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ON ESTIMATING THE DIFFUSION COEFFICIENT

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Abstract

Random processes of the diffusion type have the property that microscopic fluctuations of the trajectory make possible the identification of certain statistical parameters from one continuous observation. The paper deals with the construction of parameter estimates when observations are made at discrete but very dense time points.

DIFFUSION PROCESSES; PARAMETER ESTIMATION; LOCAL ASYMPTOTIC MIXED NORMALITY; DISCRETE OBSERVATION

1. Introduction

Asymptotic theory of estimation for parameters of diffusion processes has been developed mainly with regard to parameters in the drift coefficient (see e.g. Prakasa Rao and Rubin (1981)). The unknown parameter which is included in the diffusion coefficient of the process given as the solution of the stochastic differential equation

(1)
$$d\xi_t = a(\xi_t)dt + \sqrt{\vartheta} f(\xi_t) dW_t,$$

$t \in [0,1]$, $\xi_0 = x_0$, can be determined with probability 1 by using continuous observation of ξ_t on an arbitrarily short part of any trajectory. The estimator can be based on the relation

(2)
$$P\text{-}\lim_{n\to\infty} \sum_{k=1}^{n} (\xi_{t_k} - \xi_{t_{k-1}})^2 = \vartheta \int_0^1 f^2(\xi_t)dt,$$

where $\{\{t_k\}_{k=0}^n,\ n = 0, 1, \cdots\}$ is a sequence of divisions of the interval $[0,1]$ such that

$$\max\{(t_k - t_{k-1}),\ 0 \leqq k \leqq n\} \to 0, \quad \text{as } n \to \infty$$

(see e.g. Arató (1978)).

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But in a number of applications we have at our disposal only a discrete-time observation of the process ξ. In some of these situations the method mentioned above is applicable, but does not give us the best results. This will be illustrated in the example in Section 4 of the present paper. There it is proved that we can obtain better results by using the local asymptotic normality property (LAN) of the model (1).

But in a slightly wider class of processes given by Equation (3) the LAN condition is not available. In Section 3 the validity of the local asymptotic mixed normality (LAMN) condition for this case is proved.

The LAN condition was introduced by Le Cam in 1960 in one of his fundamental papers. The asymptotic properties of estimates in connection with the LAN condition have been studied among several others by Hájek (1972) and Le Cam (1960), (1972). These papers contain basic results about local asymptotic minimax property. Situations later occurred (some processes with dependent observations, Galton–Watson supercritical branching process, pure-birth process) in which the LAN condition was not satisfied but it can be replaced by a similar and more general LAMN condition. The LAMN condition was defined first by Jeganathan in 1979. The connection between the LAMN condition and the properties of statistical testing and estimation were investigated in Jeganathan (1981), (1982), (1983) and Swansen (1983). Jeganathan (1981), (1983) extended Le Cam's and Hájek's results to the LAMN case.

This paper refers to estimation of a parameter from a diffusion process which is observed at equidistant sampling points only. This model is treated by Dacunha-Castelle and Florens-Zmirou (1984) where a very useful expansion of the transition probability density which we employ is given. The main result of the present work consists in proving that the LAMN condition is satisfied in the case studied here. This implies the validity of the generalized local asymptotic minimax theorem and of the condition under which the lower bound is obtained (see Swansen (1983)). The validity of this condition implies the consistency and the asymptotic efficiency of the corresponding estimators. Section 4 contains two examples. The first one gives some estimators for the parameter ϑ in (1), the second deals with estimators in the linear case.

2. Assumptions, notations and definitions

Let $\{\xi_t, t \in [0,1]\}$ be a real-valued process satisfying the stochastic differential equation

$$d\xi_t = a(\xi_t, \vartheta)dt + f(\xi_t, \vartheta)dW_t, \tag{3}$$

$t \in [0,1]$, $\xi_0 = x_0$, where $\{W_t, t \in [0,1]\}$ is a standard Wiener process, ϑ is an unknown parameter from an open interval $\Theta \subset R$. Suppose that $a(x, \vartheta)$ and $f(x, \vartheta)$ are known real-valued functions continuous on $R \times \Theta$ and such that

(i) $f(x, \vartheta) > 0$ for all $(x, \vartheta) \in R \times \Theta$,
(ii) $\partial f / \partial x$, $\partial^2 f / \partial x^2$, $\partial^3 f / \partial x^3$, $\partial f / \partial \vartheta$, $\partial^2 f / \partial x \partial \vartheta$ are continuous on $R \times \Theta$.
(iii) $\partial a / \partial x$, $\partial^2 a / \partial x^2$, $\partial a / \partial \vartheta$ are continuous on $R \times \Theta$,

We assume that ξ is canonically defined on a set Ω of all continuous real-valued functions from $[0,1]$ to R. Let

$$F = \{F_t\,; F_t = \sigma(\xi_s, s \leqq t), t \in [0,1]\}$$

be the system of σ-fields induced on Ω by ξ. Denote by P_{ϑ_0} the probability measure on (Ω, F) corresponding to ξ with $\vartheta = \vartheta_0 \in \Theta$.

Now, we are concerned with a discretization of the process ξ when its values are observed at equidistant sampling points $t_k = k/n$, $k = 0, 1, \cdots, n$ for some integer $n = 1, 2, \cdots$. Let $X^n = \{X^n_k\}^n_{k=1}$, $X^n_k = \xi_{k/n}$ be the Markov chain that we observe. We shall write X_k instead of X^n_k when it is evident that n is fixed. On (Ω^n, F^n) denote by P^n_ϑ the probability measure generated by the chain X^n, where

$$\Omega^n = R^n, \quad F^n = \{F^n_k; F^n_k = \sigma(X^n_1, \cdots, X^n_k), k = 0, 1, \cdots, n\}, \quad \vartheta \in \Theta.$$

If P and Q are probability measures dP/dQ denotes the Radon–Nikodym derivative of the absolute continuous part of P with respect to Q.

Definition 1. The sequence of families $\{P^n_\vartheta, \vartheta \in \Theta\}_{n \geqq 1}$ of probability measures is said to be locally asymptotic mixed normal (LAMN) in $\vartheta \in \Theta$ if the two following conditions are satisfied.

(A1) There exist sequences $\{\Delta_n\}_{n \geqq 1}$ and $\{\Gamma_n\}_{n \geqq 1}$ of F^n-measurable random variables, $\Gamma_n > 0$ a.s., $n = 1, 2, \cdots$, and the difference

$$\log \frac{dP^n_{\vartheta_{n,h}}}{dP^n_\vartheta} - h \Delta_n \Gamma_n^{1/2} + \tfrac{1}{2} h^2 \Gamma_n$$

converges to 0 in P^n_ϑ probability for every $h > 0$, where $\vartheta_{n,h} = \vartheta + d_n h$, $\{d_n\}_{n \geqq 1}$ is a sequence of positive constants, $d_n \to 0$ as $n \to \infty$.

(A2) There exist independent random variables Δ and Γ, $\Delta \sim \mathrm{N}(0,1)$, $\Gamma > 0$ a.s. such that (Δ_n, Γ_n) converges in distribution under P^n_ϑ to (Δ, Γ) as $n \to \infty$.

Remark 1. The random variables Δ_n, Δ, Γ_n, Γ in Definition 1 are dependent on $\vartheta \in \Theta$.

Remark 2. If Γ in Definition 1 is a non-random positive constant, we obtain the local asymptotic normality condition (LAN).

3. Local asymptotic mixed normality

Proposition 1. Let ξ fulfil (3). Under assumptions (i), (ii), (iii), the sequence $\{P^n_\vartheta, \vartheta \in \Theta\}_{n \geqq 1}$ generated by ξ satisfies the LAMN condition in $\vartheta_0 \in \Theta$ with

$d_n = n^{-1/2}$,

$$\Gamma = 2\int_0^1 g^2(\xi_t, \vartheta)dt,$$

where

$$g(x, \vartheta) = \frac{\partial f}{\partial \vartheta}(x, \vartheta)[f(x, \vartheta)]^{-1}.$$

Before giving the proof of this proposition we shall present a preliminary result.

Lemma 1. (A1) from Definition 1 holds.

Proof. We consider the following transformation:

$$\eta_t^\vartheta = T_\vartheta(\xi_t) = \int_0^{\xi_t} \frac{dt}{f(x, \vartheta)} \tag{4}$$

for $\vartheta \in \Theta$, $t \in [0, 1]$. The new process η satisfies the stochastic differential equation

$$d\eta_t^\vartheta = b(\eta_t^\vartheta, \vartheta)dt + dW_t,$$

$t \in [0, 1]$, $\eta_0^\vartheta = T_\vartheta(x_0)$, $b(y, \vartheta) = a(T_\vartheta^{-1}(y), \vartheta)[f(T_\vartheta^{-1}(y), \vartheta)]^{-1} - \frac{1}{2}f'(T_\vartheta^{-1}(y), \vartheta)$ where the prime denotes the derivative with respect to x.

Let $p_\vartheta(t, x, z)$ and $q_\vartheta(t, y, v)$ be the transition probability densities of processes ξ and η respectively.

Under our assumptions b is of class C^2 as a function of y on R. From Lemma 3 in Dacunha-Castelle and Florens-Zmirou (1984) we obtain the expansion

$$q_\vartheta\left(\frac{1}{n}, Y_k^\vartheta, Y_{k+1}^\vartheta\right)$$

$$= \left(\frac{n}{2\pi}\right)^{1/2} \exp\left\{-\frac{n}{2}C_k^{(1)}(\vartheta, Y_k^\vartheta, Y_{k+1}^\vartheta) + C_k^{(2)}(\vartheta, Y_k^\vartheta, Y_{k+1}^\vartheta) + C_k^{(3)}(\vartheta, Y_k^\vartheta, Y_{k+1}^\vartheta) + O_p(n^{-3/2})\right\}, \tag{5}$$

where the symbol $O_p(n^{-3/2})$ denotes a function such that $n^{3/2}O_p(n^{-3/2})$ is bounded in probability. Denoting $Y_k^\vartheta = T_\vartheta(X_k)$ we have

$$C_k^{(1)}(\vartheta, Y_k^\vartheta, Y_{k+1}^\vartheta) = \left(\int_{k/n}^{(k+1)/n} d\eta_t^\vartheta\right)^2 = \left(\int_{k/n}^{(k+1)/n} b(\eta_t^\vartheta, \vartheta)dt + W_{k+1} - W_k\right)^2,$$

$$C_k^{(2)}(\vartheta, Y_k^\vartheta, Y_{k+1}^\vartheta) = \int_{Y_k^\vartheta}^{Y_{k+1}^\vartheta} b(y, \vartheta)dy,$$

$$C_k^{(3)}(\vartheta, Y_k^\vartheta, Y_{k+1}^\vartheta) = -\frac{1}{2n}(Y_{k+1}^\vartheta - Y_k^\vartheta)^{-1}\int_{Y_k^\vartheta}^{Y_{k+1}^\vartheta} [b(y, \vartheta) + b'(y, \vartheta)]dy.$$

Next, we shall investigate the Radon–Nikodym derivative $dP^n_\vartheta/dP^n_{\vartheta_0}$.

From the transformation formula for densities and from (5) we have

$$\frac{dP^n_\vartheta}{dP^n_{\vartheta_0}} = \prod_{k=1}^n \frac{p_\vartheta\left(\frac{1}{n}, X_{k-1}, X_k\right)}{p_{\vartheta_0}\left(\frac{1}{n}, X_{k-1}, X_k\right)}$$

$$(6) \qquad = \prod_{k=1}^n \frac{f(T_{\vartheta_0}(Y^{\vartheta_0}_k), \vartheta_0) q_\vartheta\left(\frac{1}{n}, Y^\vartheta_{k-1}, Y^\vartheta_k\right)}{f(T_\vartheta(Y^\vartheta_k), \vartheta) q_{\vartheta_0}\left(\frac{1}{n}, Y^{\vartheta_0}_{k-1}, Y^{\vartheta_0}_k\right)}$$

$$= \prod_{k=1}^n \frac{f(X_k, \vartheta_0)}{f(X_k, \vartheta)} \exp\left\{-\sum_{k=0}^{n-1}\left[\frac{n}{2} D^{(1)}_k - D^{(2)}_k - D^{(3)}_k + O_p(n^{-3/2})\right]\right\},$$

where

$$D^{(i)}_k = C^{(i)}_k(\vartheta, Y^\vartheta_k, Y^\vartheta_{k+1}) - C^{(i)}_k(\vartheta_0, Y^{\vartheta_0}_k, Y^{\vartheta_0}_{k+1}), \qquad i = 1,2,3.$$

Let us set $\vartheta = \vartheta_{n,h} = \vartheta_0 + hn^{-1/2}$. Under our hypothesis b and b' are bounded and continuous, $f > 0$ and hence we obtain

$$D^{(2)}_k = O_p(n^{-3/2}) + O_p(n^{-1/2})\delta W_k, \quad D^{(3)}_k = O_p(n^{-3/2}),$$

$k = 0, 1, \cdots, n-1$, $n = 1, 2, \cdots$, and $\delta W_k = W_{k+1} - W_k$. Using (4) and Taylor's expansion we get

$$Y^{\vartheta_{n,h}}_{k+1} - Y^{\vartheta_{n,h}}_k$$

$$= \int_{X_k}^{X_{k+1}} \frac{dx}{f(x, \vartheta_{n,h})}$$

$$= \{1 + [-g(X_k, \vartheta_0)hn^{-1/2} - (\dot{g}(X_k, \vartheta_0) - g^2(X_k, \vartheta_0))\tfrac{1}{2}hn^{-1} + O_p(n^{-3/2})]$$

$$\times [1 + O_p(1)\delta W_k]\}(Y^{\vartheta_0}_{k+1} - Y^{\vartheta_0}_k)$$

where a dot denotes the derivative with respect to ϑ. Hence we obtain

$$D^{(1)}_k = (Y^{\vartheta_{n,h}}_{k+1} - Y^{\vartheta_{n,h}}_k)^2 - (Y^{\vartheta_0}_{k+1} - Y^{\vartheta_0}_k)^2$$

$$= [-2g(X_k, \vartheta_0)hn^{-1/2} - (\dot{g}(X_k, \vartheta_0) - 2g^2(X_k, \vartheta_0))h^2n^{-1}](\delta W_k)^2$$

$$+ O_p(n^{-3/2})\delta W_k + O_p(n^{-5/2}).$$

Equation (6) then has the form

$$\frac{dP^n_{\vartheta_{n,h}}}{dP^n_\vartheta} = \prod_{k=1}^n \frac{f(X_k, \vartheta_0)}{f(X_k, \vartheta_{n,h})} \exp\left\{\sum_{k=0}^{n-1}[g(X_k, \vartheta_0)hn^{-1/2} + (\dot{g}(X_k, \vartheta_0) - g^2(X_k, \vartheta_0))h^2n^{-1}]n(\delta W_k)^2 + \sum_{k=0}^{n-1} O_p(n^{-1/2})\delta W_k + O_p(n^{-1/2})\right\}.$$

After elementary computations using Taylor's expansion for the logarithm the following expression is obtained:

$$\frac{dP^n_{\vartheta_{n,h}}}{dP^n_{\vartheta_0}} = \exp\left\{\sum_{k=0}^{n-1} hn^{-1/2} g(X_k, \vartheta_0)(n(\delta W_k)^2 - 1) - h^2 n^{-1} g^2(X_k, \vartheta_0) + R_n\right\},$$

where $R_n \to 0$ in $P^n_{\vartheta_0}$-probability as $n \to \infty$. This gives (A.1) with

$$\Gamma_n = \frac{2}{n} \sum_{k=0}^{n-1} g^2(X_k, \vartheta_0), \quad \Delta_n = \left[\sum_{k=0}^{n-1} n^{-1/2} g(X_k, \vartheta_0)(n(\delta W_k)^2 - 1)\right] \Gamma_n^{-1/2}.$$

The lemma is proved.

Proof of Proposition 1. We shall prove

$$(\Delta_n, \Gamma_n) \to (\Delta, \Gamma) \tag{7}$$

in distribution under $P^n_{\vartheta_0}$ as $n \to \infty$, where Δ is a random variable having N(0, 1) distribution, independent of the random variable $\Gamma = 2\int_0^1 g(\xi_t, \vartheta_0)dt$. With regard to $\Gamma_n \to \Gamma$ a.s. as $n \to \infty$, (7) can be replaced by

$$(\Delta_n \Gamma_n^{1/2}, \Gamma) \to (\Delta\Gamma^{1/2}, \Gamma) \tag{8}$$

in distribution under $P^n_{\vartheta_0}$ as $n \to \infty$.

Note that the paths of ξ are bounded in probability. (8) will be proved for versions $\{\xi^K_t, t \in [0, 1]\}$ of the process ξ which are determined by truncated coefficients

$$\begin{aligned} a^K(x, \vartheta) &= \min\{\max\{-K, a(x, \vartheta)\}, K\}, \\ f^K(x, \vartheta) &= \min\{\max\{-K, f(x, \vartheta)\}, K\}, \end{aligned} \tag{9}$$

$K > 0$. For large K the chain $\{X^{n,K}_k\}^n_{k=1}$ corresponding to process ξ^K can be identified with the chain $\{X^n_k\}^n_{k=1}$ with probability arbitrarily close to 1 for $n = 1, 2, \cdots$. This yields (8) in the general case. Hence we assume (9) but to simplify the notation the index K is omitted in what follows.

To establish (8) we shall use characteristic functions. We have to show that the expectation of

$$A^n = \exp(iu\Delta_n\Gamma_n^{1/2} + iv\Gamma) - \exp(-\tfrac{1}{2}u^2 + iv)\Gamma$$

tends to 0 as $n \to \infty$ for $(u, v) \in R \times R$. Denote

$$\begin{aligned} B^n_k = &\exp\left(iu\Delta^n_{k-1} - u^2 \int_{k/n}^1 g^2(\xi_t, \vartheta)dt\right) \\ &\times \left[\exp(iuY^n_{k-1}) - \exp\left(-u^2 \int_{(k-1)/n}^{k/n} g^2(\xi_t, \vartheta)dt\right)\right] \\ &\times \exp\left(iv2 \int_0^1 g^2(\xi_t, \vartheta)dt\right), \end{aligned}$$

where

$$\Delta_k^n = \sum_{k=1}^{n} n^{-1/2} g(X_k, \vartheta)(n(\delta W_k)^2 - 1) = \sum_{j=1}^{n} Y_j^n.$$

There is $A^n = \Sigma_{k=1}^n B_k^n$, and hence

$$\begin{aligned}
|EA^n| &\leqq E \sum_{k=1}^{n} |E\{E[B_k^n | F_k^n] | F_{k-1}^n\}| \\
&= E \sum_{k=1}^{n} \left| \exp\left(iu\Delta_{k-1}^n + iv2 \int_0^{(k-1)/n} g^2(\xi_t, \vartheta) dt \right) \right. \\
&\quad \times E\left\{ \exp\left(2iv \int_{(k-1)/n}^{k/n} g^2(\xi_t, \vartheta) dt \right) \right. \\
&\quad \times \left[\exp(iuY_{k-1}^n) - \exp\left(-u^2 \int_{(k-1)/n}^{k/n} g^2(\xi_t, \vartheta) dt \right) \right] \\
&\quad \times E\left[\exp\left((-\tfrac{1}{2}u^2 + iv) 2 \int_{k/n}^{1} g^2(\xi_t, \vartheta) dt \right) \Big| F_k^n \right] \left. \left. \Big| F_{k-1}^n \right\} \right| .
\end{aligned} \tag{10}$$

Since ξ is a Markov process it follows that there exists a bounded function $\varphi(x, \vartheta)$ such that

$$E\left\{ \exp\left((-\tfrac{1}{2}u^2 + iv) 2 \int_{k/n}^{1} g^2(\xi_t, \vartheta) dt \right) \Big| F_k^n \right\} = \varphi(X_k, \vartheta). \tag{11}$$

Assumptions (i), (ii), (iii) imply the existence of continuous second derivative of φ with respect to x. Let us consider the expansion

$$\begin{aligned}
\varphi(X_k, \vartheta) &= \varphi(X_{k-1}, \vartheta) + \varphi'(X_{k-1}, \vartheta) \left[\frac{1}{n} a(X_{k-1}, \vartheta) + f(X_{k-1}, \vartheta) \delta W_k \right] + n^{-3/4} R \\
&= \Phi(X_{k-1}, \vartheta) + \Psi(X_{k-1}, \vartheta) \delta W_k + n^{-3/4} R,
\end{aligned} \tag{12}$$

where R tends to 0 in quadratic mean as $n \to \infty$. Using (12), (11) and the Taylor's expansions for the exponentials in (10) we get the form

$$\begin{aligned}
|EA^n| &= E \sum_{k=1}^{n} \left| E\left\{ \left[iuY_{k-1}^n - \tfrac{1}{2}u^2 (Y_{k-1}^n) - \frac{u^2}{n} g^2(X_{k-1}, \vartheta) + n^{-5/4} R_1 \right] \right. \right. \\
&\quad \times \left(1 + i\frac{v}{n} g^2(X_{k-1}, \vartheta) + \frac{1}{n} R_2 \right) \\
&\quad \times (\Phi(X_{k-1}, \vartheta) + \Psi(X_{k-1}, \vartheta) \delta W_k + n^{-3/4} R) \left. \left. \Big| F_{k-1}^n \right\} \right| \\
&= n^{-5/4} E \sum_{k=1}^{n} |E\{S_k | F_{k-1}^n\}| \\
&\leqq n^{-5/4} E \sum_{k=1}^{n} E\{|S_k| | F_{k-1}^n\} \\
&= n^{-5/4} \sum_{k=1}^{n} E|S_k|,
\end{aligned}$$

where

(13) $$|R_1| \leqq n^{-1/4} u^3 C_1 (n(\delta W_k)^2 - 1)^3 + n^{-3/4} u^4 C_2, \qquad |R_2| \leqq \frac{v^2}{n} C_3,$$

(14) $$S_k = R_1 \left(1 + i\frac{v}{n} g^2(X_{k-1}, \vartheta) + \frac{1}{n} R_2\right) (\Phi(X_{k-1}, \vartheta) + n^{-3/4} R).$$

From (13), (14) and the boundedness of $g(x, \vartheta)$ it follows that $E|S_k|$ is bounded and hence

$$n^{-5/4} \sum_{k=1}^{n} E|S_k| \to 0, \quad \text{as } n \to \infty,$$

which completes the proof.

Whenever the LAMN condition is satisfied we can use the extension of Hájek's result as it has been presented e.g. by Swansen (1983) or Jeganathan (1983).

Let $l(\hat{\vartheta} - \vartheta)$ be a loss-function such that

(i) $l(z) = l(|z|)$, $l(0) = 0$,

(ii) $l(y) \leqq l(z)$ if $|y| \leqq |z|$,

(iii) $\int l(z + \alpha) w^{1/2} \exp(-\frac{1}{2}\beta w z^2) dz\, dG(w) < \infty$

for some $\alpha > 0$ and all $\beta > 0$, where $G(\cdot)$ is the distribution function of Γ. The following proposition is the consequence of Theorem 5.5 in Jeganathan (1983) and Proposition 1 in this paper.

Proposition 2. Let a process $\{\xi_t, t \in [0, 1]\}$ be defined by (3). Then for any sequence $\{\hat{\vartheta}_n\}_{n \geqq 1}$ of estimators based on X_k, $k = 1, 2, \cdots, n$, of parameter ϑ_0

(15) $$\begin{aligned} &\lim_{b \to \infty} \liminf_{n \to \infty} \sup_{|h| < b} E^n_{n,h}[l(n^{1/2}(\hat{\vartheta}_n - \vartheta_{n,h}))] \\ &\geqq (2\pi)^{-1/2} \int l(zw^{-1/2}) \exp(-\tfrac{1}{2}z^2) dz\, dG(w) \end{aligned}$$

holds and the lower bound is obtained only if

(16) $$n^{1/2}(\hat{\vartheta}_n - \vartheta_0) - \Delta_n \Gamma_n^{-1/2} \to 0$$

in $P^n_{\vartheta_0}$-probability as $n \to \infty$.

Remark 3. If the function $f(x, \vartheta)$ in (3) is such that $f(x, \vartheta) = \gamma(\vartheta) h(x)$ where $\gamma(\cdot)$ and $h(\cdot)$ are continuously differentiable real-valued functions then the LAMN condition in Proposition 1 can be replaced by LAN and Hájek's local minimax theorem will be valid in this case (see Hájek (1972)).

4. Two examples

1. The process ξ defined by

$$d\xi_t = a(\xi_t) dt + \sqrt{\vartheta} f(\xi_t) dW_t,$$

$t \in [0,1]$, $\xi_0 = x_0$, and $a(x)$, $f(x)$ are bounded functions, satisfying (i), (ii), (iii) (from Section 2) on R.

The family $\{P^n_\vartheta, \vartheta \in \Theta\}$ satisfies the LAN condition and

$$\log \frac{dP^n_{\vartheta_n,h}}{dP^n_{\vartheta_0}} \to \frac{h}{2\vartheta n^{1/2}} \sum_{k=1}^{n} [n(\delta W_k)^2 - 1] - \frac{h^2}{4\vartheta^2}$$

in $P^n_{\vartheta_0}$-probability as $n \to \infty$. The condition (16) has the form

$$n^{1/2}\left(\hat{\vartheta} - \vartheta_0 \sum_{k=1}^{n} (\delta W_k)^2\right) \to 0 \tag{17}$$

in $P^n_{\vartheta_0}$-probability as $n \to \infty$. It is fulfilled for example by

$$\hat{\vartheta} = \sum_{k=0}^{n-1} \frac{\left[X_{k+1} - X_k - \frac{1}{n} a(X_k)\right]^2}{f^2(X_k)} .$$

$\hat{\vartheta}$ is a consistent, asymptotically efficient estimator of ϑ_0 and the statistic

$$\frac{\vartheta - \vartheta_0}{\vartheta_0 \sqrt{2/n}}$$

has asymptotically the N(0, 1) distribution.

From (2) we obtain the estimator

$$\bar{\vartheta} = \sum_{k=0}^{n-1} \frac{[X_{k+1} - X_k]^2}{\frac{1}{n} \sum_{k=0}^{n-1} f^2(X_k)}$$

which also satisfies the condition (17).

2. Consider the linear model

$$d\xi_t = a\xi_t dt + \sqrt{\vartheta} dW_t, \tag{18}$$

$t \in [0,1]$, $\xi_0 = x_0$. Arató (1978) presented the estimator

$$\bar{\vartheta} = (1 - \bar{\rho})\bar{s},$$

where

$$\bar{\rho} = \frac{1}{\bar{s}^2(n-1)} \sum_{k=1}^{n} \eta_k \eta_{k-1}; \quad \bar{s} = \frac{1}{n} \sum_{k=1}^{n} \eta_k^2; \quad \eta_k = X_k - \frac{1}{n} \sum_{k=1}^{n} X_k$$

for which

$$E(\bar{\vartheta} - \vartheta_0) = O\left(\frac{1}{n}\right); \quad \operatorname{Var} \bar{\vartheta} = \frac{2\vartheta_0^2}{n} + o\left(\frac{1}{n}\right).$$

From the condition (17) we derive the estimator

$$\hat{\vartheta} = \frac{1}{4} \sum_{k=1}^{n} \left[X_k \left(2 + \frac{a}{n}\right) - X_{k-1} \left(2 - \frac{a}{n}\right) \right]^2$$

for which

$$E(\hat{\vartheta} - \vartheta_0) = O\left(\frac{1}{n^2}\right); \quad \operatorname{Var} \hat{\vartheta} = \frac{2\vartheta_0^2}{n} + O\left(\frac{1}{n^3}\right).$$

The unbiased estimator with minimal variance equal to $2\vartheta_0^2/n$ is obtained in the form

$$\tilde{\vartheta} = \frac{2a}{n(1-\rho^2)} \sum_{k=1}^{n} (X_k - \rho X_{k-1})^2,$$

where $\rho = \exp(-a/n)$.

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