For $s > s_0$ sufficiently large one may take $\delta = s^{-1/(l+2)}$, in which case

$$\left|\frac{1}{2\pi}\int_0^{2\pi}\cos f(x)\,dx\right| < Cs^{-1/(l+2)} \to 0, \qquad s \to \infty.$$

The second integral in (8) may be estimated in a similar way. As a consequence we obtain

$$\left|f_s\left(\frac{2\pi j_0}{m_0}, \frac{2\pi j_1}{m_1}, \cdots, \frac{2\pi j_k}{m_N}\right)\right| < Cs^{-1(l+2)} \qquad \sum_{k=0}^{N} j_k^2 \neq 0,$$

where $l$ is the maximal multiplicity of a zero of $f''(x)$ on $[0, 2\pi]$.

The other assertions of Theorem 2 are evident.

**Corollary.** *Let $\varphi(z)$ be an analytic function in an $\varepsilon$-neighborhood of the interval $[0, 2\pi + \max_{1 \leqq i \leqq N} T_i]$, $0 < T_i < \infty$, $i = 1, \cdots, N$, in the complex plane. The necessary and sufficient condition for the limit relation*

$$\mathbf{P}\{([s\varphi(\eta)]_{m_0}, [s\varphi(\eta + T_1)]_{m_1}, \cdots, [s\varphi(\eta + T_N)]_{m_N}) = (i_0, i_1, \cdots, i_N)\} \xrightarrow[s\to\infty]{} \frac{1}{\prod_{k=0}^{N} m_k}$$

*to hold for any point $(i_0, i_1, \cdots, i_N) \in \Delta_{N+1}$ with integer $i_k$ is*

$$\frac{j_0}{m_0}\varphi''(x) + \sum_{k=1}^{N}\frac{j_k}{m_k}\varphi''(x + T_k) \not\equiv 0 \quad \textit{for} \quad x \in [0, 2\pi].$$

**Example.** Let $\varphi(x) = \sin x$, $N = 1$, $T \neq k\pi$, $k = 0, \pm 1, \pm 2, \cdots$. It is not difficult to verify that the random variable $([s \cdot \sin \eta]_{m_0}, [s \cdot \sin(\eta + T)]_{m_1})$ is asymptotically uniformly distributed over the lattice points of $\Delta_2$ as $s \to \infty$.

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# ON MOMENT INEQUALITIES FOR STOCHASTIC INTEGRALS

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(*Translated by E. Lukacs*)

**1.** In this note we consider moment inequalities for stochastic integrals with respect to the Wiener process (concerning the definitions and notations, see [1]):

$$A_p\mathbf{E}\left(\int_0^T |f(s,\omega)|^2\,ds\right)^{p/2} \leqq \mathbf{E}\left|\int_0^T f(s,\omega)\,dw(s)\right|^p \leqq B_p\mathbf{E}\left(\int_0^T |f(s,\omega)|^2\,ds\right)^{p/2},$$

where $A_p$ and $B_p$ are constants depending only on $p$. The existence of these constants for degree $p > 1$ follows from results of a paper by Millar [2] dealing with the general theory of stochastic integrals in the sense of $L_p$-martingales. The method, used in [2] for the derivation of the inequalities considered, consists in the application of martingale transforms (cf. [3]) to stochastic integrals of "step" functions.

In our paper we generalize Millar's results to the case of multidimensional stochastic integrals and partly to the case of degree $p > 0$. The method for the derivation of the moment inequalities used here consists in the application of the formulae of Itô to a specially selected function and subsequent simple transformations. The result so obtained was used in the author's paper [4] where, in particular, it was shown that the mathematical expectation of the stochastic integral equals zero under the condition

$$\mathbf{E}\left(\int_0^T |f(s,\omega)|^2\,ds\right)^{1/2} < \infty, \tag{1}$$

and not only under the condition $\mathbf{E}\int_0^T |f(s,\omega)|^2\,ds < \infty$ which was known earlier.

**2.** Suppose that a standard $m$-dimensional Wiener process $w(s) = (w_i(s))$, $i = 1, \cdots, m$, is given on the probability space $(\Omega, \mathscr{F}, \mathbf{P})$. Let the random matrix function $f(s,\omega) = (f_{ij}(s,\omega))$, $i, j = 1, \cdots, n$, be such that the stochastic integral $\int_0^T f(s,\omega)\,dw(s)$ is defined taking on values in the $n$-dimensional euclidean space. We denote

$$|f(s,\omega)|^2 = \sum_{i=1}^m \sum_{j=1}^n f_{ij}^2(s,\omega).$$

**Theorem.** *If* $\mathbf{E}(\int_0^T |f(s,\omega)|^2\,ds)^{p/2} < \infty$, *then the inequalities*

$$A_p\mathbf{E}\left(\int_0^T |f(s,\omega)|^2\,ds\right)^{p/2} \leqq \mathbf{E}\left|\int_0^T f(s,\omega)\,dw(s)\right|^p, \qquad p > 1, \tag{2}$$

$$\mathbf{E}\left|\int_0^T f(s,\omega)\,dw(s)\right|^p \leqq B_p\mathbf{E}\left(\int_0^T |f(s,\omega)|^2\,ds\right)^{p/2}, \qquad p > 0, \tag{3}$$

*are valid, where $A_p$ and $B_p$ are constants which depend only on $p$.*

**3.** Proof of the theorem. We write

$$\eta(t) = \int_0^t f(s,\omega)\,dw(s), \qquad \xi(t) = \int_0^t |f(s,\omega)|^2\,ds.$$

Then, according to a formula of Itô,

$$d[\delta + c\xi(t) + |\eta(t)|^2] = (1 + c)|f(t,\omega)|^2\,dt + 2(\eta(t), f(t,\omega)\,dw(t)),$$

where $\delta$ and $c$ are some positive numbers and where, for any $p$,

$$\begin{aligned} d[\delta + c\xi(t) + |\eta(t)|^2]^{p/2} = {} & \{\tfrac{1}{2}p(1 + c)[\delta + c\xi(t) + |\eta(t)|^2]^{p/2-1}|f(t,\omega)|^2 \\ & + \tfrac{1}{2}p(p-2)[\delta + c\xi(t) + |\eta(t)|^2]^{p/2-2}|f^*(t,\omega)\eta(t)|^2\}\,dt \\ & + p[\delta + c\xi(t) + |\eta(t)|^2]^{p/2-1}(\eta(t), f(t,\omega)\,dw(t)). \end{aligned} \tag{4}$$

Here $f^*(t,\omega)$ is the transpose of the matrix $f(t,\omega)$. We shall assume that $|f(t,\omega)| \leqq K < \infty$ (this requirement is necessary only in the intermediate stage of the proof). Then, after integrating equation (4) with respect to the measure $dt \times d\mathbf{P}$, the last term becomes zero by virtue of well-known properties of stochastic integrals. We obtain

$$\begin{aligned} \mathbf{E}[\delta + c\xi(t) + |\eta(t)|^2]^{p/2} = {} & \delta^{p/2} + \tfrac{1}{2}p(1 + c)\mathbf{E}\int_0^t [\delta + c\xi(s) + |\eta(s)|^2]^{p/2-1}|f(s,\omega)|^2\,ds \\ & + \tfrac{1}{2}p(p-2)\mathbf{E}\int_0^t [\delta + c\xi(s) + |\eta(s)|^2]^{p/2-2}|f^*(s,\omega)\eta(s)|^2\,ds. \end{aligned} \tag{5}$$

From this relation all the needed inequalities are deduced. Let, for example, $0 < p \leqq 2$. Then the last term in (5) in non-positive and we have, as $\delta$ tends to zero, the obvious inequalities

$$2^{p/2-1}[c^{p/2}\mathbf{E}\xi^{p/2}(t) + \mathbf{E}|\eta(t)|^p] \leqq \mathbf{E}[c\xi(t) + |\eta(t)|^2]^{p/2}$$
$$\leqq \tfrac{1}{2}p(1 + c)\mathbf{E}\int_0^t [c\xi(s) + |\eta(s)|^2]^{p/2-1}|f(s, \omega)|^2\, ds$$
$$\leqq \tfrac{1}{2}p(1 + c)c^{p/2-1}\mathbf{E}\int_0^t \xi(s)^{p/2-1}\, d\xi(s) = (1 + c)c^{p/2-1}\mathbf{E}\xi(t)^{p/2},$$

and therefore

$$\mathbf{E}|\eta(t)|^p \leqq (c/2)^{p/2}(2/c + 2 - 2^{p/2})\mathbf{E}\xi(t)^{p/2}, \qquad 0 < p \leqq 2.$$

If $p \geqq 2$, then the last term in (5) is positive and all inequalities, written above, are reversed. Therefore,

$$\mathbf{E}|\eta(t)|^p \geqq (c/2)^{p/2}(2/c + 2 - 2^{p/2})\mathbf{E}\xi(t)^{p/2}, \qquad p \geqq 2,$$

where the constant $c$ is chosen according to the condition $0 < c < 1/(2^{p/2-1} - 1)$.

In order to obtain a lower estimate for $1 < p \leqq 2$, we use the inequality $|f^*(s, \omega)\eta(s)|^2 \leqq |f(s, \omega)|^2\eta(s)|^2$.

Then we obtain from (5), after some transformations, for $c = 0$

$$\mathbf{E}[\delta + |\eta(t)|^2]^{p/2} \geqq \delta^{p/2} + \tfrac{1}{2}p(p - 1)\mathbf{E}\int_0^t [\delta + |\eta(s)|^2]^{p/2-1}|f(s, \omega)|^2\, ds. \tag{6}$$

On the other hand, it follows from (5) that

$$\begin{aligned} &2^{p/2-1}\{c^{p/2}\mathbf{E}\xi(t)^{p/2} + \mathbf{E}[\delta + |\eta(t)|^2]^{p/2}\} \\ &\leqq \delta^{p/2} + \tfrac{1}{2}p(1 + c)\mathbf{E}\int_0^t [\delta + |\eta(s)|^2]^{p/2-1}|f(s, \omega)|^2\, ds. \end{aligned} \tag{7}$$

Combining (6) and (7) we obtain, as $\delta \to 0$,

$$\mathbf{E}|\eta(t)|^p \geqq (2c)^{p/2}\left(\frac{2 + 2c}{p - 1} - 2^{p/2}\right)^{-1}\mathbf{E}\xi(t)^{p/2}, \qquad 1 < p \leqq 2.$$

If $p \geqq 2$, the inequalities are reversed and therefore (for corresponding values of $c$)

$$\mathbf{E}|\eta(t)|^p \leqq (2c)^{p/2}\left(\frac{2 + 2c}{p - 1} - 2^{p/2}\right)^{-1}\mathbf{E}\xi(t)^{p/2}, \qquad p \geqq 2.$$

Thus inequalities (2) and (3) are proven under the assumption that the function $f(s, \omega)$ is bounded. The general case is proven by means of the standard truncation method and a passage to the limit, as it was done by M. Zakai [5] for the derivation of moments of another type.

**4.** REMARK 1. Since the estimates we obtained depend on an arbitrary constant $c$, one can obtain extreme values of the constants $A_p$ and $B_p$ corresponding to our appropriate method by varying $c$. For example, if $p = 1$ we obtain $\min B_1 = 2(2 - \sqrt{2})^{1/2} \sim -1.54$. It is possible to improve somewhat the value of these constants if for each concrete $p$, a sharper inequality is applied to the members of the relation (5).

REMARK 2. In our paper we succeeded in obtaining estimates from below only for powers $p > 1$. The following simple example shows that the absolute constant $A_1$ can only be zero. Let $f(t, \omega) = \chi\{\tau_b > t\}$, where $\chi\{D\}$ is the characteristic function of the set $D$ and where $\tau_b = \inf\{t \geqq 0 : w(t) \geqq 1 - b\sqrt{t}\}$. It follows from the result of [4] that, for $b > 0$, $\mathbf{E}_b^{1/2} = 1/b < \infty$. Assume that there exists a constant $A_1 > 0$ such that the inequality (2) of the theorem is satisfied for $p = 1$. For $T = \infty$ this inequality is, in our example, equivalent to the relations

$$A_1\mathbf{E}\tau_b^{1/2} = A_1/b \leqq \mathbf{E}|w(\tau_b)| = \mathbf{E}|1 - b\tau_b^{1/2}| \leqq 2.$$

Since $b > 0$ is arbitrary, it is clear that we obtain a contradiction.

REMARK 3. Condition (1), which is sufficient to assure that the mathematical expectation of the stochastic integral equals zero, can generally not be weakened. Indeed, let $b = 0$ in the example of Remark 2. Then $\mathbf{E}w(\tau_0) = 1$ but also $\mathbf{E}\tau_0^{1/2} = \infty$ (although $\mathbf{E}\tau_0^{1/2-\varepsilon} < \infty$ for any $\varepsilon > 0$).

In conclusion we use the occasion to express our appreciation to A. N. Shiryaev for his attention to this work.

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# ON THE FIRST PASSAGE TIME OF DIFFUSION PROCESSES THROUGH TIME-DEPENDENT BOUNDARIES

*V. A. LEBEDEV*

(*Translated by B. Seckler*)

**1.** Consider a homogeneous Markov diffusion stochastic process $\xi(t) = (\xi_1(t), \cdots, \xi_n(t))$ characterized by the system of stochastic differential equations

$$d\xi_i(t) = a_i(\xi(t))\,dt + \sum_{j=1}^{m} \sigma_{ij}(\xi(t))\,dw_j(t),$$

$i = 1, \cdots, n$, where $w_j(t), j = 1, \cdots, m$, are independent standard Wiener processes. Let

$$x = (x_1, \cdots, x_n), \qquad b_{ik}(x) = \sum_{j=1}^{m} \sigma_{ij}(x)\sigma_{kj}(x),$$

$$Lu(t, x) = \frac{\partial u(t, x)}{\partial t} + \sum_{i=1}^{n} a_i(x)\frac{\partial u(t, x)}{\partial x_i} + \frac{1}{2}\sum_{i,k=1}^{n} b_{ik}(x)\frac{\partial^2 u(t, x)}{\partial x_i \partial x_k},$$

$$L_x W(x) = \sum_{i=1}^{n} a_i(x)\frac{\partial W(x)}{\partial x_i} + \frac{1}{2}\sum_{i,k=1}^{n} b_{ik}(x)\frac{\partial^2 W(x)}{\partial x_i \partial x_k}.$$

Suppose that, for each bounded domain in $\{x\}$-space, there exists a constant $K$ for which

$$\sum_{i=1}^{n} |a_i(x) - a_i(y)| + \sum_{i=1}^{n}\sum_{j=1}^{m} |\sigma_{ij}(x) - \sigma_{ij}(y)| \leqq K|x - y|,$$

when the values $x$ and $y$ belong to this domain.

Let $E$ denote a domain of points $(t, x)$ in the half-space $\{t > T_0\}$ such that the initial point $(t_0, x_0)$ of the stochastic process $\xi(t)$ ($\xi(t_0) = x_0$) is an interior point of $E$ and let $\tau$ be the first passage time of the stochastic process $\xi(t)$ through the boundary of $E$.

Throughout, it will be assumed that $\xi(t)$ is a regular process in the domain $E$, i.e., if $\tau^{(r)}$ is the first passage time of $\xi(t)$ through the boundary $|x| = r$ with $|x_0| < r$, then $\lim_{r\uparrow\infty} \min(\tau, \tau^{(r)}) = \tau$ with probability 1. A process regular in a given domain is clearly regular in any of its subdomains. In particular, a stochastic process $\xi(t)$ is always regular in a domain bounded with respect to $x$.

**2.** Let $u(t, x)$ be a twice continuously differentiable function with respect to $x_i$ and once with respect to $t$. Then by Itô's formula,

$$du(t, \xi(t)) = Lu(t, \xi(t))\,dt + \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\partial u(t, \xi(t))}{\partial x_i}\sigma_{ij}(\xi(t))\,dw_j(t). \tag{1}$$