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CHAPTER 3

CONFORMING FINITE ELEMENT METHODS FOR SECOND-ORDER PROBLEMS

Introduction

In this chapter, we consider the problem of determining estimates in various norms of the difference $(u - u_h)$, where $u \in V$ is the solution of a second-order boundary value problem and $u_h \in V_h$ is the discrete solution obtained in a subspace V_h of V.

From Céa's lemma (Theorem 2.4.1), the best error estimate would result in exhibiting the element $\theta_h u \in V_h$ which is such that $\inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega} = \|u - \theta_h u\|_{1,\Omega}$, i.e., the projection of the solution u on the space V_h. However, such a projection is not particularly easy to work with, and it turns out that it is much more convenient to use the *X_h-interpolant $\Pi_h u$ of the solution u*, so that we shall get instead the error estimate $\|u - u_h\|_{1,\Omega} \leq C\|u - \Pi_h u\|_{1,\Omega}$.

Since we shall assume in this chapter that the set $\bar{\Omega}$ is polygonal, it can be written as a union $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$ of polygonal finite elements K, such as the ones which have been heretofore described. This in turn implies that the corresponding spaces V_h will be contained in the space V (the domains of definition of their functions are identical), i.e., that the corresponding finite element method is indeed *conforming*.

Taking into account that we are using the norm $\|\cdot\|_{1,\Omega}$ and that $(\Pi_h u)|_K = \Pi_K u$ for all $K \in \mathscr{C}_h$ (Theorem 2.3.2), we can write

$$\|u - \Pi_h u\|_{1,\Omega} = \Big(\sum_{K \in \mathcal{T}_h} \|u - \Pi_K u\|_{1,K}^2\Big)^{1/2}.$$

Therefore, the problem of finding an estimate for the error $\|u - u_h\|_{1,\Omega}$ is reduced to the problem of evaluating quantities such as $\|u - \Pi_K u\|_{1,K}$ and the solution of such "*local*" *interpolation problems* is the object of Section 3.1. In view of other future needs, we shall in fact estimate the difference $(u - \Pi_K u)$ with respect to more general norms and seminorms.

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A typical – and crucial – result in this direction is that, for a finite element (K, P_K, Σ_K) *which can be imbedded in an affine family* and whose P_K-interpolation operator leaves invariant the polynomials of degree $\leq k$ (equivalently, the inclusions $P_k(K) \subset P_K$ hold), there exists a constant C independent of K such that

$$\forall v \in H^{k+1}(K), \quad |v - \Pi_K v|_{m,K} \leq C \frac{h_K^{k+1}}{\rho_K^m} |v|_{k+1,K}, \quad 0 \leq m \leq k+1,$$

where

h_K = diameter of K,
ρ_K = supremum of the diameters of the spheres inscribed in K.

Such a result is proved (in a more general form) in Theorem 3.1.5.

One key idea in the process of getting such estimates is to go from any finite element K of an affine family to the reference finite element of the family and then back to the finite element K.

Another key is to use *a basic result about Sobolev spaces, due to J. Deny and J.L. Lions, which pervades the mathematical analysis of the finite element method*: Over the quotient space $H^{k+1}(\Omega)/P_k(\Omega)$, the semi-norm $|\cdot|_{k+1,\Omega}$ is a norm equivalent to the quotient norm. This result is proved in Theorem 3.1.1, for the more general *Sobolev spaces* $W^{m,p}(\Omega)$.

In practice, one often considers a *regular family* of finite elements, in the sense that the diameters h_K approach zero, and that there exists a constant σ independent of K such that $h_K \leq \sigma\rho_K$. For such a regular family, the previous interpolation error estimate becomes (Theorem 3.1.6)

$$|v - \Pi_K v|_{m,K} = O(h_K^{k+1-m}), \quad 0 \leq m \leq k+1.$$

Using Céa's lemma, we obtain in Section 3.2 the error estimates (Theorem 3.2.2).

$$\|u - u_h\|_{1,\Omega} \leq C\|u - \Pi_h u\|_{1,\Omega} = O(h^k), \quad \text{with } h = \max_{K \in \mathcal{T}_h} h_K,$$

under basically the same assumptions as before about the family of finite elements which make up the finite element spaces. It is worth mentioning here that, although the above error estimate is not the best, it is generally possible to show that the order of convergence *is* the best possible: In other words, it would not be improved by replacing $\Pi_h u$ by $\theta_h u$.

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Nevertheless the range of applicability of the above results is limited inasmuch as the X_h-interpolant of the solution u is defined only if some smoothness is assumed on the solution u, and likewise, the above error estimates are obtained provided the solution is sufficiently smooth ($u \in H^{k+1}(\Omega)$). Fortunately, we show in Theorem 3.2.3 that, with the minimal assumptions that the solution u is in the space $H^1(\Omega)$ and that the spaces P_K contain the space $P_1(K)$, $K \in \mathcal{T}_h$, convergence still holds, i.e., one has $\lim_{h\to 0} \|u - u_h\|_{1,\Omega} = 0$.

Next, using a method due to J.P. Aubin and J.A. Nitsche (cf. the Aubin–Nitsche lemma; Theorem 3.2.4), we show that there is in most cases an improvement in the error estimate in the norm $|\cdot|_{0,\Omega}$ in the sense that (Theorem 3.2.5)

$$|u - u_h|_{0,\Omega} = \mathrm{O}(h^{k+1}).$$

Section 3.2 ends up with the so-called *inverse inequalities* (Theorem 3.2.6).

Finally, in Section 3.3, we follow the penetrating *method of weighted norms of J.A. Nitsche*, who has recently shown that, if $u \in W^{k+1,\infty}(\Omega)$,

$$|u - u_h|_{0,\infty,\Omega} = \begin{cases} \mathrm{O}(h^{2-\epsilon}) & \text{for any } \epsilon > 0 \quad \text{if } k = 1, \\ \mathrm{O}(h^{k+1}) & \text{if } k \geq 2, \end{cases}$$

$$\|u - u_h\|_{1,\infty,\Omega} = \begin{cases} \mathrm{O}(h^{1-\epsilon}) & \text{for any } \epsilon > 0 \quad \text{if } k = 1, \\ \mathrm{O}(h^{k}) & \text{if } k \geq 2, \end{cases}$$

where $|\cdot|_{0,\infty,\Omega}$ and $\|\cdot\|_{1,\infty,\Omega}$ stand for the norms of the spaces $L^\infty(\Omega)$ and $W^{1,\infty}(\Omega)$, respectively. Restricting ourselves for brevity to the case $k = 1$, the corresponding error estimates are obtained in Theorem 3.3.7.

It is worth pointing out that *all the error estimates found in Section* 3.2 *and* 3.3 *are optimal* in the sense that, with the same regularity assumptions on the function u, one gets the same asymptotic estimates (or "almost" the same for the norms $|\cdot|_{0,\infty,\Omega}$ and $\|\cdot\|_{1,\infty,\Omega}$ when $k = 1$) when the discrete solution $u_h \in V_h$ is replaced by the X_h-interpolant $\Pi_h u \in V_h$.

3.1. Interpolation theory in Sobolev spaces

The Sobolev spaces $W^{m,p}(\Omega)$. The quotient space $W^{k+1,p}(\Omega)/P_k(\Omega)$

We shall consider the *Sobolev space* $W^{m,p}(\Omega)$ which, for any integer $m \geq 0$, and any number p satisfying $1 \leq p \leq \infty$, consists of those func-

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tions $v \in L^p(\Omega)$ for which all partial derivatives $\partial^\alpha v$ (in the distribution sense) with $|\alpha| \leq m$ belong to the space $L^p(\Omega)$. *Equipped with the norm*

$$\begin{cases} \|v\|_{m,p,\Omega} = \left(\sum_{|\alpha|\leq m} \int_\Omega |\partial^\alpha v|^p \,\mathrm{d}x\right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \|v\|_{m,\infty,\Omega} = \max_{|\alpha|\leq m} \{\operatorname{ess.sup}_{x\in\Omega} |\partial^\alpha v(x)|\} & \text{if } p = \infty, \end{cases} \tag{3.1.1}$$

the space $W^{m,p}(\Omega)$ *is a Banach space.* We shall also use the semi-norms

$$\begin{cases} |v|_{m,p,\Omega} = \left(\sum_{|\alpha|= m} \int_\Omega |\partial^\alpha v|^p \,\mathrm{d}x\right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ |v|_{m,\infty,\Omega} = \max_{|\alpha|= m} \left\{\operatorname{ess.sup}_{x\in\Omega} |\partial^\alpha v(x)|\right\} & \text{if } p = \infty. \end{cases} \tag{3.1.2}$$

The *Sobolev space* $W_0^{m,p}(\Omega)$ is the closure of the space $\mathscr{D}(\Omega)$ in the space $W^{m,p}(\Omega)$.

Given a subset A of $\mathbf{R}^n$ and given a function $v \in \mathscr{C}^m(A)$, the notation $\|v\|_{m,\infty,A}$ and $|v|_{m,\infty,A}$ will also denote the norm $\max_{|\alpha|\leq m} \sup_{x\in A} |\partial^\alpha v(x)|$ and the semi-norm $\max_{|\alpha|=m} \sup_{x\in A} |\partial^\alpha v(x)|$, respectively. Notice that

$$W^{m,2}(\Omega) = H^m(\Omega), \quad W_0^{m,2}(\Omega) = H_0^m(\Omega),$$
$$\|\cdot\|_{m,2,\Omega} = \|\cdot\|_{m,\Omega}, \quad |\cdot|_{m,2,\Omega} = |\cdot|_{m,\Omega}.$$

As usual, *the open sets Ω that will be considered in this section will be assumed to have a Lipschitz-continuous boundary.* In addition they will be assumed to be connected when needed (this assumption is used in the proof of Theorem 3.1.1).

In view of future needs, we shall record here some basic properties of the Sobolev spaces that will be often used. In what follows, the notation $X \hookrightarrow Y$ indicates that the normed linear space X is contained in the normed linear space Y with a continuous injection, and the notation $x \overset{c}{\subset} Y$ indicates in addition the compactness of the injection. Finally, for any integer $m \geq 0$ and any number $\alpha \in]0,1]$, $\mathscr{C}^{m,\alpha}(\bar{\Omega})$ denotes the space of all functions in $\mathscr{C}^m(\bar{\Omega})$ whose m-th derivatives satisfy a Hölder's condition with exponent α. Equipped with the norm

$$\|v\|_{\mathscr{C}^{m,\alpha}(\bar{\Omega})} = \|v\|_{m,\infty,\bar{\Omega}} + \max_{|\beta|=m} \sup_{\substack{x,y\in\bar{\Omega} \\ x\neq y}} \frac{|\partial^\beta v(x) - \partial^\beta v(y)|}{\|x-y\|^\alpha},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}^n$, the space $\mathscr{C}^{m,\alpha}(\bar{\Omega})$ is a Banach space.

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By the *Sobolev's imbedding theorems*, the following inclusions hold, for all integers $m \geq 0$ and all $1 \leq p \leq \infty$,

$$\left.\begin{array}{lll} W^{m,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) & \text{with } \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{m}{n}, & \text{if } m < \dfrac{n}{p}, \\ W^{m,p}(\Omega) \hookrightarrow L^{q}(\Omega) & \text{for all } q \in [1, \infty[, & \text{if } m = \dfrac{n}{p}, \\ W^{m,p}(\Omega) \hookrightarrow \mathscr{C}^{0, m-(n/p)}(\bar{\Omega}), & & \text{if } \dfrac{n}{p} < m < \dfrac{n}{p} + 1, \\ W^{m,p}(\Omega) \hookrightarrow \mathscr{C}^{0,\alpha}(\bar{\Omega}) & \text{for all } 0 < \alpha < 1, & \text{if } m = \dfrac{n}{p} + 1, \\ W^{m,p}(\Omega) \hookrightarrow \mathscr{C}^{0,1}(\bar{\Omega}), & & \text{if } \dfrac{n}{p} + 1 < m. \end{array}\right\} \quad (3.1.3)$$

By the *Kondrasov theorems*, the compact injections

$$\left.\begin{array}{lll} W^{m,p}(\Omega) \overset{c}{\subset} L^{q}(\Omega) & \text{for all } 1 \leq q < p^*, & \\ & \text{with } \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{m}{n}, & \text{if } m < \dfrac{n}{p}, \\ W^{m,p}(\Omega) \overset{c}{\subset} L^{q}(\Omega) & \text{for all } q \in [1, \infty[, & \text{if } m = \dfrac{n}{p}, \\ W^{m,p}(\Omega) \overset{c}{\subset} \mathscr{C}^{0}(\bar{\Omega}), & & \text{if } \dfrac{n}{p} < m, \end{array}\right\} \quad (3.1.4)$$

hold for all $1 \leq p \leq \infty$. The compact injection

$$H^1(\Omega) \overset{c}{\subset} L^2(\Omega)$$

(i.e., the special case $p = q = 2$) is known as the *Rellich theorem*.

Of course, analogous inclusion can be derived by "translating" the orders of derivations. Thus for instance, one has $W^{m+r,p}(\Omega) \hookrightarrow W^{r,p^*}(\Omega)$ if $m < \frac{n}{p}$, etc...

We also note that, for $1 \leq p < \infty$, one has

$$(\mathscr{C}^{\infty}(\bar{\Omega}))^{-} = W^{m,p}(\Omega).$$

Remark 3.1.1. The assumption that the boundary is Lipschitz-continuous is not always necessary for proving the above properties. For example, one can derive the compact inclusion $W^{1,p}(\Omega) \overset{c}{\subset} L^{q}(\Omega)$ for all $1 \leq q \leq p$, or the above density property, as long as the boundary of Ω is continuous and the set Ω is bounded, etc... □

Since an open set Ω with a Lipschitz-continuous boundary is bounded, it makes sense to consider the *quotient space* $W^{k+1,p}(\Omega)/P_k(\Omega)$. This space is a Banach space when it is equipped with the *quotient norm*

$$\dot{v} \in W^{k+1,p}(\Omega)/P_k(\Omega) \to \|\dot{v}\|_{k+1,p,\Omega} = \inf_{p \in P_k(\Omega)} \|v + p\|_{k+1,p,\Omega}, \tag{3.1.5}$$

where

$$\dot{v} = \{w \in W^{k+1,p}(\Omega);\quad (w - v) \in P_k(\Omega)\} \tag{3.1.6}$$

denotes the equivalence class of the element $v \in W^{k+1,p}(\Omega)$.

Then the mapping

$$\dot{v} \in W^{k+1,p}(\Omega)/P_k(\Omega) \to |\dot{v}|_{k+1,p,\Omega} = |v|_{k+1,p,\Omega} \tag{3.1.7}$$

is *a priori* only a semi-norm on the quotient space $W^{k+1,p}(\Omega)/P_k(\Omega)$, which satisfies the inequality

$$\forall \dot{v} \in W^{k+1,p}(\Omega)/P_k(\Omega), \quad |\dot{v}|_{k+1,p,\Omega} \leq \|\dot{v}\|_{k+1,p,\Omega} \tag{3.1.8}$$

(to see this, observe that, for any polynomial $p \in P_k(\Omega)$,

$$\|v + p\|_{k+1,p,\Omega} = (|v|^p_{k+1,p,\Omega} + \|v + p\|^p_{k,p,\Omega})^{1/p} \geq |v|_{k+1,p,\Omega},$$

with the standard modification for $p = \infty$). It is a fundamental result that *it is in fact a norm over the quotient space, equivalent to the quotient norm* (3.1.5), as we now prove (cf. Exercise 3.1.1 for a generalization).

Theorem 3.1.1. *There exists a constant $C(\Omega)$ such that*

$$\forall v \in W^{k+1,p}(\Omega), \quad \inf_{p \in P_k(\Omega)} \|v + p\|_{k+1,p,\Omega} \leq C(\Omega)|v|_{k+1,p,\Omega} \tag{3.1.9}$$

and consequently, one has

$$\forall \dot{v} \in W^{k+1,p}(\Omega)/P_k(\Omega), \quad \|\dot{v}\|_{k+1,p,\Omega} \leq C(\Omega)|\dot{v}|_{k+1,p,\Omega}. \tag{3.1.10}$$

Proof. Let $N = \dim P_k(\Omega)$ and let f_i, $1 \leq i \leq N$, be a basis of the dual space of $P_k(\Omega)$. Using the Hahn–Banach extension theorem, there exist continuous linear forms over the space $W^{k+1,p}(\Omega)$, again denoted f_i, $1 \leq i \leq N$, such that for any $p \in P_k(\Omega)$, we have $f_i(p) = 0$, $1 \leq i \leq N$, if and only if $p = 0$. We will show that there exists a constant $C(\Omega)$ such that

$$\forall v \in W^{k+1,p}(\Omega), \|v\|_{k+1,p,\Omega} \leq C(\Omega)\Big(|v|_{k+1,p,\Omega} + \sum_{i=1}^{N} |f_i(v)|\Big). \tag{3.1.11}$$

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Inequality (3.1.9) will then be a consequence of inequality (3.1.11): Given any function $v \in W^{k+1,p}(\Omega)$, let $q \in P_k(\Omega)$ be such that $f_i(v+q) = 0$, $1 \leq i \leq N$. Then, by (3.1.11),

$$\inf_{p \in P_k(\Omega)} \|v+p\|_{k+1,p,\Omega} \leq \|v+q\|_{k+1,p,\Omega} \leq C(\Omega)|v|_{k+1,p,\Omega},$$

which proves (3.1.9). If inequality (3.1.11) is false, there exists a sequence $(v_l)_{l=1}^{\infty}$ of functions $v_l \in W^{k+1,p}(\Omega)$, such that

$$\forall l \geq 1, \quad \|v_l\|_{k+1,p,\Omega} = 1, \quad \text{and} \quad \lim_{l \to \infty} \left(|v_l|_{k+1,p,\Omega} + \sum_{i=1}^{N} |f_i(v_l)| \right) = 0. \tag{3.1.12}$$

Since the sequence (v_l) is bounded in $W^{k+1,p}(\Omega)$, there exists a subsequence, again denoted (v_l), and a function $v \in W^{k,p}(\Omega)$, such that

$$\lim_{l \to \infty} \|v_l - v\|_{k,p,\Omega} = 0 \tag{3.1.13}$$

(this follows from the Kondrasov or Rellich theorems for $1 \leq p < \infty$ and from Ascoli's Theorem for $p = \infty$). Since, by (3.1.12),

$$\lim_{l \to \infty} |v_l|_{k+1,p,\Omega} = 0, \tag{3.1.14}$$

and since the space $W^{k+1,p}(\Omega)$ is complete, we conclude from (3.1.13) and (3.1.14) that the sequence (v_l) converges in the space $W^{k+1,p}(\Omega)$. The limit v of this sequence is such that

$$\forall \alpha \quad \text{with} \quad |\alpha| = k+1, \quad |\partial^\alpha v|_{0,p,\Omega} = \lim_{l \to \infty} |\partial^\alpha v_l|_{0,p,\Omega} = 0,$$

and thus $\partial^\alpha v = 0$ for all multi-index α with $|\alpha| = k+1$. With the connectedness of Ω, it follows from distribution theory that the function v is a polynomial of degree $\leq k$. Using (3.1.12), we have

$$f_i(v) = \lim_{l \to \infty} f_i(v_l) = 0,$$

so that we conclude that $v = 0$, from the properties of the linear forms f_i. But this contradicts the equality $\|v_l\|_{k+1,p,\Omega} = 1$ for all l. □

Error estimates for polynomial preserving operators

Our main objective in this section is to estimate the *interpolation errors* $|v - \Pi_K v|_{m,q,K}$ and $\|v - \Pi_K v\|_{m,q,K}$, where Π_K is the P_K-interpolation operator associated with some finite element. At other places, however,

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we shall need similar estimates, but for more general polynomial preserving operators, i.e., not necessarily of interpolation type. This is why we shall develop an approximation theory valid also for such general operators.

To begin with, we need a definition: We shall say that two open subsets Ω and $\hat{\Omega}$ of $\mathbf{R}^n$ are *affine-equivalent* if there exists an invertible affine mapping

$$F: \hat{x} \in \mathbf{R}^n \to F(\hat{x}) = B\hat{x} + b \in \mathbf{R}^n \tag{3.1.15}$$

such that

$$\Omega = F(\hat{\Omega}). \tag{3.1.16}$$

As in the case of affine-equivalent finite elements (compare with (2.3.16)–(2.3.17)), we shall use the correspondences

$$\hat{x} \in \hat{\Omega} \to x = F(\hat{x}) \in \Omega, \tag{3.1.17}$$

$$(\hat{v}: \hat{\Omega} \to \mathbf{R}) \to (v = \hat{v} \cdot F^{-1}: \Omega \to \mathbf{R}), \tag{3.1.18}$$

between the points $\hat{x} \in \hat{\Omega}$ and $x \in \Omega$, and between functions defined over the set $\hat{\Omega}$ and the set Ω. Notice that we have

$$\hat{v}(\hat{x}) = v(x) \tag{3.1.19}$$

for all points $\hat{x}$, x in the correspondence (3.1.17) and all functions $\hat{v}$, v in the correspondence (3.1.18).

Remark 3.1.2. In case the functions v and $\hat{v}$ are defined only almost everywhere (as in the next theorem for instance), it is understood that relation (3.1.19) is to hold for almost all points $\hat{x} \in \hat{\Omega}$, and thus for almost all points $x \in \Omega$. □

We need to know how the Sobolev semi-norms defined in (3.1.2) behave from an open set to an affine-equivalent one. This is the object of the next theorem.

Here and subsequently, $\|\cdot\|$ stands for both the Euclidean norm in $\mathbf{R}^n$ and the associated matrix norm.

Theorem 3.1.2. *Let Ω and $\hat{\Omega}$ be two affine-equivalent open subsets of $\mathbf{R}^n$. If a function v belongs to the space $W^{m,p}(\Omega)$ for some integer $m \geq 0$ and some number $p \in [1, \infty]$, the function $\hat{v} = v \cdot F$ belongs to the space*

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$W^{m,p}(\hat{\Omega})$, and in addition, there exists a constant $C = C(m, n)$ such that

$$\forall v \in W^{m,p}(\Omega), \quad |\hat{v}|_{m,p,\hat{\Omega}} \leq C\|B\|^m |\det(B)|^{-1/p} |v|_{m,p,\Omega}, \tag{3.1.20}$$

where B is the matrix occurring in the mapping F of (3.1.15).

Analogously, one has

$$\forall \hat{v} \in W^{m,p}(\hat{\Omega}), \quad |v|_{m,p,\Omega} \leq C\|B^{-1}\|^m |\det(B)|^{1/p} |\hat{v}|_{m,p,\hat{\Omega}}. \tag{3.1.21}$$

Proof. (i) Let us first assume that the function v belongs to the space $\mathscr{C}^m(\bar{\Omega})$, so that the function $\hat{v}$ belongs to the space $\mathscr{C}^m(\hat{\bar{\Omega}})$.

Since, for any multi-index α with $|\alpha| = m$, one has

$$\partial^\alpha \hat{v}(\hat{x}) = D^m \hat{v}(\hat{x})(e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_m}),$$

where the vectors e_{α_i}, $1 \leq i \leq m$, are some of the basis vectors of $\mathbf{R}^n$, we deduce that

$$|\partial^\alpha \hat{v}(\hat{x})| \leq \|D^m \hat{v}(\hat{x})\| = \sup_{\substack{\|\xi_i\| \leq 1 \\ 1 \leq i \leq m}} |D^m \hat{v}(\hat{x})(\xi_1, \xi_2, \ldots, \xi_m)|.$$

Consequently we obtain

$$\begin{aligned} |\hat{v}|_{m,p,\hat{\Omega}} &= \left(\int_{\hat{\Omega}} \sum_{|\alpha|=m} |\partial^\alpha \hat{v}(\hat{x})|^p \, d\hat{x} \right)^{1/p} \\ &\leq C_1(m, n) \left(\int_{\hat{\Omega}} \|D^m \hat{v}(\hat{x})\|^p \, d\hat{x} \right)^{1/p}, \end{aligned} \tag{3.1.22}$$

where the constant $C_1(m, n)$ may be chosen as

$$C_1(m, n) = \sup_{1 \leq p} (\operatorname{card}\{\alpha \in N^m; \quad |\alpha| = m\})^{1/p}.$$

Using the differentiation rule for composition of functions, we note that for any vectors $\xi_i \in \mathbf{R}^n$, $1 \leq i \leq m$,

$$D^m \hat{v}(\hat{x})(\xi_1, \xi_2, \ldots, \xi_m) = D^m v(x)(B\xi_1, B\xi_2, \ldots, B\xi_m),$$

so that

$$\|D^m \hat{v}(\hat{x})\| \leq \|D^m v(x)\| \, \|B\|^m,$$

and therefore,

$$\int_{\hat{\Omega}} \|D^m \hat{v}(\hat{x})\|^p \, d\hat{x} \leq \|B\|^{mp} \int_{\hat{\Omega}} \|D^m v(F(\hat{x}))\|^p \, d\hat{x}. \tag{3.1.23}$$

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Using the formula of change of variables in multiple integrals, we get

$$\int_{\hat{\Omega}} \|D^m v(F(\hat{x}))\|^p \,\mathrm{d}\hat{x} = |\det(B^{-1})| \int_{\Omega} \|D^m v(x)\|^p \,\mathrm{d}x. \tag{3.1.24}$$

Since there exists a constant $C_2(m, n)$ such that

$$\|D^m v(x)\| \leq C_2(m, n) \max_{|\alpha|=m} |\partial^\alpha v(x)|,$$

we obtain

$$\left(\int_{\Omega} \|D^m v(x)\|^p \,\mathrm{d}x\right)^{1/p} \leq C_2(m, n)|v|_{m,p,\Omega}. \tag{3.1.25}$$

Inequality (3.1.20) is then a consequence of inequalities (3.1.22), (3.1.23), (3.1.24) and (3.1.25).

(ii) To complete the proof when $p \neq \infty$, it remains to use the continuity of the linear operator ι: $v \in C^m(\bar{\Omega}) \to \hat{v} \in W^{m,p}(\hat{\Omega})$ with respect to the norms $\|\cdot\|_{m,p,\Omega}$ and $\|\cdot\|_{m,p,\hat{\Omega}}$, the denseness of the space $\mathscr{C}^m(\bar{\Omega})$ in the space $W^{m,p}(\Omega)$, and the definition of the (unique) extension of the mapping ι to the space $W^{m,p}(\Omega)$.

(iii) Let us finally consider the case $p = \infty$. A function $v \in W^{m,\infty}(\Omega)$ belongs to the spaces $W^{m,p}(\Omega)$ for all $p < \infty$ (recall that the assumption of Lipschitz-continuity of the boundary implies the boundedness of the set Ω). Therefore, by part (ii), the function $\hat{v}$ belongs to the spaces $W^{m,p}(\hat{\Omega})$ for all $p < \infty$, and there exists a constant $C(m, n)$ such that

$$\forall p \geq 1, \quad \forall \alpha \in \mathbb{N}^m, \quad |\alpha| \leq m,$$
$$|\partial^\alpha \hat{v}|_{0,p,\hat{\Omega}} \leq |\hat{v}|_{|\alpha|,p,\hat{\Omega}} \leq C(m, n)\|B\|^{|\alpha|} \sup_{1 \leq p} |\det(B)|^{-1/p} \|v\|_{m,p,\Omega}.$$

Since the upper bound on the semi-norm $|\partial^\alpha v|_{0,p,\hat{\Omega}}$ is independent of the number p, this shows that, for each $|\alpha| \leq m$, the function $\partial^\alpha \hat{v}$ is in the space $L^\infty(\Omega)$ for each $|\alpha| \leq m$. Consequently, the function $\hat{v}$ belongs to the space $W^{m,\infty}(\hat{\Omega})$. To conclude, it suffices to use inequality (3.1.21) for all $p \geq 1$ in conjunction with the property that for any function $w \in L^\infty(\Omega)$, Ω bounded, one has

$$|w|_{0,\infty,\Omega} = \lim_{p \to \infty} |w|_{0,p,\Omega}.$$

Inequality (3.1.21) is proved in a similar fashion. □

To apply Theorem 3.1.2, it is desirable to evaluate the norms $\|B\|$ and

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$\|B^{-1}\|$ in terms of simple geometric quantities. This is the object of the next theorem, where we use the following notations:

$$h = \text{diam}(\Omega),\ \hat{h} = \text{diam}(\hat{\Omega}), \tag{3.1.26}$$

$$\left.\begin{aligned} \rho &= \sup\{\text{diam}(S);\ S \text{ is a ball contained in } \Omega\}, \\ \hat{\rho} &= \sup\{\text{diam}(\hat{S});\ \hat{S} \text{ is a ball contained in } \hat{\Omega}\}. \end{aligned}\right\} \tag{3.1.27}$$

Theorem 3.1.3. *Let $\hat{\Omega}$ and $\Omega = F(\hat{\Omega})$ be two affine-equivalent open subsets of $\mathbf{R}^n$, where $F: \hat{x} \in \mathbf{R}^n \to (B\hat{x} + b) \in \mathbf{R}^n$ is an invertible affine mapping. Then the upper bounds*

$$\|B\| \leq \frac{h}{\hat{\rho}} \quad \textit{and} \quad \|B^{-1}\| \leq \frac{\hat{h}}{\rho} \tag{3.1.28}$$

hold.

Proof. We may write

$$\|B\| = \frac{1}{\hat{\rho}} \sup_{\|\xi\| = \hat{\rho}} \|B\xi\|.$$

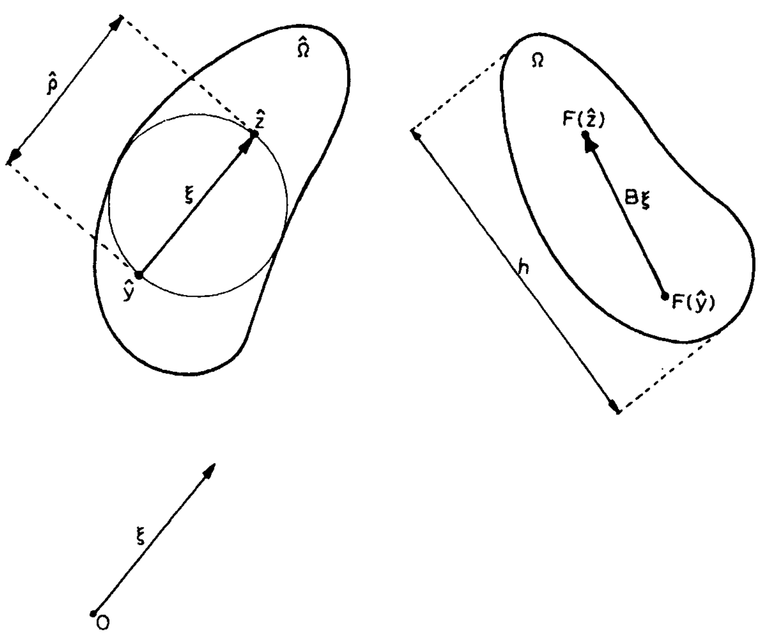

Fig. 3.1.1

Given a vector ξ satisfying $\|\xi\| = \hat{\rho}$, there exist two points $\hat{y}, \hat{z} \in \bar{\hat{\Omega}}$ such that $\hat{y} - \hat{z} = \xi$, by definition of $\hat{\rho}$ (Fig. 3.1.1). Since $B\xi = F(\hat{y}) - F(\hat{z})$ with $F(\hat{y}) \in \bar{\Omega}$, $F(\hat{z}) \in \bar{\Omega}$, we deduce that $\|B\xi\| \leq h$, and thus the first inequality (3.1.28) is proved. The other inequality is proved in a similar fashion. □

We are now in a position to prove an important property of *polynomial preserving operators*, i.e., which satisfy a relation of the form (3.1.30) below for some integer $k \geq 0$.

Theorem 3.1.4. *For some integers $k \geq 0$ and $m \geq 0$ and some numbers $p, q \in [1, \infty]$, let $W^{k+1,p}(\hat{\Omega})$ and $W^{m,q}(\hat{\Omega})$ be Sobolev spaces satisfying the inclusion*

$$W^{k+1,p}(\hat{\Omega}) \hookrightarrow W^{m,q}(\hat{\Omega}), \tag{3.1.29}$$

and let $\hat{\Pi} \in \mathscr{L}(W^{k+1,p}(\hat{\Omega});\ W^{m,q}(\hat{\Omega}))$ be a mapping such that

$$\forall \hat{p} \in P_k(\hat{\Omega}), \quad \hat{\Pi}\hat{p} = \hat{p}. \tag{3.1.30}$$

For any open set Ω which is affine-equivalent to the set $\hat{\Omega}$, let the mapping Π_Ω be defined by

$$(\Pi_\Omega v)\hat{} = \hat{\Pi}\hat{v}, \tag{3.1.31}$$

for all functions $\hat{v} \in W^{k+1,p}(\hat{\Omega})$ and $v \in W^{k+1,p}(\Omega)$ in the correspondence (3.1.18). *Then there exists a constant $C(\hat{\Pi}, \hat{\Omega})$ such that, for all affine-equivalent sets Ω,*

$$\forall v \in W^{k+1,p}(\Omega), \quad |v - \Pi_\Omega v|_{m,q,\Omega} \leq \\ \leq C(\hat{\Pi}, \hat{\Omega})(\operatorname{meas}(\Omega))^{(1/q)-(1/p)} \frac{h^{k+1}}{\rho^m} |v|_{k+1,p,\Omega}, \tag{3.1.32}$$

with h and ρ defined as in (3.1.26) *and* (3.1.27) *respectively.*

Proof. Using the polynomial invariance (3.1.30), we obtain the identity

$$\forall \hat{v} \in W^{k+1,p}(\hat{\Omega}), \quad \forall \hat{p} \in P_k(\hat{\Omega}), \quad \hat{v} - \hat{\Pi}\hat{v} = (I - \hat{\Pi})(\hat{v} + \hat{p}),$$

where I, the identity mapping from $W^{k+1,p}(\hat{\Omega})$ into $W^{m,q}(\hat{\Omega})$, is con-

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tinuous by (3.1.29). From this identity we deduce that

$$|\hat{v} - \hat{\Pi}\hat{v}|_{m,q,\hat{\Omega}} \leq \|I - \hat{\Pi}\|_{\mathscr{L}(W^{k+1,p}(\hat{\Omega}); W^{m,q}(\hat{\Omega}))} \inf_{\hat{p} \in P_k(\hat{\Omega})} \|\hat{v} + \hat{p}\|_{k+1,p,\hat{\Omega}}$$
$$\leq C(\hat{\Pi}, \hat{\Omega})|\hat{v}|_{k+1,p,\hat{\Omega}}, \tag{3.1.33}$$

by Theorem 3.1.1.

It follows from relation (3.1.31) that

$$\hat{v} - \hat{\Pi}\hat{v} = (v - \Pi_\Omega v)^{\wedge},$$

and therefore an application of Theorem 3.1.2 yields

$$|v - \Pi_\Omega v|_{m,q,\Omega} \leq C\|B^{-1}\|^m |\det(B)|^{1/q} |\hat{v} - \hat{\Pi}\hat{v}|_{m,q,\hat{\Omega}}. \tag{3.1.34}$$

By the same theorem,

$$|\hat{v}|_{k+1,p,\hat{\Omega}} \leq C\|B\|^{k+1} |\det(B)|^{-1/p} |v|_{k+1,p,\Omega}, \tag{3.1.35}$$

and thus, to obtain inequality (3.1.32), it suffices to combine inequalities (3.1.33), (3.1.34) and (3.1.35), the upper bounds $\|B\| \leq h/\hat{\rho}$ and $\|B^{-1}\| \leq \hat{h}/\rho$ (Theorem 3.1.3), and, finally, to observe that

$$|\det(B)| = \frac{\text{meas}(\Omega)}{\text{meas}(\hat{\Omega})}. \qquad \square$$

Estimates of the interpolation errors $|v - \Pi_K v|_{m,q,K}$ for affine families of finite elements

By specializing the above result to finite elements, we obtain *estimates of the interpolation errors* $|v - \Pi_K v|_{m,q,K}$ (for another approach, see Exercise 3.1.2; for a refined analysis of the dependence upon the geometry, see Exercise 3.1.4).

Theorem 3.1.5. *Let $(\hat{K}, \hat{P}, \hat{\Sigma})$ be a finite element, for which s denotes the greatest order of partial derivatives occurring in the definition of $\hat{\Sigma}$. If the following inclusions hold, for some integers $m \geq 0$ and $k \geq 0$ and for some numbers $p, q \in [1, \infty]$,*

$$W^{k+1,p}(\hat{K}) \hookrightarrow \mathscr{C}^s(\hat{K}), \tag{3.1.36}$$

$$W^{k+1,p}(\hat{K}) \hookrightarrow W^{m,q}(\hat{K}), \tag{3.1.37}$$

$$P_k(\hat{K}) \subset \hat{P} \subset W^{m,q}(\hat{K}), \tag{3.1.38}$$

there exists a constant $C(\hat{K}, \hat{P}, \hat{\Sigma})$ such that, for all affine-equivalent

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finite elements (K, P, Σ), *and all functions* $v \in W^{k+1,p}(K)$,

$$|v - \Pi_K v|_{m,q,K} \leq C(\hat{K}, \hat{P}, \hat{\Sigma})(\text{meas}(K))^{(1/q)-(1/p)} \frac{h_K^{k+1}}{\rho_K^m} |v|_{k+1,p,K}, \tag{3.1.39}$$

where $\Pi_K v$ *denotes the* P_K*-interpolant of the function* v, *and*

$$\left.\begin{aligned} &\text{meas}(K) = \mathrm{d}x - \textit{measure of } K, \\ &h_K = \textit{diam}(K), \\ &\rho_K = \textit{sup}\ \{\textit{diam}(S);\ S \textit{ is a ball contained in } K\}. \end{aligned}\right\} \tag{3.1.40}$$

Proof. The inclusion $P_k(\hat{K}) \subset \hat{P}$ in conjunction with the fact that the $\hat{P}$-interpolation operator $\hat{\Pi}$ reduces to the identity over the space $\hat{P}$ (cf. (2.3.8)) implies that

$$\forall \hat{p} \in P_k(\hat{K}), \quad \hat{\Pi}\hat{p} = \hat{p}. \tag{3.1.41}$$

Let then $\hat{v}$ be a function in the space $W^{k+1,p}(\hat{K})$, so that it belongs to the space dom $\hat{\Pi} = \mathscr{C}^s(\hat{K})$ (cf. (2.3.7)) since the inclusion $W^{k+1,p}(\hat{K}) \subset \mathscr{C}^s(\hat{K})$ holds. For definiteness, let us assume that $s = 2$ (recall that in practice, $s = 0$, 1 or 2) so that the $\hat{P}$-interpolant of the function $\hat{v}$ takes the form

$$\hat{\Pi}\hat{v} = \sum_i \hat{v}(\hat{a}_i^0)\hat{p}_i^0 + \sum_{i,k} \{D\hat{v}(\hat{a}_i^1)\hat{\xi}_{i,k}^1\}\hat{p}_{ik}^1 + \sum_{i,k,l} \{D^2\hat{v}(\hat{a}_i^2)(\hat{\xi}_{ik}^2, \hat{\xi}_{il}^2)\}\hat{p}_{ikl}^2. \tag{3.1.42}$$

We proceed to show that the linear mapping $\hat{\Pi}$: $W^{k+1,p}(\hat{K}) \to W^{m,q}(\hat{K})$ (by (3.1.38), the space $\hat{P}$ is contained in the space $W^{m,q}(\hat{K})$) is continuous: From (3.1.42), we deduce that

$$\begin{aligned} \|\hat{\Pi}\hat{v}\|_{m,q,\hat{K}} &\leq \sum_i |\hat{v}(\hat{a}_i^0)|\, \|\hat{p}_i^0\|_{m,q,\hat{K}} + \sum_{i,k} |\{D\hat{v}(\hat{a}_i^1)\hat{\xi}_{ik}^1\}|\, \|\hat{p}_{ik}^1\|_{m,q,\hat{K}} \\ &\quad + \sum_{i,k,l} |\{D^l\hat{v}(\hat{a}_i^2)(\hat{\xi}_{ik}^2, \hat{\xi}_{il}^2)\}|\, \|\hat{p}_{ikl}^2\|_{m,q,\hat{K}} \\ &\leq C(\|\hat{p}_i^0\|_{m,q,\hat{K}}, \|\hat{\xi}_{ik}^1\|\, \|\hat{p}_{ik}^1\|_{m,q,\hat{K}}, \|\hat{\xi}_{ik}^2\|\, \|\hat{\xi}_{il}^2\|\, \|\hat{p}_{ikl}^2\|_{m,q,\hat{K}})\|\hat{v}\|_{2,\infty,\hat{K}} \\ &\leq C(\hat{K}, \hat{P}, \hat{\Sigma})\|\hat{v}\|_{k+1,p,\hat{K}}, \end{aligned}$$

where in the last inequality, we have made use of the inclusion (3.1.36).

Since the P_K- and $\hat{P}$-interpolation operators are related through the

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correspondence

$$(\Pi_K v)\hat{} = \hat{\Pi}\hat{v} \quad \text{for all} \quad v \in \operatorname{dom} \Pi_K$$

(cf. (2.3.19)), we may apply Theorem 3.1.4, and inequality (3.1.39) is just a re-statement of inequality (3.1.32) in the present case. □

Remark 3.1.3. If necessary, the factor $(\text{meas}(K))^{(1/q)-(1/p)}$ may also be expressed in terms of the parameters h_K and ρ_K by means of the inequalities

$$\sigma_n \rho_K^n \leq \text{meas}(K) \leq \sigma_n h_K^n,$$

where σ_n denotes the dx-measure of the unit sphere in $\mathbf{R}^n$. □

It is possible to dispose of the parameter ρ_K in the upper bound (3.1.39) provided we restrict ourselves to finite elements which do not become "flat" in the limit, as we shall show (Theorem 3.1.6). First, we need a definition, of a purely *geometrical* nature.

We shall say that a family of finite elements (K, P_K, Σ_K) is *regular* if the following two conditions are satisfied:

(i) There exists a constant σ such that

$$\forall K, \frac{h_K}{\rho_K} \leq \sigma \tag{3.1.43}$$

(see Exercise 3.1.3 for an equivalent formulation of this condition for triangles).

(ii) The diameters h_K approach zero (in order to avoid introducing new letters, K is viewed as the parameter of the family).

For such families, the interpolation error estimate of Theorem 3.1.5 can be immediately converted into simple estimates of the *norms* $\|v - \Pi_K v\|_{m,q,K}$.

Theorem 3.1.6. *Let there be given a regular affine family of finite elements* (K, P_K, Σ_K) *whose reference finite element* $(\hat{K}, \hat{P}, \hat{\Sigma})$ *satisfies assumptions* (3.1.36), (3.1.37) *and* (3.1.38). *Then there exists a constant* $C(\hat{K}, \hat{P}, \hat{\Sigma})$ *such that, for all finite elements K in the family, and all functions* $v \in W^{k+1,p}(K)$,

$$\|v - \Pi_K v\|_{m,q,K} \leq C(\hat{K}, \hat{P}, \hat{\Sigma})(\text{meas}(K))^{(1/q)-(1/p)} h_K^{k+1-m} |v|_{k+1,p,K}. \tag{3.1.44}$$

□

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Remark 3.1.4. Only the boundedness of the diameters h_K (implied by condition (ii)) is used in the derivation of the upper bound (3.1.44). □

In order to get a more concrete understanding of such estimates, we have recorded in the next table (Fig. 3.1.2) some interpolation error estimates in the $\|\cdot\|_{m,K}$ norms ($p = q = 2$) for various finite elements which can be imbedded in affine families.

Remark 3.1.5. If the function v lacks the "optimal" regularity, inter-

$\|v - \Pi_K v\|_{m,K}$	$O(h_K^{2-m})$ $0 \leqslant m \leqslant 2$ $(k = 1)$	$O(h_K^{3-m})$ $0 \leqslant m \leqslant 3$ $(k = 2)$		$O(h_K^{4-m})$ $0 \leqslant m \leqslant 4$ $(k = 3)$		
Regularity of the function v	$H^2(K)$	$H^3(K)$		$H^4(K)$		
Upper bound on the dimension n, (to insure that $H^{k+1}(K) \hookrightarrow \mathscr{C}^s(K)$)	$n \leqslant 3$ $(s = 0)$	$n \leqslant 5$ $(s = 0)$	$n \leqslant 3$ $(s = 1)$	$n \leqslant 7$ $(s = 0)$	$n \leqslant 5$ $(s = 1)$	$n \leqslant 3$ $(s = 2)$

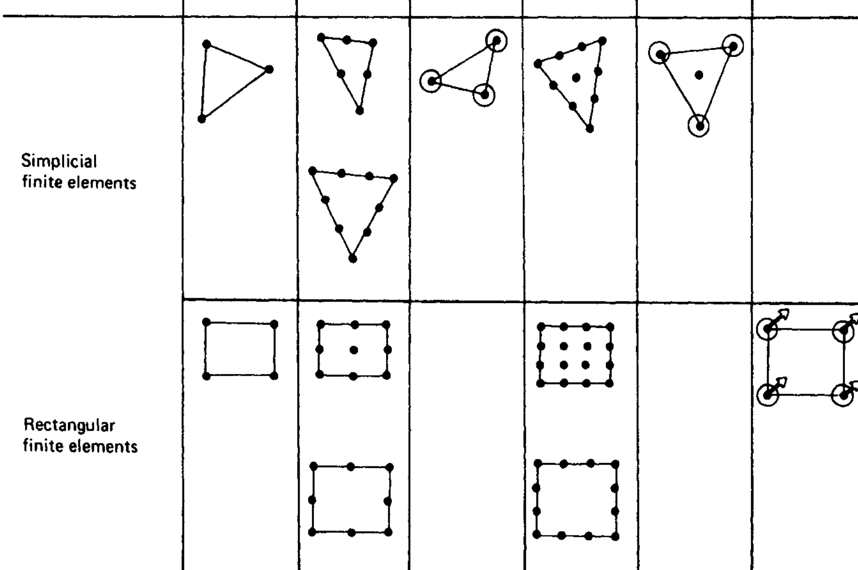

Fig. 3.1.2

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polation error estimates may still hold *provided the P_K-interpolant is still defined*, with smaller values of k, however. If we are considering Hermite triangles of type (3) for instance, if the function v is "only" in the space $H^3(K)$ and if $n \leqslant 3$, one has $\|v - \Pi_K v\|_{m,K} = O(h_K^{3-m})$ for $0 \leqslant m \leqslant 3$, and so on... □

Remark 3.1.6. Further conditions may be added in the definition of a regular family made up of a *specific* finite element. For example, this will be the case for the isoparametric n-simplex of type (2) (cf. Section 4.3) and the Hsieh–Clough–Tocher triangle (cf. Section 6.1). □

Exercises

3.1.1. The following abstract generalization of both Theorems 3.1.1 and 3.1.4 is due to L. TARTAR (unpublished) and can be found in BREZZI & MARINI (1975).

Let V be a Banach space and let V_1, V_2 and W be three normed vector spaces. Let $A_i \in \mathscr{L}(V; V_i)$, $i = 1, 2$, be two given mappings, the mapping A_1 being compact. It is assumed that there exists a constant c_0 such that

$$\forall v \in V, \quad \|v\|_V \leqslant c_0(\|A_1 v\|_{V_1} + \|A_2 v\|_{V_2}).$$

Finally, let $L \in \mathscr{L}(V; W)$ be a mapping such that

$$v \in \ker A_2 \Rightarrow Lv = 0.$$

(i) Show that the space $P = \ker A_2$ is finite-dimensional.

(ii) Show that there exists a constant c_1 such that

$$\forall v \in V, \quad \inf_{p \in P} \|v - p\|_V \leqslant c_1 \|A_2 v\|_{V_2}.$$

(iii) Deduce from (ii) that there exists a constant C such that

$$\forall v \in V, \quad \|Lv\|_W \leqslant C \|A_2 v\|_{V_2}.$$

(iv) Let (for simplicity only, we restrict ourselves to $p = q = 2$, $m = k + 1$)

$$V = W = H^{k+1}(\Omega),$$
$$V_1 = H^k(\Omega), \quad A_1 = I,$$
$$V_2 = (L^2(\Omega))^{r_{k+1}}, \quad A_2 \colon v \in H^{k+1}(\Omega) \to (\partial^\alpha v)_{|\alpha| = k+1},$$

where $\Gamma_{k+1} = \text{card}\{\alpha \in \mathbf{N}^n;\ |\alpha| = k+1\}$ and the mapping L is given by

$$L\colon v \in H^{k+1}(\Omega) \to (v - \Pi v) \in H^{k+1}(\Omega),$$

where the mapping $\Pi \in \mathscr{L}(H^{k+1}(\Omega))$ is such that

$$\forall p \in P_k(\Omega),\quad \Pi p = p.$$

Then derive an inequality similar to that of (3.1.33) (we could as well let $V_1 = L^2(\Omega)$; see below).

(v) Let

$$\begin{aligned}
&V = W = H^{k+1}(\Omega),\\
&V_1 = L^2(\Omega),\quad A_1 = I,\\
&V_2 = (L^2(\Omega))^n,\quad A_2\colon v \in H^{k+1}(\Omega)\\
&\qquad \to \{D^{k+1}v(\cdot)(e_i^{k+1})\}_{i=1}^n \in (L^2(\Omega))^n,\\
&L\colon v \in H^{k+1}(\Omega) \to (v - \Pi v) \in H^{k+1}(\Omega),
\end{aligned}$$

where the mapping $\Pi \in \mathscr{L}(H^{k+1}(\Omega))$ is such that

$$\forall p \in Q_k(\Omega),\quad \Pi p = p.$$

Then show that there exists a constant $C(\Pi, \Omega)$ such that

$$\|v - \Pi v\|_{k+1,\Omega} \leq C(\Pi, \Omega)[v]_{k+1,\Omega},$$

where

$$[v]_{k+1,\Omega} = \left(\sum_{i=1}^n \int_\Omega |D^{k+1}v(x)(e_i^{k+1})|^2\,\mathrm{d}x\right)^{1/2}.$$

[Hint: Use the following result due to N. Aronszajn and K.T. Smith, and proved in SMITH (1961): There exists a constant C such that

$$\forall v \in W^{m,p}(\Omega),\quad \|v\|_{m,p,\Omega} \leq C(|v|_{0,p,\Omega} + [v]_{m,p,\Omega}),$$

where

$$[v]_{m,p,\Omega} = \left(\sum_{i=1}^n \int_\Omega |D^m v(x)(e_i^m)|^p\,\mathrm{d}x\right)^{1/2}.]$$

3.1.2. Let (K, P_K, Σ_K) be a Lagrange finite element such that the inclusions

$$P_k(K) \subset P_K \subset \mathscr{C}^k(K)$$

hold for some integer k, and let there be given a function $v\colon K \to \mathbf{R}$ which

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will be assumed to be sufficiently smooth for all subsequent purposes.

For any integer $m \geq 0$, we let

$$|v|_{m,\infty,K} = \sup_{x \in K} \|D^m v(x)\|_{\mathscr{L}_m(\mathbf{R}^n;\,\mathbf{R})}$$

(we recall that there exists a constant $C(m, n)$ such that

$$|v|_{m,\infty,K} \leq |v|_{m,\infty,K} \leq C(m, n)|v|_{m,\infty,K}).$$

If a and x are two points in the set K (assumed here to be convex), Taylor's formula of order k reads

$$v(a) = v(x) + Dv(x)(a-x) + \cdots + \frac{1}{k!}D^k v(x)(a-x)^k + \\ + \mathscr{R}_k(v; a, x),$$

where the remainder $\mathscr{R}_k(v; a, x)$ is given by

$$\mathscr{R}_k(v; a, x) = \frac{1}{k!}\int_0^1 (1-t)^k D^{k+1} v(ta + (1-t)x)(a-x)^{k+1}\,\mathrm{d}t \\ = \frac{1}{(k+1)!}D^{k+1} v(\theta a + (1-\theta)x)(a-x)^{k+1}.$$

In fact, we shall only need the estimates

$$\mathscr{R}_k(v; a, x)| \leq \frac{1}{(k+1)!}|v|_{k+1,\infty,K}\|a-x\|^{k+1}$$

which follow from either expression of the remainder.

(i) Let p_i, $1 \leq i \leq N$, be the basis functions associated with the set $\Sigma_K = \{p(a_i),\ 1 \leq i \leq N\}$. Show that, for all $x \in K$ (Ciarlet & Raviart (1972a)).

$$D^m(\Pi_K v - v)(x) = \sum_{i=1}^N \mathscr{R}_k(v; a_i, x) D^m p_i(x), \quad 0 \leq m \leq k.$$

Notice that for $m = 0$, one obtains a *multi-point Taylor formula* (Ciarlet & Wagschal (1971)):

$$v(x) = \sum_{i=1}^N v(a_i)p_i(x) - \sum_{i=1}^N \mathscr{R}_k(v; a_i, x)p_i(x).$$

(ii) Let $(\hat{K}, \hat{P}, \hat{\Sigma})$ be an affine-equivalent finite element. Show that (with the usual notations)

$$|v - \Pi_K v|_{m,\infty,K} \leq C(\hat{K}, \hat{P}, \hat{\Sigma})\frac{h_K^{k+1}}{\rho_K^m}|v|_{k+1,\infty,K}, \quad 0 \leq m \leq k,$$

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where

$$C(\hat{K}, \hat{P}, \hat{\Sigma}) = \frac{\hat{h}^m}{(k+1)!} \sum_{i=1}^{N} |\hat{p}_i|_{m,\infty,\hat{K}}.$$

Thus, this yields an estimate of the constant which appears in (3.1.39) when $p = q = \infty$.

(iii) In the sequel, p is any number which satisfies $1 \leq p \leq \infty$ and $k+1 > (n/p)$, so that the inclusion $W^{k+1,p}(K) \hookrightarrow \mathscr{C}^0(K)$ holds. Show that

$$|\mathscr{R}_k(v; a, \cdot)|_{0,p,K} \leq \frac{1}{k!\left(k+1-\dfrac{n}{p}\right)} h_K^{k+1} |v|_{k+1,p,K},$$

for all $a \in K$.

(iv) Deduce from (iii) that

$$|v - \Pi_K v|_{m,p,K} \leq \frac{1}{k!\left(k+1-\dfrac{n}{p}\right)} \left(\sum_{i=1}^{N} |p_i|_{m,\infty,K}\right) h_K^{k+1} |v|_{k+1,p,K}.$$

(v) Assume that K is an n-simplex and that the basis functions p_i are expressed uniquely in terms of the barycentric coordinates (in this respect, cf. also Exercise 2.3.6):

$$p_i(x) = f_i(\lambda_1(x), \dots, \lambda_{n+1}(x)), \quad 1 \leq i \leq N,$$

where the (smooth) functions f_i are independent of (K, P_K, Σ_K). Then show that (ARCANGÉLI & GOUT (1976))

$$|v - \Pi_K v|_{m,p,K} \leq C(\hat{K}, \hat{P}, \hat{\Sigma}) \frac{h_K^{k+1}}{\rho_K^m} |v|_{k+1,p,K},$$

where

$$C(\hat{K}, \hat{P}, \hat{\Sigma}) = \frac{m!}{k!\left(k+1-\dfrac{n}{p}\right)} \times$$

$$\times \sum_{i=1}^{N} \max_{\substack{0 \leq \lambda_j \leq 1 \\ \Sigma_j \lambda_j = 1}} \sum_{|\alpha| = m} \frac{1}{\alpha_1! \dots \alpha_{n+1}!} |\partial^\alpha f_i(\lambda_1, \dots, \lambda_{n+1})|.$$

Therefore this provides another estimate of the constants which appear in the interpolation error estimate.

[Hint: As in CIARLET & WAGSCHAL (1971), prove and use the in-

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equalities

$$|\lambda_i|_{1,\infty,K} \leq \frac{1}{\rho_K}, \quad 1 \leq i \leq N.]$$

(vi) Using the result of (v), show that, for triangles of type (1),

$$|v - \Pi_K v|_{m,K} \leq C_m^1 \frac{h_K^2}{\rho_K^m} |v|_{2,K} \quad \text{which} \quad C_0^1 = C_1^1 = 3,$$

and that, for triangles of type (2),

$$|v - \Pi_K v|_{m,K} \leq C_m^2 \frac{h_K^3}{\rho_K^m} |v|_{3,K} \quad \text{with} \quad C_0^2 = 2, \quad C_1^2 = 6, \quad C_2^2 = 9.$$

These estimates can be further improved. See ARCANGELI & GOUT (1976) and GOUT (1976).

3.1.3. Show that for a family of triangular finite elements, condition (3.1.43) is equivalent to *Zlámal's condition* (ZLÁMAL (1968)) that there exists a constant θ_0 such that

$$\forall K, \quad \theta_K \geq \theta_0 > 0,$$

where for each triangle K, θ_K denotes the smallest angle of K.

3.1.4. The object of this problem is to study (in the special case $m = 1$ for simplicity) the improvement of JAMET (1976b) concerning the dependence of the interpolation error estimates upon the geometry of the finite elements.

(i) Let Ω be an open subset of $\mathbf{R}^n$ with $h = \operatorname{diam} \Omega$. In addition, let $\Pi_\Omega \in \mathscr{L}(W^{k(+1),p}(\Omega); W^{1,p}(\Omega))$ be a mapping such that

(a) $\Pi_\Omega p = p$ for all $p \in P_k(\Omega)$ and

(b) there exists a non zero vector $\xi \in \mathbf{R}^n$ such that if $Dv(x)\xi = 0$ for all $x \in \Omega$ for some function $v \in W^{k(+1),p}(\Omega)$, then $D\Pi_\Omega v(x)\xi = 0$ for all $x \in \Omega$.

Show that there exists a mapping $\Phi \in \mathscr{L}(W^{k,p}(\Omega); L^p(\Omega))$ such that

$$\forall v \in W^{k+1,p}(\Omega), \; D\Pi_\Omega v(\cdot)\xi = \Phi(Dv(\cdot)\xi),$$

$$\forall p \in P_{k-1}(\Omega), \quad \Phi p = p.$$

(ii) Let $\hat{\Omega}$ be an open set which is affine-equivalent to Ω and let the mapping $\hat{\Pi}$ be defined in the usual way. Using (i), show that there exists a constant $C(\hat{\Pi}, \hat{\Omega})$ such that

$$\forall v \in W^{k+1,p}(\Omega), \quad |Dv(\cdot)\xi - D\Pi_\Omega v(\cdot)\xi|_{0,p,\Omega} \leq C(\hat{\Pi}, \hat{\Omega}) h^{k(+1)} |v|_{k+1,p,\Omega}.$$

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(iii) Assume that the property of (i) is satisfied for n linearly independent vectors ξ_i, $1 \leq i \leq n$. Show that there exists a constant $C(\hat{\Pi}, \hat{\Omega})$ such that (compare with (3.1.32))

$$\forall v \in W^{k+1,p}(\Omega), \quad |v - \Pi_\Omega v|_{1,p,\Omega} \leq C(\hat{\Pi}, \hat{\Omega}) \frac{h^k}{\cos \theta} |v|_{k+1,p,\Omega},$$

where the angle θ (a function of the n vectors ξ_i, $1 \leq i \leq n$) is defined as follows: Given any vector η with $\|\eta\| = 1$, let $\theta_i(\eta) \in [0, \pi/2]$ be the angle between the directions of the vectors η and ξ_i, $1 \leq i \leq n$. Then we let

$$\theta = \max_{\eta \in \mathbb{R}^n} \{\min_{1 \leq i \leq n} \theta_i(\eta)\}.$$

(iv) Assume that $k > (n/p)$. In the case of n-simplices of type (k), show that we may choose for vectors ξ_i, $1 \leq i \leq n$, any n vectors out of the $n(n+1)/2$ vectors of the form $(a_j - a_i)$, $1 \leq i < j \leq n+1$. As a consequence show that in the case of triangles for instance, we may have $\|v - \Pi_K v\|_{1,p,\Omega} = O(h^k)$ even though Zlámal's condition is violated.

(v) Apply the previous analysis to rectangles of type (k).

3.2. Application to second-order problems over polygonal domains

Estimate of the error $\|u - u_h\|_{1,\Omega}$

Let there be given a second-order boundary value problem, posed over a space V which satisfies the usual inclusions $H_0^1(\Omega) \subset V = \bar{V} \subset H^1(\Omega)$. One basic hypothesis will be that the set $\bar{\Omega}$ is *polygonal*, essentially because such an assumption allows us to *exactly* cover the set $\bar{\Omega}$ with polygonal finite elements. Then with any such finite element, we associate a finite element space X_h. Next, we define an appropriate subspace V_h of X_h (this takes into account the boundary conditions contained in the definition of the space V) which is included in the space V, so that we are using a *conforming finite element method.* One main property that we shall assume is that *the space V_h contains the X_h-interpolant of the solution u*: See Section 2.3 where the special cases $V_h = X_h \subset V = H^1(\Omega)$ and $V_h = X_{0h} \subset V = H_0^1(\Omega)$ have been thoroughly discussed. This would also be true of a problem where $V = \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_0\}$ provided the subset Γ_0 of Γ can be written exactly as a union of faces of some finite elements. By contrast, this is not true in general of a

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nonhomogeneous Dirichlet problem. In this direction, see Exercises 3.2.1 and 3.2.2.

Throughout this section, we shall make the following assumptions, denoted (H 1), (H 2) and (H 3), whose statements will not be repeated.

(H 1) We consider a *regular family of triangulations* $\mathcal{T}_h$ in the following sense:

(i) There exists a constant σ such that

$$\forall K \in \bigcup_h \mathcal{T}_h, \quad \frac{h_K}{\rho_K} \leq \sigma. \tag{3.2.1}$$

(ii) The quantity

$$h = \max_{K \in \mathcal{T}_h} h_k \tag{3.2.2}$$

approaches zero.

In other words, *the family formed by the finite elements* (K, P_K, Σ_K), $K \in \bigcup_h \mathcal{T}_h$, *is a regular family of finite elements*, in the sense of Section 3.1.

Remark 3.2.1. There is of course an ambiguity in the meaning of h, which was first considered as a defining parameter of both families $(\mathcal{T}_h)$ and (X_h), and which was next specifically defined in (3.2.2). We have nevertheless conformed to this often followed usage. □

(H 2) All the finite elements (K, P_K, Σ_K), $K \in \bigcup_h \mathcal{T}_h$, are affine-equivalent to a single reference finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$. In other words, *the family* (K, P_K, Σ_K), $K \in \mathcal{T}_h$ *for all* h, *is an affine family of finite elements*, in the sense of Section 2.3.

(H 3) *All the finite elements* (K, P_K, Σ_K), $K \in \bigcup_h \mathcal{T}_h$, *are of class* $\mathscr{C}^0$.

We first prove an approximation property of the family (V_h) (Theorem 3.2.1), from which we derive an estimate for the *error in the norm* $\|\cdot\|_{1,\Omega}$ (Theorem 3.2.2).

In the sequel, C stands for a constant independent of h and of the various functions involved (not necessarily the same at its various occurences).

Theorem 3.2.1. *In addition to* (H 1), (H 2) *and* (H 3), *assume that there exist integers* $k \geq 0$ *and* $l \geq 0$ *with* $l \leq k$, *such that the following inclusions*

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are satisfied:

$$P_k(\hat{K}) \subset \hat{P} \subset H^l(\hat{K}), \tag{3.2.3}$$

$$H^{k+1}(\hat{K}) \hookrightarrow \mathscr{C}^s(\hat{K}), \tag{3.2.4}$$

where s *is the maximal order of partial derivatives occurring in the definitions of the set* $\hat{\Sigma}$.

Then there exists a constant C *independent of* h *such that, for any function* $v \in H^{k+1}(\Omega) \cap V$,

$$\|v - \Pi_h v\|_{m,\Omega} \leqslant Ch^{k+1-m}|v|_{k+1,\Omega}, \quad 0 \leqslant m \leqslant \min\{1, l\}, \tag{3.2.5}$$

$$\Big(\sum_{K \in \mathscr{T}_h} \|v - \Pi_h v\|^2_{m,K}\Big)^{1/2} \leqslant Ch^{k+1-m}|v|_{k+1,\Omega}, \quad 2 \leqslant m \leqslant \min\{k+1, l\} \tag{3.2.6}$$

where $\Pi_h v \in V_h$ *is the* X_h*-interpolant of the function* v.

Proof. Applying Theorem 3.1.6 with $p = q = 2$, we obtain

$$\|v - \Pi_K v\|_{m,K} \leqslant Ch_K^{k+1-m}|v|_{k+1,K}, \quad 0 \leqslant m \leqslant \min\{k+1, l\}.$$

Using the relations $(\Pi_h v)|_K = \Pi_K v$, $K \in \mathscr{T}_h$ (cf. (2.3.32)) and the inequalities $h_K \leqslant h$, $K \in \mathscr{T}_h$ (cf. (3.2.2)), we get

$$\Big(\sum_{K \in \mathscr{T}_h} \|v - \Pi_h v\|^2_{m,K}\Big)^{1/2} \leqslant Ch^{k+1-m}\Big(\sum_{K \in \mathscr{T}_h} |v|^2_{k+1,K}\Big)^{1/2}$$
$$= Ch^{k+1-m}|v|_{k+1,\Omega}, \quad 0 \leqslant m \leqslant \min\{k+1, l\}.$$

Thus inequalities (3.2.6) are proved and inequalities (3.2.5) follow by observing that

$$\Big(\sum_{K \in \mathscr{T}_h} \|v - \Pi_h v\|^2_{m,K}\Big)^{1/2} = \|v - \Pi_h v\|_{m,\Omega},$$

for $m = 0$ and for $m = 1$ (when $l \geqslant 1$) since the inclusions $\hat{P} \subset H^1(\hat{K})$ and $X_h \subset \mathscr{C}^0(\bar{\Omega})$ implies $X_h \subset H^1(\Omega)$ (Theorem 2.1.1). □

Remark 3.2.2. Analogous interpolation error estimates hold if the function v is "only" in the spaces $(\mathscr{C}^s(\bar{\Omega}) \cap \Pi_{K \in \mathscr{T}_h} H^{k+1}(K)) \cap V$. It suffices to replace the semi-norm $|v|_{k+1,\Omega}$ by the semi-norm $(\Sigma_{K \in \mathscr{T}_h} |v|^2_{k+1,K})^{1/2}$ in the right-hand sides of inequalities (3.2.5) and (3.2.6). Such more general estimates are seldom needed. □

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Theorem 3.2.2. *In addition to* (H 1), (H 2) *and* (H 3), *assume that there exists an integer $k \geqslant 1$ such that the following inclusions are satisfied*:

$$P_k(\hat{K}) \subset \hat{P} \subset H^1(\hat{K}), \tag{3.2.7}$$

$$H^{k+1}(\hat{K}) \hookrightarrow \mathscr{C}^s(\hat{K}), \tag{3.2.8}$$

where s is the maximal order of partial derivatives occurring in the definition of the set $\hat{\Sigma}$.

Then if the solution $u \in V$ of the variational problem is also in the space $H^{k+1}(\Omega)$, there exists a constant C independent of h such that

$$\|u - u_h\|_{1,\Omega} \leqslant Ch^k |u|_{k+1,\Omega}, \tag{3.2.9}$$

where $u_h \in V_h$ is the discrete solution.

Proof. It suffices to use inequality (3.2.5) with $v = u$ and $m = 1$, in conjunction with Céa's lemma (Theorem 2.4.1), which yields

$$\|u - u_h\|_{1,\Omega} \leqslant C \inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega} \leqslant C\|u - \Pi_h u\|_{1,\Omega}. \qquad \square$$

Sufficient conditions for $\lim_{h\to 0} \|u - u_h\|_{1,\Omega} = 0$

The previous result has been established under the assumptions that the solution u is *sufficiently smooth* (in $H^{k+1}(\Omega)$ for some $k \geqslant 1$) and that the X_h-interpolant $\Pi_h u$ exists (cf. the inclusion $H^{k+1}(\hat{K}) \hookrightarrow \mathscr{C}^s(\hat{K})$ which is satisfied if $k > (n/2) - 1 + s$). If these hypotheses are not valid, it is still possible to prove the convergence of the method if the solution u belongs to the space $V \cap H^1(\Omega)$ and if the "minimal" assumptions (3.2.10) below hold, using a "density argument" as we now show (one should notice that the assumption $s \leqslant 1$ in the next theorem is not a restriction *in practice* for second-order problems). For a different approach, see Exercise 3.2.3.

Theorem 3.2.3. *In addition to* (H 1), (H 2) *and* (H 3), *assume that the inclusions*

$$P_1(\hat{K}) \subset \hat{P} \subset H^1(\hat{K}) \tag{3.2.10}$$

are satisfied, and that there are no directional derivatives of order $\geqslant 2$ in the set $\hat{\Sigma}$.

Then we have

$$\lim_{h\to 0} \|u - u_h\|_{1,\Omega} = 0. \tag{3.2.11}$$

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Proof. Define the space

$$\mathcal{V} = W^{2,\infty}(\Omega) \cap V. \tag{3.2.12}$$

Since the inclusions (3.2.10) and

$$W^{2,\infty}(\hat{K}) \hookrightarrow \mathscr{C}^s(\hat{K}), \quad s = 0 \quad \text{or} \quad 1,$$
$$W^{2,\infty}(\hat{K}) \hookrightarrow H^1(\hat{K}),$$

hold, we may apply Theorem 3.1.6 with $k = 1$, $p = \infty$, $m = 1$, $q = 2$: There exists a constant C such that

$$\forall v \in \mathcal{V}, \quad \|v - \Pi_K v\|_{1,K} \leq C(\text{meas}(K))^{1/2} h_K |v|_{2,\infty,K},$$

from which we deduce that

$$\|v - \Pi_h v\|_{1,\Omega} = \Big(\sum_{K \in \mathcal{T}_h} \|v - \Pi_K v\|_{1,K}^2\Big)^{1/2} \leq Ch(\text{meas}(\Omega))^{1/2} |v|_{2,\infty,\Omega},$$

and thus we have proved that

$$\lim_{h \to 0} \|v - \Pi_h v\|_{1,\Omega} = 0. \tag{3.2.13}$$

For all h and all $v \in \mathcal{V}$, we can write

$$\|u - \Pi_h v\|_{1,\Omega} \leq \|u - v\|_{1,\Omega} + \|v - \Pi_h v\|_{1,\Omega}. \tag{3.2.14}$$

Given the solution $u \in V$ and any number $\epsilon > 0$, we first determine a function $v_\epsilon \in \mathcal{V}$ which satisfies the inequality $\|u - v_\epsilon\|_{1,\Omega} \leq \epsilon/2$. This is possible because *the space $\mathcal{V}$ is dense in the space V*. Then by (3.2.13), there exists an $h_0(\epsilon)$ such that $\|v_\epsilon - \Pi_h v_\epsilon\|_{1,\Omega} \leq \epsilon/2$ for all $h \leq h_0(\epsilon)$. In view of inequality (3.2.14), we have therefore shown that

$$\lim_{h \to 0} \inf_{v_h \in V_h} \|u - v_h\| = 0,$$

and the conclusion follows from Céa's lemma (Theorem 2.4.1). □

A close look at the above proof shows that the choice (3.2.12) of the space $\mathcal{V}$ is the result of the following requirements: On the one hand it had to be dense in the space V and on the other hand the value $k = 1$ was needed in order to apply Theorem 3.1.6 so as to obtain property (3.2.13) with the assumption $P_1(\hat{K}) \subset \hat{P}$. Therefore the space $\mathcal{V}$ had to contain derivatives of order ≤ 2 (this condition limits in turn the admissible values of s to 0 and 1) and consequently, one is naturally led to the space of the form (3.2.12). In fact, any space of the form $\mathcal{V} =$

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$W^{2,p}(\Omega) \cap V$ with p sufficiently large, would have also been acceptable, as one may verify.

Estimate of the error $|u - u_h|_{0,\Omega}$. *The Aubin–Nitsche lemma*

In theorem 3.2.2, we have given assumptions which insure that $\|u - u_h\|_{1,\Omega} = O(h^k)$ so that the *error in the norm* $|\cdot|_{0,\Omega}$, i.e., the quantity $|u - u_h|_{0,\Omega}$, is at least of the same order. Our next objective is to show that, under mild additional assumptions, one has in fact $|u - u_h|_{0,\Omega} = O(h^{k+1})$.

Let us first define an abstract setting which is well adapted to this type of improved error estimates:

In addition to the space V, with norm $\|\cdot\|$, we are given a *Hilbert space* H, with norm $|\cdot|$ and inner product $(\cdot,\cdot)$, *such that* $\bar{V} = H$ *with a continuous injection* (in the present case, we shall have typically $V = H_0^1(\Omega)$, $H^1(\Omega)$, or any closed space contained in between these two spaces, and $H = L^2(\Omega)$).

Then we shall identify the space H with its dual, so that the space H may be identified with a subspace of the dual space V' of V, as we now show:

Let $f \in H$ be given. Since $V \subset H$ with a continuous injection ι, we have

$$\forall v \in V, \quad |(f, v)| \leq |f|\,|v| \leq \|\iota\|\,|f|\,\|v\|, \tag{3.2.15}$$

and therefore the mapping $v \in V \to (f, v)$ defines an element $\tilde{f} \in V'$. The mapping $f \in H \to \tilde{f} \in V'$ is an injection for if $(f, v) = 0$ for all $v \in V$, then $(f, v) = 0$ for all $v \in H$ since V is dense in H, and thus $f = 0$. *We shall henceforth identify f and $\tilde{f}$*, i.e., we shall write

$$\forall f \in H, \quad \forall v \in V, \quad (f, v) = f(v). \tag{3.2.16}$$

We next prove an *abstract error estimate*. With the same assumptions as for the Lax–Milgram lemma (Theorem 1.1.3), we let as usual $u \in V$ and $u_h \in V_h$ denote the solutions of the variational problems

$$\forall v \in V, \quad a(u, v) = f(v), \tag{3.2.17}$$

$$\forall v_h \in V_h, \quad a(u_h, v_h) = f(v_h), \tag{3.2.18}$$

respectively. We recall that M denotes an upper bound for the norm of the bilinear form $a(\cdot,\cdot)$ (cf. (1.1.19)).

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Theorem 3.2.4 (*Aubin–Nitsche lemma*). *Let H be a Hilbert space, with norm $|\cdot|$ and inner product $(\cdot,\cdot)$, such that*

$$\bar{V} = H \quad \text{and} \quad V \hookrightarrow H. \tag{3.2.19}$$

Then one has

$$|u-u_h| \leq M\|u-u_h\|\left(\sup_{g\in H}\left\{\frac{1}{|g|}\inf_{\varphi_h\in V_h}\|\varphi_g-\varphi_h\|\right\}\right), \tag{3.2.20}$$

where, for any $g \in H$, $\varphi_g \in V$ is the unique solution of the variational problem:

$$\forall v \in V, \quad a(v, \varphi_g) = (g, v). \tag{3.2.21}$$

Proof. To estimate $|u-u_h|$, we shall use the characterization

$$|u-u_h| = \sup_{g\in H}\frac{|(g, u-u_h)|}{|g|}. \tag{3.2.22}$$

Using the identification (3.2.16), we can solve problem (3.2.21) for all $g \in H$ (the proof is exactly the same as that of the Lax–Milgram lemma). Since $(u-u_h)$ is an element of the space V, we have in particular

$$a(u-u_h, \varphi_g) = (g, u-u_h),$$

on the one hand, and we have

$$\forall \varphi_h \in V_h, \quad a(u-u_h, \varphi_h) = 0,$$

on the other, which we obtain by subtracting (3.2.17) and (3.2.18). Using the above relations, we obtain

$$\forall \varphi_h \in V_h, \quad (g, u-u_h) = a(u-u_h, \varphi_g-\varphi_h),$$

and therefore,

$$|(g, u-u_h)| \leq M\|u-u_h\| \inf_{\varphi_h\in V_h}\|\varphi_g-\varphi_h\|. \tag{3.2.23}$$

Inequality (3.2.20) is therefore a consequence of the characterization (3.2.22) and inequality (3.2.23). □

A look at the above proof shows that φ_g had to be the solution of problem (3.2.21), i.e., where the arguments are interchanged in the bilinear form. Problem (3.2.21) is a special case of the following varia-

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tional problem: Given any element $g \in V'$, find an element $\varphi \in V$ such that

$$\forall v \in V, \quad a(v, \varphi) = g(v).$$

Such a problem is called the *adjoint problem* of problem (3.2.17). Of course the two problems coincide if the bilinear form is symmetric. It is easily verified that when the variational problem (3.2.17) corresponds to a second-order boundary value problem (cf. the examples given in Section 1.2), the same is true for its adjoint problem.

As we shall see, the abstract error estimate of Theorem 3.2.4 yields an improvement in the order of convergence for a restricted class of second-order problems, which we now define: A second-order boundary value problem whose variational formulation is (3.2.17), resp. (3.2.21), is said to be *regular* if the following two conditions are satisfied:

(i) For *any* $f \in L^2(\Omega)$, resp. *any* $g \in L^2(\Omega)$, the corresponding solution u_f, resp. u_g, is in the space $H^2(\Omega) \cap V$.

(ii) There exists a constant C such that

$$\forall f \in L^2(\Omega), \quad \|u_f\|_{2,\Omega} \leq C|f|_{0,\Omega}, \tag{3.2.24}$$

$$\text{resp.}\ \forall g \in L^2(\Omega), \quad \|\varphi_g\|_{2,\Omega} \leq C|g|_{0,\Omega}. \tag{3.2.25}$$

Remark 3.2.3. Consider for instance problem (3.2.17). Then without the assumption of regularity, we simply know that (use Remark 1.1.3 and the identification (3.2.16)):

$$\forall f \in L^2(\Omega), \quad \alpha\|u_f\|_{1,\Omega} \leq \|f\|^* = \sup_{v \in V} \frac{|f(v)|}{\|v\|_{1,\Omega}}$$
$$= \sup_{v \in V} \frac{|\int f v \,\mathrm{d}x|}{\|v\|_{1,\Omega}} \leq |f|_{0,\Omega}. \qquad \square$$

Indeed, this regularity is not too restrictive a condition: For example the homogeneous Dirichlet problem and homogeneous Neumann problem associated with the data of (1.2.23) (with $g = 0$) are regular if Ω is convex and if the functions a_{ij} and a are sufficiently smooth. However, this would not be the case for the homogeneous mixed problem of (1.2.28).

We are now in a position to estimate the error in the norm $|\cdot|_{0,\Omega}$.

Theorem 3.2.5. *In addition to* (H 1), (H 2) *and* (H 3), *assume that* $s = 0$, *that the dimension* n *is* ≤ 3, *and that there exists an integer* $k \geq 1$ *such* *that the solution u is in the space* $H^{k+1}(\Omega)$ *and such that the inclusions*

$$P_k(\hat{K}) \subset \hat{P} \subset H^1(\hat{K}) \tag{3.2.26}$$

hold.

Then if the adjoint problem is regular, there exists a constant C independent of h such that

$$|u - u_h|_{0,\Omega} \leq Ch^{k+1}|u|_{k+1,\Omega}. \tag{3.2.27}$$

Proof. Since $n \leq 3$, the inclusion $H^2(\hat{K}) \hookrightarrow \mathscr{C}^0(\hat{K})$ holds (if $s = 1$, the inclusion $H^2(\hat{K}) \hookrightarrow \mathscr{C}^1(\hat{K})$ holds only if $n = 1$; this is why we have restricted ourselves to the case $s = 0$). Applying Theorem 3.2.1 and inequality (3.2.25), we obtain, for each $g \in H = L^2(\Omega)$,

$$\inf_{\varphi_h \in V_h} \|\varphi_g - \varphi_h\|_{1,\Omega} \leq \|\varphi_g - \Pi_h \varphi_g\|_{1,\Omega} \leq Ch|\varphi_g|_{2,\Omega} \leq Ch|g|_{0,\Omega}.$$

Combining the above inequality with inequality (3.2.20) yields

$$|u - u_h|_{0,\Omega} \leq Ch\|u - u_h\|_{1,\Omega},$$

and it remains to use inequality (3.2.9) of Theorem 3.2.2. □

Concluding remarks. Inverse inequalities

Although we restricted ourselves to the case of a single partial differential equation, it should be clear that *the analysis of this section includes the systems of equations of plane and three-dimensional elasticity* (cf. (1.2.40)) posed over polygonal domains. In this case, the space $\mathbf{V}_h$ is a *product* of two or three identical finite element spaces V_h: With each degree of freedom of the space V_h, one associates two or three unknowns which are the corresponding components of the approximate displacement.

The asymptotic estimates obtained in Theorems 3.2.2 and 3.2.5 are the best one could hope for, inasmuch as *the orders of convergence are the same as if the discrete solution* u_h *were replaced by the* X_h*-interpolant of the function* u: Compare (3.2.9) and (3.2.5) with $m = 1$, and (3.2.27) and (3.2.5) with $m = 0$.

Consequently, the table in Fig. 3.1.2 is also useful for getting a practical appraisal of the upper bounds of Theorems 3.2.2 and 3.2.5. For instance, one gets $\|u - u_h\|_{m,\Omega} = O(h^{2-m})$, $m = 0, 1$, with n-simplices or rectangles of type (1), or $\|u - u_h\|_{m,\Omega} = O(h^{3-m})$, $m = 0, 1$, with n-simplices of type (2) or (3′) or rectangles of type (2) or (3′), etc... Neverthe-

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less, the higher the order of convergence, the higher the assumed regularity of the solution, and this observation limits considerably the practical value of such estimates. For example, let us assume that we are using n-simplices of type (3) while the solution is "only" in the space $H^2(\Omega)$: Then the application of Theorems 3.2.2 and 3.2.5 with $k=1$ shows that one gets only $\|u-u_h\|_{m,\Omega}=O(h^{2-m})$, $m=0,1$. Therefore, unless the solution is very smooth, the use of polynomials of high degree does not improve the quality of the approximation. Interestingly, the same conclusion was also drawn through purely practical considerations, by the engineers who seldom use polynomials of degree ≥ 4 for approximating the solution of second-order boundary value problems.

To conclude this section, we shall define a simple property of a family of triangulations (of the type described at the beginning of this section), whose value lies essentially (as usual) in the consequences which we shall derive (cf. Theorem 3.2.6). Although we had no immediate need for this property in the present section, it shall be used subsequently at various places, beginning in the next section, so that it seemed appropriate to record it here.

We shall say that a family of triangulations satisfies an *inverse assumption*, in view of the *inverse inequalities* to be established in the next theorem, if there exists a constant ν such that

$$\forall K \in \bigcup_h \mathcal{T}_h, \quad \frac{h}{h_K} \leq \nu. \tag{3.2.28}$$

Notice that this is by no means a restrictive condition in practice.

For such families, we are able to estimate the equivalence constants between familiar semi-norms (we remind the reader that σ is the constant which appears in the regularity assumption; cf. (3.2.1)).

Theorem 3.2.6. *Let there be given a family of triangulations which satisfies hypotheses* (H 1), (H 2) *and an inverse assumption, and let there be given two pairs* (l,r) *and* (m,q) *with* $l,m \geq 0$ *and* $(r,q) \in [1,\infty]$ *such that*

$$l \leq m \quad \text{and} \quad \hat{P} \subset W^{l,r}(\hat{K}) \cap W^{m,q}(\hat{K}). \tag{3.2.29}$$

Then there exists a constant $C = C(\sigma,\nu,l,r,m,q)$ *such that*

$$\forall v_h \in X_h, \quad \Big(\sum_{K\in\mathcal{T}_h} |v_h|^q_{m,q,K}\Big)^{1/q} \leq \frac{C}{(h^n)^{\max\{0,(1/r)-(1/q)\}}h^{m-l}} \Big(\sum_{K\in\mathcal{T}_h} |v_h|^r_{l,r,K}\Big)^{1/r} \tag{3.2.30}$$

if $p, q < \infty$, *with*

$$\max_{K\in\mathcal{T}_h} |v_h|_{m,\infty,K} \quad \textit{in lieu of} \quad \Big(\sum_{K\in\mathcal{T}_h} |v_h|^q_{m,q,K}\Big)^{1/q} \quad \textit{if } q=\infty,$$

$$\max_{K\in\mathcal{T}_h} |v_h|_{l,\infty,K} \quad \textit{in lieu of} \quad \Big(\sum_{K\in\mathcal{T}_h} |v_h|^r_{l,r,K}\Big)^{1/r} \quad \textit{if } r=\infty.$$

Proof. Given a function $v_h \in X_h$ and a finite element $K \in \mathcal{T}_h$, we have by Theorem 3.1.2,

$$\begin{aligned} |\hat{v}_K|_{l,r,\hat{K}} &\leq C\|B_K\|^l |\det(B)|^{-1/r} |v_h|_{l,r,K}, \\ |v_h|_{m,q,K} &\leq C\|B_K^{-1}\|^m |\det(B)|^{1/q} |\hat{v}_K|_{m,q,\hat{K}}, \end{aligned} \tag{3.2.31}$$

where the function $\hat{v}_K$ is in the standard correspondence with the function $v_h|_K$.

Define the space

$$\hat{N} = \{\hat{p} \in \hat{P};\ |\hat{p}|_{l,r,\hat{K}} = 0\} = \begin{cases} \{0\} & \text{if } l = 0, \\ \hat{P} \cap P_{l-1}(\hat{K}) & \text{if } l \geq 1. \end{cases}$$

Since $l \leq m$ by assumption, the implication

$$\hat{p} \in \hat{N} \Rightarrow |\hat{p}|_{m,q,\hat{K}} = 0$$

holds and therefore the mapping

$$\dot{\hat{p}} \in \hat{P}/\hat{N} \to \|\dot{\hat{p}}\|_{m,q,\hat{K}} = \inf_{\hat{s}\in\hat{N}} |\hat{p} - \hat{s}|_{m,q,\hat{K}}$$

is a norm over the quotient space $\hat{P}/\hat{N}$. Since this quotient space is finite-dimensional, this norm is equivalent to the quotient norm $\|\cdot\|_{l,r,\hat{K}}$ and therefore there exists a constant $\hat{C} = \hat{C}(l, r, m, q)$ such that

$$\hat{p} \in \hat{P}, \quad |\hat{p}|_{m,q,\hat{K}} = \|\dot{\hat{p}}\|_{m,q,\hat{K}} \leq \hat{C}\|\dot{\hat{p}}\|_{l,r,\hat{K}} = \hat{C}|\hat{p}|_{l,r,\hat{K}}. \tag{3.2.32}$$

Taking into account the regularity hypothesis and the inverse assumption, we obtain from inequalities (3.2.31) and (3.2.32) and Theorem 3.1.3,

$$|v_h|_{m,q,K} \leq C(\sigma, \nu) \frac{(h^n)^{1/q-1/r}}{h^{m-l}} |v_h|_{l,r,K}. \tag{3.2.33}$$

Assume first that $q = \infty$, so that there exists a finite element $K_0 \in \mathcal{T}_h$ such that, using (3.2.33),

$$\max_{K\in\mathcal{T}_h} |v_h|_{m,\infty,K} = |v_h|_{m,\infty,K_0} \leq C\frac{(h^n)^{-1/r}}{h^{m-l}} |v_h|_{l,r,K_0} \leq C\frac{(h^n)^{-1/r}}{h^{m-l}} |v_h|_{l,r,\Omega}.$$

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Assume next that $q < \infty$. We deduce from inequality (3.2.33) that

$$\Big(\sum_{K\in\mathcal{T}_h} |v_h|_{m,q,K}\Big)^{1/q} \le C\frac{(h^n)^{(1/q)-(1/r)}}{h^{m-l}}\Big(\sum_{K\in\mathcal{T}_h} |v_h|_{l,r,K}^q\Big)^{1/q}.$$

Then we distinguish three cases: Either $r \le q$, so that

$$\Big(\sum_{K\in\mathcal{T}_h} |v_h|_{l,r,K}^q\Big)^{1/q} \le \Big(\sum_{K\in\mathcal{T}_h} |v_h|_{l,r,K}^r\Big)^{1/r}$$

by Jensen's inequality, or $q < r < \infty$, so that Hölder's inequality yields

$$\Big(\sum_{K\in\mathcal{T}_h} |v_h|_{l,r,K}^q\Big)^{1/q} \le \mathcal{K}_h^{(1/q)-(1/r)}\Big(\sum_{K\in\mathcal{T}_h} |v_h|_{l,r,K}^r\Big)^{1/r}$$

with

$$\mathcal{K}_h = \operatorname{card} \mathcal{T}_h \le \frac{C(\sigma, \nu)}{h^n},$$

or, finally, $r = \infty$, in which case we get

$$\Big(\sum_{K\in\mathcal{T}_h} |v_h|_{l,\infty,K}^q\Big)^{1/q} \le \mathcal{K}_h^{1/q} \max_{K\in\mathcal{T}_h} |v_h|_{l,\infty,K},$$

and inequality (3.2.30) is proved in all cases. □

Inequalities of the form (3.2.30) are of course immediately converted into inequalities involving the semi-norms $|\cdot|_{m,q,\Omega}$ or $|\cdot|_{l,r,\Omega}$ if it so happens that the inclusions $X_h \subset W^{m,q}(\Omega)$ or $X_h \subset W^{l,r}(\Omega)$ holds.

For example, let us assume that hypothesis (H 3) is satisfied and that the inclusion $\hat{P} \subset H^1(\hat{K})$ holds so that the inclusion $X_h \subset \mathscr{C}^0(\bar{\Omega}) \cap H^1(\Omega)$ holds. Then we have

$$\forall v_h \in X_h, \quad |v_h|_{0,\infty,\Omega} \le \frac{C}{h^{n/2}}|v_h|_{0,\Omega}, \tag{3.2.34}$$

$$\forall v_h \in X_h, \quad |v_h|_{1,\Omega} \le \frac{C}{h}|v_h|_{0,\Omega}, \quad \text{etc.}\ldots \tag{3.2.35}$$

If now hypothesis (H3) is satisfied and if the inclusion $\hat{P} \subset W^{1,\infty}(\hat{K})$ holds, then we get similarly

$$\forall v_h \in X_h, \quad |v_h|_{1,\infty,\Omega} \le \frac{C}{h}|v_h|_{0,\infty,\Omega}, \text{etc.}\ldots \tag{3.2.36}$$

Another observation is that similar inequalities between *norms* can be directly derived from these inequalities. For instance, we obtain

$$\forall v_h \in X_h, \quad \|v_h\|_{1,\Omega} \le \frac{C}{h}|v_h|_{0,\Omega}. \tag{3.2.37}$$

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Remark 3.2.4. Inverse inequalities can be likewise established between the above semi-norms and other semi-norms or norms, such as $\|\cdot\|_{L^p(\Gamma)}$. In this direction, see Exercise 3.2.4. □

Exercises

3.2.1. The object of this problem is to indicate a way of approximating the solution of a *nonhomogeneous Dirichlet problem* (see also the next problem) whose solution $u \in H^1(\Omega)$ satisfies (cf. Exercise 1.2.2)

$$\begin{cases} (u - u_0) \in H_0^1(\Omega), \\ \forall v \in H_0^1(\Omega), \quad a(u, v) = f(v), \end{cases}$$

where u_0 is a given function in the space $H^1(\Omega)$, and the forms $a(\cdot,\cdot)$ and $f(\cdot)$ satisfy the usual assumptions of the Lax–Milgram lemma, the bilinear form being assumed to be symmetric in addition.

Given a finite element space X_h, we let as usual

$$X_{0h} = \{v_h \in X_h; \quad v_h = 0 \quad \text{on} \quad \Gamma\}.$$

(i) Given a function $u_{0h} \in X_h$, show that the discrete problem: Find $u_h \in X_h$ such that

$$\begin{cases} u_h \in u_{0h} + X_{0h} = \{v_h \in X_h; \quad (v_h - u_{0h}) \in X_{0h}\} \\ \forall v_h \in X_{0h}, \quad a(u_h, v_h) = f(v_h), \end{cases}$$

has a unique solution.

(ii) Show that (Strang & Fix (1973, p. 200)):

$$\|u - u_h\|_{1,\Omega} \leq \sqrt{\frac{M}{\alpha}} \inf_{v_h \in (u_{0h} + X_{0h})} \|u - v_h\|.$$

(iii) Assume that the spaces X_h are made up of n-simplices of type (k). Indicate how should one choose the function u_{0h} in order that

$$\|u - u_h\|_{1,\Omega} = \mathrm{O}(h^k),$$

assuming the functions u and u_0 are sufficiently smooth.

3.2.2. This problem describes a *penalty method* for approximating the solution of a nonhomogeneous Dirichlet problem, whose solution $u \in H^1(\Omega)$ satisfies (cf. Exercise 1.2.2):

$$\begin{cases} (u - u_0) \in H_0^1(\Omega), \\ \forall v \in H_0^1(\Omega), \quad a(u, v) = f(v), \end{cases}$$

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where u_0 is a given function in the space $H^1(\Omega)$ and where for simplicity, we shall assume that

$$a(u, v) = \int_\Omega \sum_{i=1}^n \partial_i u \partial_i v \,\mathrm{d}x, \qquad f(v) = \int_\Omega fv \,\mathrm{d}x, \quad f \in L^2(\Omega).$$

In what follows, we consider a family of finite element spaces X_h. We are also given a family of real numbers $\epsilon(h) > 0$ with $\lim_{h\to 0} \epsilon(h) = 0$.

(i) Show that, for each h, the discrete problem: Find $u_h \in X_h$ such that

$$\forall v_h \in X_h, \quad a(u_h, v_h) + \frac{1}{\epsilon(h)} \int_\Gamma (u_h - u_0) v_h \,\mathrm{d}\gamma = f(v_h),$$

has a unique solution.

(ii) Assume that the solution u is in the space $H^2(\Omega)$. Show that, for all $v_h \in X_h$,

$$\begin{aligned} |u - u_h|_{1,\Omega}^2 + \frac{1}{\epsilon(h)} \|u_h - u_0\|_{L^2(\Gamma)}^2 = {} & a(u_h - u, v_h - u) \\ & + \frac{1}{\epsilon(h)} \int_\Gamma (u_h - u)(v_h - u) \,\mathrm{d}\gamma \\ & - \int_\Gamma \partial_\nu u (u_h - u) \,\mathrm{d}\gamma \\ & - \int_\Gamma \partial_\nu u (u - v_h) \,\mathrm{d}\gamma. \end{aligned}$$

Using this identity and the inequality $ab \leq \eta a^2 + (1/\eta) b^2$ valid for all $\eta > 0$, derive the following abstract error estimate: There exists a constant C independent of h such that

$$\begin{aligned} |u - u_h|_{1,\Omega} \leq C \inf_{v_h \in X_h} \Big\{ & |u - v_h|_{1,\Omega}^2 + \frac{1}{\epsilon(h)} \|u_0 - v_h\|_{L^2(\Gamma)}^2 \\ & + \epsilon(h) \|\partial_\nu u\|_{L^2(\Gamma)}^2 \Big\}^{1/2}. \end{aligned}$$

(iii) Assume that the spaces X_h are made up of n-simplices of type (k) and that the solution u is sufficiently smooth. Show that, for some constant C independent of h,

$$|u - u_h|_{1,\Omega} \leq C \Big(\frac{h^k}{\sqrt{\epsilon(h)}} + \sqrt{\epsilon(h)} \Big) \|u\|_{k+1,\Omega},$$

and thus deduce the optimal choice for $\epsilon(h)$, so as to maximize the order

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of convergence (therefore, as far as the order of convergence is concerned, the method proposed in the previous exercise is preferable).

3.2.3. The purpose of this problem is to analyze a procedure of CLÉMENT (1975) for defining an operator whose approximation properties are similar to those of the standard interpolation operator but which can be defined in more general situations.

For simplicity, we shall consider finite element spaces X_h made up of triangles of type (1), but the analysis can be extended to triangular finite elements with polynomials of higher degree.

With each vertex b_i, $1 \leq i \leq M$, of the triangulation, we associate a basis function $w_i \in X_h$ in the usual manner, i.e., one has

$$w_i(b_j) = \delta_{ij}, \quad 1 \leq i, j \leq M.$$

For each i, we set

$$S_i = \operatorname{supp} w_i.$$

Given a function $v \in L^2(\Omega)$, we let $P_i v$ denote, for each $i = 1, \ldots, M$, the projection of the function v in the space $L^2(S_i)$ over the subspace $P_1(S_i)$, i.e., one has

$$P_i v \in P_1(S_i) \quad \text{and} \quad \forall p \in P_1(S_i), \quad \int_{S_i} (v - P_i v) p \, \mathrm{d}x = 0,$$

and we set

$$r_h v = \sum_{i=1}^{M} P_i v(b_i) w_i.$$

In this fashion, we have defined a mapping

$$r_h : L^2(\Omega) \to X_h.$$

In the sequel, we consider a family of spaces X_h associated with a regular family of triangulations.

(i) Show that there exists a constant C independent of h such that

$$\forall i \in [1, M], \quad K \subset S_i \Rightarrow \operatorname{diam} S_i \leq C h_K,$$

and that there exists an integer ν independent of h such that

$$\forall i \in [1, M], \quad \operatorname{card}\{K \in \mathcal{T}_h; \quad K \subset S_i\} \leq \nu.$$

(ii) Show that there exists a constant C independent of h such that

$$\forall i \in [1, M], \quad \forall v \in H^1(S_i),$$

$$|v - P_i v|_{m,S_i} \leq C(\operatorname{diam} S_i)^{l-m}|v|_{l,S_i}, \quad 0 \leq m \leq l \leq 2.$$

(iii) Show that there exists a constant C independent of h such that

$$\forall K \in \mathcal{T}_h, \quad \forall p \in P_1(K),$$
$$|p|_{m,\infty,K} \leq C(\operatorname{meas}(K))^{-1/2} h_K^{-m}|p|_{0,K}, \quad m = 0, 1.$$

(iv) Show that there exists a constant C independent of h such that

$$\forall i \in [1, M], \quad |w_i|_{m,K} \leq C(\operatorname{meas}(K))^{1/2} h_K^{-m}, \quad m = 0, 1.$$

(v) Show that

$$\forall v \in L^2(\Omega), \quad \begin{cases} |v - r_h v|_{0,\Omega} \leq C|v|_{0,\Omega}, \\ \lim\limits_{h \to 0} |v - r_h v|_{0,\Omega} = 0, \end{cases}$$

$$\forall v \in H^1(\Omega), \quad \begin{cases} |v - r_h v|_{m,\Omega} \leq C h^{1-m}|v|_{1,\Omega}, \quad m = 0, 1, \\ \lim\limits_{h \to 0} |v - r_h v|_{1,\Omega} = 0, \end{cases}$$

$$\forall v \in H^2(\Omega), \quad \begin{cases} |v - r_h v|_{m,\Omega} \leq C h^{2-m}|v|_{2,\Omega}, \quad m = 0, 1, \\ \left(\sum\limits_{K \in \mathcal{T}_h} |v - r_h v|_{2,K}^2\right)^{1/2} \leq C|v|_{2,\Omega}, \end{cases}$$

where C denotes as usual various constants independent of h.

[Hint: Let $K \in \mathcal{T}_h$ be a triangle with vertices b_i, $1 \leq i \leq 3$. Prove the identity

$$(r_h v - v)_{|K} = (P_1 v - v)_{|K} + \sum_{i=2}^{3} (P_i v(b_i) - P_1 v(b_i)) w_{i|K}$$

and use the previous questions to estimate appropriate semi-norms of the functions $(P_1 v - v)_{|K}$ and $\Sigma_{i=2}^3 (P_i v(b_i) - P_1 v(b_i)) w_{i|K}$.]

(vi) If the function v belongs to the space $H_0^1(\Omega)$, is the function $r_h v$ in the space $X_{0h} = \{v_h \in X_h;\ v_h = 0 \text{ on } \Gamma\}$?

(vii) Apply the results of question (v) to the approximation of a second-order boundary value problem. Compare with Theorems 3.2.2 and 3.2.5.

3.2.4. Let there be given a family of triangulations which satisfies hypothesis (H 2) and an inverse assumption. It is also assumed that $\hat{P} \subset \mathscr{C}^0(\hat{K})$. Then show that for each $p \in [1, \infty]$, there exists a constant $C = C(p)$ independent of h such that

$$\forall v_h \in X_h, \quad \|v_h\|_{L^p(\Gamma)} \leq \frac{C}{h^{1/p}}|v_h|_{0,p,\Omega}.$$

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3.3. Uniform convergence

A model problem. Weighted semi-norms $|\cdot|_{\phi;m,\Omega}$

For ease of exposition, we shall simply consider the homogeneous Dirichlet problem for the operator $-\Delta$, which corresponds to the following data:

$$\begin{cases} V = H_0^1(\Omega), \\ a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \\ f(v) = \int_\Omega fv \, dx, \quad f \in L^2(\Omega). \end{cases} \tag{3.3.1}$$

Assuming that $\bar{\Omega}$ is a *convex polygonal* subset of $\mathbf{R}^2$, we shall restrict ourselves to finite element spaces X_h which are made up of *triangles of type* (1), so that the corresponding discrete problems are posed in the spaces $V_h = \{v_h \in X_h;\ v_h = 0 \text{ on } \Gamma\}$ (results concerning the use of triangles of type (k) and higher dimensions are indicated at the end of this section and in the section "Bibliography and Comments").

We shall assume once and for all that we are given a family of triangulations of the set $\bar{\Omega}$ which

(i) is *regular* and

(ii) satisfies an *inverse assumption*, i.e., there exist two constants σ and ν such that

$$\forall K \in \bigcup_h \mathcal{T}_h, \quad \frac{h_K}{\rho_K} \leq \sigma, \quad \text{and} \quad \frac{h}{h_K} \leq \nu. \tag{3.3.2}$$

Our main tool in the study of the error in the norms $|\cdot|_{0,\infty,\Omega}$ and $\|\cdot\|_{1,\infty,\Omega}$ will be the consideration of appropriate *weighted norms and semi-norms*. Accordingly, the first part of this section will be devoted to the study of those properties of such semi-norms which are of interest for our subsequent analysis (cf. Theorems 3.3.1 to 3.3.4).

Given a *weight-function* ϕ, i.e., a function which satisfies

$$\phi \in L^\infty(\Omega) \quad \text{and} \quad \phi \geq 0 \text{ a.e.} \quad \text{on} \quad \Omega, \tag{3.3.3}$$

we define, for each integer $m \geq 0$, the *weighted semi-norms*

$$v \in H^m(\Omega) \to |v|_{\phi;m,\Omega} = \left(\int_\Omega \phi \sum_{|\beta|=m} |\partial^\beta v|^2 \, dx \right)^{1/2}. \tag{3.3.4}$$

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To begin with, we observe that, *if the function* ϕ^{-1} *exists and is also in the space* $L^\infty(\Omega)$, an application of Cauchy–Schwarz inequality gives

$$\forall \alpha \in \mathbf{R}, \quad \forall u, v \in H^1(\Omega), \quad a(u, v) \leq |u|_{\phi^\alpha;1,\Omega} |v|_{\phi^{-\alpha};1,\Omega}. \tag{3.3.5}$$

Departing from the general case, we shall in fact concentrate our subsequent study on weighted semi-norms of the particular type $|\cdot|_{\phi^\alpha;m,\Omega}$, $\alpha \in \mathbf{R}$, where the function ϕ is of the form (3.3.7) below. Our first task is to extend to such weighted semi-norms the property that there exists a constant c_1, solely dependent upon the set Ω, such that

$$\forall w \in H_0^1(\Omega) \cap H^2(\Omega), \quad |w|_{2,\Omega} \leq c_1 |\Delta w|_{0,\Omega}. \tag{3.3.6}$$

Theorem 3.3.1. *There exists a constant* $C_1 = C_1(\Omega)$ *such that, for all functions* ϕ *of the form*

$$\phi : x \in \bar{\Omega} \to \phi(x) = \frac{1}{\|x - \bar{x}\|^2 + \theta^2}, \quad \theta > 0, \quad \bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathbf{R}^2, \tag{3.3.7}$$

we have

$$\forall v \in H_0^1(\Omega) \cap H^2(\Omega), \quad |v|^2_{\phi^{-1};2,\Omega} \leq C_1 (|\Delta v|^2_{\phi^{-1};0,\Omega} + |v|^2_{1,\Omega}). \tag{3.3.8}$$

Proof. Let v be an arbitrary function in the space $H_0^1(\Omega) \cap H^2(\Omega)$. Then the function

$$w = (x_1 - \bar{x}_1) v$$

also belongs to the space $H_0^1(\Omega) \cap H^2(\Omega)$, and

$$\begin{aligned}
(x_1 - \bar{x}_1)\partial_{11} v &= \partial_{11} w - 2\partial_1 v, \\
(x_1 - \bar{x}_1)\partial_{12} v &= \partial_{12} w - \partial_2 v, \\
(x_1 - \bar{x}_1)\partial_{22} v &= \partial_{22} w, \\
\Delta w &= (x_1 - \bar{x}_1)\Delta v + 2\partial_1 v.
\end{aligned}$$

Using these relations and inequality (3.3.6), we find a constant c_2 such that

$$\begin{aligned}
\int_\Omega (x_1 - \bar{x}_1)^2 \sum_{|\beta|=2} |\partial^\beta v|^2 \,\mathrm{d}x &\leq 2c_1^2 |\Delta w|^2_{0,\Omega} + 8|v|^2_{1,\Omega} \\
&\leq c_2 \left\{ \int_\Omega (x_1 - \bar{x}_1)^2 (\Delta v)^2 \,\mathrm{d}x + |v|^2_{1,\Omega} \right\}.
\end{aligned}$$

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Since we have likewise

$$\int_\Omega (x_2-\bar{x}_2)^2 \sum_{|\beta|=2} |\partial^\beta v|^2 \,dx \le c_2\left\{\int_\Omega (x_2-\bar{x}_2)^2(\Delta v)^2\,dx + |v|_{1,\Omega}^2\right\},$$

we eventually obtain

$$|v|^2_{\phi^{-1};2,\Omega} = \int_\Omega ((x_1-\bar{x}_1)^2+(x_2-\bar{x}_2)^2+\theta^2) \sum_{|\beta|=2} |\partial^\beta v|^2\,dx$$

$$\le \max\{2c_2, c_1^2\}(|\Delta v|^2_{\phi^{-1};0,\Omega} + |v|^2_{1,\Omega}),$$

and the proof is complete. □

As exemplified by the above computations, we shall depart in this section from our practice of letting the same letter C denote various constants, not necessarily the same in their various occurrences. This is due not only to the unusually large number of such constants which we shall come across, but also – and essentially – to their sometimes intricate interdependence. Therefore, constants will be numbered and, in addition, their dependence on other quantities will be made explicit when necessary. However the possible dependence upon the set Ω and the constants σ and ν of (3.3.2) will be systematically omitted. While we shall use capital letters C_i, $i \ge 1$, for constants occurring in important inequalities, small letters c_i, $i \ge 1$, will rather be reserved for intermediate computations.

In the next two theorems, we examine the relationships between the weighted semi-norms $|\cdot|_{\phi^\alpha;m,\Omega}$ (the function ϕ being as in (3.3.7)) and the standard semi-norms $|\cdot|_{m,\infty,\Omega}$. Such relationships will play a crucial role in the derivation of the eventual error estimates.

Theorem 3.3.2. *For each number $\alpha > 1$ and each integer $m \ge 0$, there exists a constant $C_2(\alpha, m)$ such that, for all functions ϕ of the form*

$$\phi: x \in \bar{\Omega} \to \phi(x) = \frac{1}{\|x-\bar{x}\|^2+\theta^2}, \quad \theta > 0, \quad \bar{x} \in \bar{\Omega}, \tag{3.3.9}$$

we have

$$\forall v \in W^{m,\infty}(\Omega), \quad |v|_{\phi^\alpha;m,\Omega} \le C_2(\alpha, m)\frac{1}{\theta^{\alpha-1}}|v|_{m,\infty,\Omega}. \tag{3.3.10}$$

For each number $\beta \in]0, 1[$, and each integer $m \ge 0$, there exists a *constant* $C_3(\beta, m)$ *such that for all functions* ϕ *of the form* (3.3.9), *we have*

$$\forall \theta \leqslant \beta, \quad \forall v \in W^{m,\infty}(\Omega), \quad |v|_{\phi;m,\Omega} \leqslant C_3(\beta, m)|\ln \theta|^{1/2}|v|_{m,\infty,\Omega}. \tag{3.3.11}$$

Proof. Clearly, one has

$$|v|_{\phi^\alpha;m,\Omega} \leqslant c_3(m)\Big(\int_\Omega \phi^\alpha \,\mathrm{d}x\Big)^{1/2}|v|_{m,\infty,\Omega}.$$

Next let $\delta = \operatorname{diam}(\Omega)$, so that

$$\int_\Omega \phi^\alpha \,\mathrm{d}x \leqslant \int_{B(\bar{x};\delta)} \phi^\alpha \,\mathrm{d}x = 2\pi \int_0^\delta \frac{\tau \,\mathrm{d}\tau}{(\tau^2+\theta^2)^\alpha}.$$

If $\alpha > 1$, write

$$\int_0^\delta \frac{\tau \,\mathrm{d}\tau}{(\tau^2+\theta^2)^\alpha} \leqslant \int_0^\infty \frac{\tau \,\mathrm{d}\tau}{(\tau^2+\theta^2)^\alpha} = \frac{1}{2(\alpha-1)\theta^{2(\alpha-1)}},$$

and inequality (3.3.10) is proved with $C_2(\alpha, m) = c_3(m)(\pi/(\alpha-1))^{1/2}$. If $\alpha = 1$, we have for $\theta \leqslant \beta < 1$,

$$\int_0^\delta \frac{\tau \,\mathrm{d}\tau}{\tau^2+\theta^2} = |\ln \theta| + \frac{1}{2}\ln(\theta^2+\delta^2)$$
$$\leqslant |\ln \theta| + \frac{1}{2}\ln(1+\delta^2) \leqslant c_4(\beta)|\ln \theta|,$$

with

$$c_4(\beta) = 1 + \frac{\ln(1+\delta^2)}{2|\ln \beta|},$$

and inequality (3.3.11) is proved with $C_3(\beta, m) = c_3(m)(2\pi c_4(\beta))^{1/2}$. □

We next obtain inequalities in the opposite direction. In order that they be useful for our subsequent purposes, however, we shall establish these inequalities only for functions in the finite element space X_h, and further, we shall restrict ourselves to weight-functions of the form ϕ or ϕ^2, with ϕ as in (3.3.9), for which (i) the parameter θ cannot approach zero too rapidly when h approaches zero (cf. (3.3.13)), and for which (ii) the points $\bar{x}$ depend upon the particular function $v_h \in X_h$ under consideration (cf. (3.3.15) and (3.3.17)).

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Theorem 3.3.3. *For each number $\gamma > 0$, there exist constants $C_4(\gamma)$ and $C_5(\gamma)$ such that, if for each h, we are given a function ϕ_h of the form*

$$\phi_h : x \in \bar{\Omega} \to \phi_h(x) = \frac{1}{\|x - x_h\|^2 + \theta_h^2}, \quad x_h \in \bar{\Omega}, \tag{3.3.12}$$

in such a way that

$$\exists \gamma > 0, \quad \forall h, \theta_h \geq \gamma h, \tag{3.3.13}$$

then (i) *we have*

$$\forall v_h \in X_h, \quad |v_h|_{0,\infty,\Omega} \leq C_4(\gamma) \frac{\theta_h^2}{h} |v_h|_{\phi_h^2;0,\Omega} \tag{3.3.14}$$

if, for each function $v_h \in X_h$, the point $x_h \in \bar{\Omega}$ in (3.3.12) *is chosen in such a way that*

$$|v_h(x_h)| = |v_h|_{0,\infty,\Omega}, \tag{3.3.15}$$

and (ii) *we have*

$$\forall v_h \in X_h, \quad |v_h|_{1,\infty,\Omega} \leq C_5(\gamma) \frac{\theta_h}{h} |v_h|_{\phi_h;1,\Omega} \tag{3.3.16}$$

if, for each function $v_h \in X_h$, the point $x_h \in \bar{\Omega}$ in (3.3.12) *is chosen in such a way that*

$$\max\{|\partial_1 v_h(x_h)|, \quad |\partial_2 v_h(x_h)|\} = |v_h|_{1,\infty,\Omega}. \tag{3.3.17}$$

Proof. (i) Let v_h be an arbitrary function in the space X_h, and let the point x_h be chosen as in (3.3.15). We can write

$$\begin{aligned}\forall x \in \bar{\Omega}, \quad |v_h|_{0,\infty,\Omega} - |v_h(x)| = |v_h(x_h)| - |v_h(x)| &\leq |v_h(x_h) - v_h(x)| \\ &\leq \sqrt{2}|v_h|_{1,\infty,\Omega}\|x - x_h\| \\ &\leq \frac{c_5}{h}|v_h|_{0,\infty,\Omega}\|x - x_h\|,\end{aligned}$$

for some constant c_5 (in the last inequality we have used the fact that the family of triangulations satisfies an inverse assumption; cf. Theorem 3.2.6). In other words,

$$\forall x \in \bar{\Omega}, \quad |v_h(x)| \geq \left(1 - \frac{c_5}{h}\|x - x_h\|\right)|v_h|_{0,\infty,\Omega},$$

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and consequently ($B(a;r) = \{x \in \mathbf{R}^2; \|x - a\| \leq r\}$),

$$|v_h|^2_{\phi_h^2;0,\Omega} \geq |v_h|^2_{0,\infty,\Omega} \int_{\bar{\Omega} \cap B(x_h; h/2c_5)} \left(\frac{1 - \frac{c_5}{h}\|x - x_h\|}{\|x - x_h\|^2 + \theta_h^2} \right)^2 \mathrm{d}x.$$

The set $\bar{\Omega}$ being polygonal, there exists a constant c_6 such that

$$\operatorname{meas}\left\{\bar{\Omega} \cap B\left(x_h; \frac{h}{2c_5}\right)\right\} \geq c_6\left(\frac{h}{2c_5}\right)^2.$$

We also have

$$\forall x \in B\left(x_h; \frac{h}{2c_5}\right), \quad 1 - \frac{c_5}{h}\|x - x_h\| \geq \frac{1}{2},$$

and

$$\forall x \in B\left(x_h; \frac{h}{2c_5}\right), \quad \frac{1}{\|x - x_h\|^2 + \theta_h^2} \geq \frac{1}{\frac{h^2}{4c_5^2} + \theta_h^2} \geq \frac{1}{\theta_h^2\left(1 + \frac{1}{4c_5^2\gamma^2}\right)},$$

by assumption (3.3.13). Combining the previous inequalities, we obtain an inequality of the form (3.3.14) with

$$C_4(\gamma) = \frac{4c_5}{\sqrt{c_6}}\left(1 + \frac{1}{4c_5^2\gamma^2}\right).$$

(ii) Let v_h be an arbitrary function in the space X_h, let the point x_h be chosen as in (3.3.17), and let $K_h \in \mathcal{T}_h$ denote a triangle which contains the point x_h. Since the gradient ∇v_h is constant over the set K_h, we deduce

$$|v_h|^2_{\phi_h;1,\Omega} \geq |v_h|^2_{1,\infty,\Omega} \int_{K_h} \frac{\mathrm{d}x}{\|x - x_h\|^2 + \phi_h^2}.$$

With this inequality and the inequalities

$$\operatorname{meas} K_h \geq c_7(\sigma, \nu)h^2,$$

$$\forall x \in K_h, \quad \frac{1}{\|x - x_h\|^2 + \theta_h^2} \geq \frac{1}{\theta_h^2\left(1 + \frac{1}{\gamma^2}\right)},$$

we obtain an inequality of the form (3.3.16) with

$$C_5(\gamma) = \sqrt{\frac{1}{c_7}\left(1 + \frac{1}{\gamma^2}\right)}. \qquad \square$$

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To conclude this analysis of weighted semi-norms, we examine in the next theorem the *interpolation error estimates* in the semi-norms $|\cdot|_{\phi_h^\alpha;m,\Omega}$ where, for each h, the function ϕ_h is of the form (3.3.12). The conclusion (cf. (3.3.20)) is that *the error estimates are exactly the same as in the case of the usual semi-norms* $|\cdot|_{m,\Omega}$ *provided the parameter* θ_h *does not approach zero too rapidly with* h (cf. (3.3.19)). Notice, however, that if the behavior of the function θ_h can be "at best" linear as in the previous theorem, the constant which appears in inequality (3.3.19) is not arbitrary, by contrast with the constant γ which appeared in inequality (3.3.13). Finally, observe that no restriction will be imposed upon the points x_h.

Theorem 3.3.4. *There exists a constant* C_6 *and, for each* $\alpha \in \mathbf{R}-\{0\}$, *there exist constants* $C_7(\alpha) > 0$ *and* $C_8(\alpha)$ *such that, if for each* h, *we are given a function* ϕ_h *of the form*

$$\phi_h: x \in \bar{\Omega} \to \phi_h(x) = \frac{1}{\|x - x_h\|^2 + \theta_h^2}, \quad x_h \in \bar{\Omega}, \tag{3.3.18}$$

in such a way that

$$\forall h, \quad \theta_h \geqslant C_7(\alpha) h, \tag{3.3.19}$$

then (i) *we have*

$$\forall v \in H^2(\Omega), \quad |v - \Pi_h v|_{\phi_h^\alpha;m,\Omega} \leqslant C_6 h^{2-m} |v|_{\phi_h^\alpha;2,\Omega}, \quad m = 0, 1, \tag{3.3.20}$$

and (ii) *we have*

$$\forall v_h \in X_h, \quad |\phi_h^\alpha v_h - \Pi_h(\phi_h^\alpha v_h)|_{\phi_h^{-\alpha};1,\Omega} \leqslant$$

$$\leqslant C_8(\alpha) \frac{h}{\theta_h} \left(|v_h|_{\phi_h^{\alpha+1};0,\Omega} + |v_h|_{\phi_h^\alpha;1,\Omega} \right). \tag{3.3.21}$$

Proof. (i) There exists a constant c_8 such that

$$\forall v \in H^2(\Omega), \ \forall K \in \mathcal{T}_h, \quad |v - \Pi_K v|_{m,K} \leqslant c_8 h^{2-m} |v|_{2,K}, \quad m = 0, 1.$$

Next, we have

$$|v - \Pi_K v|_{\phi_h^\alpha;m,K} \leqslant (\phi_h^\alpha(\bar{x}_K))^{1/2} |v - \Pi_K v|_{m,K},$$
$$|v|_{2,K} \leqslant (\phi_h^\alpha(x_K))^{-1/2} |v|_{\phi_h^\alpha;2,K},$$

where, for each $K \in \mathcal{T}_h$, the points $x_K \in K$ and $\bar{x}_K \in K$ are chosen in

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such a way that

$$0 < \phi_h^\alpha(x_K) = \inf_{x\in K} \phi_h^\alpha(x), \quad \phi_h^\alpha(\bar{x}_K) = \sup_{x\in K} \phi_h^\alpha(x).$$

Since

$$\frac{\partial_i(\phi_h^\alpha)(x)}{\phi_h^\alpha(x)} = -2\alpha \frac{(x_i - x_{hi})}{\|x - x_h\|^2 + \theta_h^2}, \quad i = 1, 2,$$

we obtain

$$\sup_{x\in\bar{\Omega}} \frac{|\partial_i(\phi_h^\alpha)(x)|}{\phi_h^\alpha(x)} \leqslant \sup_{x\in\bar{\Omega}} \frac{\|D(\phi_h^\alpha)(x)\|}{\phi_h^\alpha(x)} \leqslant 2|\alpha| \sup_{x\in\bar{\Omega}} \frac{\|x - x_h\|}{\|x - x_h\|^2 + \theta_h^2} \leqslant \frac{|\alpha|}{\theta_h},$$

and therefore

$$\phi_h^\alpha(\bar{x}_K) \leqslant \phi_h^\alpha(x_K) + |\alpha| \frac{h}{\theta_h} \phi_h^\alpha(\bar{x}_K).$$

Consequently, if we let

$$C_7(\alpha) = 2|\alpha|,$$

so that

$$\theta_h \geqslant C_7(\alpha) h \Rightarrow \forall K \in \mathcal{T}_h, \frac{\phi_h^\alpha(\bar{x}_K)}{\phi_h^\alpha(x_K)} \leqslant 2,$$

the conjunction of the above inequalities yields inequality (3.3.20) with $C_6 = \sqrt{2} c_8$.

(ii) Since the function $\phi_h^\alpha v_h$ is in the space $\mathscr{C}^0(\bar{\Omega}) = \operatorname{dom} \Pi_h$ and since the restrictions $\phi_h^\alpha v_h|_K$ belong to the space $H^2(K)$ for all $K \in \mathcal{T}_h$, the same argument as in (i) shows that

$$|\phi_h^\alpha v_h - \Pi_K(\phi_h^\alpha v_h)|_{\phi_h^{-\alpha};1,\Omega} \leqslant C_7 h \Big(\sum_{K\in\mathcal{T}_h} |\phi_h^\alpha v_h|^2_{\phi_h^{-\alpha};2,K} \Big)^{1/2}.$$

We have (recall that $v_{h|K} \in P_1(K)$ for all $K \in \mathcal{T}_h$)

$$\forall K \in \mathcal{T}_h, \quad \partial_{ij}(\phi_h^\alpha v_h) = (\partial_{ij}\phi_h^\alpha) v_h + (\partial_i \phi_h^\alpha)\partial_j v_h + (\partial_j \phi_h^\alpha)\partial_i v_h \quad \text{in} \quad K,$$

and

$$\frac{\partial_{ij}(\phi_h^\alpha)(x)}{\phi_h^\alpha(x)} = 4\alpha(\alpha+1) \frac{(x_i - x_{hi})(x_j - x_{hj})}{(\|x - x_h\|^2 + \theta_h^2)^2} - 2\alpha \frac{\delta_{ij}}{\|x - x_h\|^2 + \theta_h^2}.$$

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Using the inequalities

$$\frac{\|x-x_h\|^2}{(\|x-x_h\|^2+\theta_h^2)^2}\leq\frac{\phi_h^{1/2}(x)}{2\theta_h},\quad\text{and}\quad\frac{1}{\|x-x_h\|^2+\theta_h^2}\leq\frac{\phi_h^{1/2}(x)}{\theta_h},$$

we deduce that

$$\forall x\in\bar{\Omega},\quad|\partial_{ij}(\phi_h^\alpha)(x)|\leq\frac{2}{\theta_h}(|\alpha^2+\alpha|+|\alpha|)\phi_h^{\alpha+1/2}(x).$$

Using the above inequalities and the inequality (cf. part (i))

$$\forall x\in\bar{\Omega},\quad|\partial_i(\phi_h^\alpha)(x)|\leq\frac{|\alpha|}{\theta_h}\phi_h^\alpha(x),$$

we conclude that there exists a constant $c_9(\alpha)$ such that

$$\forall x\in K,\quad\phi_h^{-\alpha}(x)\sum_{i,j=1}^{2}|\partial_{ij}(\phi_h^\alpha v_h)(x)|^2\leq$$
$$\leq\frac{c_9(\alpha)}{\theta_h^2}\left(\phi_h^{\alpha+1}(x)|v_h(x)|^2+\phi_h^\alpha(x)\sum_{i=1}^{2}|\partial_i v_h(x)|^2\right),$$

and thus,

$$\sum_{K\in\mathcal{T}_h}|\phi_h^\alpha v_h|^2_{\phi_h^{-\alpha};2,K}=\sum_{K\in\mathcal{T}_h}\int_K\phi_h^{-\alpha}\sum_{i,j=1}^{2}|\partial_{ij}(\phi_h^\alpha v_h)(x)|^2\,dx\leq$$
$$\leq\frac{c_9(\alpha)}{\theta_h^2}(|v_h|^2_{\phi_h^{\alpha+1};0,\Omega}+|v_h|^2_{\phi_h^\alpha;1,\Omega}).$$

Therefore we have proved inequality (3.3.21), with

$$C_8(\alpha)=C_7\sqrt{c_9(\alpha)}.\qquad\square$$

Uniform boundedness of the mapping $u\to u_h$ with respect to appropriate weighted norms

After the above preliminaries, we now come to the central object of this section, i.e., the estimate of the errors $|u-u_h|_{0,\infty,\Omega}$ and $|u-u_h|_{1,\infty,\Omega}$ via *the method of weighted norms of J.A. Nitsche.* The analysis will comprise *three stages.* In the *first stage* (cf. the next theorem), we consider for each *h the projection operator*

$$P_h\colon v\in H_0^1(\Omega)\to P_h v\in V_h \tag{3.3.22}$$

associated with the inner product $a(\cdot,\cdot)$ of (3.3.1), and which is therefore

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defined for each $v \in H_0^1(\Omega)$ by the relations

$$P_h v \in V_h \quad \text{and} \quad \forall w_h \in V_h, \quad a(v - P_h v, w_h) = 0. \tag{3.3.23}$$

Thus we have in particular $u_h = P_h u$, where u_h is the discrete solution found in the space V_h and u is the solution of the problem defined in (3.3.1). We shall then show that for an appropriate choice of the parameters θ_h in the functions ϕ_h (cf. (3.3.25) and (3.3.26) below), *the mappings P_h are also bounded independently of h when both spaces $H_0^1(\Omega)$ and V_h are equipped with the weighted norm*

$$v \to (|v|^2_{\phi_h^2;0,\Omega} + |v|^2_{\phi_h;1,\Omega})^{1/2}. \tag{3.3.24}$$

Theorem 3.3.5. *There exist three constants $h_0 \in]0, 1[$, $C_9 > 0$ and C_{10} such that, if for each h, we are given a function ϕ_h of the form*

$$\phi_h : x \in \bar{\Omega} \to \phi_h(x) = \frac{1}{\|x - x_h\|^2 + \theta_h^2}, \quad x_h \in \bar{\Omega}, \tag{3.3.25}$$

in such a way that

$$\forall h, \quad \theta_h = C_9 h |\ln h|^{1/2}, \tag{3.3.26}$$

then

$$\forall h \leq h_0, \quad \forall v \in H^1(\Omega),$$
$$|P_h v|_{\phi_h^2;0,\Omega} + |P_h v|_{\phi_h;1,\Omega} \leq C_{10}(|v|_{\phi_h^2;0,\Omega} + |v|_{\phi_h;1,\Omega}). \tag{3.3.27}$$

Proof. For convenience, the proof will be divided in four steps.

(i) *There exist two constants C_{11} and C_{12} such that, if*

$$\forall h, \quad \theta_h \geq C_{11} h, \tag{3.3.28}$$

then

$$\forall v \in H_0^1(\Omega), \quad |P_h v|^2_{\phi_h;1,\Omega} \leq C_{12}(|P_h v|^2_{\phi_h^2;0,\Omega} + |v|^2_{\phi_h;1,\Omega}). \tag{3.3.29}$$

For brevity, let

$$v_h = P_h v.$$

Since

$$|v_h|^2_{\phi_h;1,\Omega} = a(v_h, \phi_h v_h) + \frac{1}{2}\int_\Omega \Delta\phi_h \, v_h^2 \, dx$$

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and

$$\forall x \in \bar{\Omega}, \quad \Delta\phi_h(x) = \frac{4(\|x - x_h\|^2 - \theta_h^2)}{\|x - x_h\|^2 + \theta_h^2}\phi_h^2(x) \leq 4\phi_h^2(x),$$

we deduce that

$$|v_h|^2_{\phi_h;1,\Omega} \leq a(v_h, \phi_h v_h) + 2|v_h|^2_{\phi_h^2;0,\Omega}. \tag{3.3.30}$$

Using relations (3.3.23) we can write

$$a(v_h, \phi_h v_h) = a(v_h - v, \phi_h v_h - \Pi_h(\phi_h v_h)) + a(v, \phi_h v_h). \tag{3.3.31}$$

An application of inequality (3.3.5) with $\alpha = 1$ shows that

$$|a(v_h - v, \phi_h v_h - \Pi_h(\phi_h v_h))| \leq$$

$$\leq (|v|_{\phi_h;1,\Omega} + |v_h|_{\phi_h;1,\Omega})|\phi_h v_h - \Pi_h(\phi_h v_h)|_{\phi_h^{-1};1,\Omega}.$$

By Theorem 3.3.4, we have that, if

$$\forall h, \quad \theta_h \geq c_{10}h, \quad \text{with } c_{10} = C_7(1), \tag{3.3.32}$$

then (cf. inequality (3.3.21) with $\alpha = 1$)

$$|\phi_h v_h - \Pi_h(\phi_h v_h)|_{\phi_h^{-1};1,\Omega} \leq c_{11}\frac{h}{\theta_h}(|v_h|_{\phi_h^2;0,\Omega} + |v_h|_{\phi_h;1,\Omega}),$$

with $c_{11} = C_8(1)$.

Combining the previous inequalities, we find that, for $\theta_h \geq c_{10}h$,

$$|a(v_h - v, \phi_h v_h - \Pi_h(\phi_h v_h))| \leq$$

$$\leq c_{11}\frac{h}{\theta_h}(|v|_{\phi_h;1,\Omega} + |v_h|_{\phi_h;1,\Omega})(|v_h|_{\phi_h^2;0,\Omega} + |v_h|_{\phi_h;1,\Omega}). \tag{3.3.33}$$

By another application of inequality (3.3.5) with $\alpha = 1$, we obtain

$$a(v, \phi_h v_h) \leq |v|_{\phi_h;1,\Omega}|\phi_h v_h|_{\phi_h^{-1};1,\Omega}. \tag{3.3.34}$$

Using the inequality

$$\forall x \in \bar{\Omega}, \quad \sum_{i=1}^{2} |\partial_i \phi_h(x)|^2 \leq 4\phi_h^3(x),$$

we find that, for some constant c_{12},

$$|\phi_h v_h|_{\phi_h^{-1};1,\Omega} \leq c_{12}(|v_h|_{\phi_h^2;0,\Omega} + |v_h|_{\phi_h;1,\Omega}). \tag{3.3.35}$$

Combining relations (3.3.30) to (3.3.35), we have found that, for $\theta_h \geq c_{10}h$,

$$|v_h|^2_{\phi_h;1,\Omega} \leq 2|v_h|^2_{\phi_h^2;0,\Omega} + c_{12}|v|_{\phi_h;1,\Omega}(|v_h|_{\phi_h^2;0,\Omega} + |v_h|_{\phi_h;1,\Omega})$$
$$+ c_{11}\frac{h}{\theta_h}(|v|_{\phi_h;1,\Omega} + |v_h|_{\phi_h;1,\Omega})(|v_h|_{\phi_h^2;0,\Omega} + |v_h|_{\phi_h;1,\Omega}),$$

i.e., an inequality of the form

$$A^2 \leq 2C^2 + c_{12}B(A+C) + c_{11}\frac{h}{\theta_h}(A+B)(A+C).$$

Assuming

$$\forall h, \quad \theta_h \geq 2c_{11}h, \tag{3.3.36}$$

we get

$$A^2 \leq 4C^2 + (1+2c_{12})BC + A((1+2c_{12})B + C)$$
$$\leq 4C^2 + \left(\frac{1}{2} + c_{12}\right)(B^2 + C^2) + \frac{A^2}{2} + (1+2c_{12})^2B^2 + C^2,$$

and therefore step (i) is proved with (cf. (3.3.32) and (3.3.36))

$$C_{11} = \max(c_{10}, 2c_{11}) \tag{3.3.37}$$

in relation (3.3.28) and

$$C_{12} = \max\{11 + 2c_{12}, (1+2c_{12})(3+4c_{12})\} \tag{3.3.38}$$

in relation (3.3.29).

(ii) *There exists a constant C_{13} such that, if we assume $\theta_h \geq C_{11}h$ (the constant C_{11} has been determined in step* (i)), *we have*

$$\forall v \in H_0^1(\Omega), |P_hv|^2_{\phi_h^2;0,\Omega} + |P_hv|^2_{\phi_h;1,\Omega} \leq$$
$$\leq C_{13}(|v|^2_{\phi_h^2;0,\Omega} + |v|^2_{\phi_h;1,\Omega} + h^2|\psi_h|^2_{\phi_h^{-1};2,\Omega}), \tag{3.3.39}$$

where, for each h, $\psi_h = \psi_h(v)$ is the solution of the variational problem:

$$\psi_h \in H_0^1(\Omega) \quad \textit{and} \quad \forall w \in H_0^1(\Omega),$$
$$\int_\Omega \nabla\psi_h \cdot \nabla w \,\mathrm{d}x = \int_\Omega \phi_h^2(P_hv)w\,\mathrm{d}x. \tag{3.3.40}$$

Notice that because the set Ω is assumed to be convex, the function

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ψ_h is in the space $H^2(\Omega)$ and therefore, it is legitimate to consider the semi-norm $|\cdot|_{\phi_h^{-1};2,\Omega}$ in inequality (3.3.39).

Using the definition of the function ψ_h, and letting again $v_h = P_h v$, we can write

$$|v_h|^2_{\phi_h^2;0,\Omega} = a(v_h - v, \psi_h - \Pi_h\psi_h) + \int_\Omega \phi_h^2 v_h v \,\mathrm{d}x. \tag{3.3.41}$$

By applying inequality (3.3.5) with $\alpha = 1$ and inequality (3.3.20) with $\alpha = -1$ (this is possible because we assume $\theta_h \geq C_{11}h$ and $C_{11} \geq c_{10} = C_7(-1)$; cf. (3.3.32) and (3.3.37)), we obtain

$$\begin{aligned} &|a(v_h - v, \psi_h - \Pi_h\psi_h)| \leq \\ &\quad \leq C_7 h(|v_h|_{\phi_h;1,\Omega} + |v|_{\phi_h;1,\Omega})|\psi_h|_{\phi_h^{-1};2,\Omega}, \end{aligned} \tag{3.3.42}$$

Next we have

$$\int_\Omega \phi_h^2 v_h v \,\mathrm{d}x \leq |v_h|_{\phi_h^2;0,\Omega}|v|_{\phi_h^2;0,\Omega} \leq \frac{1}{2}(|v_h|^2_{\phi_h^2;0,\Omega} + |v|^2_{\phi_h^2;0,\Omega}), \tag{3.3.43}$$

so that, by combining relations (3.3.41), (3.3.42) and (3.3.43), we obtain the inequality

$$\begin{aligned} |v_h|^2_{\phi_h^2;0,\Omega} \leq{}& C_7 h(|v_h|_{\phi_h;1,\Omega} + |v|_{\phi_h;1,\Omega})|\psi_h|_{\phi_h^{-1};2,\Omega} \\ &+ \frac{1}{2}(|v_h|^2_{\phi_h^2;0,\Omega} + |v|^2_{\phi_h^2;0,\Omega}), \end{aligned}$$

which in turn implies the inequality

$$\begin{aligned} \forall \delta > 0, \quad |v_h|^2_{\phi_h^2;0,\Omega} \leq{}& \delta |v_h|^2_{\phi_h;1,\Omega} + |v|^2_{\phi_h^2;0,\Omega} + |v|^2_{\phi_h;1,\Omega} \\ &+ \left(1 + \frac{1}{\delta}\right) C_7^2 h^2 |\psi_h|^2_{\phi_h^{-1};2,\Omega}. \end{aligned} \tag{3.3.44}$$

Let then $\delta = 1/(3C_{12})$, where C_{12} is the constant appearing in inequality (3.3.29). The corresponding inequality (3.3.44) added to inequality (3.3.29) times the factor $2/(3C_{12})$ yields

$$\begin{aligned} &\frac{1}{3}|v_h|^2_{\phi_h^2;0,\Omega} + \frac{1}{3C_{12}}|v_h|^2_{\phi_h;1,\Omega} \leq \\ &\quad \leq |v|^2_{\phi_h^2;0,\Omega} + \frac{5}{3}|v|^2_{\phi_h;1,\Omega} + (1 + 3C_{12})C_7^2 h^2 |\psi_h|^2_{\phi_h^{-1};2,\Omega}, \end{aligned}$$

i.e., an inequality of the form (3.3.39).

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(iii) *Given any number $\theta_0 \in]0, 1[$, there exists a constant $C_{14}(\theta_0)$ such that*

$$\forall \theta_h \in]0, \theta_0[, \quad |\psi_h|^2_{\phi_h^{-1};2,\Omega} \leq C_{14}(\theta_0) \frac{|\ln \theta_h|}{\theta_h^2} |P_h v|^2_{\phi_h^2;0,\Omega}. \tag{3.3.45}$$

Since $-\Delta\psi_h = \phi_h^2 v_h (v_h = P_h v)$, we have

$$|\Delta\psi_h|_{\phi_h^{-1};0,\Omega} = |v_h|_{\phi_h^3;0,\Omega},$$

and consequently, by Theorem 3.3.1,

$$|\psi_h|^2_{\phi_h^{-1};2,\Omega} \leq C_1(|v_h|^2_{\phi_h^3;0,\Omega} + |\psi_h|^2_{1,\Omega}). \tag{3.3.46}$$

Since $\phi_h(x) \leq 1/\theta_h^2$ for all $x \in \bar{\Omega}$, we first find that

$$|v_h|^2_{\phi_h^3;0,\Omega} \leq \frac{1}{\theta_h^2} |v_h|^2_{\phi_h^2;0,\Omega}. \tag{3.3.47}$$

To take care of the other term which appears in the right-hand side of inequality (3.3.46), we shall prove that *for each number $\theta_0 \in]0, 1[$, there exists a constant $c_{13}(\theta_0)$ such that, for all functions ϕ of the form*

$$\phi : x \in \bar{\Omega} \to \phi(x) = \frac{1}{\|x - \bar{x}\|^2 + \theta^2}, \quad \bar{x} \in \bar{\Omega}, \quad 0 < \theta \leq \theta_0, \tag{3.3.48}$$

we have

$$\forall \psi \in H_0^1(\Omega) \cap H^2(\Omega), \quad |\psi|^2_{1,\Omega} \leq c_{13}(\theta_0) \frac{|\ln \theta|}{\theta^2} |\Delta\psi|^2_{\phi^{-2};0,\Omega}. \tag{3.3.49}$$

Taking into account that

$$|\Delta\psi_h|_{\phi_h^{-2};0,\Omega} = |v_h|_{\phi_h^2;0,\Omega},$$

and applying inequalities (3.3.49) (with $\psi = \psi_h$ and $\phi = \phi_h$), (3.3.46), and (3.3.47), we then find an inequality of the form (3.3.45), with

$$C_{14}(\theta_0) = C_1\left(\frac{1}{|\ln \theta_0|} + c_{13}(\theta_0)\right). \tag{3.3.50}$$

It therefore remains to prove relation (3.3.49) (another method for proving the same relation is suggested in Exercise 3.3.1). Given an arbitrary function $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$, we have

$$\begin{aligned} |\psi|^2_{1,\Omega} &= -\int_\Omega \psi\,\Delta\psi \,\mathrm{d}x \leq |\Delta\psi|_{\phi^{-2};0,\Omega} |\psi|_{\phi^2;0,\Omega} \\ &\leq \frac{1}{2} \frac{|\ln \theta|}{\theta^2} |\Delta\psi|^2_{\phi^{-2};0,\Omega} + \frac{1}{2} \frac{\theta^2}{|\ln \theta|} |\psi|^2_{\phi^2;0,\Omega}. \end{aligned} \tag{3.3.51}$$

Let then G denote the Green's function associated with the operator $-\Delta$ in Ω and the boundary condition $v = 0$ on Ω, so that

$$|\psi|^2_{\phi^2;0,\Omega} = \int_\Omega \phi^2(x)\left|\int_\Omega G(x,\xi)\Delta\psi(\xi)\,\mathrm{d}\xi\right|^2 \mathrm{d}x$$
$$\leq \int_\Omega \phi^{-2}(\xi)|\Delta\psi(\xi)|^2\left\{\int_\Omega \phi^2(x)G(x,\xi)\times\right.$$
$$\left.\times\left\{\int_\Omega \phi^2(\eta)G(x,\eta)\,\mathrm{d}\eta\right\}\mathrm{d}x\right\}\mathrm{d}\xi. \tag{3.3.52}$$

There exists a constant c_{14} such that (cf. for example STAKGOLD (1968, p. 143))

$$\forall x, y \in \bar{\Omega}, \quad x \neq y, \quad 0 \leq G(x,y) \leq c_{14}|\ln\|x-y\||. \tag{3.3.53}$$

Using this inequality, we proceed to show that for arbitrary points $x, \bar{x} \in \Omega$ and for any number θ with $0 < \theta \leq \theta_0 < 1$, there exists a constant $c_{15}(\theta_0)$ such that

$$\int_\Omega \varphi^2(\eta)G(x,\eta)\,\mathrm{d}\eta = \int_\Omega \frac{G(x,\eta)}{(\|\eta-\bar{x}\|^2+\theta^2)^2}\,\mathrm{d}\eta \leq c_{15}(\theta_0)\frac{|\ln\theta|}{\theta^2}. \tag{3.3.54}$$

To see this, write

$$\int_\Omega \frac{|\ln\|x-\eta\||}{(\|\eta-\bar{x}\|^2+\theta^2)^2}\,\mathrm{d}\eta = \sum_{\lambda=1}^{3}\int_{\Omega_\lambda}\frac{|\ln\|x-\eta\||}{(\|\eta-\bar{x}\|^2+\theta^2)^2}\,\mathrm{d}\eta,$$

where

$$\Omega_1 = \Omega_1(x,\theta) = \{\eta \in \Omega;\ \|\eta - x\| \leq \theta\},$$
$$\Omega_2 = \Omega_2(x,\theta) = \{\eta \in \Omega;\ \theta \leq \|\eta - x\| \leq 1\},$$
$$\Omega_3 = \Omega_3(x) = \{\eta \in \Omega;\ 1 \leq \|\eta - x\|\}.$$

We then obtain the following inequalities (observe that the last two inequalities make sense only if the sets Ω_2 and Ω_3 are not empty, and that we have diam $\Omega \geq 1$ if the set Ω_3 is not empty):

$$\int_{\Omega_1}\frac{|\ln\|x-\eta\||}{(\|\eta-\bar{x}\|^2+\theta^2)^2}\,\mathrm{d}\eta \leq -\frac{1}{\theta^4}\int_{\Omega_1}\ln\|\eta-x\|\,\mathrm{d}\eta$$
$$= -\frac{1}{\theta^4}\int_{B(0;\theta)}\ln\|\xi\|\,\mathrm{d}\xi$$
$$= \frac{\pi}{\theta^2}\left(\frac{1}{2}-\ln\theta\right) \leq \pi\left(1+\frac{1}{2|\ln\theta_0|}\right)\frac{|\ln\theta|}{\theta^2},$$

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$$\begin{aligned}\int_{\Omega_2} \frac{|\ln \|x-\eta\||}{(\|\eta-\bar{x}\|^2+\theta^2)^2}\,\mathrm{d}\eta &\leq |\ln \theta| \int_{\Omega_2} \frac{\mathrm{d}\eta}{(\|\eta-\bar{x}\|^2+\theta^2)^2}\\ &\leq |\ln \theta| \int_{\mathbb{R}^2} \frac{\mathrm{d}\eta}{(\|\eta-\bar{x}\|^2+\theta^2)^2}\\ &= \frac{|\ln \theta|}{\theta^2} \int_{\mathbb{R}^2} \frac{\mathrm{d}\xi}{(1+\|\xi\|^2)^2} = \pi \frac{|\ln \theta|}{\theta^2},\\ \int_{\Omega_3} \frac{|\ln \|x-\eta\||}{(\|\eta-\bar{x}\|^2+\theta^2)^2}\,\mathrm{d}\eta &\leq \ln(\operatorname{diam} \Omega) \int_{\Omega_3} \frac{\mathrm{d}\eta}{(\|\eta-\bar{x}\|^2+\theta^2)^2}\\ &\leq \ln(\operatorname{diam} \Omega) \int_{\mathbb{R}^2} \frac{\mathrm{d}\eta}{(\|\eta-\bar{x}\|^2+\theta^2)^2}\\ &= \frac{\pi}{\theta^2} \ln(\operatorname{diam} \Omega) \leq \left(\frac{\pi \ln(\operatorname{diam} \Omega)}{|\ln \theta_0|}\right) \frac{|\ln \theta|}{\theta^2}.\end{aligned}$$

Consequently, inequality (3.3.54) is proved, with

$$c_{15}(\theta_0) = \pi c_{14}\left\{2 + \frac{1+2\ln(\operatorname{diam} \Omega)}{2|\ln \theta_0|}\right\}.$$

The conjunction of inequalities (3.3.51) to (3.3.54) then implies inequality (3.3.49) with

$$c_{13}(\theta_0) = \frac{1}{2}(1 + c_{15}^2(\theta_0)).$$

(iv) *It remains to combine the results of steps* (ii) *and* (iii): We have determined constants C_{11}, C_{13} and $C_{14}(\theta_0)$ for each $\theta_0 \in]0, 1[$ such that (cf. inequalities (3.3.39) and (3.3.45))

$$\begin{aligned}C_{11}h \leq \theta_h \leq C_0 < 1 \Rightarrow |P_h v|^2_{\phi_h^2;0,\Omega} + |P_h v|^2_{\phi_h;1,\Omega} \leq\\ \leq C_{13}(|v|^2_{\phi_h^2;0,\Omega} + |v|^2_{\phi_h;1,\Omega}) + C_{13}C_{14}(\theta_0)\frac{|\ln \theta_h| h^2}{\theta_h^2}|P_h v|^2_{\phi_h^2,0,\Omega}.\end{aligned} \tag{3.3.55}$$

Let for example $\theta_0 = \frac{1}{2}$ and let

$$\theta_h = C_9 h|\ln h|^{1/2} \quad \text{with} \quad C_9 = 2\left(C_{13}C_{14}\left(\frac{1}{2}\right)\right)^{1/2}. \tag{3.3.56}$$

Then there exists a number $h_0 \in]0, 1[$ such that

$$h \leq h_0 \Rightarrow \begin{cases} C_{11}h \leq \theta_h \leq \frac{1}{2} = \theta_0, \\ |\ln \theta_h| \leq 2|\ln h|. \end{cases} \tag{3.3.57}$$

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This being the case, we have found an inequality of type (3.3.27) with

$$C_{10} = 2\sqrt{C_{13}}. \tag{3.3.58}$$

□

Estimates of the errors $|u - u_h|_{0,\infty,\Omega}$ *and* $|u - u_h|_{1,\infty,\Omega}$.
Nitsche's method of weighted norms

We next develop the *second stage* of our analysis. Using the inequalities (cf. Theorems 3.3.2 and 3.3.3) between the semi-norms $|\cdot|_{m,\infty,\Omega}$, $m = 0, 1$, and the weighted semi-norms which appear in inequality (3.3.27), we show in the next theorem that *the projection operators* P_h of (3.3.22), considered as acting from the subspace $H_0^1(\Omega) \cap W^{1,\infty}(\Omega)$ of the space $H_0^1(\Omega)$ onto the space V_h, *are bounded independently of h when the space* $H_0^1(\Omega) \cap W^{1,\infty}(\Omega)$ *is equipped with the norm*

$$v \to |v|_{0,\infty,\Omega} + h|\ln h|\,|v|_{1,\infty,\Omega} \tag{3.3.59}$$

and the space V_h *is equipped with the norm*

$$v \to |\ln h|^{-1/2}|v|_{0,\infty,\Omega} + h|v|_{1,\infty,\Omega}. \tag{3.3.60}$$

Remark 3.3.1. Such norms may be viewed as "weighted $W^{1,\infty}(\Omega)$-like" norms. □

Theorem 3.3.6. *There exists a constant* C_{15} *such that*

$$\begin{aligned}
&\forall h \leq h_0, \quad \forall v \in H_0^1(\Omega) \cap W^{1,\infty}(\Omega),\\
&\quad |\ln h|^{-1/2}|P_h v|_{0,\infty,\Omega} + h|P_h v|_{1,\infty,\Omega} \leq\\
&\quad \leq C_{15}(|v|_{0,\infty,\Omega} + h|\ln h|\,|v|_{1,\infty,\Omega}),
\end{aligned} \tag{3.3.61}$$

where the constant $h_0 > 0$ *has been determined in Theorem* 3.3.5.

Proof. Let there be given a function v in the space $H_0^1(\Omega) \cap W^{1,\infty}(\Omega)$. For each $h \leq h_0$, we define the function

$$\phi_{0h}: x \in \bar{\Omega} \to \phi_{0h}(x) = \frac{1}{\|x - x_h^0\|^2 + \theta_h^2}, \tag{3.3.62}$$

with

$$|P_h v(x_h^0)| = |P_h v|_{0,\infty,\Omega} \quad \text{and} \quad \theta_h = C_9 h|\ln h|^{1/2}, \tag{3.3.63}$$

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where h_0 and C_9 are the constants determined in Theorem 3.3.5. Since $\theta_h \geq C_{11}h$ for $h \leq h_0$ (cf. (3.3.57), we may apply inequality (3.3.14)): There exists a constant

$$c_{16} = C_4(C_{11}) \tag{3.3.64}$$

such that

$$|P_h v|_{0,\infty,\Omega} \leq c_{16}\frac{\theta_h^2}{h}|P_h v|_{\phi_{0h}^2;0,\Omega}. \tag{3.3.65}$$

By inequality (3.3.27),

$$|P_h v|_{\phi_{0h}^2;0,\Omega} \leq C_{10}(|v|_{\phi_{0h}^2;0,\Omega} + |v|_{\phi_{0h};1,\Omega}), \tag{3.3.66}$$

and by inequalities (3.3.10) and (3.3.11), there exists a constant ($\theta_h \leq \theta_0 = \frac{1}{2}$ for $h \leq h_0$; cf. (3.3.57))

$$c_{17} = \max\{C_2(2,0), \quad C_3(\tfrac{1}{2}, 1)\} \tag{3.3.67}$$

such that

$$|v|_{\phi_{0h}^2;0,\Omega} + |v|_{\phi_{0h};1,\Omega} \leq c_{17}\left(\frac{1}{\theta_h}|v|_{0,\infty,\Omega} + |\ln\theta_h|^{1/2}|v|_{1,\infty,\Omega}\right). \tag{3.3.68}$$

Combining inequalities (3.3.65) to (3.3.68), we find that

$$|P_h v|_{0,\infty,\Omega} \leq C_{10}c_{16}c_{17}\left(\frac{\theta_h}{h}|v|_{0,\infty,\Omega} + \frac{\theta_h^2|\ln\theta_h|^{1/2}}{h}|v|_{1,\infty,\Omega}\right).$$

Using the relation $\theta_h = C_9 h|\ln h|^{1/2}$ (cf. (3.3.26)) and the inequality $|\ln\theta_h| \leq 2|\ln h|$ (cf. (3.3.57)), we eventually get, for all $h \leq h_0$,

$$|\ln h|^{-1/2}|P_h v|_{0,\infty,\Omega} \leq c_{18}(|v|_{0,\infty,\Omega} + h|\ln h|\,|v|_{1,\infty,\Omega}), \tag{3.3.69}$$

with

$$c_{18} = C_{10}c_{16}c_{17}\max\{C_9, \sqrt{2}\,C_9^2\}. \tag{3.3.70}$$

Likewise, for each $h \leq h_0$, define the function

$$\phi_{1h} : x \in \bar{\Omega} \to \phi_{1h}(x) = \frac{1}{\|x - x_h^1\|^2 + \theta_h^2}, \tag{3.3.71}$$

with

$$\max\{|\partial_1 P_h v(x_h^1), \partial_2 P_h v(x_h^1)|\} = |v_h|_{1,\infty,\Omega} \text{ and } \theta_h = C_9 h|\ln h|^{1/2}. \tag{3.3.72}$$

Then there exists (cf. inequality (3.3.16)) a constant

$$c_{19} = C_5(C_{11}) \tag{3.3.73}$$

such that

$$|P_h v|_{1,\infty,\Omega} \leq c_{19} \frac{\theta_h}{h} |P_h v|_{\phi_{1h};1,\Omega}, \tag{3.3.74}$$

and, by inequality (3.3.27),

$$|P_h v|_{\phi_{1h};1,\Omega} \leq C_{10}(|v|_{\phi_{1h}^2;0,\Omega} + |v|_{\phi_{1h};1,\Omega}). \tag{3.3.75}$$

Then, arguing as before, we get, for all $h \leq h_0$,

$$h|P_h v|_{1,\infty,\Omega} \leq c_{20}(|v|_{0,\infty,\Omega} + h|\ln h|\,|v|_{1,\infty,\Omega}) \tag{3.3.76}$$

with

$$c_{20} = C_{10}c_{17}c_{19}\max\{1, \sqrt{2}\,C_9\}. \tag{3.3.77}$$

The conjunction of inequalities (3.3.69) and (3.3.76) implies inequality (3.3.61) with

$$C_{15} = c_{18} + c_{20}.$$ □

Remark 3.3.2. In Theorem 3.3.5, the behavior of the function θ_h was somehow "bounded below" by a constant times $(h|\ln h|^{1/2})$. The key to the success of the present argument was that such a function θ_h tends nevertheless sufficiently rapidly vers zero with h so as to produce the right factors (as functions of h) in the inequalities (3.3.69) and (3.3.76). □

In the *third* – and final – *stage* of our study, the uniform boundedness of the projection mappings P_h which we just established allows us in turn to easily derive the desired error estimates (recall that the discrete solution u_h is nothing but the projection $P_h u$ of the solution u).

Theorem 3.3.7. *Assume that the solution $u \in H_0^1(\Omega)$ of the boundary value problem associated with the data* (3.3.1) *is also in the space $W^{2,\infty}(\Omega)$.*

Then there exists a constant C independent of h such that

$$|u - u_h|_{0,\infty,\Omega} \leq Ch^2|\ln h|^{3/2}|u|_{2,\infty,\Omega}, \tag{3.3.78}$$

$$|u - u_h|_{1,\infty,\Omega} \leq Ch|\ln h|\,|u|_{2,\infty,\Omega}. \tag{3.3.79}$$

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Proof. The norm of the identity mapping acting from the space $H_0^1(\Omega) \cap W^{1,\infty}(\Omega)$ equipped with the norm of (3.3.59) into the same space, but equipped with the norm of (3.3.60), is bounded above by $|\ln h_0|^{-1/2}$ for all $h \leq h_0' = \min\{h_0, 1/e\}$.

Next we have the identity

$$\forall v_h \in V_h, \quad u - u_h = u - P_h u = (I - P_h)(u - v_h),$$

so that we infer from Theorem 3.3.6 that, for all $h \leq h_0'$,

$$|\ln h|^{-1/2}|u - u_h|_{0,\infty,\Omega} + h|u - u_h|_{1,\infty,\Omega} \leq$$
$$\leq (|\ln h_0|^{-1/2} + C_{15}) \inf_{v_h \in V_h} (|u - v_h|_{0,\infty,\Omega} + h|\ln h|\,|u - v_h|_{1,\infty,\Omega})$$

Since there exists a constant c_{21} such that

$$\inf_{v_h \in V_h} (|u - v_h|_{0,\infty,\Omega} + h|\ln h|\,|u - v_h|_{1,\infty,\Omega}) \leq c_{21} h^2 |\ln h|\,|u|_{2,\infty,\Omega},$$

inequalities (3.3.78) and (3.3.79) follow with

$$C = c_{21}(|\ln h_0|^{-1/2} + c_{15}). \qquad \square$$

In fact, the error estimate of (3.3.78) is not optimal: J.A. NITSCHE (1976b) gets the improved error bound

$$|u - u_h|_{0,\infty,\Omega} \leq Ch^2|\ln h|\,|u|_{2,\infty,\Omega}, \tag{3.3.80}$$

at the expense, however, of a technical refinement in the argument, special to triangles of type (1). At any rate, the discrepancy between (3.3.78) and (3.3.80) is somehow insignificant: Both error estimates (3.3.78) and (3.3.80) show an $O(h^{2-\epsilon})$ convergence for any $\epsilon > 0$.

To conclude, we point out that all the essential features of *Nitsche's method of weighted norms* have been presented: Indeed, the extension to more general cases proceeds along the same lines. In particular, the use of higher-order polynomial spaces (i.e., $P_K = P_k(K)$ for some $k \geq 2$, n arbitrary) yields a simplification in that the "$|\ln h|$" term present for $k = 1$ disappears in the norms then considered. Thus inequality (3.3.61) is replaced by an inequality of the simpler form (cf. NITSCHE (1975))

$$|P_h v|_{0,\infty,\Omega} + h|P_h v|_{1,\infty,\Omega} \leq C(|v|_{0,\infty,\Omega} + h|v|_{1,\infty,\Omega}). \tag{3.3.81}$$

Such inequalities are obtained after inequalities reminiscent of that of (3.3.27) have been established for appropriate weighted norms of the form $|\cdot|_{\phi_h^{\alpha+1};0,\Omega} + |\cdot|_{\phi_h^{\alpha};1,\Omega}$, $(n/2) < \alpha < (n/2) + 1$, with functions ϕ_h again defined as in (3.3.25).

Exercises

3.3.1. Following NITSCHE (1977), the object of this problem is to provide another proof of inequality (3.3.49), i.e., that for any $\theta_0 \in]0, 1[$, there exists a constant $c(\theta_0)$ such that, for any function ϕ of the form

$$\phi : x \in \bar{\Omega} \to \phi(x) = \frac{1}{\|x - \bar{x}\|^2 + \theta^2}, \qquad \bar{x} \in \bar{\Omega}, \quad 0 < \theta \leqslant \theta_0,$$

we have

$$\forall \psi \in H_0^1(\Omega) \cap H^2(\Omega), \quad |\psi|_{1,\Omega}^2 \leqslant c(\theta_0) \frac{|\ln \theta|}{\theta^2} |\Delta\psi|_{\phi^{-2};0,\Omega}^2.$$

(i) Let

$$\lambda(\Omega) = \inf_{\psi \in H_0^1(\Omega) \cap H^2(\Omega)} \frac{|\Delta\psi|_{\phi^{-2};0,\Omega}^2}{|\psi|_{1,\Omega}^2}$$

and show that $\lambda(\Omega)$ is the smallest eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda \phi^2 u \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma, \end{cases}$$

so that $\lambda(\Omega)$ is a strictly positive quantity (references about eigenvalue problems can be found in the section "Additional Bibliography and Comments" at the end of Chapter 4).

(ii) Let $\tilde{\Omega} = B(\bar{x};\ \mathrm{diam}\,(\Omega))$ and show by a direct computation that

$$\frac{1}{\lambda(\tilde{\Omega})} \leqslant c(\theta_0) \frac{|\ln \theta|}{\theta^2}.$$

(iii) Conclude, by using the implication

$$\Omega_1 \subset \Omega_2 \Rightarrow \lambda(\Omega_2) \leqslant \lambda(\Omega_1).$$

3.3.2. The object of this exercise is to show how an error estimate in the norm $|\cdot|_{0,\infty,\Omega}$ can be quickly derived, once one is willing to accept a poorer order of convergence than that obtained in Theorem 3.3.7. The terminology is the same as in Section 3.2.

In addition to (H1), (H2) and (H3), assume that $s = 0$, that the

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dimension n is $\leqslant 3$, that the family of triangulations satisfies an inverse assumption, and that the inclusions

$$P_1(\hat{K}) \subset \hat{P} \subset H^1(\hat{K})$$

hold (notice that by hypothesis (H3), we also have $\hat{P} \subset L^\infty(\hat{K})$).

Let then u_h be the corresponding discrete solution which approximates the solution u of a general second order boundary value problem of the type considered in Section 3.2. Show that if the adjoint problem is regular, and if $u \in H^2(\Omega) \cap V$, there exists a constant C independent of h such that

$$|u - u_h|_{0,\infty,\Omega} \leqslant Ch|u|_{2,\Omega} \text{ if } n = 2,$$
$$|u - u_h|_{0,\infty,\Omega} \leqslant C\sqrt{h}\,|u|_{2,\Omega} \text{ if } n = 3.$$

[Hint: write $|u - u_h|_{0,\infty,\Omega} \leqslant |u_h - \Pi_h u|_{0,\infty,\Omega} + |u - \Pi_h u|_{0,\infty,\Omega}$ and use an appropriate inverse inequality for the first term.]

Bibliography and comments

3.1. The content of this section is essentially based on, and slightly improved upon, CIARLET & RAVIART (1972a). In particular, R. Arcangéli suggested the simpler proof of inequality (3.1.33) given here.

For reference about the Sobolev spaces $W^{m,p}(\Omega)$ and their various properties, see ADAMS (1975), LIONS (1962), NEČAS (1967), ODEN & REDDY (1976a, chapter 3). Theorem 3.1.1 was originally proved in DENY & LIONS (1953–1954) for open sets which satisfy the "cone property" (such sets are slightly more general than those with Lipschitz-continuous boundaries). An abstract extension of this lemma is indicated in Exercise 3.1.1.

There has been considerable interest in interpolation theory and approximation theory in several variables during the past decade, one reason behind this recent interest being the need of such theories for studying convergence properties of finite element methods. Special mention must be made however of the pioneering works of PÓLYA (1952) and SYNGE (1957), for what we call here rectangles of type (1) and triangles of type (1), respectively.

The "classical" approach consists in obtaining error estimates in C^m-norms. In this direction, see the contributions of BARNHILL &

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GREGORY (1976b), BARNHILL & WHITEMAN (1973), BIRKHOFF (1971, 1972), BIRKHOFF, SCHULTZ & VARGA (1968), CARLSON & HALL (1973), CIARLET & RAVIART (1972a), CIARLET & WAGSCHAL (1971), COATMÉLEC (1966), LEAF & KAPER (1974), NICOLAIDES (1972, 1973), NIELSON (1973), SCHULTZ (1969b, 1973), STRANG (1971, 1972a), ŽENÍŠEK (1970, 1973), ZLÁMAL (1968, 1970). Although in most cases a special role is played by the canonical Cartesian coordinates, a more powerful coordinate-free approach, using Fréchet derivatives, can be developed, such as in CIARLET & RAVIART (1972a), CIARLET & WAGSCHAL (1971), where the interpolation error estimates are obtained as corollaries of multi-point Taylor formulas (Exercise 3.1.2). See also COATMÉLEC (1966). Another frequently used tool is the kernel Theorem of SARD (1963).

Some authors have considered the problem of estimating the constants which appear in the interpolation error estimates. See ARCANGÉLI & GOUT (1976) (cf. Exercise 3.1.2), ATTÉIA (1977), BARNHILL & WHITEMAN (1973), GOUT (1976), MEINGUET (1975), MEINGUET & DESCLOUX (1977).

The approach in Sobolev spaces which has been followed here has been given much attention. In this respect, we quote the fundamental contributions of BRAMBLE & HILBERT (1970, 1971), BRAMBLE & ZLÁMAL (1970). Other relevant references are AUBIN (1967a, 1967b, 1968a, 1968b, 1972), BABUŠKA (1970, 1972b), BIRKHOFF, SCHULTZ & VARGA (1968), BRAMBLE (1970), CIARLET & RAVIART (1972a), FIX & STRANG (1969), DI GUGLIELMO (1970), HEDSTROM & VARGA (1971), KOUKAL (1973), NITSCHE (1969, 1970), SCHULTZ (1969b), VARGA (1971).

Interesting connections with standard spline theory can be found in ATTÉIA (1975), MANSFIELD (1972b), NIELSON (1973) and, especially, DUCHON (1976a, 1976b).

The dependence of the interpolation error estimates upon the geometry of the element (through the parameters h_K and ρ_K) generalize Zlámal's condition, as given in ZLÁMAL (1968, 1970), and the "uniformity condition" of STRANG (1972a). JAMET (1976a) has recently shown (cf. Exercise 3.1.4) that, for some finite elements at least, the regularity condition given in (3.1.43) can be replaced by a less stringent one. In a special case, the same condition has been simultaneously and independently found by BABUŠKA & AZIZ (1976). In essence, it amounts to saying, in case of triangles, that no angle of the triangle should approach π in the limit while by the present analysis no angle should

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approach 0 in the limit. Incidentally, this was already observed by SYNGE (1957).

3.2. and 3.3. There exists a very large literature on various possible error estimates one can get for conforming finite element method and here we shall merely record several lists of references, depending upon the viewpoints.

We shall first observe that almost all the papers previously referred to in Section 3.1 also contributed to the error analysis in the norm $\|\cdot\|_{1,\Omega}$, inasmuch as this simply requires a straightforward application of Céa's lemma, just as we did in Theorem 3.2.2.

Even though the X_h-interpolation operator cannot be defined for lack of regularity of the function to be approximated, an approximation theory can still be developed, as in CLÉMENT (1975), HILBERT (1973), PINI (1974), STRANG (1972a). See Exercise 3.2.3 where we have indicated the approach of Ph. Clément.

Historically, the first proof of convergence of a finite element method, albeit in a special case, seems to be due to FRIEDRICHS (1962). Early works on convergence in the engineering literature are JOHNSON & MCLAY (1968), MCLAY (1963), OLIVEIRA (1968, 1969).

The reader who wishes to get general introductions to, and surveys on, the various aspects of the convergence of the finite element method may consult BIRKHOFF & FIX (1974), CARLSON & HALL (1971), CIARLET (1973), FELIPPA & CLOUGH (1970), KIKUCHI (1975c), ODEN (1975), STRANG (1972a, 1974b), THOMÉE (1973a), VEIDINGER (1974), ZLÁMAL (1973c).

Using a priori estimates (in various norms) on the solution (cf. NEČAS (1967) and KONDRAT'EV (1967)), it is possible to get error estimates which depend solely on the data of the problem. See BRAMBLE & ZLÁMAL (1970), NITSCHE (1970), OGANESJAN & RUKHOVETS (1969). In the case of the equation $-\Delta u = f$ over a rectangle, BARNHILL & GREGORY (1976a) obtain theoretical values for the constants which appear in the error estimate, which are realistic, as shown in BARNHILL, BROWN, MCQUEEN & MITCHELL (1976).

"Nonuniform" error estimates are obtained in BABUŠKA & KELLOG (1975), HELFRICH (1976). The case of indefinite bilinear forms is considered in CLÉMENT (1974), SCHATZ (1974). DOUGLAS, DUPONT & WHEELER (1974a) give estimates for the flux on the boundary. HOPPE (1973) has suggested the use of piecewise harmonic polynomials, and his idea has been justified by RABIER (1977). See also BABUŠKA (1974a), ROSE (1975).

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There are various ways of treating nonhomogeneous Dirichlet boundary conditions. The most straightforward method is suggested in Exercise 3.2.1. See AUBIN (1972), STRANG & FIX (1973, Section 4.4), THOMÉE (1973a). Lagrange multipliers may also be used as in BABUŠKA (1973a), as well as penalty techniques (cf. Exercise 3.2.2) as in BABUŠKA (1973b). See also the section "Bibliography and Comments", Section 4.4.

For domains with corners or, more generally, for problems where the solution presents singularities, see BARNHILL & WHITEMAN (1973, 1975), BABUŠKA (1972a, 1974b, 1976), BABUŠKA & ROSENZWEIG (1972), BARSOUM (1976), CIARLET, NATTERER & VARGA (1970), CROUZEIX & THOMAS (1973), DAILEY & PIERCE (1972), FIX (1969), FIX, GULATI & WAKOFF (1973), FRIED & YANG (1972), HENNART & MUND (1976), NITSCHE (1976a), SCHATZ & WAHLBIN (1976a), SCOTT (1973b), STRANG & FIX (1973, Chapter 8), THATCHER (1976), VEIDINGER (1972), WAIT & MITCHELL (1971). Recent references in the engineering literature are HENSHELL & SHAW (1975), YAMAMOTO & SUMI (1976).

For further results concerning the error estimates in the norm $\|\cdot\|_{1,\Omega}$, see BABUŠKA & AZIZ (1972, Section 6.4) where it is notably discussed whether they are the best possible, using the theory of n-widths.

Many "abstract" finite element methods, or variants thereof, have been considered, by AUBIN (1967b, 1972), BABUŠKA (1970, 1971a, 1971b, 1972b), FIX & STRANG (1969), DI GUGLIELMO (1971), MOCK (1976), STRANG (1971), STRANG & FIX (1971).

The inverse inequalities established in Section 3.2 are found in many places. See notably DESCLOUX (1973).

The technique which yields the error estimate in the norm $|\cdot|_{0,\Omega}$ was developed independently by AUBIN (1967b) and NITSCHE (1968), and also by OGANESJAN & RUKHOVETS (1969). See KIKUCHI (1975c) for a generalization.

The subject of uniform convergence has a (relatively) long story. In one dimension, we mention NITSCHE (1969), CIARLET (1968), CIARLET & VARGA (1970), and the recent contributions of DOUGLAS & DUPONT (1973, 1976b), DOUGLAS, DUPONT & WAHLBIN (1975b), NATTERER (1977). For special types of triangulations in higher dimensions, see BRAMBLE, NITSCHE & SCHATZ (1975), BRAMBLE & SCHATZ (1976), BRAMBLE & THOMÉE (1974), DOUGLAS, DUPONT & WHEELER (1974b), NATTERER (1975b).

The first contribution to the general case is that of NITSCHE (1970). Then CIARLET & RAVIART (1973) improved the analysis of J.A. Nitsche

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by using a discrete maximum principle introduced in CIARLET (1970). More specifically, CIARLET & RAVIART (1973) have considered finite element approximations of general second-order nonhomogeneous Dirichlet problems posed over polygonal domains in $\mathbf{R}^n$. Then the discrete problem is said to satisfy a *discrete maximum principle* if one has

$$f \leqslant 0 \Rightarrow \max_{x\in\bar{\Omega}} u_h(x) \leqslant \max\{0, \max_{x\in\Gamma} u_h(x)\}.$$

In the case of the operator $(-\Delta u + au)$ with $a \geqslant 0$ and $n = 2$, it is shown that the discrete maximum principle holds for h small enough if there exists $\epsilon > 0$ such that all the angles of all the triangles found in all the triangulations are $\leqslant [(\pi/2) - \epsilon]$ (in case $a = 0$, it suffices that the angles of the triangles be $\leqslant \pi/2$). Returning to the general case, it is shown that when the discrete problems satisfy a maximum principle, one has

$$\lim_{u\to 0} |u - u_h|_{0,\infty,\Omega} = 0 \quad \text{if } u \in W^{1,p}(\Omega) \text{ with } p > n,$$

$$|u - u_h|_{0,\infty,\Omega} = O(h) \quad \text{if } u \in W^{2,p}(\Omega) \text{ with } 2p > n,$$

i.e., there was still a loss of one in the expected order of convergence.

Recently, NATTERER (1975a), NITSCHE (1975, 1976b, 1977) and SCOTT (1976a) obtained simultaneously optimal (or nearly optimal) orders of convergence. The greatest generality is achieved in the particularly penetrating analysis of J.A. Nitsche, which we have followed in Section 3.3 (the proof of inequality (3.3.49) is that of RANNACHER (1977)).

While weighted Sobolev norms are also introduced by F. Natterer, R. Scott's main tool is a careful analysis of the approximation of the Green's function. The uniform boundedness in appropriate norms of particular Hilbertian projections, on which J.A. Nitsche's argument is essentially based, was also noticed by DOUGLAS, DUPONT & WAHLBIN (1975a) who have proved (albeit through a different approach) the boundedness in the norms $|\cdot|_{0,q,\Omega}$, $1 \leqslant q \leqslant \infty$, of the projections, with respect to the inner-product of the space $L^2(\Omega)$, onto certain finite element spaces.

J.A. Nitsche's technique has since then been successfully extended in several directions, notably to more general second-order boundary value problems by RANNACHER (1976b), to the obstacle problem by J.A. Nitsche himself, to the minimal surface problem and other nonlinear problems by J. Frehse and R. Rannacher (cf. Chapter 5), to plates by R. Rannacher (cf. Chapter 6), to mixed methods by R. Scholz (cf. Chapter 7).

There is currently a wide interest in obtaining various refinements of the error estimates, such as "interior" estimates, superconvergence results, etc.... In addition to the previously quoted reference, we mention BRAMBLE & SCHATZ (1974, 1976), BRAMBLE & THOMÉE (1974), DESCLOUX (1975), DESCLOUX & NASSIF (1977), DOUGLAS & DUPONT (1973, 1976a), NITSCHE (1972a), NITSCHE & SCHATZ (1974), SCHATZ & WAHLBIN (1976, 1977).

A little explored direction of research is that of the optimal choice of triangulation: For a given number of finite elements of a specific type, the problem consists in finding the "best" triangulation so as to minimize the error in some sense. For references in this direction, see CARROLL & BARKER (1973), MCNEICE & MARCAL (1973), PRAGER (1975), RAJAGOPALAN (1976), TURCKE & MCNEICE (1972).

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