Numer. Math. 59, 107–127 (1991)

Numerische Mathematik

On the asymptotic exactness of error estimators for linear triangular finite elements

Ricardo Durán*,, María Amelia Muschietti*, and Rodolfo Rodríguez*****

Departamento de Matemática, Facultad de Ciencias Exactas, UNLP. C.C. 172, 1900 - La Plata, Argentina

Received October 5, 1990

Summary. This paper deals with a-posteriori error estimates for piecewise linear finite element approximations of elliptic problems. We analyze two estimators based on recovery operators for the gradient of the approximate solution. By using superconvergence results we prove their asymptotic exactness under regularity assumptions on the mesh and the solution.

One of the estimators can be easily computed in terms of the jumps of the gradient of the finite element approximation. This estimator is equivalent to the error in the energy norm under rather general conditions. However, we show that for the asymptotic exactness, the regularity assumption on the mesh is not merely technical. While doing this, we analyze the relation between superconvergence and asymptotic exactness for some particular examples.

Subject classifications. AMS(MOS) 65N30, 65N15, CR: G1.8

1 Introduction

In recent years, considerable interest has been shown in a posteriori error estimates and adaptive refinement for finite element approximations of second order elliptic problems.

In the one variable case, a rather complete theory has been developed by Babuška and Rheinboldt [5, 7, 8]. In particular, they have proven that, under suitable regularity assumptions on the solution, several estimators are asymptotically exact in the energy norm. The estimators they have considered are essentially of two kinds. The first one is based on the solution of local problems while the second one, on the computation of residuals and jumps of the approximate solution.

These ideas have been generalized to problems in two variables [6, 3, 9], but the analysis is much more complicated in this case.

* Member of CONICET

** Current address: Department of Mathematics, University of Maryland, College Park, MD 20742, USA

*** Current address: Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA

Offprint requests to: R. Rodríguez

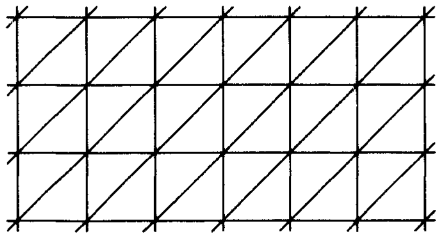

Fig. 1

Estimators based on the jumps of the normal derivative of the approximate solution can be constructed and analyzed by means of superconvergence results. In particular, for rectangular elements, the results of Zlámal [25] can be used to device several error estimators [1, 12], and moreover to show that they are asymptotically exact [12]. These estimators can be viewed as generalizations of a one-dimensional estimator of Babuška and Reinboldt [8].

Here we consider the case of linear triangular elements. Superconvergent recovery of the gradient for particular meshes has been considered by several authors. A fundamental result was obtained by Oganesjan and Ruhovec [19] for uniform meshes (like those in Fig. 1) that states that the difference between the gradient of the approximate solution u_h and the gradient of the Lagrange interpolation u^I of the exact solution u is of higher order than the error itself whenever the solution is regular enough, that is,

$$\|\nabla(u_h-u^I)\|_{0,\Omega}=O(h^{1+\varepsilon}) \tag{1.1}$$

for some $\varepsilon>0$. We say that there is superconvergence whenever (1.1) holds.

Later, several authors ([17, 16, 15, 2]) generalized (1.1) to other triangular meshes. In particular, Wheeler and Whiteman [23] proved that the superconvergence result has a local character.

In order to construct an asymptotically exact error estimator, we define a recovery operator for the gradient, that is, an operator G such that Gu_h is a higher order approximation of ∇u than ∇u_h itself; namely,

$$\|\nabla u-Gu_h\|_{0,\Omega}=O(h^{1+\varepsilon})$$

for some $\varepsilon>0$. Given G, we define the error estimator ε by

$$\varepsilon:=Gu_h-\nabla u_h.$$

In the case of a uniform mesh like that in Fig. 1, an operator G based on the interpolation of the average of the two gradients in the triangles sharing a common edge was introduced in [24].

In this work, we introduce a recovery operator G based on quadratic isoparametric interpolation. We show that

$$\|\nabla u-Gu^I\|_{0,\Omega_0}\leqq Ch^2\|u\|_{3,\Omega_1}, \tag{1.3}$$

for subregions $\Omega_0 \Subset \Omega_1 \subset \Omega$ in which the mesh satisfies a regularity assumption (we call it quasi-parallelism and, intuitively, it means that the meshes are higher order perturbations of uniform meshes).

The superconvergence result (1.1) is also valid for quasi-parallel meshes [23], therefore, the error estimator based on G is asymptotically exact in the sense that the relative error

$$\frac{\|\varepsilon - \nabla e\|_{0,\Omega_0}}{\|\nabla e\|_{0,\Omega_0}} \xrightarrow[h \to 0]{} 0 \tag{1.4}$$

whenever the H^1-seminorm of the error $e := u - u_h$ is properly $O(h)$. (Note that (1.4) implies asymptotical exactness as defined by Babuška and Rheinboldt [8].)

Afterwards, we calculate the value of Gu^{I} at each edge midpoint and we show that it is a weighted average of the restrictions of ∇u^{I} to the triangles containing that point. So, by interpolating linearly those values in each triangle, we obtain another operator $\tilde{G}$ which is a generalization of the operator introduced in [24] for uniform meshes.

We also prove that

$$\|\nabla u - \tilde{G} u^{\mathrm{I}}\|_{0,\Omega_0} \leqq C h^2 \|u\|_{3,\Omega_1} \tag{1.5}$$

and so, defining $\tilde{\varepsilon} := \tilde{G} u_h - \nabla u_h$, we obtain a result analogous to (1.4) for this estimator.

The estimator $\tilde{\varepsilon}$ can be easily computed in terms of the jumps of the normal derivative of the approximate solution. Estimators of this kind are actually in use [14, 18, 20, 22].

On the other hand, the techniques by Babuška and Miller [4] can be applied for any general triangular mesh, regular in the usual sense, and for any problem whose solution $u \in H^1(\Omega)$, to show that the estimator $\tilde{\varepsilon}$ is equivalent to the error in the energy norm; namely, there exist constants C_1 and C_2 such that

$$C_1 \|\tilde{\varepsilon}\|_{0,\Omega} \leqq \|\nabla e\|_{0,\Omega} \leqq C_2 \|\tilde{\varepsilon}\|_{0,\Omega}.$$

Therefore, it is natural to ask if the asymptotic exactness is satisfied in regions where the meshes are not quasi-parallel. We show that this is not true. Specifically, we consider a subregion where the meshes are of the so called criss-cross type. For the case of Laplace equation, numerical evidence of the lack of superconvergence for this kind of meshes was presented by Levine [16]. We give here a proof of this fact and, by showing that (1.5) holds in these meshes for a family of functions, we provide examples where $\tilde{\varepsilon}$ is not asymptotically exact.

Finally we analyze some particular problems and show that the asymptotical exactness of $\tilde{\varepsilon}$ fails even in cases where there is superconvergence. Also we use these examples to show that $\tilde{\varepsilon}$ is not asymptotically exact even in the weaker sense of Babuška and Rheinboldt [8]. As a conclusion, we can say that the assumption on the meshes is not only a technical matter; that is, the asymptotical exactness of $\tilde{\varepsilon}$ does not hold in general.

The remainder of the paper is organized as follows. In Sect. 2 we introduce the model problem and notations; Sect. 3 deals with recovery operators and error estimators, and finally in Sect. 4 we analyze the case of criss-cross meshes.

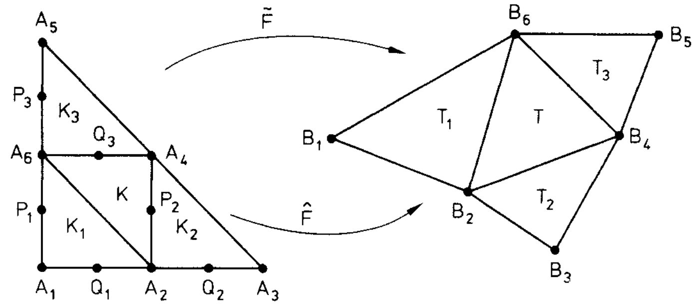

Fig. 2

2 Model problem and notations

Let Ω be a bounded polygon in $\mathbb{R}^2$. We consider elliptic problems in which the function u satisfies

$$(2.1) \qquad -\operatorname{div}(a\nabla u)+bu=f, \quad \text{in } \Omega,$$

with boundary conditions of the Dirichlet type, where a is a Lipschitz function such that $a(x)\geqq a_0>0$ and $b\geqq 0$ are bounded functions.

We use the standard notation for Sobolev spaces, $H^m(D)$, $W^{m,\infty}(D)$ for $m\geqq 0$ and $H_0^1(D)$, and the usual norms and seminorms $\|\cdot\|_{m,D}$, $|\cdot|_{m,D}$, $\|\cdot\|_{m,\infty,D}$ and $|\cdot|_{m,\infty,D}$.

Let $\{\mathscr{T}_h\}$ be a regular family of triangulations of Ω, where, as usual, h stands for the mesh size. Let $u_h\in V_h:=\{v\in H^1(\Omega)\colon v|_T\in\mathscr{P}_1, \forall\, T\in\mathscr{T}_h\}$ be the piecewise linear finite element approximate solution of problem (2.1). ($\mathscr{P}_m$ denotes the set of polynomials of degree not greater than m). Let $e:=u-u_h$ denote the error of this approximation.

From now on C will denote a constant independent of h and u, but not necessarily the same at each occurrence.

3 Recovery operators and error estimators

First, we define a recovery operator based on quadratic isoparametric interpolation. In order to do this, let us introduce some notation. Given an interior element $T\in\mathscr{T}_h$ we denote by T_i, $i=1, 2, 3$ its neighbor triangles. Let K and K_i be reference triangles as in Fig. 2. We set

$$T^*:=T\cup\bigcup_{i=1}^{3} T_i \quad\text{and}\quad K^*:=K\cup\bigcup_{i=1}^{3} K_i$$

and we define two transformations on K^*, a quadratic $\tilde{F}$ and a piecewise linear $\hat{F}$ both satisfying

$$\tilde{F}(A_i)=\hat{F}(A_i)=B_i, \qquad i=1, \ldots, 6,$$

with the notation of Fig. 2.

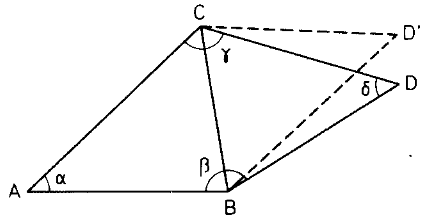

Fig. 3

We want to define a recovery operator G such that Gu^I be a better approximation to ∇u than ∇u^I. For $v \in V_h$ we define Gv locally in such a way that $Gv|_T$ takes into account only the information of the six nodal values $v(A_i)$, $i=1, \ldots, 6$.

In order to make clear the idea behind the definition of G, let us first remark that in the simpler case that $T=K$ and $T_i=K_i$, $i=1, 2, 3$, we would define for any $v \in V_h$ and $x \in K$,

$$Gv(x) := \nabla(I_2 v)(x),$$

where I_2 is the standard quadratic interpolation operator at the nodes A_i, $i=1, \ldots, 6$.

To extend this definition of G to the general case, we need to assume that $\tilde{F}\colon K^* \to \mathbb{R}^2$ is a one to one correspondence with a smooth inverse defined on $\tilde{T} := \tilde{F}(K^*)$. In such a case, for $v \in V_h$ we first define for $x \in T^*$,

$$G_T v(x) := \nabla((I_2 \hat{v}) \circ \tilde{F}^{-1})(\tilde{F}(\hat{F}^{-1}(x))), \tag{3.1}$$

where $\hat{v} = v \circ \hat{F}$, and then for $x \in T$,

$$Gv(x) := G_T v(x).$$

For a boundary triangle T we choose an interior element S neighbor of T and repeat the construction using $T^* := S^*$ to define $G_T v := G_S v$.

In order to prove the asymptotic exactness of the estimator defined by means of this recovery operator, we need to assume a further regularity assumption on the meshes.

Given a subregion $\Omega_1 \subset \Omega$ we say that the meshes of the family $\{\mathcal{T}_h\}$ are quasi-parallel on Ω_1 if for any triangle $T \in \mathcal{T}_h$ such that $T \subset \Omega_1$, and for any neighbor $T' \in \mathcal{T}_h$ sharing an edge with T (see Fig. 3), the opposite angles of the quadrilateral $T \cup T'$ differ in $O(h)$. That is, according to the notation of Fig. 3,

$$|\alpha - \delta| = O(h), \qquad |\beta - \gamma| = O(h).$$

An analogous assumption is that if D' is the point such that $ABCD'$ is a parallelogram then $\operatorname{dist}(D, D') = O(h^2)$.

It is easy to show that quasi-parallelism is equivalent to the usual regularity for the isoparametric quadratic triangle $\tilde{T}$. Under this hypothesis, $\tilde{F}$ has a smooth inverse (see [10]). Therefore, G is well defined for quasi-parallel meshes but this is not a necessary assumption. In fact, exception made of some degenerate meshes, $\tilde{F}$ will have always a smooth inverse and hence G will be well defined.

On the other hand there are several different assumptions implying quasi-parallelism that have been used in previous works about superconvergence in linear triangular elements; for instance those in [16] and [23]. In particular, whenever the nodes of the meshes in $\mathcal{T}_h$ are images of the nodes of uniform meshes under a fixed diffeomorphism, they are quasi-parallel.

From now on, we assume that we have a family of quasi-parallel meshes on a subregion $\Omega_1 \subset \Omega$ and we shall prove some properties concerning the recovery operator G.

Lemma 3.1. *For any element $T \subset \Omega_1$ the following estimate holds,*

$$\|G_T v\|_{0,T^*} \leqq C \|\nabla v\|_{0,T^*}, \qquad \forall v \in V_h. \tag{3.2}$$

Proof. We have,

$$\|G_T v\|_{0,\infty,T^*} = \|\nabla(I_2 \hat{v} \circ \tilde{F}^{-1})\|_{0,\infty,\tilde{T}} \leqq \|D\tilde{F}^{-1}\|_{0,\infty,\tilde{T}} \|\nabla(I_2 \hat{v})\|_{0,\infty,K^*}, \tag{3.3}$$

where $D\tilde{F}^{-1}$ is the jacobian matrix of $\tilde{F}^{-1}$. Now,

$$\|\nabla(I_2 \hat{v})\|_{0,\infty,K^*} \leqq C \|\nabla \hat{v}\|_{0,\infty,K^*}; \tag{3.4}$$

in fact, with the notation of Fig. 3, it is easily seen that,

$$\frac{\partial}{\partial x} I_2 \hat{v}(P_i) = \frac{\partial \hat{v}}{\partial x}(P_i), \qquad i = 1, 2, 3$$

and

$$\frac{\partial}{\partial y} I_2 \hat{v}(Q_i) = \frac{\partial \hat{v}}{\partial y}(Q_i), \qquad i = 1, 2, 3$$

and therefore $\nabla(I_2 \hat{v})$ can be thought of as a linear interpolant of $\nabla \hat{v}$ in K^* and so (3.4) holds.

On the other hand,

$$\|\nabla \hat{v}\|_{0,\infty,K^*} \leqq \|D\hat{F}\|_{0,\infty,K^*} \|\nabla v\|_{0,\infty,T^*}.$$

Therefore, from (3.3) we obtain,

$$\|G_T v\|_{0,\infty,T^*} \leqq C \|D\tilde{F}^{-1}\|_{0,\infty,\tilde{T}} \|D\hat{F}\|_{0,\infty,K^*} \|\nabla v\|_{0,\infty,T^*}. \tag{3.5}$$

Now, it is known that (see [10]),

$$\|D\tilde{F}^{-1}\|_{0,\infty,\tilde{T}} \geqq \frac{C}{h} \quad \text{and} \quad \|D\hat{F}\|_{0,\infty,K^*} \leqq Ch.$$

Therefore (3.5) yields

$$\|G_T v\|_{0,\infty,T^*} \leqq C \|\nabla v\|_{0,\infty,T^*}$$

and using inverse inequalities we obtain (3.2). □

Lemma 3.2. *Let $T \subset \Omega_1$ and let $v \in H^3(\tilde{T})$, then*

$$\|\nabla v - G v^{\mathrm{I}}\|_{0,T} \leqq C h^2 \|v\|_{3,\tilde{T}}. \tag{3.6}$$

Proof. Since $I_2(\hat{v}^{\mathrm{I}}) = I_2\hat{v}$ we have,

$$\begin{aligned}(3.7)\qquad \|\nabla v - G v^{\mathrm{I}}\|_{0,T} &= \|\nabla v - \nabla(I_2\hat{v}\circ\tilde{F}^{-1})\circ(\tilde{F}\circ\hat{F}^{-1})\|_{0,T}\\ &\leqq \|\nabla v - \nabla(I_2\hat{v}\circ\tilde{F}^{-1})\|_{0,T}\\ &\quad + \|\nabla(I_2\hat{v}\circ\tilde{F}^{-1}) - \nabla(I_2\hat{v}\circ\tilde{F}^{-1})\circ(\tilde{F}\circ\hat{F}^{-1})\|_{0,T}.\end{aligned}$$

Now,

$$\begin{aligned}(3.8)\qquad &\|\nabla(I_2\hat{v}\circ\tilde{F}^{-1}) - \nabla(I_2\hat{v}\circ\tilde{F}^{-1})\circ(\tilde{F}\circ\hat{F}^{-1})\|_{0,T}\\ &\leqq |T|^{1/2}\|\nabla(I_2\hat{v}\circ\tilde{F}^{-1}) - \nabla(I_2\hat{v}\circ\tilde{F}^{-1})\circ(\tilde{F}\circ\hat{F}^{-1})\|_{0,\infty,T}\\ &\leqq |T|^{1/2}|I_2\hat{v}\circ\tilde{F}^{-1}|_{2,\infty,\tilde{T}}\sup_{x\in T}|x - \tilde{F}\circ\hat{F}^{-1}(x)|.\end{aligned}$$

But, under our hypothesis it is easily seen that,

$$\sup_{x\in T}|x - \tilde{F}\circ\hat{F}^{-1}(x)| \leqq C h^2$$

and using an inverse inequality for $I_2\hat{v}\circ\tilde{F}^{-1}$ we obtain from (3.8) that the second term on the right hand side of (3.7) is bounded by

$$C h^2 |I_2\hat{v}\circ\tilde{F}^{-1}|_{2,\tilde{T}} \leqq C h^2 \|v\|_{2,\tilde{T}},$$

where the last inequality follows from the isoparametric interpolation theory [10].

Finally, to bound the first term on the right hand side of (3.7) we use again the known results for isoparametric interpolation and we obtain (3.6). □

Let us now define the error estimator

$$\varepsilon := G u_h - \nabla u_h.$$

Let $\Omega_0 \Subset \Omega_1$ be subdomains of Ω and assume as before that the mesh is quasi-parallel in Ω_1. It has been proven [23] that, for h small enough,

$$\|\nabla u^{\mathrm{I}} - \nabla u_h\|_{0,\Omega_0} \leqq C(h^2\|u\|_{3,\Omega_1} + \|e\|_{0,\Omega_1}).$$

Therefore, applying Lemmas 3.1 and 3.2 we have,

$$\begin{aligned}\|\nabla u - G u_h\|_{0,\Omega_0} &\leqq \|\nabla u - G u^{\mathrm{I}}\|_{0,\Omega_0} + \|G(u^{\mathrm{I}} - u_h)\|_{0,\Omega_0}\\ &\leqq C(h^2\|u\|_{3,\Omega_1} + \|e\|_{0,\Omega_1})\end{aligned}$$

and consequently

$$\|\varepsilon - \nabla e\|_{0,\Omega_0} \leqq C(h^2\|u\|_{3,\Omega_1} + \|e\|_{0,\Omega_1}).$$

Remark 3.1. The last formula shows that $\|\varepsilon - \nabla e\|_{0,\Omega_0} = O(h^{1+\varepsilon})$, whenever

$$(3.9)\qquad \|e\|_{0,\Omega_1} = O(h^{1+\varepsilon}) \qquad \text{for some } \varepsilon\in(0,1].$$

For a convex polygon, if the solution $u\in H^2(\Omega)$, then it is easily seen that (3.9) holds. For a nonconvex polygon, (3.9) also holds for instance for the Laplace operator with $f\in L^2(\Omega)$ and homogeneous Dirichlet boundary conditions. In fact,

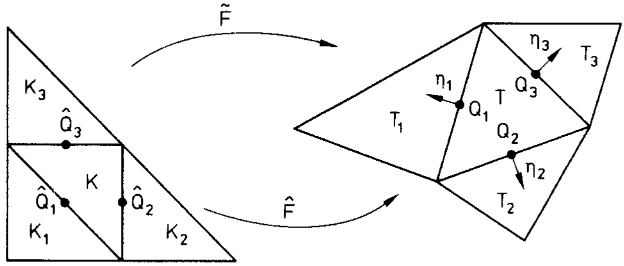

Fig. 4

in such a case, the solution of (2.1) belongs to $H^{1+s}(\Omega)$ for all $s<\pi/\omega$ where ω is the largest angle of the polygon (see Grisvard [13]) and $\|e\|_{0,\Omega_1} \leqq Ch^{2s}|u|_{1+s,\Omega}$.

From now on we assume that (3.9) holds and also that the error is properly $O(h)$ in Ω_0; that is,

$$|e|_{1,\Omega_0} \geqq Ch \tag{3.10}$$

for some constant C depending on u but not on h. This last requirement is not very restrictive, being satisfied in all but trivial cases (see [4]).

Therefore, if the solution is in $H^3(\Omega_1)$, then the estimator ε is asymptotically exact in Ω_0, that is,

$$\frac{\|\varepsilon-\nabla e\|_{0,\Omega_0}}{\|\nabla e\|_{0,\Omega_0}} = O(h^{\varepsilon}).$$

In order to define a simpler error estimator, let us compute the values of Gv, for $v \in V_h$, at the midpoint of any interior edge.

Lemma 3.3. *Denoting by T^- and T^+ two adjacent triangles and Q the midpoint of the common side, for any $v \in V_h$ we have,*

$$Gv(Q) = \frac{|T^-|}{|T^-|+|T^+|}\nabla v|_{T^-} + \frac{|T^+|}{|T^-|+|T^+|}\nabla v|_{T^+}. \tag{3.11}$$

Proof. We use the notation of Fig. 4.

Applying the chain rule to (3.1) we have

$$G_T v(Q_i) = [(D\tilde{F}^{-1})(\tilde{F}(\hat{Q}_i))][\nabla(I_2\hat{v})(\hat{Q}_i)].$$

Now let us observe that for any quadratic function g defined on K^*,

$$\nabla g(\hat{Q}_i) = \frac{\nabla(g^{\mathrm{I}}|_K)+\nabla(g^{\mathrm{I}}|_{K_i})}{2};$$

hence

$$\nabla(I_2\hat{v})(\hat{Q}_i) = \frac{\nabla\hat{v}|_K + \nabla\hat{v}|_{K_i}}{2}$$

and also

$$(D\tilde{F})(\hat{Q}_i) = \frac{D\hat{F}|_K + D\hat{F}|_{K_i}}{2}.$$

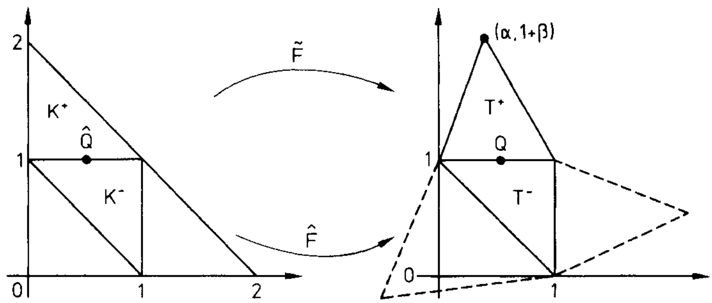

Fig. 5

Therefore,

$$G_T v(Q_i) = [(D\tilde{F})(\hat{Q}_i)]^{-1} \frac{\nabla \hat{v}|_K + \nabla \hat{v}|_{K_i}}{2} \tag{3.12}$$

$$= (D\hat{F}|_K + D\hat{F}|_{K_i})^{-1} (\nabla \hat{v}|_K + \nabla \hat{v}|_{K_i})$$

depends only on $v|_T$ and $v|_{T_i}$.

In the simpler case in which $\tilde{F}$ is affine (and hence $\hat{F} = \tilde{F}$), (3.11) holds. In fact,

$$G_T v(Q_i) = (D\hat{F})^{-1} \frac{\nabla \hat{v}|_K + \nabla \hat{v}|_{K_i}}{2}$$

and $\nabla \hat{v} = (D\hat{F})(\nabla v \circ \hat{F})$; hence

$$G_T v(Q_i) = \frac{\nabla v|_K + \nabla v|_{K_i}}{2}.$$

In view of this, we may assume that $T = K$ and restrict ourselves to consider the case described in Fig. 5 for any $\alpha \in \mathbb{R}$ and $\beta > 0$.

In this case, a straightforward computation yields

$$(D\tilde{F})(\hat{Q}) \left(\frac{|T^-|}{|T^-| + |T^+|} \nabla v|_{T^-} + \frac{|T^+|}{|T^-| + |T^+|} \nabla v|_{T^+} \right) = \frac{\nabla \hat{v}|_{K^-} + \nabla \hat{v}|_{K^+}}{2}$$

and so, by using (3.12) we conclude the lemma. □

In particular, (3.11) shows that Gv is continuous at those midpoints. This fact allows us to define a second recovery operator:

$$\tilde{G}\colon V_h \to [L^2(\Omega)]^2$$

by interpolating the values of Gv at the midpoints of the sides by a piecewise linear function, that is,

$$\tilde{G} v|_T \in \mathscr{P}_1 \times \mathscr{P}_1, \quad \forall T \in \mathscr{T}_h$$

and

$$\tilde{G} v(Q) = G v(Q), \quad \forall Q \in \mathscr{M},$$

where $\mathscr{M}$ is the set of midpoints of all the edges of the mesh.

Lemma 3.4. *Let $T \subset \Omega_1$ and $v \in V_h$, then*

$$\|\tilde{G}v\|_{0,T} \leqq C \|\nabla v\|_{0,T^*}. \tag{3.13}$$

Proof. Since $\tilde{G}v$ is piecewise linear and interpolates Gv at $Q \in \mathscr{M}$ we have,

$$\|\tilde{G}v\|_{0,\infty,T} \leqq C \|Gv\|_{0,\infty,T}$$

therefore (3.13) follows from (3.2) by using inverse inequalities. □

Lemma 3.5. *Let $T \subset \Omega_1$ and $v \in H^3(\tilde{T})$, then*

$$\|\nabla v - \tilde{G}v^{\mathrm{I}})\|_{0,T} \leqq C h^2 \|v\|_{3,\tilde{T}}. \tag{3.14}$$

Proof. Let $\tilde{\nabla} v$ be the nonconforming piecewise linear interpolation of ∇v, that is,

$$\tilde{\nabla} v := \sum_{Q \in \mathscr{M}} \nabla v(Q)\, \varphi_Q,$$

where φ_Q is piecewise linear, $\varphi_Q(Q) = 1$ and $\varphi_Q(P) = 0$ for $P \in \mathscr{M}$, $P \neq Q$.
Then, given $T \in \mathscr{T}_h$, we have

$$\|\nabla v - \tilde{G}v^{\mathrm{I}}\|_{0,T} \leqq \|\nabla v - \tilde{\nabla} v\|_{0,T} + \|\tilde{\nabla} v - \tilde{G}u^{\mathrm{I}}\|_{0,T} \tag{3.15}$$

and from the standard interpolation theory we know that

$$\|\nabla v - \tilde{\nabla} v\|_{0,T} \leqq C h^2 \|v\|_{3,T}. \tag{3.16}$$

For the second term on the right hand side of (3.15) we use an inverse inequality and the fact that $\|\varphi_Q\|_{0,\infty,T} \leqq C$ to obtain,

$$\begin{aligned}
\|\tilde{\nabla} v - \tilde{G}v^{\mathrm{I}}|_{0,T} &= \left\| \sum_{i=1}^{3} [\tilde{\nabla} v(Q_i) - Gv^{\mathrm{I}}(Q_i)]\, \varphi_{Q_i} \right\|_{0,T} \\
&\leqq \sum_{i=1}^{3} \|\tilde{\nabla} v - Gv^{\mathrm{I}}\|_{0,\infty,T} \|\varphi_{Q_i}\|_{0,T} \\
&\leqq C \sum_{i=1}^{3} \frac{1}{h_T} \|\tilde{\nabla} v - Gv^{\mathrm{I}}\|_{0,T}\, h_T \|\varphi_{Q_i}\|_{0,\infty,T} \\
&\leqq C \|\tilde{\nabla} v - Gv^{\mathrm{I}}\|_{0,T},
\end{aligned}$$

where Q_i are the midpoints of the sides of T and h_T its diameter. Therefore, (3.14) follows from (3.6) and the triangular inequality. □

The recovery operator $\tilde{G}$ provides a simple error estimator which is based on the jumps of the normal derivative of the approximate solution. Indeed, if η_i is the outer normal to T in Q_i in Fig. 4, and we denote the jump of a function ϕ at Q_i by $[\![\phi]\!]_{Q_i} = \phi|_T(Q_i) - \phi|_{T_i}(Q_i)$, then for any $v \in V_h$ we have

$$\tilde{G}v(Q_i) = \nabla v|_T + \frac{|T_i|}{|T| + |T_i|} \left[\!\left[\frac{\partial v}{\partial \eta_i} \right]\!\right]_{Q_i} \eta_i .$$

Therefore, if we define

$$\tilde{\varepsilon} =: \tilde{G} u_h - \nabla u_h ,$$

we have for any interior triangle T

$$\tilde{\varepsilon}(Q_i) = \frac{|T_i|}{|T| + |T_i|} \left[\!\left[\frac{\partial u_h}{\partial \eta_i} \right]\!\right]_{Q_i} \eta_i$$

and since $\tilde{\varepsilon}$ is linear on T we obtain

$$\begin{aligned} \|\tilde{\varepsilon}\|_{0,T}^2 &= \frac{|T|}{3} (|\tilde{\varepsilon}(Q_1)|^2 + |\tilde{\varepsilon}(Q_2)|^2 + |\tilde{\varepsilon}(Q_3)|^2) \\ &= \frac{|T|}{3} \sum_{i=1}^{3} \left(\frac{|T_i|}{|T| + |T_i|} \right)^2 \left[\!\left[\frac{\partial u_h}{\partial \eta_i} \right]\!\right]_{Q_i}^2 . \end{aligned}$$

Now, using Lemmas 3.4 and 3.5 we may conclude that if the solution is regular enough then the estimator $\tilde{\varepsilon}$ is asymptotically exact. That is, if $u \in H^3(\Omega_1)$ and if (3.9) and (3.10) hold, then

$$\frac{\|\tilde{\varepsilon} - \nabla e\|_{0,\Omega_0}}{\|\nabla e\|_{0,\Omega_0}} = O(h^{\varepsilon}).$$

Remark 3.3 By using the techniques developed by Babuska and Miller [4] it is possible to prove that the estimator $\tilde{\varepsilon}$ is equivalent to ∇e in the L^2-norm for very general meshes and without any extra assumption of regularity of the solution. Namely, if the family of meshes $\mathscr{T}_h$ is regular in the sense of [10], $u \in H^1(\Omega)$ and (3.10) holds, then there exist two constants C_1 and C_2 such that

$$C_1 \|\tilde{\varepsilon}\|_{0,\Omega} \leqq \|\nabla e\|_{0,\Omega} \leqq C_2 \|\tilde{\varepsilon}\|_{0,\Omega} . \quad \square$$

4 Non asymptotic exactness of $\tilde{\varepsilon}$

Estimators like $\tilde{\varepsilon}$ based on the jumps of $\frac{\partial u_h}{\partial \eta}$ are widely used in practical computations (v.g. [14, 18, 20, 21, 22]). We have just proved that $\tilde{\varepsilon}$ is asymptotically exact in those regions where the meshes are quasi-parallel; therefore, it arises naturally the question of whether or not this extra regularity of the triangulations is essential.

In this section, we analyze the behavior of the estimator $\tilde{\varepsilon}$ in regions where the meshes are very regular but not quasi-parallel. Specifically, we consider

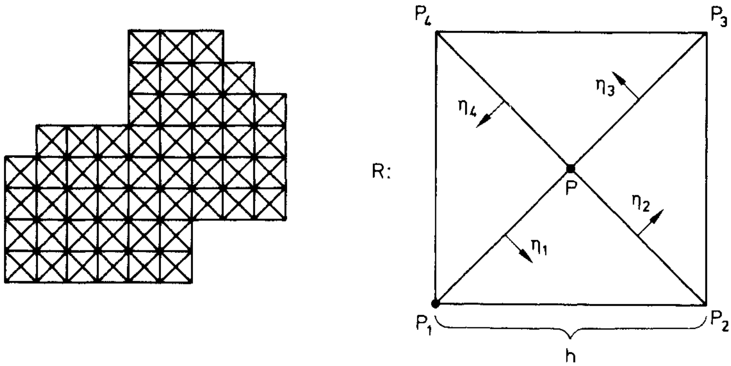

Fig. 6

a region where the meshes are of the criss-cross type and uniform (see Fig. 6) and the differential operator is the Laplacian.

Numerical evidence of the lack of superconvergence for this kind of meshes was presented by Levine [16]. First, we give here a proof of this fact. Let us denote by R the square in Fig. 6. In what follows, we use the notation in that figure.

Lemma 4.1. *Let $v \in H^3(R)$ and $\varphi \in V_h$ be the basis function corresponding to the node P, then*

$$|\int_R (\nabla u - \nabla v^{\mathrm{I}}) \cdot \nabla \varphi - \tfrac{1}{6} \int_R \Delta v| \leqq C h^2 |v|_{3,R}. \tag{4.1}$$

Proof. For $v \in H^3(R)$ let $Lv := \int_R (\nabla v - \nabla v^{\mathrm{I}}) \cdot \nabla \varphi - \frac{1}{6} \int_R \Delta v$. First we prove that $Lv = 0$ for any $v \in \mathscr{P}_2$. Since $\varphi = 0$ on ∂R we have

$$\int_R \nabla v \cdot \nabla \varphi = -\int_R \Delta v \varphi = -\Delta v \int_R \varphi = -\frac{h^2}{3} \Delta v. \tag{4.2}$$

On the other hand,

$$\int_R \nabla v^{\mathrm{I}} \nabla \varphi = \sum_{T \subset R} \int_T \nabla v^{\mathrm{I}} \nabla \varphi = \sum_{T \subset R} \int_{\partial T} \varphi \frac{\partial v^{\mathrm{I}}}{\partial n} = -\sum_{i=1}^{4} \int_{[P,P_i]} \varphi \left[\!\left[\frac{\partial v^{\mathrm{I}}}{\partial \eta_i} \right]\!\right]_{[P,P_i]},$$

where η_i is the unit normal to the side $[P, P_i]$ as in Fig. 6. Now,

$$\left[\!\left[\frac{\partial v^{\mathrm{I}}}{\partial \eta_i} \right]\!\right]_{[P,P_i]} = \begin{cases} \dfrac{v(P_2) - 2v(P) + v(P_4)}{h/\sqrt{2}}, & i = 1, 3 \\[2ex] \dfrac{v(P_3) - 2v(P) + v(P_1)}{h/\sqrt{2}}, & i = 2, 4 \end{cases}$$

and $\int_{[P,P_i]} \varphi = \frac{h}{2\sqrt{2}}$, $i=1, 2, 3, 4$. Hence

$$\int_R \nabla v^{\mathrm{I}} \cdot \nabla \varphi = \left[4v(P) - \sum_{i=1}^{4} v(P_i)\right] = -\frac{h^2}{2} \Delta v, \tag{4.3}$$

where we used that $v \in \mathscr{P}_2$. Therefore, subtracting (4.3) from (4.2) we have

$$\int_R (\nabla v - \nabla v^{\mathrm{I}}) \cdot \nabla \varphi = \frac{h^2}{6} \Delta v = \tfrac{1}{6} \int_R \Delta v,$$

and so $Lv=0$ for any $v \in \mathscr{P}_2$. On the other hand, by using that $\|\nabla \varphi\|_{0,R} = 2$, the usual interpolation theory and the Cauchy-Schwarz inequality we have

$$|Lv| \leqq \|\nabla \varphi\|_{0,R} \, \|\nabla u - \nabla v^{\mathrm{I}}\|_{0,R} + \tfrac{1}{6} \left|\int_R \Delta v\right| \leqq Ch|v|_{2,R} + \frac{h}{6} |v|_{2,R}.$$

Therefore, a standard application of Bramble-Hilbert Lemma (see for instance [11]) yields

$$|Lv| \leqq Ch^2 |v|_{3,R}$$

for any $v \in H^3(R)$ thus concluding the proof. □

Theorem 4.1. *Let $\Omega_0 \subset \Omega$ be a region where the mesh is like in Fig. 6. Let $u \in H^3(\Omega_0)$ be such that $\left|\int_{\Omega_0} \Delta u\right| \geqq \alpha$ for some constant $\alpha > 0$. Then,*

$$\|\nabla u_h - \nabla u^{\mathrm{I}}\|_{0,\Omega_0} \geqq \frac{h\alpha}{12\sqrt{|\Omega_0|}} - Ch^2 |u|_{3,\Omega_0}.$$

Proof. Let $R \subset \Omega_0$ and $\varphi \in V_h$ as in the previous lemma. Then

$$\begin{aligned} \int_R (\nabla u_h - \nabla u^{\mathrm{I}}) \cdot \nabla \varphi &= \int_\Omega (\nabla u_h - \nabla u^{\mathrm{I}}) \cdot \nabla \varphi = \int_\Omega (\nabla u - \nabla u^{\mathrm{I}}) \cdot \nabla \varphi \\ &= \int_R (\nabla u - \nabla u^{\mathrm{I}}) \cdot \nabla \varphi = \tfrac{1}{6} \int_R \Delta u + \delta_R, \end{aligned}$$

with $|\delta_R| \leqq Ch^2 |u|_{3,R}$ because of Lemma 4.1. Therefore,

$$\|\nabla u_h - \nabla u^{\mathrm{I}}\|_{0,R} \, \|\nabla \varphi\|_{0,R} \geqq \left|\int_R (\nabla u_h - \nabla u^{\mathrm{I}}) \cdot \nabla \varphi\right| = \tfrac{1}{6} \left|\int_R \Delta u + \delta_R\right|$$

and using that $\|\nabla \varphi\|_{0,R} = 2$ we have

$$\begin{aligned} \|\nabla u_h - \nabla u^{\mathrm{I}}\|_{0,\Omega_0}^2 &= \sum_{R \subset \Omega_0} \|\nabla u_h - \nabla u^{\mathrm{I}}\|_{0,R}^2 \\ &\geqq \frac{1}{144} \sum_{R \subset \Omega_0} \left(\int_R \Delta u + \delta_R\right)^2 \geqq \frac{1}{144 \operatorname{card}\{R \subset \Omega_0\}} \left(\int_{\Omega_0} \Delta u + \sum_{R \subset \Omega_0} \delta_R\right)^2. \end{aligned}$$

Now $\operatorname{card}\{R\subset\Omega_0\}=\frac{|\Omega_0|}{h^2}$; so,

$$\|\nabla u_h-\nabla u^{\mathrm{I}}\|_{0,\Omega}\geqq\frac{h}{12\sqrt{|\Omega_0|}}\Big(\Big|\int_{\Omega_0}\Delta u\Big|-\Big|\sum_{R\subset\Omega_0}\delta_R\Big|\Big)$$
$$\geqq\frac{h}{12\sqrt{|\Omega_0|}}\Big|\int_{\Omega_0}\Delta u\Big|-Ch^2|u|_{3,\Omega_0},$$

which concludes the theorem. □

This theorem shows that on any subregion where the mesh is of the criss-cross type and for problems with rather general solutions, there is no superconvergence in the sense of (1.1).

In order to apply Theorem 4.1 to the analysis of the estimator $\tilde{\varepsilon}$ we also need the following property.

Lemma 4.2. *Under the assumptions of Theorem* 4.1 *we have,*

$$\|\tilde{G}u_h-\tilde{G}u^{\mathrm{I}}\|_{0,\Omega_0}\geqq\frac{h\alpha}{12\sqrt{3|\Omega_0|}}-Ch^2|u|_{3,\Omega_0}.$$

Proof. Let R be as in Fig. 6, then it is enough to prove that for any $v\in V_h$,

$$\|\tilde{G}v\|_{0,R}\geqq\frac{1}{\sqrt{3}}\|\nabla v\|_{0,R}.$$

Some simple calculations show that for any $v\in V_h$,

$$\int_R|\nabla v|^2=\sum_{i=1}^{4}[v(P_i)-v(P)]^2,$$

$$\int_R|\tilde{G}v|^2\geqq\frac{h^2}{6}\sum_{i=1}^{4}|\tilde{G}v(Q_i)|^2,$$

where Q_i is the midpoint of the side $[P, P_i]$, and

$$|\tilde{G}v(Q_i)|^2\geqq\frac{2[v(P_i)-v(P)]^2}{h^2},\qquad i=1,2,3,4.$$

So the lemma holds. □

We want to exhibit some particular problems for which $\tilde{\varepsilon}$ is not asymptotically exact. Note that this is not an immediate consequence of the previous lemma since for nonquasiparallel meshes $\tilde{G}(u^{\mathrm{I}})$ is not necessarily a superconvergent approximation of ∇u. In the next lemma, we estimate the difference $\nabla u-\tilde{G}(u^{\mathrm{I}})$.

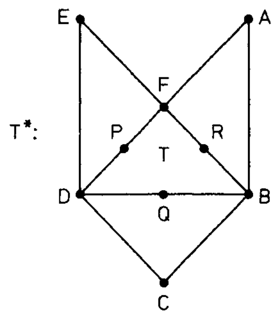

Fig. 7

Lemma 4.3. *Let T be an interior element and T^* the union of T with its neighbor elements as in Fig. 7. Let* $Wv := \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2}$. *If* $v \in H^3(T^*)$, *then*

$$\left| \|\nabla v - \tilde{G}(v^{\mathrm{I}})\|_{0,T} - \frac{h}{8\sqrt{3}} \Big| \int_{[D,B]} Wv \Big| \right| \leqq C h^2 |v|_{3,T^*}.$$

Proof. Let $C_v := \int_{[D,B]} Wv$ and $E_v \colon T^* \to \mathbb{R}^2$ be the function defined by

$$E_v(S) := C_v \frac{S-Q}{2\sqrt{2}h}, \qquad S \in T^*.$$

Let us define an operator $L \colon H^3(T^*) \to [L^2(T)]^2$ by $Lv := \nabla v - \tilde{G} v^{\mathrm{I}} - E_v$. First, we prove that $Lv = 0$ for any $v \in \mathscr{P}_2$. Indeed, in this case $Lv \in (\mathscr{P}_1 \times \mathscr{P}_1)$ and so it is sufficient to prove that Lv vanishes at P, Q and R.

After some computations we obtain for $v \in \mathscr{P}_2$:

$$\nabla v(P) = \begin{pmatrix} 3\dfrac{v(F)-v(D)}{2h} - \dfrac{v(E)-v(C)}{2h} \\ \dfrac{v(F)-v(D)}{2h} + \dfrac{v(E)-v(C)}{2h} \end{pmatrix}, \tag{4.4}$$

$$\nabla v(Q) = \begin{pmatrix} \dfrac{v(B)-v(D)}{h} \\ \dfrac{v(F)-v(C)}{h} \end{pmatrix},$$

$$\nabla v(R) = \begin{pmatrix} -3\dfrac{v(F)-v(B)}{2h} + \dfrac{v(A)-v(C)}{2h} \\ \dfrac{v(F)-v(B)}{2h} + \dfrac{v(A)-v(C)}{2h} \end{pmatrix},$$

and for any $w \in V_h$:

$$\tilde{G}w(P)=\begin{pmatrix} \dfrac{w(F)-w(D)}{h}+\dfrac{w(B)-w(E)}{2h} \\ \dfrac{w(F)-w(D)}{h}-\dfrac{w(B)-w(E)}{2h} \end{pmatrix}, \tag{4.5}$$

$$\tilde{G}w(Q)=\begin{pmatrix} \dfrac{w(B)-w(D)}{h} \\ \dfrac{w(F)-w(C)}{h} \end{pmatrix},$$

$$\tilde{G}w(R)=\begin{pmatrix} -\dfrac{w(F)-w(B)}{h}+\dfrac{w(A)-w(D)}{2h} \\ \dfrac{w(F)-w(B)}{h}+\dfrac{w(A)-w(D)}{2h} \end{pmatrix}.$$

Therefore, subtracting (4.5) with $w=v^{\mathrm{I}}$ from (4.4) we obtain:

$$(\nabla v-\tilde{G}v^{\mathrm{I}})(P)=\begin{pmatrix} \dfrac{v(F)-v(D)}{2h}+\dfrac{v(G)-v(B)}{2h} \\ -\dfrac{v(F)-v(D)}{2h}-\dfrac{v(C)-v(B)}{2h} \end{pmatrix},$$

$$(\nabla v-\tilde{G}v^{\mathrm{I}})(Q)=\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$(\nabla u-\tilde{G}v^{\mathrm{I}})(R)=\begin{pmatrix} -\dfrac{v(F)-v(B)}{2h}-\dfrac{v(C)-v(D)}{2h} \\ -\dfrac{v(F)-v(B)}{2h}-\dfrac{v(C)-v(D)}{2h} \end{pmatrix}.$$

Now, since $v\in\mathscr{P}_2$, Wv is costant and we have

$$Wv=\frac{v(B)-2v(Q)+v(D)}{(h/2)^2}-\frac{v(F)-2v(Q)+v(C)}{(h/2)^2};$$

hence

$$(\nabla v-\tilde{G}v^{\mathrm{I}})(P)=\frac{h}{8}\,Wv\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and

$$(\nabla v-\tilde{G}v^{\mathrm{I}})(R)=\frac{h}{8}\,Wv\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Moreover, $C_v=hWv$ and so, computing the values of E_v, we see that $(\nabla v-\tilde{G}v^{\mathrm{I}})$ and E_v coincide at P, Q and R. Therefore, $Lv=0$ for $v\in\mathscr{P}_2$.

On the other hand, since $\|\tilde{G}v^{\mathrm{I}}\|_{0,T}\leqq C\|\nabla v^{\mathrm{I}}\|_{0,T^*}$, it is easily seen that

$$\|Lv\|_{0,T}\leqq\|\nabla v\|_{0,T}+\|\tilde{G}v\|_{0,T}+\frac{h^{3/2}}{8\sqrt{3}}\|Wv\|_{0,[D,B]}\leqq C(|v|_{1,T^*}+h|v|_{2,T^*}+h^2|v|_{3,T^*}),$$

where we used a trace theorem to bound the last term. Now, a standard application of the Bramble-Hilbert Lemma yields,

$$\|Lv\|_{0,T} \leqq Ch^2 |v|_{3,T^*}. \tag{4.6}$$

and since

$$\|E_v\|_{0,T} = \frac{h}{8\sqrt{3}} |C_v|,$$

the lemma follows from (4.6). □

Lemma 4.3 says, in particular, that $\tilde{G}v^I$ is a superconvergent approximation to ∇v for smooth functions such that $Wv=0$. Collecting all the lemmas we have the following theorem which says that, in general, $\tilde{\varepsilon}$ is not asymptotically exact in the sense that the relative error does not tend to zero.

Theorem 4.2. *Let $\Omega_0 \Subset \Omega_1 \subset \Omega$ be subregions where the meshes are of the criss-cross type like in Fig. 6. Let $u \in H^3(\Omega_1) \cap H^2(\Omega)$ be such that $Wu=0$ and $|\int_{\Omega_0} \Delta u| \geqq \alpha$ for some $\alpha > 0$. Then, for h small enough,*

$$\frac{\|\tilde{\varepsilon} - \nabla e\|_{0,\Omega_0}}{\|\nabla e\|_{0,\Omega_0}} \geqq \beta$$

for some positive constant β.

Proof. For h small enough $\cup\{T^*: T \subset \Omega_0\} \subset \Omega_1$; then we may apply Lemmas 4.2 and 4.3:

$$\begin{aligned}
\|\tilde{\varepsilon} - \nabla e\|_{0,\Omega_0} &= \|\tilde{G}u_h - \nabla u\|_{0,\Omega_0} \\
&\geqq \|\tilde{G}u_h - \tilde{G}u^I\|_{0,\Omega_0} - \|\tilde{G}u^I - \nabla u\|_{0,\Omega_0} \\
&\geqq \frac{h\alpha}{12\sqrt{3}|\Omega_0|} - Ch^2 |u|_{3,\Omega_1}.
\end{aligned}$$

Now, since $u \in H^2(\Omega)$, then $\|e\|_{1,\Omega_0} \leqq Ch|u|_{2,\Omega}$, and hence the theorem is proved. □

A trivial example of a function satisfying the hypothesis of the Theorem 4.2 is $u(x, y) = x^2 + y^2$. Note that in this case $\tilde{G}(u^I) = \nabla u$.

So, we have shown that the estimator $\tilde{\varepsilon}$ is not asymptotically exact in general. However, there are still two natural questions.

i) Is $\tilde{\varepsilon}$ asymptotically exact whenever there is superconvergence in the sense of (1.1)?

ii) It is asymptotically exact in the weaker sense of [8]? I.e.: does the so called effectivity index

$$\theta_{\Omega_0} := \frac{\|\tilde{\varepsilon}\|_{0,\Omega_0}}{\|\nabla e\|_{0,\Omega_0}} \xrightarrow[h \to 0]{} 1?$$

We are going to show that the answer to these two questions is negative. In order to do that, let us consider a problem such that the solution $u \in \mathscr{P}_2$

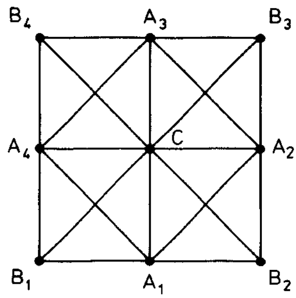

Fig. 8

and meshes which are criss-cross in the whole Ω. In this simple case we may calculate explicitly u_h.

For each square R as in Fig. 6 the equation corresponding to the node P gives:

$$u_h(P)=\frac{h^2}{12}f+\frac{1}{4}\sum_{i=1}^{4} u_h(P_i). \tag{4.7}$$

Considering a patch like that in Fig. 8 and eliminating the unknowns corresponding to the middle point of each square we obtain:

$$\frac{1}{2}\,\frac{4u_h(C)-\sum_{i=1}^{4} u_h(A_i)}{h^2}+\frac{1}{2}\,\frac{4u_h(C)-\sum_{i=1}^{4} u_h(B_i)}{2h^2}=f.$$

Therefore, the difference scheme associated with the linear elements in a criss-cross mesh is an average of two five-points finite difference schemes, which, as it is well known, are exact for $u\in\mathscr{P}_2$. Hence, for each square R (with the notation of Fig. 6) we have

$$u_h(P_i)=u(P_i), \qquad i=1,2,3,4,$$

and, by using (4.7), we obtain

$$u_h(P)=\tfrac{1}{12}\left(8u(P)+\sum_{i=1}^{4} u(P_i)\right)=u(P)+\tfrac{1}{24}h^2\Delta u.$$

Using these values of u_h we obtain (with the notation of Fig. 7)

$$\nabla u_h|_T=\begin{pmatrix} \dfrac{u(B)-u(D)}{h} \\ \dfrac{2u(F)-u(B)-u(D)}{h}+\dfrac{h}{12}\Delta u \end{pmatrix}, \tag{4.8}$$

and using the values of ∇u of (4.4) we obtain after some simple calculations:

$$\nabla e(P)=\frac{h}{8}\begin{pmatrix}-\Delta u-Wu+2\dfrac{\partial^2 u}{\partial x\,\partial y}\\ \frac{1}{3}\Delta u+Wu-2\dfrac{\partial^2 u}{\partial x\,\partial y}\end{pmatrix},$$

$$\nabla e(Q)=\frac{h}{4}\begin{pmatrix}0\\ \frac{1}{3}\Delta u+Wu\end{pmatrix},$$

$$\nabla e(R)=\frac{h}{8}\begin{pmatrix}\Delta u+Wu+2\dfrac{\partial^2 u}{\partial x\,\partial y}\\ \frac{1}{3}\Delta u+Wu+2\dfrac{\partial^2 u}{\partial x\,\partial y}\end{pmatrix}.$$

On the other hand, using (4.5) for $w=u_h$, it is easily seen that:

$$\tilde{G}u_h(P)=\begin{pmatrix}\dfrac{u(F)-u(D)}{h}+\dfrac{u(B)-u(E)}{2h}+\dfrac{h}{24}\Delta u\\ \dfrac{u(F)-u(D)}{h}-\dfrac{u(B)-u(E)}{2h}+\dfrac{h}{24}\Delta u\end{pmatrix}, \tag{4.9}$$

$$\tilde{G}u_h(Q)=\begin{pmatrix}\dfrac{u(B)-u(D)}{h}\\ \dfrac{u(F)-u(C)}{h}\end{pmatrix},$$

$$\tilde{G}u_h(R)=\begin{pmatrix}-\dfrac{u(F)-u(B)}{h}+\dfrac{u(A)-u(D)}{2h}-\dfrac{h}{24}\Delta u\\ \dfrac{u(F)-u(B)}{h}+\dfrac{u(A)-u(D)}{2h}+\dfrac{h}{24}\Delta u\end{pmatrix},$$

and subtracting (4.8) from (4.9) we obtain:

$$\tilde{\varepsilon}(P)=\frac{h}{4}\begin{pmatrix}-\frac{1}{3}\Delta u+\dfrac{\partial^2 u}{\partial x\,\partial y}\\ \frac{1}{3}\Delta u-\dfrac{\partial^2 u}{\partial x\,\partial y}\end{pmatrix},$$

$$\tilde{\varepsilon}(Q)=\frac{h}{4}\begin{pmatrix}0\\ -\frac{1}{3}\Delta u+Wu\end{pmatrix},$$

$$\tilde{\varepsilon}(R)=\frac{h}{4}\begin{pmatrix}\frac{1}{3}\Delta u+\dfrac{\partial^2 u}{\partial x\,\partial y}\\ \frac{1}{3}\Delta u+\dfrac{\partial^2 u}{\partial x\,\partial y}\end{pmatrix}.$$

Now, let us consider two examples. First, for $u(x, y) = x^2 - y^2$, we have

$$\|\tilde{\varepsilon}\|_{0,T} = \frac{h^2}{\sqrt{12}}, \quad \text{and} \quad \|\nabla e\|_{0,T} = \frac{h^2}{\sqrt{6}}.$$

Therefore, for any subregion Ω_0 as in Fig. 6, $\theta_{\Omega_0} = \frac{\sqrt{2}}{2} \neq 1$, independently of the mesh size h. Note that in this case there is superconvergence (moreover $u_h \equiv u^I$); however the estimator $\tilde{\varepsilon}$ is not asymptotically exact.

Secondly, for $u(x, y) = x^2 + y^2$, we have

$$\|\tilde{\varepsilon}\|_{0,T} = \frac{h^2\sqrt{5}}{6\sqrt{3}}, \quad \text{and} \quad \|\nabla e\|_{0,T} = \frac{h^2}{\sqrt{18}},$$

so, $\theta_{\Omega_0} = \frac{\sqrt{30}}{6}$. Therefore, in both examples the estimator $\tilde{\varepsilon}$ is not asymptotically exact even in the weaker sense of [8].

Acknowledgements. We want to thank R. Nochetto for many helpful discussions. Part of this work was performed while the first author was visiting the Center for Applied Mathematics at Purdue University. He wishes to express his gratitude to this institution for its hospitality.

References

1. Ainsworth, M., Craig, A.W.: A posteriori error estimators in the finite element method. Num. Anal. Rep. NA88/03, Durham, England, 1988
2. Andreev, A.B. Lazarov, R.D.: Superconvergence of the gradient for quadratic triangular finite elements. Numer. Methods for PDEs, **4**, 15–32 (1988)
3. Babuška, I., Miller, A.: A-posteriori error estimates and adaptive techniques for the finite element method. Tech. Note BN-968, IPST, University of Maryland, 1981
4. Babuška, I., Miller, A.: A feedback finite element method with a posteriori error estimation: Part I. The finite element method and some basic properties of the a posteriori error estimator. Comp. Methods Appl. Mech. Engrg. **61**, 1–40 (1987)
5. Babuška, I., Rheinboldt, W.C.: A posteriori error estimators in the finite element method. Internat. J. Numer. Meth. Eng. **12**, 1597–1615 (1978)
6. Babuška, I., Rheinboldt, W.C.: Error estimates for adaptive finite element computations. SIAM J. Numer. Anal. **15**, 736–754 (1978)
7. Babuška, I., Rheinboldt, W.C.: Analysis of optimal finite element meshes in R^1. Math. Comp. **33**, 435–463 (1979)
8. Babuška, I., Rheinboldt, W.C.: A posteriori error analysis of finite element solutions for one-dimensional problems. SIAM J. Numer. Anal. **18**, 565–589 (1981)
9. Bank, R.E., Weiser, A.: Some a posteriori error estimators for elliptic partial differential equations. Math. Comp. **44**, 283–301 (1985)
10. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. Amsterdam: North Holland 1978
11. Durán, R.: On polynomial approximation in Sobolev spaces. SIAM J. Numer. Anal. **5**, 985–988 (1983)
12. Durán, R., Muschietti, M.A., Rodríguez, R.: Asymptotically exact error estimators for rectangular finite elements. (To appear)
13. Grisvard, P.: Elliptic Problems in Nonsmooth Domains. Boston: Pitman 1985
14. Hinton, E., Ozakca, M., Rao, N.V.R.: Adaptative analysis of thin shells using facet elements. CR/950/90, University College of Swansea (1990)

15. Křížek, M., Neittaanmäki, P.: Superconvergence phenomenom in the finite element method arising from averaging gradients. Numer. Math. **45**, 105–116 (1984)
16. Levine, N.: Stress sampling points for linear triangles in the finite element method. IMA J. Numer. Anal. **5**, 407–427 (1985)
17. Lin, Q., Lü, T.: Asymptotic expansions for finite element approximation of elliptic problems on polygonal domains. In: Computing Methods in Applied Sciences and Engineering, Proc. Sixth Int. Conf. Versailles, 1983. L.N. in Comp. Sci., North-Holland, INRIA, pp. 317–321 (1984)
18. Mesztenyi, C., Szymczak, W.: FEARS user's manual for UNIVAC 1100. Tech. Note BN-991, IPST, University of Maryland (1982)
19. Oganesjan, L.A., Ruhovec, L.A.: Investigation of the convergence rate of variational-difference schemes for elliptic second order equations in a two dimensional domain with a smooth boundary. Z. Vychisl. Mat. i Mat. Fiz. **9**, 1102–1120 (1969) [Translation in USSR Comput. Math. Math. Phys. **9**, 153–188 (1969)]
20. Rivara, M.C.: EXPDES user's manual. Catholic Univ., Leuven, Belgium (1984)
21. Rivara, M.C.: Adaptive multigrid software for the finite element method. Ph.D. dissertation, Catholic. Univ., Leuven, Belgium (1984)
22. Verfürth, R.: A posteriori error estimators for the Stokes equations. Numer. Math. **55**, 309–325 (1989)
23. Wheeler, M.F., Whiteman, J.R.: Superconvergent recovery of gradients on subdomains from piecewise linear finite element approximations. Numer. Methods for PDEs, **3**, 357–374 (1987)
24. Whiteman, J.R., Goodsell, G.: Some gradient superconvergence results in the finite element method. In: Numerical Analysis and Parallel Processing. Lecture Notes in Math. # 1397. Berlin Heidelberg New York: Springer 182–259
25. Zlámal, M.: Some superconvergence results in the finite element method. In: Mathematical Aspects of the Finite Element Method, Lecture Notes in Math. # 606, pp. 353–362. Berlin Heidelberg New York: Springer 1977