EXACT ASYMPTOTIC ESTIMATES OF BROWNIAN PATH VARIATION

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1. Summary. A standard Brownian motion in R^d, the Euclidean space of d dimensions, will be denoted by $X(t, \omega)$ or $X(t)$ when we do not need to be explicit about the point ω in the underlying probability space. We consider sample paths, that is, properties of $X(t, \omega)$ as a function of t for fixed ω. It is well known that almost all paths $X(t)$ are everywhere continuous but not of bounded variation on finite intervals and that the square variation is "almost" finite on a fixed interval and becomes constant under suitable restrictions. To be precise, if

$$\pi_n = \{0 = t_{n,0} < y_{n,1} < \cdots < t_{n,k_n} = 1\}$$

is a fixed sequence of partitions of [0, 1] such that π_{n+1} contains all the division points of π_n and

$$\sigma(\pi_n) = \max_i (t_{n,i} - t_{n,i-1}) \to 0 \quad \text{as} \quad n \to \infty,$$

then

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} |X(t_{n,i}) - X(t_{n,i-1})|^2 = d \tag{1.1}$$

with probability 1. This result is due to Lévy [11] but it can be most easily proved by a martingale argument (see Doob [4]).

If $Q(\delta)$ denotes the class of partitions π of (0, 1) such that $\sigma(\pi) < \delta$, we could ask about the size of $V_2(X, \pi)$ as π ranges over $Q(\delta)$, where

$$V_2(X, \pi) = \sum_{i=1}^{k} |X(t_i) - X(t_{i-1})|^2$$

when $\pi = \{0 = t_0 < t_1 < \cdots < t_k = 1\}$. Lévy showed in [12] that $V_2(X, \pi)$ was unbounded for $\pi \in Q(\delta)$ while the corresponding $V_\alpha(X, \pi)$ for $\alpha > 2$ remains bounded with probability 1. The modulus of continuity for Brownian motion (see Lemma 2.1) leads easily to a proof that

$$V_\psi(X, \pi) = \sum_{i=1}^{k} \psi(|X(t_i) - X(t_{i-1})|)$$

is bounded for all π whenever $\psi(s) = s^2/\log^* s$, where $\log^* s = \max\{1, |\log s|\}$. Although this function $\psi(s)$ is bigger as $s \to 0$ than s^α for any $\alpha > 2$, it is not

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the biggest function which leads to a bounded ψ-variation. We show that the "correct" function is

$$\psi_1(s) = s^2/2 \log^* \log^* s$$

in the sense that

$$W_{\psi_1}(X; 0, 1) = \sup_{\pi} V_{\psi_1}(X, \pi) \tag{1.2}$$

is a finite random variable; but (1.2) will be infinite if ψ_1 is replaced by any function ψ such that $\psi(s)/\psi_1(s) \to +\infty$ as $s \to 0$. This result is obtained in §4 as a corollary to Theorem 1 in which we prove that, with probability 1,

$$\lim_{\delta \to 0} [\sup_{\pi \in Q(\delta)} V_{\psi_1}(X, \pi)] = 1.$$

Since it is clear that the small intervals of π are the cause of $V_2(X, \pi)$ being unbounded, we could define a proper random variable

$$S_k(X) = \sup_{\pi \in \mathcal{P}_k} V_2(X, \pi),$$

where $\mathcal{P}_k$ is the class of partitions π for which $\min (t_i - t_{i-1}) \geq k^{-1}$. Greenwood [10] and Stackelberg [15] obtained bounds on the growth of S_k as $k \to \infty$ and W. E. Pruitt (see Math. Review of [15]) pointed out that the modulus of continuity of the path leads to $S_k = O (\log k)$. In §5 we apply the methods we have developed to show that $S_k \sim 2 \log \log k$ as $k \to \infty$ with probability 1.

Section 6 is devoted to an examination of the small values of $V_2(X, \pi)$. For $d = 1$, as was pointed out by Goffman and Loughlin [9], since $X(t)$ is continuous we must have

$$\inf_{\pi} V_{\psi}(X, \pi) = 0$$

for any $\psi(s)$ such that $s^{-1}\psi(s) \to 0$ as $s \to 0$. To get over this problem Goffman and Loughlin defined the *weak* ψ-variation by

$$\inf_{\pi} \sum_{i=1}^{k} \psi\{R(X; t_{i-1}, t_i)\} = U_{\psi}(X; 0, 1), \tag{1.3}$$

where

$$R(X; a, b) = \sup_{a \leq s \leq t < b} |X(t) - X(s)| \tag{1.4}$$

and they showed, using properties of the graph of Brownian motion obtained in [16], that the weak variation is positive for $\psi(s) = s^{\alpha}$ for any $\alpha < 2$. We show that the right function for measuring the weak variation is

$$\psi_3(s) = s^2 \log^* \log^* s$$

and obtain an exact asymptotic limit for the left-hand side of (1.3) when π is restricted to $Q(\delta)$ and $\delta \to 0$. We can also find the asymptotic behaviour of

$$T_k = \inf_{\pi \varepsilon \mathcal{P}_k} \sum_{i=1}^{k} \{R(X; t_{i-1}, t_i)\}^2,$$

though a lack of knowledge of the distribution of $R(X; t_{i-1}, t_i)$ when $d \geq 2$ means that we have an exact answer only for $d = 1$.

It is natural to ask whether there are analogues of our results for general processes with stationary, independent increments. One would not expect exact results like Theorems 1 or 2 to be valid for strong variation because the upper tail of the distribution of $|X(t)|$ is not thin except in the Brownian motion case. One could use the boundedness of the variational sum $V_\psi(X, \pi)$ to define a section in the partial ordering of functions $\psi(s)$ determined by growth rate as $s \to 0$ and then try to characterize the upper and lower functions of the section. This is an interesting problem but we have not made significant progress towards its solution. Instead we obtain analogues of Theorem 3 for the weak variation of some processes. Since the lower tail of the distribution of $R(X; 0, 1)$ will always be small if the Lévy measure is infinite, we would expect that there is always a "right" function $\psi(s)$ to replace ψ_3 in (1.3). However, we are only able to specify this correct function in cases where the fine structure of the sample path is already well understood, namely, (i) all stable processes other than the asymmetric Cauchy process, (ii) processes with orthogonal components of type (i), and (iii) all subordinators. The results for cases (i) and (iii) are contained in Theorems 6 and 5.

Section 2 lists notation and the main results needed which are proved elsewhere and §3 gives some preliminary lemmas which may be of independent interest.

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2. Notation and estimates. We will use the modulus sign to denote both the distance $|x - y|$ between two points of R^d and the Lebesgue measure $|E|$ of a subset $E \subset R^d$. We use c to denote a finite positive constant whose precise value is unimportant; the value of c may be different in each statement or equation. $\{a_i\}$, $\{b_i\}$, $\{q_i\}$ will denote constants whose values are known.

It is assumed that $X(t)$ is a standard Hunt process in R^d with stationary independent increments, $X(t + h) - X(t)$, which are normally distributed with density

$$(2\pi h)^{-\frac{1}{2}d} \exp(-|x|^2/2h) \tag{2.1}$$

and that $X(0) = 0$. Brownian motion satisfies the scaling property that $r^{-\frac{1}{2}}X(rt)$ is a version of the same process as $X(t)$ for every $r > 0$. We continually use the Markov property and the strong Markov property usually without explicit mention. Our goal is to obtain "almost sure" results, that is, we are interested in properties of $X(t, \omega)$ as a function of t which determine events of probability 1 in the underlying probability space Ω of the process. To obtain such results we need good estimates of the probabilities of a large selection of different events which occur with small probability; our first step is to organize these estimates.

Define the random variables

$$M(h) = \sup_{0 \leq t \leq h} |X(t)| \quad \text{and} \quad R(h) = \sup_{0 \leq s < t < h} |X(t) - X(s)|$$

and note that $R(X; a, b)$ defined in (1.4) has the same distribution as $R(b - a)$. Further

$$|X(h)| \leq M(h) \leq R(h) \leq 2M(h). \tag{2.2}$$

It can be easily deduced from the form of the density (2.1) that

$$P\{|X(h)| > \lambda h^{\frac{1}{2}}\} \sim a_d \lambda^{d-2} e^{-\frac{1}{2}\lambda^2} \text{ as } \lambda \to \infty \tag{2.3}$$

for suitable constants a_d ($a_1 = (2/\pi)^{\frac{1}{2}}$). For each d the exact distribution of $M(h)$ can be found by inverting the first passage time distribution obtained in [3], but the distribution of $R(h)$ seems to be known only for $d = 1$ where it was computed by Feller [7]. For $d = 1$ Feller's result leads to

$$P\{R(h) > \lambda h^{\frac{1}{2}}\} \sim 4a_1 \lambda^{-1} e^{-\frac{1}{2}\lambda^2} \text{ as } \lambda \to \infty \tag{2.4}$$

and for the lower tail

$$P\{R(h) < \lambda^{-1} h^{\frac{1}{2}}\} \sim 8\lambda^2 e^{-\frac{1}{2}\pi^2\lambda^2} \text{ as } \lambda \to \infty. \tag{2.5}$$

The upper tail estimate extends to general dimension d by using

$$P\{R(h) > \lambda h^{\frac{1}{2}}\} \sim 4P\{\, |X(h)| > \lambda h^{\frac{1}{2}}\} \text{ as } \lambda \to \infty$$

which could be proved by tightening an argument of W. E. Pruitt which gives for all λ, $h > 0$ and $d \geq 1$

$$P\{R(h) > \lambda h^{\frac{1}{2}}\} \leq 2P\{M(h) > \lambda h^{\frac{1}{2}}\} \leq 4P\{|X(h)| > \lambda h^{\frac{1}{2}}\}. \tag{2.6}$$

We indicate the proof of the first half of (2.6); the second half can be proved by a similar (somewhat easier) argument. Because of the scaling property it is sufficient to consider the case $h = 1$. Let τ be the stopping time defined by

$$\tau = \inf_{s > 0} \{R(s) > \lambda\}.$$

Put $E_\lambda = \{\tau < 1\} = \{R(1) > \lambda\}$. Since the path is continuous, for each $\omega \in E_\lambda$ there exists a first time $s(\omega)$ satisfying $0 \leq s(\omega) < \tau(\omega)$ and

$$|X(\tau) - X(s)| = \lambda$$

with the whole of $X(t, \omega)$ for $0 \leq t \leq \tau$ contained in the closed sphere $S(\omega)$ with center $X(s)$ and radius λ. Note that τ, s and the direction of the vector $X(\tau) - X(s) = v(\omega)$ are all determined by the σ-field F_τ, so we can apply the strong Markov property to obtain independence for the increments in the path after time τ. In particular, for $\omega \varepsilon E_\lambda$ the plane through 0 normal to $v(\omega)$ divides R^d into two half-spaces each equally likely to contain $X(1 - \tau) - X(\tau)$. This means that $P\{X(1) \varepsilon H(\omega)/E_\lambda\} = \frac{1}{2}$, where $H(\omega)$ is the half space not containing $S(\omega)$ but bounded by a plane tangent to $S(\omega)$ at $X(\tau)$. If we now define

$$M'(1) = \sup_{0 \le s \le 1} |X(s) - X(1)|,$$

then

$$X(1) \varepsilon H(\omega) \Rightarrow |X(1) - X(s)| > \lambda \Rightarrow M'(1) > \lambda$$

so that

$$P\{M'(1) > \lambda | E_\lambda\} \ge \tfrac{1}{2}.$$

But clearly

$$\{M'(1) > \lambda\} \subset E_\lambda$$

so that

$$P\{M(1) > \lambda\} = P\{M'(1) > \lambda\} \ge \tfrac{1}{2} P\{R(1) > \lambda\}.$$

The inequalities (2.6) are of no help in estimating the lower tails. For the lower tail of $M(h)$ we can apply the results of [3] to obtain

$$P\{M(h) < \lambda^{-1} h^{\frac{1}{2}}\} \sim b_d e^{-q_d \lambda^2} \quad \text{as} \quad \lambda \to \infty, \tag{2.7}$$

where b_d, q_d are known positive constants ($b_1 = 8/\pi^2$, $q_1 = \pi^2/8$) and for the lower tail of $R(h)$ when $d \ge 2$ we have to be content with applying (2.2) to (2.7). As we will not be considering any very delicate results in this paper we will only use the following crude estimates.

For any $\epsilon > 0$ and d a fixed dimension there is a λ_0 such that if $\lambda > \lambda_0$, then all of Equations (2.8) to (2.11) are valid.

$$\begin{aligned} e^{-\frac{1}{2}(1+\epsilon)\lambda^2} &< P\{|X(h)| > \lambda h^{\frac{1}{2}}\} \le P\{M(h) > \lambda h^{\frac{1}{2}}\} \\ &\le P\{R(h) > \lambda h^{\frac{1}{2}}\} < e^{-\frac{1}{2}(1-\epsilon)\lambda^2} \end{aligned} \tag{2.8}$$

$$e^{-q_d(1+\epsilon)\lambda^2} < P\{M(h) < \lambda^{-1} h^{\frac{1}{2}}\} < e^{-q_d(1-\epsilon)\lambda^2} \tag{2.9}$$

For $d = 1$

$$e^{-\frac{1}{2}\pi^2(1+\epsilon)\lambda^2} < P\{R(h) < \lambda^{-1} h^{\frac{1}{2}}\} < e^{-\frac{1}{2}\pi^2(1-\epsilon)\lambda^2}. \tag{2.10}$$

For $d \ge 2$

$$e^{-4q_d(1+\epsilon)\lambda^2} < P\{R(h) < \lambda^{-1} h^{\frac{1}{2}}\} < e^{-q_d(1-\epsilon)\lambda^2}. \tag{2.11}$$

To avoid trivialities we use

$$\log^* s = \max(1, |\log s|)$$

and note that

$$k(s) = \log^* \log^* s = \log \log (1/s) \text{ when } 0 < s < e^{-e}.$$

Put

$$\varphi_1(s) = (2sk(s))^{\frac{1}{2}} \qquad \psi_1(s) = s^2/(2k(s)) \tag{2.12}$$

$$\varphi_2(s) = s^{\frac{1}{2}} \qquad \psi_2(s) = s^2 \tag{2.13}$$

$$\varphi_3(s) = (s/k(s))^{\frac{1}{2}} \qquad \psi_3(s) = s^2k(s). \tag{2.14}$$

Note that each of these functions is monotone in s and each of the pairs φ_i, ψ_i are asymptotically inverse as $s \to 0$ in the sense that

$$\varphi_i\{\psi_i(s)\} \sim s, \quad \psi_i\{\varphi_i(s)\} \sim s \text{ as } s \to 0. \tag{2.15}$$

The local growth condition at a fixed point t is given by the standard law of the iterated logarithm: with probability 1

$$\limsup_{h \to 0} \frac{|X(t + h) - X(t)|}{\varphi_1(h)} = \limsup_{h \to 0} \frac{R(X; t, t + h)}{\varphi_1(h)} = 1. \tag{2.16}$$

An application of (2.15) to Theorem 4(i) of [3] gives

$$\liminf_{h \to 0} \frac{M(X; t, t + h)}{\varphi_3(h)} = q_d^{-\frac{1}{2}}, \tag{2.17}$$

where

$$M(X; t, t + h) = \sup_{t \le s < t+h} |X(s) - X(t)|.$$

Standard arguments also lead to

$$\liminf_{h \to 0} \frac{R(X; t, t + h)}{\varphi_3(h)} = f_d , \tag{2.18}$$

where the f_d's are finite positive constants ($f_1 = \sqrt{2}/\pi$ and for $d \ge 2$, $(\frac{1}{2})q_d^{-\frac{1}{2}} \le f_d \le q_d^{-\frac{1}{2}}$). Since the proofs of iterated logarithm results use only the independence of increments over disjoint intervals, they remain valid when the interval $(t, t + h)$ is replaced by $(t - h, t)$. Estimates of the growth which are valid uniformly for all t can be found for each of the random variables $X(t + h) - X(t)$, $M(X; t, t + h)$ and $R(X; t, t + h)$. We will require only the upper asymptotic bound due to Lévy [11] which we state as the following lemma.

LEMMA 2.1. *For a Brownian motion in R^d there is probability* 1 *that if* $\varphi_4(h) = (2h \log^* h)^{\frac{1}{2}}$, *then*

$$\limsup_{h \to 0} \sup_{0 \le t \le 1} \frac{X(t + h) - X(t)}{\varphi_4(h)} = 1. \tag{2.19}$$

Note that the numerator in (2.19) can be replaced by any of $|X(t+h) - X(t)|$, $M(X; t, t+h)$, $R(X; t, t+h)$ without changing the result.

3. Preliminary results. In the present paper we are considering behaviour over an interval of positive length so that the usual zero-one laws do not help to show the existence of a constant once bounds have been proved. Instead we formalize a result which was used in [19] for similar purposes.

LEMMA 3.1. *If $Y(t)$ is a 1-dimensional process with stationary independent increments and Y is strictly increasing with continuous paths almost surely, then there is a finite positive constant c such that with probability 1, $Y(t) = ct$ for all $t > 0$.*

In the iterated logarithm results (2.16)–(2.18) we need quantitative estimates of the probability of not getting close to the asymptotic value over a prescribed small interval. This sort of estimate is contained in [14; Lemma 8], [8; Lemma 7] and several other papers concerned with the exact Hausdorff measure of sample paths. We do not give details of the argument again as they are very similar to those used in the above papers. A useful form of the result is the following lemma.

LEMMA 3.2. *For a Brownian motion $X(t)$ in R^d and given $\epsilon > 0$ there is a constant $\gamma = \gamma(\epsilon, d)$ and an integer n_0 such that for any t and $n \geq n_0$*

$$\text{(i)} \qquad P\left\{ \sup_{1 \leq i \leq n^{9/10}} \frac{|X(t + i/n) - X(t)|}{\varphi_1(i/n)} < 1 - \epsilon \right\} < e^{-(\log n)^\gamma}$$

$$\text{(ii)} \qquad P\left\{ \inf_{1 \leq i \leq n^{9/10}} \frac{R(X; t, t + i/n)}{\varphi_3(i/n)} > q_d^{-\frac{1}{2}} + \epsilon \right\} < e^{-(\log n)^\gamma}.$$

Our final preliminary result amounts to the observation that all the iterated logarithm results have a "two sided" version. We give a proof in one case and state another.

LEMMA 3.3. *If $X(t)$ is a Brownian path in R^d and $t > 0$ is any fixed time, then with probability 1*

$$\lim_{\delta \to 0} \left[\sup_{\substack{u \geq 0, v \geq 0 \\ 0 < u+v < \delta}} \frac{|X(t+u) - X(t-v)|}{\varphi_1(u+v)} \right] = 1.$$

Proof. By the standard one-sided result (2.16) it is clear that $H_\delta(t)$ is monotone in δ and $\lim_{\delta \to 0} H_\delta(t) \geq 1$, where

$$H_\delta(t) = \sup_{\substack{u \geq 0, v \geq 0 \\ 0 < u+v < \delta}} \frac{|X(t+u) - X(t-v)|}{\varphi_1(u+v)} = \sup_{\substack{u \geq 0, v \geq 0 \\ 0 < u+v < \delta}} \frac{R(X; t-v, t+u)}{\varphi_1(u+v)}.$$

It is therefore sufficient to prove that for any $\epsilon > 0$ there is probability 1 that for sufficiently small δ

$$H_\delta(t) < 1 + 5\epsilon. \tag{3.1}$$

We may assume $\epsilon < \frac{1}{5}$. By (2.16) and its left-handed analogue we can pick $\delta_1 > 0$ such that whenever $0 < u + v < \delta_1$

$$|X(t + u) - X(t)| < (1 + \epsilon)\varphi_1(u)$$

$$|X(t) - X(t - v)| < (1 + \epsilon)\varphi_1(v).$$

Hence if $0 < u + v < \delta_1$ and $v < \epsilon^2 u$, we have

$$\begin{aligned} |X(t + u) - X(t - v)| &< (1 + \epsilon)\varphi_1(u) + (1 + \epsilon)\varphi_1(v) \\ &\le (1 + \epsilon)(\varphi_1(u) + 2\epsilon\varphi_1(u)) \\ &< (1 + 5\epsilon)\varphi_1(u) < (1 + 5\epsilon)\varphi_1(u + v) \end{aligned}$$

and a similar argument works for $u < \epsilon^2 v$.

We now adapt the usual technique as follows. Let $u_n = v_n = \exp(-n^\alpha)$ with $\alpha = 1 - \epsilon < 1$. Pick n_0 such that

$$\frac{\varphi_1(u_n)}{\varphi_1(u_{n+1})} < 1 + \epsilon \quad \text{for} \quad n \ge n_0 \tag{3.2}$$

and pick $\delta_2 > 0$ such that if $u + v < \delta_2$,

$$u_{n+1} < u \le u_n \quad \text{and} \quad v_{m+1} < v \le v_m ,$$

then $n, m \ge n_0$. With these conditions

$$\frac{|X(t + u) - X(t - v)|}{\varphi_1(u + v)} \ge 1 + 5\epsilon \Rightarrow \frac{R(X; t - v_m, t + u_n)}{\varphi_1(u_{n+1} + v_{m+1})} \ge 1 + 5\epsilon$$

which in turn implies by (3.2) that the event $E_{n,m}$ occurs where

$$E_{n,m} = \left\{\omega: \frac{R(X; t - v_m, t + u_n)}{\varphi_1(u_n + v_m)} > 1 + 3\epsilon\right\}.$$

An easy computation using (2.8) shows that for $m \ge n \ge n_1$

$$P(E_{n,m}) < n^{-1-2\epsilon}.$$

For a fixed integer n, if k is the smallest integer such that $u_{n+k}/u_n < \epsilon^2$, then

$$k < cn^\epsilon$$

for some c depending on ϵ so that

$$P\left(\bigcup_{m=n}^{n+k} E_{n,m}\right) < cn^{-1-\epsilon}.$$

An application of Borel–Cautelli shows that with probability 1 there is an n_2 such that none of $E_{n,m}$ occur with $n + k \ge m \ge n \ge n_2$. Similarly there is an integer n_3 such that none of $E_{n,m}$ occur with $m + k \ge n \ge m \ge n_3$. If we now take $\delta > 0$ but smaller than δ_1, δ_2, u_{n_2}, v_{n_2}, we have (3.1) in all possible cases.

LEMMA 3.4. *For a Brownian path $X(t)$ in R^d and a fixed time t there is probability* 1 *that*

(i) *if $d = 1$, then*

$$\lim_{\delta\to 0} \inf_{\substack{u\geq 0, v\geq 0\\ 0<u+v<\delta}} \frac{R(X;\, t - v,\, t + u)}{\varphi_3(u + v)} = \sqrt{2}/\pi$$

and

(ii) *if $d \geq 2$, then*

$$\lim_{\delta\to 0} \inf_{\substack{u\geq 0, v\geq 0\\ 0<u+v<\delta}} \frac{R(X;\, t - v,\, t + u)}{\varphi_3(u + v)} \geq \tfrac{1}{2}q_d^{-\frac{1}{2}}.$$

Remark 1. In Lemma 3.4 (ii) it seems certain that the lim inf should be the same as for the one-sided result and that both should have the value $\frac{1}{2}q_d^{-\frac{1}{2}}$, but insufficient is known about the lower tail of $R(h)$ to prove this.

Remark 2. It is clear that the proof of Lemma 3.3 can be adapted to derive two-sided local laws from one-sided laws whenever the relevant tail is a negative exponential power. However the more delicate division into upper and lower functions will not be the same for two-sided growth as for one-sided growth. We do not require any such very delicate results in the present paper, so we make no attempt to derive the appropriate integral test for two-sided growth.

4. Strong variation of Brownian motion. The object of this section is to show that $\psi_1(s) = s^2/\log^* \log^* s$ is the correct function for measuring the variation of $X(t)$. If we put $V_{\psi_1}(X, \pi) = \sum_{i=1}^{k} \psi_1 \{|X(t_i) - X(t_{i-1})|\}$ whenever $\pi = \{0 = t_0 < t_1 < \cdots < t_k = 1\}$, we want to prove that $V_{\psi_1}(X, \pi)$ is bounded for all π but that it becomes unbounded if we replace ψ_1 by any function ψ such that $\psi(s)/\psi_1(s) \to \infty$ as $s \to 0$.

THEOREM 1. *If $Q(\delta)$ is the class of finite partitions π of* (0, 1) *for which the mesh $\sigma(\pi) < \delta$ and if ψ_1 is defined by* (2.12), *then almost all Brownian paths $X(t)$ in R^d satisfy*

$$\lim_{\delta\to 0} [\sup_{\pi\in Q(\delta)} V_{\psi_1}(X, \pi)] = 1.$$

Proof. To obtain the lower bound we use a Vitali covering argument. For a fixed $\epsilon > 0$ let

$$E_\delta = \{t \in (0, 1): \psi_1\{|X(t + h) - X(t)|\} > (1 - \epsilon)h \text{ for some } h \in (0, \delta)\}.$$

Then (2.16) shows that for each fixed t, δ, $P\{t \in E_\delta\} = 1$. A Fubini argument applied to the product of P in Ω and Lebesgue measure in (0, 1) shows that

$$P\{\,|E_\delta| = 1\} = 1.$$

But then $E = \bigcap_{1 \geq \delta > 0} E_\delta = \bigcap_{n=1}^{\infty} E_{1/n}$ so that

$$P\{\,|E| = 1\} = 1$$

and for each t in E there are arbitrarily small intervals $[t, t + h]$ such that

$$\psi_1\{|X(t + h) - X(t)|\} > (1 - \epsilon)h; \tag{4.1}$$

the family of all intervals satisfying (4.1) is a Vitali covering of the set E. We can therefore pick a finite subcollection of intervals of length less than δ which are disjoint but have total length at least $1 - \epsilon$. Let π be any partition of $[0, 1]$ in $Q(\delta)$ which includes in its subintervals all the finite disjoint sets, of total length at least $(1 - \epsilon)$, of intervals $(t_i, t_i + h_i)$ satisfying (4.1). Then

$$\begin{aligned} V_{\psi_1}(X, \pi) &\geq \sum_i \psi_1\{|X(t_i + h_i) - X(t_i)|\} \\ &> (1 - \epsilon) \sum h_i \quad \text{by} \quad (4.1) \\ &> (1 - \epsilon)^2. \end{aligned}$$

Hence for each $\epsilon > 0$ and $\delta > 0$ we have

$$P\{\sup_{\pi \in Q(\delta)} V_{\psi_1}(X, \pi) > (1 - \epsilon)^2\} = 1$$

and allowing ϵ and then δ to decrease to zero through a countable set gives

$$P\{\lim_{\delta \to 0} \sup_{\pi \in Q(\delta)} V_{\psi_1}(X, \pi) \geq 1\} = 1. \tag{4.2}$$

To obtain the opposite inequality is more difficult as we now have to consider all possible fine partitions and show that *none* of them can give rise to a large $V_{\psi_1}(X, \pi)$. For each $\epsilon > 0$ it is clearly sufficient to show that with probability 1 there is a $\delta > 0$ (depending on the path) such that

$$\sup_{\pi \in Q(\delta)} V_{\psi_1}(X, \pi) < 1 + 10\epsilon. \tag{4.3}$$

For any partition π we divide its subintervals into three classes.

(a) (t_{i-1}, t_i) is *good* if $\psi_1\{|X(t_i) - X(t_{i-1})|\} < (1 + \epsilon)(t_i - t_{i-1})$
(b) (t_{i-1}, t_i) is *medium* if $(1+\epsilon)(t_i - t_{i-1}) \leq \psi_1\{|X(t_i) - X(t_{i-1})|\} < 5(t_i - t_{i-1})$
(c) (t_{i-1}, t_i) is *bad* if $\psi_1\{|X(t_i) - X(t_{i-1})|\} \geq 5(t_i - t_{i-1})$

The sum defining $V_{\psi_1}(X, \pi)$ splits into the three parts $\sum'$, $\sum''$ and $\sum'''$ corresponding to the good, medium and bad subintervals respectively. Since the total length of good intervals is at most 1 we have immediately

$$\sum{}' \psi_1\{|X(t_i) - X(t_{i-1})|\} < 1 + \epsilon. \tag{4.4}$$

To deal with $\sum''$ it is sufficient to show that the total length of good intervals becomes close to 1 when $\pi \in Q(\delta)$ and δ is small enough. Put

$$\begin{aligned} R_{\epsilon,\eta} = \{t \in (0, 1) : \psi_1\{R(X; t - u, t + v)\} < 1 + \epsilon \text{ for all } (u, v) \\ \text{satisfying } u \geq 0, v \geq 0, 0 < u + v < \eta\}. \end{aligned}$$

By Lemma 3.3 we know that for fixed t

$$P\{t \in R_{\epsilon,\eta}\} \to 1 \quad \text{as} \quad \eta \to 0.$$

A Fubini argument applied to the indicator function of $R_{\epsilon,\eta}$ in $\Omega \times [0, 1]$ now yields

$$P\{\ |R_{\epsilon,\eta}| \to 1 \quad \text{as} \quad \eta \to 0\} = 1.$$

Hence with probability 1 we can find δ_1 such that

$$|R_{\epsilon,\eta}| > 1 - \epsilon$$

for all $\eta \leq \delta_1$. Hence if $\pi \in Q(\delta_1)$, each subinterval (t_{i-1}, t_i) which contains a point of R_{ϵ,δ_1} will be good and the total length of good subintervals is at least $1 - \epsilon$. Hence if $\sigma(\pi) < \delta_1$, then

$$\sum{}'' \psi_1\{|X(t_i) - X(t_{i-1})|\} < 5\epsilon. \tag{4.5}$$

It remains to show that the contribution of the bad intervals can also be made small. We do this by approximating each bad interval by a slightly larger one in a restricted family. Put

$$h_n = \exp(-n^{1-\epsilon})$$

and note that $h_{n+1}/h_n \to 1$ as $n \to \infty$. Let $\mathcal{J}_n$ consist of the class of intervals

$$J_{i,k} = [(in^{-\epsilon} + k)h_n, (in^{-\epsilon} + k + 1)h_n], \quad 0 \leq i \leq n^\epsilon, 0 \leq k \leq h_n^{-1}.$$

The number of intervals in the class $\mathcal{J}_n$ is about $n^\epsilon h_n^{-1}$, and we can find $\delta_2 > 0$ such that for any $0 \leq s < t \leq 1$ and $t - s < \delta_2$ we can find an interval

$$J_{i,k} = [s_n, s_n + h_n] \supset (s, t)$$

with $J_{i,k}$ in the class $\mathcal{J}_n$ but

$$|J_{i,k}| < (5/4)(t - s). \tag{4.6}$$

If the interval (s, t) is bad, it follows that

$$\psi_1\{R(X; s_n, s_n + h_n)\} \geq \psi_1\{|X(t) - X(s)|\} \geq 5\,|t - s| > 4h_n.$$

But by (2.8) for any s

$$P\{\psi_1\{R(X; s, s + h_n)\} > 4h_n\} < n^{-4+\epsilon}$$

so that if Z_n denotes the number of the intervals of $\mathcal{J}_n$ which satisfy

$$\psi_1\{R(X; s, s + h_n)\} > 4h_n, \tag{4.7}$$

then

$$E\{Z_n\} < n^\epsilon h_n^{-1} n^{-4+\epsilon}.$$

Since we may assume without loss of generality that $\epsilon < \frac{1}{4}$, an application of the Borel–Cantelli lemma now shows that with probability 1 there exists

n_0 such that

$$Z_n \leq n^{-2-\epsilon}h_n^{-1} \quad \text{for all} \quad n \geq n_0 . \tag{4.8}$$

With probability 1 we can choose $\delta_3 > 0$ by Lemma 2.1 such that for $0 \leq s < t \leq 1$ and $t - s < \delta_3$

$$|X(t) - X(s)|^2 < 3(t - s) \log^* (t - s). \tag{4.9}$$

Thus if $n \geq n_0$ and $h_n < \delta_3$, we have

$$\psi_1\{R(X; s, s + h_n)\} \leq 3h_n \log^* h_n$$

for every s, so that the sum of $\psi_1\{R(X; s, s + h_n)\}$ over the intervals of $\mathcal{J}_n$ which satisfy (4.7) is at most

$$n^{-2-\epsilon}h_n^{-1}3h_n \log^* h_n = 3n^{-1-2\epsilon}.$$

Now pick $n_1 > n_0$ such that $\sum_{n=n_1}^{\infty} 3n^{-1-2\epsilon} < \epsilon$, and we see that the sum of $\psi_1\{R(X; s, s + h_n)\}$ over all the intervals of $\mathcal{J}_n$ satisfying (4.7) for $n \geq n_1$ is at most ϵ. Since each small bad interval of π is contained in at least one of the $\mathcal{J}_n$ satisfying (4.7), we have proved that there exists δ_4 such that if $\pi \in Q(\delta_4)$, then

$$\sum\nolimits''' \psi_1\{|X(t_i) - X(t_{i-1})|\} < \epsilon.$$

This together with (4.4) and (4.5) establishes (4.3) for any π such that $\sigma(\pi) < \delta = \min (\delta_1 , \delta_2 , \delta_3 , \delta_4)$.

Remark. It is clear that we have also proved

$$\lim_{\delta\to 0} [\sup_{\pi \in Q(\delta)} \sum \psi_1\{R(X; t_{i-1} , t_i)\}] = 1$$

almost surely.

Now recall the definition of strong variation

$$W_\psi(X; 0, 1) = \sup_{\text{all } \pi} \sum \psi\{|X(t_i) - X(t_{i-1})|\}.$$

COROLLARY. *The strong variation $W_{\psi_1}(X; 0, 1)$ is a random variable satisfying $P\{1 \leq W_{\psi_1} < +\infty\} = 1$. If $\psi(s)$ satisfies $\psi(s)/\psi_1(s) \to +\infty$ as $s \to 0$, then W_ψ is infinite almost surely.*

Proof. The last sentence in the statement and $P\{W_{\psi_1} \geq 1\} = 1$ follow immediately from Theorem 1. To show that W_{ψ_1} is finite we first choose δ (depending on the path) such that $V_{\psi_1}(X, \pi) < 2$ when $\sigma(\pi) < \delta$. For a general partition π the number of the intervals of length at least δ cannot be more than δ^{-1} and each of these contributes at most $\psi_1\{R(X; 0, 1)\}$ to the sum. Hence for any π

$$V_{\psi_1}(X, \pi) < 2 + \delta^{-1}\psi_1\{R(X; 0, 1)\}.$$

Remark. It seems to be a difficult problem to find the distribution of the random variable $W_{\psi_1}(X; 0, 1)$.

5. The square variation in terms of the mesh. In the proof of the corollary to Theorem 1 we used the fact that $V_\psi(X, \pi)$ is bounded above for all those π whose subintervals have length at least δ. Hence

$$S_k(X) = \sup_{\pi \in \mathcal{P}_k} V_{\psi_2}(X, \pi) \tag{5.1}$$

is a random variable, where $\psi_2(s) = s^2$ and $\mathcal{P}_k$ is the class of partitions π of $(0, 1)$ satisfying

$$\min (t_i - t_{i-1}) \geq k^{-1}.$$

Greenwood [10] suggested the problem of finding the asymptotic growth rate of S_k ; this is solved precisely by the following theorem.

THEOREM 2. *Almost all Brownian paths $X(t)$ in R^d satisfy*

$$\lim_{k \to \infty} S_k(X)/\log \log k = 2.$$

Proof. The argument used to establish Theorem 1 requires only minor alterations to show that almost all paths satisfy

$$\lim_{\delta \to 0} \left[\sup_{\pi \in Q(\delta)} \sum_{t_i \in \pi} \frac{|X(t_i) - X(t_{i-1})|^2}{\log^* \log^* (t_i - t_{i-1})} \right] = 2.$$

For any $\epsilon > 0$ we can therefore find δ (depending on the path) such that for every $\pi \in Q(\delta)$

$$\sum_{t_i \in \pi} \frac{|X(t_i) - X(t_{i-1})|^2}{\log^* \log^* (t_i - t_{i-1})} < 2 + \epsilon.$$

If $(t_i - t_{i-1}) \geq k^{-1}$, $\log^* \log^* (t_i - t_{i-1}) \leq \log \log k$ so that

$$\sup_{\pi \in Q(\delta) \cap \mathcal{P}_k} V_{\psi_2}(X, \pi) < (2 + \epsilon) \log \log k. \tag{5.2}$$

But now if $\pi \in \mathcal{P}_k$ and contains some intervals of length at least δ, then the total contribution from such intervals is at most

$$\delta^{-1}\{R(X; 0, 1)\}^2 < \epsilon \log \log k \tag{5.3}$$

for $k \geq k_0 = k_0(\omega)$. (5.2) and (5.3) give

$$\frac{S_k(X)}{\log \log k} \leq 2 + 2\epsilon$$

for $k \geq k_0$. Since ϵ is arbitrary we have therefore deduced that almost all paths satisfy

$$\limsup_{k \to \infty} \frac{S_k(X)}{\log \log k} \leq 2. \tag{5.4}$$

To get a result in the other direction we cannot use a Vitali theorem since we want covers by intervals of length at least k^{-1} and we will also want the intervals of the partition to have lengths which are not too much greater than k^{-1}; we have to achieve such partitions with large $V_{\psi_2}(X, \pi)$ for every sufficiently large integer k. Fortunately log log k grows very slowly so we can be satisfied with quite crude estimates. Consider first the subsequence of integers $n_r = 2^r$. Given $\epsilon > 0$ for all sufficiently large r we show that there exists a π in which all the subintervals are of the form $(in_r^{-1}, (i + j)n_r^{-1})$ with i, j positive integers and

$$V_{\psi_2}(X, \pi) > (2 - 6\epsilon) \log \log n_r . \tag{5.5}$$

Since ϵ is arbitrary this clearly implies that

$$\liminf_{r\to\infty} \frac{S_{2^r}(X)}{\log \log 2^r} \geq 2.$$

Using the monotonicity of S_k then gives

$$\liminf_{k\to\infty} \frac{S_k(X)}{\log \log k} \geq 2$$

which completes the proof of the theorem with (5.4). It remains to establish that with probability 1, (5.5) is true for all sufficiently large r.

Suppose T_r is the number of integers i such that

$$\sup_{1 \leq j \leq n_r^{9/10}} \frac{\left|X\left(\frac{i+j}{n_r}\right) - X\left(\frac{i}{n_r}\right)\right|^2}{\frac{j}{n_r} \log \log n_r} < 2 - 2\epsilon. \tag{5.6}$$

For $r \geq r_1$ the slow growth of log log k ensures that (5.6) implies

$$\sup_{1 \leq j \leq n_r^{9/10}} \frac{\left|X\left(\frac{i+j}{n_r}\right) - X\left(\frac{i}{n_r}\right)\right|^2}{\frac{j}{n_r} \log \log \frac{n_r}{j}} < 2 - \epsilon$$

and by Lemma 3.2(i) this event has probability at most exp $(-r^\gamma)$ provided $r \geq r_2$. Hence the event

$$\{T_r > \epsilon n_r\}$$

occurs finitely often, so we can find r_3 such that $T_r \leq \epsilon n_r$ for all $r \geq r_3$. Now for $r \geq r_3$ we can form a partition π using mostly intervals with endpoints in_r^{-1} for which V_{ψ_2} is large. To do this start at 0 and pick the division points by induction. If $t_i = in_r^{-1}$ has been picked and i is such that (5.6) is false, pick the first integer j such that

$$\left|X\left(\frac{i+j}{n_r}\right) - X\left(\frac{i}{n_r}\right)\right|^2 \geq (2 - 2\epsilon) \frac{j}{n_r} \log \log n_r \tag{5.7}$$

and put $t_{i+1} = (i + j)n_r^{-1}$. On the other hand, if i is such that (5.6) is true, put $t_{i+1} = (i + 1)n_r^{-1}$. Since $T_r \geq \epsilon n_r$, the total length of intervals satisfying

(5.7) will be at least $(1 - 2\epsilon)$ so that (5.5) is established for this particular π. This completes the proof.

6. Weak variation of Brownian motion. In § 4 we examined the large values (as π varies) of

$$V_\psi'(X, \pi) = \sum \psi\{R(X; t_{i-1}, t_i)\}$$

which turn out to be the same as the large values of $V_\psi(X, \pi)$. Our object in the present section is to look at the small values of $V_\psi'(X, \pi)$ in two different ways. The result corresponding to Theorem 1 is the following.

THEOREM 3. *For a Brownian path $X(t)$ in R^d there is probability 1 that*

$$\lim_{\delta \to 0} [\inf_{\pi \varepsilon Q(\delta)} V_{\psi_3}'(X, \pi)] = \lambda_d ,$$

where λ_d is a finite positive constant ($\lambda_1 = 2\pi^{-2}$).

Proof. We first assume $d = 1$ and then indicate how the argument can be completed for $d \geq 2$. The argument for the lower bound is similar to that used in Theorem 1 to show that the total length of "good" intervals is close to one when $\sigma(\pi)$ is small. Lemma 3.4(i) together with a Fubini argument shows that if δ is small enough, then

$$\left|\left\{t \varepsilon (0, 1): \inf_{\substack{u \geq 0, v \geq 0 \\ 0 < u+v < \delta}} \frac{\psi_2\{R(X; t-u, t+v)\}}{u+v} > (\lambda_1 - \epsilon)\right\}\right| > 1 - \epsilon. \tag{6.1}$$

This implies that the contribution to $V_{\psi_3}'(X, \pi)$ will be large for most intervals of π; to be precise, if $\sigma(\pi) < \delta$, then

$$V_{\psi_3}'(X, \pi) > (\lambda_1 - \epsilon)(1 - \epsilon).$$

Since ϵ is arbitrary we have with probability 1

$$\lim_{\delta \to 0} [\inf_{\pi \varepsilon Q(\delta)} V_{\psi_3}'(X, \pi)] \geq \lambda_1 . \tag{6.2}$$

The argument in the other direction is a simplified version of that used for the lower bound in Theorem 2. For a fixed integer n we call the division point in^{-1} good if there is an integer j, $1 \leq j \leq n^{9/10}$, such that

$$\psi_3\{R(X; in^{-1}, (i + j)n^{-1})\} < (\lambda_1 + \epsilon)jn^{-1}; \tag{6.3}$$

otherwise in^{-1} is called bad. By Lemma 3.2(ii) the number of bad points Y_n is a random variable such that

$$E\{Y_n\} < n \exp\{-(\log n)^\gamma\} \tag{6.4}$$

for some suitable $\gamma > 0$. By Lemma 2.1 there is an integer n_1 such that for $n \geq n_1$ we have

$$\psi_3\{R(X; in^{-1}, (i + 1)n^{-1})\} < 3n^{-1} \log n \log \log n. \tag{6.5}$$

By a Borel–Cantelli argument applied to the estimate (6.4) there is an integer n_2 such that if $n = 2^k > n_2$, then

$$Y_n < n\ (\log n)^{-2}.$$

Hence for n of the form 2^k and large enough we can find a partition π in which there are not more than $n(\log n)^{-2}$ bad subintervals each of length n^{-1} and the good subintervals satisfy (6.3) and have length less than $n^{-1/10}$. For such a π

$$V'_{\psi_3}(X; \pi) = \sum{}' \psi_3\{R(X; t_i, t_{i-1})\} + \sum{}'' \psi_3\{R(X; t_i, t_{i-1})\},$$

where $\sum'$ denotes the sum over intervals satisfying (6.3) and $\sum''$ denotes the sum over bad intervals which still satisfy (6.5). Hence

$$V'_{\psi_3}(X; \pi) \leq \lambda_1 + \epsilon + \frac{3 \log \log n}{\log n}$$

$$< \lambda_1 + 2\epsilon \quad \text{if} \quad n \quad \text{is large enough.}$$

Since ϵ is arbitrary this, together with (6.2), completes the proof for $d = 1$.

The same arguments for $d \geq 2$ yield

$$(4q_d)^{-1} \leq \lim_{\delta \to 0} [\inf_{\pi \in Q(\delta)} V'_{\psi_3}(X, \pi)] \leq q_d^{-1} \tag{6.6}$$

with probability 1. But now it is clear that we can form a process

$$Y(s) = \lim_{\delta \to 0} [\inf_{\pi_s \in Q(\delta)} V'_{\psi_3}(X, \pi_s)],$$

where π_s is a dissection of the interval $(0, s)$; this process Y is continuous and strictly increasing by (6.6) but has independent increments. By Lemma 3.1 there exists a constant λ_d such that $(4\ q_d)^{-1} \leq \lambda_d \leq q_d^{-1}$ and so $Y(s) = \lambda_d s$. In particular $Y(1) = \lambda_d$.

COROLLARY. *The weak ψ_3-variation of X over* $(0, 1)$, $U_{\psi_3}(X; 0, 1)$, *defined by* (1.3) *is a random variable satisfying*

$$P\{0 < U_{\psi_3}(X; 0, 1) \leq \lambda_d\} = 1.$$

If $\psi(s)/\psi_3(s) \to 0$ as $s \to 0$, then $U_\psi(X; 0, 1) = 0$ with probability 1.

Proof. The only statement which does not follow immediately from the theorem is that $U_{\psi_3}(X; 0, 1) > 0$ with probability 1. To prove this first choose δ such that

$$\inf_{\pi \in Q(\delta)} V'_{\psi_3}(X, \pi) > \tfrac{1}{2}\lambda_d .$$

Then if $t - s > \delta$, $0 \leq s < t \leq 1$, we have

$$R(X; s, t) \geq \tau > 0$$

so that if π contains at least one interval (s, t) of length greater than δ, then

$$V'_{\psi_3}(X, \pi) > \psi_3\{R(X; x, t)\} \geq \tau^2 > 0.$$

Hence

$$U_{\psi_2}(X; 0, 1) \geq \min(\tau^2, \tfrac{1}{2}\lambda_d) > 0.$$

The same changes which yielded Theorem 2 from Theorem 1 can be applied to the small variation to give the following theorem.

THEOREM 4. *If $X(t)$ is a Brownian path in R^d, there is probability* 1 *that*

$$\lim_{n\to\infty} [\inf_{\pi \varepsilon \mathcal{P}_n} (\log \log n) V'_{\psi_2}(X; \pi)] = \rho_d ,$$

where ρ_d is a finite positive constant ($\rho_1 = 2\pi^{-2}$).

Although it is clear that we have asked the natural interesting questions about the small values of $V_\psi(X; \pi)$ when $d = 1$, there are other questions which may be more natural when d is large. For $d = 2$ or 3 the path has double points (see [6]) so it seems plausible that

$$\inf_{\pi \varepsilon Q(\delta)} V_\psi(X; \pi) = 0 \tag{6.7}$$

for every $\psi(s)$ such that $s^{-1}\psi(s) \to 0$ as $s \to 0$ though I have not proved this rigorously. For $d = 4$ double points just fail to happen, while for $d \geq 5$ the process "escapes" rapidly from each point (see [5]). I can prove that for $d \geq 5$ if $\psi(s) = s^2 (\log^* s)^r$, then $\inf_{\pi \varepsilon Q(\delta)} V_\psi(X, \pi) \to \infty$ as $\delta \to 0$ when $r > 2(d - 4)^{-1}$; but I have not been able to characterize the functions ψ for which (6.7) is true.

7. Weak variation for other processes. For a survey of sample path properties and definitions relating to processes with stationary independent increments the reader is referred to [18]. If the Lévy measure of the distribution is infinite, the jumps will be everywhere dense in time and, in a certain sense, will dominate the behaviour of the large values of $V_\psi(X, \pi)$. (See Blumenthal and Getoor [1] and [2] for early results on variation and Millar [13] for some recent results.) We were able to get precise information about sup $V_\psi(X, \pi)$ in the case of Brownian motion because the upper tail is negative exponential; this is not true for other processes so that one would not expect any "correct" function for $W_\psi(X; 0, 1)$. The lower tail of the distribution of $R(h)$ will always be thin when the Lévy measure is infinite, but its exact asymptotic form is only known for a few special classes of processes. We prove an analogue of Theorem 3 for two important special cases.

For any subordinator Fristedt and Pruitt [8] gave a construction of a precise function $f(s)$ (explicitly defined in terms of the exponent of the process) such that the Hausdorff f-measure of the path set $P(0, 1)$ is a constant c with probability 1. This same function f is correct for measuring the weak variation.

THEOREM 5. *If $X(t)$ is a subordinator corresponding to an infinite Lévy measure and if f is a Hausdorff measure function such that $f - m\{P(0, 1)\} = c$ with probability* 1, *then for almost all paths*

$$\lim_{\delta\to 0} [\inf_{\pi \varepsilon Q(\delta)} V'_f(X, \pi)] = c.$$

Remark. When we defined $R(X; a, b)$ in (1.4) we were careful to omit the value $X(b)$ from the set considered. This made no difference in the Brownian motion case (where the paths are continuous) but is vital now as it allows us not to "cover" the jumps. We will see in the course of the proof that our definition of weak variation is such that for any $\delta > 0$ no jump of size greater than δ contributes anything when we are looking for dissections π which make $V_f'(X, \pi)$ small.

Proof. Let $\bar{P}(0, 1)$ be the closure of the path set $P(0, 1)$, where

$$P(s, t) = \{x \in R^1 : x(\tau) = x \quad \text{for some} \quad \tau, s \le \tau \le t\}.$$

Then $\bar{P}(0, 1)$ differs from $P(0, 1)$ by a countable set, namely, the set of values of $X(t - 0)$ at the jumps $t = t_k$ of the process. Hence, almost surely

$$f - m\,\bar{P}(0, 1) = f - m\,P(0, 1) = c.$$

Since the process $X(t)$ is strictly increasing, for any $\delta > 0$ there is an $\eta > 0$ (depending on the path) such that

$$X(t) - X(s) < \eta \Rightarrow t - s < \delta. \tag{7.1}$$

For any $\epsilon > 0$ we can cover $\bar{P}(0, 1)$ by a set of open intervals I_j with $|I_j| < \eta$ and

$$\sum f(|I_j|) < c + \epsilon.$$

Since $\bar{P}(0, 1)$ is compact we may assume that the covering is finite. Now replace the open intervals I_j, one at a time, by half open intervals

$$K_j = [\alpha_j, \beta_j) \subset I_j$$

with the property that α_j is the largest real number for which the covering property is preserved and β_j is the right-hand endpoint of I_j (K_j will be empty if I_j is not needed for the covering). We can assume that the intervals K_j are disjoint, cover $\bar{P}(0, 1)$ and satisfy

$$\sum_{j=1}^{n} f(|K_j|) < c + \epsilon.$$

But if $s_j = \inf\{t : X(t) \in K_j\}$ and $t_j = \sup\{t : X(t) \in K_j\}$, then it is clear that

$$P(s_j, t_j) \subset K_j$$

so that

$$R(X; s_j, t_j) \le |K_j|$$

and the intervals $[s_j, t_j)$ form a finite dissection π of $(0, 1)$. Since $|K_j| < \eta$, (7.1) gives $\sigma(\pi) < \delta$. Thus with probability 1 we have found a π in $Q(\delta)$ such that

$$V_f'(X; \pi) < c + \epsilon.$$

Since ϵ and δ are arbitrary, we have proved that

$$\lim_{\delta\to 0} [\inf_{\pi \varepsilon Q(\delta)} V_f'(X, \pi)] \le c \tag{7.2}$$

almost surely.

In order to show that f-measure is small, we have to produce an economical covering by sets of diameter less than η for every $\eta > 0$. The converse of (7.1) is false because of the jumps, so we need to show that we can make $V_f'(X, \pi)$ small while using π such that *all* the $R(X; t_{i-1}, t_i)$ are small. Note that

$$X(1) = \sum_{0 \le t \le 1} J_t ,$$

where $J_t = X(t) - X(t - 0)$ is the jump at time t. If we arrange the jumps in order of decreasing size, then we can find a finite set $s_1, s_2, \cdots, s_k$ of discontinuity points (which will include all jumps greater than η) for which

$$\sum_{i=1}^{k} J_{s_i} > X(1) - \eta. \tag{7.3}$$

Now choose δ small enough to ensure that no interval (s, t) of length less than 2δ contains more than one of the set s_i, $i = 1, 2, \cdots, k$, and in addition

$$\begin{aligned} &f\{R(X; s_i - \delta, s_i)\} < \tfrac{1}{2}f(\rho) \\ &f\{R(X; s_i, s_i + \delta)\} < \tfrac{1}{2}f(\rho) \end{aligned} \qquad i = 1, 2, \cdots, k \tag{7.4}$$

where ρ is the smallest of the J_{s_i}, $1 \le i \le k$.

Now if π is such that $\sigma(\pi) < \delta$ and (t_{i-1}, t_i) is a typical subinterval of π, then either (t_{i-1}, t_i) contains none of the s_j as interior points in which case (7.3) implies that

$$X(t_i - 0) - X(t_{i-1}) < \eta \tag{7.5}$$

or $t_{i-1} < s_j < t_i$ for exactly one s_j in which case

$$f\{R(X; t_{i-1}, s_j)\} + f\{R(X; s_j, t_i)\} < f(\rho) < f\{R(X; t_{i-1}, t_i)\}$$

and we can therefore reduce $V_f'(X; \pi)$ by adding the point s_j to the partition.

Hence, for any $\epsilon > 0$, $\eta > 0$ we can choose a $\pi \varepsilon Q(\delta)$ which contains all the jumps s_j and is such that

$$V_f(X; \pi) \le \lim_{\delta\to 0} [\inf_{\pi \varepsilon Q(\delta)} V_f'(X, \pi)] + \epsilon.$$

By (7.5) the intervals

$$K_i = [X(t_{i-1}, X(t_i - 0))$$

form a covering of $P(0, 1)$ by intervals of length less than η such that

$$\sum_{i=1}^{n} f(|K_i|) = V_f(X; \pi).$$

We have therefore proved that

$$f - mP(0, 1) \leq \lim_{\delta \to 0} [\inf_{\pi \in Q(\delta)} V_f'(X, \pi)]$$

and this together with (7.2) completes the proof of the theorem.

We now look at stable processes which satisfy a scaling property. They were classified in [17] according to the value of the continuous transition density $p(t, x)$ at $x = 0$; a stable process is of type A if $p(t, 0) > 0$ and otherwise it is of type B. For a suitable line the projection on that line of a type B process will be a subordinator. This means that the lower tail of the distribution of $R(1)$ is of smaller order for type B processes than for type A processes, so for a given order $\alpha < 1$ we would expect the correct function for measuring weak variation to be smaller for the type B process than for the type A process. Asymptotic bounds for the lower tail of $M(h)$, from which one can deduce bounds for the tail of $R(h)$ by using (2.2), were obtained in [17].

THEOREM 6. *Suppose $X(t)$ is a stable process of index α in R^d. Put*

$$\psi_4(s) = \begin{cases} s^\alpha \log^* \log^* s & \text{if the process is of type A} \\ s^\alpha (\log^* \log^* s)^{1-\alpha} & \text{if the process is of type B.} \end{cases}$$

Then there is a finite positive constant c such that

$$\lim_{\delta \to 0} [\inf_{\pi \in Q(\delta)} V_{\psi_4}'(X; \pi)] = c$$

with probability 1.

Proof. A positive lower bound can be obtained by exactly the same argument as was used in Theorem 3 by using a two-sided estimate for the small values of $R(X; t - u, t + v)$ which will have the same form as the one-sided asymptotic laws obtained in Theorem 4(iii) and Theorem 5(iii) of [17]. To obtain a finite upper bound we need to modify the argument used to deal with the "bad" points i/n, because we have no result corresponding to Lemma 2.1 for Brownian motion to which we can appeal. We can use instead an idea which yielded economical small covers of the path to give finite Hausdorff measure.

For the upper bound there is no loss in generality in assuming $\alpha < d$; since if $\alpha \geq d$, we could take three independent copies in orthogonal subspaces of R^{3d} and this process in R^{3d} would have the same index α and larger variations $V_{\psi_4}'(X, \pi)$. The condition $\alpha < d$ implies that the process is transient. Cover the space with a mesh $\wedge_r$ of dyadic cubes of side 2^{-r} and centers at $(i_1 2^{-r}, i_2 2^{-r}, \cdots, i_d 2^{-r})$. By Lemma 6.1 of [14] if S_r is the number of cubes of $\wedge_r$ entered by $P(0, 1)$, we have

$$E\{S_r\} < c2^{rd}. \tag{7.6}$$

Let C_r be a typical cube of $\wedge_r$ which is hit by $P(0, 1)$ and suppose τ_1 is the first hitting time of C_r. For integers $k \geq 2$ let C_r^k be the cube with the same center as C_r and side length k times as long and let $[s_k, t_k)$ be the largest interval such that $s_k \leq \tau_1 < t_k$ and

$$P(s_k, t_k) \subset C_r^k.$$

Then

$$R(X; s_k, t_k) \leq k2^{-r} d^{\frac{1}{2}}. \tag{7.7}$$

Now if c is small enough, there is a $\gamma > 0$ such that

$$E = \left\{ \inf_{2 \leq k < 2^{\frac{1}{2}r}} \frac{\psi_4\{R(X; s_k, t_k)\}}{t_k - s_k} > c \right\}$$

will happen with probability at most $\exp(-r^\gamma)$. This can be proved directly or by inverting Lemma 4.5 of [14] and using (2.2). A cube C_r is said to have a bad entry at τ_1 if this event E happens. In this case we use the interval (s_2, t_2) and the estimate (7.7). If E does not happen, then we use an interval (s_k, t_k), $2 \leq k \leq 2^{\frac{1}{2}r}$, for which

$$\psi_4\{R(X; s_k, t_k)\} \leq c(t_k - s_k). \tag{7.8}$$

Given an entry τ_1 to C_r, we obtain in this way a cube C_r^k and a $t_k > \tau_1$ such that $X(t) \in C_r^k$ for $\tau_1 \leq t < t_k$; the original C_r may be hit again for times $t \geq t_k$, but since the process is transient, this will happen with probability less than $p < 1$ so that the number of such re-entries τ_i to C_r is a random variable with distribution bounded by the geometric distribution. Hence if T_r denotes the total number of entries or re-entries to cubes of the lattice $\wedge_r$, we can strengthen (7.6) to

$$E\{T_r\} < c2^{-r\alpha}.$$

If Z_r is the total number of "bad" entries to cubes of $\wedge_r$, then

$$E\{Z_r\} < c2^{-r\alpha} \exp(-r^\gamma).$$

A Borel–Cantelli argument now shows that there exists n_0 such that if $r \geq n_0$, then

$$Z_r < 2^{-r\alpha} (\log r)^{-2}$$

and therefore by (7.7) the total contribution of all "bad" entries into cubes C_r is at most

$$\sum\nolimits'' \psi_4\{R(X; s_2, t_2)\} < c\, \frac{\log^* \log^* 2^{-r}}{(\log r)^2} < \epsilon \tag{7.9}$$

provided $r \geq n_1$. Using (s_k, t_k) obtained from all the entries or re-entries of cubes C_r gives a covering of $(0, 1)$. We can replace this by a finite collection $\mathfrak{C}$ of intervals which cover no point more than twice. For any $\delta > 0$ we can find $\eta > 0$ such that

$$R(X; s, t) < \eta \Rightarrow (t - s) < \delta$$

so that if $r \geq n_2$, we will have for all entries

$$t_k - s_k < \delta.$$

Hence, if we form π by taking all the endpoints of this finite collection $\mathcal{C}$, we will have $\pi \varepsilon Q(\delta)$ and

$$\sum{}' \psi_4\{R(X; s_k, t_k)\} \leq c \sum (t_k - s_k) < 2c$$

using (7.8). This together with (7.9) yields

$$V'_{\psi_4}(X; \pi) < 2c + \epsilon$$

and we have a finite upper bound for

$$\lim_{\delta \to 0} [\inf_{\pi \varepsilon Q(\delta)} V'_{\psi}(X; \pi)].$$

An application of Lemma 3.1 is now sufficient to complete the proof of the theorem.

COROLLARY. *If $X(t)$ is a stable process in R^d and if ψ_4 is the function defined in the theorem, then*

$$P\{0 < U_{\psi_4}(X; 0, 1) \leq c\} = 1,$$

where $U_{\psi_4}(X; 0, 1)$ denotes the weak ψ_4-variation of $P(0, 1)$ defined in (1.3).

Remark. The correct function ψ_4 for measuring the weak variation of a stable process is precisely the same as the correct Hausdorff measure function for $P(0, 1)$ found in [17] when $\alpha < d$. When the process is transient it seems likely that the two problems of finding the right function for weak variation and the right function for Hausdorff measure of the path will be equivalent for any independent increment process though the constant will only be the same for processes which are close to subordinators.

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