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A multivariate central limit theorem for continuous local martingales

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Abstract

A theorem on the weak convergence of a properly normalized multivariate continuous local martingale is proved. The time-change theorem used for this purpose allows for short and transparent arguments.

Keywords: Multivariate central limit theorem; Continuous martingales; Weak convergence; Time-change device; Nested filtrations; Stable convergence

1. Introduction

In this paper we study the convergence of a d-dimensional continuous local martingale M as 'time' tends to infinity. We suppose that there exist normalizing matrices K_t such that as $t \to \infty$, we have $\|K_t\| \to 0$ and

$$K_t \langle M \rangle_t K_t^{\mathrm{T}} \xrightarrow{\mathrm{P}} \eta\eta^{\mathrm{T}}, \tag{1}$$

where η is some random matrix (T denotes transposition and $\|\cdot\|$ is a certain matrix norm, see the beginning of Section 4). Our main result, Theorem 4.1, states that under this condition, we have weak convergence of the normalized martingale $K_t M_t$ to a mixture of normals.

Recently, a similar result has been reported by Küchler and Sørensen (1999) (see also the book Küchler and Sørensen (1997)). In their setup, unlike in the present paper, M is a square integrable martingale (not necessarily continuous) with covariance matrices $\Sigma_t = \mathrm{E}(M_t M_t^{\mathrm{T}})$, that determine the normalization in (1) via the additional assumption that there exists a positive-definite limit of $K_t \Sigma_t K_t^{\mathrm{T}}$ as $t \to \infty$. The latter assumption is typically tedious to verify in practice. It seems therefore worthwhile to notice once again that in the special case of our concern, when M is continuous, all we need is assumption (1). This is a non-trivial improvement. Our result shows for instance that the analysis needed on p. 489 of Küchler and Sørensen (1999) to check condition (d) of their Theorem 2.1 is unnecessary.

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The result of Küchler and Sørensen is in fact a Cramér–Wold extension of a one-dimensional result in Feigin (1985). We will use the same device in the course of proving Theorem 4.1 to reduce the statement of the theorem to a statement about one-dimensional local martingales. However, our basic one-dimensional results rely on totally different arguments than those of Feigin (1985). The fact that we focus on continuous martingales allows for using a so-called *time-change device*, by viewing each one-dimensional continuous local martingale as a time-changed Brownian motion. In this way, a statement about continuous local martingales reduces to the corresponding statement about Brownian motions. The time-change device is a quite powerful tool in general and indeed, it leads to short and transparent proofs of the one-dimensional results presented in Section 3.

This paper is organized as follows. In Section 2, we introduce some notation and we recall the time-change theorem mentioned above. In Section 3, we focus first on the simplest particular case of a one-dimensional martingale whose quadratic variation satisfies condition (1) with deterministic η (or rather its one-dimensional analogue (5) in Section 3). A simple time-change argument yields a limit result in this case, see (8) below. In the remainder of Section 3 we discuss how to handle the general case of a random limit η in (5). This leads us to treating the so-called *nested* sequences of local martingales, see Corollary 3.2 for a limit result on such sequences that is a consequence of Theorem 3.1 on nested Brownian motions. This result provides the main argument in the proof of our Theorem 4.1 in Section 4.

2. Preliminaries

We will consider continuous random processes as random elements of the space $C[0,\infty)$. This is the space of all continuous functions $f:[0,\infty)\to\mathbb{R}$, endowed with the metric

$$d(f,g)=\sum_{n=1}^{\infty}2^{-n}\left(\max_{t\leqslant n}|f(t)-g(t)|\wedge 1\right).$$

Under this metric, $C[0,\infty)$ is a Polish space. By $C_0^+[0,\infty)$ we denote the subspace of $C[0,\infty)$ that consists of all functions f that are non-decreasing and start in 0, i.e. $f(0)=0$. Since $C_0^+[0,\infty)$ is closed in $C[0,\infty)$, it is a Polish space. We will use the fact that the following maps are continuous ($\mathscr{X}$ is an arbitrary Polish space):

$$\phi:C[0,\infty)\times[0,\infty)\to\mathbb{R},\quad \phi(f,t)=f(t), \tag{2}$$

$$\psi:C[0,\infty)\times[0,\infty)\to C[0,\infty),\quad \psi(f,t)=f(t+\cdot)-f(\cdot), \tag{3}$$

$$\xi:C[0,\infty)\times[0,\infty)\times\mathscr{X}\to\mathbb{R}\times\mathscr{X},\quad \xi(f,t,x)=(f(t),x). \tag{4}$$

Weak convergence and convergence in probability in Polish spaces are denoted by the symbols $\rightsquigarrow$ and $\xrightarrow{\mathrm{P}}$, respectively.

In this paper we consider local martingales $M=\{M_t\}_{t\geqslant 0}$ with continuous sample paths $t\mapsto M_t$. All local martingales M are assumed to start in 0, i.e. $M_0=0$. Throughout this section and the next, local martingales are one-dimensional. We assume that all filtrations satisfy the so-called *usual conditions* (see Karatzas and Shreve, 1991, Definition 2.25). This technical assumption assures the existence of the quadratic variation process $\langle M\rangle$ of a continuous local martingale M. We can consider a one-dimensional continuous local martingale M as a random element of $C[0,\infty)$ and its quadratic variation $\langle M\rangle$ as a random element of $C_0^+[0,\infty)$.

The following well-known theorem will play a central role in the sequel. It states that each continuous local martingale can be embedded in a Brownian motion. A proof of this theorem can be found in Karatzas and Shreve (1991, Theorem 3.4.6 and Problem 3.4.7).

Theorem 2.1 (Time-change theorem). *Let $M=(M_t,\mathscr{F}_t\colon t\geqslant 0)$ be a continuous local martingale and for $s\geqslant 0$, define*

$$\tau_s=\inf\{t\geqslant 0\colon \langle M\rangle_t>s\},\quad \mathscr{G}_s=\mathscr{F}_{\tau_s}.$$

The underlying probability space can be suitably extended in order to support a Brownian motion W with respect to the filtration $\{\mathscr{G}_t\}$, such that a.s.

$$M_t=W_{\langle M\rangle_t},\quad \forall t\geqslant 0.$$

Remark 2.2. See Karatzas and Shreve (1991, Remark 4.1 on p. 169) for the exact construction of the extended probability space. It is important to note that the extension does not change the law of the local martingale. In this paper we study properties of sequences of such laws and therefore we may assume that each continuous local martingale M in question is embedded in a Brownian motion W in the sense of the above theorem. We will call W *the Brownian motion corresponding to* M.

3. Nested local martingales

The main argument used in the course of the proof of Theorem 4.1 will be presented at the end of this section, see Corollary 3.2 concerning nested sequences of continuous local martingales. In order to explain why it is useful to treat such nested sequences, we first consider the following special case.

Let M be a one-dimensional continuous local martingale. Suppose that for a certain non-negative number η and positive numbers k_t

$$\frac{\langle M\rangle_t}{k_t}\xrightarrow{\mathrm{P}}\eta\quad \text{as } t\to\infty. \tag{5}$$

Let W be the Brownian motion corresponding to M. For each $t\geqslant 0$ define the process W^t by putting $W^t_s=W_{k_ts}/\sqrt{k_t}$, for all $s\geqslant 0$. Then the scaling property of Brownian motion implies that each process W^t is again a Brownian motion. For all $t\geqslant 0$ we have

$$\frac{M_t}{\sqrt{k_t}}=W^t_{\langle M\rangle_t/k_t}.$$

Each W^t is a Brownian motion, so we have $W^t\rightsquigarrow B$, where B is a Brownian motion. Since η is deterministic, we have the implication

$$\left[W^t\rightsquigarrow B,\ \frac{\langle M\rangle_t}{k_t}\xrightarrow{\mathrm{P}}\eta\right]\Rightarrow\left[\left(W^t,\frac{\langle M\rangle_t}{k_t}\right)\rightsquigarrow(B,\eta)\right]. \tag{6}$$

By the continuous mapping theorem it thus follows that for any continuous map ϕ

$$\phi\left(W^t,\frac{\langle M\rangle_t}{k_t}\right)\rightsquigarrow\phi(B,\eta). \tag{7}$$

In the special case of the map ϕ defined by (2) the left-hand side is equal to $M_t/\sqrt{k_t}$ and the right-hand side equals B_η, so (7) yields

$$\frac{M_t}{\sqrt{k_t}}\rightsquigarrow \mathrm{N}(0,\eta). \tag{8}$$

Hence in this simple case the time-change device already gives us a desired result, a central limit theorem for the normalized martingale $M_t/\sqrt{k_t}$. But when η is random, the matter is more complicated. Then it is not a priori clear whether or not we have implication (6). We will prove Theorem 3.1 which tells us that owing to the special *nesting relation* between the Brownian motions W^t, they are asymptotically independent of $\langle M\rangle_t/k_t$. This means that in the case of a random η implication (6) also holds, with B a Brownian motion that is independent of η.

It will be convenient to formulate the nesting condition in terms of filtrations. For all $n \in \mathbb{N}$, let $\{\mathscr{F}_t^n\}_{t\geqslant 0}$ be a filtration on the probability space $(\Omega, \mathscr{F}, P)$. Following Feigin (1985), we call the sequence $\{\mathscr{F}_t^n\}_{t\geqslant 0}$ *nested* if there exists a sequence $t_n \downarrow 0$ such that

$$\mathscr{F}^n_{t_n} \subseteq \mathscr{F}^{n+1}_{t_{n+1}}$$

for all $n \in \mathbb{N}$, and

$$\bigvee_{n=1}^{\infty} \mathscr{F}^n_{t_n^n} = \bigvee_{n=1}^{\infty} \mathscr{F}^n_{\infty}.$$

Any sequence $t_n \downarrow 0$ for which these conditions are satisfied is called an N-sequence. A sequence of adapted processes $X^n = (X_t^n, \mathscr{F}_t^n\colon t\geqslant 0)$ on $(\Omega, \mathscr{F}, P)$ is called nested if the corresponding filtrations $\{\mathscr{F}_t^n\}_{t\geqslant 0}$ are nested. As in Feigin (1985) and Hall and Heyde (1980), this nesting condition will lead to so-called *stable* limit results (see Remark 4.2).

We will obtain a limit result for nested continuous local martingales as a corollary of the following theorem concerning nested Brownian motions, that turn out to be asymptotically independent of any other random element.

Theorem 3.1. *Let $W^n = (W_t^n, \mathscr{F}_t^n\colon t\geqslant 0)$ be a sequence of Brownian motions on the probability space $(\Omega, \mathscr{F}, P)$. If for all $n \in \mathbb{N}$ there exists an $\{\mathscr{F}_t^n\}$-stopping time τ_n such that*
(i) $\tau_n \xrightarrow{\mathrm{P}} 0$,
(ii) $\mathscr{F}^n_{\tau_n} \subseteq \mathscr{F}^{n+1}_{\tau_{n+1}}$ $\forall n \in \mathbb{N}$,
(iii) $\bigvee_{n=1}^{\infty} \mathscr{F}^n_{\tau_n} = \bigvee_{n=1}^{\infty} \mathscr{F}^n_{\infty}$,
then, for all random elements X on $(\Omega, \mathscr{F}, P)$ with values in a Polish space $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$, we have $(W^n, X) \rightsquigarrow (W, X)$, where W is a Brownian motion that is independent of X.

Proof. For $n \in \mathbb{N}$ define the process V^n by

$$V_t^n = W^n_{\tau_n + t} - W^n_{\tau_n}, \quad t \geqslant 0.$$

Note that we have

$$W^n - V^n = \phi(W^n, \tau_n) - \psi(W^n, \tau_n), \tag{9}$$

where ϕ and ψ are given by (2) and (3), respectively. Of course, the processes W^n converge weakly to a Brownian motion W. Together with assumption (i) this implies that $(W^n, \tau_n) \rightsquigarrow (W, 0)$. Hence, using (9) and the continuous mapping theorem we see that $W^n - V^n \rightsquigarrow 0$. It thus suffices to show that $(V^n, X) \rightsquigarrow (W, X)$, where W is a Brownian motion, independent of X.

We will show that for all W-continuity sets $A \in \mathscr{B}(C[0,\infty))$ and X-continuity sets $B \in \mathscr{B}(\mathscr{X})$, we have

$$P(V^n \in A,\ X \in B) \to P(W \in A)P(X \in B)$$

(this is sufficient, see Theorem 3.1 of Billingsley (1968)). The fact that $W^n - V^n \rightsquigarrow 0$ implies in particular that V^n converges weakly to a Brownian motion. Hence, by the portmanteau theorem, we have

$$P(V^n \in A) \to P(W \in A)$$

for all W-continuity sets $A \in \mathscr{B}(C[0,\infty))$. In view of the inequality

$$|P(V^n \in A,\ X \in B) - P(W \in A)P(X \in B)|$$

$$\leqslant |P(V^n \in A,\ X \in B) - P(V^n \in A)P(X \in B)|$$

$$+ |P(V^n \in A)P(X \in B) - P(W \in A)P(X \in B)|$$

it remains to show that $|P(V^n \in A,\ X \in B) - P(V^n \in A)P(X \in B)| \to 0$.

For notational convenience, put $\mathscr{G} = \bigvee_{n=1}^{\infty} \mathscr{F}_{\infty}^n$. From assumptions (ii) and (iii) it follows, by the martingale convergence theorem, that for all $B \in \mathscr{B}(\mathscr{X})$

$$P(X \in B \mid \mathscr{F}_{\tau_n}^n) \xrightarrow{L_1} P(X \in B \mid \mathscr{G}).$$

Consequently, we have for all $A \in \mathscr{B}(C[0,\infty))$ and $B \in \mathscr{B}(\mathscr{X})$

$$|E[1_{\{V^n \in A\}} P(X \in B \mid \mathscr{F}_{\tau_n}^n)] - E[1_{\{V^n \in A\}} P(X \in B \mid \mathscr{G})]| \to 0.$$

By the strong Markov property, V^n is independent of $\mathscr{F}_{\tau_n}^n$. This implies that the first expectation in the preceding display is equal to $P(V^n \in A)P(X \in B)$. The $\mathscr{G}$-measurability of V^n implies that the second expectation is equal to $P(V^n \in A, X \in B)$. $\square$

Corollary 3.2. *Let $(M_t^n, \mathscr{F}_t^n\colon t \geqslant 0)$ be a nested sequence of continuous local martingales and suppose that there exists an N-sequence t_n such that*

$$\langle M^n \rangle_{t_n} \xrightarrow{\mathrm{P}} 0.$$

Let $t \geqslant 0$ be fixed. If there exists a non-negative random variable C, such that

$$\langle M^n \rangle_t \xrightarrow{\mathrm{P}} C, \tag{10}$$

then, for each random element X defined on $\Omega, \mathscr{F}, P$ with values in some Polish vector space $\mathscr{X}$, we have

$$(M_t^n, X) \rightsquigarrow (W_C, X),$$

where W is a Brownian motion that is independent of (C, X).

Proof. Let $(W_t^n, \mathscr{G}_t^n\colon t \geqslant 0)$ be the Brownian motion corresponding to $(M_t^n, \mathscr{F}_t^n\colon t \geqslant 0)$ and define $\tau_n = \langle M^n \rangle_{t_n}$. Then τ_n is a $\{\mathscr{G}_t^n\}$-stopping time (see the time-change theorem). By construction, all conditions of the preceding theorem are satisfied. It then follows from this theorem and (10) that

$$(W^n, \langle M_t^n \rangle, X) = (W^n, C, X) + (0, \langle M_t^n \rangle - C, 0) \rightsquigarrow (W, C, X),$$

where W is a Brownian motion that is independent of the pair (C, X). Now write $(M_t^n, X) = \xi(W^n, \langle M_t^n \rangle, X)$, with ξ the continuous map defined in (4) and apply the continuous mapping theorem. We get $(M_t^n, X) = \xi(W^n, \langle M_t^n \rangle, X) \rightsquigarrow \xi(W, C, X) = (W_C, X)$. $\square$

4. The main theorem

In this section we prove the main result of the paper. If A is an $n \times m$ matrix we write $\|A\| = \sup\{|Ax|\colon x \in \mathbb{R}^m, |x| = 1\}$. The conditions of the theorem involve matrices K_t of which we require that $\|K_t\| \to 0$. Since all norms on a Euclidean space generate the same topology, this is equivalent to the condition that each entry of K_t converges to 0. As usual, $\mathrm{N}_d(0, \Sigma)$ denotes a d-dimensional normal distribution with mean vector 0 and covariance matrix Σ.

Theorem 4.1. *Let $(M_t, \mathscr{F}_t\colon t \geqslant 0)$ be a d-dimensional continuous local martingale. If there exist invertible, non-random $d \times d$-matrices $(K_t\colon t \geqslant 0)$ such that as $t \to \infty$*

(i) $K_t \langle M \rangle_t K_t^{\mathrm{T}} \xrightarrow{\mathrm{P}} \eta\eta^{\mathrm{T}}$

where η is a random $d \times d$-matrix,

(ii) $\|K_t\| \to 0$,

then, for each $\mathbb{R}^k$-valued random vector X defined on the same probability space as M, we have

$$(K_t M_t, X) \rightsquigarrow (\eta Z, X) \quad \text{as } t \to \infty,$$

where $Z \sim \mathrm{N}_d(0, I)$ and Z is independent of (η, X).

Remark 4.2. In terms of stable convergence (see e.g. Aldous and Eagleson, 1978), we may reformulate the statement of the theorem as follows:

$$K_t M_t \rightsquigarrow V \quad \text{(stably)},$$

where V has characteristic function $u \mapsto E \exp[-\frac{1}{2} u^{\mathrm{T}} \eta \eta^{\mathrm{T}} u]$.

Proof of Theorem 4.1. First observe that for all $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^k$, we have

$$E e^{\mathrm{i} x^{\mathrm{T}} \eta Z + \mathrm{i} y^{\mathrm{T}} X} = E e^{-(1/2) x^{\mathrm{T}} \eta \eta^{\mathrm{T}} x + \mathrm{i} y^{\mathrm{T}} X}.$$

So in terms of characteristic functions we have to prove that for all $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^k$

$$E e^{\mathrm{i} x^{\mathrm{T}} K_t M_t + \mathrm{i} y^{\mathrm{T}} X} \to E e^{-(1/2) x^{\mathrm{T}} \eta \eta^{\mathrm{T}} x + \mathrm{i} y^{\mathrm{T}} X} \quad \text{as } t \to \infty.$$

That is, we need to prove

$$(x^{\mathrm{T}} K_t M_t, Y) \rightsquigarrow (x^{\mathrm{T}} \eta Z, Y) \quad \text{as } t \to \infty \tag{11}$$

for all $x \in \mathbb{R}^d$ and all real-valued random variables Y, where $Z \sim \mathrm{N}_d(0, I)$ and Z is independent of (η, Y).

Let a_n be an arbitrary sequence such that $a_n \to \infty$. We introduce the one-dimensional continuous processes M^n as follows:

$$M_t^n = x^{\mathrm{T}} K_{a_n} M_{a_n t}, \quad t \geqslant 0.$$

Observe that for all $n \in \mathbb{N}$, M^n is a continuous local martingale with respect to the filtration $\{\mathscr{F}_{a_n t}\}$ and that

$$\langle M^n \rangle_t = x^{\mathrm{T}} K_{a_n} \langle M \rangle_{a_n t} K_{a_n}^{\mathrm{T}} x, \quad t \geqslant 0. \tag{12}$$

In this notation (11) reduces to

$$(M_1^n, Y) \rightsquigarrow (x^{\mathrm{T}} \eta Z, Y). \tag{13}$$

In order to prove (13) we will show that every subsequence a_{l_n} of a_n has a further subsequence a_{k_n}, such that

$$(M_1^{k_n}, Y) \rightsquigarrow (x^{\mathrm{T}} \eta Z, Y).$$

We can choose a subsequence a_{k_n} of a_{l_n} and numbers $0 < t_n \downarrow 0$, so that

$$a_{k_n} t_n \uparrow \infty \quad \text{and} \quad \|K_{a_{k_n}} K_{a_{k_n} t_n}^{-1}\| \to 0. \tag{14}$$

Indeed, since $\|K_{a_{l_n}}\| \to 0$ and $1 = \|I\| \leqslant \|K_{a_{l_n}}\| \; \|K_{a_{l_n}}^{-1}\|$, we have $\|K_{a_{l_n}}^{-1}\| \to \infty$. So we can choose the subsequence a_{k_n} in such a way that the following inequalities are satisfied:

$$\|K_{a_{k_n}}\| \leqslant \frac{1}{n \|K_{a_{l_n}}^{-1}\|} \quad \text{and} \quad a_{k_n} \geqslant n a_{l_n}. \tag{15}$$

Now put $t_n = a_{l_n} / a_{k_n}$. By the second of the inequalities we have $t_n \leqslant 1/n$, so $t_n \downarrow 0$. Moreover, $a_{k_n} t_n = a_{l_n}$, which tends to ∞ in view of the first condition in (14). As for the second condition in (14), it is satisfied as well since by the inequality in (15)

$$\|K_{a_{k_n}} K_{a_{k_n} t_n}^{-1}\| \leqslant \|K_{a_{k_n}}\| \; \|K_{a_{k_n} t_n}^{-1}\| \leqslant \frac{1}{n},$$

which means that the sequences a_{k_n} and t_n possess the desired properties.

We are going to apply Corollary 3.2 to the local martingales M^{k_n}. We saw already that M^{k_n} is a continuous local martingale w.r.t. the filtration $\{\mathscr{F}_{a_{k_n}t}\}$, so it is clear that the M^{k_n} are nested. By the first relation in (14), t_n is an N-sequence. Moreover, by (12) we have

$$\begin{aligned}\|\langle M^{k_n}\rangle_{t_n}\| &= \|x^{\mathrm{T}}K_{a_{k_n}}\langle M\rangle_{a_{k_n}t_n}K_{a_{k_n}}^{\mathrm{T}}x\|\\ &= \|x^{\mathrm{T}}(K_{a_{k_n}}K_{a_{k_n}t_n}^{-1})K_{a_{k_n}t_n}\langle M\rangle_{a_{k_n}t_n}K_{a_{k_n}t_n}^{\mathrm{T}}(K_{a_{k_n}}K_{a_{k_n}t_n}^{-1})^{\mathrm{T}}x\|\\ &\leqslant \|K_{a_{k_n}}K_{a_{k_n}t_n}^{-1}\|^2\|K_{a_{k_n}t_n}\langle M\rangle_{a_{k_n}t_n}K_{a_{k_n}t_n}^{\mathrm{T}}\||x|^2.\end{aligned}$$

So it follows by the second relation in (14) and by assumption (i) that $\langle M^{k_n}\rangle_{t_n}\xrightarrow{\mathrm{P}}0$.

The preceding paragraph shows that the assertion of Corollary 3.2 can be applied to the local martingales M^{k_n}. To this end, observe that by assumption (i)

$$\langle M^{k_n}\rangle_1 = x^{\mathrm{T}}K_{a_{k_n}}\langle M\rangle_{a_{k_n}}K_{a_{k_n}}^{\mathrm{T}}x\xrightarrow{\mathrm{P}}x^{\mathrm{T}}\eta\eta^{\mathrm{T}}x.$$

It then follows from the corollary that

$$(M_1^{k_n},Y)\rightsquigarrow(W_{x^{\mathrm{T}}\eta\eta^{\mathrm{T}}x},Y),$$

where W is a Brownian motion, independent of $(x^{\mathrm{T}}\eta\eta^{\mathrm{T}}x,Y)$. Finally, use the independence of W and $(x^{\mathrm{T}}\eta\eta^{\mathrm{T}}x,Y)$ to see that $(W_{x^{\mathrm{T}}\eta\eta^{\mathrm{T}}x},Y)$ has the same distribution as $(x^{\mathrm{T}}\eta Z,Y)$, where $Z\sim\mathrm{N}_d(0,I)$ and Z is independent of (η,Y). $\square$

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