Chapter 3

Singular value inequalities

3.0 Introduction and historical remarks

Singular values and the singular value decomposition play an important role in high-quality statistical computations and in schemes for data compression based on approximating a given matrix with one of lower rank. They also play a central role in the theory of unitarily invariant norms. Many modern computational algorithms are based on singular value computations because the problem of computing the singular values of a general matrix (like the problem of computing the eigenvalues of a Hermitian matrix) is well-conditioned; for numerous examples see [GVl].

There is a rich abundance of inequalities involving singular values, and a selection from among them is the primary focus of this chapter. But approximations and inequalities were not the original motivation for the study of singular values. Nineteenth-century differential geometers and algebraists wanted to know how to determine whether two real bilinear forms

$$\varphi_A(x,y) = \sum_{i,\,j=1}^{n} a_{ij}x_iy_j \text{ and } \varphi_B(x,y) = \sum_{i,\,j=1}^{n} b_{ij}x_iy_j, \tag{3.0.1}$$
$$A = [a_{ij}],\ B = [b_{ij}] \in M_n(\mathbb{R}),\ x = [x_i],\ y = [y_i] \in \mathbb{R}^n$$

were equivalent under independent real orthogonal substitutions, that is, whether there are real orthogonal Q_1, $Q_2 \in M_n(\mathbb{R})$ such that $\varphi_A(x,y) = \varphi_B(Q_1x, Q_2y)$ for all x, $y \in \mathbb{R}^n$. One could approach this problem by finding a canonical form to which any such bilinear form can be reduced by orthogonal substitutions, or by finding a complete set of invariants for a bilinear form under orthogonal substitutions. In 1873, the Italian differential geometer E. Beltrami did both, followed, independently, by the French algebraist C. Jordan in 1874.

Beltrami discovered that for each real $A \in M_n(\mathbb{R})$ there are always real orthogonal $Q_1, Q_2 \in M_n(\mathbb{R})$ such that

$$Q_1^T A Q_2 = \Sigma = \text{diag}(\sigma_1(A), \ldots, \sigma_n(A)) \tag{3.0.2}$$

is a nonnegative diagonal matrix, where $\sigma_1(A)^2 \geq \cdots \geq \sigma_n(A)^2$ are the eigenvalues of AA^T (and also of A^TA). Moreover, he found that the (orthonormal) columns of Q_1 and Q_2 are eigenvectors of AA^T and A^TA, respectively. Although Beltrami proposed no terminology for the elements of his canonical form, this is what we now call the *singular value decomposition* for a real square matrix; the *singular values* of A are the numbers $\sigma_1(A) \geq \cdots \geq \sigma_n(A) \geq 0$. The diagonal bilinear form

$$\varphi_{Q_1^T A Q_2}(\xi, \eta) = \sigma_1(A)\xi_1\eta_1 + \cdots + \sigma_n(A)\xi_n\eta_n \tag{3.0.3}$$

gives a convenient canonical form to which any real bilinear form $\varphi_A(x,y)$ can be reduced by independent orthogonal substitutions, and the eigenvalues of AA^T are a complete set of invariants for this reduction.

Jordan came to the same canonical form as Beltrami, but from a very different point of view. He found that the (necessarily real) eigenvalues of the $2n$-by-$2n$ real symmetric matrix

$$\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \tag{3.0.4}$$

are paired by sign, and that its n largest eigenvalues are the desired coefficients $\sigma_1(A), \ldots, \sigma_n(A)$ of the canonical form (3.0.3) (see Problem 2). Jordan's proof starts by observing that the largest singular value of A is the maximum of the bilinear form x^TAy subject to the constraints $x^Tx = y^Ty = 1$, and concludes with an elegant step-by-step deflation in the spirit of our proof of Theorem (3.1.1). The block matrix (3.0.4) has been rediscovered repeatedly by later researchers, and plays a key role in relating eigenvalue results for Hermitian matrices to singular value results for general matrices.

Apparently unaware of the work of Beltrami and Jordan, J. J. Sylvester (1889/90) discovered, and gave yet a third proof for, the factorization (3.0.2) for real square matrices. He termed the coefficients $\sigma_i(A)$ in (3.0.3) the *canonical multipliers* of the bilinear form $\varphi_A(x,y)$. The leading idea for Sylvester's proof was "to regard a finite orthogonal substitution as the product of an infinite number of infinitesimal ones."

In 1902, L. Autonne showed that every nonsingular complex $A \in M_n$ can be written as $A = UP$, where $U \in M_n$ is unitary and $P \in M_n$ is positive definite. Autonne returned to these ideas in 1913/15 and used the fact that A^*A and AA^* are similar [See Problem 1(a) in Section (3.1)] to show that any square complex matrix $A \in M_n$ (singular or nonsingular) can be written as $A = V\Sigma W^*$, where $V, W \in M_n$ are unitary and Σ is a nonnegative diagonal matrix; he gave no name to the diagonal entries of Σ. Autonne recognized that the positive semidefinite factor Σ is determined essentially uniquely by A, but that V and W are not, and he determined the set of all possible unitary factors V, W associated with A [see Theorem (3.1.1′)]. He also recognized that the unitary factors V and W could be chosen to be real orthogonal if A is real, thus obtaining the theorem of Beltrami, Jordan, and Sylvester as a special case; however, Autonne was apparently unaware of their priority for this result. He realized that by writing $A = V\Sigma W^* = (VW^*)(W\Sigma W^*) = (V\Sigma V^*)(VW^*)$ he could generalize his 1902 polar decomposition to the square singular case. Although Autonne did not consider the singular value decomposition of a nonsquare A in his 1915 paper, the general case follows easily from the square case; see Problem 1(b) in Section (3.1).

In his 1915 paper, Autonne also considered special forms that can be achieved for the singular value decomposition of A under various assumptions on A, for example, unitary, normal, real, coninvolutory ($\bar{A} = A^{-1}$), and symmetric. In the latter case, Autonne's discovery that a complex square symmetric A can always be written as $A = U\Sigma U^T$ for some unitary U and nonnegative diagonal Σ gives him priority for a useful (and repeatedly rediscovered) result often attributed in the literature to Schur (1945) or Takagi (1925) (see Section (4.4) of [HJ]). To be precise, one must note that Autonne actually presented a proof for the factorization $A = U\Sigma U^T$ only for nonsingular A, but his comments suggest that he knew that the assumption of nonsingularity was inessential. In any event, the general case follows easily from the nonsingular case; see Problem 5.

The pioneering work on the singular value decomposition in Autonne's 77-page 1915 paper seems to have been completely overlooked by later researchers. In the classic 1933 survey [Mac], Autonne's 1915 paper is referenced, but the singular value decomposition for square complex matrices (MacDuffee's Theorem 41.6) is stated so as to suggest (incorrectly) to the reader that Autonne had proved it only for nonsingular matrices.

In a 1930 paper, Browne cited Autonne's 1913 announcement and used his factorization $A = V\Sigma W^*$ (perhaps for the first time) to prove inequalities for the spectral radius of Hermitian and general square matrices. Browne attached no name to the diagonal entries of Σ, and referred to them merely as "the square roots...of the characteristic roots of AA^*."

In 1931, Wintner and Murnaghan rediscovered the polar decomposition of a nonsingular square complex matrix as well as a special case (nonsingular A) of Autonne's observation that one may always write $A = PU = UQ$ (the same unitary U) for possibly different positive definite P and Q; they also noted that one may choose $P = Q$ if and only if A is normal. Wintner and Murnaghan seemed unaware of any prior work on the polar decomposition.

A complete version of the polar decomposition for a rectangular complex matrix (essentially Theorem (3.1.9) herein) was published in 1935 by Williamson, who credited both Autonne (1902) and Wintner-Murnaghan (1931) for prior solution of the square nonsingular case; Autonne's 1915 solution of the general square case is not cited. Williamson's proof, like Autonne's, starts with the spectral decompositions of the Hermitian matrices AA^* and A^*A. Williamson did not mention the singular value decomposition, so there is no recognition that the general singular value decomposition for a rectangular complex matrix follows immediately from his result.

Finally, in 1939, Eckart and Young gave a clear and complete statement of the singular value decomposition for a rectangular complex matrix [Theorem (3.1.1)], crediting Autonne (1913) and Sylvester (1890) for their prior solutions of the square complex and real cases. The Eckart-Young proof is self-contained and is essentially the same as Williamson's (1930); they do not seem to recognize that the rectangular case follows easily from the square case. Eckart and Young give no indication of being aware of Williamson's result and do not mention the polar decomposition, so there is no recognition that their theorem implies a general polar decomposition for rectangular complex matrices. They give no special name to the nonnegative square roots of "the characteristic values of AA^*," and they view the factorization $A = V\Sigma W^*$ as a generalization of the "principal axis transformation" for Hermitian matrices.

While algebraists were developing the singular value and polar decompositions for finite matrices, there was a parallel and apparently quite independent development of related ideas by researchers in the theory of integral equations. In 1907, E. Schmidt published a general theory of real integral equations, in which he considered both symmetric and nonsymmetric kernels. In his study of the nonsymmetric case, Schmidt introduced a pair of integral equations of the form

$$\varphi(s) = \lambda \int_a^b K(s,t)\, \psi(t)\, dt$$

and (3.0.5)

$$\psi(s) = \lambda \int_a^b K(t,s)\, \varphi(t)\, dt$$

where the functions $\varphi(s)$ and $\psi(s)$ are not identically zero. Schmidt showed that the scalar λ must be real since λ^2 is an eigenvalue of the symmetric (and positive semidefinite) kernel

$$H(s,t) = \int_a^b K(s,\tau)\, K(t,\tau)\, d\tau$$

If one thinks of $K(s,t)$ as an analog of a matrix A, then $H(s,t)$ is an analog of AA^T. Traditionally, the "eigenvalue" parameter λ in the integral equation literature is the reciprocal of what matrix theorists call an eigenvalue. Recognizing that such scalars λ together with their associated pairs of functions $\varphi(s)$ and $\psi(s)$ are, for many purposes, the natural generalization to the nonsymmetric case of the eigenvalues and eigenfunctions that play a key role in the theory of integral equations with symmetric kernels, Schmidt called λ an "eigenvalue" and the associated pair of functions $\varphi(s)$ and $\psi(s)$ "adjoint eigenfunctions" associated with λ.

Picard (1910) further developed Schmidt's theory of nonsymmetric kernels but, at least in the symmetric case, refers to Schmidt's "eigenvalues" as *singular values* (valeurs singulières). Perhaps in an effort to avoid confusion between Schmidt's two different uses of the term "eigenvalue," later researchers in integral equations seem to have adopted the term "singular value" to refer to the parameter λ in (3.0.5). In a 1937 survey, for example, Smithies refers to "singular values, i.e., E. Schmidt's eigen-values of the nonsymmetric kernel, and not to the eigen-values in the ordinary sense." Smithies also notes that he had been "unable to establish any direct connection between the orders of magnitude of the eigen-values and the singular values when the kernel is not symmetric." Establishing such a connection seems to have remained a theme of Smithies' research for many years, and Smithies' student Chang (1949) succeeded in establishing an indirect connection: Convergence of an infinite series of given powers of the singular values of an integral kernel implies convergence of the infinite series of the same powers of the absolute values of the eigenvalues. Weyl (1949) then showed that there was actually a direct inequality between partial sums of Chang's two series, and the modern theory of singular value inequalities was born.

Although it is not at all concerned with the singular value decomposition or polar decomposition, a seminal 1939 paper of Von Neumann made vital use of facts about singular values in showing that a norm $\| A \|$ on $M_{m,n}$ is unitarily invariant if and only if it is a symmetric gauge function of the square roots of "the proper values of AA^*" [Theorem (3.5.18)]. Despite his familiarity with the integral equation and operator theory literature, Von Neumann never uses the term "singular value," and hence his pages are speckled with square roots applied to the eigenvalues of AA^*. His primary tools are duality and convexity, and since the basic triangle inequality for singular value sums [Corollary (3.4.3)] was not discovered until 1951, Von Neumann's proof is both long and ingenious.

During 1949-50, a remarkable series of papers in the *Proceedings of the National Academy of Sciences* (*U.S.*) established all of the basic inequalities involving singular values and eigenvalues that are the main focus of this chapter. Let $\lambda_1,\ldots,\lambda_n$ and $\sigma_1,\ldots,\sigma_n$ denote the eigenvalues and singular values of a given square matrix, ordered so that $|\lambda_1| \geq \cdots \geq |\lambda_n|$ and $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$. Apparently motivated by Chang's (1949) analytic results in the theory of integral equations, Weyl (1949) showed that $|\lambda_1 \cdots \lambda_k| \leq \sigma_1 \cdots \sigma_k$ for $k = 1,\ldots, n$ [Theorem (3.3.2)] and deduced that $\varphi(|\lambda_1|) + \cdots + \varphi(|\lambda_k|) \leq \varphi(\sigma_1) + \cdots + \varphi(\sigma_k)$ for $k = 1,\ldots, n$, for any increasing function φ on $[0,\infty)$ such that $\varphi(e^t)$ is convex on $(-\infty,\infty)$; Weyl was particularly interested in $\varphi(s) = s^p$, $p > 0$, since this special case completely explained (and enormously simplified) Chang's results. Ky Fan (1949, 1950) introduced initial versions of variational characterizations of eigenvalue and singular value sums that became basic for much of the later work in this area. Pólya (1950) gave an alternative proof of a key lemma in Weyl's 1949 paper. Pólya's insight [incorporated into Problem 23 of Section (3.3)] led to extensive and fruitful use of properties of doubly stochastic matrices and majorization in the study of singular value inequalities. None of these papers uses the term "singular value;" instead, they speak of the "two kinds of eigenvalues" of a matrix, namely, the eigenvalues of A and those of A^*A.

In a 1950 paper, A. Horn begins by saying, "In this note I wish to present a theorem on the singular values of a product of completely continuous operators on Hilbert space.... The singular values of an operator K are the positive square roots of the eigen-values of K^*K, where K^* is the adjoint of K." He then shows that

$$\sigma_1(AB) \cdots \sigma_k(AB) \le \sigma_1(A) \cdots \sigma_k(A)\sigma_1(B) \cdots \sigma_k(B)$$

for all $k = 1, 2, \ldots$ [Theorem (3.3.4)] and gives the additive inequalities in Theorem (3.3.14).

In the following year, Ky Fan (1951) extended the work in his 1949-50 papers and obtained the fundamental variational characterization of singular value sums [Theorem (3.4.1)] that enabled him to prove basic inequalities such as Theorem (3.3.16), Corollary (3.4.3), and the celebrated Fan dominance theorem [Corollary (3.5.9(a)(b))]. He also revisited Von Neumann's characterization of all unitarily invariant norms [Theorem (3.5.18)] and showed how it follows easily from his new insights. A novel and fundamental feature of Fan's variational characterizations of singular values is that they are quasilinear functions of A itself, not via A^*A. They are surely the foundation for all of the modern theory of singular value inequalities.

In 1954, A. Horn proved that Weyl's 1949 inequalities are sufficient for the existence of a matrix with prescribed singular values and eigenvalues, and stimulated a long series of other investigations to ascertain whether inequalities originally derived as necessary conditions are sufficiently strong to characterize exactly the properties under study. In this 1954 paper, which, unlike his 1950 paper, is clearly a paper on matrix theory rather than operator theory, Horn uses "singular values" in the context of matrices, a designation that seems to have become the standard terminology for matrix theorists writing in English. In the Russian literature one sees singular values of a matrix or operator referred to as *s-numbers* [GoKr]; this terminology is also used in [Pie].

Problems

1. Use Beltrami's theorem to show that the two real bilinear forms in (3.0.1) are equivalent under independent real orthogonal substitutions if and only if AA^T and BB^T have the same eigenvalues.

2. Let $A \in M_{m,n}$ be given, let $q = \min\{m,n\}$, and let

$$\mathcal{A} = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \in M_{m+n}$$

Verify that the characteristic polynomial of $\mathcal{A}$ can be evaluated as

$$\det(tI_{m+n} - \mathcal{A}) = \det \begin{bmatrix} I_m & t^{-1}A \\ 0 & I_n \end{bmatrix} \det \begin{bmatrix} tI_m & -A \\ -A^* & tI_n \end{bmatrix}$$

$$= \det \begin{bmatrix} tI_m - t^{-1}AA^* & 0 \\ -A^* & tI_n \end{bmatrix} = t^{n-m} \det(t^2 I_m - AA^*)$$

and conclude that the eigenvalues of $\mathcal{A}$ are $\pm\sigma_1(A),\ldots,\pm\sigma_q(A)$ and $|m-n|$ additional zeroes. This is a generalization of Jordan's observation about the significance of the matrix (3.0.4).

3. Suppose $A \in M_{m,n}$ can be written as $A = V\Sigma W^*$, where $V = [v_{ij}] \in M_m$ and $W = [w_{ij}] \in M_n$ are unitary, and $\Sigma = [\sigma_{ij}] \in M_{m,n}$ has $\sigma_{ij} = 0$ for all $i \neq j$, $\sigma_{ii} \geq 0$ for $i = 1,\ldots, \min\{m,n\}$.

(a) Verify Browne's upper bound on the singular values σ_{ii}:

$$\sigma_{ii} = \Big| \sum_{k=1}^{n} \sum_{j=1}^{m} a_{jk}\bar{v}_{ji}w_{ki} \Big| \leq \sum_{k=1}^{n} \sum_{j=1}^{m} |a_{jk}| \; |v_{ji}w_{ki}| \tag{†}$$

$$\leq \tfrac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{m} |a_{jk}| \, (|v_{ji}|^2 + |w_{ki}|^2)$$

$$\leq \tfrac{1}{2}\Big[\max_j \sum_{k=1}^{n} |a_{jk}| + \max_k \sum_{j=1}^{m} |a_{jk}| \Big]$$

$$= \tfrac{1}{2}\Big[|||A|||_\infty + |||A|||_1 \Big]$$

where $|||A|||_\infty$ and $|||A|||_1$ denote the maximum absolute row and column sum norms, respectively. Actually, Browne considered only the square case $m = n$ because he was interested in obtaining eigenvalue bounds. He knew that every eigenvalue λ of A satisfies $\min_i \sigma_{ii} \leq |\lambda| \leq \max_i \sigma_{ii}$, and concluded that $\rho(A) \leq \tfrac{1}{2}(|||A|||_\infty + |||A|||_1)$. He was apparently unaware of the better bound $\rho(A) \leq \min\{ |||A|||_\infty, |||A|||_1\}$.

(b) Use the Cauchy-Schwarz inequality at the point (†) in the preceding argument to give the better bounds

$$\sigma_{ii} \leq \left[\max_k \sum_{j=1}^{m} |a_{jk}|^{2p} \right]^{\frac{1}{2}} \left[\max_j \sum_{k=1}^{n} |a_{jk}|^{2(1-p)} \right]^{\frac{1}{2}}$$

for any $p \in [0,1]$. What is this for $p = \frac{1}{2}$, for $p = 0$, and for $p = 1$? Show that Browne's inequality in (a) follows from the case $p = \frac{1}{2}$.

(c) Verify Browne's corollary of the bound in (a): If $A \in M_n$ is Hermitian, then $\rho(A) \leq |||A|||_\infty$. He was apparently unaware that this bound holds for all $A \in M_n$, Hermitian or not.

4. In their 1939 paper, Eckart and Young found a criterion for simultaneous unitary equivalence of two given matrices $A, B \in M_{m,n}$ to real diagonal forms $V^*AW = E \geq 0$, $V^*BW = F$, namely, the two products AB^* and B^*A are both Hermitian.

(a) What does this say when $m = n$ and A and B are both Hermitian?

(b) Verify the necessity and sufficiency of the criterion.

(c) The Eckart-Young criterion can be weakened slightly: AB^* and B^*A are both normal if and only if there exist unitary V and W such that V^*AW and V^*BW are both diagonal (but not necessarily real); see Problem 26 in Section (7.3) of [HJ].

(d) Show that one can write $A = VEW^*$ and $B = VFW^*$ with E and F both nonnegative diagonal if and only if AB^* and B^*A are both positive semidefinite.

5. Let $A \in M_n$ be a given complex symmetric matrix. Assume that one can always write $A = U\Sigma U^T$ when A is nonsingular, where $U, \Sigma \in M_n$, U is unitary, and Σ is nonnegative diagonal. Provide details for the following two arguments to show that one can also factorize a singular A in this way.

(a) Let $u_1, \ldots, u_\nu \in \mathbb{C}^n$ be an orthonormal basis for the nullspace of A, let $U_2 \equiv [u_1 \ \ldots \ u_\nu] \in M_{n,\nu}$, and let $U = [U_1 \ \ U_2] \in M_n$ be unitary. Show that $U^TAU = A_1 \oplus 0$, where $A_1 \in M_{n-\nu}$ is nonsingular. Now write $A_1 = V_1\Sigma_1 W_1^*$ and obtain the desired factorization for A.

(b) Consider $A_\epsilon = A + \epsilon I = U_\epsilon \Sigma_\epsilon U_\epsilon^T$ for all sufficiently small $\epsilon > 0$ and use the selection principle in Lemma (2.1.8) of [HJ].

Notes and Further Reading. The following classical papers mentioned in this section mark noteworthy milestones in the development of the theory of singular values and the singular value or polar decomposition: E. Beltrami, Sulle Funzioni Bilineari, *Giornale de Matematiche* 11 (1873), 98-106. C. Jordan, Mémoire sur les Formes Bilinéaires, *J. Math. Pures Appl.* (2) 19 (1874), 35-54. J. J. Sylvester, Sur la réduction biorthogonale d'une forme linéo-linéaire à sa forme canonique, *Comptes Rendus Acad. Sci. Paris* 108 (1889), 651-653. J. J. Sylvester, On the Reduction of a Bilinear Quantic of the n^{th} Order to the Form of a Sum of n Products by a Double Orthogonal Substitution, *Messenger of Mathematics* 19 (1890), 42-46. L. Autonne, Sur les Groupes Linéaires, Réels et Orthogonaux, *Bull. Soc. Math. France* 30 (1902), 121-134. E. Schmidt, Zur Theorie der linearen und nichtlinearen Integralgleichungen, *Math. Annalen* 63 (1907), 433-476. É. Picard, Sur un Théorème Général Relatif aux Équations Intégrales de Première Espèce et sur quelques Problèmes de Physique Mathématique, *Rend. Circ. Mat. Palermo* 29 (1910), 79-97. L. Autonne, Sur les Matrices Hypohermitiennes et les Unitaires, *Comptes Rendus Acad. Sci. Paris* 156 (1913), 858-862; this is a short announcement of some of the results in the following detailed paper. L. Autonne, Sur les Matrices Hypohermitiennes et sur les Matrices Unitaires, *Ann. Univ. Lyon, Nouvelle Série I*, Fasc. 38 (1915), 1-77. E. T. Browne, The Characteristic Roots of a Matrix, *Bull. Amer. Math. Soc.* 36 (1930), 705-710. A. Wintner and F. D. Murnaghan, On a Polar Representation of Non-singular Square Matrices, *Proc. National Acad. Sciences* (*U.S.*) 17 (1931), 676-678. J. Williamson, A Polar Representation of Singular Matrices, *Bull. Amer. Math. Soc.* 41 (1935), 118-123. F. Smithies, Eigenvalues and Singular Values of Integral Equations, *Proc. London Math. Soc.* (2) 43 (1937), 255-279. J. Von Neumann, Some Matrix-Inequalities and Metrization of Matric-Space, *Tomsk Univ. Rev.* 1 (1937), 286-300; reprinted in Vol. 4 of Von Neumann's *Collected Works*, A. H. Taub, ed., Macmillan, N.Y., 1962. C. Eckart and G. Young, A Principal Axis Transformation for Non-Hermitian Matrices, *Bull. Amer. Math. Soc.* 45 (1939), 118-121. S. H. Chang, On the Distribution of the Characteristic Values and Singular Values of Linear Integral Equations, *Trans. Amer. Math. Soc.* 67 (1949), 351-367. H. Weyl, Inequalities Between the Two Kinds of Eigenvalues of a Linear Transformation, *Proc. National Acad. Sciences* (*U.S*) 35 (1949), 408-411. K. Fan, On a Theorem of Weyl Concerning Eigenvalues of Linear Transformations. I, *Proc. National Acad. Sciences* (*U.S.*) 35 (1949), 652-655. K. Fan, On a Theorem of Weyl Concerning Eigenvalues of Linear Transformations. II, *Proc. National Acad. Sciences* (*U.S.*) 36 (1950), 31-35. G. Pólya, Remark on Weyl's Note "Inequalities Between the Two Kinds of

Eigenvalues of a Linear Transformation," *Proc. National Acad. Sciences* 36 (*U.S.*) (1950), 49-51. A. Horn, On the Singular Values of a Product of Completely Continuous Operators, *Proc. National Acad. Sciences* (*U.S.*) 36 (1950), 374-375. K. Fan, Maximum Properties and Inequalities for the Eigenvalues of Completely Continuous Operators, *Proc. National Acad. Sciences* (*U.S.*) 37 (1951), 760-766. A. Horn, On the Eigenvalues of a Matrix with Prescribed Singular Values, *Proc. Amer. Math. Soc.* 5 (1954), 4-7.

3.1 The singular value decomposition

There are many ways to approach a proof of the singular value decomposition of a matrix $A \in M_{m,n}$. A natural route is via the spectral theorem applied to the positive semidefinite matrices AA^* and A^*A (see Problem 1); we present a proof that emphasizes the normlike properties of the singular values.

3.1.1 Theorem. Let $A \in M_{m,n}$ be given, and let $q = \min\{m,n\}$. There is a matrix $\Sigma = [\sigma_{ij}] \in M_{m,n}$ with $\sigma_{ij} = 0$ for all $i \neq j$, and $\sigma_{11} \geq \sigma_{22} \geq \cdots \geq \sigma_{qq} \geq 0$, and there are two unitary matrices $V \in M_m$ and $W \in M_n$ such that $A = V\Sigma W^*$. If $A \in M_{m,n}(\mathbb{R})$, then V and W may be taken to be real orthogonal matrices.

Proof: The Euclidean unit sphere in $\mathbb{C}^n$ is a compact set and the function $f(x) = \|Ax\|_2$ is a continuous real-valued function, so the Weierstrass theorem guarantees that there is some unit vector $w \in \mathbb{C}^n$ such that $\|Aw\|_2 = \max\{\|Ax\|_2 \colon x \in \mathbb{C}^n, \|x\|_2 = 1\}$. If $\|Aw\|_2 = 0$, then $A = 0$ and the asserted factorization is trivial, with $\Sigma = 0$ and any unitary matrices $V \in M_m$, $W \in M_n$. If $\|Aw\|_2 \neq 0$, set $\sigma_1 \equiv \|Aw\|_2$ and form the unit vector $v \equiv Aw/\sigma_1 \in \mathbb{C}^m$. There are $m-1$ orthonormal vectors $v_2, \ldots, v_m \in \mathbb{C}^m$ so that $V_1 = [v \; v_2 \; \cdots \; v_m] \in M_m$ is unitary and there are $n-1$ orthonormal vectors $w_2, \ldots, w_n \in \mathbb{C}^n$ so that $W_1 = [w \; w_2 \; \cdots \; w_n] \in M_n$ is unitary. Then

$$\bar{A}_1 \equiv V_1^* A W_1 = \begin{bmatrix} v^* \\ v_2^* \\ \vdots \\ v_m^* \end{bmatrix} [Aw \;\; Aw_2 \; \ldots \; Aw_n]$$

$$= \begin{bmatrix} v^* \\ v_2^* \\ \vdots \\ v_m^* \end{bmatrix} [\sigma_1 v \; Aw_2 \; \dots \; Aw_n]$$

$$= \begin{bmatrix} \sigma_1 & v^* A w_2 & \cdots & v^* A w_n \\ 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{bmatrix}, \quad A_2 \in M_{m-1,n-1}$$

$$= \begin{bmatrix} \sigma_1 & z^* \\ 0 & A_2 \end{bmatrix}, \quad z \equiv \begin{bmatrix} v^* A w_2 \\ \vdots \\ v^* A w_n \end{bmatrix} \in \mathbb{C}^{n-1}$$

Now consider the unit vector $\zeta \equiv \begin{bmatrix} \sigma_1 \\ z \end{bmatrix} / (\sigma_1^2 + z^* z)^{\frac{1}{2}} \in \mathbb{C}^n$ and compute

$$\begin{aligned} \|A(W_1\zeta)\|_2^2 &= \|(V_1^* A W_1)\zeta\|_2^2 = \|\tilde{A}_1 \zeta\|_2^2 \\ &= [(\sigma_1^2 + z^* z)^2 + \|A_2 z\|_2^2]/(\sigma_1^2 + z^* z) \\ &\geq \sigma_1^2 + z^* z \end{aligned}$$

which is strictly greater than σ_1^2 if $z \neq 0$. Since this would contradict the maximality of $\sigma_1 = \max\{\|Ax\|_2 \colon x \in \mathbb{C}^n, \|x\|_2 = 1\}$, we conclude that $z = 0$ and $\tilde{A}_1 = V_1^* A W_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & A_2 \end{bmatrix}$. Now repeat this argument on $A_2 \in M_{m-1,n-1}$ and its successors to obtain the desired unitary matrices V and W as products of the unitary matrices that accomplish the reduction at each step. The matrix $\Sigma = [\sigma_{ij}] \in M_{m,n}$ has $\sigma_{ii} = \sigma_i$ for $i = 1, \dots, q$. If $m \leq n$, $AA^* = (V\Sigma W^*)(W\Sigma^T V^*) = V\Sigma\Sigma^T V^*$ and $\Sigma\Sigma^T = \operatorname{diag}(\sigma_1^2, \dots, \sigma_q^2)$, so the nonnegative numbers $\{\sigma_i\}$ are uniquely determined by A; if $m > n$, consider A^*A to arrive at the same conclusion. If A is real, all the calculations required can be performed over the reals, in which case the real unitary matrices V and W are real orthogonal. $\square$

The construction used in the preceding proof shows that if $w \in \mathbb{C}^n$ is a unit vector such that $\|Aw\|_2 = \max\{\|Ax\|_2 \colon x \in \mathbb{C}^n, \|x\|_2 = 1\}$, then the $n-1$ dimensional subspace of $\mathbb{C}^n$ that is orthogonal to w is mapped by A into a subspace of $\mathbb{C}^n$ that is orthogonal to Aw. It is this geometric fact that permits the successive unitary equivalences that eventually reduce A to Σ.

In the singular value decomposition $A = V\Sigma W^*$ given in Theorem (3.1.1), the quantities $\sigma_i(A) = \sigma_{ii}$, $i = 1, 2,\ldots, q = \min\{m,n\}$, are the *singular values* of the matrix $A \in M_{m,n}$. We shall usually arrange the singular values of a matrix in decreasing (nonincreasing) order $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_q(A) \geq 0$. The number of positive singular values of A is evidently equal to the rank of A. The columns of the unitary matrix W are *right singular vectors* of A; the columns of V are *left singular vectors* of A.

If $A \in M_n$ is normal, consider a spectral decomposition $A = U\Lambda U^*$ with a unitary $U \in M_n$, $\Lambda = \mathrm{diag}(\lambda_1,\ldots, \lambda_n)$, and $|\lambda_1| \geq \cdots \geq |\lambda_n|$. If we write $\Sigma \equiv \mathrm{diag}(|\lambda_1|,\ldots, |\lambda_n|)$, then $\Lambda = D\Sigma$, where $D = \mathrm{diag}(e^{i\theta_1},\ldots, e^{i\theta_n})$ is a diagonal unitary matrix. Thus, $A = U\Lambda U^* = (UD)\Sigma U^* = V\Sigma W^*$, which is a singular value decomposition of A with $V = UD$ and $W = U$. Also, we see that the singular values of a normal matrix are just the absolute values of its eigenvalues. This observation has a converse (see Problem 19) and a generalization to diagonalizable matrices (see Problem 45).

Notice that the singular values of $A = V\Sigma W^*$ are exactly the nonnegative square roots of the q largest eigenvalues of either $A^*A = W\Sigma^T\Sigma W^*$ or $AA^* = V\Sigma\Sigma^T V^*$; the remaining eigenvalues of A^*A and AA^*, if any, are all zero. Consequently, the ordered singular values of A are uniquely determined by A, and the singular values of A and A^* are the same. Furthermore, the singular values of $U_1 A U_2$ are the same as those of A whenever U_1 and U_2 are unitary matrices of appropriate sizes; this expresses the *unitary invariance* (invariance under unitary equivalence) of the set of singular values of a complex matrix.

Unlike the diagonal factor Σ in a singular value decomposition $A = V\Sigma W^*$, the left and right unitary factors V and W are never uniquely determined; the degree of nonuniqueness depends on the multiplicities of the singular values. Let $s_1 > s_2 > \cdots > s_k > 0$ denote the distinct positive singular values of $A \in M_{m,n}$ with respective positive multiplicities $\mu_1, \mu_2,\ldots, \mu_k$. Then $\mu_1 + \cdots + \mu_k = r = \mathrm{rank}\, A \leq q \equiv \min\{m,n\}$,

$$AA^* = V\Sigma\Sigma^T V^* = V[s_1^2 I_{\mu_1} \oplus \cdots \oplus s_k^2 I_{\mu_k} \oplus 0_{m-r}]V^* \equiv VS_1V^*$$

and

$$A^*A = W\Sigma^T\Sigma W^* = W[s_1^2 I_{\mu_1} \oplus \cdots \oplus s_k^2 I_{\mu_k} \oplus 0_{n-r}]W^* \equiv WS_2W^*$$

where $I_{\mu_j} \in M_{\mu_j}$ are identity matrices for $j = 1,\ldots, k$, $0_{m-r} \in M_{m-r}$, $0_{n-r} \in M_{n-r}$

are zero matrices, and S_1, S_2 are the indicated diagonal matrices. If $\hat{V} \in M_m$ and $\hat{W} \in M_n$ are unitary matrices such that $A = \hat{V}\Sigma\hat{W}^*$, then

$$AA^* = VS_1V^* = \hat{V}S_1\hat{V}^*, \text{ so } S_1(V^*\hat{V}) = (V^*\hat{V})S_1$$

and

$$A^*A = WS_2W^* = \hat{W}S_2\hat{W}^*, \text{ so } S_2(W^*\hat{W}) = (W^*\hat{W})S_2$$

that is, $V^*\hat{V}$ and $W^*\hat{W}$ each commutes with a diagonal matrix with equal diagonal entries grouped together. A simple calculation (see (0.9.1) in [HJ]) reveals the basic fact that $V^*\hat{V}$ and $W^*\hat{W}$ must be block diagonal with diagonal blocks (necessarily unitary) conformal to those of S_1 and S_2, that is,

$$\hat{V} = V[V_1 \oplus \cdots \oplus V_k \oplus \tilde{V}] \text{ and } \hat{W} = W[W_1 \oplus \cdots \oplus W_k \oplus \tilde{W}]$$

for some unitary V_i, $W_i \in M_{\mu_i}$, $i = 1,\ldots, k$, $\tilde{V} \in M_{m-r}$, $\tilde{W} \in M_{n-r}$. But since $\hat{V}\Sigma\hat{W}^* = V\Sigma W^*$, it follows that $V_i = W_i$ for $i = 1,\ldots, k$. Thus, we can characterize the set of all possible left and right unitary factors in a singular value decomposition as follows:

3.1.1′ Theorem. Under the assumptions of Theorem (3.1.1), suppose that the distinct nonzero singular values of A are $s_1 > \cdots > s_k > 0$, with respective multiplicities $\mu_1,\ldots, \mu_k \geq 1$. Let $\mu_1 + \cdots + \mu_k = r$ and let $A = V\Sigma W^*$ be a given singular value decomposition with $\Sigma = \text{diag}(s_1 I_{\mu_1},\ldots, s_k I_{\mu_k}, 0_{q-r}) \in M_{m,n}$. Let $\hat{V} \in M_m$ and $\hat{W} \in M_n$ be given unitary matrices. Then $A = \hat{V}\Sigma\hat{W}^*$ if and only if there are unitary matrices $U_i \in M_{\mu_i}$, $i = 1,\ldots, k$, $\tilde{V} \in M_{m-r}$, and $\tilde{W} \in M_{n-r}$ such that

$$\hat{V} = V[U_1 \oplus \cdots \oplus U_k \oplus \tilde{V}] \text{ and } \hat{W} = W[U_1 \oplus \cdots \oplus U_k \oplus \tilde{W}]$$

One useful special case of the preceding characterization occurs when $m = n$ and $k = r = n$ or $n - 1$, that is, A is square and has distinct singular values. In this case, there are diagonal unitary matrices $D_1 = \text{diag}(e^{i\theta_1},\ldots, e^{i\theta_{n-1}}, d_1)$ and $D_2 = \text{diag}(e^{i\theta_1},\ldots, e^{i\theta_{n-1}}, d_2)$ such that $\hat{V} = VD_1$ and $\hat{W} =$

WD_2; if $r = n$ (A is nonsingular), then $d_1 = d_2$ and $D_1 = D_2$.

The construction used in the proof of Theorem (3.1.1) shows that

$\sigma_1(A) = \max \{\|Ax\|_2 \colon x \in \mathbb{C}^n, \|x\|_2 = 1\}$, so $\sigma_1 = \|Aw_1\|_2$ for some unit vector $w_1 \in \mathbb{C}^n$

$\sigma_2(A) = \max \{\|Ax\|_2 \colon x \in \mathbb{C}^n, \|x\|_2 = 1, x \perp w_1\}$, so $\sigma_2 = \|Aw_2\|_2$ for some unit vector $w_2 \in \mathbb{C}^n$ such that $w_2 \perp w_1$

$\vdots$

$\sigma_k(A) = \max \{\|Ax\|_2 \colon x \in \mathbb{C}^n, \|x\|_2 = 1, x \perp w_1, \ldots, w_{k-1}\}$, so $\sigma_k = \|Aw_k\|_2$ for some unit vector $w_k \in \mathbb{C}^n$ such that $w_k \perp w_1, \ldots, w_{k-1}$

$\vdots$

Thus, $\sigma_1(A) = |||A|||_2$, the spectral norm of A, and each singular value is the norm of A as a mapping restricted to a suitable subspace of $\mathbb{C}^n$. The similarity of this observation to the situation that holds for square Hermitian matrices is not merely superficial, and there is an analog of the Courant-Fischer theorem for singular values.

3.1.2 Theorem. Let $A \in M_{m,n}$ be given, let $\sigma_1(A) \geq \sigma_2(A) \geq \cdots$ be the ordered singular values of A, and let k be a given integer with $1 \leq k \leq \min\{m,n\}$. Then

$$\sigma_k(A) = \min_{w_1, \ldots, w_{k-1} \in \mathbb{C}^n} \max_{\substack{x \in \mathbb{C}^n \\ \|x\|_2 = 1 \\ x \perp w_1, \ldots, w_{k-1}}} \|Ax\|_2 \tag{3.1.2a}$$

$$= \max_{w_1, \ldots, w_{n-k} \in \mathbb{C}^n} \min_{\substack{x \in \mathbb{C}^n \\ \|x\|_2 = 1 \\ x \perp w_1, \ldots, w_{n-k}}} \|Ax\|_2 \tag{3.1.2b}$$

$$= \min_{\substack{S \subset \mathbb{C}^n \\ \dim S = n-k+1}} \max_{\substack{x \in S \\ \|x\|_2 = 1}} \|Ax\|_2 \tag{3.1.2c}$$

$$= \max_{\substack{S \subset \mathbb{C}^n \\ \dim S = k}} \min_{\substack{x \in S \\ \|x\|_2 = 1}} \|Ax\|_2 \tag{3.1.2d}$$

where the outer optimizations in (3.1.2(c,d)) are over all subspaces S with the indicated dimension.

Proof: Use the "min-max" half of the Courant-Fischer theorem ((4.2.12) in [HJ]) to characterize the decreasingly ordered eigenvalues of A^*A:

$$\lambda_k(A^*A) = \min_{w_1,\ldots,w_{k-1}\in\mathbb{C}^n} \max_{\substack{0\neq x\in\mathbb{C} \\ x\perp w_1,\ldots,w_{k-1}}} \frac{x^*(A^*A)x}{x^*x}$$

$$= \min_{w_1,\ldots,w_{k-1}\in\mathbb{C}^n} \max_{\substack{x\in\mathbb{C}^n \\ \|x\|_2=1 \\ x\perp w_1,\ldots,w_{k-1}}} \|Ax\|_2^2$$

which proves (3.1.2a) since $\lambda_k(A^*A) = \sigma_k(A)^2$. The same argument with the max-min half of the Courant-Fischer theorem ((4.2.13) in [HJ]) proves the characterization (3.1.2b). The alternative formulations (3.1.2(c,d)) are equivalent versions of (3.1.2(a,b)), in which the specification of x via a stated number of orthogonality constraints is replaced by specification of x via membership in a subspace—the orthogonal complement of the span of the constraints. $\square$

The Courant-Fischer theorem implies useful interlacing theorems for eigenvalues of Hermitian matrices, so it is not surprising that interlacing theorems for singular values of complex matrices follow from the preceding variational characterization. The proof of the following result is formally identical to the proof of the classical inclusion principle given for Theorem (4.3.15) in [HJ]. For a different proof, see Theorem (7.3.9) in [HJ].

3.1.3 Corollary. Let $A\in M_{m,n}$ be given, and let A_r denote a submatrix of A obtained by deleting a total of r rows and/or columns from A. Then

$$\sigma_k(A) \geq \sigma_k(A_r) \geq \sigma_{k+r}(A), \quad k = 1,\ldots, \min\{m,n\} \tag{3.1.4}$$

where for $X\in M_{p,q}$ we set $\sigma_j(X)\equiv 0$ if $j>\min\{p,q\}$.

Proof: It suffices to consider the case $r=1$, in which any one row or column is deleted, and to show that $\sigma_k(A)\geq\sigma_k(A_1)\geq\sigma_{k+1}(A)$. The general case then follows by repeated application of these inequalities. If A_1 is formed from A by deleting column s, denote by e_s the standard unit basis vector with a 1 in position s. If $x \in \mathbb{C}^n$, denote by $\xi \in \mathbb{C}^{n-1}$ the vector obtained by deleting entry s from x. Now use (3.1.2a) to write

$$\begin{aligned}
\sigma_k(A) &= \min_{w_1,\ldots,w_{k-1} \in \mathbb{C}^n} \ \max_{\substack{x \in \mathbb{C}^n \\ \|x\|_2 = 1 \\ x \perp w_1,\ldots,w_{k-1}}} \|Ax\|_2 \\
&\geq \min_{w_1,\ldots,w_{k-1} \in \mathbb{C}^n} \ \max_{\substack{x \in \mathbb{C}^n \\ \|x\|_2 = 1 \\ x \perp w_1,\ldots,w_{k-1}, e_s}} \|Ax\|_2 \\
&= \min_{\omega_1,\ldots,\omega_{k-1} \in \mathbb{C}^{n-1}} \ \max_{\substack{\xi \in \mathbb{C}^{n-1} \\ \|\xi\|_2 = 1 \\ \xi \perp \omega_1,\ldots,\omega_{k-1}}} \|A_1\xi\|_2 = \sigma_k(A_1)
\end{aligned}$$

For the second inequality, use (3.1.2b) to write

$$\begin{aligned}
\sigma_{k+1}(A) &= \max_{w_1,\ldots,w_{n-k-1} \in \mathbb{C}^n} \ \min_{\substack{x \in \mathbb{C}^n \\ \|x\|_2 = 1 \\ x \perp w_1,\ldots,w_{n-k-1}}} \|Ax\|_2 \\
&\leq \max_{w_1,\ldots,w_{n-k-1} \in \mathbb{C}^n} \ \min_{\substack{x \in \mathbb{C}^n \\ \|x\|_2 = 1 \\ x \perp w_1,\ldots,w_{n-k-1}, e_s}} \|Ax\|_2 \\
&= \max_{\omega_1,\ldots,\omega_{n-k-1} \in \mathbb{C}^{n-1}} \ \min_{\substack{\xi \in \mathbb{C}^{n-1} \\ \|\xi\|_2 = 1 \\ \xi \perp \omega_1,\ldots,\omega_{n-k-1}}} \|A_1\xi\|_2 = \sigma_k(A_1)
\end{aligned}$$

If a row of A is deleted, apply the same argument to A^*, which has the same singular values as A. $\square$

By combining Theorem (3.1.2) with the Courant-Fischer theorem for Hermitian matrices, we can obtain useful inequalities between individual singular values of a matrix and eigenvalues of its Hermitian part.

3.1.5 Corollary. Let $A \in M_n$ be given, let $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$ denote its ordered singular values, let $H(A) = \frac{1}{2}(A + A^*)$, and let $\{\lambda_i(H(A))\}$ denote the algebraically decreasingly ordered eigenvalues of $H(A)$, $\lambda_1(H(A)) \geq \cdots \geq \lambda_n(H(A))$. Then

$$\sigma_k(A) \geq \lambda_k(H(A)) \text{ for } k = 1,\dots, n \tag{3.1.6a}$$

More generally,

$$\sigma_k(A) \geq \lambda_k(H(UAV)) \text{ for all } k = 1,\dots, n \text{ and all unitary } U, V \in M_n \tag{3.1.6b}$$

Proof: For any unit vector $x \in \mathbb{C}^n$, we have

$$\begin{aligned} x^* H(A) x &= \tfrac{1}{2}(x^* A x + x^* A^* x) = \operatorname{Re} x^* A x \leq |x^* A x| \\ &\leq \|x\|_2 \, \|Ax\|_2 = \|Ax\|_2 \end{aligned}$$

Thus,

$$\begin{aligned} \lambda_k(H(A)) &= \min_{w_1, \dots, w_{k-1} \in \mathbb{C}^n} \; \max_{\substack{x \in \mathbb{C}^n \\ \|x\|_2 = 1 \\ x \perp w_1, \dots, w_{k-1}}} x^* H(A) x \\ &\leq \min_{w_1, \dots, w_{k-1} \in \mathbb{C}^n} \; \max_{\substack{x \in \mathbb{C}^n \\ \|x\|_2 = 1 \\ x \perp w_1, \dots, w_{k-1}}} \|Ax\|_2 = \sigma_k(A) \end{aligned}$$

The more general assertion (3.1.6b) follows from (3.1.6a) since $\sigma_k(A) = \sigma_k(UAV)$ for every $U, V \in M_n$. $\square$

We now consider several matrix representations associated with the singular value decomposition. One way to phrase the spectral theorem for normal matrices is to say that every n-by-n normal matrix is a complex linear combination of n pairwise orthogonal Hermitian projections. An analogous result, valid for all matrices, follows immediately from the singular value decomposition.

3.1.7 Definition. A matrix $P \in M_{m,n}$ is said to be a *rank r partial isometry* if $\sigma_1(P) = \cdots = \sigma_r(P) = 1$ and $\sigma_{r+1}(P) = \cdots = \sigma_q(P) = 0$, where $q \equiv \min\{m,n\}$. Two partial isometries $P, Q \in M_{m,n}$ (of unspecified rank) are said to be *orthogonal* if $P^*Q = 0$ and $PQ^* = 0$.

The asserted representations in the following theorem are readily verified.

3.1.8 Theorem. Let $A \in M_{m,n}$ have singular value decomposition $A = V\Sigma W^*$ with $V = [v_1 \ \ldots \ v_m] \in M_m$ and $W = [w_1 \ \ldots \ w_n] \in M_n$ unitary, and $\Sigma = [\sigma_{ij}] \in M_{m,n}$ with $\sigma_1 = \sigma_{11} \geq \cdots \geq \sigma_q = \sigma_{qq} \geq 0$ and $q = \min\{m,n\}$. Then

(a) $A = \sigma_1 P_1 + \cdots + \sigma_q P_q$ is a nonnegative linear combination of mutually orthogonal rank one partial isometries, with $P_i = v_i w_i^*$ for $i = 1,\ldots, q$.

(b) $A = \mu_1 K_1 + \cdots + \mu_q K_q$ is a nonnegative linear combination of partial isometries with rank $K_i = i$, $i = 1,\ldots, q$, such that

(1) $\mu_i = \sigma_i - \sigma_{i+1}$ for $i = 1,\ldots, q-1$, $\mu_q = \sigma_q$;
(2) $\mu_i + \cdots + \mu_q = \sigma_i$ for $i = 1,\ldots, q$; and
(3) $K_i = VE_iW^*$ for $i = 1,\ldots, q$ in which the first i columns of $E_i \in M_{m,n}$ are the respective unit basis vectors $e_1,\ldots, e_i$ and the remaining $n - i$ columns are zero.

Another useful representation that follows immediately from the singular value decomposition is the *polar decomposition*.

3.1.9 Theorem. Let $A \in M_{m,n}$ be given.

(a) If $n \geq m$, then $A = PY$, where $P \in M_m$ is positive semidefinite, $P^2 = AA^*$, and $Y \in M_{m,n}$ has orthonormal rows.
(b) If $m \geq n$, then $A = XQ$, where $Q \in M_n$ is positive semidefinite, $Q^2 = A^*A$, and $X \in M_{m,n}$ has orthonormal columns.
(c) If $m = n$, then $A = PU = UQ$, where $U \in M_n$ is unitary, $P, Q \in M_n$ are positive semidefinite, $P^2 = AA^*$, and $Q^2 = A^*A$.

In all cases, the positive semidefinite factors P and Q are uniquely determined by A and their eigenvalues are the same as the singular values of A.

Proof: If $n \geq m$ and $A = V\Sigma W^*$ is a singular value decomposition, write $\Sigma = [S \;\; 0]$ and $W = [W_1 \;\; W_2]$, where $S = \mathrm{diag}(\sigma_1(A),\ldots,\sigma_m(A)) \in M_m$ and $W_1 \in M_{n,m}$. Then $A = V[S \;\; 0][W_1 \;\; W_2]^* = VSW_1^* = (VSV^*)(VW_1^*)$. Notice that $P \equiv VSV^*$ is positive semidefinite and $Y \equiv VW_1^*$ satisfies $YY^* = VW_1^*W_1V^* = VIV^* = I$, so Y has orthonormal rows. The assertions in (b) follow from applying (a) to A^*. For (c), notice that $A = V\Sigma W^* = (V\Sigma V^*)(VW^*) = (VW^*)(W\Sigma W^*)$, so we may take $P = V\Sigma V^*$, $Q = W\Sigma W^*$, and $U = VW^*$. $\square$

Exercise. In the square case (c) of Theorem (3.1.9), use the characterization in Theorem (3.1.1′) to show that all possible unitary factors U (left or right) in the polar decomposition of $A \in M_n$ are of the form $U = V[I \oplus \tilde{U}]W^*$, where the unitary matrix $\tilde{U} \in M_{n-r}$ is arbitrary, $r = \mathrm{rank}\, A$, and $A = V\Sigma W^*$ is a given singular value decomposition. In particular, conclude that all three factors P, Q, and U are uniquely determined when A is nonsingular.

Problems

1. Provide the details for the following proof of the singular value decomposition:

(a) Let $A \in M_n$ be given. Then AA^* and A^*A are both normal and have the same eigenvalues, so they are unitarily similar. Let $U \in M_n$ be unitary and such that $A^*A = U(AA^*)U^*$. Then UA is normal, so there is a unitary $X \in M_n$ and a diagonal $\Lambda \in M_n$ such that $UA = X\Lambda X^*$. Write $\Lambda = \Sigma D$, where $\Sigma = |\Lambda|$ is nonnegative and D is a diagonal unitary matrix. Then $A = V\Sigma W^*$ with $V = U^*X$, $W = DX^*$. This is essentially the approach to the singular value decomposition used in L. Autonne's 1915 paper cited in Section (3.0).

(b) If $A \in M_{m,n}$ with $m > n$, let $u_1,\ldots,u_\nu$ be an orthonormal basis for the nullspace of A^*, so $\nu \geq m-n$. Let $U_2 \equiv [u_1 \; \ldots \; u_{m-n}] \in M_{m,m-n}$ and let $U = [U_1 \;\; U_2] \in M_m$ be unitary. Then

$$U^*A = \begin{bmatrix} A_1 \\ 0 \end{bmatrix}$$

with $A_1 \in M_n$, so

$$A = U\begin{bmatrix} V\Sigma W^* \\ 0 \end{bmatrix} = U(V \oplus I)\begin{bmatrix} \Sigma \\ 0 \end{bmatrix} W^*$$

If $m < n$, apply this result to A^*.

2. Let $A \in M_{m,n}$. Explain why the rank of A is exactly the number of its nonzero singular values.

3. If $A \in M_{m,n}$ and $A = V\Sigma W^*$ is a singular value decomposition, what are singular value decompositions of A^*, A^T, $\bar{A}$ and, if $m = n$ and A is nonsingular, A^{-1}? Conclude that the singular values of A, A^*, A^T, and $\bar{A}$ are all the same, and, if $A \in M_n$ is nonsingular, the singular values of A are the reciprocals of the singular values of A^{-1}.

4. Let $A = [a_{ij}] \in M_{m,n}$ have a singular value decomposition $A = V\Sigma W^*$ with unitary $V = [v_{ij}] \in M_m$ and $W = [w_{ij}] \in M_n$, and let $q = \min\{m,n\}$.

(a) Show that each $a_{ij} = v_{i1}\bar{w}_{j1}\sigma_1(A) + \cdots + v_{iq}\bar{w}_{jq}\sigma_q(A)$.

(b) Use the representation in (a) to show that

$$\sum_{i=1}^{q} |a_{ii}| \leq \sum_{k=1}^{q}\sum_{i=1}^{q} |v_{ik}w_{ik}|\ \sigma_k(A) \leq \sum_{k=1}^{q} \sigma_k(A) \tag{3.1.10a}$$

(c) When $m = n$, use (a) and the conditions for equality in the Cauchy-Schwarz inequality to show that

$$\operatorname{Re}\operatorname{tr} A \leq \sum_{i=1}^{n} \sigma_i(A) \tag{3.1.10b}$$

with equality if and only if A is positive semidefinite.

(d) When $m = n$, let $A = U\Delta U^*$ with U unitary and Δ upper triangular, let $\lambda_1(A),\ldots,\lambda_n(A)$ be the main diagonal entries of Δ, and let $D \equiv \operatorname{diag}(e^{i\theta_1},\ldots,e^{i\theta_n})$ with each $\theta_k \in \mathbb{R}$. What are the eigenvalues and singular values of the matrix $AUDU^*$? Use (c) to show that

$$\operatorname{Re}\operatorname{tr} A \leq |\operatorname{tr} A| \leq \sum_{i=1}^{n} |\lambda_i(A)| \leq \sum_{i=1}^{n} \sigma_i(A) \tag{3.1.10c}$$

See (3.3.35) and (3.3.13a,b) for generalizations of these inequalities.

5. Recall that a matrix $C \in M_n$ is a *contraction* if $\sigma_1(C) \leq 1$ (and hence $0 \leq$

$\sigma_i(C) \leq 1$ for all $i = 1, 2,\ldots, n$). All matrices in this problem are in M_n.
(a) Show that the unitary matrices are the only rank n (partial) isometries in M_n.
(b) Show that any finite product of contractions is a contraction.
(c) Is a product of partial isometries a partial isometry?
(d) Describe all the rank r partial isometries and contractions in M_n in terms of their singular value decompositions.
(e) Show that $C \in M_n$ is a rank one partial isometry if and only if $C = xy^*$ for some unit vectors $x, y \in \mathbb{C}^n$.
(f) For $1 \leq r < n$, show that every rank r partial isometry in M_n is a convex combination of two distinct unitary matrices in M_n.
(g) Use Theorem (3.1.8) to show that every matrix in M_n is a finite nonnegative linear combination of unitary matrices in M_n.
(h) Use Theorem (3.1.8(b)) to show that a given $A \in M_n$ is a contraction if and only if it is a finite convex combination of unitary matrices in M_n. See Problem 27 for another approach to this result.

6. Let $A \in M_{m,n}$. Show that $\sigma_1(A) = \max\{|x^*Ay|: x, y \in \mathbb{C}^n$ are unit vectors$\}$. If $m = n$, show that $\sigma_1(A) = \max\{|\mathrm{tr}\, AC|: C \in M_n$ is a rank one partial isometry$\}$. See Theorem (3.4.1) for an important generalization of this result.

7. Let $A \in M_n$ be given and let r be a given integer with $1 \leq r \leq n$. Show that there is a partial isometry $C_r \in M_n$ such that $\sigma_1(A) + \cdots + \sigma_r(A) = \mathrm{tr}\, AC_r$. What is this when $r = 1$? Compare with Problem 6. See Theorem (3.3.1) for a proof that $\sigma_1 + \cdots + \sigma_r = \max\{|\mathrm{tr}\, AC_r|: C_r \in M_n$ is a rank r partial isometry$\}$.

8. Let $A \in M_n$. A vector $x \in \mathbb{C}^n$ such that $Ax = x$ is called a *fixed point* of A; the nonzero fixed points of A are just its eigenvectors corresponding to the eigenvalue $\lambda = 1$.

(a) If A is a contraction, show that every fixed point of A is a fixed point of A^*.

(b) Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ to show that the assertion in (a) is not generally true if A is not a contraction.

9. Let $A \in M_n$, and let $U, V \in M_n$ be unitary. Show that the singular values of A and UAV are the same. Are the eigenvalues the same? The eigenvalues of A^2 are the squares of the eigenvalues of A. Is this true for the singular values as well?

10. Let $A \in M_n$.

(a) Show that $\sigma_1(A) \geq \rho(A) = \max\{|\lambda|: \lambda \text{ is an eigenvalue of } A\}$.

(b) Let $x, y \in \mathbb{C}^n$ and let $A \equiv xy^*$. Calculate $\sigma_1(A)$ and $\rho(A)$. What does the general inequality $\sigma_1(A) \geq \rho(A)$ give in this case?

11. Let $A \in M_n$. Show that $\sigma_1(A) \cdots \sigma_n(A) = |\det A| = |\lambda_1(A) \cdots \lambda_n(A)|$.

12. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in M_n$. What are the eigenvalues and singular values of A? Verify that $\sigma_1(A) \geq \rho(A)$ and $\sigma_1(A)\sigma_2(A) = |\lambda_1(A)\lambda_2(A)| = |\det A|$.

13. Show that all the singular values of a matrix $A \in M_n$ are equal if and only if A is a scalar multiple of a unitary matrix.

14. Let $1 \leq i \leq n$ be a given integer and consider the function $\sigma_i(\cdot)$: $M_n \to \mathbb{R}^+$. Show that $\sigma_i(cA) = |c|\,\sigma_i(A)$ for all $A \in M_n$ and all $c \in \mathbb{C}$, and $\sigma_i(A) \geq 0$ for all $A \in M_n$. Is $\sigma_i(\cdot)$ a norm on M_n? Is it a seminorm?

15. Verify the assertions in Theorem (3.1.8).

16. State an analog of (3.1.8b) for normal matrices in M_n.

17. If $A \in M_n$ is positive semidefinite, show that the eigenvalues and singular values of A are the same.

18. Let $A = [a_{ij}] \in M_n$ have eigenvalues $\{\lambda_i(A)\}$ and singular values $\{\sigma_i(A)\}$, and let $A = U\Delta U^* = U(\Lambda + T)U^*$ be a Schur upper triangularization of A, that is, U is unitary, $\Lambda = \operatorname{diag}(\lambda_1(A),\ldots, \lambda_n(A))$, and $T = [t_{ij}]$ is strictly upper triangular. Let $\|A\|_2 = (\operatorname{tr} A^*A)^{\frac{1}{2}}$ denote the Frobenius norm of A.

(a) Show that

$$\|A\|_2^2 = \sum_{i=1}^{n} \sigma_i(A)^2 = \|\Lambda\|_2^2 + \|T\|_2^2 \geq \sum_{i=1}^{n} |\lambda_i(A)|^2 \qquad (3.1.11)$$

with equality if and only if A is normal. Although different Schur upper triangularizations of A can result in different strictly upper triangular parts T in (3.1.11), notice that the quantity

$$\|T\|_2 = \left[\sum_{i=1}^{n} \sigma_i(A)^2 - \sum_{i=1}^{n} |\lambda_i(A)|^2\right]^{\frac{1}{2}} \qquad (3.1.12)$$

has the same value for all of them. This is called the *defect from normality* of A *with respect to the Frobenius norm.*

(b) Let $x_1,\ldots, x_n \geq 0$ be given and let $a_{i,i+1} = \sqrt{x_i}$, $i = 1,\ldots, n-1$, $a_{n1} = \sqrt{x_n}$, and all other $a_{ij} = 0$. Show that the eigenvalues of A are the n nth roots of $(x_1 \cdots x_n)^{\frac{1}{2}}$ and derive the arithmetic-geometric mean inequality $x_1 + \cdots + x_n \geq n(x_1 \cdots x_n)^{1/n}$ from the inequality in (a).

19. Let $A \in M_n$ have singular values $\sigma_1(A) \geq \cdots \geq \sigma_n(A) \geq 0$ and eigenvalues $\{\lambda_i(A)\}$ with $|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|$. Show that A is normal if and only if $\sigma_i(A) = |\lambda_i(A)|$ for all $i = 1,\ldots, n$. More generally, if $\sigma_i(A) = |\lambda_i(A)|$ for $i = 1,\ldots, k$, show that A is unitarily similar to $D \oplus B$, where $D = \text{diag}(\lambda_1,\ldots, \lambda_k)$, $B \in M_{n-k}$, and $\sigma_k(A) = |\lambda_k(A)| \geq \sigma_1(B)$.

20. Using the notation of Theorem (3.1.1), let $W = [w_1 \ \ldots \ w_n]$ and $V = [v_1 \ \ldots \ v_m]$, where the orthonormal sets $\{v_i\} \subset \mathbb{C}^m$ and $\{w_i\} \subset \mathbb{C}^n$ are left and right singular vectors of A, respectively. Show that $Aw_i = \sigma_i v_i$, $A^* v_i = \sigma_i w_i$, $\|A^* v_i\|_2 = \sigma_i$, and $\|Aw_i\|_2 = \sigma_i$ for $i = 1,\ldots, \min\{m,n\}$. These are the matrix analogs of E. Schmidt's integral equations (3.0.5).

21. Provide details for the following proof of the singular value decomposition that relies on the spectral theorem for Hermitian matrices (Theorem (4.1.5) in [HJ]): Let $A \in M_{m,n}$ have rank r. Then $A^*A \in M_n$ is Hermitian and positive semidefinite, so $A^*A = U(\Lambda^2 \oplus 0)U^* = UD(I_r \oplus 0)DU^*$, where $U \in M_n$ is unitary, $\Lambda \in M_r$ is diagonal and positive definite, and $D = \Lambda \oplus I_{n-r} \in M_n$. Then $(AUD^{-1})^*(AUD^{-1}) = I_r \oplus 0$, so $AUD^{-1} = [V_1 \ \ 0]$, where $V_1 \in M_{m,r}$ has orthonormal columns. If $V = [V_1 \ \ V_2] \in M_m$ is unitary, then $A = [V_1 \ \ 0]DU^* = V(\Lambda \oplus 0)U^* = V\Sigma W^*$.

22. Deduce the singular value decomposition (3.1.1) from the polar decomposition (3.1.9).

23. According to Corollary (3.1.3), deleting some rows or columns of a matrix may decrease some of its singular values. However, if the rows or columns deleted are all zero, show that the nonzero singular values are unchanged, and some previously zero singular values are deleted.

24. Let $x \in \mathbb{C}^n$ be given. If one thinks of x as an n-by-1 matrix, that is, $x \in M_{n,1}$, what is its singular value decomposition? What is the singular value?

25. Suppose $A \in M_n$ and $\sigma_1(A) \leq 1$, that is, A is a contraction. If $\text{H}(A) \equiv \frac{1}{2}(A + A^*) = I$, show that $A = I$.

26. Let $U \in M_n$ be a given unitary matrix and suppose $U = \frac{1}{2}(A + B)$ for some contractions $A, B \in M_n$. Show that $A = B = U$.

27. Let B_n denote the unit ball of the spectral norm in M_n, that is, $B_n = \{A \in M_n\colon\ \sigma_1(A) \leq 1\}$ is the set of contractions in M_n. Like the unit ball of any norm, B_n is a convex set.

(a) If $A \in B_n$ and if A is an extreme point of B_n, show that A is unitary.

(b) If $U \in M_n$ is unitary, use Problem 26 to show that U is an extreme point of B_n.

(c) Conclude that the extreme points of B_n are exactly the unitary matrices in M_n.

(d) Use (c) to show that $A \in B_n$ if and only if A is a finite convex combination of unitary matrices. See Problem 5 as well as Problem 4 of Section (3.2) for different approaches to this result.

28. Let $A \in M_n$ be given and let $r = \text{rank } A$. Show that A is normal if and only if there is a set $\{x_1, \ldots, x_r\} \subset \mathbb{C}^n$ of r orthonormal vectors such that $|x_i^* A x_i| = \sigma_i(A)$ for $i = 1, \ldots, r$.

29. Let $A \in M_n$, $B \in M_m$. Show that the set of singular values of the direct sum $A \oplus B$ is the union of the sets of singular values of A and B, including multiplicities. Show that $\sigma_1(A \oplus B) = \max\{\sigma_1(A), \sigma_1(B)\}$.

30. Let $A \in M_n$ be given. Use (3.1.4) to show that $\sigma_1(A) \geq$ maximum Euclidean column or row length in A and $\sigma_n(A) \leq$ minimum Euclidean column or row length in A. If A is nonsingular, conclude that $\kappa(A) = \sigma_1(A)/\sigma_n(A)$, the spectral condition number of A, is bounded from below by the ratio of the largest to smallest Euclidean lengths of the set of rows and columns of A. Thus, if a system of linear equations $Ax = b$ is poorly scaled (that is, its ratio of largest to smallest row and column norms is large), then the system must be ill conditioned. This *sufficient* condition for ill conditioning is not *necessary*, however. Give an example of an ill-conditioned A for which all the rows and columns have nearly the same norm.

31. Let $A \in M_n$ be given and have singular values $\sigma_1(A) \geq \cdots \geq \sigma_n(A) \geq 0$. If $\mathrm{H}(A) = \frac{1}{2}(A + A^*)$ is positive semidefinite and if $\alpha \geq 0$ is a given scalar, show that $\sigma_i(A + \alpha I)^2 \geq \sigma_i(A)^2 + \alpha^2$ for $i = 1, \ldots, n$. Show by example that this need not be true, and indeed $\sigma_i(A + \alpha I) < \sigma_i(A)$ is possible, if $\mathrm{H}(A)$ is not positive semidefinite.

32. Suppose $A \in M_n$ is skew-symmetric, that is, $A = -A^T$. Show that the nonzero singular values of A are of the form $s_1, s_1, s_2, s_2, \ldots, s_k, s_k$, where

$s_1,\ldots, s_k > 0$, that is, A has even rank and its nonzero singular values occur in pairs.

33. The purpose of this problem is to use the singular value decomposition to show that any complex symmetric $A \in M_n$ can be written as $A = U\Sigma U^T$ for some unitary $U \in M_n$ and nonnegative diagonal $\Sigma \in M_n$. The approach suggested is essentially the one used by L. Autonne in his 1915 paper cited in Section (3.0).

(a) Let $D = d_1 I_{n_1} \oplus \cdots \oplus d_r I_{n_r} \in M_n$ be diagonal with $|d_1| > \cdots > |d_r| \geq 0$ and $n_1 + \cdots + n_r = n$, let a unitary $U \in M_n$ be given, and suppose $DU = U^T D$. Show that $U = U_1 \oplus \cdots \oplus U_r$, where each $U_i \in M_{n_i}$ is unitary and $U_i = U_i^T$ if $d_i \neq 0$.

(b) Let $B \in M_n$ be a given diagonalizable matrix. Show that there is a polynomial $p(t)$ of degree at most $n-1$ such that $C \equiv p(B)$ satisfies $C^2 = B$. If, in addition, $B = B^T$, then $C = C^T$. Using the notation and hypothesis of part (a), show that there is a unitary $Z \in M_n$ such that $Z^2 = U$ and $DZ = Z^T D$.

(c) Let $A \in M_n$ be given and let $A = V\Sigma W^*$ be a given singular value decomposition for A. If $A = A^T$, show that $\Sigma(V^T W) = (V^T W)^T \Sigma$. Show that there is a unitary Z such that $V = \overline{W}(Z^T)^2$ and $Z^T\Sigma = \Sigma Z$, and show that $A = (\overline{W}Z^T)\Sigma(\overline{W}Z^T)^T = U\Sigma U^T$, as desired.

34. The interlacing inequalities (3.1.4) have been shown to be necessary constraints on the singular values of a submatrix of given size, relative to the singular values of the overall matrix. They are also known to be sufficient, in the sense that they precisely characterize the possible ranges of singular values for submatrices (of a given size) of a matrix with given singular values. The purpose of this problem is to demonstrate their sufficiency in an interesting special case.

(a) Let $A \in M_n$ have singular values $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$. Denote the singular values of the upper left $(n-1)$-by-$(n-1)$ principal submatrix of A by $s_1 \geq \cdots \geq s_n \geq 0$. Show that

$$\sigma_1 \geq s_1 \geq \sigma_3,\quad \sigma_2 \geq s_2 \geq \sigma_4,\ldots,\quad \sigma_{n-2} \geq s_{n-2} \geq \sigma_n,\quad \sigma_{n-1} \geq s_{n-1} \geq 0 \qquad (3.1.13)$$

(b) Let $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ be given and let $\Sigma \equiv \mathrm{diag}(\sigma_1,\ldots, \sigma_n)$. Explain why $\{U\Sigma U^T\colon\ U \in M_n \text{ is unitary}\}$ is exactly the set of symmetric matrices in M_n with singular values $\sigma_1,\ldots, \sigma_n$.

(c) Let $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ and $s_1 \geq \cdots \geq s_{n-1} \geq 0$ satisfy the interlacing inequalities (3.1.13). Show that the set of $n-1$ numbers $\{s_1, -s_2, s_3, -s_4, \ldots\}$ interlaces the set of n numbers $\{\sigma_1, -\sigma_2, \sigma_3, -\sigma_4, \ldots\}$ (in the sense of the hypothesis of Theorem (4.3.10) in [HJ]) when both are put into algebraically decreasing order. Conclude that there is a real symmetric $Q \in M_n(\mathbb{R})$ such that the upper left $(n-1)$-by-$(n-1)$ principal submatrix of $Q\,\mathrm{diag}(\sigma_1, -\sigma_2, \sigma_3, -\sigma_4, \ldots)\,Q^T \equiv A$ has eigenvalues $s_1, -s_2, s_3, -s_4, \ldots$. Explain why the singular values of A are $\sigma_1, \ldots, \sigma_n$ and the singular values of the principal submatrix are $s_1, \ldots, s_n$.

(d) Let $A \in M_n$ be a given symmetric matrix with singular values $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ [for example, $A = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$], and let $s_1 \geq \cdots \geq s_{n-1} \geq 0$ be given. Explain why there is a unitary $U \in M_n$ such that UAU^T has singular values $\sigma_1, \ldots, \sigma_n$ and its upper left $(n-1)$-by-$(n-1)$ principal submatrix has singular values $s_1, \ldots, s_{n-1}$ if and only if the interlacing inequalities (3.1.13) are satisfied.

35. Let $A \in M_{m,n}$ be given, suppose it has $q = \min\{m,n\}$ distinct singular values, let $A = V\Sigma W^*$ be a singular value decomposition, and let $A = \hat{V}\Sigma\hat{W}^*$ be another given singular value decomposition. If $m \leq n$, show that there is a diagonal unitary $D \in M_m$ such that $\hat{V} = VD$. If $m \geq n$, show that there is a diagonal unitary $D \in M_n$ such that $\hat{W} = WD$. In these two cases, how are the other unitary factors related?

36. Let $A \in M_n$ be given, let $\mathrm{H}(A) = \frac{1}{2}(A + A^*)$ denote the Hermitian part of A, and let $\mathrm{S}(A) = \frac{1}{2}(A - A^*)$ denote the skew-Hermitian part of A. Order the eigenvalues of $\mathrm{H}(A)$ and $\mathrm{S}(A)$ so that $\lambda_1(\mathrm{H}(A)) \geq \cdots \geq \lambda_n(\mathrm{H}(A))$ and $\lambda_1(i\mathrm{S}(A)) \geq \cdots \geq \lambda_n(i\mathrm{S}(A))$. Use Corollary (3.1.5) to show that

$$\sigma_k(A) \geq \max\{\lambda_k(\mathrm{H}(A)), -\lambda_{n-k+1}(\mathrm{H}(A)), \lambda_k(i\mathrm{S}(A)), -\lambda_{n-k+1}(i\mathrm{S}(A))\} \tag{3.1.14}$$

for $k = 1, \ldots, n$.

37. Recall the Loewner partial order on n-by-n Hermitian matrices: $A \succeq B$ if and only if $A - B$ is positive semidefinite. For $A \in M_n$, we write $|A| \equiv (A^*A)^{\frac{1}{2}}$.

(a) If $A \in M_n$ has a singular value decomposition $A = V\Sigma W^*$, use Corollary (3.1.5) to show that $|A| = W\Sigma W^* \succeq 0$. Show that $|UA| = |A|$ for any unitary $U \in M_n$.

(b) If $\mathrm{H}(A) = \frac{1}{2}(A + A^*)$ has a spectral decomposition $\mathrm{H}(A) = U\Lambda U^*$

with a unitary $U \in M_n$, $\Lambda = \operatorname{diag}(\lambda_1(A),\dots, \lambda_n(A))$, and $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$, and if $A = V\Sigma W^*$ is a singular value decomposition with $\Sigma = \operatorname{diag}(\sigma_1(A),\dots, \sigma_n(A))$, use (3.1.6a) to show that $\Sigma \succeq \Lambda$, $U\Sigma U^* \succeq \mathrm{H}(A)$, and $(UW^*)|A|(UW^*)^* \succeq \mathrm{H}(A)$, where $Z \equiv UW^*$ is unitary. Conclude that for each given $X \in M_n$ there is some unitary $Z \in M_n$ (Z depends on X) such that $Z|X|Z^* \succeq \mathrm{H}(X)$.

(c) Show that Corollary (3.1.5) is equivalent to the assertion that for each given $X \in M_n$ there is some unitary $Z \in M_n$ such that $Z|X|Z^* \succeq \mathrm{H}(X)$.

(d) For each given $X \in M_n$, why is there some unitary $U \in M_n$ such that $UX \succeq 0$?

(e) Let $A, B \in M_n$ be given. Show that there are unitary $U_1, U_2 \in M_n$ such that

$$|A + B| \preceq U_1|A|U_1^* + U_2|B|U_2^* \tag{3.1.15}$$

This is often called the *matrix-valued triangle inequality*.

(f) Show by example that $|A + B| \preceq |A| + |B|$ is not generally true.

38. Let $A, B \in M_{m,n}$ be given. Show that there exist unitary $V \in M_m$ and $W \in M_n$ such that $A = VBW$ if and only if A and B have the same singular values. Thus, the set of singular values of a matrix is a complete set of invariants for the equivalence relation of unitary equivalence on $M_{m,n}$.

39. Let $A \in M_n$ be given, and let $\mathrm{H}(A) = \frac{1}{2}(A + A^*)$ and $\mathrm{S}(A) = \frac{1}{2}(A - A^*)$ denote its Hermitian and skew-Hermitian parts, respectively. The singular values of $\mathrm{H}(A)$ and $\mathrm{S}(A)$ (which are the absolute values of their eigenvalues), respectively, are sometimes called the *real singular values* and *imaginary singular values* of A; in this context, the ordinary singular values are sometimes called the *absolute singular values* of A. Show that

$$\sum_{i=1}^{n} \sigma_i(A)^2 = \sum_{i=1}^{n} \sigma_i(\mathrm{H}(A))^2 + \sum_{i=1}^{n} \sigma_i(\mathrm{S}(A))^2$$

40. Let $A \in M_{m,n}$ be given, let $q = \min\{m,n\}$, and let $S = \operatorname{diag}(\sigma_1(A),\dots, \sigma_q(A))$. Use the singular value decomposition to show that the Hermitian block matrix

$$\mathcal{A} = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \in M_{m+n}$$

is unitarily similar to the diagonal matrix $D \oplus (-D) \oplus 0_{m+n-2q} \in M_{m+n}$. Thus, the algebraically ordered eigenvalues of $\mathcal{A}$ are

$$\sigma_1(A) \geq \cdots \geq \sigma_q(A) \geq \underbrace{0 = \cdots = 0}_{(m+n-2q \text{ times})} \geq -\sigma_q(A) \geq \cdots \geq -\sigma_1(A)$$

This observation, which was the foundation of Jordan's 1874 development of the singular value decomposition, can be useful in converting results about Hermitian matrices into results about singular values, and vice versa; see Sections (7.3-4) of [HJ] for examples.

41. Give an example of $A, B \in M_2$ for which the singular values of AB and BA are different. What about the eigenvalues?

42. Let $A \in M_n$ be given. Use Theorem (3.1.1) to show that:
(a) There exists a unitary $U \in M_n$ such that $A^* = UAU$.
(b) There exists a unitary $U \in M_n$ such that AU is Hermitian (even positive semidefinite).

43. Let $A \in M_n$ be given.
(a) Use Theorem (3.1.1) to show that there is an $\mathcal{A} \in M_n$ that is unitarily similar to A and satisfies $\mathcal{A}^*\mathcal{A} = \Sigma^2 = \text{diag}(\sigma_1(A)^2,\ldots, \sigma_n(A)^2)$. Let $x \in \mathbb{C}^n$ be a unit vector such that $\mathcal{A}x = \lambda x$. Compute $\|\mathcal{A}x\|_2^2 = \|\lambda x\|_2^2$ and show that any eigenvalue $\lambda(A)$ of A satisfies $\sigma_n(A) \leq |\lambda(A)| \leq \sigma_1(A)$.
(b) Use the fact that $\rho(A) \leq |||A|||_2$ (simply because the spectral norm is a matrix norm) to show that $\sigma_n(A) \leq |\lambda(A)| \leq \sigma_1(A)$ for every eigenvalue $\lambda(A)$ of A.

44. Let $C \in M_n$ be a contraction. Use the singular value decomposition to show that $(I - CC^*)^{\frac{1}{2}}C = C(I - C^*C)^{\frac{1}{2}}$, where each square root is the unique positive semidefinite square root of the indicated matrix. This identity is useful when working with the representation (1.5.21) for the unit ball of the numerical radius norm.

45. Let $A, B \in M_n$ be given and suppose $A = SBS^{-1}$ for some nonsingular $S \in M_n$. Let $\kappa_2(S) \equiv \sigma_1(S)/\sigma_n(S)$ denote the spectral condition number of S. Use Theorem (4.5.9) in [HJ] to show that

$$\sigma_k(B)/\kappa_2(S) \leq \sigma_k(A) \leq \kappa_2(S)\sigma_k(B), \quad k = 1,\ldots, n \tag{3.1.16}$$

When A is diagonalizable and $A = S\Lambda S^{-1}$ with $\Lambda = \text{diag}(\lambda_1(A),\ldots, \lambda_n(A))$ and $|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|$, use these bounds to show that

$$|\lambda_k(A)|/\kappa_2(S) \leq \sigma_k(A) \leq \kappa_2(S)|\lambda_k(A)|, \quad k = 1,\ldots, n \tag{3.1.17}$$

When A is normal, use (3.1.17) to show that its singular values are the absolute values of its eigenvalues. See Problem 31 in Section (3.3) for another approach to the inequalities (3.1.16-17).

46. Show that $\sigma > 0$ is a singular value of $A \in M_{m,n}$ if and only if the block matrix $\begin{bmatrix} A & -\sigma I \\ -\sigma I & A^* \end{bmatrix} \in M_{m+n}$ is singular.

Notes and Further Readings. The interlacing inequalities between the singular values of a matrix and its submatrices are basic and very useful facts. For the original proof that the interlacing inequalities (3.1.4) describe exactly the set of all possible singular values of a submatrix of given size, given the singular values of the overall matrix, see R. C. Thompson, Principal Submatrices IX: Interlacing Inequalities for Singular Values of Submatrices, *Linear Algebra Appl.* 5 (1972), 1-12. For a different proof, see J. F. Queiró, On the Interlacing Property for Singular Values and Eigenvalues, *Linear Algebra Appl.* 97 (1987), 23-28.

3.2 Weak majorization and doubly substochastic matrices

Because the singular value decomposition $A = V\Sigma W^*$ is a natural generalization of the spectral decomposition $A = U\Lambda U^*$ for a square Hermitian or normal matrix, familiar properties of Hermitian or normal matrices can point the way toward interesting results for general matrices. For example, if $A = [a_{ij}] \in M_n$ is normal and $A = U\Lambda U^*$ with a unitary $U = [u_{ij}] \in M_n$ and diagonal $\Lambda = \operatorname{diag}(\lambda_1,\ldots, \lambda_n)$, direct computation shows that the vector $a = [a_{ii}] \in \mathbb{C}^n$ of main diagonal entries of A and the vector $\lambda(A) = [\lambda_i(A)] \in \mathbb{C}^n$ of eigenvalues of A are related by the transformation

$$a = S\lambda(A) \tag{3.2.1}$$

in which $S = [\,|u_{ij}|^2\,] = [U \circ \bar{U}] \in M_n$ is doubly stochastic (see Theorem (4.3.33) in [HJ]), that is, S has nonnegative entries and all its row and column sums are one.

Let $A = [a_{ij}] \in M_{m,n}$ have a singular value decomposition $A = V\Sigma W^*$, where $V = [v_{ij}] \in M_m$ and $W = [w_{ij}] \in M_n$ are unitary and $\Sigma = [\sigma_{ij}] \in M_{m,n}$ has $\sigma_{ii} = \sigma_i(A)$ for $i = 1,\ldots, q \equiv \min\{m,n\}$. A calculation reveals that the vector of diagonal entries $a = [a_{ii}] \in \mathbb{C}^q$ of A and its vector of singular values $\sigma(A) = [\sigma_i(A)] \in \mathbb{C}^q$ are related by the transformation

$$a = Z\sigma(A) \tag{3.2.2}$$

where $Z = [v_{ij}\bar{w}_{ij}] \in M_q$ is the Hadamard product of the upper-left q-by-q principal submatrices of the unitary matrices V and W. If we let $Q \equiv |Z| = [\,|v_{ij}w_{ij}|\,]$, then we have the entrywise inequalities

$$|a| = [\,|a_i|\,] \leq |Z|\,\sigma(A) = Q\,\sigma(A)$$

Notice that the row sums of the nonnegative matrix Q are at most one since

$$\left[\sum_{j=1}^{q} |v_{ij}w_{ij}|\right]^2 \leq \sum_{j=1}^{q} |v_{ij}|^2 \sum_{j=1}^{q} |w_{ij}|^2 \leq \sum_{j=1}^{m} |v_{ij}|^2 \sum_{j=1}^{n} |w_{ij}|^2 = 1$$

for all $i = 1,\ldots, q$; the same argument shows that the column sums of Q are at most one as well.

3.2.3 Definition. A matrix $Q \in M_n(\mathbb{R})$ is said to be *doubly substochastic* if its entries are nonnegative and all its row and column sums are at most one.

The set of doubly substochastic n-by-n matrices is clearly a convex set that contains all doubly stochastic n-by-n matrices. It is useful to know that it is generated in a simple way by the set of doubly stochastic $2n$-by-$2n$ matrices. Let $Q \in M_n$ be doubly substochastic, let $e \equiv [1, \ldots, 1]^T \in \mathbb{R}^n$, and let $D_r = \operatorname{diag}(Qe)$ and $D_c = \operatorname{diag}(Q^Te)$ be diagonal matrices containing the row and column sums of Q. The doubly stochastic $2n$-by-$2n$ matrix

$$\begin{bmatrix} Q & I - D_r \\ I - D_c & Q^T \end{bmatrix} \tag{3.2.4}$$

is a dilation of Q. Conversely, it is evident that any square submatrix of a doubly stochastic matrix is doubly substochastic.

The doubly stochastic matrices are the convex hull of the permutation matrices (Birkhoff's theorem (8.7.1) in [HJ]). There is an analogous characterization of the doubly substochastic matrices.

3.2.5 Definition. A matrix $P \in M_n(\mathbb{R})$ is said to be a *partial permutation matrix* if it has at most one nonzero entry in each row and column, and these nonzero entries (if any) are all 1.

It is evident that every partial permutation matrix is doubly substochastic and can be obtained (perhaps in more than one way) by replacing some 1 entries in a permutation matrix by zeroes. Moreover, every square submatrix of a permutation matrix is doubly substochastic.

If $Q \in M_n(\mathbb{R})$ is a given doubly substochastic matrix, construct the $2n$-by-$2n$ doubly stochastic dilation (3.2.4) and use Birkhoff's theorem to express it as a convex combination of $2n$-by-$2n$ permutation matrices. Then Q is the same convex combination of the upper-left n-by-n principal submatrices of these permutation matrices, each of which is doubly substochastic. The conclusion is a doubly substochastic analog of Birkhoff's theorem: Every doubly substochastic matrix is a finite convex combination of partial permutation matrices. Conversely, a finite convex combination of partial permutation matrices is evidently doubly substochastic.

As a final observation, suppose a given doubly substochastic matrix Q is expressed as a convex combination of partial permutation matrices. In each partial permutation matrix summand, replace some zero entries by ones to make it a permutation matrix. The resulting convex combination of permutation matrices is a doubly stochastic matrix, all of whose entries are not less than those of Q. Conversely, a matrix $Q \in M_n(\mathbb{R})$ such that $0 \leq Q \leq S$ for some doubly stochastic $S \in M_n(\mathbb{R})$ is evidently doubly substochastic. We summarize the preceding observations as follows:

3.2.6 Theorem. Let $Q \in M_n(\mathbb{R})$ be a given matrix with nonnegative entries. The following are equivalent:

(a) Q is doubly substochastic.
(b) Q has a doubly stochastic dilation, that is, Q is an upper left principal submatrix of a doubly stochastic matrix.
(c) Q is a finite convex combination of partial permutation matrices.
(d) There is a doubly stochastic $S \in M_n(\mathbb{R})$ such that $0 \leq Q \leq S$.

There is an intimate connection between doubly stochastic matrices and (strong) majorization inequalities between the entries of two real n-vectors; see Section (4.3) of [HJ]. If $\lambda = [\lambda_i]$ and $a = [a_i]$ are two given real n-vectors, their entries may be re-indexed in algebraically decreasing order

$$\lambda_{[1]} \geq \cdots \geq \lambda_{[n]} \text{ and } a_{[1]} \geq \cdots \geq a_{[n]}$$

or in algebraically increasing order

$$\lambda_{(1)} \leq \cdots \leq \lambda_{(n)} \text{ and } a_{(1)} \leq \cdots \leq a_{(n)}$$

The inequalities

$$\sum_{i=1}^{k} a_{[i]} \leq \sum_{i=1}^{k} \lambda_{[i]} \text{ for } k = 1,\ldots, n \text{ with equality for } k = n \tag{3.2.7a}$$

are easily seen to be equivalent to the inequalities

$$\sum_{i=1}^{k} a_{(i)} \geq \sum_{i=1}^{k} \lambda_{(i)} \text{ for } k = 1,\ldots, n \text{ with equality for } k = n \tag{3.2.7b}$$

These equivalent families of (strong) majorization inequalities characterize the relationship between the main diagonal entries and eigenvalues of a Hermitian matrix (Theorem (4.3.26) in [HJ]) and are equivalent to the existence of a doubly stochastic $S \in M_n(\mathbb{R})$ such that $a = S\lambda$. It is not surprising that a generalized kind of majorization is intimately connected with doubly substochastic matrices.

3.2.8 Definition. Let $x = [x_i]$, $y = [y_i] \in \mathbb{R}^n$ be given vectors, and denote their algebraically decreasingly ordered entries by $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$. We say that *y weakly majorizes x* if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \text{ for } k = 1,\ldots, n \tag{3.2.9}$$

Notice that in weak majorization there is no requirement that the sums of all the entries of x and y be equal.

3.2.10 Theorem. Let $x = [x_i]$, $y = [y_i] \in \mathbb{R}^n$ be given vectors with nonnegative entries. Then y weakly majorizes x if and only if there is a doubly substochastic $Q \in M_n(\mathbb{R})$ such that $x = Qy$.

Proof: If $Q \in M_n(\mathbb{R})$ is doubly substochastic and $Qy = x$ with $x, y \geq 0$, let $S \in M_n(\mathbb{R})$ be doubly stochastic and such that $0 \leq Q \leq S$. If we adopt the notation of Definition (3.2.8) to denote the algebraically decreasingly ordered entries of a real vector, and if $(Qy)_{i_1} = (Qy)_{[1]}, \ldots, (Qy)_{i_k} = (Qy)_{[k]}$, then

$$\sum_{i=1}^{k} (Qy)_{[i]} = \sum_{j=1}^{k} (Qy)_{i_j} \leq \sum_{j=1}^{k} (Sy)_{i_j} \leq \sum_{i=1}^{k} (Sy)_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$$

for $k = 1, \ldots, n$; we invoke Theorem (4.3.33) in [HJ] for the last inequality. Thus, y weakly majorizes $Qy = x$.

Conversely, suppose y weakly majorizes x and $x, y \geq 0$. If $x = 0$, let $Q \equiv 0$. If $x \neq 0$, let ϵ_x and ϵ_y denote the smallest positive entries of x and y, respectively, set $\delta \equiv (y_1 - x_1) + \cdots + (y_n - x_n) \geq 0$, and let m be any positive integer such that $\delta/m \leq \min\{\epsilon_x, \epsilon_y\}$. Let

$$\xi \equiv [x_1, \ldots, x_n, \delta/m, \ldots, \delta/m]^T \in \mathbb{R}^{n+m} \text{ and}$$
$$\eta \equiv [y_1, \ldots, y_n, 0, \ldots, 0]^T \in \mathbb{R}^{n+m}$$

Then

$$\sum_{i=1}^{k} \xi_{[i]} \leq \sum_{i=1}^{k} \eta_{[i]} \text{ for } k = 1, \ldots, m+n$$

with equality for $k = m + n$. Thus, there is a strong majorization relationship between ξ and η. By Theorem (4.3.33) in [HJ] there is a doubly stochastic $S \in M_{m+n}(\mathbb{R})$ such that $\xi = S\eta$. If we let Q denote the upper left n-by-n principal submatrix of S, then Q is doubly substochastic and $x = Qy$. $\square$

3.2.11 Corollary. Let $x = [x_i]$, $y = [y_i] \in \mathbb{R}^n$ be given. Then y weakly majorizes x if and only if there is a doubly stochastic $S \in M_n(\mathbb{R})$ such that the entrywise inequalities $x \leq Sy$ hold.

Proof: If there is a doubly stochastic S such that $x \leq Sy$, then since there is a (strong) majorization relationship between Sy and y, x must be weakly majorized by y. Conversely, suppose x is weakly majorized by y, let $e =$

$[1, \ldots, 1]^T \in \mathbb{R}^n$, and choose $\kappa \geq 0$ so that $x + \kappa e \geq 0$ and $y + \kappa e \geq 0$. Theorem (3.2.10) guarantees that there is a doubly substochastic $Q \in M_n(\mathbb{R})$ such that $x + \kappa e = Q(y + \kappa e)$. Theorem (3.2.6(d)) ensures that there is a doubly stochastic S such that $Q \leq S$. Then $x + \kappa e = Q(y + \kappa e) \leq Sy + \kappa e$, so $x \leq Sy$. □

Problems

1. The matrix (3.2.4) shows that every n-by-n doubly substochastic matrix Q has a $2n$-by-$2n$ doubly stochastic dilation. Consider $Q = 0$ to show that the dilation sometimes cannot be smaller than $2n$-by-$2n$.

2. Let $\hat{V}$ and $\hat{W}$ denote any q-by-q submatrices of given unitary matrices $V \in M_m$, $W \in M_n$, $q \leq \min\{m,n\}$. Show that $|\hat{V} \circ \hat{W}|$ (entrywise absolute values) is doubly substochastic.

3. Let $A = [a_{ij}] \in M_{m,n}$ be given and let $q = \min\{m,n\}$. Let $a = [a_{11}, \ldots, a_{qq}]^T \in \mathbb{C}^q$ and $\sigma(A) = [\sigma_1(A), \ldots, \sigma_q(A)]^T \in \mathbb{R}^q$ denote the vectors of main diagonal entries and singular values of A, respectively, with $\sigma_1(A) \geq \cdots \geq \sigma_q(A)$. Let $|a|_{[1]} \geq \cdots \geq |a|_{[q]}$ denote the absolutely decreasingly ordered main diagonal entries of A and let $A = V\Sigma W^*$ be a singular value decomposition of A. Use the discussion at the beginning of this section and Theorem (3.2.10) to verify the weak majorization

$$|a|_{[1]} + \cdots + |a|_{[k]} \leq \sigma_1(A) + \cdots + \sigma_k(A) \text{ for } k = 1, \ldots, q$$

What does this say if $A \in M_n$ is positive semidefinite? For a different proof of these inequalities and some related results, see Problem 21 in Section (3.3).

4. (a) Consider the partial permutation matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Notice that it is a convex combination of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. A *generalized permutation matrix* is obtained from a permutation matrix by replacing each 1 entry by an entry (real or complex) with absolute value 1. Show that each partial permutation matrix in M_n is a convex combination of two real generalized permutation matrices.
(b) If $G \in M_n$ is a generalized permutation matrix and if $\sigma = [\sigma_1, \ldots, \sigma_n]^T \in \mathbb{R}^n$ is a nonnegative vector, show that the singular values of $\operatorname{diag}(G\sigma)$ are $\sigma_1, \ldots, \sigma_n$.
(c) Let $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ and $s_1 \geq \cdots \geq s_n \geq 0$ be given and suppose $s_1 + \cdots$

$+ s_k \leq \sigma_1 + \cdots + \sigma_k$ for $k = 1, \ldots, n$, that is, the values $\{\sigma_i\}$ weakly majorize the values $\{s_i\}$. Show that $\mathrm{diag}(s_1, \ldots, s_n)$ is a finite convex combination of matrices, each of which has the same singular values $\sigma_1, \ldots, \sigma_n$.

(d) Suppose given matrices $A_1, \ldots, A_m \in M_n$ all have the same singular values $\sigma_1 \geq \cdots \geq \sigma_n$. For each $i = 1, \ldots, m$, let $a^{(i)} \in \mathbb{C}^n$ denote the vector of main diagonal entries of A_i, that is, $a^{(i)} \equiv \mathrm{diag}(A_i)$. If $\mu_1, \ldots, \mu_n$ are nonnegative real numbers such that $\mu_1 + \cdots + \mu_n = 1$, show that the entries of the vector $\mu_1 |a^{(1)}| + \cdots + \mu_n |a^{(n)}|$ are weakly majorized by the entries of the vector $\sigma = [\sigma_1, \ldots, \sigma_n]^T$.

(e) Let $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ and $A \in M_n$ be given. Prove that A is in the convex hull of the set of all n-by-n complex matrices that have the same singular values $\sigma_1, \ldots, \sigma_n$ if and only if the singular values of A are weakly majorized by the given values $\{\sigma_i\}$, that is, $\sigma_1(A) + \cdots + \sigma_k(A) \leq \sigma_1 + \cdots + \sigma_k$ for all $k = 1, \ldots, n$.

(f) Recall that $A \in M_n$ is a *contraction* if $\sigma_1(A) \leq 1$; the set of contractions in M_n is the unit ball of the spectral norm. Use (e) to show that A is a contraction if and only if it is a finite convex combination of unitary matrices. See Problems 4 and 27 of Section (3.1) for different approaches to this result.

5. A given norm $\|\cdot\|$ on $\mathbb{F}^n \equiv \mathbb{R}^n$ or $\mathbb{C}^n$ is *absolute* if $\|x\| = \| \, |x| \, \|$ for all $x \in \mathbb{F}^n$ (entry-wise absolute values); it is *permutation-invariant* if $\|x\| = \|Px\|$ for all $x \in \mathbb{F}^n$ and every permutation matrix $P \in M_n(\mathbb{R})$. It is known (see Theorem (5.5.10) in [HJ]) that $\|\cdot\|$ is absolute if and only if it is *monotone*, that is, $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$. Let $x = [x_i]$, $y = [y_i] \in \mathbb{F}^n$ be given, and suppose $|y|$ weakly majorizes $|x|$. Use Corollary (3.2.11) to show that $\|x\| \leq \|y\|$ for every absolute permutation-invariant norm $\|\cdot\|$ on $\mathbb{R}^n$.

6. Notice that there is an analogy between the dilation (3.2.4) and the dilation discussed in Problem 21 of Section (1.6). There is also a doubly stochastic dilation of a given doubly substochastic matrix that has a property analogous to the power dilation property considered in Problem 24 of Section (1.6). If $Q \in M_n$ is a given doubly substochastic matrix, and if k is a given positive integer, let D_r and D_c be defined as in (3.2.4) and define the $(k+1)$-by-$(k+1)$ block matrix $S = [S_{ij}] \in M_{n(k+1)}$, each $S_{ij} \in M_n$, as follows: $S_{11} = Q$, $S_{12} = I - D_r$, $S_{k+1,1} = I - D_c$, $S_{k+1,2} = Q^T$, $S_{i,i+1} = I$ for $i = 2, \ldots, k$, and all other blocks are zero. Show that $S \in M_{n(k+1)}$ is doubly stochastic and that $S^m = \begin{bmatrix} Q^m & * \\ * & * \end{bmatrix}$ for $m = 1, 2, \ldots, k$. What is this for $k = 1$?

Notes and Further Readings. For a modern survey of the ideas discussed in this section, with numerous references, see T. Ando, Majorization, Doubly Stochastic Matrices, and Comparison of Eigenvalues, *Linear Algebra Appl.* 118 (1989), 163-248. See also [MOl]. Sharp bounds on the size of a doubly stochastic S with the power dilation property discussed in Problem 6 are given in the paper by R. C. Thompson and C.-C. T. Kuo cited in Section (1.6): If $Q = [q_{ij}] \in M_n$ is doubly substochastic, let δ denote the least integer greater than $n - \Sigma_{i,j}\, q_{ij}$. Then there is a doubly stochastic $S \in M_{n+\mu}$ such that $S^m = \begin{bmatrix} Q^m & * \\ * & * \end{bmatrix}$ for $m = 1, 2,\ldots, k$ if and only if $\mu \geq k\delta$.

3.3 Basic inequalities for singular values and eigenvalues

It is clear that one cannot prescribe completely independently the eigenvalues and singular values of a matrix $A \in M_n$. For example, if some singular value of A is zero, then A is singular and hence it must have at least one zero eigenvalue. A basic result in this section is a necessary condition (which we show to be sufficient as well in Section (3.6)) indicating the interdependence between the singular values and eigenvalues of a square matrix. We also develop useful inequalities for singular values of products and sums of matrices.

The basic inequalities between singular values and eigenvalues, and the singular value inequalities for products of matrices are both consequences of the following inequality, which follows readily from unitary invariance and the interlacing property for singular values of a submatrix of a given matrix.

3.3.1 Lemma. Let $C \in M_{m,n}$, $V_k \in M_{m,k}$, and $W_k \in M_{n,k}$ be given, where $k \leq \min\{m,n\}$ and V_k, W_k have orthonormal columns. Then

(a) $\sigma_i(V_k^* C W_k) \leq \sigma_i(C)$, $i = 1,\ldots, k$, and

(b) $|\det V_k^* C W_k| \leq \sigma_1(C) \cdots \sigma_k(C)$.

Proof: Since the respective columns of V_k and W_k can be extended to orthonormal bases of $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, there are unitary matrices $V \in M_m$ and $W \in M_n$ such that $V = [V_k\ \ *]$ and $W = [W_k\ \ *]$. Since $V_k^* C W_k$ is the upper left k-by-k submatrix of $V^* C W$, the interlacing inequalities (3.1.4) and unitary invariance of singular values ensure that $\sigma_i(V_k^* C W_k) \leq \sigma_i(V^* C W) = \sigma_i(C)$, $i = 1,\ldots, k$, and hence $|\det V_k^* C W_k| = \sigma_1(V_k^* C W_k) \cdots$

$\sigma_k(V_k^* C W_k) \le \sigma_1(C) \cdots \sigma_k(C)$. □

Important necessary conditions relating the singular values and eigenvalues now follow readily in the following result, first proved by H. Weyl in 1949.

3.3.2 Theorem. Let $A \in M_n$ have singular values $\sigma_1(A) \ge \cdots \ge \sigma_n(A) \ge 0$ and eigenvalues $\{\lambda_1(A), \ldots, \lambda_n(A)\} \subset \mathbb{C}$ ordered so that $|\lambda_1(A)| \ge \cdots \ge |\lambda_n(A)|$. Then

$$|\lambda_1(A) \cdots \lambda_k(A)| \le \sigma_1(A) \cdots \sigma_k(A) \text{ for } k = 1, \ldots, n,$$
$$\text{with equality for } k = n \qquad (3.3.3)$$

Proof: By the Schur triangularization theorem, there is a unitary $U \in M_n$ such that $U^* A U = \Delta$ is upper triangular and $\operatorname{diag} \Delta = (\lambda_1, \ldots, \lambda_n)$. Let $U_k \in M_{n,k}$ denote the first k columns of U, and compute

$$U^* A U = [U_k \;\; *]^* \, A \, [U_k \;\; *] = \begin{bmatrix} U_k^* A U_k & * \\ * & * \end{bmatrix} = \Delta$$

Thus, $U_k^* A U_k = \Delta_k$ is upper triangular since it is the upper left k-by-k principal submatrix of Δ, and $\operatorname{diag} \Delta_k = (\lambda_1, \ldots, \lambda_k)$. Now apply the lemma with $C = A$ and $V_k = W_k = U_k$ to conclude that

$$|\lambda_1(A) \cdots \lambda_k(A)| = |\det \Delta_k| = |\det U_k^* A U_k| \le \sigma_1(A) \cdots \sigma_k(A)$$

When $k = n$, one sees readily from the singular value decomposition that $|\det A| = \sigma_1(A) \cdots \sigma_n(A)$, and, since $\det A = \lambda_1(A) \cdots \lambda_n(A)$, we are done. □

If the singular values of two matrices are known, what can one say about the singular values of their product? A useful answer is another immediate consequence of Lemma (3.3.1), a theorem about singular values of matrix products first proved by A. Horn in 1950.

3.3.4 Theorem. Let $A \in M_{m,p}$ and $B \in M_{p,n}$ be given, let $q \equiv \min\{n,p,m\}$, and denote the ordered singular values of A, B, and AB by $\sigma_1(A) \ge \cdots \ge \sigma_{\min\{m,p\}}(A) \ge 0$, $\quad \sigma_1(B) \ge \cdots \ge \sigma_{\min\{p,n\}}(B) \ge 0$, $\quad$ and

$\sigma_1(AB) \geq \cdots \geq \sigma_{min\{m,n\}}(AB) \geq 0$. Then

$$\prod_{i=1}^{k} \sigma_i(AB) \leq \prod_{i=1}^{k} \sigma_i(A)\sigma_i(B), \quad k = 1,\dots, q \tag{3.3.5}$$

If $n = p = m$, then equality holds in (3.3.5) for $k = n$.

Proof: Let $AB = V\Sigma W^*$ be a singular value decomposition of the product AB, and let $V_k \in M_{m,k}$ and $W_k \in M_{n,k}$ denote the first k columns of V and W, respectively. Then $V_k^*(AB)W_k = \text{diag}(\sigma_1(AB),\dots, \sigma_k(AB))$ because it is the upper left k-by-k submatrix of $V^*(AB)W = \Sigma$. Since $p \geq k$, use the polar decomposition (3.1.9(b)) to write the product $BW_k \in M_{p,k}$ as $BW_k = X_kQ$, where $X_k \in M_{p,k}$ has orthonormal columns, $Q \in M_k$ is positive semidefinite, $Q^2 = (BW_k)^*(BW_k) = W_k^*B^*BW_k$, and hence $\det Q^2 = \det W_k^*(B^*B)W_k \leq \sigma_1(B^*B) \cdots \sigma_k(B^*B) = \sigma_1(B)^2 \cdots \sigma_k(B)^2$ by Lemma (3.3.1). Use Lemma (3.3.1) again to compute

$$\begin{aligned}\sigma_1(AB) \cdots \sigma_k(AB) &= |\det V_k^*(AB)W_k| = |\det V_k^*AX_kQ| \\ &= |\det V_k^*AX_k \det Q| \leq (\sigma_1(A) \cdots \sigma_k(A))(\sigma_1(B) \cdots \sigma_k(B))\end{aligned} \tag{3.3.6}$$

If $n = p = m$, then $\sigma_1(AB) \cdots \sigma_n(AB) = |\det AB| = |\det A|\,|\det B| = \sigma_1(A) \cdots \sigma_n(A)\, \sigma_1(B) \cdots \sigma_n(B)$. □

The inequalities (3.3.3) and (3.3.5) are a kind of multiplicative majorization, and if A is nonsingular we can take logarithms in (3.3.3) to obtain equivalent ordinary (strong) majorization inequalities

$$\sum_{i=1}^{k} \log|\lambda_i(A)| \leq \sum_{i=1}^{k} \log \sigma_i(A),\ k = 1,\dots, n, \text{ with equality for } k = n \tag{3.3.7}$$

Since our next goal is to show that these inequalities may be exponentiated term-by-term to obtain weak majorization inequalities such as

$$\sum_{i=1}^{k} |\lambda_i(A)| \leq \sum_{i=1}^{k} \sigma_i(A), \quad k = 1,\dots, n$$

it is convenient to establish that there is a wide class of functions $f(t)$ (including $f(t) = e^t$) that preserve systems of inequalities such as (3.3.7); such functions have been termed *Schur-convex* or *isotone*. The next lemma shows that any increasing convex function with appropriate domain preserves weak majorization.

3.3.8 Lemma. Let $x_1,\ldots, x_n$, $y_1,\ldots, y_n$ be $2n$ given real numbers such that $x_1 \geq x_2 \geq \cdots \geq x_n$, $y_1 \geq y_2 \geq \cdots \geq y_n$, and

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad k = 1,\ldots, n \tag{3.3.9a}$$

Let $f(\cdot)$ be a given real-valued function on the interval $[a,b] \equiv [\min\{x_n, y_n\}, y_1]$. If $f(\cdot)$ is increasing and convex on $[a,b]$, then $f(x_1) \geq \cdots \geq f(x_n)$, $f(y_1) \geq \cdots \geq f(y_n)$, and

$$\sum_{i=1}^{k} f(x_i) \leq \sum_{i=1}^{k} f(y_i), \quad k = 1,\ldots, n \tag{3.3.9b}$$

Write $x \equiv [x_i]$, $y \equiv [y_i] \in \mathbb{R}^n$. For any $z \equiv [z_i] \in \mathbb{R}^n$ with all $z_i \in [a,b]$, write $f(z) \equiv [f(z_i)] \in \mathbb{R}^n$ and let $f(z)_{[1]} \geq \cdots \geq f(z)_{[n]}$ denote an algebraically decreasingly ordered rearrangement of the entries of $f(z)$. If equality holds for $k = n$ in (3.3.9a) and if $f(\cdot)$ is convex (but not necessarily increasing) on $[a,b] = [y_n, y_1]$, then

$$\sum_{i=1}^{k} f(x)_{[i]} \leq \sum_{i=1}^{k} f(y)_{[i]}, \; k = 1,\ldots, n-1 \text{ and } \sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i) \tag{3.3.9c}$$

Proof: We need to show that $f(y)$ weakly majorizes $f(x)$. Corollary (3.2.11) ensures that there is a doubly stochastic $S \in M_n(\mathbb{R})$ such that $x \leq Sy$, and if equality holds for $k = n$ in (3.3.9a) we may choose S so that $x = Sy$. In the former case, monotonicity of $f(\cdot)$ ensures that $f(x) \leq f(Sy)$, and in the latter case, $f(x) = f(Sy)$. In both cases, $f(Sy)$ weakly majorizes $f(x)$. Thus, it suffices to show that $f(Sy)$ is weakly majorized by $f(y)$. Use Birkhoff's Theorem to write

$$S = \alpha_1 P_1 + \cdots + \alpha_m P_m$$

where all $\alpha_i > 0$, $\alpha_1 + \cdots + \alpha_m = 1$, and each $P_i \in M_n(\mathbb{R})$ is a permutation matrix. Convexity of $f(\cdot)$ ensures that

$$f(Sy) = f(\sum_{i=1}^{m} \alpha_i P_i y) \leq \sum_{i=1}^{m} \alpha_i f(P_i y) = \sum_{i=1}^{m} \alpha_i P_i f(y) = Sf(y)$$

which implies that $f(Sy)$ is weakly majorized by $Sf(y)$. Since there is a strong majorization relationship between $Sf(y)$ and $f(y)$, and since weak majorization is a transitive relation, $f(Sy)$ is weakly majorized by $f(y)$. □

3.3.10 Corollary. Let $\alpha_1,\ldots, \alpha_n$, $\beta_1,\ldots, \beta_n$ be $2n$ given nonnegative real numbers such that $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$, $\beta_1 \geq \cdots \geq \beta_n \geq 0$, and

$$\prod_{i=1}^{k} \alpha_i \leq \prod_{i=1}^{k} \beta_i, \quad k = 1,\ldots, n \tag{3.3.11}$$

Let $f(\cdot)$ be a given real-valued function on the interval $[a,b] \equiv [\min\{\alpha_n,\beta_n\},\beta_1]$ and define $\varphi(t) \equiv f(e^t)$. Then

(i) We have the weak majorization

$$\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i, \quad k = 1,\ldots, n \tag{3.3.12a}$$

(ii) If $\alpha_1 = 0$, set $p \equiv 0$; otherwise, set $p \equiv \max\{i: \alpha_i > 0,\ i \in \{1,\ldots, n\}\}$. If $p = 0$, assume $f(\beta_i) \geq f(0)$ for all $i = 1,\ldots, n$; otherwise, assume that $f(\cdot)$ is increasing on $[a,b]$ and that $\varphi(\cdot)$ is convex on $[\ln \min\{\alpha_p,\beta_p\},\ln \beta_1]$. Then

$$\sum_{i=1}^{k} f(\alpha_i) \leq \sum_{i=1}^{k} f(\beta_i) \quad \text{for } k = 1,\ldots, n \tag{3.3.12b}$$

(iii) If $\alpha_n > 0$ and equality holds for $k = n$ in (3.3.11), assume that $\varphi(\cdot)$ is convex (but not necessarily increasing) on $[\ln \beta_n,\ln \beta_1]$. Then

$$\sum_{i=1}^{n} f(\alpha_i) \leq \sum_{i=1}^{n} f(\beta_i) \tag{3.3.12c}$$

Proof: The inequalities (3.3.12a) follow from (3.3.12b) with $f(t) = t$, so it suffices to prove (3.3.12b,c). First consider the case $p = n$, in which all α_i and β_i are positive. Then (3.3.11) is equivalent to

$$\sum_{i=1}^{k} \log \alpha_i \leq \sum_{i=1}^{k} \log \beta_i, \qquad k = 1,\ldots, n$$

The inequalities (3.3.12b,c) now follow from Lemma (3.3.8) using $\varphi(\cdot)$. Now suppose that $p < n$. The case $p = 0$ is trivial (since our hypotheses ensure that $f(\beta_i) \geq f(0) = f(\alpha_i)$ for all $i = 1,\ldots, n$), so assume that $1 \leq p < n$. The validity of (3.3.12b) for $k = 1,\ldots, p$ has already been established in the first case considered, so we need to consider only $k = p + 1,\ldots, n$. Since monotonicity of $f(\cdot)$ implies that $f(\beta_i) \geq f(0) = f(\alpha_i)$ for all $i = p + 1,\ldots, n$, we conclude that

$$\sum_{i=1}^{p+r} f(\beta_i) = \sum_{i=1}^{p} f(\beta_i) + \sum_{i=p+1}^{p+r} f(\beta_i) \geq \sum_{i=1}^{p} f(\alpha_i) + \sum_{i=p+1}^{p+r} f(0)$$

$$= \sum_{i=1}^{p+r} f(\alpha_i) \quad \text{for } r = 1,\ldots, n-p$$

$\square$

Although the multiplicative inequalities (3.3.11) imply the additive inequalities (3.3.12), the reverse implication is false; see Problem 3. The following results now follow from Theorem (3.3.2) and Corollary (3.3.10).

3.3.13 Theorem. Let $A \in M_n$ have ordered singular values $\sigma_1(A) \geq \cdots \geq \sigma_n(A) \geq 0$ and eigenvalues $\{\lambda_1(A),\ldots, \lambda_n(A)\}$ ordered so that $|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|$. Then

(a) $\sum_{i=1}^{k} |\lambda_i(A)| \le \sum_{i=1}^{k} \sigma_i(A)$ for $k = 1,\dots, n$

In particular,

(a′) $|\mathrm{tr}\, A| \le \sum_{i=1}^{n} \sigma_i(A)$

(b) $\sum_{i=1}^{k} |\lambda_i(A)|^p \le \sum_{i=1}^{k} \sigma_i(A)^p$ for $k = 1,\dots, n$ and any $p > 0$

More generally, let $f(\cdot)$ be a given real-valued function on $[0,\infty)$ and define $\varphi(t) \equiv f(e^t)$. If $f(\cdot)$ is increasing on $[0,\infty)$ and $\varphi(\cdot)$ is convex on $(-\infty,\infty)$, then

(c) $\sum_{i=1}^{k} f(|\lambda_i(A)|) \le \sum_{i=1}^{k} f(\sigma_i(A))$ for $k = 1,\dots, n$

If $\varphi(\cdot)$ is convex (but not necessarily increasing) on $(-\infty,\infty)$, and if either A is nonsingular or $f(\cdot)$ is continuous on $[0,\infty)$, then

(d) $\sum_{i=1}^{n} f(|\lambda_i(A)|) \le \sum_{i=1}^{n} f(\sigma_i(A))$

In particular, if A is nonsingular, then

(e) $\sum_{i=1}^{n} |\lambda_i(A)|^p \le \sum_{i=1}^{n} \sigma_i(A)^p$ for all $p \in \mathbb{R}$

The same reasoning permits us to deduce a host of other inequalities from the inequalities (3.3.5) for the singular values of a product.

3.3.14 Theorem. Let $A \in M_{n,r}$ and $B \in M_{r,m}$ be given, let $q \equiv \min\{n,r,m\}$, and denote the ordered singular values of A, B, and AB by $\sigma_1(A) \ge \cdots \ge \sigma_{min\{n,r\}} \ge 0$, $\sigma_1(B) \ge \cdots \ge \sigma_{min\{r,m\}}(B) \ge 0$, and $\sigma_1(AB) \ge \cdots \ge \sigma_{min\{n,m\}}(AB) \ge 0$. Then

(a) $$\sum_{i=1}^{k} \sigma_i(AB) \le \sum_{i=1}^{k} \sigma_i(A)\sigma_i(B) \quad \text{for } k = 1,\dots, q$$

(b) $$\sum_{i=1}^{k} [\sigma_i(AB)]^p \le \sum_{i=1}^{k} [\sigma_i(A)\sigma_i(B)]^p \quad \text{for } k = 1,\dots, q \text{ and any } p > 0$$

More generally, let $f(\cdot)$ be a given real-valued function on $[0,\infty)$ and define $\varphi(t) \equiv f(e^t)$. If $f(\cdot)$ is increasing on $[0,\infty)$ and $\varphi(\cdot)$ is convex on $(-\infty,\infty)$, then

(c) $$\sum_{i=1}^{k} f(\sigma_i(AB)) \le \sum_{i=1}^{k} f(\sigma_i(A)\sigma_i(B)) \quad \text{for } k = 1,\dots, q$$

If $m = n = r$, if $\varphi(\cdot)$ is convex (but not necessarily increasing) on $(-\infty,\infty)$, and if either A and B are nonsingular or $f(\cdot)$ is continuous on $[0,\infty)$, then

(d) $$\sum_{i=1}^{n} f(\sigma_i(AB)) \le \sum_{i=1}^{n} f(\sigma_i(A)\sigma_i(B))$$

In particular, if $m = n = r$ and if A and B are nonsingular, then

(e) $$\sum_{i=1}^{n} \sigma_i(AB)^p \le \sum_{i=1}^{n} \sigma_i(A)^p\, \sigma_i(B)^p \quad \text{for all } p \in \mathbb{R}$$

For analogs of the inequalities (a) for the Hadamard product, see Theorems (5.5.4) and (5.6.2).

We have so far been concentrating on inequalities involving sums and products of *consecutive* singular values and eigenvalues, as in the Weyl inequalities (3.3.3). There are many other useful inequalities, however, and as an example we consider singular value analogs of the Weyl inequalities for eigenvalues of sums of Hermitian matrices (Theorem (4.3.7) in [HJ]). A preliminary lemma is useful in the proof we give.

3.3.15 Lemma. Let $A \in M_{m,n}$ be given, let $q = \min\{m,n\}$, let A have a singular value decomposition $A = V\Sigma W^*$, and partition $W = [w_1 \; \dots \; w_n]$

according to its columns, the right singular vectors of A, whose corresponding singular values are $\sigma_1(A) \geq \sigma_2(A) \geq \cdots$. If $S \equiv \text{Span}\,\{w_i, \ldots, w_n\}$, then $\max\,\{\|Ax\|_2\colon x \in S, \|x\|_2 = 1\} = \sigma_i(A)$, $i = 1, \ldots, q$.

Proof: If $x = \alpha_i w_i + \cdots + \alpha_n w_n$ and $|\alpha_i|^2 + \cdots + |\alpha_n|^2 = 1$, then $\|Ax\|_2^2 = \|V\Sigma W^* x\|_2^2 = \|\Sigma W^* x\|_2^2 = \sigma_i(A)^2|\alpha_i|^2 + \cdots + \sigma_q(A)^2|\alpha_q|^2 \leq \sigma_i(A)^2$. $\square$

3.3.16 Theorem. Let $A, B \in M_{m,n}$ be given and let $q = \min\,\{m,n\}$. The following inequalities hold for the decreasingly ordered singular values of A, B, $A + B$, and AB^*:

$$\text{(a)}\quad \sigma_{i+j-1}(A + B) \leq \sigma_i(A) + \sigma_j(B) \tag{3.3.17}$$

$$\text{(b)}\quad \sigma_{i+j-1}(AB^*) \leq \sigma_i(A)\sigma_j(B) \tag{3.3.18}$$

for $1 \leq i, j \leq q$ and $i + j \leq q + 1$. In particular,

$$\text{(c)}\quad |\sigma_i(A + B) - \sigma_i(A)| \leq \sigma_1(B) \quad \text{for } i = 1, \ldots, q \tag{3.3.19}$$

and

$$\text{(d)}\quad \sigma_i(AB^*) \leq \sigma_i(A)\sigma_1(B) \quad \text{for } i = 1, \ldots, q \tag{3.3.20}$$

Proof: Let $A = V\Sigma_A W^*$ and $B = X\Sigma_B Y^*$ be singular value decompositions of A and B with unitary $W = [w_1 \ \ldots \ w_n]$, $Y = [y_1 \ \ldots \ y_n] \in M_n$ and unitary $V = [v_1 \ \ldots \ v_m]$, $X = [x_1 \ \ldots \ x_n] \in M_m$. Let i and j be positive integers with $1 \leq i, j \leq q$ and $i + j \leq q + 1$.

First consider the sum inequalities (3.3.17). Define $S' \equiv \text{Span}\,\{w_i, \ldots, w_n\}$ and $S'' \equiv \text{Span}\,\{y_j, \ldots, y_n\}$; notice that $\dim S' = n - i + 1$ and $\dim S'' = n - j + 1$. Then

$$\begin{aligned}
\nu \equiv \dim(S' \cap S'') &= \dim S' + \dim S'' - \dim(S' + S'') \\
&= (n - i + 1) + (n - j + 1) - \dim(S' + S'') \\
&\geq (n - i + 1) + (n - j + 1) - n = n - (i + j - 1) + 1 \geq 1
\end{aligned}$$

because of the bounds assumed for i and j. Thus, the subspace $S' \cap S''$ has positive dimension ν, $n - \nu + 1 \leq i + j - 1$, and we can use (3.1.2(c)) and Lemma (3.3.15) to compute

$$\sigma_{i+j-1}(A + B) \leq \sigma_{n-\nu+1}(A + B)$$

$$= \min_{\substack{S \subset \mathbb{C}^n \\ \dim S = \nu}} \max_{\substack{x \in S \\ \|x\|_2 = 1}} \|(A+B)x\|_2$$

$$\leq \max_{\substack{x \in S' \cap S'' \\ \|x\|_2 = 1}} \|(A+B)x\|_2$$

$$\leq \max_{\substack{x \in S' \cap S'' \\ \|x\|_2 = 1}} \|Ax\|_2 + \max_{\substack{x \in S' \cap S'' \\ \|x\|_2 = 1}} \|Bx\|_2$$

$$\leq \max_{\substack{x \in S' \\ \|x\|_2 = 1}} \|Ax\|_2 + \max_{\substack{x \in S'' \\ \|x\|_2 = 1}} \|Bx\|_2 = \sigma_i(A) + \sigma_j(B)$$

Now consider the product inequalities (3.3.18). Use the polar decomposition (3.1.9(c)) to write $AB^* = UQ$, where $U \in M_m$ is unitary and $Q \in M_m$ is positive semidefinite and has the same singular values (which are also its eigenvalues) as AB^*. Let

$$S' = \text{Span}\,\{U^* v_i, \ldots, U^* v_n\} \text{ and } S'' = \text{Span}\,\{x_j, \ldots, x_n\}$$

so $\nu \equiv \dim(S' \cap S'') \geq n - (i + j - 1) + 1 \geq 1$, as before. Since $Q = U^* A B^*$, we have $x^* Q x = x^* U^* A B^* x = (A^* U x)^* (B^* x) \leq \|A^* U x\|_2 \, \|B^* x\|_2$ for any $x \in \mathbb{C}^n$ and hence we can use (3.1.2(c)), the Courant-Fischer theorem (4.2.11) in [HJ], and Lemma (3.3.15) to compute

$$\sigma_{i+j-1}(AB^*) = \sigma_{i+j-1}(Q) \leq \sigma_{n-\nu+1}(Q)$$

$$= \min_{\substack{S \subset \mathbb{C}^n \\ \dim S = \nu}} \max_{\substack{x \in S \\ \|x\|_2 = 1}} x^* Q x$$

$$\leq \max_{\substack{x \in S' \cap S'' \\ \|x\|_2 = 1}} x^* Q x \leq \max_{\substack{x \in S' \cap S'' \\ \|x\|_2 = 1}} \|A^* U x\|_2 \, \|B^* x\|_2$$

$$\leq \max_{\substack{x \in S' \cap S'' \\ \|x\|_2 = 1}} \|A^* U x\|_2 \max_{\substack{x \in S' \cap S'' \\ \|x\|_2 = 1}} \|B^* x\|_2$$

$$\leq \max_{\substack{x \in S' \\ \|x\|_2=1}} \|A^* Ux\|_2 \max_{\substack{x \in S'' \\ \|x\|_2=1}} \|B^*x\|_2 = \sigma_i(A)\, \sigma_j(B)$$

Notice that the increasingly ordered eigenvalues $\lambda_1(Q) \leq \cdots \leq \lambda_n(Q)$ are related to the decreasingly ordered singular values $\sigma_1(Q) \geq \cdots \geq \sigma_n(Q)$ by $\lambda_i(Q) = \sigma_{n-i+1}(Q)$.

The inequalities $\sigma_i(A + B) \leq \sigma_i(A) + \sigma_1(B)$ and $\sigma_i(AB^*) \leq \sigma_i(A)\sigma_1(B)$ follow from setting $j = 1$ in (a) and (b). The two-sided bound in the additive case now follows from observing that $\sigma_i(A) = \sigma_i([A + B] - B) \leq \sigma_i(A + B) + \sigma_1(-B) = \sigma_i(A + B) + \sigma_1(B)$. □

As a final application in this section of the interlacing inequalities for singular values, we consider the following generalization of the familiar and useful fact that lim $\sigma_1(A^m)^{1/m} = |\lambda_1(A)| = \rho(A)$ as $m \to \infty$ for any $A \in M_n$. This is a theorem of Yamamoto, first proved in 1967.

3.3.21 Theorem. Let $A \in M_n$ be given, and let $\sigma_1(A) \geq \cdots \geq \sigma_n(A)$ and $\{\lambda_1(A),\dots, \lambda_n(A)\}$ denote its singular values and eigenvalues, respectively, with $|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|$. Then

$$\lim_{m\to\infty} [\sigma_i(A^m)]^{1/m} = |\lambda_i(A)| \text{ for } i = 1,\dots, n$$

Proof: The case $i = 1$ is a special case of a theorem valid for all prenorms on M_n (see Corollaries (5.7.10) and (5.6.14) in [HJ]). The case $i = n$ is trivial if A is singular, and follows in the nonsingular case by applying the case $i = 1$ to A^{-1}, since $\sigma_1(A^{-1}) = 1/\sigma_n(A)$ and $|\lambda_1(A^{-1})| = |1/\lambda_n(A)|$. Use the Schur triangularization theorem to write $A = U\Delta U^*$ with $U, \Delta \in M_n$, U unitary, and Δ upper triangular with diag $\Delta = (\lambda_1,\dots, \lambda_n)$. For $B = [b_{ij}] \in M_n$, let $B_{[i]}$ denote the i-by-i upper left principal submatrix of B, and let $B_{<i>} \in M_{n-i+1}$ denote the lower right $(n-i+1)$-by-$(n-i+1)$ principal submatrix of B; notice that the entry b_{ii} is in the lower right corner of $B_{[i]}$ and in the upper left corner of $B_{<i>}$.

The upper triangular structure of Δ ensures that $(\Delta^m)_{[i]} = (\Delta_{[i]})^m$ and $(\Delta^m)_{<i>} = (\Delta_{<i>})^m$, as well as $|\lambda_i(\Delta_{[i]})| = |\lambda_i(A)|$ and $|\lambda_1(\Delta_{<i>})| = |\lambda_i(A)|$. Thus, the left half of the interlacing inequalities (3.1.4) gives the lower bound

$$\sigma_i(A^m) = \sigma_i(\Delta^m) \geq \sigma_i((\Delta^m)_{[i]}) = \sigma_i((\Delta_{[i]})^m)$$

Now write $B_m \equiv [0 \;\; (\Delta^m)_{<i>}] \in M_{n-i+1,n}$ and $T_m \equiv [(\Delta^m)_{[i]} \;\; *] \in M_{i,n}$, so

$$\Delta^m = \begin{bmatrix} T_m \\ B_m \end{bmatrix}$$

Notice that B_m is obtained from Δ^m by deleting $i-1$ rows, and that the singular values of B_m and $(\Delta^m)_{<i>}$ are the same since $(\Delta^m)_{<i>}$ is obtained from B_m by deleting zero columns. We can now obtain an upper bound by using the right half of (3.1.4) with $r = i-1$ and $k = 1$:

$$\sigma_i(A^m) = \sigma_i(\Delta^m) = \sigma_{1+(i-1)}(\Delta^m) \leq \sigma_1(B_m)$$

$$= \sigma_1((\Delta^m)_{<i>}) = \sigma_1((\Delta_{<i>})^m)$$

Thus, we have the two-sided bounds

$$\sigma_i((\Delta_{[i]})^m)^{1/m} \leq \sigma_i(A^m)^{1/m} \leq \sigma_1((\Delta_{<i>})^m)^{1/m}$$

which yield the desired result since $\sigma_i((\Delta_{[i]})^m)^{1/m} \to |\lambda_i(\Delta_{[i]})| = |\lambda_i(A)|$ and $\sigma_1((\Delta_{<i>})^m)^{1/m} \to |\lambda_1(\Delta_{<i>})| = |\lambda_i(A)|$ as $m \to \infty$. $\square$

Problems

1. Explain why the case $k = 1$ in (3.3.3) and (3.3.5) are familiar results.

2. Use (3.3.3) to show that if $A \in M_n$ has rank r, then A has at least $n-r$ zero eigenvalues, $1 \leq r < n$. Could it have more than $n-r$ zero eigenvalues?

3. Let $\alpha_1 \geq \cdots \geq \alpha_n \geq 0$ and $\beta_1 \geq \cdots \geq \beta_n \geq 0$ be two given ordered sets of nonnegative real numbers.
   (a) For $n = 2$, prove by direct calculation that the multiplicative inequalities $\alpha_1 \leq \beta_1$ and $\alpha_1\alpha_2 \leq \beta_1\beta_2$ imply the additive inequalities $\alpha_1 \leq \beta_1$ and $\alpha_1 + \alpha_2 \leq \beta_1 + \beta_2$.
   (b) For $n = 2$, consider $\alpha_1 = 1$, $\alpha_2 = \frac{1}{2}$, $\beta_1 = 5/4$, $\beta_2 = \frac{1}{4}$ and show that the additive inequalities do not imply the multiplicative inequalities. Thus, there is more intrinsic information in the inequalities (3.3.3), (3.3.5), and (3.3.11) than in the inequalities (3.3.13(a)), (3.3.14(a)), and (3.3.12), respectively.

4. Let $A_1, A_2, \ldots, A_m \in M_n$ for some integer $m \geq 2$. Using the notation of (3.3.14), show that

$$\sum_{i=1}^{k} \sigma_i(A_1 \cdots A_m) \leq \sum_{i=1}^{k} \sigma_i(A_1) \cdots \sigma_i(A_m) \text{ for } k = 1, \ldots, n \quad (3.3.22)$$

5. Let $A, B \in M_n$.
(a) Why is $\sigma_1(AB) \leq \sigma_1(A)\sigma_1(B)$?
(b) Consider $A = B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ to show that $\sigma_2(AB) > \sigma_2(A)\sigma_2(B)$ is possible.
(c) Does the example in (b) contradict the known inequality $\sigma_1(AB) + \sigma_2(AB) \leq \sigma_1(A)\sigma_1(B) + \sigma_2(A)\sigma_2(B)$? What are the values of all the terms of this inequality for the example in (b)?

6. Prove Theorem (3.3.14).

7. Let $A, B \in M_n$ be given. Show that

$$\prod_{i=1}^{k} (\alpha + t|\lambda_i(A)|) \leq \prod_{i=1}^{k} [\alpha + t\sigma_i(A)] \quad (3.3.23)$$

and

$$\prod_{i=1}^{k} [\alpha + t\sigma_i(AB)] \leq \prod_{i=1}^{k} [\alpha + t\sigma_i(A)\sigma_i(B)] \quad (3.3.24)$$

for all $\alpha, t > 0$ and all $k = 1, \ldots, n$. For $\alpha = 1$, these inequalities are useful in the theory of integral equations.

8. Provide details for the following proof of Weyl's theorem (3.3.2) that avoids direct use of Lemma (3.3.1): Using the notation of (3.3.2), let $U^*AU = \Delta$ be upper triangular with diag $\Delta = [\lambda_1, \ldots, \lambda_n]^T$, and write $\Delta = [\Delta_1 \;\; \Delta_2]$ and $\Delta_1 = \begin{bmatrix} \Delta_3 \\ 0 \end{bmatrix}$ with $\Delta_1 \in M_{n,k}$ and $\Delta_3 \in M_k$. Then $\sigma_i(A) = \sigma_i(\Delta) \geq \sigma_i(\Delta_1) = \sigma_i(\Delta_3)$ for $i = 1, \ldots, k$, so $\sigma_1(A) \cdots \sigma_k(A) \geq |\det \Delta_3| = |\lambda_1(A) \cdots \lambda_k(A)|$.

9. Deduce Weyl's theorem (3.3.2) from the A. Horn singular value product theorem (3.3.4).

10. Let $A, B \in M_{m,n}$ be given and let $q = \min\{m,n\}$.
(a) Use the inequality (3.3.13(a)) and the product inequalities (3.3.14(a)) to prove *Von Neumann's trace theorem*

$$|\operatorname{tr} A^*B| \le \sum_{i=1}^{q} \sigma_i(A)\sigma_i(B) \tag{3.3.25}$$

(b) Use (a) and the singular value decomposition to show that $\max\{|\operatorname{tr} UA^*VB|: U \in M_n \text{ and } V \in M_m \text{ are unitary}\} = \Sigma_i \sigma_i(A)\sigma_i(B)$. Show that there are choices of U and V that yield the maximum and make both $(UA^*)(VB)$ and $(VB)(UA^*)$ positive semidefinite; see Theorem (7.4.10) in [HJ] for an important related result.

11. Use (3.3.17) to explain why the problem of computing singular values of a given matrix is intrinsically well conditioned. Contrast with the analogous situation for eigenvalues; see (6.3.3–4) in [HJ].

12. Prove a two-sided multiplicative bound analogous to the two-sided additive bound in (3.3.19): If $A, B \in M_n$, then $\sigma_1(B)\sigma_i(A) \ge \sigma_i(AB) \ge \sigma_n(B)\sigma_i(A)$ for $i = 1, \ldots, n$. If $B = I + E$, verify the multiplicative perturbation bounds

$$[1 - \sigma_1(E)]\sigma_i(A) \le \sigma_i(A[I+E]) \le [1 + \sigma_1(E)]\sigma_i(A), \quad i = 1, \ldots, n \tag{3.3.26}$$

What relative perturbation bounds on $\sigma_i(A[I+E])/\sigma_i(A)$ does this give for $i = 1, \ldots, \operatorname{rank} A$? What does this say for $i > \operatorname{rank} A$?

13. Let $A \in M_{m,n}$ and $X \in M_{n,k}$ be given with $k \le \min\{m,n\}$. Show that $\det(X^*A^*AX) \le [\sigma_1(A) \cdots \sigma_k(A)]^2 \det X^*X$.

14. Let $A \in M_n$. Show that $|\lambda_1(A) \cdots \lambda_k(A)| = \sigma_1(A) \cdots \sigma_k(A)$ for all $k = 1, \ldots, n$ if and only if A is normal.

15. Suppose a given real-valued function $\varphi(u) \equiv \varphi(u_1, \ldots, u_k)$ of k scalar variables has continuous first partial derivatives in the domain $\mathcal{D}_k(L) \equiv \{u = [u_i] \in \mathbb{R}^k: L < u_k \le u_{k-1} \le \cdots \le u_2 \le u_1 < \infty\}$, where $L \ge -\infty$ is given. Let $\alpha = [\alpha_i] \in \mathcal{D}_k(L)$ and $\beta = [\beta_i] \in \mathcal{D}_k(L)$ be given, so the real vectors α and β have algebraically decreasingly ordered entries.

(a) Sketch $\mathcal{D}_k(L)$ for $k = 2$ and finite L; for $L = -\infty$.

(b) Show that $\mathcal{D}_k(L)$ is convex.

(c) Explain why $f(t) \equiv \varphi((1-t)\alpha + t\beta)$ is defined and continuously differentiable for $t \in [0,1]$, and

$$\varphi(\beta) - \varphi(\alpha) = \int_0^1 f'(t)\, dt$$

Show that

$$f'(t) = (\beta - \alpha)^T \nabla\varphi = \sum_{i=1}^{k} (\beta_i - \alpha_i) \frac{\partial \varphi}{\partial u_i}$$

$$= (\beta_1 - \alpha_1) \frac{\partial \varphi}{\partial u_1} + \sum_{i=2}^{k-1} \Big[\sum_{j=1}^{i-1} (\beta_j - \alpha_j) \Big] \Big[\frac{\partial \varphi}{\partial u_i} - \frac{\partial \varphi}{\partial u_{i+1}} \Big]$$

$$+ \frac{\partial \varphi}{\partial u_k} \sum_{i=1}^{k} (\beta_i - \alpha_i)$$

for all $t \in [0,1]$.

(d) Now suppose that α is weakly majorized by β, that is,

$$\sum_{j=1}^{i} \alpha_j \le \sum_{j=1}^{i} \beta_j, \quad i = 1, \ldots, k$$

and suppose that $\varphi(\cdot)$ satisfies the inequalities

$$\frac{\partial \varphi}{\partial u_1} \ge \frac{\partial \varphi}{\partial u_2} \ge \cdots \ge \frac{\partial \varphi}{\partial u_k} \ge 0 \text{ at every point in } \mathcal{D}_k(L) \tag{3.3.27}$$

that is, the vector $\nabla\varphi(u)$ is a point in $\mathcal{D}_k(0)$ for every $u \in \mathcal{D}_k(L)$. Show that $\varphi(\alpha) \le \varphi(\beta)$, so φ is monotone with respect to weak majorization.

(e) Let $f: \mathbb{R} \to \mathbb{R}$ be a given increasing convex and continuously differentiable function. Show that $\varphi(u) \equiv f(u_1) + \cdots + f(u_k)$ satisfies the inequalities (3.3.27) on $\mathcal{D}_k(-\infty)$. Use a smoothing argument to deduce the result in Lemma (3.3.8), which does not require that $f(\cdot)$ be continuously differentiable.

16. Suppose a given real-valued function $F(u) \equiv F(u_1, \ldots, u_k)$ of k scalar variables has continuous first partial derivatives in the domain $\mathcal{D}_k(0) \equiv \{u = [u_i] \in \mathbb{R}^k\colon\ 0 \le u_k \le u_{k-1} \le \cdots \le u_2 \le u_1 < \infty\}$, and suppose $u \circ \nabla F(u) \in \mathcal{D}_k(0)$ for every $u \in \mathcal{D}_k(0)$, that is,

$$u_1 \frac{\partial F}{\partial u_1} \ge u_2 \frac{\partial F}{\partial u_2} \ge \cdots \ge u_k \frac{\partial F}{\partial u_k} \ge 0 \text{ at every point } u \in \mathcal{D}_k(0) \tag{3.3.28}$$

(a) Show that the function $\varphi(u) \equiv F(e^{u_1},\ldots, e^{u_k})$ satisfies the hypotheses in Problem 15 on $\mathcal{D}_k(-\infty)$ and satisfies the inequalities (3.3.27).
(b) Using the notation of Theorem (3.3.2), show that $F(|\lambda_1(A)|,\ldots, |\lambda_k(A)|) \le F(\sigma_1(A),\ldots, \sigma_k(A))$ for $k = 1,\ldots, n$.
(c) Using the notation of Theorem (3.3.4), show that $F(\sigma_1(AB),\ldots, \sigma_k(AB)) \le F(\sigma_1(A)\sigma_1(B),\ldots, \sigma_k(A)\sigma_k(B))$ for $k = 1,\ldots, \min\{m,n,p\}$.
(d) Let the real-valued function $f(t)$ be increasing and continuously differentiable on $[0,\infty)$ and suppose $f(e^t)$ is convex on $(-\infty,\infty)$. Show that $F(u) = F(u_1,\ldots, u_k) \equiv f(e^{u_1}) + \cdots + f(e^{u_k})$ satisfies the required smoothness conditions and the inequalities (3.3.28). Conclude that the additive weak majorization relations (3.3.13-14(d)) follow from the product inequalities (3.3.2,4) and the general inequalities in (b) and (c).
(e) Let $S_j(u_1,\ldots, u_k)$ denote the jth elementary symmetric function of the k scalar variables $u_1,\ldots, u_k$; see (1.2.9) in [HJ]. Show that $S_j(u_1,\ldots, u_k)$ is a smooth function that satisfies the inequalities (3.3.28).
(f) Using the notation of Theorems (3.3.2,4), conclude that each elementary symmetric function $S_j(u_1,\ldots, u_k)$ satisfies

$$S_j(|\lambda_1(A)|,\ldots, |\lambda_k(A)|) \le S_j(\sigma_1(A),\ldots, \sigma_k(A))$$

and

$$S_j(\sigma_1(AB),\ldots, \sigma_k(AB)) \le S_j(\sigma_1(A)\sigma_1(B),\ldots, \sigma_k(A)\sigma_k(B))$$

for all appropriate values of k and all $j = 1,\ldots, k$. What do these inequalities say when $j = 1$?

17. Let $A \in M_n$ be given. Using the notation of Theorem (3.3.2), show that $|\lambda_1(A) \cdots \lambda_k(A)| \le \sigma_1(A^m)^{1/m} \cdots \sigma_k(A^m)^{1/m}$, $k = 1,\ldots, n$, $m = 1, 2,\ldots$. If $f(t)$ is an increasing continuously differentiable real-valued function on $[0,\infty)$, and if $f(e^t)$ is convex on $(-\infty,\infty)$, show that

$$\sum_{i=1}^{k} f(|\lambda_i(A)|) \le \sum_{i=1}^{k} f(\sigma_i(A^m)^{1/m}), \quad k = 1,\ldots, n,\ m = 1, 2,\ldots \tag{3.3.29}$$

which is a generalization of (3.3.13(d)). What happens as $m \to \infty$?

18. Let $A = [a_{ij}], B \in M_{m,n}$ be given, let $q = \min\{m,n\}$, let $[A,B]_F \equiv \operatorname{tr} B^*A$ denote the Frobenius inner product on $M_{m,n}$, and let $\|A\|_2 \equiv [A,A]_F^{1/2}$ denote the Frobenius norm (the l_2 norm) on $M_{m,n}$.

(a) Show that

$$\|A\|_2 = \left[\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right]^{\frac{1}{2}} = \left[\sum_{i=1}^{q}\sigma_i(A)^2\right]^{\frac{1}{2}} \tag{3.3.30}$$

(b) Show that

$$|[A,B]_F| \le \sigma_1(A)\sigma_1(B) + \cdots + \sigma_q(A)\sigma_q(B) \le \|A\|_2\,\|B\|_2 \tag{3.3.31}$$

which gives an enhancement to the Cauchy-Schwarz inequality for the Frobenius inner product.

(c) Show that

$$\left[\sum_{i=1}^{q}[\sigma_i(A) - \sigma_i(B)]^2\right]^{\frac{1}{2}} \le \|A - B\|_2 \tag{3.3.32}$$

which may be thought of as a simultaneous perturbation bound on all of the singular values. How is this related to the perturbation bound (3.3.19) for one singular value? Discuss the analogy with the Hoffman-Wielandt theorem (Theorem (6.3.5) in [HJ]).

19. Ordinary singular values are intimately connected with the Euclidean norm, but many properties of singular values hold for other functions of a matrix connected with arbitrary norms. Let $\|\cdot\|_a$ and $\|\cdot\|_b$ denote two given norms on $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, and let $q = \min\{m,n\}$. For $A \in M_{m,n}$, define

$$\sigma_k(A; \|\cdot\|_a, \|\cdot\|_b) \equiv \min_{\substack{S \subset \mathbb{C}^n \\ \dim S = n-k+1}} \; \max_{\substack{x \in S \\ \|x\|_b = 1}} \|Ax\|_a, \quad k = 1,\dots, q$$

where S ranges over all subspaces with the indicated dimension.

(a) Show that $\sigma_k(A; \|\cdot\|_a, \|\cdot\|_b) = 0$ if $k > q$.

(b) Show that $\sigma_1(A; \|\cdot\|_a, \|\cdot\|_b)$ is a norm on $M_{m,n}$.

(c) Show that

$$\sigma_{i+j-1}(A + B; \|\cdot\|_a, \|\cdot\|_b) \le \sigma_i(A; \|\cdot\|_a, \|\cdot\|_b) + \sigma_j(B; \|\cdot\|_a, \|\cdot\|_b)$$

for all $A, B \in M_{m,n}$, $1 \le i, j \le q$, and $i + j \le q + 1$.

(d) Verify the perturbation bounds

$$|\sigma_i(A+B; \|\cdot\|_a, \|\cdot\|_b) - \sigma_i(A; \|\cdot\|_a, \|\cdot\|_b)| \le \sigma_1(B; \|\cdot\|_a, \|\cdot\|_b)$$

for all $A, B \in M_{m,n}$ and all $i = 1,\ldots, q$.

20. Let $A \in M_n$ be given. There are three natural Hermitian matrices associated with A: A^*A, AA^*, and $\mathrm{H}(A) \equiv \frac{1}{2}(A + A^*)$. Weyl's theorem (3.3.2) and A. Horn's theorem (3.6.6) characterize the relationship between the eigenvalues of A and those of A^*A and AA^*. A theorem of L. Mirsky characterizes the relationship between the eigenvalues of A and $\mathrm{H}(A)$. Let $\{\lambda_i(A)\}$ and $\{\lambda_i(\mathrm{H}(A))\}$ denote the eigenvalues of A and $\mathrm{H}(A)$, respectively, ordered so that $\mathrm{Re}\,\lambda_1(A) \ge \cdots \ge \mathrm{Re}\,\lambda_n(A)$ and $\lambda_1(\mathrm{H}(A)) \ge \cdots \ge \lambda_n(\mathrm{H}(A))$.

(a) Show that

$$\sum_{i=1}^{k} \mathrm{Re}\,\lambda_i(A) \le \sum_{i=1}^{k} \lambda_i(\mathrm{H}(A)), \quad k = 1,\ldots, n, \text{ with equality for } k = n \tag{3.3.33}$$

(b) Conversely, if $2n$ given scalars $\{\lambda_i\}_{i=1}^n \subset \mathbb{C}$ and $\{\eta_i\}_{i=1}^n \subset \mathbb{R}$ are ordered so that $\mathrm{Re}\,\lambda_1 \ge \cdots \ge \mathrm{Re}\,\lambda_n$ and $\eta_1 \ge \cdots \ge \eta_n$, and if

$$\sum_{i=1}^{k} \mathrm{Re}\,\lambda_i \le \sum_{i=1}^{k} \eta_i, \quad k = 1,\ldots, n, \text{ with equality for } k = n$$

show that there is some $A \in M_n$ such that $\{\lambda_i\}$ is the set of eigenvalues of A and $\{\eta_i\}$ is the set of eigenvalues of $\mathrm{H}(A)$.

(c) Use (3.1.6a) to show that

$$\sum_{i=1}^{k} \mathrm{Re}\,\lambda_i(A) \le \sum_{i=1}^{k} \lambda_i(\mathrm{H}(A)) \le \sum_{i=1}^{k} \sigma_i(A) \quad \text{for } k = 1,\ldots, n \tag{3.3.34a}$$

If f is a real-valued convex function on the interval $[\lambda_n(\mathrm{H}(A)), \lambda_1(\mathrm{H}(A))]$, show that

$$\sum_{i=1}^{n} f(\mathrm{Re}\,\lambda_i(A)) \le \sum_{i=1}^{n} f(\lambda_i(\mathrm{H}(A))) \tag{3.3.34b}$$

If f is a real-valued increasing convex function on the interval

$[\lambda_n(\mathrm{H}(A)), \sigma_1(A)]$, show that

$$\sum_{i=1}^{k} f(\mathrm{Re}\,\lambda_i(A)) \le \sum_{i=1}^{k} f(\lambda_i(\mathrm{H}(A))) \le \sum_{i=1}^{k} f(\sigma_i(A)) \quad \text{for } k = 1,\dots, n \tag{3.3.34c}$$

(d) Let $H \in M_n$ be a given Hermitian matrix, and write $H = \mathrm{Re}\,H + i\,\mathrm{Im}\,H$, where $\mathrm{Re}\,H = (H + \bar{H})/2$ is real symmetric and $\mathrm{Im}\,H = (H - \bar{H})/2i$ is real skew-symmetric. Let $\mathrm{Re}\,H = QDQ^T$, where $Q \in M_n(\mathbb{R})$ is real orthogonal and $D = \mathrm{diag}(\alpha_1,\dots, \alpha_n)$ is real diagonal. Show that $\alpha_1,\dots, \alpha_n$ are the main diagonal entries of Q^THQ, and conclude, as in (a), that there is a strong majorization relationship between the eigenvalues of $\mathrm{Re}\,H$ and the eigenvalues of H.

(e) Let $\lambda_1(\mathrm{Re}\,\mathrm{H}(A)) \ge \cdots \ge \lambda_n(\mathrm{Re}\,\mathrm{H}(A))$ denote the algebraically decreasingly ordered eigenvalues of $\mathrm{Re}\,\mathrm{H}(A) = \frac{1}{2}(\mathrm{H}(A) + \overline{\mathrm{H}(A)})$. Use (d) to show that $\mathrm{Re}\,\lambda_i(A)$ can be replaced by $\lambda_i(\mathrm{Re}\,\mathrm{H}(A))$ in (3.3.33) and (3.3.34a-c).

21. Let $A = [a_{ij}] \in M_{m,n}$ be given and let $q = \min\{m,n\}$. Let $a \equiv [a_{11}, \dots, a_{qq}]^T \in \mathbb{C}^q$ and $\sigma(A) \equiv [\sigma_1(A), \dots, \sigma_q(A)]^T \in \mathbb{R}^q$ denote the vectors of main diagonal entries and singular values of A, respectively, with $\sigma_1(A) \ge \cdots \ge \sigma_q(A)$. Let $|a|_{[1]} \ge \cdots \ge |a|_{[q]}$ denote the absolutely decreasingly ordered main diagonal entries of A.

(a) Use the basic inequalities (3.3.13(a)) or the inequalities (3.1.10a) (which follow directly from the singular value decomposition) and the interlacing inequalities (3.1.4) to show that

$$|a|_{[1]} + \cdots + |a|_{[k]} \le \sigma_1(A) + \cdots + \sigma_k(A) \quad \text{for } k = 1,\dots, q \tag{3.3.35}$$

For a different proof of these necessary conditions, see Problem 3 in Section (3.2).

(b) Use Lemma (3.3.8) to deduce the weaker necessary conditions

$$|a|_{[1]}^2 + \cdots + |a|_{[k]}^2 \le \sigma_1(A)^2 + \cdots + \sigma_k(A)^2 \quad \text{for } k = 1,\dots, q \tag{3.3.36}$$

Consider $\alpha_1 = 1$, $\alpha_2 = 1/\sqrt{2}$, $s_1 = \sqrt{5}/2$, $s_2 = \frac{1}{2}$ to show that there are values satisfying (3.3.36) that do not satisfy (3.3.35), and hence cannot be the main diagonal entries and singular values of a complex matrix.

(c) Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2$. Show that $[\sigma_1(A) - \sigma_2(A)]^2 = \operatorname{tr} A^*A - 2|\det A| = |a|^2 + |b|^2 + |c|^2 + |d|^2 - 2|ad - bc| \geq (|a| - |d|)^2 + (|b| - |c|)^2 \geq (|a| - |d|)^2$. Using the notation in (a), conclude that

$$|a|_{[1]} - |a|_{[2]} \leq \sigma_1(A) - \sigma_2(A) \text{ for all } A \in M_2 \tag{3.3.37}$$

(d) Consider $\alpha_1 = s_1 = 1$, $\alpha_2 = \frac{1}{2}$, $s_2 = 3/4$ to show that there are potential values for the main diagonal entries and singular values of a 2-by-2 complex matrix that satisfy the necessary conditions (3.3.35) (and hence satisfy (3.3.36) as well) but do not satisfy the necessary conditions (3.3.37).

(e) Let $\sigma_1 \geq \sigma_2 \geq 0$ and $a_1, a_2 \in \mathbb{C}$ be given with $|a_1| \geq |a_2|$. Suppose $|a_1| + |a_2| \leq \sigma_1 + \sigma_2$ and $|a_1| - |a_2| \leq \sigma_1 - \sigma_2$. Square these inequalities, show that $\pm 2[|a_1 a_2| - \sigma_1\sigma_2| \leq \sigma_1^2 + \sigma_2^2 - (|a_1|^2 + |a_2|^2)$, and conclude that $||a_1 a_2| - \sigma_1\sigma_2| \leq 2||a_1 a_2| - \sigma_1\sigma_2| \leq \sigma_1^2 + \sigma_2^2 - (|a_1|^2 + |a_2|^2)$. Sketch the loci of $xy = ||a_1 a_2| - \sigma_1\sigma_2|$ and $x^2 + y^2 = \sigma_1^2 + \sigma_2^2 - (|a_1|^2 + |a_2|^2)$, $x, y \in \mathbb{R}$, and explain why there exist real ξ, η such that $\xi\eta = |a_1 a_2| - \sigma_1\sigma_2$ and $\xi^2 + \eta^2 + |a_1|^2 + |a_2|^2 = \sigma_1^2 + \sigma_2^2$. Set

$$B = \begin{bmatrix} |a_1| & \xi \\ \eta & |a_2| \end{bmatrix}$$

show that $\operatorname{tr} B^*B = \sigma_1^2 + \sigma_2^2$ and $\det B = \sigma_1\sigma_2$, and conclude that B has singular values σ_1 and σ_2. Construct a diagonal unitary D such that $A = DB$ has main diagonal entries a_1, a_2 and singular values σ_1, σ_2.

(f) Explain why there is some $A \in M_2$ with given main diagonal entries $a_1, a_2 \in \mathbb{C}$ and singular values $\sigma_1 \geq \sigma_2 \geq 0$ if and only if the inequalities (3.3.35) (with $n = 2$) and (3.3.37) are satisfied. Of what theorem for general $n = 2, 3, \ldots$ do you think this is a special case?

22. We have shown that (weak) multiplicative majorization inequalities of the form (3.3.5) imply additive weak majorization inequalities of the form (3.3.14). However, not every weak additive majorization comes from a multiplicative one. Show by example that (3.3.5) is false if the ordinary product AB is replaced by the Hadamard product $A \circ B$. Nevertheless, Theorem (5.5.4) says that (3.3.14) does hold when AB is replaced by $A \circ B$.

23. Using the notation of Lemma (3.3.8), provide details for the following alternative proof that avoids using facts about doubly substochastic matrices and relies only on the basic relationship between strong majorization and doubly stochastic matrices. State carefully the hypotheses and conclusions appropriate to this argument, and notice that one may now have to assume that $f(\cdot)$ is defined, monotone, and convex on an interval that extends to the left of the point $\min\{x_n, y_n\}$. If $\delta_k \equiv \Sigma_{i=1}^k (y_i - x_i) > 0$, choose points x'_{k+1} and y'_{k+1} such that $x_k \geq x'_{k+1}$, $y_k \geq y'_{k+1}$, and $\delta_k = x'_{k+1} - y'_{k+1}$. Let $x'_i = x_i$ and $y'_i = y_i$ for $i = 1, \ldots, k$ and let $x' \equiv [x'_i]$, $y' \equiv [y'_i] \in \mathbb{R}^{k+1}$. By strong majorization, there is a doubly stochastic $S \in M_{k+1}(\mathbb{R})$ such that $x' = Sy'$, so $f(x_1) + \cdots + f(x_k) + f(x'_{k+1}) \leq f(y_1) + \cdots + f(y_k) + f(y'_{k+1})$. Now use $f(y'_{k+1}) - f(x'_{k+1}) \leq 0$.

24. Based on Lemma (3.3.8), how can the hypotheses of the domain of the function $f(\cdot)$ in Theorems (3.3.13-14) be weakened?

25. What do Theorems (3.3.13-14(d)) say for $f(t) = 1/t$? Can you get the same result from (3.3.13-14(a))? What about $f(t) = t^{-\frac{1}{2}}$? If $A, B \in M_n$ are nonsingular, show that

$$\left[\sum_{i=1}^{n} \sigma_i(AB)^{-\frac{1}{2}}\right]^2 \leq \left[\sum_{i=1}^{n} \sigma_i(A^{-1})\right]\left[\sum_{i=1}^{n} \sigma_i(B^{-1})\right] \tag{3.3.38}$$

26. Let $A \in M_n$ be given, and let m be a given positive integer.

(a) Show that

$$\sum_{i=1}^{k} \sigma_i(A^m) \leq \sum_{i=1}^{k} \sigma_i(A)^m \quad \text{for } k = 1, \ldots, n$$

(b) Show that

$$\sum_{i=1}^{k} \sigma_i(A^m)^p \leq \sum_{i=1}^{k} \sigma_i(A)^{mp} \quad \text{for } k = 1, \ldots, n \tag{3.3.39}$$

for all $p > 0$.

(c) Show that

$$|\operatorname{tr} A^{2m}| \leq \operatorname{tr}[(A^m)^* A^m] \leq \operatorname{tr}[(A^*A)^m] \text{ for all } m = 1, 2, \ldots \tag{3.3.40}$$

The role of this inequality in statistical mechanics is mentioned in Section (6.5); see Corollary (6.5.32) for some other functions $f(\cdot)$ that satisfy inequalities of the form $f(A^{2m}) \leq f([A^*A]^m)$.

27. Let $A = [a_{ij}] \in M_n$ be a given Hermitian matrix with algebraically decreasingly ordered eigenvalues $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ and main diagonal entries $a_1 \geq \cdots \geq a_n$. Let $f(t)$ be a given convex function on the interval $[\lambda_n(A), \lambda_1(A)]$.

(a) Show that

$$\sum_{i=1}^{n} f(a_{ii}) \leq \sum_{i=1}^{n} f(\lambda_i(A)) \tag{3.3.41}$$

and, if $f(t)$ is also an increasing function,

$$\sum_{i=1}^{k} f(a_{ii}) \leq \sum_{i=1}^{k} f(\lambda_i(A)) \text{ for } k = 1,\ldots, n \tag{3.3.42}$$

(b) Let $\{x_1,\ldots, x_n\} \subset \mathbb{C}^n$ be a given orthonormal set, and let the values $x_1^*Ax_1,\ldots, x_n^*Ax_n$ be algebraically ordered as $\alpha_1 \geq \cdots \geq \alpha_n$. Show that

$$\sum_{i=1}^{n} f(x_i^*Ax_i) \leq \sum_{i=1}^{n} f(\lambda_i(A)) \tag{3.3.43}$$

and, if $f(\cdot)$ is also increasing,

$$\sum_{i=1}^{k} f(\alpha_i) \leq \sum_{i=1}^{k} f(\lambda_i(A)) \text{ for } k = 1,\ldots, n \tag{3.3.44}$$

28. Let $A \in M_n$ be given. Show that every eigenvalue λ of A satisfies $\sigma_1(A) \geq |\lambda| \geq \sigma_n(A)$.

29. Let $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ be a given twice differentiable real-valued function on $(a,b) \subset \mathbb{R}$, and let $\varphi(t) \equiv f(e^t)$. Show that if $f(t)$ is increasing and convex, then $\varphi(t)$ is convex. Conversely, consider $f(t) = 1/t$ and $f(t) = 1/t^{\frac{1}{2}}$ on $(0,\infty)$ to show that $\varphi(t)$ can be convex and f can be either decreasing or concave (but not both at any one point). Comment on the relative strengths of the hypotheses of parts (c) and (d) of Theorems (3.3.13-14).

30. Let $A, B \in M_n$ be given *commuting* matrices. If all eigenvalues are arranged in decreasing absolute value as in Theorem (3.3.21), show that

$$|\lambda_k(AB)| \le |\lambda_{k-j+1}(A)|\ |\lambda_j(B)|,\ j = 1,\dots, k,\ k = 1,\dots, n \qquad (3.3.45)$$

which implies

$$|\lambda_k(AB)| \le \min\{\rho(A)\,|\lambda_k(B)|, \rho(B)\,|\lambda_k(A)|\},\ k = 1,\dots, n \qquad (3.3.46a)$$

and

$$\rho(AB) \le \rho(A)\,\rho(B) \qquad (3.3.46b)$$

If, in addition, both A and B are nonsingular, deduce that

$$|\lambda_k(AB)| \ge |\lambda_{k+j-1}(A)|\ |\lambda_{n-j+1}(B)|,\quad j = 1,\dots, n-k+1,\ k = 1,\dots, n \qquad (3.3.47a)$$

and

$$|\lambda_n(AB)| \ge |\lambda_n(A)|\ |\lambda_n(B)| \qquad (3.3.47b)$$

Give examples to show that these inequalities need not hold if the hypothesis of commutativity is dropped.

31. Let $A \in M_n$ be given, let $|\lambda_1(A)| \ge \cdots \ge |\lambda_n(A)|$ denote its absolutely decreasingly ordered eigenvalues, and let $\sigma_1(A) \ge \cdots \ge \sigma_n(A)$ denote its decreasingly ordered singular values. If A is normal, we know that $|\lambda_i(A)| = \sigma_i(A)$ for $i = 1,\dots, n$, but if A is "almost normal," can we say that $|\lambda_i(A)| \doteq \sigma_i(A)$? The *defect from normality* of A with respect to a given unitarily invariant norm $\|\cdot\|$ is defined in terms of the set of all Schur unitary triangularizations of A:

$$\delta(A;\|\cdot\|) = \inf\{\|T\|:\ A = U(\Lambda + T)U^*,\ U \text{ is unitary, } \Lambda \text{ is diagonal, and } T \text{ is strictly upper triangular}\} \qquad (3.3.48)$$

One might consider A to be "almost normal" if $\delta(A;\|\cdot\|)$ is small. This measure of nonnormality is perhaps most familiar when $\|\cdot\|$ is the Frobenius norm $\|\cdot\|_2$.

(a) Show that

$$\delta(A;\|\cdot\|_2)^2 = \sum_{i=1}^{n}[\sigma_i(A)^2 - |\lambda_i(A)|^2] \tag{3.3.49}$$

Conclude that all $|\lambda_i(A)| \doteq \sigma_i(A)$ if and only if A is "almost normal" in the sense that $\delta(A;\|\cdot\|_2)$ is small.
(b) Consider the spectral norm $|||\cdot|||_2$. Use (3.3.19) and a Schur upper triangularization $A = U(\Lambda + T)U^*$ to show that

$$|\sigma_i(A) - |\lambda_i(A)|| \le \delta(A;|||\cdot|||_2) \text{ for } i = 1,\ldots, n \tag{3.3.50}$$

Thus, if A is "almost normal" in the sense that $\delta(A;|||\cdot|||_2)$ is small, then all $\sigma_i(A) \doteq |\lambda_i(A)|$.
(c) Suppose $A = SBS^{-1}$ for some nonsingular $S \in M_n$. Use (3.3.20) to show that

$$\sigma_i(B)/\kappa_2(S) \le |\sigma_i(A)| \le \kappa_2(S)\sigma_i(B) \text{ for } i = 1,\ldots, n \tag{3.3.51}$$

where $\kappa_2(S) \equiv |||S|||_2\,|||S^{-1}|||_2$ denotes the spectral condition number of S. What does this say when A is normal? See Problem 45 in Section (3.1) for another approach to the inequalities (3.3.51).

Notes and Further Readings. The basic theorems (3.3.2) and (3.3.4) were first published in the classic papers of H. Weyl (1949) and A. Horn (1950) cited at the end of Section (3.0). The proof we give for Theorem (3.3.2) is similar to the proof of the lemma in Section (2.4) of J. P. O. Silberstein, On Eigenvalues and Inverse Singular Values of Compact Linear Operators in Hilbert Space, *Proc. Cambridge Phil. Soc.* 49 (1953), 201-212. Theorem (3.3.4) can be generalized to give inequalities for products of arbitrarily chosen singular values, not just the consecutive ones: If A, $B \in M_n$ and indices $1 \le i_1 < i_2 < \cdots < i_k \le n$ are given, then

$$\begin{aligned}\sigma_{i_1}(AB)\cdots\sigma_{i_k}(AB) \le \min\{&\sigma_{i_1}(A)\cdots\sigma_{i_k}(A)\sigma_1(B)\cdots\sigma_k(B),\\ &\sigma_1(A)\cdots\sigma_k(A)\sigma_{i_1}(B)\cdots\sigma_{i_k}(B)\}\end{aligned} \tag{3.3.52}$$

For a proof and a discussion of related results for arbitrary sums of eigenvalues of sums of Hermitian matrices, see the appendix to [Gan 86], pp. 621-624, or Section 4 of [AmMo]. There are also generalizations of the inequalities (3.3.18): If $A, B \in M_n$ and indices $1 \leq i_1 < \cdots < i_k \leq n$, $1 \leq j_1 < \cdots < j_k \leq n$ are given with $i_k + j_k - k \leq n$, then

$$\prod_{r=1}^{k} \sigma_{i_r + j_r - r}(AB) \leq \prod_{r=1}^{k} \sigma_{i_r}(A)\, \sigma_{j_r}(B) \tag{3.3.53}$$

For a proof, see R. C. Thompson, On the Singular Values of Matrix Products-II, *Scripta Math.* 29 (1973), 111-114.

The discovery of inequalities of the form (3.3.13b) seems to have been stimulated by results in the theory of integral equations. In the 1949 paper cited at the end of Section (3.0), S.-H. Chang showed with a long analytic argument that if the infinite series $\Sigma_{i=1}^{\infty} \sigma_i(K)^p$ of powers of the singular values of an L^2 kernel K is convergent for some $p > 0$, then the series $\Sigma_{i=1}^{\infty} |\lambda_i(K)|^p$ of the same powers of the absolute eigenvalues of K is also convergent, but Chang had no inequality between these two series. Apparently stimulated by Chang's results, Weyl discovered the basic inequalities in Theorems (3.3.2) and (3.3.13).

The proof given for Yamamoto's theorem (3.3.21) was developed in C. R. Johnson and P. Nylen, Yamamoto's Theorem for Generalized Singular Values, *Linear Algebra Appl.* 128 (1990), 147-158 (where generalizations to other types of singular values are given) and also in R. Mathias, Two Theorems on Singular Values and Eigenvalues, *Amer. Math. Monthly* 97 (1990), 47-50, from which the approach to Weyl's theorem given in Problem 8 is also taken. There is always a version of the right half of the interlacing inequalities (3.1.4) for the generalized singular values associated with two arbitrary norms, as defined in Problem 19; if the two norms have a general monotonicity property, one also gets a version of the left half—see M.-J. Sodupe, Interlacing Inequalities for Generalized Singular Values, *Linear Algebra Appl.* 87 (1987), 85-92. The fact that the weak majorization conditions (3.3.35), together with a general version of (3.3.37), namely, $|a|_{[1]} + \cdots + |a|_{[n-1]} - |a|_{[n]} \leq \sigma_1 + \cdots + \sigma_{n-1} - \sigma_n$, are both necessary and sufficient for $2n$ given scalars $a_1, \ldots, a_n \in \mathbb{C}$ and $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ to be the main diagonal entries and singular values of some n-by-n complex matrix A is proved in R. C. Thompson, Singular Values, Diagonal Elements, and Convexity, *SIAM J. Appl. Math.* 32 (1977), 39-63. If, in addition, one wants A to be symmetric, then there are some additional conditions that must be satisfied; these are described in R. C. Thompson, Singular Values and Diagonal Elements of Complex Symmetric Matrices, *Linear Algebra Appl.* 26 (1979), 65-106. For a characterization of the cases in which equality holds in the inequalities (3.3.3), (3.3.13a), (3.3.33), or (3.3.35) for some value of k, see C.-K. Li, Matrices with Some Extremal Properties, *Linear Algebra Appl.* 101 (1988), 255-267.

3.4 Sums of singular values: the Ky Fan k-norms

One way to approach majorization inequalities for the eigenvalues of sums of Hermitian matrices is via the variational characterization

$$\lambda_1(A) + \cdots + \lambda_k(A)$$
$$= \max\{\mathrm{tr}\, U_k^* A U_k \colon U_k \in M_{n,k} \text{ and } U_k^* U_k = I\},\ k = 1,\dots, n$$

in which the eigenvalues of the Hermitian matrix $A \in M_n$ are arranged in algebraically decreasing order $\lambda_1 \geq \cdots \geq \lambda_n$ (see Corollary (4.3.18) and Theorem (4.3.27) in [HJ]). An analogous characterization of the sum of the k largest singular values has the pleasant consequence that it makes evident that this sum is a norm. The following result is a natural generalization of Problem 6 in Section (3.1).

3.4.1 Theorem. Let $A \in M_{m,n}$ have singular values $\sigma_1(A) \geq \cdots \geq \sigma_q(A) \geq 0$, where $q = \min\{m,n\}$. For each $k = 1,\dots, q$ we have

$$\sum_{i=1}^{k} \sigma_i(A) = \max\{|\mathrm{tr}\, X^* A Y| \colon X \in M_{m,k},\ Y \in M_{n,k},\ X^*X = I = Y^*Y\}$$

$$= \max\{|\mathrm{tr}\, AC| \colon C \in M_{n,m} \text{ is a rank } k \text{ partial isometry}\}$$

Proof: If $X \in M_{m,k}$ and $Y \in M_{n,k}$ satisfy $X^*X = I = Y^*Y$, notice that $\mathrm{tr}\, X^*AY = \mathrm{tr}\, AYX^*$ and $C \equiv YX^* \in M_{n,m}$ has $C^*C = XY^*YX^* = XX^*$. Since the k largest singular values of XX^* are the same as those of $X^*X = I \in M_k$, we conclude that $C = YX^*$ is a rank k partial isometry. Conversely, if $C \in M_{n,m}$ is a given rank k partial isometry, then the singular value decomposition of C is

$$C = V\Sigma W^* = [V_k \ *] \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W_k^* \\ * \end{bmatrix} = V_k W_k^*$$

where $I_k \in M_k$ is an identity matrix, and $V_k \in M_{n,k}$ and $W_k \in M_{m,k}$ are the first k columns of the unitary matrices $V \in M_n$ and $W \in M_n$, respectively. Thus the two asserted variational formulae are equivalent and it suffices to show that the second equals the indicated singular value sum. But this is an immediate consequence of (3.3.13a) and (3.3.14a). Compute

$$|\operatorname{tr} AC| = \Big|\sum_{i=1}^{m} \lambda_i(AC)\Big| \le \sum_{i=1}^{m} |\lambda_i(AC)| \le \sum_{i=1}^{m} \sigma_i(AC)$$

$$\le \sum_{i=1}^{q} \sigma_i(A)\sigma_i(C) = \sum_{i=1}^{k} \sigma_i(A) \tag{3.4.2}$$

in which all indicated eigenvalues $\{\lambda_i\}$ and singular values $\{\sigma_i\}$ are arranged in decreasing absolute value. If $A = V\Sigma W^*$ is a singular value decomposition of A, let

$$C_{max} \equiv WP_kV^*, \text{ where } P_k \equiv \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \in M_{n,m} \text{ and } I_k \in M_k$$

Then $\operatorname{tr} AC_{max} = \operatorname{tr} V\Sigma W^* WP_kV^* = \operatorname{tr} V\Sigma P_kV^* = \operatorname{tr} \Sigma P_k = \sigma_1(A) + \cdots + \sigma_k(A)$, so the upper bound in (3.4.2) can be achieved. $\square$

The preceding result is useful as a quasilinear characterization of the sum of the k largest singular values and it implies majorization relations for sums of singular values of sums of (not necessarily square) matrices.

3.4.3 Corollary. Let $A, B \in M_{m,n}$ have respective ordered singular values $\sigma_1(A) \ge \cdots \ge \sigma_q(A) \ge 0$ and $\sigma_1(B) \ge \cdots \ge \sigma_q(B) \ge 0$, $q \equiv \min\{m,n\}$, and let $\sigma_1(A+B) \ge \cdots \ge \sigma_q(A+B) \ge 0$ be the ordered singular values of $A+B$. Then

$$\sum_{i=1}^{k} \sigma_i(A+B) \le \sum_{i=1}^{k} \sigma_i(A) + \sum_{i=1}^{k} \sigma_i(B), \quad k = 1, \ldots, q$$

Proof: Let $P_{n,m;k}$ denote the set of rank k partial isometries in $M_{n,m}$. Use

(3.4.1) and observe that

$$\begin{aligned}
&\sum_{i=1}^{k} \sigma_i(A+B) \\
&\quad = \max\{|\operatorname{tr}(A+B)C| : C \in P_{n,m;k}\} \\
&\quad = \max\{|\operatorname{tr}(AC+BC)| : C \in P_{n,m;k}\} \\
&\quad \leq \max\{|\operatorname{tr}(AC)| + |\operatorname{tr}(BC)| : C \in P_{n,m;k}\} \\
&\quad \leq \max\{|\operatorname{tr}(AC)| : C \in P_{n,m;k}\} + \max\{|\operatorname{tr}(BC)| : C \in P_{n,m;k}\} \\
&\quad = \sum_{i=1}^{k} \sigma_i(A) + \sum_{i=1}^{k} \sigma_i(B)
\end{aligned}$$ □

Except for the largest singular value, individual singular values need not obey the triangle inequality. For example, with $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, it is clear that $\sigma_2(A+B) = \sigma_2(I) = 1 \nleq \sigma_2(A) + \sigma_2(B) = 0$. Nevertheless, the previous result says that the sum of the k largest singular values does obey the triangle inequality. Actually, somewhat more is true.

3.4.4 Corollary. For $A \in M_{m,n}$, let $q \equiv \min\{m,n\}$, and let $N_k(A) \equiv \sigma_1(A) + \cdots + \sigma_k(A)$ denote the sum of the k largest singular values of A. Then

(a) $N_k(\cdot)$ is a norm on $M_{m,n}$ for $k = 1,\ldots, q$.

(b) When $m = n$, $N_k(\cdot)$ is a matrix norm on M_n for $k = 1,\ldots, n$.

Proof: To prove (a), we must show that $N_k(\cdot)$ is a positive homogeneous function on $M_{m,n}$ that satisfies the triangle inequality (see (5.1.1) in [HJ]). It is clear that $N_k(A) \geq 0$, and since $N_k(A) \geq \sigma_1(A) = |||A|||_2$ is the spectral norm of A it is also clear that $N_k(A) = 0$ if and only if $A = 0$. If $c \in \mathbb{C}$ is a given scalar, then $(cA)^*(cA) = |c|^2 A^*A$, so $\sigma_i(cA) = |c|\, \sigma_i(A)$ for $i = 1,\ldots, q$. Since (3.4.3) says that $N_k(A+B) \leq N_k(A) + N_k(B)$, we conclude that $N_k(\cdot)$ is a norm on $M_{m,n}$. Now let $m = n$. To show that $N_k(\cdot)$ is a matrix norm on M_n, we must show that $N_k(AB) \le N_k(A)N_k(B)$ for all $A, B \in M_n$. This follows immediately from (3.3.14a) since

$$N_k(AB) = \sum_{i=1}^{k} \sigma_i(AB) \le \sum_{i=1}^{k} \sigma_i(A)\sigma_i(B) \le \sum_{i=1}^{k} \sigma_i(A) \sum_{j=1}^{k} \sigma_j(B)$$

$$= N_k(A)N_k(B)$$ □

The function $N_k(A) \equiv \sigma_1(A) + \cdots + \sigma_k(A)$ is often called the *Ky Fan k-norm*. For a different approach to the preceding result, via VonNeumann's theory of unitarily invariant norms, see (7.4.43) and (7.4.54) in [HJ].

The following inequality for singular value sums is very useful in matrix approximation problems; see Problem 18 in Section (3.5).

3.4.5 Theorem. Let $A, B \in M_{m,n}$ be given, and suppose A, B, and $A - B$ have decreasingly ordered singular values $\sigma_1(A) \ge \cdots \ge \sigma_q(A)$, $\sigma_1(B) \ge \cdots \ge \sigma_q(B)$, and $\sigma_1(A-B) \ge \cdots \ge \sigma_q(A-B)$, where $q = \min\{m,n\}$. Define $s_i(A,B) \equiv |\sigma_i(A) - \sigma_i(B)|$, $i = 1,\ldots, q$, and let $s_{[1]}(A,B) \ge \cdots \ge s_{[q]}(A,B)$ denote a decreasingly ordered rearrangement of the values of $s_i(A,B)$. Then

$$\sum_{i=1}^{k} s_{[i]}(A,B) \le \sum_{i=1}^{k} \sigma_i(A-B) \text{ for } k = 1,\ldots, q \tag{3.4.6}$$

That is, if we denote the vectors of decreasingly ordered singular values of A, B, and $A - B$ by $\sigma(A)$, $\sigma(B)$, and $\sigma(A-B) \in \mathbb{R}^q$, then the entries of $|\sigma(A) - \sigma(B)|$ are weakly majorized by the entries of $\sigma(A-B)$.

Proof: Consider the $(m+n)$-by-$(m+n)$ Hermitian block matrices

$$\tilde{A} \equiv \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}, \; \tilde{B} \equiv \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$$

and use Jordan's observation (3.0.4) that the eigenvalues of $\tilde{A}$ are $\pm\sigma_1(A),\ldots, \pm\sigma_q(A)$, together with $m + n - 2q$ zeroes, and similarly for $\tilde{B}$ (see Theorem (7.3.7) in [HJ]). Denote the vectors of algebraically decreasingly ordered eigenvalues of $\tilde{A}$, $\tilde{B}$, and $\tilde{A} - \tilde{B}$ by $\lambda(\tilde{A})$, $\lambda(\tilde{B})$, and $\lambda(\tilde{A} - \tilde{B}) \in \mathbb{R}^{m+n}$. Notice that the entries of $\lambda(\tilde{A}) - \lambda(\tilde{B})$ are $\pm(\sigma_1(A) - \sigma_1(B)),\ldots, \pm(\sigma_q(A) - \sigma_q(B))$

together with $m+n-2q$ additional zero entries. Thus, the q algebraically largest entries of $\lambda(\tilde{A})-\lambda(\tilde{B})$ comprise the set $\{|\sigma_1(A)-\sigma_1(B)|,\ldots, |\sigma_q(A)-\sigma_q(B)|\}$. The q algebraically largest entries of $\lambda(\tilde{A}-\tilde{B})$ comprise the set $\{\sigma_1(\tilde{A}-\tilde{B}),\ldots, \sigma_q(\tilde{A}-\tilde{B})\}$. Since (4.3.27) in [HJ] guarantees that there is a (strong) majorization relationship between all the entries of $\lambda(\tilde{A}-\tilde{B})$ and all the entries of $\lambda(\tilde{A})-\lambda(\tilde{B})$ (this also follows from Corollary (3.4.3); see (3.4.11b) in Problem 8), there is a weak majorization between the sets of q largest entries of these two vectors: The q algebraically largest entries of $\lambda(\tilde{A})-\lambda(\tilde{B})$ are weakly majorized by the q algebraically largest entries of $\lambda(\tilde{A}-\tilde{B})$, which is exactly what the inequalities (3.4.6) assert. $\square$

Problems

1. Let $A, B\in M_n$ be rank k and rank r Hermitian projections ($A^*=A$ and $A^2=A$), respectively. Show that AB is a contraction with rank at most $\min\{k,r\}$ that need not be Hermitian or a projection (an idempotent).

2. Consider the function $L_k(A)\equiv|\lambda_1(A)|+\cdots+|\lambda_k(A)|$ on M_n, $k=1,\ldots,n$; the eigenvalues are arranged in decreasing order $|\lambda_1|\geq\cdots\geq|\lambda_n|$. Is $L_k(\cdot)$ a norm on M_n? Why?

3. Define the *Ky Fan p-k norm* on $M_{m,n}$ by $N_{k;p}(A)\equiv[\sigma_1(A)^p+\cdots+\sigma_k(A)^p]^{1/p}$, $p\geq 1$, $k=1,\ldots,\min\{m,n\}$.

(a) Use Corollary (3.4.3) and Lemma (3.3.8) to show that

$$f(\sigma_1(A+B))+\cdots+f(\sigma_k(A+B))$$
$$\leq f(\sigma_1(A)+\sigma_1(B))+\cdots+f(\sigma_k(A)+\sigma_k(B))$$

for $k=1,\ldots,\min\{m,n\}$ for every real-valued increasing convex function $f(\cdot)$ on $[0,\infty)$.

(b) Take $f(t)\equiv t^p$ and use Minkowski's inequality to show that $N_{k;p}(\cdot)$ is a norm on $M_{m,n}$ for all $p\geq 1$ and all $k=1,\ldots,q=\min\{m,n\}$. When $k=q$, $N_{q;p}(\cdot)$ is often called the *Schatten p-norm*.

(c) When $m=n$, use Corollary (3.3.14(c)) to show that all the Ky Fan p-k norms are matrix norms on M_n.

4. Let $A\in M_n$ be given, let $e\equiv[1,\ldots,1]^T\in\mathbb{R}^n$, and let $r(X)$ denote the numerical radius norm on M_n.

(a) Show that the Ky Fan n-norm, often called the *trace norm*, has the variational characterization

$$N_n(A) = \max \{ |\operatorname{tr} AU| : \ U \in M_n \text{ is unitary} \}$$

$$= \max \{ |e^T(A \circ U)e| : \ U \in M_n \text{ is unitary} \} \qquad (3.4.7)$$

(b) Explain why the point $\operatorname{tr}(AU)$ is in the field of values of nAU and use Problem 23(b) in Section (1.5) to show that $N_n(A) \leq 4nr(A)$.
(c) Explain why the point $e^T(A \circ U)e$ is in the field of values of $nA \circ U$ and use Corollary (1.7.24) to show that $N_n(A) \leq n\, r(A)$. Show that this bound is sharp for every $n = 1, 2, \ldots$.

5. Let $A \in M_n$ be given. Use Theorem (3.4.1) to show that

$$\Big| \sum_{i=1}^{k} x_i^* U A x_i \Big| \leq \sigma_1(A) + \cdots + \sigma_k(A), \quad k = 1, \ldots, n \qquad (3.4.8)$$

for any unitary $U \in M_n$ and any orthonormal set $\{x_1, \ldots, x_k\} \subset \mathbb{C}^n$. For each k, show that equality is possible for some choice of U and $\{x_i\}$.

6. Let $A \in M_n$ be given, let $n_1, \ldots, n_m$ and $\nu_1, \ldots, \nu_m$ be given positive integers with $n_1 + \cdots + n_m = n = \nu_1 + \cdots + \nu_m$, and let $I_{n_i} \in M_{n_i}$ and $I_{\nu_i} \in M_{\nu_i}$ be identity matrices for $i = 1, \ldots, m$. Consider the simple families of mutually orthogonal Hermitian projections

$$\Lambda_1 = I_{n_1} \oplus 0_{n-n_1}, \Lambda_2 = 0_{n_1} \oplus I_{n_2} \oplus 0_{n-n_1-n_2}, \ldots, \Lambda_m = 0_{n-n_m} \oplus I_{n_m}$$

and

$$D_1 = I_{\nu_1} \oplus 0_{n-\nu_1}, D_2 = 0_{\nu_1} \oplus I_{\nu_2} \oplus 0_{n-\nu_1-\nu_2}, \ldots, D_m = 0_{n-\nu_m} \oplus I_{\nu_m}$$

and let $\hat{A} \equiv \Lambda_1 A D_1 + \cdots + \Lambda_m A D_m$.

(a) Notice that $\hat{A} = A_1 \oplus \cdots \oplus A_m$, where each $A_i \in M_{n_i, \nu_i}$ is the submatrix of A corresponding to the nonzero rows of Λ_i and nonzero columns of D_i.

(b) Show that the set of singular values of $\hat{A}$ is the union of the sets of singular values of $A_1, \ldots, A_m$, including multiplicities.

(c) If a given singular value $\sigma_i(\hat{A})$ is a singular value of A_k, show that corresponding left and right singular vectors $x^{(i)}$ and $y^{(i)}$ of $\hat{A}$ (that is,

$\hat{A}x^{(i)} = \sigma_k(\hat{A})y^{(i)}$ and $\|x^{(i)}\|_2 = \|y^{(i)}\|_2 = 1$) can be chosen so that all entries of $x^{(i)}$ and $y^{(i)}$ not corresponding to the diagonal 1 entries in Λ_i and D_i, respectively, are zero.

(d) If left and right singular vectors $x^{(1)},\ldots, x^{(k)}$ and $y^{(1)},\ldots, y^{(k)}$ of $\hat{A}$ corresponding to the singular values $\sigma_1(\hat{A}) \geq \cdots \geq \sigma_k(\hat{A})$ are chosen as in (c), use Problem 5 to show that

$$\sum_{i=1}^{k} \sigma_i(\hat{A}) = \sum_{i=1}^{k} x_i^* A y_i \leq \sum_{i=1}^{k} \sigma_i(A) \text{ for } k = 1,\ldots, n \tag{3.4.9}$$

(e) Conclude that the singular values of $\hat{A}$ are weakly majorized by those of A.

7. Let $P_1,\ldots, P_m \in M_n$ be m given mutually orthogonal Hermitian projections, that is, $P_i = P_i^*$, $P_i^2 = P_i$ for $i = 1,\ldots, m$, $P_iP_j = 0$ for all $i \neq j$, and $\Sigma_i P_i = I$. Let $Q_1,\ldots, Q_m \in M_n$ be m given mutually orthogonal Hermitian projections. The linear operator $A \rightarrow \hat{A} \equiv P_1AQ_1 + \cdots + P_mAQ_m$ on M_n is sometimes called a *pinching* or *diagonal cell operator* because of the form of the special case considered in Problem 6.

(a) Show that

$$\sigma_1(\hat{A}) + \cdots + \sigma_k(\hat{A}) \leq \sigma_1(A) + \cdots + \sigma_k(A) \text{ for } k = 1,\ldots, n \tag{3.4.10}$$

for every $A \in M_n$, so the singular values of $\hat{A}$ are always weakly majorized by those of A; that is, every diagonal cell operator on M_n is norm decreasing with respect to the fundamental Ky Fan k-norms.

(b) As in Problem 3, conclude that $f(\sigma_1(\hat{A})) + \cdots + f(\sigma_k(\hat{A})) \leq f(\sigma_1(A)) + \cdots + f(\sigma_k(A))$ for $k = 1,\ldots, n$ for every real-valued $f(\cdot)$ on $[0,\infty)$ that is increasing and convex.

8. Let $A, B \in M_n$ be given Hermitian matrices, and let the eigenvalues $\{\lambda_i(A)\}$, $\{\lambda_i(B)\}$, and $\{\lambda_i(A + B)\}$ be arranged in algebraically decreasing order $\lambda_1 \geq \cdots \geq \lambda_n$. Use Corollary (3.4.3) to deduce the strong majorization inequalities

$$\sum_{i=1}^{k} \lambda_i(A + B) \leq \sum_{i=1}^{k} [\lambda_i(A) + \lambda_i(B)] \text{ for } k = 1,\ldots, n$$
$$\text{with equality for } k = n \tag{3.4.11a}$$

Conclude that

$$\sum_{i=1}^{k}[\lambda_i(A)-\lambda_i(B)] \le \sum_{i=1}^{k}\lambda_i(A-B) \quad \text{for } k=1,\ldots,n \tag{3.4.11b}$$

9. Corollary (3.4.3), the triangle inequality for the fundamental Ky Fan k-norms, follows easily from the variational characterization (3.4.1), in whose proof we used the A. Horn singular value majorization inequalities (3.3.14a) for the ordinary matrix product to derive (3.4.2). The purpose of this problem is to give a direct proof of Theorem (3.4.1) that does not rely on the inequalities (3.3.14a). The following argument uses nothing about singular values except the singular value decomposition (3.1.1). Let $x=[x_i]$, $y=[y_i]\in\mathbb{R}^n$ be given real nonnegative vectors with $x_1\ge\cdots\ge x_n\ge 0$ and $y_1\ge\cdots\ge y_n\ge 0$.

(a) Show that $x_1y_2+x_2y_1=x_1y_1+x_2y_2-(x_1-x_2)(y_1-y_2)\le x_1y_1+x_2y_2$.

(b) For any permutation matrix $P\in M_n(\mathbb{R})$, show that $x^TPy\le x^Ty$.

(c) For any doubly stochastic $S\in M_n(\mathbb{R})$, use Birkhoff's theorem (8.7.1) in [HJ] to show that $x^TSy\le x^Ty$.

(d) For any doubly substochastic $Q\in M_n(\mathbb{R})$, use Theorem (3.2.6) to show that $x^TQy\le x^Ty$.

(e) Using the notation of Theorem (3.4.1) and a singular value decomposition $A=V\Sigma W^*$, show that

$$\max\{|\mathrm{tr}(X^*AY)|:\ X\in M_{m,k},\ Y\in M_{n,k},\ X^*X=I,\ Y^*Y=I\}$$

$$=\max\{|\mathrm{tr}(X^*\Sigma Y)|:\ X\in M_{m,k},\ Y\in M_{n,k},\ X^*X=I,\ Y^*Y=I\}$$

If $X=[x_{ij}]\in M_{m,k}$ and $Y=[y_{ij}]\in M_{n,k}$ have orthonormal columns, show that $\mathrm{tr}(X^*\Sigma Y)=\sigma(A)^TZ\eta$, where $\eta=[\eta_i]\in\mathbb{R}^q$ has $\eta_1=\cdots=\eta_k=1$, $\eta_{k+1}=\cdots=\eta_q=0$, $\sigma(A)=[\sigma_1(A),\ldots,\sigma_q(A)]^T\in\mathbb{R}^q$, and $Z\equiv[\bar{x}_{ij}y_{ij}]_{i,j=1}^{q}$.

(f) Now use Problem 2 in Section (3.2) to conclude that

$$\max\{|\mathrm{tr}(X^*AY)|:\ X\in M_{m,k},\ Y\in M_{n,k},\ X^*X=I,\ Y^*Y=I\}$$

$$\le\sigma(A)^T\eta=\sigma_1(A)+\cdots+\sigma_k(A) \tag{3.4.12}$$

(g) Exhibit X and Y for which equality is achieved in (3.4.12).

Further Reading. The first proof of Theorem (3.4.1) is in the 1951 paper by Ky Fan cited at the end of Section (3.0).

3.5 Singular values and unitarily invariant norms

A norm $\|\cdot\|$ on $M_{m,n}$ is *unitarily invariant* if $\|UAV\| = \|A\|$ for all unitary $U \in M_m$ and $V \in M_n$, and for all $A \in M_{m,n}$. It is clear from the singular value decomposition that if $\|\cdot\|$ is a unitarily invariant norm, then $\|A\| = \|V\Sigma W^*\| = \|\Sigma(A)\|$ is a function only of the singular values of A. The nature and properties of this function are the primary focus of this section.

If $\|\cdot\|$ is a given unitarily invariant norm on $M_{m,n}$, there is a natural sense in which it induces a unitarily invariant norm on $M_{r,s}$ for any r, s with $1 \le r \le m$ and $1 \le s \le n$: For $A \in M_{r,s}$, define $\|A\| \equiv \|\mathcal{A}\|$, where

$$\mathcal{A} \equiv \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \in M_{m,n} \tag{3.5.0}$$

has been augmented by zero blocks to fill out its size to m-by-n. One checks that the norm on $M_{r,s}$ defined in this way is both well defined and unitarily invariant.

3.5.1 Lemma. Let $\|\cdot\|$ be a given norm on $M_{m,n}$ and let

$$\|A\|^D \equiv \max\{|\text{tr } AC^*|:\ C \in M_{m,n},\ \|C\| = 1\} \tag{3.5.2}$$

denote its dual norm with respect to the Frobenius inner product. Then $\|\cdot\|$ is unitarily invariant if and only if $\|\cdot\|^D$ is unitarily invariant.

Proof: Suppose $\|\cdot\|$ is unitarily invariant, and let $U \in M_m$, $V \in M_n$ be unitary. Then

$$\begin{aligned}\|UAV\|^D &= \max\{|\text{tr}(UAV)C^*|:\ C \in M_{m,n},\ \|C\| = 1\} \\ &= \max\{|\text{tr}(A[U^*CV^*]^*)|:\ C \in M_{m,n},\ \|C\| = 1\} \\ &= \max\{|\text{tr}(AE^*)|:\ E \in M_{m,n},\ \|UEV\| = 1\} \\ &= \max\{|\text{tr}(AE^*)|:\ E \in M_{m,n},\ \|E\| = 1\}\end{aligned}$$

$$= \|A\|^D$$

The converse follows from the preceding argument and the *duality theorem* for norms: $(\|A\|^D)^D = \|A\|$ (see Theorem (5.5.14) in [HJ]). □

Notice that the duality theorem gives the representation

$$\|A\| = \max\{|\mathrm{tr}\, AC^*|:\ C \in M_{m,n}, \|C\|^D = 1\} \tag{3.5.3}$$

3.5.4 Definition. We write $\mathbb{R}^n_{+\downarrow} \equiv \{x = [x_i] \in \mathbb{R}^n:\ x_1 \geq \cdots \geq x_n \geq 0\}$. For $X \in M_{m,n}$ and $\alpha = [\alpha_i] \in \mathbb{R}^q_{+\downarrow}$, we write

$$\|X\|_\alpha \equiv \alpha_1\sigma_1(X) + \cdots + \alpha_q\sigma_q(X)$$

where $q \equiv \min\{m,n\}$.

Because of the ordering convention we have adopted for the vector of singular values $\sigma(X)$, notice that we always have $\sigma(X) \in \mathbb{R}^q_{+\downarrow}$, where $X \in M_{m,n}$ and $q = \min\{m,n\}$.

3.5.5 Theorem. Let m, n be given positive integers and let $q \equiv \min\{m,n\}$.

(a) For each given nonzero $\alpha \in \mathbb{R}^q_{+\downarrow}$, $\|A\|_\alpha \equiv \alpha_1\sigma_1(A) + \cdots + \alpha_q\sigma_q(A)$ is a unitarily invariant norm on $M_{m,n}$.

(b) For each given unitarily invariant norm $\|\cdot\|$ on $M_{m,n}$ there is a compact set $\mathcal{N}(\|\cdot\|) \subset \mathbb{R}^q_{+\downarrow}$ such that

$$\|A\| = \max\{\|A\|_\alpha:\ \alpha \in \mathcal{N}(\|\cdot\|)\} \tag{3.5.6}$$

for all $A \in M_{m,n}$. One may take the set $\mathcal{N}(\|\cdot\|)$ to be

$$\mathcal{N}(\|\cdot\|) = \{\sigma(X):\ \|X\|^D = 1\} \tag{3.5.7}$$

Proof: Assertion (a) is a simple consequence of summation by parts:

$$\|A\|_\alpha = \sum_{i=1}^{q} \alpha_i\sigma_i(A) = \sum_{i=1}^{q-1}(\alpha_i - \alpha_{i+1})N_i(A) + \alpha_q N_q(A) \tag{3.5.8}$$

where $N_k(A) = \sigma_1(A) + \cdots + \sigma_k(A)$ is the Ky Fan k-norm. Since the entries of α are monotone decreasing, (3.5.8) represents $\|A\|_\alpha$ as a nonnegative linear combination of the (unitarily invariant) Ky Fan k-norms, in which at least one coefficient is positive if $\alpha \neq 0$. To prove (b), let $U_1, U_2 \in M_n$ and $V_1, V_2 \in M_m$ be unitary. Then for any $C \in M_{m,n}$ with $\|C\|^D = 1$, unitary invariance and (3.5.3) give

$$\|A\| = \|V_2^* V_1 A U_1 U_2^*\|$$

$$\geq |\mathrm{tr}[(V_2^* V_1 A U_1 U_2^*)C^*]| = |\mathrm{tr}[(V_1 A U_1)(V_2 C U_2)^*]|$$

Now use the singular value decompositions of A and C to select the unitary matrices U_1, V_1, U_2, V_2 so that $V_1 A U_1 = \Sigma(A)$ and $V_2 C U_2 = \Sigma(C)$, from which it follows that

$$\|A\|_{\sigma(C)} = \sum_{i=1}^{q} \sigma_i(A)\sigma_i(C) \leq \|A\| \text{ whenever } C \in M_{m,n} \text{ and } \|C\|^D = 1$$

Using (3.5.3) again, select $C_0 \in M_{m,n}$ such that $\|C_0\|^D = 1$ and

$$\|A\| = \max\{|\mathrm{tr}\, AC^*|:\ C \in M_{m,n}, \|C\|^D = 1\} = |\mathrm{tr}\, AC_0^*|$$

$$= \left|\sum_{i=1}^{m} \lambda_i(AC_0^*)\right| \leq \sum_{i=1}^{m} \sigma_i(AC_0^*) \leq \sum_{i=1}^{m} \sigma_i(A)\sigma_i(C_0)$$

$$= \|A\|_{\sigma(C_0)} \leq \|A\|$$

Thus, $\|A\| = \|A\|_{\sigma(C_0)}$ and we have shown that

$$\|A\| = \max\{\|A\|_{\sigma(C)}:\ C \in M_{m,n}, \|C\|^D = 1\}$$

as asserted in (b). $\square$

There are many useful consequences of the characterization (3.5.6) of unitarily invariant norms.

3.5.9 Corollary. Let $A, B \in M_{m,n}$ be given and let $q = \min\{m,n\}$. The following are equivalent:

(a) $\|A\| \leq \|B\|$ for every unitarily invariant norm $\|\cdot\|$ on $M_{m,n}$.

(b) $N_k(A) \leq N_k(B)$ for $k = 1,\dots, q$, where $N_k(X) \equiv \sigma_1(X) + \cdots + \sigma_k(X)$ denotes the Ky Fan k-norm.

(c) $\|A\|_\alpha \leq \|B\|_\alpha$ for all $\alpha \in \mathbb{R}^q_{+\downarrow}$.

Proof: Since each Ky Fan k-norm $N_k(\cdot)$ is unitarily invariant, (a) implies (b). The identity (3.5.8) shows that (b) implies (c), and the representation (3.5.6) shows that (c) implies (a). $\square$

Since the conditions (b) of the preceding corollary are just the assertion that $\sigma(A)$ is weakly majorized by $\sigma(B)$, we see that every unitarily invariant norm is monotone with respect to the partial order on $M_{m,n}$ induced by weak majorization of the vectors of singular values.

3.5.10 Corollary. Let $\|\cdot\|$ be a given unitarily invariant norm, and let $E_{11} \in M_{m,n}$ have the entry 1 in position 1,1 and zeros elsewhere. Then

(a) $\|AB^*\| \leq \sigma_1(A)\,\|B\|$ for all $A, B \in M_{m,n}$, and

(b) $\|A\| \geq \sigma_1(A)\|E_{11}\|$ for all $A \in M_{m,n}$.

Proof: For each $k = 1,\dots, q \equiv \min\{m,n\}$ we have

$$N_k(AB^*) = \sum_{i=1}^{k} \sigma_i(AB^*) \leq \sum_{i=1}^{k} \sigma_1(A)\sigma_i(B^*)$$

$$= \sigma_1(A)\sum_{i=1}^{k} \sigma_i(B) = \sigma_1(A)N_k(B) = N_k(\sigma_1(A)B)$$

and

$$N_k(\sigma_1(A)E_{11}) = \sigma_1(A) \leq \sum_{i=1}^{k} \sigma_i(A) = N_k(A)$$

Then (a) and (b) follow from the equivalence of (3.5.9a,b). $\square$

The following generalization of Corollary (3.5.9) is a final easy consequence of the characterization (3.5.6) of unitarily invariant norms.

3.5.11 Corollary. Let $f(t_1,\ldots,t_k)\colon \mathbb{R}_+^k \to \mathbb{R}_+$ be a given nonnegative-valued function of k nonnegative real variables that is increasing in each variable separately, that is,

$$f(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_k) \le f(t_1,\ldots,t_{i-1},t_i+\epsilon,t_{i+1},\ldots,t_k)$$

for all $\epsilon, t_1,\ldots,t_k \ge 0$ and all $i=1,\ldots,k$. Let $A, B_1,\ldots,B_k \in M_{m,n}$ be given, and let $q=\min\{m,n\}$. Then $\|A\| \le f(\|B_1\|,\ldots,\|B_k\|)$ for all unitarily invariant norms $\|\cdot\|$ if and only if $\|A\|_\alpha \le f(\|B_1\|_\alpha,\ldots,\|B_m\|_\alpha)$ for all $\alpha \in \mathbb{R}_{+\downarrow}^q$.

Proof: The forward implication is immediate. For the reverse implication, let $\alpha_0 \in \mathbb{R}_{+\downarrow}^q$ be given and use the characterization (3.5.6) and the monotonicity of f in each variable to write

$$f(\|B_1\|,\ldots,\|B_k\|) = f(\max \|B_1\|_\alpha,\ldots,\max \|B_k\|_\alpha)$$

$$\ge f(\|B_1\|_{\alpha_0},\ldots,\|B_k\|_{\alpha_0}) \ge \|A\|_{\alpha_0}$$

Since $\alpha_0 \in \mathbb{R}_{+\downarrow}^q$ was arbitrary, we conclude that

$$\|A\| = \max \|A\|_\alpha \le f(\|B_1\|,\ldots,\|B_k\|)$$

In every case, the indicated maximum is taken over all α in the compact set $\mathcal{N}(\|\cdot\|)$ for which (3.5.6) holds, for example, the set described by (3.5.7). $\square$

We now wish to use Corollary (3.5.11) to derive a general inequality for unitarily invariant norms. To do so, we need the following characterization of positive semidefinite 2-by-2 block matrices.

3.5.12 Lemma. Let $L \in M_m$, $M \in M_n$, and $X \in M_{m,n}$ be given. Then

$$\begin{bmatrix} L & X \\ X^* & M \end{bmatrix} \in M_{m+n} \tag{3.5.13}$$

is positive semidefinite if and only if L and M are positive semidefinite and there is a contraction $C \in M_{m,n}$ (that is, $\sigma_1(C) \leq 1$) such that $X = L^{\frac{1}{2}} C M^{\frac{1}{2}}$.

Proof: Suppose L and M are positive definite. By Theorem (7.7.7) in [HJ], the block matrix (3.5.13) is positive semidefinite if and only if

$$1 \geq \rho(X^* L^{-1} X M^{-1}) = \rho(M^{-\frac{1}{2}} X^* L^{-1} X M^{-\frac{1}{2}})$$

$$= \rho[(L^{-\frac{1}{2}} X M^{-\frac{1}{2}})^* (L^{-\frac{1}{2}} X M^{-\frac{1}{2}})] = \sigma_1(L^{-\frac{1}{2}} X M^{-\frac{1}{2}})^2$$

Setting $C \equiv L^{-\frac{1}{2}} X M^{-\frac{1}{2}}$, we have $X = L^{\frac{1}{2}} C M^{\frac{1}{2}}$, as desired. The general case now follows from a limiting argument. $\square$

Now suppose that a given block matrix of the form (3.5.13) is positive semidefinite with $X \in M_{m,n}$ and write $X = L^{\frac{1}{2}} C M^{\frac{1}{2}}$. Then

$$\prod_{i=1}^{k} \sigma_i(X) = \prod_{i=1}^{k} \sigma_i(L^{\frac{1}{2}} C M^{\frac{1}{2}}) \leq \prod_{i=1}^{k} \sigma_i(L^{\frac{1}{2}}) \sigma_i(C) \sigma_i(M^{\frac{1}{2}})$$

$$\leq \prod_{i=1}^{k} \sigma_i(L)^{\frac{1}{2}} \sigma_i(M)^{\frac{1}{2}} \quad \text{for } k = 1, \ldots, q = \min\{m,n\}$$

and hence

$$\prod_{i=1}^{k} \sigma_i(X)^p \leq \prod_{i=1}^{k} \sigma_i(L)^{p/2} \sigma_i(M)^{p/2} \text{ for } k = 1, \ldots, q$$

for any $p > 0$. Moreover, for any $\alpha = [\alpha_i] \in \mathbb{R}^q_{+\downarrow}$ we also have

$$\prod_{i=1}^{k} \alpha_i \sigma_i(X)^p \leq \prod_{i=1}^{k} \alpha_i \sigma_i(L)^{p/2} \sigma_i(M)^{p/2} \text{ for } k = 1, \ldots, q \qquad (3.5.14)$$

Corollary (3.3.10) now gives the inequalities

$$\sum_{i=1}^{q} \alpha_i \sigma_i(X)^p \leq \sum_{i=1}^{q} (\alpha_i^{\frac{1}{2}} \sigma_i(L)^{p/2})(\alpha_i^{\frac{1}{2}} \sigma_i(M)^{p/2})$$

$$\leq \left[\sum_{i=1}^{q} \alpha_i \sigma_i(L)^p\right]^{\frac{1}{2}} \left[\sum_{i=1}^{q} \alpha_i \sigma_i(M)^p\right]^{\frac{1}{2}}$$

These inequalities say that

$$\|(X^*X)^{p/2}\|_\alpha \leq [\,\|L^p\|_\alpha \|M^p\|_\alpha]^{\frac{1}{2}}$$

for all $p > 0$ and all $\alpha \in \mathbb{R}^q_{+\downarrow}$. If we now apply Corollary (3.5.11) with $f(t_1, t_2) = (t_1 t_2)^{\frac{1}{2}}$, we have proved the following:

3.5.15 Theorem. Let $L \in M_m$, $M \in M_n$, and $X \in M_{m,n}$ be such that the block matrix

$$\begin{bmatrix} L & X \\ X^* & M \end{bmatrix} \in M_{m+n}$$

is positive semidefinite. Then

$$\|(X^*X)^{p/2}\|^2 \leq \|L^p\|\,\|M^p\| \qquad (3.5.16)$$

for every $p > 0$ and every unitarily invariant norm $\|\cdot\|$.

Some special cases of this inequality are considered in Problems 7-10. In addition to the quasilinearization (3.5.6), there is another useful characterization of unitarily invariant norms, originally due to Von Neumann.

3.5.17 Definition. A function $g\colon \mathbb{R}^q \to \mathbb{R}_+$ is said to be a *symmetric gauge function* if

(a) $g(x)$ is a norm on $\mathbb{R}^q$;
(b) $g(x) = g(|x|)$ for all $x \in \mathbb{R}^q$, where $|x| = [|x_i|]$; and
(c) $g(x) = g(Px)$ for all $x \in \mathbb{R}^q$ and every permutation matrix $P \in M_q(\mathbb{R})$.

Thus, a symmetric gauge function on $\mathbb{R}^q$ is an absolute permutation-invariant norm. Since a norm on $\mathbb{R}^q$ is absolute if and only if it is monotone (Theorem (5.5.10) in [HJ]), it follows that a symmetric gauge function is also a monotone norm.

3.5.18 Theorem. If $\|\cdot\|$ is a given unitarily invariant norm on $M_{m,n}$, then there is a symmetric gauge function $g(\cdot)$ on $\mathbb{R}^q$ such that $\|A\| = g(\sigma(A))$ for all $A \in M_{m,n}$. Conversely, if $g(\cdot)$ is a given symmetric gauge function on $\mathbb{R}^q$, then $\|A\| \equiv g(\sigma(A))$ is a unitarily invariant norm on $M_{m,n}$.

Proof: If $\|\cdot\|$ is a given unitarily invariant norm on $M_{m,n}$, for $x \in \mathbb{R}^q$ define $g(x) \equiv \|X\|$, where $X = [x_{ij}] \in M_{m,n}$ has $x_{ii} = x_i$, $i = 1,\ldots, q$, and all other entries are zero. That $g(\cdot)$ is a norm on $\mathbb{R}^q$ follows from the fact that $\|\cdot\|$ is a norm. Absoluteness and permutation-invariance of $g(\cdot)$ on $\mathbb{R}^q$ follow from unitary invariance of $\|\cdot\|$: Consider $X \to DX$ or $X \to PXQ$, where D is a diagonal unitary matrix and P, Q are permutation matrices.

Conversely, if $g(\cdot)$ is a given symmetric gauge function on $\mathbb{R}^q$, for $A \in M_{m,n}$ define $\|A\| \equiv g(\sigma(A))$. Positive definiteness and homogeneity of $\|\cdot\|$ follow from the fact that $g(\cdot)$ is a norm, while unitary invariance of $\|\cdot\|$ follows from unitary invariance of singular values. To show that $\|\cdot\|$ satisfies the triangle inequality, let $A, B \in M_{m,n}$ be given and use Corollary (3.4.3) to observe that the entries of the vector $\sigma(A+B)$ are weakly majorized by the entries of $\sigma(A) + \sigma(B)$. Then Corollary (3.2.11) guarantees that there is a doubly stochastic matrix $S \in M_q(\mathbb{R})$ such that $\sigma(A+B) \leq S[\sigma(A) + \sigma(B)]$, and $S = \mu_1 P_1 + \cdots + \mu_N P_N$ can be written as a convex combination of permutation matrices. Now use monotonicity, the triangle inequality, and permutation invariance of $g(\cdot)$ to conclude that

$$
\begin{aligned}
\|A+B\| &\equiv g(\sigma(A+B)) \leq g(S[\sigma(A)+\sigma(B)]) \\
&\leq g(S\sigma(A)) + g(S\sigma(B)) \\
&\leq \sum_{i=1}^{N} \mu_i [g(P_i\sigma(A)) + g(P_i\sigma(B))] \\
&= \sum_{i=1}^{N} \mu_i [g(\sigma(A)) + g(\sigma(B))] = g(\sigma(A)) + g(\sigma(B)) \\
&= \|A\| + \|B\|
\end{aligned}
$$

$\square$

Problems

1. Let $\|\cdot\|$ be a given unitarily invariant norm on $M_{m,n}$. For given inte-

gers r, s with $1 \le r \le m$ and $1 \le s \le n$, and for any $A \in M_{r,s}$, define $\|A\| \equiv \|\mathcal{A}\|$, where $\mathcal{A} \in M_{m,n}$ is the block matrix (3.5.0). Explain why this function $\|\cdot\|: M_{r,s} \to \mathbb{R}$ is well defined, is a norm, and is unitarily invariant.

2. Let $\|\cdot\|$ be a given unitarily invariant norm on $M_{m,n}$ and let $q = \min\{m,n\}$. Use Theorem (3.5.5) and the identity (3.5.8) to show that

$$\delta\, \sigma_1(A) \equiv \delta\, N_1(A) \le \|A\| \le \delta\, N_q(A) \equiv \delta\,[\sigma_1(A) + \cdots + \sigma_q(A)] \quad (3.5.19)$$

for all $A \in M_{m,n}$, where $\delta \equiv \max\{\sigma_1(C)\colon C \in M_{m,n} \text{ and } \|C\|^D = 1\}$ is a geometric factor that is independent of A.

3. Let $\|\cdot\|$ be a given unitarily invariant norm on M_n. Recall that $\|\cdot\|$ is a *matrix norm* if $\|AB\| \le \|A\|\,\|B\|$ for all $A, B \in M_n$. Let $E_{11} \in M_n$ denote the matrix with 1,1 entry 1 and all other entries 0. Show that the following are equivalent:

(a) $\|\cdot\|$ is a matrix norm on M_n.

(b) $\|A^m\| \le \|A\|^m$ for all $A \in M_n$ and all $m = 1,2,\ldots$.

(c) $\|A^2\| \le \|A\|^2$ for all $A \in M_n$.

(d) $\|E_{11}\| \ge 1$.

(e) $\|A\| \ge \sigma_1(A)$ for all $A \in M_n$.

4. Let $A \in M_{m,n}$ be given and let $\|\cdot\|$ be a given norm on $M_{m,n}$.
(a) If $\|\cdot\|$ is unitarily invariant, show that $\|BAC\| \le \sigma_1(B)\sigma_1(C)\|A\|$ for all $B \in M_m$ and all $C \in M_n$. In particular, if B and C are contractions, then $\|BAC\| \le \|A\|$.
(b) A norm $\|\cdot\|$ on $M_{m,n}$ is said to be *symmetric* if

$$\|BAC\| \le \sigma_1(B)\sigma_1(C)\,\|A\| \text{ for all } B \in M_m \text{ and all } C \in M_n$$

Show that $\|\cdot\|$ is symmetric if and only if it is unitarily invariant.

5. Use Theorem (3.5.18) to show that the Schatten p-norms $[\sigma_1(A)^p + \cdots + \sigma_q(A)^p]^{1/p}$ and, more generally, the Ky Fan p-k norms $[\sigma_1(A)^p + \cdots + \sigma_k(A)^p]^{1/p}$ are unitarily invariant norms on $M_{m,n}$, $q = \min\{m,n\}$, $1 \le k \le q$, $p \ge 1$.

6. Let $A \in M_{m,n}$ be given, and let $|A| \equiv (A^*A)^{\frac{1}{2}}$ denote the unique positive semidefinite square root of A^*A, which is the positive semidefinite

factor in the polar decomposition $A = U|A|$. Show that $\|A\| = \|\,|A|\,\|$ for every unitarily invariant norm $\|\cdot\|$ on $M_{m,n}$.

7. Let $A, B \in M_{m,n}$ be given, and let $\|\cdot\|$ be a given unitarily invariant norm. Apply (3.5.16) to $[A \;\; B]^*[A \;\; B]$ and show that

$$\|\,|A^*B|^p\|^2 \leq \|(A^*A)^p\|\,\|(B^*B)^p\| \text{ for all } p > 0 \tag{3.5.20}$$

where $|X|$ is defined in Problem 6. In particular, deduce the inequality

$$\|\,|A^*B|^{\frac{1}{2}}\|^2 \leq \|A\|\,\|B\| \tag{3.5.21}$$

and a Cauchy-Schwarz inequality for ordinary products and unitarily invariant norms

$$\|A^*B\|^2 \leq \|A^*A\|\,\|B^*B\| \tag{3.5.22}$$

What does this say when $n = 1$? If a given 2-by-2 block matrix $A \equiv [A_{ij}]_{i,j=1}^2$ is positive semidefinite, with $A_{11} \in M_k$ and $A_{22} \in M_{n-k}$, show that its (unitarily invariant) *norm compression* $A_{\|\cdot\|} \equiv [\,\|A_{ij}\|\,]_{i,j=1}^2 \in M_2$ is positive semidefinite. Show by example that a unitarily invariant norm compression of a positive semidefinite 3-by-3 block matrix need not be positive semidefinite.

8. Let $A, B \in M_{m,n}$ be given and let $\|\cdot\|$ be a given unitarily invariant norm. Apply (3.5.16) to the Hadamard product of $[A^* \;\; I]^*[A^* \;\; I]$ and $[I \;\; B]^*[I \;\; B]$ and show that

$$\|\,|A \circ B|^p\|^2 \leq \|[(AA^*) \circ I]^p\|\,\|[(B^*B) \circ I]^p\| \text{ for all } p > 0 \tag{3.5.23}$$

where $|A|$ is defined as in Problem 6. In particular, deduce a Cauchy-Schwarz inequality for Hadamard products and unitarily invariant norms

$$\|A \circ B\|^2 \leq \|(AA^*) \circ I\|\,\|(B^*B) \circ I\| \leq \|AA^*\|\,\|B^*B\| \tag{3.5.24}$$

What does this say when $n = 1$ and $\|\cdot\|$ is the spectral norm?

9. Let $A, B \in M_n$ be given, and suppose $A = X^*Y$ for some $X, Y \in M_{r,n}$. Consider the positive semidefinite matrices $[X \;\; Y]^*[X \;\; Y]$ and

$$\begin{bmatrix} \sigma_1(B)I & B \\ B^* & \sigma_1(B)I \end{bmatrix}$$

and their Hadamard product. Use Theorem (3.5.15) to show that

$$\| A \circ B \| \leq \sigma_1(B) c_1(X) c_1(Y) \| I \| \tag{3.5.25}$$

for any unitarily invariant norm $\|\cdot\|$ on M_n, where $c_1(Z)$ denotes the maximum Euclidean column length of $Z \in M_{r,n}$. Take $\|\cdot\|$ to be the spectral norm and deduce Schur's inequality $\sigma_1(A \circ B) \leq \sigma_1(A)\sigma_1(B)$. If $A = [a_{ij}]$ is positive semidefinite, take $X = Y = A^{\frac{1}{2}}$ and deduce another inequality of Schur: $\sigma_1(A \circ B) \leq \sigma_1(B) \max_i a_{ii}$.

10. If $A \in M_n$ is normal, show that the 2-by-2 block matrix $\begin{bmatrix} |A| & A \\ A^* & |A| \end{bmatrix}$ is positive semidefinite, where $|A|$ is defined in Problem 6. Use Theorem (3.5.15) to show that

$$\| A \circ B \| \leq \| \, |A| \circ |B| \, \| \tag{3.5.26}$$

for all normal $A, B \in M_n$ and every unitarily invariant norm $\|\cdot\|$ on M_n. The hypothesis of normality of both A and B is essential here; there are positive definite A and nonnormal B for which this inequality is false for some unitarily invariant norm.

11. Let $\|\cdot\|$ be a given norm on M_n, and define $\nu(A) \equiv \|A^*A\|^{\frac{1}{2}}$.
(a) If the norm $\|\cdot\|$ satisfies the inequality

$$\|A^*B\|^2 \leq \|A^*A\| \, \|B^*B\| \text{ for all } A, B \in M_n \tag{3.5.27}$$

show that $\nu(A)$ is a norm on M_n. In particular, conclude that $\nu(A)$ is a unitarily invariant norm on M_n whenever $\|\cdot\|$ is a unitarily invariant norm on M_n. Explain why the Schatten p-norms $[\sigma_1(A)^p + \cdots + \sigma_n(A)^p]^{1/p}$ are norms of this type for $p \geq 2$.
(b) Show that the l_1 norm $\|A\|_1 = \Sigma_{i,j}|a_{ij}|$ does *not* satisfy (3.5.27), but that the l_∞ norm $\|A\|_\infty = \max_{i,j} |a_{ij}|$ *does* satisfy (3.5.27); neither of these norms is unitarily invariant.
(c) Show that the set of norms on M_n that satisfy (3.5.27) is a convex set that is strictly larger than the set of all unitarily invariant norms,

but does not include all norms on M_n.

12. Let $\|\cdot\|$ be a given unitarily invariant norm on M_n. For all $A \in M_n$, show that

$$\|A\| = \min\{\|B^*B\|^{\frac{1}{2}}\,\|C^*C\|^{\frac{1}{2}}:\ B, C \in M_n \text{ and } A = BC\} \quad (3.5.28)$$

13. For $A \in M_{m,n}$, let $\hat{A}$ be any submatrix of A. Show that $\|\hat{A}\| \le \|A\|$ for every unitarily invariant norm on $M_{m,n}$, where $\|\hat{A}\|$ is defined as in Problem 1. Give an example of a norm for which this inequality does not always hold.

14. If $A, B \in M_n$ are positive semidefinite and $A \succeq B \succeq 0$, show that $\|A\| \ge \|B\|$ for every unitarily invariant norm $\|\cdot\|$ on M_n.

15. A norm $\nu(\cdot)$ on M_n is said to be *unitary similarity invariant* if $\nu(A) = \nu(UAU^*)$ for all $A \in M_n$ and all unitary $U \in M_n$. Notice that every unitarily invariant norm is unitary similarity invariant.
(a) For any unitarily invariant norm $\|\cdot\|$ on M_n, show that $N(A) \equiv \|A\| + |\operatorname{tr} A|$ is a unitary similarity invariant norm on M_n that is not unitarily invariant.
(b) Show that the numerical radius $r(A)$ is a unitary similarity invariant norm on M_n that is not unitarily invariant.

16. Let $\|\cdot\|$ be a given unitary similarity invariant norm on M_n. Show that the function $\nu(A) \equiv \|\,|A|\,\|$ is a unitary similarity invariant function on M_n that is always a prenorm, and is a norm if and only if $\|X\| \le \|Y\|$ whenever $X, Y \in M_n$ are positive semidefinite and $Y \succeq X \succeq 0$. Here, $|A|$ is defined as in Problem 6.

17. Show that the dual of the Ky Fan k-norm $N_k(A) = \sigma_1(A) + \cdots + \sigma_k(A)$ on $M_{m,n}$ is

$$N_k(A)^D = \max\{N_1(A),\ N_q(A)/k\} \text{ for } k = 1,\ldots, q = \min\{m,n\} \quad (3.5.29)$$

In particular, conclude that the spectral norm $\sigma_1(A) = |||A|||_2$ and the trace norm $N_q(A) = \sigma_1(A) + \cdots + \sigma_q(A)$ are dual norms.

18. Let $A = [a_{ij}],\ B = [b_{ij}] \in M_{m,n}$ be given and let $q = \min\{m,n\}$. Define the diagonal matrix $\Sigma(A) = [\sigma_{ij}] \in M_{m,n}$ by $\sigma_{ii} = \sigma_i(A)$, all other $\sigma_{ij} = 0$, where $\sigma_1(A) \ge \cdots \ge \sigma_q(A)$ are the decreasingly ordered singular values of A. Let $\sigma(A) = [\sigma_i(A)]_{i=1}^q \in \mathbb{R}^q$ denote the vector of decreasingly ordered singular values of A; define $\Sigma(B)$ and $\sigma(B)$ similarly.

(a) Show that the singular values of $\Sigma(A) - \Sigma(B)$ are the entries of the vector $|\sigma(A) - \sigma(B)|$ (which are not necessarily decreasingly ordered).
(b) Use Corollary (3.5.9) and Theorem (3.4.5) to show that

$$\|A - B\| \geq \|\Sigma(A) - \Sigma(B)\| \geq |\, \|A\| - \|B\| \,| \tag{3.5.30}$$

for every unitarily invariant norm $\|\cdot\|$ on $M_{m,n}$.
(c) If $B \in M_{m,n}$ is any matrix with rank $B = r \leq q$, show that

$$\|A - B\| \geq \|\text{diag}(0,\ldots, 0, \sigma_{r+1}(A),\ldots, \sigma_q(A))\| \tag{3.5.31}$$

Explain why this has the lower bound $\sigma_q(A)\|E_{11}\|$ if $r < q$. Using the singular value decomposition $A = V\Sigma(A)W^*$, exhibit a matrix B_r for which this inequality is an equality.
(d) Conclude that for a given $A \in M_{m,n}$ and for each given $r = 1,\ldots,$ rank A there is some $B_r \in M_{m,n}$ that is a best rank r approximation to A, simultaneously with respect to every unitarily invariant norm $\|\cdot\|$ on $M_{m,n}$.
(e) Let $m = n$ and show that $\|A - B\| \geq \sigma_n(A)\|E_{11}\|$ for every singular $B \in M_n$, where $E_{11} \in M_n$ has a 1 entry in position 1,1 and zeros elsewhere. Show that this lower bound is sharp; that is, exhibit a singular B for which this inequality is an equality.
(f) Derive the following singular value bounds from (3.5.30):

$$\max_{1 \leq i \leq q} |\sigma_i(A) - \sigma_i(B)| \leq |||A - B|||_2 \equiv \sigma_1(A - B) \tag{3.5.32}$$

$$\sum_{i=1}^{q} [\sigma_i(A) - \sigma_i(B)]^2 \leq \|A - B\|_2^2 = \sum_{i,j} |a_{ij} - b_{ij}|^2 \tag{3.5.33}$$

$$\sum_{i=1}^{q} |\sigma_i(A) - \sigma_i(B)| \leq \|A - B\|_{tr} \equiv \sum_{i=1}^{q} \sigma_i(A - B) \tag{3.5.34}$$

19. For any given $A \in M_{m,n}$ with singular values $\sigma_1(A) \geq \sigma_2(A) \geq \cdots$ show that

$$\sigma_k(A) = \inf\{\, |||A - B|||_2 \colon\ B \in M_{m,n} \text{ and rank } B < k\} \tag{3.5.35}$$

for $k = 2,\ldots, \min\{m,n\}$. This characterization of singular values extends naturally to infinite-dimensional Banach spaces, whereas the notion of eigenvalues of A^*A does not; see (2.3.1) of [Pie].

20. Let $\|\cdot\|$ be a given norm on M_n. We say that $\|\cdot\|$ is *self-adjoint* if $\|A\| = \|A^*\|$ for all $A \in M_n$; that $\|\cdot\|$ is an *induced norm* if there is some norm $\nu(\cdot)$ on $\mathbb{C}^n$ such that $\|A\| = \max\{\nu(Ax): x \in \mathbb{C}^n \text{ and } \nu(x) = 1\}$ for all $A \in M_n$; and that $\|\cdot\|$ is *derived from an inner product* if there is an inner product $<\cdot,\cdot>$ on $M_n \times M_n$ such that $\|A\| = <A,A>^{\frac{1}{2}}$ for all $A \in M_n$. The spectral norm $|||A|||_2 = \sigma_1(A)$ is induced by the Euclidean norm on $\mathbb{C}^n$, and the Frobenius norm $\|A\|_2 = (\text{tr } AA^*)^{\frac{1}{2}}$ is derived from the Frobenius inner product; both are unitarily invariant and self-adjoint. According to Theorem (5.6.36) in [HJ], the spectral norm is the only norm on M_n that is both self-adjoint and induced.

(a) Show that $\|\cdot\|$ is both unitarily invariant and induced if and only if $\|\cdot\| = |||\cdot|||_2$.

(b) Let $g(x)$ be a given symmetric gauge function on $\mathbb{R}^n$. Suppose $g(x)$ is derived from an inner product, so $g(x) = x^TBx$ for some positive definite $B \in M_n$. Show that $g(e_i) = g(e_j)$ and $g(e_i + e_j) = g(e_i - e_j)$ for all $i \neq j$. Conclude that $B = \beta I$ for some $\beta > 0$ and $g(x) = \beta^{\frac{1}{2}}\|x\|_2$.

(c) Show that $\|\cdot\|$ is both unitarily invariant and derived from an inner product if and only if $\|\cdot\| = \gamma\|\cdot\|_2$ for some $\gamma > 0$.

21. Let φ: $M_n \rightarrow M_n$ be a given function such that:

(1) $\varphi(A)$ and A have the same singular values for all $A \in M_n$;

(2) $\varphi(\alpha A + \beta B) = \alpha\varphi(A) + \beta\varphi(B)$ for all $A, B \in M_n$ and all $\alpha, \beta \in \mathbb{R}$;

(3) $\varphi(\varphi(A)) = A$ for all $A \in M_n$.

Let $\mathcal{I}(\varphi) \equiv \{A \in M_n: \varphi(A) = A\}$ denote the set of matrices that are invariant under φ. Show that:

(a) $\frac{1}{2}(A + \varphi(A)) \in \mathcal{I}(\varphi)$ for all $A \in M_n$.

(b) $\mathcal{I}(\varphi)$ is a nonempty real linear subspace of M_n.

(c) For all $B \in \mathcal{I}(\varphi)$ and for every unitarily invariant norm $\|\cdot\|$ on M_n, $\|A - \frac{1}{2}(A + \varphi(A))\| = \|\frac{1}{2}(A - B) - \frac{1}{2}\varphi(A - B)\| \leq \|A - B\|$.

Conclude that $\frac{1}{2}(A + \varphi(A))$ is a best approximation to A among all matrices in $\mathcal{I}(\varphi)$ with respect to every unitarily invariant norm on M_n. Consider $\varphi(A) = \pm A^*$, $\pm A^T$, and $\pm \bar{A}$ and verify that the following are best approximations to a given $A \in M_n$ by matrices in the given class with respect to every unitarily invariant norm on M_n: $\frac{1}{2}(A + A^*)$, Hermitian; $\frac{1}{2}(A - A^*)$, skew-Hermitian; $\frac{1}{2}(A + A^T)$, symmetric; $\frac{1}{2}(A - A^T)$, skew-symmetric; $\frac{1}{2}(A + \bar{A})$, real; $\frac{1}{2}(A - \bar{A})$, purely imaginary.

22. Let $A = [A_{ij}]_{i,j=1}^2 \in M_n$ be a given positive semidefinite block matrix with $A_{11} \in M_m$ and $A_{22} \in M_{n-m}$. For a given unitarily invariant norm $\|\cdot\|$ on M_n, define $\|\cdot\|$ on M_k, $k \leq n$, as in Problem 1.

(a) Use Corollary (3.5.11) to show that $\|A\| \leq \|A_{11}\| + \|A_{22}\|$ for every unitarily invariant norm $\|\cdot\|$ on M_n if and only if $N_k(A) \leq N_k(A_{11}) + N_k(A_{22})$ for $k = 1,\ldots, n$.

(b) Prove the inequalities in (a) for the Ky Fan k-norms; conclude that $\|A\| \leq \|A_{11}\| + \|A_{22}\|$ for every unitarily invariant norm $\|\cdot\|$ on M_n.

(c) Show that $\| A_{11} \oplus A_{22} \| \leq \|A\| \leq \|A_{11}\| + \|A_{22}\|$ for every unitarily invariant norm $\|\cdot\|$ on M_n. What does this say for the Frobenius norm? the spectral norm? the trace norm?

(d) If $A = [a_{ij}]_{i,j=1}^n$, show that $\|\mathrm{diag}(a_{11},\ldots, a_{nn})\| \leq \|A\| \leq \|E_{11}\| \operatorname{tr} A$ for every unitarily invariant norm $\|\cdot\|$ on M_n.

Notes and Further Readings. Results related to (3.5.6) and (3.5.16) are in R. A. Horn and R. Mathias, Cauchy-Schwarz Inequalities Associated with Positive Semidefinite Matrices, *Linear Algebra Appl.* 142 (1990), 63–82. Ky Fan proved the equivalence of Corollary (3.5.9a,b) in 1951; this basic result is often called the *Fan dominance theorem*. The bound (3.5.31) (for the Frobenius norm) was discovered in 1907 by E. Schmidt [see Section 18 of Schmidt's paper cited in Section (3.0)], who wanted to approximate a function of two variables by a sum of products of two univariate functions; it was rediscovered in the context of factorial theories in psychology by C. Eckart and G. Young, The Approximation of One Matrix by Another of Lower Rank, *Psychometrika* 1 (1936), 211-218. The results in Problems 3, 20, and 21 are in C.-K. Li and N.-K. Tsing, On the Unitarily Invariant Norms and Some Related Results, *Linear Multilinear Algebra* 20 (1987), 107-119.

3.6 Sufficiency of Weyl's product inequalities

The eigenvalue-singular value product inequalities in Weyl's theorem (3.3.2) raise the natural question of whether the inequalities (3.3.3) exactly characterize the relationship between the eigenvalues and singular values of a matrix. Alfred Horn proved in 1954 that they do.

The inequalities (3.3.3) are a kind of multiplicative majorization, and if A is nonsingular we can take logarithms to obtain the ordinary majorization inequalities.

$$\sum_{i=1}^{k} \log|\lambda_i(A)| \le \sum_{i=1}^{k} \log \sigma_i(A), \quad k=1,\ldots,n, \text{ with equality for } k=n \quad (3.6.1)$$

Since majorization inequalities of this type are the exact relationship between the eigenvalues and main diagonal entries of a Hermitian matrix, it is not surprising that this fact is useful in establishing the converse of Theorem (3.3.2). The following lemma of Mirsky is useful in this effort.

3.6.2 Lemma. Let $A \in M_n$ be positive definite and have decreasingly ordered eigenvalues $\lambda_1(A) \ge \cdots \ge \lambda_n(A) > 0$. For $k=1,\ldots,n$, let A_k denote the upper left k-by-k principal submatrix of A, and let the positive real numbers $r_i(A)$ be defined recursively by

$$r_1(A)\cdots r_k(A) \equiv \det A_k, \quad k=1,\ldots,n \quad (3.6.3)$$

Let the decreasingly ordered values of $r_1(A),\ldots,r_n(A)$ be denoted by $r_{[1]} \ge \cdots \ge r_{[n]}$. Then

$$r_{[1]}(A)\cdots r_{[k]}(A) \le \lambda_1(A)\cdots\lambda_k(A) \text{ for } k=1,\ldots,n, \text{ with equality for } k=n \quad (3.6.4)$$

and A can be represented as $A = \Delta^*\Delta$, where $\Delta = [d_{ij}] \in M_n$ is upper triangular with main diagonal entries $d_{ii} = +\sqrt{r_i(A)}$ and singular values $\sigma_i(\Delta) = +\sqrt{\lambda_i(A)}$ for $i=1,\ldots,n$, and Δ may be taken to be real if A is real.

Conversely, let $\lambda_1 \ge \cdots \ge \lambda_n > 0$ and $r_1,\ldots,r_n$ be $2n$ given positive numbers and let $r_{[1]} \ge \cdots \ge r_{[n]} > 0$ denote the decreasingly ordered values of $r_1,\ldots,r_n$. If

$$r_{[1]}\cdots r_{[k]} \le \lambda_1\cdots\lambda_k \text{ for } k=1,\ldots,n, \text{ with equality for } k=n \quad (3.6.4')$$

then there exists a real upper triangular matrix $\Delta = [d_{ij}] \in M_n(\mathbb{R})$ with main diagonal entries $d_{ii} = +\sqrt{r_i}$ and singular values $\sigma_i(\Delta) = +\sqrt{\lambda_i}$ for $i=1,\ldots,n$.

The real symmetric positive definite matrix $A \equiv \Delta^T\Delta$ satisfies

$$r_1 \cdots r_k = \det A_k \text{ for } k = 1,\dots, n \tag{3.6.3$'$}$$

Proof: For the forward assertion, let $A = \Delta^*\Delta$ be a Cholesky factorization of A with an upper triangular $\Delta = [d_{ij}] \in M_n$ with positive main diagonal entries $\{d_{11},\dots, d_{nn}\}$, which are the eigenvalues of Δ. Moreover, Δ may be taken to be real if A is real; see Corollary (7.2.9) in [HJ]. Since $\Delta_k^*\Delta_k = A_k$ and $r_1(A) \cdots r_k(A) = \det A_k = (d_{11} \cdots d_{kk})^2$ for $k = 1,\dots, n$, it is clear that $d_{ii} = r_i(A)^{\frac{1}{2}}$ for $i = 1,\dots, n$. Moreover, $\sigma_i(\Delta)^2 = \lambda_i(\Delta^*\Delta) = \lambda_i(A)$ for $i = 1,\dots, n$. Using Weyl's inequalities (3.3.2), we have

$$r_{[1]}(A) \cdots r_{[k]}(A) = \lambda_1(\Delta)^2 \cdots \lambda_k(\Delta)^2$$
$$\leq \sigma_1(\Delta)^2 \cdots \sigma_k(\Delta)^2 = \lambda_1(A) \cdots \lambda_k(A) \text{ for } k = 1,\dots, n$$

The case $n = 1$ of the converse is trivial, so assume that it holds for matrices of all orders up to and including $n-1$ and proceed by induction. The given inequalities (3.6.4$'$) are equivalent to

$$\sum_{i=1}^{k} \log r_{[i]} \leq \sum_{i=1}^{k} \log \lambda_i, \quad k = 1,\dots, n, \text{ with equality for } k = n$$

Whenever one has such a majorization, it is always possible to find (see Lemma (4.3.28) in [HJ]) real numbers $\gamma_1,\dots, \gamma_{n-1}$ satisfying the interlacing inequalities

$$\log \lambda_1 \geq \gamma_1 \geq \log \lambda_2 \geq \gamma_2 \geq \cdots \geq \log \lambda_{n-1} \geq \gamma_{n-1} \geq \log \lambda_n$$

as well as the majorization inequalities

$$\sum_{i=1}^{k} \log r_{[i]} \leq \sum_{i=1}^{k} \gamma_i, \quad k = 1,\dots, n-1, \text{ with equality for } k = n-1$$

Then

$$\lambda_1 \geq e^{\gamma_1} \geq \lambda_2 \geq e^{\gamma_2} \geq \cdots \geq \lambda_{n-1} \geq e^{\gamma_{n-1}} \geq \lambda_n > 0 \tag{3.6.5}$$

and Lemma (3.3.8) with $f(t) = e^t$ gives

$$\eta_{[1]} \cdots \eta_{[k]} \le e^{\gamma_1} \cdots e^{\gamma_k}, \quad k = 1, \ldots, n-1, \text{ with equality for } k = n-1$$

so by the induction hypothesis there exists a real symmetric positive definite matrix $\hat{A} \in M_{n-1}$ with eigenvalues $e^{\gamma_1}, \ldots, e^{\gamma_{n-1}}$ and such that $r_1 \cdots r_k = \det \hat{A}_k$, $k = 1, \ldots, n-1$. Since interlacing inequalities of the type (3.6.5) exactly characterize the possible set of eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ of a matrix $A \in M_n$ that can be obtained from $\hat{A} \in M_{n-1}$ by bordering it with a single row and column (see Theorem (4.3.10) in [HJ]), we know that there exists a real vector $y \in \mathbb{R}^n$ and a positive real number a such that the eigenvalues of the real symmetric matrix

$$A \equiv \begin{bmatrix} \hat{A} & y \\ y^T & a \end{bmatrix} \in M_n$$

are $\lambda_1, \ldots, \lambda_n$. Since $\lambda_1 \cdots \lambda_n = r_1 \cdots r_n$ is given, we must have $\det A_n = \det A = \lambda_1 \cdots \lambda_n = r_1 \cdots r_n$, as required. The final assertion about the factorization $A = \Delta^T \Delta$ and the properties of Δ have already been established in the proof of the forward assertion. $\square$

3.6.6 Theorem. Let $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n \ge 0$ be n given nonnegative real numbers, let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be given with $|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_n|$, and suppose that

$$\sigma_1 \cdots \sigma_k \ge |\lambda_1 \cdots \lambda_k| \text{ for } k = 1, \ldots, n, \text{ with equality for } k = n \tag{3.6.7}$$

Then there exists an upper triangular matrix $\Delta \in M_n$ whose main diagonal entries (its eigenvalues) are $\lambda_1, \ldots, \lambda_n$ (in any order) and whose singular values are $\sigma_1, \ldots, \sigma_n$. The matrix Δ may be taken to be real if $\lambda_1, \ldots, \lambda_n$ are real.

Proof: First consider the case in which all the values $\{\sigma_i\}$ and $\{\lambda_i\}$ are positive real numbers. In this event, $\sigma_1^2 \cdots \sigma_k^2 \ge \lambda_1^2 \cdots \lambda_k^2$ for $k = 1, \ldots, n$, with equality for $k = n$. Thus, the converse half of Lemma (3.6.2) guarantees that there is a real upper triangular matrix $\Delta = [d_{ij}] \in M_n(\mathbb{R})$ with main diagonal entries $d_{ii} = +\sqrt{\lambda_i^2} = \lambda_i$ and singular values $\sigma_i(\Delta) = +\sqrt{\sigma_i^2} = \sigma_i$, $i = 1, \ldots, n$.

Now suppose $\{\lambda_i\}$ are all nonnegative and $\lambda_n = 0$, so $\sigma_n = 0$ as well. If $\lambda_2 = \cdots = \lambda_n = 0$, then $\sigma_1 \geq \lambda_1$ and

$$\Delta \equiv \begin{bmatrix} \lambda_1 & \beta & 0 & \cdots & 0 \\ \vdots & 0 & \sigma_2 & & \vdots \\ & & 0 & \ddots & \\ & & & \ddots & \sigma_{n-1} \\ 0 & & \cdots & & 0 \end{bmatrix} \in M_n(\mathbb{R}),\ \beta \equiv (\sigma_1^2 - \lambda_1^2)^{1/2} \geq 0$$

has the desired eigenvalues; it has the desired singular values since

$$\Delta\Delta^T = \operatorname{diag}(\lambda_1^2 + \beta^2 = \sigma_1^2, \sigma_2^2,\ldots, \sigma_{n-1}^2, 0)$$

Now suppose $\lambda_m > \lambda_{m+1} = \cdots = \lambda_n = 0$ with $2 \leq m < n$. Then $\sigma_{m-1} \geq \sigma_m \geq (\lambda_1 \cdots \lambda_m)/(\sigma_1 \cdots \sigma_{m-1}) \equiv \alpha > 0$ and we know there is a real $B \in M_m$ with eigenvalues $\lambda_1,\ldots, \lambda_m$ and singular values $\sigma_1,\ldots, \sigma_{m-1}, \alpha$. Let Q_1 be a real orthogonal matrix such that $Q_1(BB^T)Q_1^T = \operatorname{diag}(\sigma_1^2,\ldots, \sigma_{m-1}^2, \alpha^2) = CC^T$, where $C = [c_{ij}] \equiv Q_1 B Q_1^T$. Define

$$A = [a_{ij}] \equiv \left[\begin{array}{c:c} C & \begin{matrix}\beta \\ 0\end{matrix} \\ \hdashline 0 & \begin{matrix} & \sigma_{m+1} & & \\ & & \ddots & \\ & & & \sigma_{n-1} \\ & & & 0 \end{matrix} \end{array}\right] \in M_n(\mathbb{R}),\ \beta \equiv (\sigma_m^2 - \alpha^2)^{1/2}$$

That is, $a_{ij} = c_{ij}$ if $1 \leq i, j \leq m$, $a_{m,m+1} = \beta$, $a_{i,i+1} = \sigma_i$ for $i = m+1,\ldots, n-1$, and all other entries of A are zero. Then the eigenvalues of A are $\lambda_1,\ldots, \lambda_m$ (the eigenvalues of C) together with $n - m$ zeros, as desired, and

$$AA^T = \operatorname{diag}(\sigma_1^2,\ldots, \sigma_{m-1}^2, \alpha^2 + \beta^2 = \sigma_m^2, \sigma_{m+1}^2 \ldots, \sigma_{n-1}^2, 0)$$

so A has the desired singular values. If Q_2 is a real orthogonal matrix such that $\Delta \equiv Q_2 A Q_2^T$ is upper triangular, then Δ is a matrix with the asserted properties, since it has the same eigenvalues and singular values as A.

Finally, suppose nonnegative $\{\sigma_i\}$ and complex $\{\lambda_i\}$ satisfy the inequalities (3.6.7), and let $\lambda_k = e^{i\theta_k}|\lambda_k|$ for $k = 1,\ldots, n$, $\theta_k \in \mathbb{R}$. Let $A \in M_n(\mathbb{R})$ be upper triangular with eigenvalues $|\lambda_1|,\ldots, |\lambda_n|$ and singular values $\sigma_1,\ldots, \sigma_n$, set $D \equiv \operatorname{diag}(e^{i\theta_1},\ldots, e^{i\theta_n})$ and take $\Delta \equiv DB$. If $\{\lambda_i\}$ are all real, then D has diagonal entries ± 1 and hence it and Δ are real in this case.

We have now constructed an upper triangular matrix Δ with given singular values and eigenvalues satisfying (3.6.7), in which the main diagonal entries of Δ occur with decreasing moduli. By Schur's triangularization theorem (Theorem (2.3.1) in [HJ]), Δ is unitarily similar to an upper triangular matrix in which the eigenvalues appear on the main diagonal in any prescribed order, and the unitary similarity does not change the singular values. □

Problems

1. Let $\lambda_1 \geq \cdots \geq \lambda_n > 0$ and $\sigma_1 \geq \cdots \geq \sigma_n > 0$ be $2n$ given positive numbers. Explain why the following three statements are equivalent:

(a) There is some $A \in M_n$ with eigenvalues $\{\lambda_1,\ldots, \lambda_n\}$ and singular values $\{\sigma_1,\ldots, \sigma_n\}$.

(b) $\lambda_1 \cdots \lambda_k \leq \sigma_1 \cdots \sigma_k$ for $k = 1,\ldots, n$, with equality for $k = n$.

(c) There is a positive definite $A \in M_n$ with eigenvalues $\{\sigma_1,\ldots, \sigma_n\}$ and such that $\lambda_1 \cdots \lambda_k = \det A_k$ for $k = 1,\ldots, n$, where A_k denotes the upper left k-by-k principal submatrix of A.

2. (a) Let

$$A = \begin{bmatrix} r & a \\ 0 & r \end{bmatrix} \text{ and } Q_\varphi = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

where $r > 0$ and $a \in \mathbb{R}$ are given, and let $\theta \in \mathbb{R}$ be given. Show that $\varphi \in \mathbb{R}$ can be chosen so that tr $AQ_\varphi = 2r\cos\theta$, in which case the eigenvalues of AQ_φ are $r\cos\theta \pm ir\sin\theta$ and the singular values of AQ_φ are the same as those of A.
(b) Using the notation and hypotheses of Theorem (3.6.6), use (a) to show that if any nonreal values λ_i occur only in complex conjugate pairs, then there exists a real upper Hessenberg $\Delta \in M_n(\mathbb{R})$ with eigenvalues $\lambda_1,\ldots, \lambda_n$ and singular values $\sigma_1,\ldots, \sigma_n$. Moreover, Δ may be taken to have the block upper triangular form

$$\Delta = \begin{bmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_k \end{bmatrix}$$

where each block A_i is a real 1-by-1 matrix or a real 2-by-2 matrix with nonreal complex conjugate eigenvalues.

Notes and Further Readings. Theorem (3.6.6) was first proved in A. Horn, On the Eigenvalues of a Matrix with Prescribed Singular Values, *Proc. Amer. Math. Soc.* 5 (1954), 4-7; the proof we present is adapted from A. Mirsky, Remarks on an Existence Theorem in Matrix Theory Due to A. Horn, *Monatsheft Math.* 63 (1959), 241-243.

3.7 Inclusion intervals for singular values

Since the squares of the singular values of $A \in M_{m,n}$ are essentially the singular values of the Hermitian matrices AA^* and A^*A, inclusion intervals for the singular values of A can be developed by applying Geršgorin regions, norm bounds, and other devices developed in Chapter 6 of [HJ] to AA^* and A^*A. Unfortunately, this approach typically gives bounds in terms of complicated sums of products of pairs of entries of A. Our objective in this section is to develop readily computable upper and lower bounds for individual singular values that are simple functions of the entries of A, rather than their squares and pairwise products.

The upper bounds that we present are motivated by the observation that, for $A = [a_{ij}] \in M_n$, $\sigma_1^2(A) = \rho(A^*A)$, the spectral radius or largest eigenvalue of A^*A. Moreover, the spectral radius of a matrix is bounded from above by the value of any matrix norm (Theorem (5.6.9) in [HJ]), so we have the bound

$$\sigma_1(A) = [\rho(A^*A)]^{\frac{1}{2}} \leq [\,|||\,A^*A\,|||\,]^{\frac{1}{2}} \leq [\,|||\,A^*\,|||\;|||\,A\,|||\,]^{\frac{1}{2}} \tag{3.7.1}$$

for any matrix norm $|||\cdot|||$. For the particular choice $|||\cdot||| \equiv |||\cdot|||_1$, the maximum column sum matrix norm (see (5.6.4) in [HJ]), we have $|||\,A^*\,|||_1 = |||\,A\,|||_\infty$, the maximum row sum matrix norm ((5.6.5) in [HJ]), and hence we have the upper bound

$$\sigma_1(A) \leq \sqrt{|||\,A\,|||_\infty\,|||\,A\,|||_1} = \Big\{ \Big[\max_i \sum_{j=1} |a_{ij}|\Big] \Big[\max_j \sum_{i=1} |a_{ij}|\Big] \Big\}^{\frac{1}{2}} \tag{3.7.2}$$

in terms of the largest absolute row and column sums. If $A \in M_{m,n}$ is not square, application of (3.7.2) to the square matrix $[A \;\; 0] \in M_m$ (if $m > n$) or $\begin{bmatrix} A \\ 0 \end{bmatrix} \in M_n$ (if $m < n$) shows that the bound (3.7.2) is valid for any $A = [a_{ij}] \in M_{m,n}$.

To see how the estimate (3.7.2) can be used to obtain an upper bound on $\sigma_2(A)$, delete from A any row or column whose absolute sum gives the larger of the two quantities $|||A|||_1$ or $|||A|||_\infty$ and denote the resulting matrix by $A_{(1)}$. Notice that $A_{(1)} \in M_{m-1,n}$ if $|||A|||_\infty \geq |||A|||_1$ and a row was deleted, or $A_{(1)} \in M_{m,n-1}$ if $|||A|||_1 \geq |||A|||_\infty$ and a column was deleted. The interlacing property for singular values (Corollary (3.1.3)) ensures that

$$\sigma_1(A) \geq \sigma_1(A_{(1)}) \geq \sigma_2(A) \geq \sigma_2(A_{(1)}) \geq \cdots$$

so

$$\sigma_2(A) \leq \sigma_1(A_{(1)}) \leq [\,|||A_{(1)}|||_1 \; |||A_{(1)}|||_\infty\,]^{\frac{1}{2}} \tag{3.7.3}$$

Although the bound in (3.7.3) is readily computed, there is another, though weaker, bound that is even easier to compute.

3.7.4 Definition. Let $A = [a_{ij}] \in M_{m,n}$ be given. Arrange the $m+n$ quantities

$$\sum_{j=1}^{n} |a_{ij}|, \quad i = 1,\ldots, m, \text{ and } \sum_{i=1}^{m} |a_{ij}|, \quad j = 1,\ldots, n$$

in decreasing order and denote the resulting $m+n$ *ordered absolute line sums* of A by

$$l_1(A) \geq l_2(A) \geq \cdots \geq l_{m+n}(A) \geq 0$$

Observe that $l_1(A) = \max\{|||A|||_1, |||A|||_\infty\}$ and $l_2(A) \geq \min\{|||A|||_1, |||A|||_\infty\}$, so

$$|||A|||_1 \; |||A|||_\infty \leq l_1(A) l_2(A) \tag{3.7.5}$$

Moreover, after deleting a row or column of A whose absolute sum gives the value $l_1(A)$, we have $l_1(A_{(1)}) \leq l_2(A)$, $l_2(A_{(1)}) \leq l_3(A)$, and, more generally,

$$l_i(A_{(1)}) \leq l_{i+1}(A), \quad i = 1,\ldots, m+n-1, \; A \in M_{m,n} \tag{3.7.6}$$

Thus, we have the bounds

$$\sigma_2(A) \leq \sigma_1(A_{(1)}) \leq [\,|||A_{(1)}|||_1 \, |||A_{(1)}|||_\infty\,]^{\frac{1}{2}} \leq [l_1(A_{(1)}) l_2(A_{(1)})]^{\frac{1}{2}}$$

$$\leq [l_2(A) l_3(A)]^{\frac{1}{2}}$$

The process leading to these bounds can be iterated to give upper bounds for all the singular values. We have already verified the cases $k = 1, 2$ of the following theorem.

3.7.7 Theorem. Let $A \in M_{m,n}$ be given and define $A_{(0)} \equiv A$. For $k = 1, 2, \ldots$ form $A_{(k+1)}$ by deleting from $A_{(k)}$ a row or column corresponding to the largest absolute line sum $l_1(A_{(k)})$. Then

$$\begin{aligned}
\sigma_k(A) &\leq \sigma_1(A_{(k-1)}) \leq [\, |||A_{(k-1)}|||_1 \, |||A_{(k-1)}|||_\infty \,]^{\frac{1}{2}} \\
&\leq \left[l_1(A_{(k-1)}) l_2(A_{(k-1)})\right]^{\frac{1}{2}} \leq \left[l_2(A_{(k-2)}) l_3(A_{(k-2)})\right]^{\frac{1}{2}} \\
&\leq \cdots \\
&\leq \left[l_r(A_{(k-r)}) l_{r+1}(A_{(k-r)})\right]^{\frac{1}{2}} \leq \left[l_k(A) l_{k+1}(A)\right]^{\frac{1}{2}} \qquad (3.7.8)
\end{aligned}$$

for $k = 1, \ldots, \min\{m,n\}$. The norms $|||X|||_1$ and $|||X|||_\infty$ are the maximum absolute column and row sums of X, respectively.

Proof: Since $A_{(k)}$ is formed from A by deleting a total of k rows or columns, the interlacing property (3.1.4) gives the upper bound $\sigma_{k+1}(A) \leq \sigma_1(A_{(k)})$. The further bounds

$$\sigma_1(A_{(k)}) \leq [\, |||A_{(k)}|||_1 \, |||A_{(k)}|||_\infty \,]^{\frac{1}{2}} \leq \left[l_1(A_{(k)}) l_2(A_{(k)})\right]^{\frac{1}{2}}$$

are obtained by applying (3.7.1) and (3.7.5) to $A_{(k)}$. The final chain of bounds involving successive absolute line sums of $A_{(k-1)}, A_{(k-2)}, \ldots, A_{(1)}, A_{(0)} = A$ follow from (3.7.6) and the observation that $(A_{(k)})_{(1)} = A_{(k+1)}$:

$$l_i(A_{(k)}) \leq l_{i+1}(A_{(k-1)}) \leq \cdots \leq l_{i+r}(A_{(k-r)}) \leq \cdots \leq l_{i+k}(A) \qquad \square$$

The weakest of the bounds in (3.7.8) involves the quantities $l_1(A), \ldots, l_{\min\{m,n\}+1}(A)$, which can all be computed directly from the absolute row and column sums of A without any recursion. The better bounds, involving the submatrices $A_{(r)}$ and their two largest absolute line sums, are a little more work to compute but can be surprisingly good.

Exercise. Consider the square matrix

$$A = [a_{ij}] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & \\ & & 0 & \\ 1 & & & \end{bmatrix} \in M_n,\ n \geq 3$$

for which $a_{ij} = 1$ if either $i = 1$ or $j = 1$, and $a_{ij} = 0$ otherwise. Verify the entries in the following table, which was obtained by deleting the first row of A to produce $A_{(1)}$ and then deleting the first column of $A_{(1)}$ to produce $A_{(2)}$. How good are the upper bounds that are obtained for $\sigma_2(A)$ and $\sigma_3(A)$?

$k=$	1	2	3	4
$l_k(A)$	n	n	1	1
$l_k(A_{(1)})$	n-1	1	1	
$l_k(A_{(2)})$	0	0		
$\lVert\lvert A_{(k-1)} \rVert\rvert_1$	n	n-1		
$\lVert\lvert A_{(k-1)} \rVert\rvert_\infty$	n	1		
$[l_k(A)\, l_{k+1}(A)]^{\frac{1}{2}}$	n	$n^{\frac{1}{2}}$	1	
$[l_{k-1}(A_{(1)})\, l_k(A_{(1)})]^{\frac{1}{2}}$		$(n\text{-}1)^{\frac{1}{2}}$	1	
$[l_{k-2}(A_{(2)})\, l_{k-1}(A_{(2)})]^{\frac{1}{2}}$			0	
$[\lVert\lvert A_{(k-1)} \rVert\rvert_1 \lVert\lvert A_{(k-1)} \rVert\rvert_\infty]^{\frac{1}{2}}$	n	$(n\text{-}1)^{\frac{1}{2}}$	0	
$\sigma_k(A)$	$(n\text{-}\frac{3}{4})^{\frac{1}{2}}+\frac{1}{2}$	$(n\text{-}\frac{3}{4})^{\frac{1}{2}}\text{-}\frac{1}{2}$	0	

The lower bounds that we present are motivated by recalling from Corollary (3.1.5) that, for $A = [a_{ij}] \in M_n$, the smallest singular value of A has the lower bound $\sigma_n(A) \geq \lambda_{min}(\mathrm{H}(A))$. Since all the eigenvalues of the Hermitian matrix $\mathrm{H}(A) = [h_{ij}] = [\frac{1}{2}(a_{ij} + \bar{a}_{ji})]$ are contained in the union of Geršgorin intervals

$$\bigcup_{k=1}^{n} \left\{ x \in \mathbb{R}\colon |x - \mathrm{Re}\, a_{kk}| = |x - h_{kk}| \leq \sum_{j \neq k} |h_{kj}| = \tfrac{1}{2} \sum_{j \neq k} |a_{kj} + \bar{a}_{jk}| \right\}$$

(see Theorem (6.1.1) in [HJ]), we have the simple lower bound

$$\sigma_n(A) \geq \lambda_n(\mathrm{H}(A)) \geq \min_{1 \leq k \leq n} \left\{ \mathrm{Re}\, a_{kk} - \tfrac{1}{2} \sum_{j \neq k} |a_{kj} + \bar{a}_{jk}| \right\} \tag{3.7.9}$$

in which $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the algebraically decreasingly ordered eigenvalues of $\mathrm{H}(A)$. Notice that (3.7.9) is a trivial bound if any main diagonal entry of A has negative real part, an inadequacy that is easy to remedy since the singular values $\sigma_k(A)$ are unitarily invariant functions of A.

3.7.10 Definition. For $A = [a_{ij}] \in M_{m,n}$ define $D_A = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \in M_m$, where $e^{i\theta_k} a_{kk} = |a_{kk}|$ if $a_{kk} \neq 0$ and $\theta_k \equiv 0$ if $a_{kk} = 0$, $k = 1, \ldots, m$.

Since the main diagonal entries of $D_A A$ are nonnegative, (3.7.9) gives the lower bounds

$$\sigma_n(A) = \sigma_n(D_A A) \geq \lambda_n(\mathrm{H}(D_A A))$$

$$\geq \min_{1 \leq k \leq n} \left\{ |a_{kk}| - \tfrac{1}{2} \sum_{j \neq k} |e^{i\theta_k} a_{kj} + e^{-i\theta_j} \bar{a}_{jk}| \right\} \tag{3.7.11a}$$

$$\geq \min_{1 \leq k \leq n} \left\{ |a_{kk}| - \tfrac{1}{2} \sum_{j \neq k} (|a_{kj}| + |a_{jk}|) \right\}$$

$$= \min_{1 \leq k \leq n} \{ |a_{kk}| - \tfrac{1}{2}[R_k'(A) + C_k'(A)] \} \tag{3.7.11b}$$

in which we use the triangle inequality and write

$$R_k'(A) \equiv \sum_{j \neq k} |a_{kj}|, \quad C_k'(A) \equiv \sum_{i \neq k} |a_{ik}|, \quad A = [a_{ij}] \in M_{m,n}$$

as in Definition (1.5.1), for the deleted (that is, the term $|a_{kk}|$ is omitted) absolute kth row and column sums of A. Since there may be some cancellation between $e^{i\theta_k} a_{kj}$ and $e^{-i\theta_j} \bar{a}_{jk}$, the lower bound given by (3.7.11a) is generally better than that given by (3.7.11b), but the latter has the advantage of simplicity since it involves only $2n$ deleted column and row sums of A rather than all the $n(n-1)$ off-diagonal entries of $D_A A$.

In order to extend the bounds in (3.7.11a,b) to all the singular values, it is convenient to introduce notation for the left-hand endpoints of the Geršgorin intervals of $\mathrm{H}(D_A A)$.

3.7.12 Definition. Let $A = [a_{ij}] \in M_{m,n}$ be given. Arrange the n *row deficits* of $\mathrm{H}(D_A A)$,

$$d_k(\mathrm{H}(D_A A)) \equiv |a_{kk}| - \tfrac{1}{2}R'_k(\mathrm{H}(D_A A)),\ k = 1,\ldots, n$$

in decreasing order and denote the resulting *ordered row deficits* of A by

$$\alpha_1(A) \geq \alpha_2(A) \geq \cdots \geq \alpha_n(A) = \min_k \{|a_{kk}| - R'_k(\mathrm{H}(D_A A))\}$$

Notice that $\alpha_k(A) < 0$ is possible, and the lower bound (3.7.11a) may be restated as

$$\sigma_n(A) \geq \lambda_n(\mathrm{H}(D_A A)) \geq \alpha_n(A), \quad A \in M_n \tag{3.7.13}$$

To see how the estimate (3.7.13) can be used to obtain a lower bound on $\sigma_{n-1}(A)$, delete row p *and* column p from A, for p such that the p th row deficit of $\mathrm{H}(D_A A)$ has the least possible value $d_p(\mathrm{H}(D_A A)) \equiv \alpha_n(A)$; denote the resulting matrix by $A_{[n-1]} \in M_{n-1}$. Then $A_{[n-1]}$ is a principal submatrix of A and $\mathrm{H}(D_{A_{[n-1]}} A_{[n-1]})$ is a principal submatrix of $\mathrm{H}(D_A A)$, so the interlacing eigenvalues theorem for Hermitian matrices (Theorem (4.3.15) in [HJ]) ensures that

$$\lambda_{n-1}(\mathrm{H}(D_A A)) \geq \lambda_{n-1}(\mathrm{H}(D_A A)_{[n-1]})$$

(notice that $\mathrm{H}(X_{[k]}) = \mathrm{H}(X)_{[k]}$). Moreover, the respective ordered row deficits of A are not greater than those of its principal submatrix $A_{[n-1]}$, that is

$$\alpha_k(A_{[n-1]}) \geq \alpha_k(A), \quad k = 1,\ldots, n-1, \quad A \in M_n \tag{3.7.14}$$

and $\sigma_{n-1}(A) = \sigma_{n-1}(D_A A) \geq \lambda_{n-1}(\mathrm{H}(D_A A))$ by (3.1.6b). Combining these facts with (3.7.13) applied to $A_{[n-1]} \in M_{n-1}$ gives the bound

$$\sigma_{n-1}(A) \geq \lambda_{n-1}(\mathrm{H}(D_A A)) \geq \lambda_{n-1}(\mathrm{H}(D_A A)_{[n-1]})$$

$$\geq \alpha_{n-1}(A_{[n-1]}) \geq \alpha_{n-1}(A)$$

The method used to develop this lower bound for $\alpha_{n-1}(A)$ can be iterated to give lower bounds for all of the singular values. We have already verified the cases $k = n$, $n-1$ of the following theorem.

3.7.15 Theorem. Let $A \in M_n$ be given and define $A_{[n]} \equiv A$. For $k = n$, $n-1$, $n-2$,... calculate the least ordered row deficit $\alpha_k(A_{[k]})$ and suppose it is obtained by using row i_k of $\mathrm{H}((D_A A)_{[k]})$. Form $A_{[k-1]}$ by deleting both row i_k and column i_k from $A_{[k]}$. Then

$$\sigma_k(A) \geq \alpha_k(A_{[k]}) \geq \alpha_k(A_{[k+1]}) \geq \cdots \geq \alpha_k(A_{[n-1]})$$

$$\geq \alpha_k(A), \qquad k = n, n-1, n-2, \ldots, 2, 1 \tag{3.7.16}$$

Proof: Since all the concepts needed to prove the general case have already occurred in our discussion of the case $k = n-1$, the following chain of inequalities and citations provides an outline of the proof. A crucial, if simple, point is that $\mathrm{H}((D_A A)_{[k]}) \in \mathrm{M}_k$ is a principal submatrix of $\mathrm{H}(D_A A)$ because $A_{[k]} \in M_k$ is a principal submatrix of A. We continue to write the eigenvalues of $\mathrm{H}(\cdot)$ in decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots$.

$$\begin{aligned}
\sigma_k(A) &= \sigma_k(D_A A) && \text{[by unitary invariance]}\\
&\geq \lambda_k(\mathrm{H}(D_A A)) && \text{[by (3.1.6a)]}\\
&\geq \lambda_k(\mathrm{H}(D_A A)_{[k]}) && \text{[by Hermitian eigenvalue interlacing]}\\
&\geq \alpha_k(A_{[k]}) && \text{[by (3.7.13)]}\\
&\geq \alpha_k(A_{[k+1]}) \geq \alpha_k(A_{[k+2]}) \geq \cdots \geq \alpha_k(A) && \text{[by (3.7.14)]}
\end{aligned}$$

$\square$

Although the preceding theorem has been stated for square complex matrices, it can readily be applied to nonsquare matrices by the familiar device of augmenting the given matrix with a suitable block of zeroes to make it square. We illustrate this observation with an important special case.

3.7.17 Corollary. Let $A = [a_{ij}] \in M_{m,n}$ be given, let $q = \min\{m,n\}$, and let $D_A = \operatorname{diag}(e^{i\theta_1},\dots,e^{i\theta_q}) \in M_q$ be defined by $e^{i\theta_k} a_{kk} = |a_{kk}| \neq 0$ and $\theta_k = 0$ if $a_{kk} = 0$, $k = 1,\dots, q$. Then

$$\sigma_q(A) \geq \begin{cases} \min\limits_{1 \leq k \leq n} \left\{ |a_{kk}| - \tfrac{1}{2} \sum\limits_{\substack{j=1 \\ j \neq k}}^{n} |e^{i\theta_k} a_{kj} + e^{-i\theta_j} \bar{a}_{jk}| - \tfrac{1}{2} \sum\limits_{j=n+1}^{m} |a_{jk}| \right\} & \text{if } m \geq n \\ \min\limits_{1 \leq k \leq m} \left\{ |a_{kk}| - \tfrac{1}{2} \sum\limits_{\substack{j=1 \\ j \neq k}}^{m} |e^{i\theta_k} a_{kj} + e^{-i\theta_j} \bar{a}_{jk}| - \tfrac{1}{2} \sum\limits_{j=m+1}^{n} |a_{kj}| \right\} & \text{if } m \leq n \end{cases}$$

$$\geq \min_{1 \leq k \leq q} \{ |a_{kk}| - \tfrac{1}{2} [R_k'(A) + C_k'(A)] \}$$

Proof: There is no loss of generality if we assume that $m \geq n$, so let $B = [A \;\; 0] \in M_m$. Then $q = n$ and $\sigma_n(A) = \sigma_n(B) \geq \alpha_n(B)$. Because the lower right $(m-n)$-by-$(m-n)$ submatrix of $\mathrm{H}(B)$ is a block of zeroes, the corresponding row deficits

$$d_k(\mathrm{H}(D_B B)) = -R_k'(\mathrm{H}(D_B B)), \quad k = n+1,\dots, m$$

are all nonpositive. Now let

$$\tilde{\alpha}_n(B) \equiv \min\{d_1(\mathrm{H}(D_B B)),\dots, d_n(\mathrm{H}(D_B B))\}$$

If $\tilde{\alpha}_n(B) \geq 0$ then $\sigma_n(B) \geq \alpha_n(B) = \tilde{\alpha}_n(B)$. If $\tilde{\alpha}_n(B) < 0$ then it need not be true that $\alpha_n(B) = \tilde{\alpha}_n(B)$, but the inequality $\sigma_n(B) \geq \tilde{\alpha}_n(B)$ is trivially satisfied. Thus, in either case we have the desired lower bound

$$\sigma_n(A) = \sigma_n(B) \geq \tilde{\alpha}_n(B)$$

$$= \min_{1 \leq k \leq n} \left\{ |a_{kk}| - \tfrac{1}{2} \sum_{\substack{j \neq k \\ j=1}}^{n} |e^{i\theta_k} a_{kj} + e^{-i\theta_j} \bar{a}_{jk}| - \tfrac{1}{2} \sum_{j=n+1}^{m} |a_{jk}| \right\}$$

$$\geq \min_{1 \leq k \leq n} \left\{ |a_{kk}| - \tfrac{1}{2} \sum_{\substack{j \neq k \\ j=1}}^{n} (|a_{jk}| + |a_{kj}|) - \tfrac{1}{2} \sum_{j=n+1}^{m} |a_{jk}| \right\}$$

$$= \min_{1 \leq k \leq n} \{ |a_{kk}| - \tfrac{1}{2} [R_k'(A) + C_k'(A)] \} \qquad \square$$

Problems

1. What upper bound is obtained for $\sigma_1(A)$ if the Frobenius norm $\|A\|_2 = [\mathrm{tr}\, A^*A]^{\frac{1}{2}}$ is used in (3.7.1)?

2. Let $A \in M_n$ be a given nonsingular matrix, and let $\kappa(A) = \sigma_1(A)/\sigma_n(A)$ denote its spectral condition number. Using the notation of definitions (3.7.4,12), show that

$$\frac{\alpha_1(A)}{\sqrt{l_n(A)\, l_{n+1}(A)}} \leq \kappa(A) \leq \frac{\sqrt{l_1(A)\, l_2(A)}}{\alpha_n(A)} \tag{3.7.18}$$

Apply this to $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, for which $\kappa(A) \approx 2.62$. How good are the bounds?

3. For $A = [a_{ij}] \in M_n$, show that

$$\sigma_n(A) \geq \min_{1 \leq k \leq n} \tfrac{1}{2} \left[[4|a_{kk}|^2 + (R_k' - C_k')^2]^{\frac{1}{2}} - [R_k' + C_k'] \right] \tag{3.7.19}$$

Compare this lower bound with (3.7.11).

Notes and Further Readings. Most of the results in this section are taken from C. R. Johnson, A Geršgorin-Type Lower Bound for the Smallest Singular Value, *Linear Algebra Appl.* 112 (1989), 1-7. The lower bound in (3.7.19) resulted from a discussion with A. Hoffman.

3.8 Singular value weak majorization for bilinear products

Since the basic weak majorization inequalities (3.3.14a) for the ordinary matrix product have an analog for the Hadamard product (see Theorem

(5.5.4)), it is natural to try to characterize the bilinear products for which these inequalities hold. For convenience, we consider square matrices and use $\bullet$: $M_n \times M_n \rightarrow M_n$ to denote a bilinear function, which we interpret as a "product" $(A,B) \rightarrow A \bullet B$. Examples of such bilinear products are:

AB (ordinary product)

AB^T

A^TB

$A \circ B$ (Hadamard product)

$A \circ B^T$

$A^T \circ B$

$UAVBW$ for given U, V, $W \in M_n$

and any linear combination of these.

If $A = [a_{ij}] \in M_n$ is given and if $E_{ij} \in M_n$ has entry 1 in position i,j and zero entries elsewhere, notice that $\operatorname{tr} AE_{ij} = a_{ji}$. Thus, a given bilinear product $\bullet$: $M_n \times M_n \rightarrow M_n$ is completely determined by the values of $\operatorname{tr}(A \bullet B)C$ for all A, B, $C \in M_n$. Associated with any bilinear product $\bullet$ are two bilinear products $\bullet_L$ and $\bullet_R$ characterized by the adjoint-like identities

$$\operatorname{tr}(A \bullet_L B)C = \operatorname{tr} C(A \bullet_L B) = \operatorname{tr}(C \bullet A)B \text{ for all } A, B, C \in M_n \tag{3.8.1}$$

and

$$\operatorname{tr}(A \bullet_R B)C = \operatorname{tr} A(B \bullet C) \text{ for all } A, B, C \in M_n \tag{3.8.2}$$

The notations $\bullet_L$ and $\bullet_R$ are intended to remind us that the parentheses and position of $\bullet$ are moved one position to the left (respectively, right) in (3.8.1,2).

Exercise. If $\bullet$ is the usual matrix product $A \bullet B = AB$, use the identities (3.8.1,2) to show that $A \bullet_L B = AB$ and $A \bullet_R B = AB$.
Exercise. If $\bullet$ is the Hadamard product $A \bullet B = A \circ B$, show that $A \circ_L B = A^T \circ B$ and $A \circ_R B = A \circ B^T$.
Exercise. If $A \bullet B = UAVBW$ for given U, V, $W \in M_n$, show that $A \bullet_L B = VAWBU$ and $A \bullet_R B = WAUBV$.

3.8.3 **Theorem.** Let $\bullet$: $M_n \times M_n \rightarrow M_n$ be a given bilinear product. The following are equivalent:

(a) For all $A, B \in M_n$,

$$\sum_{i=1}^{k} \sigma_i(A \bullet B) \leq \sum_{i=1}^{k} \sigma_i(A)\sigma_i(B) \text{ for all } k = 1,\dots, n \tag{3.8.4}$$

(b) For all $A, B \in M_n$,

$$\sigma_1(A \bullet B) \leq \sigma_1(A)\sigma_1(B) \tag{3.8.5a}$$

$$\sum_{i=1}^{n} \sigma_i(A \bullet B) \leq \sigma_1(A)\sum_{i=1}^{n} \sigma_i(B) \tag{3.8.5b}$$

$$\sum_{i=1}^{n} \sigma_i(A \bullet B) \leq \sigma_1(B)\sum_{i=1}^{n} \sigma_i(A) \tag{3.8.5c}$$

(c) For all $A, B \in M_n$,

$$\sigma_1(A \bullet B) \leq \sigma_1(A)\sigma_1(B) \tag{3.8.6a}$$

$$\sigma_1(A \bullet_L B) \leq \sigma_1(A)\sigma_1(B) \tag{3.8.6b}$$

$$\sigma_1(A \bullet_R B) \leq \sigma_1(A)\sigma_1(B) \tag{3.8.6c}$$

(d) $|\operatorname{tr} P(Q \bullet R)| \leq 1$ for all partial isometries $P, Q, R \in M_n$ such that $\min\{\operatorname{rank} P, \operatorname{rank} Q, \operatorname{rank} R\} = 1$.

(e) $|\operatorname{tr} P(Q \bullet R)| \leq \min\{\operatorname{rank} P, \operatorname{rank} Q, \operatorname{rank} R\}$ for all partial isometries $P, Q, R \in M_n$.

Proof: The inequalities in (b) follow easily from (3.8.4) with $k = 1$ and $k = n$. Assuming (b), use Theorem (3.4.1), Theorem (3.3.13a), and (3.3.18) to compute

$$\sigma_1(A \bullet_L B) = \max\{|\operatorname{tr}(A \bullet_L B)C| : C \in M_n, \operatorname{rank} C = 1, \sigma_1(C) \leq 1\}$$

$$= \max\{|\operatorname{tr} B(C \bullet A)| : C \in M_n, \operatorname{rank} C = 1, \sigma_1(C) \leq 1\}$$

$$\leq \max\left\{\sum_{i=1}^{n}\sigma_i(B(C\bullet A)):\ C\in M_n,\ \text{rank } C=1,\ \sigma_1(C)\leq 1\right\}$$

$$\leq \max\left\{\sigma_1(B)\sum_{i=1}^{n}\sigma_i(C\bullet A):\ C\in M_n,\ \text{rank } C=1,\ \sigma_1(C)\leq 1\right\}$$

$$\leq \max\left\{\sigma_1(B)\sigma_1(A)\sum_{i=1}^{n}\sigma_i(C):\ C\in M_n,\ \text{rank } C=1,\ \sigma_1(C)\leq 1\right\}$$

$$=\sigma_1(A)\sigma_1(B)$$

The assertion (3.8.6c) for $\sigma_1(A\bullet_R B)$ follows in the same way. Now assume (c) and let $P, Q, R\in M_n$ be partial isometries with $\min\{\text{rank } P, \text{rank } Q, \text{rank } R\}=1$. If rank $P=1$, use (3.3.13a) again and (3.3.14a) to compute

$$|\text{tr } P(Q\bullet R)| \leq \sum_{i=1}^{n}\sigma_i(P(Q\bullet R))$$

$$\leq \sum_{i=1}^{n}\sigma_i(P)\sigma_i(Q\bullet R)=\sigma_1(P)\sigma_1(Q\bullet R)$$

$$=\sigma_1(Q\bullet R)\leq\sigma_1(Q)\sigma_1(R)=1$$

If rank $Q=1$ (respectively, rank $R=1$), one uses (3.8.6b) (respectively, (3.8.6c)) to reach the same conclusion.

Now assume (d) and suppose $r=\text{rank } P\leq\min\{\text{rank } Q, \text{rank } R\}$. Use Theorem (3.1.8a) to write $P=P_1+\cdots+P_r$ as a sum of mutually orthogonal rank one partial isometries and compute

$$|\text{tr } P(Q\bullet R)|=\left|\text{tr}\sum_{i=1}^{r}P_i(Q\bullet R)\right|=\left|\sum_{i=1}^{r}\text{tr } P_i(Q\bullet R)\right|$$

$$\leq\sum_{i=1}^{r}|\text{tr } P_i(Q\bullet R)|\leq r$$

If Q or R has the minimum rank, the same computation using linearity of $Q \bullet R$ in each factor leads to the desired conclusion.

Finally, assume (e) and use Theorem (3.1.8b) to write

$$A = \sum_{i=1}^{n} \alpha_i Q_i \text{ and } B = \sum_{i=1}^{n} \beta_i R_i$$

as a nonnegative linear combination of partial isometries, in which rank $Q_i =$ rank $R_i = i$ for $i = 1,\ldots, n$ and

$$\sum_{i=j}^{n} \alpha_i = \sigma_j(A) \text{ and } \sum_{i=j}^{n} \beta_i = \sigma_j(B) \text{ for } j = 1,\ldots, n$$

Let P be a given rank k partial isometry and compute

$$\begin{aligned} |\mathrm{tr}\, P(A \bullet B)| &= \Big|\mathrm{tr}\, P\Big[\Big[\sum_{i=1}^{n} \alpha_i Q_i\Big] \bullet \Big[\sum_{j=1}^{n} \beta_j R_j\Big]\Big]\Big| \\ &= \Big|\mathrm{tr}\, P\Big[\sum_{i,j=1}^{n} \alpha_i \beta_j Q_i \bullet R_j\Big]\Big| \\ &\le \sum_{i,j=1}^{n} \alpha_i \beta_j \,|\mathrm{tr}\, P(Q_i \bullet R_j)| \\ &\le \sum_{i,j=1}^{n} \alpha_i \beta_j \min\{k,i,j\} \\ &= \sum_{l=1}^{k} \Big[\sum_{i=l}^{n} \alpha_i\Big]\Big[\sum_{j=l}^{n} \beta_j\Big] \\ &= \sum_{l=1}^{k} \sigma_l(A)\sigma_l(B) \end{aligned}$$

The desired inequality now follows from Theorem (3.4.1). □

The criteria (3.8.6a-c) permit one to verify the basic weak majorization inequalities (3.8.4) for a given bilinear product $\bullet$ by checking simple submultiplicativity of the three products $\bullet$, $\bullet_L$, and $\bullet_R$ for the spectral norm. Since $\bullet = \bullet_L = \bullet_R$ for the ordinary product $A \bullet B = AB$, the criteria (3.8.6a-c) reduce to the single criterion $\sigma_1(AB) \le \sigma_1(A)\, \sigma_1(B)$, which is just the fact that the spectral norm is a matrix norm.

For the Hadamard product $\circ$, we have $A \circ_L B = A^T \circ B$ and $A \circ_R B = A \circ B^T$. Since $\sigma_1(X^T) = \sigma_1(X)$ for all $X \in M_n$, verifying the criteria (3.8.6a-c) for the Hadamard product reduces to checking that $\sigma_1(A \circ B) \le \sigma_1(A)\, \sigma_1(B)$, a fact that was proved by Schur in 1911 (see Problem 31 in Section (4.2), Theorem (5.5.1), and Problem 7 in Section (5.5) for four different proofs of this basic inequality).

There are interesting and natural bilinear products that do *not* satisfy the basic weak majorization inequalities (3.8.4); see Problem 3.

Problems

1. Show that $(\bullet_L)_L = \bullet_R$, $(\bullet_R)_R = \bullet_L$, and $(\bullet_R)_L = (\bullet_L)_R = \bullet$ for any bilinear product $\bullet\colon M_n \times M_n \to M_n$.

2. Show that any bilinear product $\bullet\colon M_n \times M_n \to M_n$ may be represented as

$$A \bullet B = \sum_{k=1}^{N} U_k A V_k B W_k \quad \text{for all } A, B \in M_n \tag{3.8.7}$$

where U_k, V_k, and W_k are determined by $\bullet$, and $N \le n^6$.

3. Let $n = pq$ and consider the block matrices $A = [A_{ij}]_{i,j=1}^{p}$, $B = [B_{ij}]_{i,j=1}^{p} \in M_n$, where each $A_{ij}, B_{ij} \in M_q$. Consider the two bilinear products $\square_1$ and $\square_2$ defined on $M_n \times M_n$ by

$$A \square_1 B = [A_{ij} B_{ij}]_{i,j=1}^{p} \tag{3.8.8}$$

and

$$A \square_2 B = \Big[\sum_{k=1}^{q} A_{ik} B_{kj} \Big]_{i,j=1}^{p} \tag{3.8.9}$$

(a) When $p = 1$, show that $\square_1$ is the usual matrix product and $\square_2$ is the Hadamard product, while if $q = 1$ then $\square_1$ is the Hadamard product and $\square_2$ is the usual matrix product. Thus, for a nonprime n, we have two natural discrete families of bilinear products on $M_n \times M_n$ that include the usual and Hadamard products.
(b) Use the defining identities (3.8.1,2) to show that

$$A\,(\square_1)_L\,B = A^{bt}\,\square_1\,B \text{ and } A\,(\square_1)_R\,B = A\,\square_1\,B^{bt} \tag{3.8.10a}$$

and

$$A\,(\square_2)_L\,B = (A^{bt})^T\,\square_2\,B \text{ and } A\,(\square_2)_R\,B = A\,\square_2\,(B^{bt})^T \tag{3.8.10b}$$

where $A^{bt} \equiv [A_{ji}]_{i,j=1}^p$ denotes the *block transpose*; notice that $(A^{bt})^T = [A_{ij}^T]_{i,j=1}^p$, that is, each block in $(A^{bt})^T$ is the ordinary transpose of the corresponding block in A.
(c) Consider $p = q = 2$, $n = 4$,

$$A = \begin{bmatrix} E_{21} & E_{11} \\ E_{22} & E_{12} \end{bmatrix} \text{ and } B = \begin{bmatrix} E_{12} & E_{22} \\ E_{11} & E_{21} \end{bmatrix}$$

and verify that $\square_1$ does not satisfy the inequalities (3.8.6b,c).
(d) Consider $p = q = 2$, $n = 4$,

$$A = \begin{bmatrix} E_{12} & E_{22} \\ E_{11} & E_{21} \end{bmatrix} \text{ and } B = \begin{bmatrix} E_{21} & E_{11} \\ E_{22} & E_{12} \end{bmatrix}$$

and verify that $\square_2$ does not satisfy the inequalities (3.8.6b,c).
(e) Use the method in Problem 7 of Section (5.5) to show that $\square_1$ and $\square_2$ satisfy the inequality (3.8.6a).

Thus, the bilinear products $\square_1$ and $\square_2$ are always submultiplicative with respect to the spectral norm but do not always satisfy the basic weak majorization inequalities (3.8.4).

Notes and Further Readings. The results in this section are taken from R. A. Horn, R. Mathias, and Y. Nakamura, Inequalities for Unitarily Invariant Norms and Bilinear Matrix Products, *Linear Multilinear Algebra* 30 (1991), 303-314, which contains additional results about the bilinear products $\square_1$ and $\square_2$. For example, if we let $\bullet$ denote either $\square_1$ or $\square_2$, there is a common generalization of the inequalities (3.5.22) and (3.5.24) for the ordinary and Hadamard products:

$$\|A \bullet B\|^2 \leq \|A^*A\| \, \|B^*B\| \tag{3.8.11}$$

for all unitarily invariant norms $\|\cdot\|$ on M_n. Moreover, there is also a generalization of the submultiplicativity criterion in Problem 3 of Section (3.5): For a given unitarily invariant norm $\|\cdot\|$, $\|A \bullet B\| \leq \|A\| \, \|B\|$ for all $A, B \in M_n$ if and only if $\|A\| \geq \sigma_1(A)$ for all $A \in M_n$. Thus, it is possible for a bilinear product to share many important properties with the ordinary (and Hadamard) product without satisfying the weak majorization inequalities (3.8.4). See also C. R. Johnson and P. Nylen, Converse to a Singular Value Inequality Common to the Hadamard and Conventional Product, *Linear Multilinear Algebra* 27 (1990), 167-187.