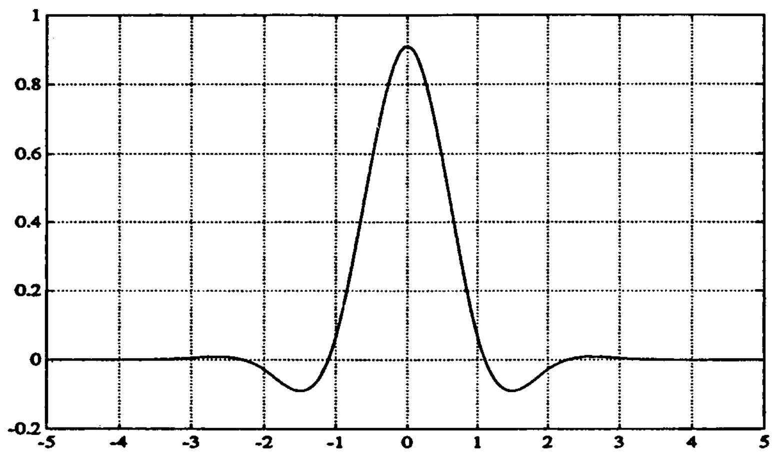


Figure 11.5.1

For the qualocation method used, superconvergence of order $O(h^2)$ at midpoints is known. The following gives his numerical results. The numbers on the right in the second to fourth columns are the observed rates of convergence.

$N = 1/h$	$\|p - p_h^Q\|_{L_\infty(\Gamma)}$		$\max_i \|p - p_h^Q\| (t_{i+1/2})$		$\|p - K_h * p_h^Q\|_{L_\infty(\Gamma)}$	
16	8.17		0.59		0.92	
32	4.22	0.95	0.24	1.28	0.426–1	4.43
64	2.08	1.02	0.61–1	2.00	0.957–3	5.48
128	1.05	0.99	0.154–1	1.99	0.197–4	5.60
256	0.52	1.00	0.385–2	2.00	0.435–6	5.50
512	0.26	1.00	0.962–3	2.00	0.107–7	5.34

Considering the pointwise global error, by trivial post-processing Tran, with a little help from friends, has managed to create a fifth order accurate approximation out of a first order accurate one. Not bad, or what?!

11.6. Remarks, including some other averaging methods.

i) The theory as given in Theorem 11.3.1 applies to the L_2-projection in locally translation invariant cases. Now, most favorably, we may have

$$\|u - u_h\|_{W_2^{-r}(\mathcal{D})} \le Ch^{2r} \tag{11.6.1}$$

and thus pick up the same rate of convergence in the pointwise norm for approximations to any derivative of u.

ii) Likewise, Theorem 11.3.1 applies to time-continuous Galerkin approximations to parabolic problems, cf. e.g. [Bramble, Schatz, Thomée and Wahlbin 1977, Sections 6 and 7], [Thomée 1980a] and [Thomée 1984, Chapter 6].

iii) To put the result of [Tran 1993] in perspective, we offer the following references to previous work on integral equations. In [Richter 1978], "natural" (no averaging of any kind) superconvergence was considered for Fredholm integral equations of the second kind. [Sloan, Burn and Datyner 1975] introduced the idea of "iteration", cf. [Nyström 1930], which we have seen in its two-point boundary value problem variant in Section 1.12.b. [Chandler 1980] gave an analysis for the K-operator as applied to the second kind Fredholm integral equation

$$(I - T)\varphi = f, \tag{11.6.2}$$

where T is a compact operator. Thus, for the Galerkin method, he treated the case of "the L_2-projection with compact perturbations". In analogy, Tran treats the H^s projection, with compact perturbations, the case of $s = -1/2$ being the one we have given some details about above.

iv) A minor variation on the K-operator in a special case in the plane occurs in [Oganesyan and Rukhovetz 1979, pp. 94 and 189]. It involves forming the average

$$\widetilde{u}_h(x) = \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h} u_h(x+y)\,dy_1 dy_2. \tag{11.6.3}$$

For piecewise linears on a triangulation which is uniform in a neighborhood Ω_0 of Ω_1, it is then not hard to show that

$$\|\nabla(u - \widetilde{u}_h)\|_{L_\infty(\Omega_1)} \le Ch^2 \ln 1/h, \tag{11.6.4}$$

provided $u \in W_\infty^3(\Omega_0)$ and provided outside influences permit. Note that, from Chapter 8, we know that already applying simple centered difference quotients gives the same result.

v) The iteration technique in 1.12.a for finding the outflow derivative in a two-point boundary value problem generalizes, in a sense, to higher dimensions. Let

$$Lv = -div(\alpha(x)\nabla v) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a_i(x)v) + a(x)v, \tag{11.6.5}$$

where $\alpha(x)$ is a scalar function, and let u be the solution to the Dirichlet problem

$$\begin{aligned} Lu &= f \text{ in } \mathcal{D}, \\ u &= 0 \text{ on } \Gamma = \partial\mathcal{D}. \end{aligned} \tag{11.6.6}$$

For v smooth enough we then have by the divergence theorem,

$$\left\langle \alpha(x)\frac{\partial u}{\partial n}, v \right\rangle = A(u,v) - (f,v), \tag{11.6.7}$$

where $\langle v, w\rangle = \int_\Gamma vw d\sigma$, $A(v,w)$ is the natural bilinear form associated with L, and $(v,w) = \int_{\mathcal{D}} vwdx$.

Assume now for simplicity of description that $\mathcal{D}$ is a polyhedral domain. On $\mathcal{D}$ we have finite element spaces $S_h(\mathcal{D})$, which match Γ but which are *not* zero on Γ. Letting $\overset{\circ}{S}_h(\mathcal{D})$ denote those functions in $S_h(\mathcal{D})$ that *do* vanish on Γ, we find $u_h \in \overset{\circ}{S}_h(\mathcal{D})$ by

$$A(u_h, \chi) = (f, \chi), \text{ for } \chi \in \overset{\circ}{S}_h(\mathcal{D}). \tag{11.6.8}$$

To form an approximation to the outflow derivative, or, rather, to $p(x) = \alpha(x)\frac{\partial u}{\partial n}$, let $p_h \in S_h\big|_\Gamma$ be given by the analogue of (11.6.7),

$$\langle p_h, \chi\rangle = A(u_h, \widetilde{\chi}) - (f, \widetilde{\chi}), \text{ for } \chi \in S_h\big|_\Gamma \tag{11.6.9}$$

where $\widetilde{\chi}$ denotes any extension of $\chi \in S_h\big|_\Gamma$ to $S_h(\mathcal{D})$.

By (11.6.8), p_h is independent of the particular extension $\widetilde{\chi}$ used. One may thus use the "simplest" one, cutting it down to be zero "as fast as possible".

The extra work in computing p_h is commensurate to doing an L_2–projection over Γ.

In these lectures we shall not give an analysis of the error in $p - p_h$, but shall merely refer to [Douglas, Dupont and Wheeler 1974a], [Carey, Humprey and Wheeler 1981], [Carey 1982], [Carey, Chow and Seager 1985], [Chow and Lazarov 1989] and [Pehlivanov, Lazarov, Carey and Chow 1992] for some investigations.

A rather intriguing phenomenon is reported in some of the computational investigations above, see [Carey, Chow and Seager 1985]. To whit: Consider the polygonal situation in the plane.

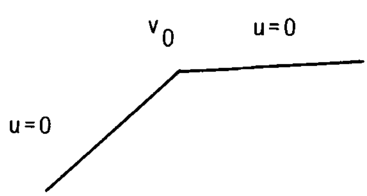


Figure 11.6.1

Since $u = 0$ on Γ, it follows that $\nabla u(v_0) = 0$, at the vertex v_0, if smooth. It appears that the accuracy in approximating the outflow derivative near v_0 suffers unless one further imposes that the functions $\chi \in S_h|_\Gamma$ in (11.6.9) also vanish at v_0.

vi) In a spirit somewhat similar to v), the Lagrange multiplier method (cf. [Babuška 1973]) incorporates a separate approximation to the outflow derivative in its basic formulation. We refer to [Bramble 1981] for some results about this approximation.

11.7. A superconvergent "global" averaging technique for function values.

This techique was introduced in [Louis 1979]. It applies to *any* mesh and *any* finite elements provided a suitable negative norm estimate for $u - u_h$ is known. It involves, however, integration of u_h against a kernel in a unit size neighborhood of the point of interest, and, it also requires knowledge of an exact fundamental solution for the underlying partial differential equation.

As an example we consider the Dirichlet problem for Poisson's equation in the plane,

$$\begin{aligned} -\Delta u &= f \text{ in } \mathcal{D}, \\ u &= 0 \text{ on } \partial\mathcal{D}. \end{aligned} \tag{11.7.1}$$

Let x_0 be a fixed point in $\mathcal{D}$ and

$$B(x_0, R) = \{x : |x - x_0| \leq R\} \subseteq \mathcal{D}, \tag{11.7.2}$$

where R is fixed, independent of h in particular. Let

$$\omega(x) = -\frac{1}{2\pi}\ln|x| + \psi(x) \tag{11.7.3}$$

where $\psi(x) \in \mathcal{C}^\infty$ is chosen so that

$$\omega(x) = \partial\omega(x)/\partial n = 0, \text{ for } |x| = R. \tag{11.7.4}$$

[Louis 1979] gives the example

$$\psi(x) = -\frac{1}{8\pi R^4}(|x|^4 - 4R^2|x|^2 + (3 - 4\ln R)R^4). \tag{11.7.5}$$

Note that it is not required that $\omega(x) \in \mathcal{C}^\infty$, merely that ψ does.

By Green's second formula and (11.7.1), (11.7.4),

$$u(x_0) = \int_{B(x_0,R)} f(y)\omega(x_0 - y) + u(y)\Delta\psi(x_0 - y)dy. \tag{11.7.6}$$

With u_h any approximation to (11.7.1), we then define

$$\widetilde{u}_h(x_0) = \int_{B(x_0,R)} f(y)\omega(x_0 - y) + u_h(y)\Delta\psi(x_0 - y)dy. \tag{11.7.7}$$

We assume that the integrals are exact (or, done to sufficiently high order of accuracy). We then have

$$(u - \widetilde{u}_h)(x_0) = \int_{B(x_0,R)} (u - u_h)(y)\Delta\psi(x_0 - y)dy. \tag{11.7.8}$$

Since $\psi \in \mathcal{C}^\infty$ and R is fixed, it is clear that the error in $(u - \widetilde{u}_h)(x_0)$ is governed by whatever negative norm accuracy $u - u_h$ enjoys, cf. Section 6.2.

Finally, it is not hard to take into account the dependence on R of the error $(u - \widetilde{u}_h)(x_0)$.

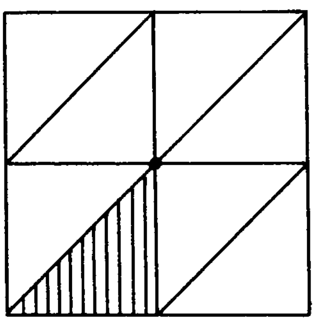
Chapter 12. A computational investigation of superconvergence for first derivatives in the plane.

We shall review the paper [Babuška, Strouboulis, Upadhyay and Gangaraj 1993] and consider their results in light of our theories in Chapters 6 and 7.

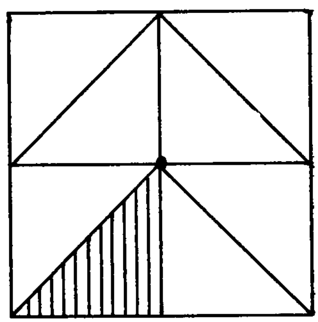
The investigation reduces the problem of finding certain superconvergence points to that of finding the zeroes of certain polynomials. For each element pattern considered and each basic finite element, *only a finite number of polynomials are involved.* I.e., to find those superconvergence points, *there is no asymptotics w.r.t. h involved.* In this sense, the investigation may be called a "computational proof".

12.1. Introduction.

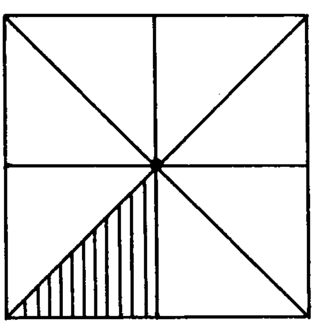
Only so–called "natural" superconvergence points will be considered, i.e., no averaging, however simple, will be taken into account, cf. Remark 12.3.1 below. Only straight evaluation of the derivative of u_h at the point in question is allowed. Let us give details of the set–up for triangular elements. For each of four basic mesh–patterns considered later in the numerical examples we have a $2h \times 2h$ "master cell" centered at x^0 $(= \bullet)$.



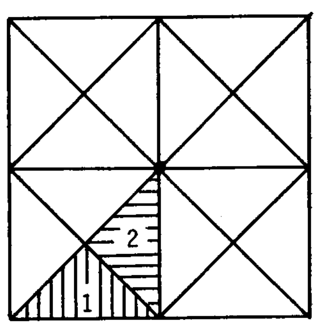
Regular Pattern



Chevron Pattern



Union Jack Pattern



Criss Cross Pattern

Figure 12.1.1

(The shaded triangles will be used later, in Section 12.3.)

In a neighborhood Ω_0 of x^0 the mesh is given by $2h$–periodic repetitions of one of these master cells, and so is translation invariant in each axis–direction by $2h$ (or h in some directions and cases).

The finite element spaces considered will be continuous piecewise polynomials of total polynomial degree $r-1$ on each triangle. (C^1 or higher finite elements will not be considered.)

Remark 12.1.1. What is important in the set–up of the master cells is that "certain things match up over opposite boundaries". Expanding on this, in our C^0, total degree $r-1$, triangular case we have a "standard" interpolation operator Int_h which interpolates the values of a function at the "principal lattice points", illustrated here in the case of linears, quadratics, cubics and quartics.

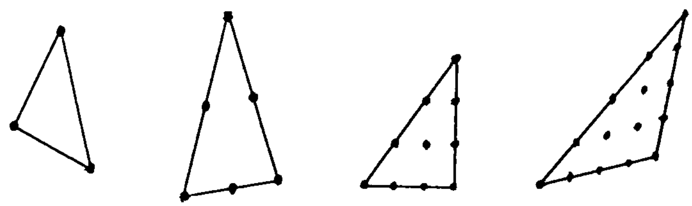


Figure 12.1.2

(See [Nicolaides 1972 and 1973].) Note that the number of points on each edge are enough to determine completely the interpolant on that edge; hence C^0 continuity is assured.

Furthermore, in all of our four pictorial examples, interpolation points on opposite sides in the master cell match up so that (taking $x_0 = (0,0)$),

$$Int_h(f(\cdot + 2h, \cdot))(-h, x_2) = Int_h(f(\cdot, \cdot))(h, x_2)$$

and correspondingly in the x_2 direction. (In our technical work, this will enter in the proof of Lemma 12.2.1.)

We could thus also consider master cells of the form (still for C^0 total degree polynomials, say)

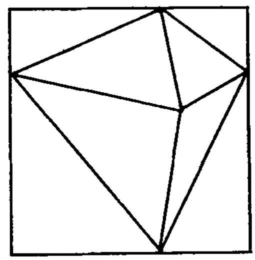


Figure 12.1.3

but *not* of the form

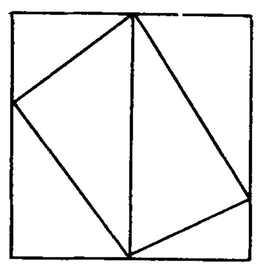


Figure 12.1.4

□

Now assume that we have a basic finite element approximation $u_h \in S_h(\Omega_0)$ to a function u which is sufficiently smooth on Ω_0. Taking here only the case of the Poisson equation, we assume

$$D(u - u_h, \chi) = 0, \text{ for } \chi \in S_h^{comp}(\Omega_0), \tag{12.1.1}$$

where $D(v, w) = \int \nabla v \cdot \nabla w$. We shall further assume that

$$\|u - u_h\|_{L_\infty(\Omega_0)} \leq Ch^{r-L}. \tag{12.1.2}$$

Of course, this error is influenced from what happens outside of Ω_0. Various pollution effects have been gotten rid of (at least beyond an order loss of h^L). E.g., if the problem is over the basic domain $\Omega_0 \subset\subset \mathcal{D}$ and $\mathcal{D}$ has reentrant corners, the mesh may have been suitably refined at such corners, cf. Example 6.3.4.

We are now interested in superconvergence for either of the derivatives $\frac{\partial}{\partial x_i}$, $i = 1$ or 2. With

$$\Omega_1 = \{x : |x - x^0| \leq H\},\ \Omega_0 = \{x : |x - x^0| \leq 2H\} \tag{12.1.3}$$

and

$$H = h^\delta,\ 0 < \delta < 1, \tag{12.1.4}$$

we shall show in Section 2 that

$$\frac{\partial}{\partial x_i}(u - u_h)(x) = \frac{\partial}{\partial x_i}\psi(x) + R_i(x),\ i = 1, 2,\ x \in \Omega_1, \tag{12.1.5}$$

where

$$\|R_i\|_{L_\infty(\Omega_1)} \leq C(h^{r-1+\delta} + h^{r-L-\delta}) \leq Ch^{r-1+\min(\delta,1-L-\delta)}. \tag{12.1.6}$$

Provided $L+\delta < 1$, the term $R_i(x)$ is thus "superconvergent" for first derivatives. Note that this is so even if the size H of the domain Ω_0 where the mesh is built up in a regular fashion from the master cell is very small, $\delta \simeq 1$; then of course one has to be more stringent about pollution control, $L \simeq 0$. In other words, supercovergence is treated here as a local phenomenon.

As for the function ψ, it is based on computations involving *only the master cell.* In short, Taylor–expand u to polynomial degree r at x^0 to get Q. Interpolate into S_h to form $\rho = Q - Int_h Q$. The effect of this is, as we shall prove in Section 12.2, among others to knock away the terms of order $\leq r-1$ in Q, and to make ρ periodic of period $2h$. Hence, one needs to do it only over the master cell. Finally, over the master cell alone, find a $2h$–periodic finite element approximation to ρ. The function ψ will then be their difference, periodically extended. (See (12.2.25), (12.2.20,21) and (12.2.16) below for exact definition of ψ.)

From (12.1.5), (12.1.6) and the simple nature of ψ, cf. Remark 12.2.1, provided $L + \delta < 1$, we shall find that points x with a fixed relative position with respect to the master cell are superconvergent for $\frac{\partial}{\partial x_i}$ on Ω_1 if and only if $\frac{\partial \psi}{\partial x_i}(x) = 0$. The computational experiments, to be reported in Section 3, then uses for Q any polynomial of degree r that may occur in a Taylor–expansion for u, that is, in general any monomial of degree p, and runs through these monomials and checks for $\frac{\partial \psi}{\partial x_i} = 0$. (Exactly how this check is done is not reported. It seems that interval arithmetic is not used.) Any such points found common to all monomials are then declared to be superconvergence points for that derivative. (In case one considers only, say, harmonic functions u, one need only consider harmonic polynomials Q of degree r in the computational experiments.)

12.2. Proof of (12.1.5), (12.1.6) and precise definition of the principal error term ψ.

Recall that $\Omega_1 \subset\subset \Omega_0$ are specific squares about the central point x^0, of side $2H$, $4H$, respectively, with $H = h^\delta$. For simplicity we also assume that the $2h$ periodic extensions of the master cell fit Ω_1 and Ω_0 exactly. We shall denote in general

$$c(x, h) = \{y : |y - x| \leq h\}. \tag{12.2.1}$$

Then $c(x^0, h)$ is the master cell and, with $x^{(i,j)} = (x_1^0 + 2ih, x_2^0 + 2jh)$, i, j integers, $c(x^{(i,j)}, h)$ are the translated cells.

Let Q be the r^{th} order Taylor–expansion for u at x^0. Then

$$\|u - Q\|_{W^s_\infty(\Omega_0)} \leq CH^{r+1-s}, \text{ for } 0 \leq s \leq r+1. \tag{12.2.2}$$

As a tool for the present analysis (it will not enter into the final ψ) we introduce the Neumann projection N_hQ of Q into $S_h(\Omega_0)$, i.e.,

$$D(Q - N_hQ, \chi) = 0, \text{ for } \chi \in S_h(\Omega_0), \tag{12.2.3}$$

$$\int_{\Omega_0} (Q - N_hQ) = 0. \tag{12.2.4}$$

We write

$$u - u_h = (Q - N_hQ) + [(u - Q) - (u_h - N_hQ)] \tag{12.2.5}$$

so that

$$\frac{\partial}{\partial x_i}(u - u_h)(x) = \frac{\partial}{\partial x_i}(Q - N_hQ)(x) + R_i^1(x), \text{ for } \chi \in \Omega_1, \tag{12.2.6}$$

where

$$\|R_i^1\|_{L_\infty(\Omega_1)} \leq \|(u - Q) - (u_h - N_hQ)\|_{W^1_\infty(\Omega_1)}. \tag{12.2.7}$$

From (12.1.1) and (12.2.3) we have

$$D((u - Q) - (u_h - N_hQ), \chi) = 0, \text{ for } \chi \in S_h^{comp}(\Omega_0) \tag{12.2.8}$$

so that by Theorem 5.5.1,

$$\begin{aligned} &\|R_i^1\|_{L_\infty(\Omega_1)} \\ &\leq C \min_{\chi \in S_h(\Omega_0)} \left(\|(u - Q) - \chi\|_{W^1_\infty(\Omega_0)} + H^{-1}\|(u - Q) - \chi\|_{L_\infty(\Omega_0)} \right) \\ &\quad + CH^{-2}\|(u - Q) - (u_h - N_hQ)\|_{L_2(\Omega_0)}. \end{aligned} \tag{12.2.9}$$

Here, using for χ the interpolant of $(u - Q)$ into $S_h(\Omega_0)$,

$$\|(u - Q) - \chi\|_{W^1_\infty(\Omega_0)} \leq Ch^{r-1}\|u - Q\|_{W^r_\infty(\Omega_0)} \leq Ch^{r-1}H, \tag{12.2.10}$$

where we used (12.2.2) in the last step. Similarly,

$$H^{-1}\|(u - Q) - \chi\|_{L_\infty(\Omega_0)} \leq Ch^r \leq Ch^{r-1}H. \tag{12.2.11}$$

Thus,

$$\|R_i^1\|_{L_\infty(\Omega_1)} \leq Ch^{r-1}H + H^{-2}\|(u - Q) - (u_h - N_hQ)\|_{L_2(\Omega_0)}. \tag{12.2.12}$$

By our assumption (12.1.2),

$$\|u - u_h\|_{L_2(\Omega_0)} \leq CH\|u - u_h\|_{L_\infty(\Omega_0)} \leq CHh^{r-L} \tag{12.2.13}$$

and by standard L_2 estimates (which are easily checked to be valid also on a small square),

$$\|Q - N_hQ\|_{L_2(\Omega_0)} \leq Ch^r\|Q\|_{W^r_2(\Omega_0)} \leq Ch^rH\|Q\|_{W^r_\infty(\Omega_0)} \leq CHh^{r-L}. \tag{12.2.14}$$

Hence (12.2.12) gives

$$\|R_i^1\|_{L_\infty(\Omega_1)} \leq C(h^{r-1}H + h^{r-L}H^{-1}). \tag{12.2.15}$$

We next prepare to attack $\frac{\partial}{\partial x_i}(Q - N_h Q)(x)$ in (12.2.6).

Let Int_h denote the standard interpolation operator into $S_h(\Omega_0)$ (cf. Remark 12.2.2 below), and let

$$\rho = Q - Int_h Q. \tag{12.2.16}$$

Let further $H^1_{PER}(\Omega_0)$ denote the $2h$–periodic functions in $W^1_2(\Omega_0)$. A central observation is the following:

Lemma 12.2.1. $\rho \in H^1_{PER}(\Omega_0)$.

Proof: Certainly, $\rho \in W^1_2(\Omega_0)$. Since the situation is built up by translates of the master cell, the interpolant is translation invariant by $2h$, i.e.,

$$Int_h(f(\cdot + x^{(i,j)}))(x) = Int_h(f(\cdot))(x + x^{(i,j)}), \tag{12.2.17}$$

cf. Remark 12.1.1. Noting that for Q a polynomial of degree r,

$$Q(x) = Q(x + x^{(i,j)}) - P_{ij}(x) \tag{12.2.18}$$

with $P_{ij}(x)$ a polynomial of degree $r-1$, which is reproduced by Int_h, we have

$$\begin{aligned} Q(x) &- Int_h(Q(\cdot))(x) \\ &= Q(x + x^{(i,j)}) - P_{ij}(x) - Int_h(Q(\cdot))(x) \\ &= Q(x + x^{(i,j)}) - Int_h((P_{ij} + Q)(\cdot))(x) \\ &= Q(x + x^{(i,j)}) - Int_h(Q(\cdot + x^{(i,j)}))(x) \\ &= Q(x + x^{(i,j)}) - Int_h(Q(\cdot))(x + x^{(i,j)}), \end{aligned} \tag{12.2.19}$$

which proves the lemma. □

Let now $S_h^{PER}(c(x^0,h))$ ($c(x^0,h)$ is the master cell, remember) denote the functions in $S_h(c(x^0,h))$ which are $2h$–periodic, i.e., function values match up at boundaries. For any $\varphi \in H^1_{PER}(\Omega_0)$ we define, *which is done on the master cell alone*, $PP(\varphi) \in S_h^{PER}(c(x^0,h))$ (PP for periodic projection) by

$$D_{c(x^0,h)}(\varphi - PP(\varphi), \chi) = 0, \text{ for } \chi \in S_h^{PER}(c(x^0,h)), \tag{12.2.20}$$

$$\int_{c(x^0,h)} \varphi - PP(\varphi) = 0. \tag{12.2.21}$$

We also let $PP(\varphi)$ denote its $2h$–periodic extension, which is then in $H^1_{PER}(\Omega_0)$. Another crucial result is the following.

Lemma 12.2.2. *For* $\varphi \in H^1_{PER}(\Omega_0)$, *we have*

$$D(\varphi - PP(\varphi), \chi) = 0, \text{ for all } \chi \in S_h^{comp}(\Omega_0). \tag{12.2.22}$$

Proof: For γ the relevant index set, we write in obvious notation,

$$D(PP(\varphi) - \varphi, \chi) = \sum_{(i,j)\in\gamma} D_{c(x^{(i,j)}h)}(PP(\varphi) - \varphi, \chi). \tag{12.2.23}$$

By the periodicity of $PP(\varphi)$ and φ then,

$$\sum_{(i,j)\in\gamma} D_{c(x^{(i,j)},h)}(PP(\varphi)-\varphi,\chi) = D_{c(x^0,h)}\Big(PP(\varphi)-\varphi,\sum_{(i,j)\in\gamma}\chi(x+x^{(i,j)})\Big). \tag{12.2.24}$$

Since $\sum_{(i,j)\in\gamma}\chi(x+x^{(i,j)}) \in S_h^{PER}(c(x^0,h))$, we get (12.2.22) from (12.2.20). □

After these preparations, we attack $Q - N_hQ$ in (12.2.6). We set

$$\psi = \rho - PP(\rho), \ \rho \text{ given in } (12.2.16), \tag{12.2.25}$$

and write

$$Q - N_hQ = \psi + [Q - N_hQ - \psi]. \tag{12.2.26}$$

Here, by (12.2.16),

$$Q - \psi = Q - (\rho - PP(\rho)) = Int_hQ + PP(\rho), \tag{12.2.27}$$

which shows that

$$Q - N_hQ - \psi = Int_hQ + PP(\rho) - N_hQ \in S_h(\Omega_0). \tag{12.2.28}$$

Further, from Lemma 12.2.2 and (12.2.3),

$$D(Q - N_hQ - \psi,\chi) = 0, \text{ for } \chi \in S_h^{comp}(\Omega_0). \tag{12.2.29}$$

From (12.2.28) and Theorem 5.5.1 then,

$$\begin{aligned}\|Q-N_hQ-\psi\|_{W^1_\infty(\Omega_1)} &\le CH^{-2}\|Q-N_hQ-\psi\|_{L_2(\Omega_0)}\\ &\le CH^{-2}(\|Q-N_hQ\|_{L_2(\Omega_0)} + \|\psi\|_{L_2(\Omega_0)}).\end{aligned} \tag{12.2.30}$$

As before, cf. (12.2.14),

$$H^{-2}\|Q-N_hQ\|_{L_2(\Omega_0)} \le Ch^{r-L}H^{-1}. \tag{12.2.31}$$

Further, by a standard duality argument (albeit on a small region, periodic functions), since $\psi = \rho - PP(\rho)$, and $\rho = Q - Int_hQ$,

$$\begin{aligned}H^{-2}\|\psi\|_{L_2(\Omega_0)} &\le CH^{-2}h\|\psi\|_{W_2^1(\Omega_0)}\\ &\le CH^{-2}h\|\rho\|_{W_2^1(\Omega_0)}\\ &\le CH^{-2}hh^{r-1}\|Q\|_{W_2^r(\Omega_0)}\\ &\le Ch^rH^{-1}\|Q\|_{W_\infty^r(\Omega_0)}\\ &\le Ch^{r-L}H^{-1}.\end{aligned} \tag{12.2.32}$$

All in all from the above,

$$\|Q-N_hQ-\psi\|_{W^1_\infty(\Omega_1)} \le Ch^{r-L}H^{-1}. \tag{12.2.33}$$

It follows from this and (12.2.6), (12.2.15) that

$$\frac{\partial}{\partial x_i}(u-u_h)(x) = \frac{\partial\psi}{\partial x_i}(x) + R_i(x), \text{ for } x\in\Omega, \tag{12.2.34}$$

where

$$\|R_i\|_{L_\infty(\Omega_1)} \le C(h^{r-1}H + h^{r-L}H^{-1}) = C(h^{r-1+\delta} + h^{r-L-\delta}), \tag{12.2.35}$$

which is the desired result, already stated as (12.1.5), (12.1.6).

Remark 12.2.1. By scaling, one finds that $\frac{\partial\psi}{\partial x_i}(x) = h^{r-1}\frac{\partial\widehat{\psi}}{\partial\widehat{x}}(\widehat{x})$ where $\widehat{}$ denotes quantities on the master cell scaled to unit size. Then, a point x of fixed relative position in the master cell is superconvergent iff $\frac{\partial\psi}{\partial x_i}(x) = 0$. □

Remark 12.2.2. In our presentation above, we have, without commenting much on it, taken Int_h to be the "standard" interpolation operator, well known in the cases at hand, see Remark 12.1.1. Actually, the proof works the same with other interpolation operators, provided they respect Remark 12.1.1 on opposite boundaries. To illustrate, consider the analogous situation in one dimension and C^0 quadratics: The "standard" would be to interpolate at mesh and midpoints, but one may equally well interpolate at meshpoints and a quarter point (or, whatever). In this connection, note that the application of the periodic projection PP in forming ψ removes the differences between interpolation operators so that the superconvergence points found do not depend on this ambiguity in defining the interpolation operator.

As it happens, some of the superconvergence points for derivatives found in the computational experiments do coincide with some of those for the "standard" interpolant. The theory here obviously (from the above comments in this remark) does not say so. The reader may wish to revisit Section 7.4.i in this regard. □

Remark 12.2.3. Although presented here only for the case of the Dirichlet form, the arguments in this section carry over without much essential change to the case of any constant coefficient operator. The case of smooth variable coefficient operators is somewhat messier: One may freeze the coefficients at x^0 and then one has to consider, at quite a few places, what happens between the original form and the frozen one. (The case of $-div(a(x_1, x_2)\nabla u) + \cdots$ can also be handled analogously to Section 6.4, extended to a general size H, of course.) □

12.3. Results of computational studies, with comments.

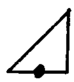
We shall give the computational results of [Babuška, Strouboulis, Upadhyay and Gangaray 1993] and comment in some detail on their findings in light of our theory in these notes, mainly in light of our symmetry theory of Chapter 7 and also the tensor product theory of Chapter 6.

Considering the Poisson equation, running all four triangular pattern cases as depicted in Section 12.1, all cases $2 \le r \le 8$, running through all possible monomials of degree r for Q, and declaring a point superconvergent for $\frac{\partial}{\partial x_1}$ for x lying in the shaded triangles in the figures of master cells pictured in Section 12.1 if $\frac{\partial\psi}{\partial x_1}(x) = 0$ for *all* such monomials, they found:

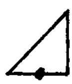
For the regular pattern:

$r = 2:$  midpoint of the side parallel to x_1–axis

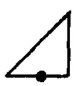
$r = 3:$  Gauss points

$r = 4:$  midpoint, as for $r = 2$

$r = 5:$ no superconvergent points for $\dfrac{\partial u}{\partial x_1}$

$r = 6:$ 

$r = 7:$ no superconvergence points for $\dfrac{\partial u}{\partial x_1}$

$r = 8:$ 

For r even, these midpoints results are known by our symmetry arguments, since the average from both sides in the case of $\frac{\partial}{\partial x_1}$ now coincides with $\frac{\partial u_h}{\partial x_1}$. Note that this is *not* so at the symmetric mesh points; there one must use averaging to obtain superconvergence also in $\partial/\partial x_1$. For $r = 3$, the result is in Remark 7.4.ii.

For the Chevron pattern, the midpoints of sides for r even stay as superconvergent points for $\frac{\partial}{\partial x_1}$. (The Gauss points for $r = 3$ are lost.) Since the Chevron–pattern has the midpoints of x_1–parallel edges as symmetry points, our theory of symmetry explains these findings as well. (Since the mesh points are not symmetry points, the result for $r = 3$ cannot be deduced as in Remark 7.4.ii. Indeed, according to these computations, it is lost.)

As for the *Union Jack mesh*, no superconvergence points for $\frac{\partial}{\partial x_1}$ were found. Since midpoints are no longer symmetry points, we know that at the only symmetry points present, the meshpoints, we have to average. So, the results shouldn't show in the present numerical experiment, and they didn't.

As for the *Criss–Cross grid*, for the shaded triangle labelled 1 in the figure, the superconvergence was at the midpoints on the side parallel to x_1 for $\frac{\partial}{\partial x_1}$, for r even. For $r = 3$, we have the same Gauss points as before, but for r odd, $r \geq 5$, no superconvergence points. Again, our symmetry theory has predicted them all (and predicts more if averaging is allowed).

For the triangle labelled 2 in the Criss–Cross grid, for $\frac{\partial}{\partial x_1}$ our symmetry theory wouldn't predict any superconvergence points without averaging and, none were found in the numerical experiments.

The investigation in [Babuška, Strouboulis, Upadhyay and Gangaraj 1993] also considers rectangular elements. They treat $\mathcal{C}^0$–elements of type:

(12.3.1) Tensor product elements; i.e., on each rectangle,

$$\chi(x_1, x_2) = \sum_{\substack{i,j \\ 0 \le i,j \le r-1}} c_{ij} x_1^i x_2^j.$$

(12.3.2) Serendipity (trunc) elements,

$$\chi(x_1, x_2) = \sum_{\substack{i,j \\ 0 \le i+j \le r-1}} c_{ij} x_1^i x_2^j + c_{r-1,1} x_1^{r-1} x_2 + c_{1,r-1} x_1 x_2^{r-1}.$$

(12.3.3) Intermediate degree $r-1$ (incomplete degree r)

$$\chi(x_1, x_2) = \sum_{\substack{i,j \\ 0 \le i+j \le r-1}} c_{ij} x_1^i x_2^j + \sum_{k=0}^{r-2} c_{r-1-k,k+1} x_1^{r-1-k} x_2^{k+1}.$$

(These elements are missing x_1^r and x_2^r from containing all polynomials of total degree r.)

For the *tensor–product elements*, they found the x_1 Gauss lines as superconvergent lines. No surprise, exactly what our one–dimensional theory combined with the tensor product theory of Section 6.2 (and 6.4, for generalization) shows. (The theory is given in Chapter 6 only for $H = 1$. The generalization presently needed is not hard.) Note also that the meshes do not have to translation invariant for the theory in Chapter 6.

The very same superconvergence points as for tensor–products occurred for the *intermediate (incomplete) family* in (12.3.3), they reported (without much detail given). For $r = 3$, i.e., incomplete cubics, cf. Section 7.4.iii; otherwise our symmetry theory in Chapter 7 in these lectures is silent, except for the center of the square and midpoints of the x_1–parallel sides for r even, in the absence of averaging.

Finally, for the case of the *Serendipity family* in (12.3.2), for $r = 2$ and 3, the superconvergence points are as for tensor products. (For $r = 2$, all three elements, tensor products, intermediate and Serendipity, are the same; for $r = 3$, Serendipity and intermediate are the same. So, this is expected, cf. Chapter 6 and Section 7.4.)

For the *Serendipity family with* $r = 4$, for $\frac{\partial}{\partial x_1}$, a line of superconvergence and four interior points were found:

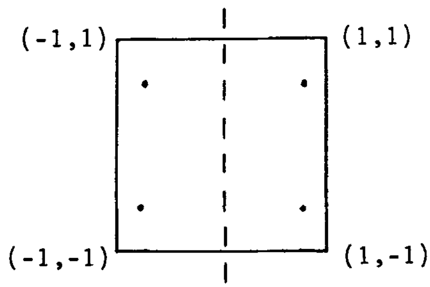


Figure 12.3.1

The four points are at $(\pm.7746\ldots, \pm.5774\ldots)$. Note that, by elementary symmetry arguments, they *cannot* also be superconvergent for $\frac{\partial}{\partial x_2}$ and so, being interior points, our symmetry theory will not predict them.

For the *Serendipity family with* $r \geq 5$, no superconvergence points for $\frac{\partial}{\partial x_1}$ were reported for r odd. For $r = 6$ and 8, the center of the square and the midpoints of the x_1–parallel sides were found, exactly as our symmetry theory says (in the absence of averaging).

In conclusion, our theories in these lectures have explained all superconvergence results found in [Babuška, Strouboulis, Upadhyay and Gangaray 1993] for general solutions of Poisson's equation *except*

a) Most of those found in the intermediate (incomplete) families on squares, and

b) The results found in the Serendipity family on a square for $r = 4$, except for the center point and the midpoints of the x_1–parallel sides.

The paper also reported on the case when u in itself is a harmonic function. In that case, only the harmonic polynomials Q need be tested. Now plenty of superconvergence points for derivatives were found, for any r (not merely even r), and the number of such points increasing with r. Of course, our theory in these lectures has not even thought about this very interesting case! Similarly, [Babuška, Strouboulis, Upadhyay and Ganagaraj 1994] investigates superconvergence in homogeneous solutions of the equations of elasticity.

Remark 12.3.1. As remarked in the beginning of Section 12.1, we have not taken into account averaging from two adjoining elements at a point on their common boundary, as in (7.2.12). It is trivial to do so in (12.1.5), and corresponding computational determination of all "averaged" superconvergence points could be performed. □

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(For the guiding principles behind this list of references, see the Preface.)

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