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Superconvergence of The Derivative Patch Recovery Technique and *A Posteriori* Error Estimation

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Abstract

The derivative patch recovery technique developed by Zienkiewicz and Zhu [1] - [3] for the finite element method is analyzed. It is shown that, for one dimensional problems and two dimensional problems using tensor product elements, the patch recovery technique yields superconvergence recovery for the derivatives. Consequently, the error estimator based on the recovered derivative is asymptotically exact.

Key words. superconvergence, element patch, derivative recovery, Gauss point, Lobatto point.

AMS(MOS) subject classifications. 65N30, 65N15

1. Introduction. It has been observed for some time that, for certain classes of the problems, the rate of convergence of the nodal values of the finite element solution and/or the derivatives of the finite element solution at some special points exceeds the possible global rate. This phenomenon has been termed "superconvergence" and has been analyzed mathematically because of its practical importance in the finite element computations [4] - [12].

It has also been found that the superconvergent solution values can be achieved by means of various recovery (or post-processing) techniques [13] - [19]. The most widely used recovery technique is averaging. The advantage of such recovery techniques is that they are able to produce superconvergence recovery of the finite element approximation not only at special points, but also at nodal points, in a subdomain and sometimes on the boundary or even in the whole domain. The majority of the recovery techniques proposed in the literature are designed for superconvergent recovery of the derivatives. For the literature regarding superconvergence and recovery techniques we refer to [20] - [22] and the references therein.

In 1992, Zienkiewicz and Zhu introduced a new derivative recovery technique known as superconvergent patch recovery (SPR) [1] - [3]. The technique recovers the solution quantities in an element from element patches surrounding the nodes of the element. The technique is applicable to practical problems because of its cost effectiveness and its flexibility on irregular meshes and general geometries. The numerical results presented in [1] - [3] shown that SPR provides superconvergent recovery on certain regular meshes and provides recovery with much improved accuracy on general meshes.

Very recently, the notion of $\eta\%$-superconvergence was introduced by Babuška et al. [23] - [25] in the study of superconvergence phenomenon. The introduction of $\eta\%$-superconvergence extends the classical definition of superconvergence and allows superconvergence phenomenon to be studied for larger set of problems and on general meshes. It is, therefore, more useful for practical computations. In their study, it is found that the superconvergence points as well as the superconvergent recovery techniques are very sensitive to the set of solutions, the element space and the geometric pattern of the meshes. Superconvergence may not exist for many problems under classical definition of superconvergence even when regular mesh patterns are used. The study of superconvergence is further complicated when general meshes are considered under classical definition of superconvergence. Nevertheless, they concluded that it is always beneficial to use SPR in the practical computation to achieve more accurate recovered solutions, i.e., the $\eta\%$-superconvergent solutions. We refer to [23] - [25] for more details on $\eta\%$-superconvergence.

In this paper, we present an analysis of the SPR under the definition of classical superconvergence. It is proved that, for our one dimensional model problem and two dimensional model problem using tensor product elements on rectangular meshes, SPR recovers a superconvergence derivative field. We also demonstrate that the corresponding Zienkiewicz-Zhu (ZZ) error estimator is asymptotically exact. We refer to [1] - [3], [26], [27] for numerical performance of SPR and ZZ error estimator on general meshes and refer to [25] for $\eta\%$-superconvergence analysis of SPR. We also refer to [28] for analysis of different versions of SPR and refer to [29] for analysis of SPR on curved isoparametric quadrilateral meshes.

The results obtained in our analysis distinguish from the conventional superconvergence results in the literature are two folded: Firstly, the regularity requirement on the exact solution u is optimal in the sense that we only need $u \in H^{r+2}$ in order to have $O(h^{r+1})$

convergent rate for the recovered gradient in the L^2-norm. The existing analysis of the superconvergence results at the Gauss points, however, requires $u \in W_\infty^{r+2}$ to assure the same convergence rate. Toward this end, a new approach is introduced in our analysis. Secondly, the superconvergence results in this work is global as long as the exact solution is sufficiently smooth. For more details, the readers are referred to Theorem 2.3, Remark 2.5, Remark 2.6, Theorem 3.4, and Remark 3.5 in the following sections.

We also note that an unsmooth boundary may cause solution singularity and the finite element method would suffer from "pollution". Further, nonsmooth data will result in singular behavior of the solution. Hence many practical problems may not satisfy our regularity condition (although the recovery technique is still applicable). In the case of a problem with singular solution, local mesh refinement and adaptive procedure are usually applied which will complicate the analysis. Therefore, a local error estimate may be needed. Such analysis is not considered in the current study. In this direction, we refer the readers to [12], [23] - [25].

We shall use the conventional notations for the Sobolev spaces and norms in the analysis. For example, $\|\cdot\|_{0,\Omega}$ denotes the L^2-norm on $L^2(\Omega)$. The index $0, \Omega$ will be dropped whenever there is no confusion. We use C as a genetic constant which is not necessarily the same at each occurrence.

The outline of the paper is as follows: In Section 2 we study the superconvergence recovery of SPR for one dimensional problems. In Section 3 we extend the one dimensional results to two dimensional problems on the rectangular meshes. As an application of SPR, in Section 4 we show that, under the same conditions given in Section 2 and Section 3, the ZZ error estimator is asymptotically exact when SPR is employed.

2. Recovery in 1-D.

Consider the following two-point boundary value problem as our model problem.

$$\begin{aligned} -u'' + bu &= f \quad \text{in } I = (0,1), \\ u(0) = u(1) &= 0, \end{aligned} \tag{2.1}$$

where $b \geq 0$. We shall assume that b and f are as smooth as necessary on $\bar{I}$ for our analysis to carry through.

The week formulation of (2.1) is: Find $u \in H_0^1(I)$ such that

$$a(u,v) = (u', v') + (bu, v) = (f, v) \quad \forall v \in H_0^1(I), \tag{2.2}$$

where $(\cdot,\cdot)$ is the inner product on $L^2(I)$ defined by

$$(f,g)=\int_0^1 f(x)g(x)dx.$$

Remark 2.1 We could consider more general two-point boundary value problems. The reason we choose (2.1) is that it exhibits the most features of the general case while it does not involve too much technical difficulties.

As preparation, we first prove a theorem for the interpolation property of polynomials for our later use. We start from some notations. Let $L_r(x)$ be the Legendre polynomial of degree r on $[-1,1]$. It is well known that $L_r(x)$ has r roots and $L_r'(x)$ has $r-1$ roots in $(-1,1)$. Denote by $g_1^{(r)},\cdots,g_r^{(r)}$, the roots of $L_r(x)$, and $l_1^{(r)},\cdots,l_{r-1}^{(r)}$, the roots of $L_r'(x)$ with $l_0^{(r)}=-1$, $l_r^{(r)}=1$.

Definition 2.1 $g_j^{(r)}$, $j=1,\cdots,r$, are called the Gauss points of rth order, and $l_j^{(r)}$, $j=0,1,\cdots,r$, are called the Lobatto points of rth order.

Theorem 2.1: *Let u be a polynomial of degree $r+1$, let u_I be its Lagrange interpolation at $r+1$ Lobatto points on $[-1,1]$. Then $u'(g_j^{(r)})=u_I'(g_j^{(r)})$, $j=1,2,\cdots,r$.*

Proof : We first establish the following identity

$$(1-x^2)L_r'(x)+\frac{r(r+1)}{2r+1}[L_{r+1}(x)-L_{r-1}(x)]=0. \tag{2.3}$$

Towards this end, the following Legendre polynomials identities [30] are used:

$$L_{r+1}'(x)-L_{r-1}'(x)=(2r+1)L_r(x),\quad r=1,2,\cdots \tag{2.4}$$

$$[(1-x^2)L_r'(x)]'+r(r+1)L_r(x)=0,\quad r=0,1,\cdots \tag{2.5}$$

Substituting (2.4) into (2.5) yields

$$[(1-x^2)L_r'(x)]'+\frac{r(r+1)}{2r+1}[L_{r+1}'(x)-L_{r-1}'(x)]=0,\quad r=1,2,\cdots \tag{2.6}$$

Then (2.3) follows by integrating (2.6) and observing that $L_{r+1}(1)-L_{r-1}(1)=0$.

Making use of (2.3), the Lobatto points can also be defined as the roots of the polynomial $L_{r+1}(x)-L_{r-1}(x)$ on $[-1,1]$. Define

$$\omega(x)=\prod_{i=0}^{r}(x-l_i^{(r)}),$$

then $\omega(x) = \alpha[L_{r+1}(x) - L_{r-1}(x)]$, where α is a constant, $r = 1, 2, \cdots$. From (2.4),

$$\omega'(x) = \alpha[L'_{r+1}(x) - L'_{r-1}(x)] = \alpha(2r+1)L_r(x), \quad r = 1, 2, \cdots$$

and hence

$$\omega'(g_j^{(r)}) = 0, \quad j = 1, 2, \cdots, r. \tag{2.7}$$

Since any polynomial u of degree $r+1$ can be written as

$$u(x) = u_I(x) + c\omega(x), \quad c = \frac{u^{(r+1)}(x)}{(r+1)!}.$$

Therefore, we obtain

$$u'(g_j^{(r)}) = u'_I(g_j^{(r)}), \quad i = 1, 2, \cdots, r,$$

by virtue of (2.7). ∎

Theorem 2.1 demonstrates that the polynomial interpolation of degree r at $r+1$ Lobatto points yields the exact derivative at the Gauss point for polynomials of degree $r+1$. This superconvergence property of the polynomial interpolation play an essential rule in the superconvergence analysis of the finite element method. We state it as a theorem, although it has often been used implicitly in the literature.

Let Δ: $0 = x_0 < x_1 < \cdots < x_N = 1$ be a partition of I, and denote $I_i = (x_{i-1}, x_i)$, $h_i = x_i - x_{i-1}$, $h = \max_{1 \le i \le N} h_i$. The Gauss and the Lobatto points on I_i are defined as the affine transformations of $g_j^{(r)}$ and $l_j^{(r)}$ to I_i, respectively,

$$G_{ij} = \frac{1}{2}(x_{i-1} + x_i + h_i g_j^{(r)}), \quad j = 1, \cdots, r,$$

$$L_{ij} = \frac{1}{2}(x_{i-1} + x_i + h_i l_j^{(r)}), \quad j = 0, 1, \cdots, r.$$

Here the index r on G_{ij} and L_{ij} is dropped in order to simplify the notation. Now we are ready to define,

$$S^r(\Delta) = \{v \in H^1(I), \quad v|_{I_i} \in P_r(I_i), \quad i = 1, \cdots, N\}, \quad S_0^r(\Delta) = H_0^1(I) \cap S^r(\Delta).$$

$S_0^r(\Delta)$ is the finite element space which includes piecewise polynomials of degree no more than r.

The following is a direct consequence of Theorem 2.1.

Corollary 2.1: *Let u be a polynomial of degree $r+1$, let $u_I \in S^r(\Delta)$ be its piecewise Lagrange interpolant at the Lobatto points of every I_i, $i = 1, \cdots, N$. Then $u'(G_{ij}) = u'_I(G_{ij})$, $i = 1, \cdots, N$, $j = 1, \cdots, r$.*

The finite element approximation of (2.2) is to find $u_h \in S_0^r(\Delta)$ such that

$$a(u_h, v) = (f, v) \quad \forall v \in S_0^r(\Delta), \tag{2.8}$$

and the optimal error estimate of the finite element solution in the L^2 norm is (e.g., [31])

$$h\|u' - u_h'\| + \|u - u_h\| \le Ch^{r+1}\|u\|_{r+1}. \tag{2.9}$$

One important feature in the standard superconvergence analysis is the relationship between the finite element solution and the Ritz projection.

Definition 2.2 For any $u \in H_0^1(I)$, its Ritz projection is $u_r \in S_0^r(\Delta)$ such that

$$(u' - u_r', v') = 0, \quad \forall v \in S_0^r(\Delta). \tag{2.10}$$

The following lemma is a simplified version of the more general theory (see, e.g., [12]) for the finite element method for two-point boundary value problems.

Lemma 2.1: *Let u and u_h be the solution of (2.2) and (2.8), respectively. Let u_r be the Ritz projection of u. Then*

$$\|u_h' - u_r'\| \le Ch^{r+1}\|u\|_{r+1}, \tag{2.11}$$

where C is a constant independent of h.

Proof : Subtracting (2.8) from (2.2) yields

$$(u' - u_h', v') + (b(u - u_h), v) = 0 \quad \forall v \in S_0^r(\Delta), \tag{2.12}$$

and subtracting (2.12) from (2.10) gives

$$(u_h' - u_r', v') = (b(u - u_h), v). \tag{2.13}$$

Let $v = u_h - u_r$ in (2.13), and we have

$$\|u_h' - u_r'\|^2 = (b(u - u_h), u_h - u_r) \le \|b\|_\infty\|u - u_h\|\|u_h - u_r\| \le \|b\|_\infty\|u - u_h\|\|u_h' - u_r'\|.$$

Here we have applied the Poicaré inequality to the last step. Hence from the error estimate (2.9), we have

$$\|u_h' - u_r'\| \le \|b\|_\infty\|u - u_h\| \le Ch^{r+1}\|u\|_{r+1}.$$

■

Remark 2.2 We see that the difference between the derivatives of the finite element solution and the Ritz projection is of the higher order (comparing with (2.9)). The importance of this result is that if the superconvergence of SPR could be achieved for Ritz projection, then it could be easily extended to the finite element solution.

Lemma 2.2: *Let $u \in H_0^1(I)$ be a polynomial of degree $r+1$, let $u_r \in S_0^r(\Delta)$ be its Ritz projection, and let $u_I \in S_0^r(\Delta)$ be its piecewise Lagrange interpolant at the Lobatto points of every I_i, $i = 1, \cdots, N$. Then $u_r = u_I$.*

Proof : It only needs to prove that

$$(u' - u_I', v') = 0, \quad \forall v \in S_0^r(\Delta).$$

Then by (2.10) and the uniqueness of the Ritz projection, we shall have $u_I = u_r$.

Since on each I_i, $i = 1, \cdots, N$, $(u' - u_I')v'$ is a polynomial of degree no more than $2r-1$, r-point Gaussian quadrature rule will give exact integration. Hence,

$$\begin{aligned}(u' - u_I', v') &= \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (u' - u_I')(x) v'(x) dx \\ &= \sum_{i=1}^{N} h_i \sum_{j=1}^{r} \hat{A}_j (u' - u_I')(G_{ij}) v'(G_{ij}) = 0,\end{aligned}$$

by applying Corollary 2.1. Here $\hat{A}_j > 0$, $j = 1, \cdots, r$ are the Gauss-Legendre weights of the quadrature rule. ■

This lemma claims that for a polynomial of degree $r+1$, its Ritz projection in the finite element space of order r (the space of piecewise polynomials of degree r) is simply the piecewise polynomial interpolation of the exact solution at the Lobatto points. An immediate consequence of this fact is that the difference between the finite element solution and the piecewise polynomial interpolation at the Lobatto points is one order higher than the usual approximation error.

For linear element, the Lobatto points are nodal points, so the Lobatto interpolation is the conventional Lagrange interpolation; for the quadratic element, the Lobatto points are nodal points and the mid-points, again, the Lobatto interpolation is the conventional interpolation. But for higher order element, Lobatto interpolation is not equal to the conventional evenly spaced interpolation. For example, for the cubic element, Lobatto interpolation points on the subinterval I_i are

$$x_{i-1}, \quad x_{i-1} + \frac{h_i}{2}(1 - \frac{1}{\sqrt{3}}), \quad x_i - \frac{h_i}{2}(1 - \frac{1}{\sqrt{3}}), \quad x_i,$$

while the evenly spaced interpolation points are

$$x_{i-1}, \quad x_{i-1} + \frac{h_i}{3}, \quad x_i - \frac{h_i}{3}, \quad x_i.$$

In the following, we shall focus our discussion on the patch recovery procedure for the Ritz projection u_r instead of the finite element solution u_h for reasons stated in Remark 2.2.

In general, u_r' is a piecewise polynomial of degree $r-1$ and is discontinuous at the nodal points x_i, $1 \le i \le N-1$. The recovered derivative by SPR is a continuous piecewise polynomial of degree r (same as u_r itself), $Ru_r' \in S^r(\Delta)$, it is uniquely determined by its values at the Lobatto points. The values of the recovered derivative at the Lobatto points are obtained by the following least-squares fitting procedure. On the element patch

$$S_i = I_i \cup \{x_i\} \cup I_{i+1},$$

consider a polynomial of degree r,

$$p_r^*(x) = (1, x, \cdots, x^r)\boldsymbol{a},$$

and $\boldsymbol{a} = (a_0, a_1, \cdots, a_r)^T$ is computed by fitting, in the least-squares sense, u_r' at $2r$ Gauss points $\{G_{ij}, G_{i+1,j}\}_{j=1}^{r}$ in S_i, $i = 1, \cdots, N-1$. Then the values of Ru_r' at the Lobatto points are the values of p_r^* at the same points, i.e.,

$$Ru_r'(L_{ij}) = p_r^*(L_{ij}), \quad j = 1, \cdots, r; \quad Ru_r'(L_{i+1,j}) = p_r^*(L_{i+1,j}), \quad j = 0, \cdots, r-1. \tag{2.14}$$

Note that there is an overlapping for adjacent element patches, i.e., $S_{i-1} \cap S_i = I_i$, $1 \le i \le N-1$. If different patches result in different recoveries on I_i, the averaging is applied (see [1] for more details). But if the exact solution is a polynomial of degree $r+1$, we shall show that two recoveries from the adjacent patches are the same.

Theorem 2.2: *Let $u \in H_0^1(I)$ be a polynomial of degree $r+1$, let $u_r \in S_0^r(\Delta)$ be its Ritz projection. Then $Ru_r' = u'$.*

Proof : Let $u_I \in S^r(\Delta)$ be the piecewise polynomial interpolation of u at the Lobatto points. From Lemma 2.2 and Corollary 2.1,

$$u_r'(G_{ij}) = u_I'(G_{ij}) = u'(G_{ij}), \quad j = 1, \cdots, r, \quad i = 1, \cdots, N.$$

Hence p_r^* fits u', in the least square sense, at $2r$ distinct points on the element patch S_i. But both p_r^* and u' are polynomials of degree r, they are then identical on S_i, $i = 1, \cdots, N-1$, therefore,

$$p_r^*(L_{ij}) = u'(L_{ij}), \quad j = 1, \cdots, r; \quad p_r^*(L_{i+1,j}) = u'(L_{i+1,j}), \quad j = 0, \cdots, r-1.$$

Recall (2.14), and we obtain

$$Ru_r'(L_{ij}) = u'(L_{ij}), \quad j = 1, \cdots, r; \quad Ru_r'(L_{i+1,j}) = u'(L_{i+1,j}), \quad j = 0, \cdots, r-1.$$

For $r = 1$ (in which case u is a linear function), this means that $Ru'_r = u'$ at all nodal points, then they are identical on I. For $r \geq 2$, we have $Ru'_r = u'$ on each S_i since they are both polynomials of degree r and are equal at $2r-1$ distinct points (notice that $L_{ir} = x_i = L_{i+1,0}$). Consequently, $Ru'_r = u'$ on the entire interval I. ■

Remark 2.3. From the proof of Theorem 2.2, we see that in order to recover u', it is sufficient to fit u'_r at no less than $r + 1$ (instead of $2r$) Gaussian points on each element patch. This fact has been observed in the numerical computations, see [1].

Remark 2.4. Theorem 2.2 shows that if the exact solution is a polynomial of degree $r+1$, SPR recovers the exact derivative for the Ritz projection in $S_0^r(\Delta)$.

Applying the Bramble-Hilbert lemma [31] as in the standard superconvergence analysis, Theorem 2.2 gives

Corollary 2.2: *Assume that $u \in H^{r+2}(I) \cap H_0^1(I)$, and let $u_r \in S_0^r(\Delta)$ be its Ritz projection. Then*

$$\|u' - Ru'_r\| \leq Ch^{r+1}|u|_{r+2},$$

where C is a constant independent of h.

The significance of this fact is: SPR provides a superconvergence recovery.

Theorem 2.3: *Let $u \in H^{r+2}(I) \cap H_0^1(I)$ be the solution of (2.2), and let $u_h \in S_0^r(\Delta)$ be the solution of (2.8). Then*

$$\|u' - Ru'_h\| \leq Ch^{r+1}\|u\|_{r+2},$$

where C is a constant independent of h.

Proof : Note that for any quasi-uniform mesh, R is a bounded linear operator with an upper bound independent of h. Using the triangular inequality, Corollary 2.2, and Lemma 2.1, we have

$$\begin{aligned} \|u' - Ru'_h\| &\leq \|u' - Ru'_r\| + \|Ru'_r - Ru'_h\| \\ &\leq Ch^{r+1}|u|_{r+2} + \|R\|\|u'_r - u'_h\| \leq Ch^{r+1}\|u\|_{r+2}. \end{aligned}$$

■

Remark 2.5. In Theorem 2.3, regularity requirement on the exact solution u is optimal. If the standard superconvergence results at the Gauss points had been used, then the regularity requirement would be $\|u\|_{r+2,\infty}$ instead of $\|u\|_{r+2}$.

Remark 2.6. Different from the traditional superconvergence result, the superconvergence obtained here is global as long as the exact solution is sufficiently smooth. This global superconvergence result has been observed in the numerical experiments [1].

In order to have a better understanding of Theorem 2.2, we examine several simple examples. We shall consider the model problem (2.14) with $b = 0$ in which case the finite element solution u_h equals to the Ritz projection u_r.

Example 1. $f = 2$. The exact solution is $1 - x^2$. We use two linear elements with the mesh points: $-1, \gamma, 1$. Note that when $\gamma = 0$, it is the uniform mesh. The finite element solution is:

$$u_h(x) = \begin{cases} (1-\gamma)(1+x) & -1 \le x \le \gamma, \\ (1+\gamma)(1-x) & \gamma \le x \le 1. \end{cases}$$

So,

$$u_h'(x) = \begin{cases} 1-\gamma & -1 \le x \le \gamma, \\ -(1+\gamma) & \gamma \le x \le 1. \end{cases}$$

We have only one element patch that contains 2 elements. SPR constructs $Ru_h'(x) = a_0 + a_1 x$ to fit $u_h'(x)$ at the two Gauss points: $(\gamma - 1)/2$ and $(\gamma + 1)/2$. Hence,

$$Ru_h'(\frac{\gamma-1}{2}) = a_0 + a_1\frac{\gamma-1}{2} = 1-\gamma, \quad Ru_h'(\frac{\gamma+1}{2}) = a_0 + a_1\frac{\gamma+1}{2} = -(1+\gamma).$$

We then find $a_0 = 0$ and $a_1 = -2$, and $Ru_h'(x) = -2x = u'(x)$, an exact recovery for the derivative which verifies Theorem 2.2.

Usually, the exact derivative recovery will be lost if Ru_h' fit u_h' at some other points. For example, when $\gamma = 0$, choose sample points $-1/3$ and $1/3$ (instead of $-1/2$ and $1/2$), and we have $Ru_h'(x) = -3x$. This recovers $u'(x)$ at the nodal point $x = 0$ only!

Example 2. $f = 6x$. The exact solution is $x - x^3$. We use two quadratic elements with mesh points: $-1, 0, 1$. The finite element solution is:

$$u_h(x) = \begin{cases} 3(x+1/2)^2/2 - 3/8 & -1 \le x \le 0, \\ -3(x-1/2)^2/2 + 3/8 & 0 \le x \le 1. \end{cases}$$

So,

$$u_h'(x) = \begin{cases} 3(x+1/2) & -1 \le x \le 0, \\ -3(x-1/2) & 0 \le x \le 1. \end{cases}$$

Again, we have only one element patch which contains 2 elements. SPR constructs $Ru_h'(x) = a_0 + a_1 x + a_2 x^2$ to fit $u_h'(x)$, in the least-squares sense, at the four Gauss points:

$$-\frac{1}{2}(1+\frac{1}{\sqrt{3}}), \quad -\frac{1}{2}(1-\frac{1}{\sqrt{3}}), \quad \frac{1}{2}(1-\frac{1}{\sqrt{3}}), \quad \frac{1}{2}(1+\frac{1}{\sqrt{3}}).$$

This least-squares procedure yields: $a_0 = 1, a_1 = 0, a_2 = -3$, and hence $Ru_h'(x) = 1 - 3x^2 = u'(x)$, an exact recovery for the derivative as proved in Theorem 2.2.

Example 3. $f = 12x^2$. The exact solution is $1 - x^4$. We know from Theorem 2.2 that we will recover the exact derivative by applying SPR to the cubic finite element method. But what will happen if we use the quadratic element? i.e., what if we still use the same finite element space as we did in Example 2? After some manipulation, we find the finite element solution in the same space as in Example 2 is:

$$u_h(x) = \begin{cases} 1 + x + 9[-(x+1/2)^2 + 1/4]/5 & -1 \le x \le 0, \\ 1 - x + 9[-(x-1/2)^2 + 1/4]/5 & 0 \le x \le 1. \end{cases}$$

So,

$$u_h'(x) = \begin{cases} 1 - 18(x+1/2)/5 & -1 \le x \le 0, \\ -1 - 18(x-1/2)/5 & 0 \le x \le 1. \end{cases}$$

Proceeding as in Example 2, we find that $a_0 = 0, a_1 = -48/5, a_2 = 0$, and hence $Ru_h'(x) = -48x/5$. Obviously, this is not the exact derivative $u'(x) = -4x^3$, but they are equal at the nodal point $x = 0$. We see that for the quartic polynomial, when quadratic element is used, the SPR procedure yields the exact derivative at the nodal point. In [1], an $O(h^4)$ convergence of the nodal derivative values at interior nodes for the quadratic element with SPR recovery was reported. Our analysis here seems to support that observation.

3. Recovery in 2-D.

As in Section 2, we only consider a simple model problem. But the analysis and the results may be generalized to other second order elliptic problems.

The counterpart of the problem (2.1) in two dimension is:

$$\begin{aligned} -\Delta u + bu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where Ω is a bounded domain with a Lipschitz continuous boundary and b is a nonnegative function. Again we assume that b and f are sufficiently smooth to permit our analysis to carry through.

The weak formulation of (3.1) is: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (\nabla u, \nabla v) + (bu, v) = (f, v) \quad \forall v \in H_0^1(\Omega), \tag{3.2}$$

where $(\cdot, \cdot)$ is the inner produce on $L^2(\Omega)$ defined by

$$(f, g) = \int_\Omega fg.$$

Again we need some preliminary results for polynomials. Let $P(r)$ denote the class of polynomials of degree r and let $Q(r)$ denote the class of polynomials of degree r in each variable separately. In other words, any element $p \in P(r)$ has the form of

$$p(x,y) = \sum_{i+j=0}^{r} c_{ij} x^i y^j,$$

whereas any $q \in Q(r)$ will be of the form

$$q(x,y) = \sum_{i=0}^{r} \sum_{j=0}^{r} c_{ij} x^i y^j.$$

Lemma 3.1: *Let $q \in Q(r)$ and $q(x_i, y_j) = 0$ at $(r+1) \times (r+1)$ distinct points $\{(x_i, y_j)\}_{i,j=1}^{r+1}$. Then $q \equiv 0$.*

Proof : Fix $y = y_j$, $1 \le j \le r+1$, $q(x, y_j)$ is a polynomial in x of degree no more than r and vanishes at $r+1$ distinct points: x_i, $i = 1, 2, \cdots, r+1$. Then $q(x, y_j) \equiv 0$, $1 \le j \le r+1$.

Now for any fixed $\bar{x}$, $q(\bar{x}, y)$ is a polynomial in y of degree no more than r and vanishes at $r+1$ distinct points: y_j, $j = 1, 2, \cdots, r+1$. Then $p(\bar{x}, y) \equiv 0$. Since $\bar{x}$ is arbitrary, $q \equiv 0$.
■

Next, define the Gauss and the Lobatto points of degree r on $[-1,1] \times [-1,1]$, respectively

$$\{(g_i^{(r)}, g_j^{(r)})\}_{i,j=1}^{r}, \quad \{(l_i^{(r)}, l_j^{(r)})\}_{i,j=0}^{r}.$$

The following theorem is the two dimension counterpart of Theorem 2.1.

Theorem 3.1: *Assume that $u \in Q(r) \cup \{x^{r+1}, y^{r+1}\}$. Let $u_I \in Q(r)$ be its Lagrange interpolant at $(r+1) \times (r+1)$ Lobatto points on $[-1,1] \times [-1,1]$. Then*

$$\frac{\partial u}{\partial x}(g_j^{(r)}, y) = \frac{\partial u_I}{\partial x}(g_j^{(r)}, y), \quad y \in [-1,1], \quad j = 1, \cdots, r; \tag{3.3}$$

$$\frac{\partial u}{\partial x}(x, g_j^{(r)}) = \frac{\partial u_I}{\partial y}(x, g_j^{(r)}), \quad x \in [-1,1], \quad j = 1, \cdots, r. \tag{3.4}$$

Proof : Fix $y = l_i^{(r)}$, $0 \le i \le r$, $u(x, l_i^{(r)})$ is a polynomial in x of degree no more than $r+1$ and $u_I(x, l_i^{(r)})$ is its Lagrange interpolant at $r+1$ Lobatto points on $[-1,1]$. By Theorem 2.1,

$$\frac{\partial u}{\partial x}(g_j^{(r)}, l_i^{(r)}) = \frac{\partial u_I}{\partial x}(g_j^{(r)}, l_i^{(r)}), \quad j = 1, \cdots, r, \quad 0 \le i \le r. \tag{3.5}$$

Since both $\dfrac{\partial u}{\partial x}(g_j^{(r)}, y)$ and $\dfrac{\partial u_I}{\partial x}(g_j^{(r)}, y)$ are polynomials in y of degree no more than r, and they are equal at $r+1$ distinct points $l_i^{(r)}$, $i = 0, 1, \cdots, r$, they are then identical, i.e.,

$$\frac{\partial u}{\partial x}(g_j^{(r)}, y) = \frac{\partial u_I}{\partial x}(g_j^{(r)}, y), \quad y \in [-1, 1], \quad j = 1, \cdots, r.$$

The proof for (3.4) is similar. ■

Remark 3.1. We see that $\nabla u_I = \nabla u$ along the Gauss lines. This type of the superconvergence for mixed finite element methods was discussed in [11].

Remark 3.2. Since $P(r+1) \subset Q(r) \cup \{x^{r+1}, y^{r+1}\}$, the conclusion in Theorem 3.1 holds for $u \in P(r+1)$ which is essential for the superconvergence analysis. But the same conclusion may not be true if $u \in Q(r+1)$, and instead, the following theorem hold.

Theorem 3.2: *Assume that $u \in Q(r+1)$. Let u_I be its Lagrange interpolant at $(r+1)\times(r+1)$ Lobatto points on $[-1, 1] \times [-1, 1]$. Then*

$$\frac{\partial^2 u}{\partial x \partial y}(g_j^{(r)}, g_i^{(r)}) = \frac{\partial^2 u_I}{\partial x \partial y}(g_j^{(r)}, g_i^{(r)}), \quad i, j = 1, \cdots, r.$$

Proof: Starting from (3.5), $\dfrac{\partial u}{\partial x}(g_j^{(r)}, y)$ is a polynomial in y of degree $r+1$ and $\dfrac{\partial u_I}{\partial x}(g_j^{(r)}, y)$ is its Lagrange interpolant at $r+1$ Lobatto points $l_i^{(r)}$, $i = 0, 1, \cdots, r$. By Theorem 2.1,

$$\frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial x}(g_j^{(r)}, y)\Big)\Big|_{y=g_i^{(r)}} = \frac{\partial}{\partial y}\Big(\frac{\partial u_I}{\partial x}(g_j^{(r)}, y)\Big)\Big|_{y=g_i^{(r)}},$$

or

$$\frac{\partial^2 u}{\partial x \partial y}(g_j^{(r)}, g_i^{(r)}) = \frac{\partial^2 u_I}{\partial x \partial y}(g_j^{(r)}, g_i^{(r)}), \quad i, j = 1, \cdots, r.$$

■

Now we assume that Ω is the unit square and let $\Delta = \Delta_x \times \Delta_y$ be a partition of Ω with

$$\Delta_x : \quad 0 = x_0 < x_1 < \cdots < x_{N_x} = 1; \quad \Delta_y : \quad 0 = y_0 < y_1 < \cdots < y_{N_y} = 1.$$

Denote $e_{ij} = I_i \times J_j$ with $I_i = (x_{i-1}, x_i)$, $J_i = (y_{j-1}, y_j)$, and define the Gauss and the Lobatto points on e_{ij} as:

$$(G_{ik}^x, G_{jl}^y) = \frac{1}{2}(x_{i-1} + x_i + h_i g_k^{(r)}, y_{j-1} + y_j + h'_j g_l^{(r)}), \quad k, l = 1, \cdots, r,$$

$$(L_{ik}^x, L_{jl}^y) = \frac{1}{2}(x_{i-1} + x_i + h_i l_k^{(r)}, y_{j-1} + y_j + h'_j l_l^{(r)}), \quad k, l = 0, 1, \cdots, r,$$

where $h_i = x_i - x_{i-1}$, $h'_j = y_j - y_{j-1}$. We define $h = \max_{1\le i\le N_x, 1\le j\le N_y}(h_i, h'_j)$, and also define

$$S^r(\Delta) = \{v \in H^1(\Omega), \quad v|_{e_{ij}} \in Q(r), \quad 1 \le i \le N_x, \quad 1 \le j \le N_y\},$$

$$S_0^r(\Delta) = H_0^1(\Omega) \cap S^r(\Delta).$$

$S_0^r(\Delta)$ is the finite element space which includes piecewise polynomials of degree no more than r in each variable.

As Corollary 2.1 to Theorem 2.1, the following result is a direct consequence of Theorem 3.1.

Corollary 3.1: *Assume that $u \in Q(r) \cup \{x^{r+1}, y^{r+1}\}$. Let $u_I \in S^r(\Delta)$ be its piecewise Lagrange interpolant at the Lobatto points of every e_{ij}, $i = 1, \cdots, N_x$, $j = 1, \cdots, N_y$. Then*

$$\frac{\partial u}{\partial x}(G_{ik}^x, y) = \frac{\partial u_I}{\partial x}(G_{ik}^x, y), \quad k = 1, \cdots, r;$$

$$\frac{\partial u}{\partial x}(x, G_{jl}^y) = \frac{\partial u_I}{\partial y}(x, G_{jl}^y), \quad l = 1, \cdots, r.$$

The finite element method and the standard error estimate are exactly the same in two dimension as stated in (2.8) and (2.9) for one dimensional case, i.e., the finite element approximation of (3.2) is to find $u_h \in S_0^r(\Delta)$ such that

$$a(u_h, v) = (f, v) \quad \forall v \in S_0^r(\Delta), \tag{3.6}$$

and the optimal error estimate in the L^2 norm is [31]

$$h\|\nabla u - \nabla u_h\| + \|u - u_h\| \le Ch^{r+1}\|u\|_{r+1}. \tag{3.7}$$

Similar as in the one dimensional case, we can define the Ritz projection in two dimension.

Definition 2.2 For any $u \in H_0^1(\Omega)$, its Ritz projection is $u_r \in S_0^r(\Delta)$ such that

$$(\nabla(u - u_r), \nabla v) = 0, \quad \forall v \in S_0^r(\Delta). \tag{3.8}$$

The following result is almost identical with Lemma 2.1 in one dimensional case.

Lemma 3.1: *Let u and u_h be the solutions of (3.2) and (3.6), and let u_r be the Ritz projection of u. Then*

$$\|\nabla(u_h - u_r)\| \le Ch^{r+1}\|u\|_{r+1}, \tag{3.9}$$

where C is a constant independent of h.

Proof : The proof is similar to that for Lemma 2.1 by noting that the derivative sign is replaced by ∇. ■

As in the one dimensional case, we shall now focus our discussion on the Ritz projection u_r (See Remark 2.2).

Lemma 3.2: *Let $u \in H_0^1(\Omega)$ be in $Q(r) \cup \{x^{r+1}, y^{r+1}\}$, let $u_r \in S_0^r(\Delta)$ be its Ritz projection, and let $u_I \in S_0^r(\Delta)$ be its piecewise polynomial interpolant at the Lobatto points of every e_{ij}, $i = 1, \cdots, N_x$, $j = 1, \cdots, N_y$. Then $u_r = u_I$.*

Proof : Using (3.8) and the uniqueness of the Ritz projection, we only need to prove that

$$(\nabla(u - u_I), \nabla v) = 0, \quad \forall v \in S_0^r(\Delta). \tag{3.10}$$

Consider for any $v \in S_0^r(\Delta)$,

$$\begin{aligned}(\nabla(u - u_I), \nabla v)_{ij} &= \int_{e_{ij}} \nabla(u - u_r)\nabla v dxdy \\ &= \int_{e_{ij}} \frac{\partial}{\partial x}(u - u_I)\frac{\partial v}{\partial x}dxdy + \int_{e_{ij}} \frac{\partial}{\partial y}(u - u_I)\frac{\partial v}{\partial y}dxdy.\end{aligned} \tag{3.11}$$

For any fixed y, $\frac{\partial}{\partial x}(u - u_I)\frac{\partial v}{\partial x}$ is a polynomial in x of degree no more than $2r - 1$, thus,

$$\int_{e_{ij}} \frac{\partial}{\partial x}(u - u_I)\frac{\partial v}{\partial x}dxdy = h_i \sum_{k=1}^{r} \hat{A}_k \int_{J_j} \Big[\frac{\partial}{\partial x}(u - u_I)\frac{\partial v}{\partial x}\Big](G_{ik}^x, y)dy = 0, \tag{3.12}$$

by Corollary 3.1. Similarly,

$$\int_{e_{ij}} \frac{\partial}{\partial y}(u - u_I)\frac{\partial v}{\partial y}dxdy = 0. \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.11),

$$(\nabla(u - u_I), \nabla v)_{ij} = 0. \tag{3.14}$$

(3.10) follows by summing up (3.14) over $i = 1, \cdots, N_x$, $j = 1, \cdots, N_y$. ■

Lemma 3.2 is the 2-D version of Lemma 2.2 which implies that for a polynomial of degree $r + 1$ ($u \in P(r + 1)$), its Ritz projection in the finite element space of order r is its Lobatto interpolation of degree r.

Same as in the one dimensional situation, we discuss the patch recovery procedure for the Ritz projection. The recovered gradient by SPR is a continuous piecewise polynomial of degree r in each variable, $R\nabla u_h = (R\frac{\partial u_h}{\partial x}, R\frac{\partial u_h}{\partial y}) \in S^r(\Delta)^2$, and its values at the Lobatto points are determined by a similar least-squares procedure described in Section 2. On the element patch (Figure 1)

$$S_{ij} = \bigcup_{\alpha=i,i+1;\beta=j,j+1} e_{\alpha\beta} \cup \{(x_i, y), (x, y_j) : x_{i-1} < x < x_{i+1}, y_{j-1} < y < y_{j+1}\},$$

we are now looking for a vector polynomial of degree r in each variable, i.e.,

$$\boldsymbol{q}_r^* = (1, x, y, \cdots, x^r y^r)(\boldsymbol{a}, \boldsymbol{b}),$$

where $m = (r+1)^2$ and $\boldsymbol{a} = (a_1, \cdots a_m)^T$, $\boldsymbol{b} = (b_1, \cdots, b_m)^T$ are determined by fitting, in the least-squares sense, ∇u_h at $2r \times 2r$ Gaussian points

$$\{(G_{ik}^x, G_{jl}^y), (G_{i+1,k}^x, G_{jl}^y), (G_{i+1,k}^x, G_{j+1,l}^y), (G_{ik}^x, G_{j+1,l}^y)\}_{k,l=1}^r,$$

in S_{ij}, $1 \le i \le N_x - 1$, $1 \le j \le N_y - 1$. If we denote the set of $(2r-1) \times (2r-1)$ Lobatto points on the element patch by N_{ij}, the values of $R\nabla u_h$ at $(x, y) \in N_{ij}$ is then given by the values of $\boldsymbol{q}_r^*$ at the same points, i.e.,

$$R\nabla u_h(x, y) = \boldsymbol{q}_r^*(x, y), \quad \forall (x, y) \in N_{ij}. \tag{3.15}$$

Here again overlapping occurs for adjacent element patches, for example, $S_{i-1,j} \cap S_{ij} \cap S_{i,j-1} \cap S_{i-1,j-1} = e_{ij}$, $2 \le i \le N_x - 1$, $2 \le j \le N_y - 1$. If on e_{ij} different patches result in different recoveries, simple averaging is applied (cf. [1] for details). We shall demonstrate in the following that if $u \in Q(r) \cup \{x^{r+1}, y^{r+1}\}$, identical recoveries are obtained from adjacent patches.

Theorem 3.3: *Let $u \in H_0^1(\Omega)$ be in $Q(r) \cup \{x^{r+1}, y^{r+1}\}$, let $u_r \in S_0^r(\Delta)$ be its Ritz projection. Then $R\nabla u_r = \nabla u$.*

Proof : Let $u_I \in S^r(\Delta)$ be the piecewise polynomial interpolation of u at the Lobatto points. From Lemma 3.2 and Corollary 3.1, for any e_{ij}, $1 \le i \le N_x$, $1 \le j \le N_y$,

$$\nabla u_r(G_{\alpha k}^x, G_{\beta l}^y) = \nabla u_I(G_{\alpha k}^x, G_{\beta l}^y) = \nabla u(G_{\alpha k}^x, G_{\beta l}^y),$$

$k, l = 1, \cdots, r$, $\alpha = i+1, i, \beta = j+1, j$. Hence, $\boldsymbol{q}_r^*$ fits ∇u, in the least square sense, at $2r \times 2r$ distinct points on S_{ij}. Since both $\boldsymbol{q}_r^*$ and ∇u belong to $Q(r) \times Q(r)$, they are then identical on S_{ij} by Lemma 3.1, especially,

$$\boldsymbol{q}_r^*(x, y) = \nabla u(x, y), \quad \forall (x, y) \in N_{ij}.$$

Recall (3.15), we then have

$$R\nabla u_r(x,y) = \nabla u(x,y), \quad \forall (x,y) \in N_{ij}.$$

For $r = 1$ (in which case ∇u is piecewise linear), this means that $R\nabla u_h(x,y) = \nabla u(x,y)$ at all nodal points and hence they are identical on Ω. For $r \geq 2$, we have $R\nabla u_r = \nabla u$ on each S_{ij} (since they are both belong to $Q(r) \times Q(r)$ and are equal at $(2r-1) \times (2r-1)$ distinct points). Therefore, they are equal on the whole Ω. ■

Remark 3.3. From the proof of Theorem 3.3, we see that, in order to recover the exact gradient, it is sufficient to choose no less than $(r+1) \times (r+1)$ (instead of $2r \times 2r$) Gaussian points in the element patch.

Remark 3.4. Theorem 3.3 is the two dimensional counterpart of Theorem 2.2, it implies that if the exact solution is a polynomial of degree $r+1$ ($u \in P(r+1)$), then SPR recovers the exact gradient for the Ritz projection in $S_0^r(\Delta)$, and hence it is a superconvergence recovery.

Same as in one dimensional case the following result can be obtained from Theorem 3.3 by applying the Bramble-Hilbert lemma.

Corollary 3.2: *Assume that $u \in H^{r+2}(\Omega) \cap H_0^1(\Omega)$, and let $u_r \in S_0^r(\Delta)$ be its Ritz projection. Then*

$$\|\nabla u - R\nabla u_r\| \leq Ch^{r+1}|u|_{r+2},$$

where C is a constant independent of h.

Finally, we have for the two dimensional problem, the following result.

Theorem 3.4: *Let $u \in H^{r+2}(\Omega) \cap H_0^1(\Omega)$ be the solution of (3.2), and let $u_h \in S_0^r(\Delta)$ be the solution of (3.6). Then*

$$\|\nabla u - R\nabla u_h\| \leq Ch^{r+1}\|u\|_{r+2},$$

where C is a constant independent of h.

Proof : The proof is similar to that for Theorem 2.3. Note that for any quasi-uniform mesh, R is a bounded linear operator with an upper bound independent of h. By the triangular inequality, Corollary 3.2, and Lemma 3.1, we have

$$\begin{aligned} \|\nabla u - R\nabla u_h\| &\leq \|\nabla u - R\nabla u_r\| + \|R\nabla u_r - R\nabla u_h\| \\ &\leq Ch^{r+1}|u|_{r+2} + \|R\|\|\nabla(u_r - u_h)\| \leq Ch^{r+1}\|u\|_{r+2}. \end{aligned}$$

■

Remark 3.5. As in Theorem 2.3, the regularity requirement here in Theorem 3.4 is optimal and the superconvergence result is global as long as the exact solution is sufficiently smooth (See Remark 2.5 and 2.6). This global superconvergence has also been observed in the numerical experiments [1].

4. A posteriori error estimates. In this section, we shall discuss the *a-posteriori* error estimator based on SPR, such error estimator is often referred as ZZ error estimator. For illustration purpose, we use the L^2-norm. Similar argument can be applied to other norms wherever superconvergence recovery is achieved.

By using SPR, we obtain $R\nabla u_h$, which converges to ∇u at a rate one order higher than ∇u_h, i.e.,

$$\|\nabla u - R\nabla u_h\| \le Ch^{r+1}\|u\|_{r+2}. \tag{4.1}$$

Using $R\nabla u_h$, we have the following ZZ *a posteriori* error estimator:

$$\epsilon = \|R\nabla u_h - \nabla u_h\|.$$

Define H, a subspace of $H_0^1(\Omega)$ as

$$H = \{u \in H_0^1(\Omega) \cap H^{r+2}(\Omega), \quad \|\nabla u - \nabla u_h\| \ge \alpha h^r |u|_{r+1} \text{ for some } \alpha > 0\}.$$

The error estimator ϵ is said to be asymptotically exact on H if

$$\frac{\epsilon}{\|\nabla u - \nabla u_h\|} \longrightarrow 1, \quad \text{as } h \to 0 \quad \forall u \in H.$$

Theorem 4.1: *Under the same conditions used in Theorem 2.3 and Theorem 3.4, the error estimator ϵ is asymptotically exact on H.*

Proof : For any fixed u in H, we have, by the triangle inequality, the definition of H, and (4.1),

$$\begin{aligned}
\frac{\epsilon}{\|\nabla u - \nabla u_h\|} &= \frac{\|R\nabla u_h - \nabla u_h\|}{\|\nabla u - \nabla u_h\|} \\
&\le \frac{\|\nabla u - \nabla u_h\| + \|R\nabla u_h - \nabla u\|}{\|\nabla u - \nabla u_h\|} \\
&\le 1 + \frac{Ch^{r+1}\|u\|_{r+2}}{\alpha h^r |u|_{r+1}} = 1 + \bar{C}h.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\epsilon}{\|\nabla u - \nabla u_h\|} &\ge \frac{\|\nabla u - \nabla u_h\| - \|R\nabla u_h - \nabla u\|}{\|\nabla u - \nabla u_h\|} \\
&\ge 1 - \frac{Ch^{r+1}\|u\|_{r+2}}{\alpha h^r |u|_{r+1}} = 1 - \bar{C}h.
\end{aligned}$$

Hence,

$$\frac{\epsilon}{\|\nabla u - \nabla u_h\|} \longrightarrow 1, \quad \text{as } h \to 0 \quad \forall u \in H,$$

■

Remark 4.1. The extension of the analysis to 3-D tensor product case is straight-forward.

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