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On some properties of Markov chain Monte Carlo simulation methods based on the particle filter

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ABSTRACT

Andrieu et al. (2010) prove that Markov chain Monte Carlo samplers still converge to the correct posterior distribution of the model parameters when the likelihood estimated by the particle filter (with a finite number of particles) is used instead of the likelihood. A critical issue for performance is the choice of the number of particles. We add the following contributions. First, we provide analytically derived, practical guidelines on the optimal number of particles to use. Second, we show that a fully adapted auxiliary particle filter is unbiased and can drastically decrease computing time compared to a standard particle filter. Third, we introduce a new estimator of the likelihood based on the output of the auxiliary particle filter and use the framework of Del Moral (2004) to provide a direct proof of the unbiasedness of the estimator. Fourth, we show that the results in the article apply more generally to Markov chain Monte Carlo sampling schemes with the likelihood estimated in an unbiased manner.

1. Introduction

Many time series models that are of interest to econometricians can be expressed in state space form. When the likelihood of such models cannot be obtained analytically, Bayesian methods of estimation methods become attractive as it is often straightforward to use Markov chain Monte Carlo (MCMC) simulation methods to carry out the inference (see, for example, Cappé et al., 2005). The particle filter provides a general approach for filtering in state space models and in particular gives an unbiased estimate of the likelihood. An early use of the particle filter within a Markov chain Monte Carlo framework is by Fernández-Villaverde and Rubio-Ramírez (2007) who applied it to macroeconomic models as an approximate approach for obtaining the posterior distribution of the parameters. See Creal (2012) for an extensive review of the use of particle filters in econometrics. Andrieu et al. (2010) utilize the unbiasedness of the estimated likelihood to show that the particle filter offers a general approach for Bayesian inference by MCMC for such models. Their MCMC scheme is based on an auxiliary variable interpretation of the estimated likelihood, and they call their methodology particle filter MCMC (PMCMC). Flury and Shephard (2011) apply the results of Andrieu et al. (2010) to interesting econometric examples such as DSGE models and Gallant et al. (2011) apply the methodology to the Bayesian estimation of a dynamic game.

Our paper builds on the work of Andrieu et al. (2010) and makes a number of contributions. The first is to explore the properties of PMCMC, and in particular to give guidance for the optimal number of particles, N, to use. More particles increase the acceptance rate of a PMCMC sampler but at an increased computational cost. To obtain analytic results on an optimal trade-off we make some simplifying assumptions which are given and discussed in Section 3. Our analysis shows that the performance of the PMCMC is governed by the standard deviation of the error in the log of the estimated likelihood (at a particular parameter ordinate) and that the optimal value of this standard deviation is around 1, with a relatively benign region for this standard deviation being between 0.5 and 1.5. Once the standard deviation exceeds 2, performance decreases exponentially with an increase in the square of the standard deviation. Practical guidelines for choosing a reasonable value for N are given at the beginning of Section 3 and in Section 3.4. In practice, in Section 5, we estimate the overall computing time, taking into account both N and the mixing in the PMCMC method. We find that the computing time is minimized by choosing N so that the standard deviation of the log-likelihood, at a posterior central value, is around 0.92.

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The second contribution is to introduce a new estimator of the likelihood based on the output of an auxiliary particle filter and use the framework of Del Moral (2004) to provide a direct proof of the unbiasedness of the resulting estimator.

The third contribution is to show empirically that the log of the estimated likelihood obtained by fully adapted auxiliary particle filters can have a much smaller standard deviation than the standard deviation obtained using the standard particle filter of Gordon et al. (1993), especially when the signal to noise ratio is high. Our analytic results then suggest that it may be sufficient to use far fewer particles using adapted particle filters, especially fully adapted particle filters, to obtain the same statistical accuracy as the standard particle filter. We note that it is very important to carry out particle filtering as efficiently as possible when the sample size T is large. This issue is discussed at the end of Section 3.4.

The fourth contribution is to show that the results in the article apply more generally to MCMC sampling schemes with the likelihood estimated unbiasedly, and in particular when the likelihood is estimated by importance sampling.

We use simulated and real examples to illustrate a number of methodological issues addressed in the article. In particular, we show that our theory for choosing the optimal number of particles, which is developed under idealized conditions, can give a good guide to actual practice.

Particle filtering (also known as sequential Monte Carlo) was proposed by Gordon et al. (1993) for online filtering and prediction of nonlinear or non-Gaussian state space models. The first use of particle filtering in econometrics appeared in Kim et al. (1998) and Pitt and Shephard (1999). The auxiliary particle filter method was introduced by Pitt and Shephard (1999) to improve the performance of the standard particle filter when the observation equation is informative relative to the state equations, that is when the signal to noise ratio is moderate to high. There is an extensive literature on online filtering using the particle filter, see for example Doucet et al. (2001), and Del Moral et al. (2006). Our article considers only the standard particle filter of Gordon et al. (1993) and the adapted particle filters proposed by Pitt and Shephard (1999).

The literature on using the particle filter to learn about model parameters is more limited. Malik and Pitt (2011) propose the smooth particle filter to estimate the parameters of a state space model using maximum likelihood. Andrieu et al. (2010) provide a framework for off-line parameter learning using the particle filter. Flury and Shephard (2011) apply PMCMC to some econometric examples using single parameter random walk proposals for off-line Bayesian inference. Storvik (2002), Lopes et al. (2011) and Carvalho et al. (2010) consider online parameter learning. This is particularly effective when sufficient statistics for the parameters, from the latent states, are available. In addition, Carvalho et al. (2010) find that fully adapted particle filters in combination with sufficient statistics lead to efficient algorithms. In this article, we also find that adapted particle filter methods lead to much more efficient algorithms. In this article we confine ourselves to considering off-line methods.

2. An auxiliary variable representation of the estimated likelihood and posterior inference

We denote all observations as $y = y_{1:T} = (y_1', \ldots, y_T')'$ and the parameters as θ. The likelihood of the observations, which we consider to be analytically intractable, will be denoted as $p(y|\theta)$. This section shows how an unbiased estimate of the likelihood gives an auxiliary variable representation of the posterior for θ that is suitable for MCMC simulation. We illustrate with two examples, importance sampling and the particle filter.

2.1. Examples

Importance sampling. We shall briefly outline the method of importance sampling for obtaining unbiased estimates of the likelihood. Importance sampling methods are widely used in Bayesian econometrics, (e.g. Geweke, 1989). For likelihood estimation, we shall assume that we have a latent variable problem so that the data $y = y_{1:T} = (y_1', \ldots, y_T')'$ is generated conditionally upon latent variables $x = x_{1:T} = (x_1', \ldots, x_T')'$ and θ so that $y \sim p(y|\theta; x)$. We will assume that the latent variables are themselves generated as $x \sim p(x|\theta)$. In this case, the likelihood estimator is given as

$$\widehat{p}_N(y|\theta, u) = N^{-1} \sum_{k=1}^{N} \omega(x^k; \theta),$$

$$\text{where } \omega(x; \theta) = p(y|\theta; x)p(x|\theta)/g(x|\theta; y) \tag{1}$$

and the x^k are independent, identically distributed samples from the proposal density $g(x|\theta; y)$. In (1), u consists of all the random variables used to generate $x^1, \ldots, x^N$. It is straightforward to verify that the likelihood estimator is unbiased as $E_{g(x|\theta;y)}[\omega(x; \theta)] = p(y|\theta)$.

Particle filters. Consider a state space model with observation equation $p(y_t|x_t; \theta)$ and state transition equation $p(x_t|x_{t-1}; \theta)$, where y_t and x_t are the observation and the state at time t and θ is a vector of unknown parameters. The distribution of the initial state is $p(x_0|\theta)$. See Cappé et al. (2005) for a modern treatment of state space models. The particle filter obtains (possibly weighted) samples from the filtering density $p(x_t|y_{1:t}; \theta)$, and estimates of the prediction density of the observations $p(y_t|y_{1:t-1}; \theta)$ through time. The product of the prediction densities for the observations over time provides the likelihood $p(y|\theta)$.

Let $\widehat{p}_N(y_t|y_{1:t-1}; \theta, u)$ be the estimate of $p(y_t|y_{1:t-1}; \theta)$ obtained from the particle filter. Then $\widehat{p}_N(y|\theta, u) = \prod_{t=1}^{T} \widehat{p}_N(y_t|y_{1:t-1}; \theta, u)$ is the estimate of $p(y|\theta)$. The vector u consists of all the random variables used in the resampling part of the method as well as in the transition density. Details of the auxiliary sampling-importance-resampling (ASIR) filter of Pitt and Shephard (1999) and the construction of the prediction density estimates $\widehat{p}_N(y_t|y_{1:t-1}; \theta, u)$ are given in Appendix A.2. Theorem 1 of Appendix A.3 shows that $\widehat{p}_N(y|\theta, u)$ is unbiased for the likelihood $p(y|\theta)$.

2.2. Auxiliary variable representation

In general, suppose that $\widehat{p}_N(y|\theta, u)$ is an estimate of the likelihood, where N is a parameter associated with forming the estimate of the likelihood, such as the number of samples in importance sampling or the number of particles in the particle filter. Let u consist of all the random variables used in the construction of the estimate $\widehat{p}_N(y|\theta, u)$. We can consider u to be canonical, by which we mean that the distribution of u is chosen to be the same for all problems and not dependent on the parameters. For example, without any loss of generality, Section 2 of Flury and Shephard (2011), considers u to consist of independent identically distributed standard uniform variates. The vector u is typically high dimensional. Formally, the dimension of u also depends upon N although for notational convenience we will write the density from which u arises as $p(u)$. We note that this is the density of the random variates used in the construction of the estimator, and not a prior or a proposal density. We note that $\widehat{p}_N(y|\theta, u)$ is not necessarily a density in y when it is obtained using the auxiliary particle filter described in Appendix A.2.

We now assume that $\widehat{p}_N(y|\theta, u)$ is an unbiased estimator of the likelihood, i.e.

$$\int \widehat{p}_N(y|\theta, u)p(u)du = p(y|\theta). \tag{2}$$

The unbiasedness of $\widehat{p}_N(y|\theta, u)$ means that it is possible to write down a joint density in θ and u which admits the correct marginal density for θ as $\pi(\theta)$, the posterior density. The explanation below follows that in Section 2 of Flury and Shephard (2011).

Let $p(\theta)$ be the prior for θ and $\pi(\theta)$ its posterior. We define the joint posterior density $\pi_N(\theta, u)$ of θ and u as

$$\begin{aligned}\pi_N(\theta, u) &= \widehat{p}_N(y|\theta, u)p(\theta)p(u)/p(y)\\ &= \pi(\theta)p(u) \times \widehat{p}_N(y|\theta, u)/p(y|\theta)\end{aligned} \tag{3}$$

where $p(y) = \int p(y|\theta)p(\theta)d\theta$, the true marginal likelihood. Because $\widehat{p}_N(y|\theta, u)$ is unbiased, the joint density $\pi_N(\theta, u)$ integrates to one and its marginal density in θ is the posterior density $\pi(\theta)$. This means that it is relatively straightforward to implement a Markov chain Monte Carlo scheme, where the target density is $\pi_N(\theta, u)$ and we use the unnormalized form in the middle equation at (3) within a Metropolis expression as in Section 3.1.

3. Analysis of a simplified particle filter Markov chain Monte Carlo sampling scheme

This section attempts to give guidance on the number of particles, N, to choose in a particle filter Markov chain Monte Carlo (PMCMC) sampling scheme. The guidance can also be applied to the choice of the number of samples for an importance sampler within an MCMC scheme. Essentially the goal is to balance two competing costs. If N is taken to be large then we will be estimating the likelihood quite precisely and this will result in mixing (shown through the autocorrelation in θ) of the Markov chain which will be almost as fast as if we knew the likelihood. However, the cost of doing this is that we take a large value of N and so computing each likelihood estimate $\widehat{p}_N(y|\theta, u)$ is expensive. On the other hand, a small value of N will result in cheap evaluations of $\widehat{p}_N(y|\theta, u)$ but possibly at the cost of slow mixing (relative to knowing the likelihood) as indicated by high autocorrelation in the draws of θ. The latter problem (N that are too small) can be particularly costly as the PMCMC algorithm is not geometrically ergodic, as discussed in Section 3.3, (see also Theorem 8 of Andrieu and Roberts, 2009). In practical terms this means that the resulting chain can occasionally become sticky, retaining the same value for very long periods. This is also problematic because, for values of N that are too small, the chain can appear to be progressing well (for the first 10,000 draws say) and then suddenly becomes stuck for long periods. This phenomenon is noted by Flury and Shephard (2011), Section 5, who observed, for a state space form model and small N, that it was necessary to run the PMCMC scheme for 10^6 iterations in order to observe this stickiness, whereas the first 10,000 draws gave the misleading impression that the chain was mixing rapidly. Therefore, there are gains in being able to choose N in a reasonably sensible manner prior to conducting a full and possibly expensive PMCMC analysis.

Sections 3.2 and 3.3 develop an approach which, with certain simplifying assumptions, allows an explicit tradeoff between the cost of running the particle filter and the mixing of the resulting Markov chain. The results of our analysis indicate that we should choose N so that the variance of the resulting log-likelihood error is around 0.85. Of course, in practice this variance will not be constant as it is a function of the parameters as well as a decreasing function of N. Section 3.4 suggests ways of setting N in practice. Section 4 examines how well the assumptions we make hold in practice.

3.1. MCMC inference using the unbiased estimated likelihood

The target density for posterior inference is $\pi_N(\theta, u)$ given by (3). It is therefore possible to use a Metropolis–Hastings simulation method to generate samples from the target density as follows. We note that here we are generating the high dimensional random variate u from the associated density $p(u)$. This arises automatically when we generate the likelihood estimator by importance sampling or particle filter methods. As a consequence we can regard u as being proposed from $p(u)$ and this density cancels in the Metropolis expression in (4).

Suppose we have a joint Markov chain in (θ_j, u_j) arising from $\pi_N(\theta, u)$. Then to move to the next step of the chain, i.e. (θ_{j+1}, u_{j+1}) we propose u^* from $p(u)$ and θ^* from a proposal density $q(\theta|\theta_j)$. We then take $(\theta_{j+1}, u_{j+1}) = (\theta^*, u^*)$ with probability,

$$\alpha(\theta_j, u_j; \theta^*, u^*) = \min\left\{1, \frac{\widehat{p}_N(y|\theta^*, u^*)p(\theta^*)}{\widehat{p}_N(y|\theta_j, u_j)p(\theta_j)}\frac{q(\theta_j|\theta^*)}{q(\theta^*|\theta_j)}\right\}, \tag{4}$$

and take $(\theta_{j+1}, u_{j+1}) = (\theta_j, u_j)$ otherwise. It is informative to note this Metropolis expression is identical to what we would obtain using the likelihood function, (see e.g. Chib and Greenberg, 1995), except for the appearance of the estimated likelihood rather than the likelihood $p(y|\theta)$ in Eq. (4). In practical terms it should be noted that we do not normally retain or record the values of u. Instead we record the current value of the likelihood estimator $\widehat{p}_N(y|\theta_j, u_j)$ and denote the new proposed value as $\widehat{p}_N(y|\theta^*, u^*)$.

3.2. Scalar representation in the MCMC scheme

Let $z = \log\widehat{p}_N(y|\theta, u) - \log p(y|\theta)$ be the error in the log likelihood estimate. We first show that to analyze the MCMC scheme of Section 3.1 that involves the high dimensional auxiliary vector u and θ, it is entirely equivalent in theory to consider the Markov chain as operating on the scalar z and θ. We note that z is a many to one scalar function of u for any given y and θ, which we write as $z = \psi(u; \theta)$, and suppress the fixed values of y from this expression. Let $g_N(z|\theta)$ be the density of z given θ when u is generated from $p(u)$ and $z = \psi(u; \theta)$. This reduction in terms of z is useful as we know something about the properties of this density $g_N(z|\theta)$ as N becomes large. We note that whilst we can think of the MCMC scheme operating on this joint space in theory, in practice we cannot evaluate z as the log-likelihood $\log p(y|\theta)$ is unknown for any θ.

Let $\pi_N(\theta, z)$ be the joint density of θ and z obtained from $\pi_N(\theta, u)$, which is the invariant density arising from the MCMC scheme on θ and u. The following lemma expresses the densities $\pi_N(z|\theta)$ and $\pi_N(\theta, z)$ in terms of $g_N(z|\theta)$.

Lemma 1. *Using the notation above,*

(i) $\pi_N(z|\theta) = \exp(z)g_N(z|\theta)$.
(ii) $\pi_N(\theta, z) = \pi(\theta)\exp(z)g_N(z|\theta)$.
(iii) $E_{g_N(z|\theta)}(\exp(z)|\theta) = 1$.

We can now think of proposing θ^* from a proposal density $q(\theta|\theta_j)$ and z^* from $g_N(z|\theta^*)$ (by transforming from u^*) where the current values are (θ_j, z_j), accepting the proposed pair with probability

$$\alpha(\theta_j, z_j; \theta^*, z^*) = \min\left\{1, \frac{\exp(z^*)\pi(\theta^*)}{\exp(z_j)\pi(\theta_j)}\frac{q(\theta_j|\theta^*)}{q(\theta^*|\theta_j)}\right\}. \tag{5}$$

In practice, we use the entirely equivalent expression (4) by noting that

$$\pi(\theta)\exp(z) = \widehat{p}_N(y|\theta, u)p(\theta)/p(y). \tag{6}$$

As the proposal for θ and the acceptance criterion are the same as used in the criterion (4), the reduced chain in our object of interest $\{\theta_j\}$ remains preserved.

The advantage of regarding the chain as operating on θ and the scalar z is that we can use our knowledge of the properties of z to inform us of how rapidly or slowly the reduced chain in $\{\theta_j\}$ mixes in sampling from the invariant density $\pi(\theta)$.

3.3. Asymptotic approximation of $g_N(z|\theta)$ and $\pi_N(z|\theta)$

Let $\sigma^2(\theta, N) = \mathrm{Var}\{\psi(u;\theta)\} = \mathrm{Var}_{g_N(z|\theta)}(z)$, where z and $g_N(z|\theta)$ are defined in the previous section, with y and θ held fixed and u arising from $p(u)$. Lemma 2 gives conditions under which $g_N(z|\theta)$ and $\pi_N(z|\theta)$ tend to normality as N increases and shows that $\sigma^2(\theta, N)$ is the only parameter that affects their distributions for large N.

Lemma 2. *Suppose that $\widehat{p}_N(y|\theta, u)$ is an unbiased estimator of $p(y|\theta)$ and*

$$\sqrt{N}\left(\widehat{p}_N(y|\theta, u) - p(y|\theta)\right) \xrightarrow{d} \mathcal{N}(0, \lambda^2(\theta)), \tag{7}$$

where $\mathcal{N}(a, b^2)$ is a normal density with mean a and variance b^2. Let $\gamma^2(\theta) = \lambda^2(\theta)/p(y|\theta)^2$.

(i) *$N\sigma^2(\theta, N) \to \gamma^2(\theta)$ as N becomes large.*

(ii) *For given θ, suppose that $z_N \sim g_N(z|\theta)$. Then, as N becomes large,*

$$\sqrt{N}\left(\frac{z_N + \frac{\gamma^2(\theta)}{2N}}{\gamma(\theta)}\right) \xrightarrow{d} \mathcal{N}(0, 1).$$

(iii) *For given θ, suppose that $z_N \sim \pi_N(z|\theta)$. Then, as N becomes large,*

$$\sqrt{N}\left(\frac{z_N - \frac{\gamma^2(\theta)}{2N}}{\gamma(\theta)}\right) \xrightarrow{d} \mathcal{N}(0, 1).$$

The second order bias correction $-\gamma^2(\theta)/2N$ in part (ii) of Lemma 2 ensures that $E_{g_N(z|\theta)}(\exp(z)|\theta) = 1$ for the asymptotic approximation and matches the finite sample result in part (iii) of Lemma 1. Lemma 1 also implies that $E_{\pi_N(z|\theta)}(\exp(z)|\theta) = E_{g_N(z|\theta)}(\exp(2z)|\theta)$. The second order bias correction $\gamma^2(\theta)/2N$ in part (iii) of Lemma 2 ensures that $E_{\pi_N(z|\theta)}(\exp(z)|\theta) = E_{g_N(z|\theta)}(\exp(2z)|\theta)$ for the asymptotic approximation as well.

Importance sampling. For importance sampling we have that $\widehat{p}_N(y|\theta, u) \xrightarrow{p} p(y|\theta)$ as $N \longrightarrow \infty$ (page 114 Geweke, 2005). If the proposal density is sufficiently heavy tailed for the variance of the weights to exist, so that $E_{g(x|\theta;y)}[\omega(x;\theta)^2]$ is finite, then the Lindeberg–Levy central limit theorem applies and

$$\sqrt{N}\left\{\widehat{p}_N(y|\theta, u) - p(y|\theta)\right\} \xrightarrow{d} \mathcal{N}(0, \lambda^2_{IS}(\theta));$$

where the variance $\lambda^2_{IS}(\theta) = E_{g(x|\theta;y)}[\omega(x;\theta)^2] - p(y|\theta)^2$ (e.g. Cappé et al., 2005, p. 287).

Particle filter. The results of Proposition 9.4.1, page 301 of Del Moral (2004) and Proposition 2 of Del Moral et al. (2006) indicate that a central limit theorem applies to the likelihood estimator $\widehat{p}_N(y|\theta, u)$ obtained by the particle filter so that

$$\sqrt{N}\left\{\widehat{p}_N(y|\theta, u) - p(y|\theta)\right\} \xrightarrow{d} \mathcal{N}(0, \lambda^2_{PF}(\theta)),$$

where an explicit expression for $\lambda^2_{PF}(\theta)$ is given on page 301 of Del Moral (2004). We do not give this expression here as it is impractical to compute it in our further work as explained in Section 3.4.

From now on we shall just consider a single move in the PMCMC scheme representing the current state as (θ', z') which we consider as distributed according to the invariant joint distribution $\pi_N(\theta, z)$ given by Lemma 1(ii). We now make three assumptions that make the analysis tractable.

Assumption 1. The proposal density for θ is its posterior distribution $\pi(\theta)$, i.e., $q(\theta|\theta') = \pi(\theta)$.

This assumption allows us to separate out the effect of the particle filter on the sampling scheme from that of the quality of the proposal density for θ. It also allows us to study the properties of the MCMC sampling schemes under ideal conditions. The joint proposal density is then $\pi(\theta)g_N(z|\theta)$. Noting equation (6), the Metropolis–Hastings expression at Eq. (5) reduces to

$$\alpha(\theta', z'; \theta^*, z^*) = \min\{1, \exp(z^* - z')\}. \tag{8}$$

When N is large, both the proposed value z^* and the current value z' will be close to zero and so proposals will be accepted frequently. In fact it can be seen that under Assumption 1 the resulting PMCMC scheme is a Markov chain on z where z^* is proposed from $g_N(z)$ and the current value z' arises from $\pi_N(z)$ where

$$g_N(z) = \int g_N(z|\theta)\pi(\theta)d\theta \quad \text{and} \quad \pi_N(z) = \int \pi_N(z|\theta)\pi(\theta)d\theta.$$

We can then express the Metropolis term at (8) as $\alpha(z'; z^*)$.

Assumption 2. For a given θ and $\tau^2 > 0$, let the number of particles N be a function of θ and τ^2, which we write as $N = N(\theta, \tau^2)$, such that $N(\theta, \tau^2) = \gamma^2(\theta)/\tau^2$. The term $\gamma^2(\theta)$ is defined in Lemma 2.

We note that the notation $N(\theta, \tau^2)$ means the number of particles as a function of the parameters θ and τ^2 and not a normal distribution.

Constructing such an 'idealized' choice of the number of particles allows us to keep the variance of z constant across different values of θ. Thus, suppose that the target variance is τ^2. Then $\sigma^2(\theta, N(\theta, \tau^2)) = \sigma^2(\theta', N(\theta', \tau^2)) = \tau^2$ for all θ. This makes it possible to discuss various summaries of the sampling scheme such as the probability of acceptance of the independent Metropolis–Hastings proposal, the inefficiency factors, the computing time and the optimal choice of N as functions of σ only. Because of our choice of N, from now on we will use σ^2 as both the variance of z and as the function $\sigma^2(\theta, N)$. Thus, we will have that $\sigma^2(\theta, N(\theta, \tau^2 = \sigma^2)) = \sigma^2$.

Assumption 3. Both $g_N(z|\theta)$ and $\pi_N(z|\theta)$ are normal as given by (the asymptotic in N) parts (ii) and (iii) of Lemma 2.

Assumptions 2 and 3 mean that both $g_N(z|\theta)$ and $\pi_N(z|\theta)$ are normal and only functions of $\sigma^2(\theta, N(\theta, \sigma^2)) = \sigma^2$. Because we deal with the difference $z^* - z'$ in the Metropolis–Hastings expression (8) we can, without loss of generality, add $\sigma^2/2$ to the mean of z in the densities $g_N(z|\theta)$ and $\pi_N(z|\theta)$ so that Assumption 3 becomes (with dependence on N now omitted)

$$g(z|\sigma) = \frac{1}{\sigma}\phi\left(\frac{z}{\sigma}\right), \qquad \pi(z|\sigma) = \frac{1}{\sigma}\phi\left(\frac{z - \sigma^2}{\sigma}\right), \tag{9}$$

where $\phi(\cdot)$ is the standard normal probability density function. Note again, that we may think of $g(z|\sigma)$ as the density of the proposal and $\pi(z|\sigma)$ as the density of the accepted (current) values of z. Let

$$\Pr(A|z', \sigma) = \int \alpha(z'; z)g(z|\sigma)dz$$

be the probability of accepting the proposal conditional on z', where $\alpha(z'; z)$ is given by the right hand side of (8) and let

$$\Pr(A|\sigma) = \int \Pr(A|z)\pi(z|\sigma)dz$$

be the unconditional probability of accepting the proposal. The following results hold.

Lemma 3. *Under Assumptions 1–3, and with* $z' = \psi(u'; \theta')$,

(i)

$$\Pr(A|z', \sigma) = \Phi\left(-\frac{z'}{\sigma}\right) + \exp\left(-z' + \frac{\sigma^2}{2}\right)\Phi\left(\frac{z'}{\sigma} - \sigma\right),$$

where $\Phi(\cdot)$ *is the standard normal cumulative distribution function.*

(ii)

$$\Pr(A|\sigma) = 2\Phi\left(-\frac{\sigma}{\sqrt{2}}\right) = 2\left\{1 - \Phi\left(\frac{\sigma}{\sqrt{2}}\right)\right\}.$$

Part (i) *thus gives the probability of accepting the proposal given* z' *and Part* (ii) *gives the unconditional probability of accepting the proposal. Fig. 4 plots* $\Pr(A|\sigma)$ *against* σ *and shows that the acceptance rate is virtually 0 if* σ *exceeds 3.*

Corollary 1. $\Pr(A|Z', \sigma) \to 0$ *as* $z' \to \infty$. *This means that the Markov chain is not uniformly ergodic as the probability cannot be uniformly bounded away from zero, (see Roberts and Tweedie, 1996, Proposition 5.1, p 103).*

A more general result is established for the MCMC methods applied to importance sampling problems in Theorem 8 of Andrieu and Roberts (2009). This property means that for large σ (small N), the PMCMC scheme may reject many proposals for some values of the current state leading to stickiness in the resulting chain.

We now derive an expression for the inefficiency $\mathrm{IF}(\sigma)$ of the sampling scheme as a function of σ. The inefficiency of the sampling scheme is the factor by which it is necessary to increase the number of iterations in the Markov chain Monte Carlo in order to obtain the same accuracy as a sampling scheme that generates independent iterates. Suppose we are interested in posterior inference about some scalar functional $\zeta = h(\theta)$ of θ. Let $\{\theta_j, j = 1, \ldots, M\}$ be the iterates of the particle filter Markov chain Monte Carlo sampling scheme after it has converged. Suppose that $\rho_\tau(\sigma)$ is the autocorrelation between ζ_j and $\zeta_{j+\tau}$. Then $\mathrm{IF}(\sigma)$ is defined as

$$\mathrm{IF}(\sigma) = 1 + 2\sum_{\tau=1}^{\infty} \rho_\tau(\sigma). \tag{10}$$

$\mathrm{IF}(\sigma)$ is also known as the integrated autocorrelation time (IACT). Let $\bar{\zeta}$ be the sample mean of the iterates $\zeta_j, j = 1, \ldots, M$, which we use as an estimate of the posterior mean of $h(\theta)$. Then,

$$M\mathrm{Var}(\bar{\zeta}) \to \mathrm{Var}(\zeta|y)\mathrm{IF}(\sigma) \quad \text{as } M \to \infty.$$

We note that if the ζ_j are independent then $\rho_j(\sigma) = 0$ for $j \geq 1$ and $\mathrm{IF}(\sigma) = 1$. The following lemma gives a computable expression for $\mathrm{IF}(\sigma)$.

Lemma 4. *Under the assumptions in this section,*

$$\mathrm{IF}(\sigma) = \int \frac{1 + p^*(w, \sigma)}{1 - p^*(w, \sigma)}\phi(w)dw \tag{11}$$

where $p^*(w, \sigma) = \Phi(w + \sigma) - \exp(-w\sigma - \sigma^2/2)\Phi(w)$. *This result also means that* $\mathrm{IF}(\sigma)$ *is invariant to the functional* ζ.

Thus, under our assumptions, the posterior mean of $N(\theta, \sigma^2)$ for given σ is

$$E_{\pi(\theta)}[N] = E_{\pi(\theta)}[\gamma^2(\theta)/\sigma^2] = \overline{\gamma^2}/\sigma^2.$$

Therefore, taking into account statistical inefficiency, the computing time is proportional to $\mathrm{IF}(\sigma)/\sigma^2$. Thus, without loss of generality (for our purposes) we take the computing time as $\mathrm{CT}(\sigma) = \mathrm{IF}(\sigma)/\sigma^2$. This expression $CT(\sigma)$ explicitly takes into account both the expected cost of computing the estimator and the cost of slow mixing in the PMCMC method.

The next lemma gives the minimizing value for σ, as well as the behavior of $\mathrm{IF}(\sigma)$ and $\mathrm{CT}(\sigma)$ for large σ.

Lemma 5. (i) *The computing time* $CT(\sigma)$ *is minimized for* $\sigma = 0.92$. *For this value of* σ, $IF(\sigma) = 4.54$ *and* $\Pr(A|\sigma) = 0.5153$.

(ii) *For* $\sigma^2 \to \infty$

$$\frac{IF(\sigma)}{2\exp(\sigma^2) - 1} \to 1 \tag{12}$$

$$\frac{\sigma^2 CT(\sigma)}{2\exp(\sigma^2) - 1} \to 1 \tag{13}$$

which means that for σ *large*

$$IF(\sigma) \approx 2\exp(\sigma^2) - 1 \quad \text{and} \quad CT(\sigma) \approx \frac{2\exp(\sigma^2) - 1}{\sigma^2}.$$

It is straightforward to verify that Eqs. (12) and (13) also hold for $\sigma \to 0$ *because* $IF(\sigma) \to 1$ *as* $\sigma \to 0$.

It is necessary to interpret the results in Eqs. (12) and (13) with some care. The asymptotics rely on $\sigma^2 \to \infty$ in which case the number of particles tends to 0, so that at extreme values of σ this approximation and the exact analytics may not be accurate because the Gaussian approximations for $g_N(z|\sigma^2)$ and $\pi_N(z|\sigma)$ given in (9) may not hold for a small number of particles. However, over the range of σ of practical importance (practically it is problematic to have $\sigma > 3$) these approximations may be extremely good as shown in Fig. 1.

Fig. 1 shows that for $\sigma > 2.5$ both $\mathrm{IF}(\sigma)$ and $\mathrm{CT}(\sigma)$ are large and that small changes in σ change both $\mathrm{IF}(\sigma)$ and $\mathrm{CT}(\sigma)$ appreciably. The figure also shows that for σ in the range 0.5–1.5, small changes in σ result in relatively small changes in $\mathrm{CT}(\sigma)$. This is important because even if we estimate σ with a small error so that we are not at the optimal σ, the effect on $\mathrm{CT}(\sigma)$ will not be serious.

3.4. Computational considerations

In the previous section we choose the number of particles $N(\theta, \sigma^2)$ as a function of θ for a given σ so that $\sigma^2(\theta, N(\theta, \sigma^2)) = \sigma^2$ and then derive the optimal choice of σ to minimize the computing time. We can follow this prescription when the exact or asymptotic variance of the error in the estimated likelihood can be estimated in a straightforward and fast way. For example, for standard importance samplers the asymptotic variance can be quickly estimated from the existing output. Using the notation of Eq. (1)

$$\mathrm{Var}\left\{\frac{\widehat{p}_N(y|\theta; u)}{p(y|\theta)}\right\} = \frac{1}{N}\mathrm{Var}_{g(x)}\left\{\frac{\omega(x; \theta)}{p(y|\theta)}\right\}$$

$$\simeq N\,\mathrm{Var}_{g(x)}\left\{\frac{\omega(x; \theta)}{\sum_{k=1}^{N}\omega(x^k; \theta)}\right\} \simeq \sum_{k=1}^{N}\pi_k^2 - \frac{1}{N},$$

where $\pi_k = \omega(x^k; \theta)/\sum_{j=1}^{N}\omega(x^j; \theta)$. Asymptotically (in N) this also gives the variance of the log of the estimator. This is a well established result (pp. 34–36 of Liu, 2001) and suggests that when importance sampling is used with a PMCMC approach it is possible to adjust N with the proposed θ to keep the resulting variance constant (ideally at $0.92^2 = 0.84$).

Whilst the form of the variance of the likelihood estimator is known, in theory, for the particle filter estimator (Del Moral, 2004, p. 301), a similar practical and fast (of order $N \times T$) variance estimator is not currently available. Thus, in practice it is infeasible to obtain the (asymptotic) variance of the error in the estimate of the log likelihood because obtaining estimates of the variation $\gamma^2(\theta)$ for each parameter ordinate θ is prohibitively expensive.

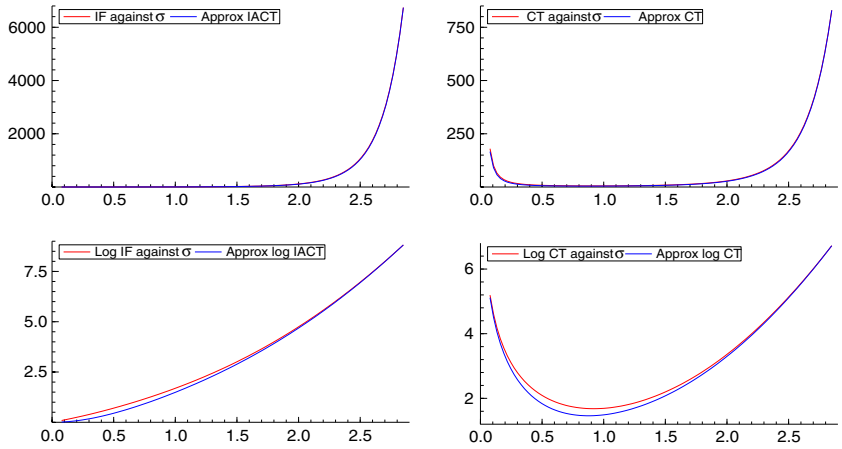


Fig. 1. Plots of IF(σ), CT(σ), log IF(σ) and log CT(σ) against σ. Also shown as dashed lines are the approximations for large σ given in part (ii) of Lemma 5.

We now present a practical way of choosing the optimal N (N_{opt}) and informally discuss how to relate the results for the inefficiency and computing time in Section 3.2 to the empirical measures of inefficiency and computing time we would obtain if Assumptions 1 and 3 hold. Our solution is to choose N_{opt} constant (i.e., it does not depend on θ) so that the standard deviation of the particle filter log-likelihood estimator evaluated at a central value ($\overline{\theta}$) of the posterior distribution of θ is 0.92.

A reasonable strategy to obtain $\overline{\theta}$ and choose N_{opt}, which we have pursued, is to first run a short MCMC scheme for large N to determine an approximate value for $\overline{\theta}$, the posterior mean. We then run several (e.g. $R = 100$) independent particle filters for a fixed starting value of N, say N_s, obtaining an estimator of the likelihood, $\widehat{p}^{(i)}_{N_s}(y|\overline{\theta})$, $i = 1, \ldots, R$ for each. We record the variance of the log of the the likelihood estimator, in the standard manner, as

$$\widehat{\sigma}^2(\overline{\theta}, N_s) = \frac{1}{R}\sum_{i=1}^{R}\left\{\log \widehat{p}^{(i)}_{N_s}(y|\overline{\theta})\right\}^2 - \left\{\frac{1}{R}\sum_{i=1}^{R}\log \widehat{p}^{(i)}_{N_s}(y|\overline{\theta})\right\}^2.$$

We then choose

$$N_{opt} = N_s \times \frac{\widehat{\sigma}^2(\overline{\theta}, N_s)}{0.92^2}.$$

For some models, it may be possible to have a close approximation to the central value $\overline{\theta}$ of θ, without running the initial MCMC scheme. If there is an approximating model to the true model for which the parameters may be quickly estimated (by maximum likelihood or methods of moments estimators) then we can use such estimators to approximate $\overline{\theta}$, used for the calculation of the variance expression above. An example of approximating models is the state space form approximation to the stochastic volatility model (Harvey et al., 1994).

We now informally relate the inefficiency and computing time measures defined in the previous section to the actual measures that we use in the reported empirical work. For any θ we define

$$\sigma^2_{\text{opt}}(\theta) = \sigma^2(\theta, N_{\text{opt}}) \simeq \frac{\gamma^2(\theta)}{N_{\text{opt}}} = \frac{\gamma^2(\theta)}{N_s} \times \frac{0.92^2}{\widehat{\sigma}^2(\overline{\theta}, N_s)}$$
$$\simeq \frac{\gamma^2(\theta)}{\gamma^2(\overline{\theta})} \times 0.92^2 = \frac{\overline{\gamma}^2(\theta)}{\overline{\gamma}^2(\overline{\theta})} \times 0.92^2,$$

where $\overline{\gamma}^2(\theta) = \gamma^2(\theta)/T$ where T is the length of the time series. By Theorem 1.4 of Cerou et al. (2011), the variance of $\widehat{p}_N(y|\theta, u)/(y|\theta)$ is bounded linearly with T (see also p. 12 of Flury and Shephard, 2011, and Table 4). As a consequence the function $\overline{\gamma}^2(\theta)$ will not grow with T but will be approximately constant as a function of the length of time series. The variance of $\log\{\widehat{p}_N(y|\theta, u)/p(y|\theta)\}$ also increases linearly with T.

Crucially, it is important, for finite T that $\sigma^2_{\text{opt}}(\theta)$ does not vary outside the range $(0.25, 2.25)$ during the MCMC or equivalently $\sigma_{\text{opt}}(\theta)$ is between 0.5 and 1.5, as noted in Section 3.2 and seen in Fig. 1. Within this region the computing time should vary little.

To understand whether $\sigma^2_{\text{opt}}(\theta)$ is likely to lie in the range above we now consider the posterior distribution of $\sigma^2_{\text{opt}}(\theta)$ for finite T, and in particular when T is large. Clearly, under the usual regularity conditions for $\pi(\theta)$, the posterior $\text{Var}(\theta|y) \propto 1/T$ and we shall assume that $\overline{\theta} = E(\theta|y)$ and so the variance of $\sigma^2_{\text{opt}}(\theta)$, typically a smooth function of θ will also reduce at rate $1/T$, so that as $T \to \infty$

$$\sigma^2_{\text{opt}}(\theta)|y \xrightarrow{d} \mathcal{N}\left(0.92^2, \frac{V(\overline{\theta})}{T}\right)$$

for some function V of θ, which means that when T becomes large, $\sigma^2_{\text{opt}}(\theta)$ lies in the required range.

We complete this section by giving an informal argument on why the computational load of PMCMC is $O(T^2)$. It has been noted that the variance of $\log\{\widehat{p}_N(y|\theta, u)/(y|\theta)\}$ increases linearly with T. As the computational load of the particle filter with N_{opt} particles is $O(N_{\text{opt}}T)$, it follows that the corresponding computational load is then approximately $O(T^2)$ if we are to keep constant the variance in the error of the log likelihood.

4. Performance of the theory in the 'finite' N case

This section compares the large sample (in N) results developed above with results for a small to moderate number of particles N for three models. All three are signal plus noise models for which it is possible to run the standard particle filter and the fully adapted particle filter (described in Appendix A.2). The finite sample results are obtained by simulation using using a fixed value of θ with several different choices for the number of particles. We take a fixed value of θ as this fixes $\sigma^2(\theta, N) = \gamma^2(\theta)/N$ and leads to the ideal setting corresponding to the "perfect proposal" in Assumption 1 where the Metropolis–Hastings acceptance probability is given by (8). Our simulation results show that a close correspondence between asymptotic theory and finite N performance for $\log \widehat{p}_N(y|\theta, u)$, even for small N. We note that similar results to those in Section 3 can be obtained for $\widehat{p}_N(y|\theta, u)$. However, we show below that much larger values of N are required for both the standard particle filter and the fully adapted particle filter for $\widehat{p}_N(y|\theta, u)$ to be normally distributed. This shows that checking how well the asymptotic theory performs for the finite N case as we do in this section is scientifically both prudent and necessary.

For the first order autoregressive (AR(1)) model plus noise example the true log likelihood is obtained by the Kalman filter. For the other two models the true log likelihood used to construct the z from the particle filter was approximated by the sample mean of $M = 50{,}000$ unbiased estimates, where we construct each unbiased estimate using $N = 10{,}000$ particles for the standard particle filter and $N = 500$ particles for the fully adapted particle filter.

4.1. AR(1) plus noise model

We consider the AR(1) plus noise model as a simple example to compare the relative performance of the standard SIR method and the fully adapted particle filter (FAPF). The model is

$$\begin{aligned} &y_t = x_t + \sigma_\varepsilon \varepsilon_t, \qquad x_{t+1} = \phi x_t + \sigma_\eta \eta_t, \\ &x_0 \sim \mathcal{N}(0, \sigma_\eta^2/(1-\phi^2)) \end{aligned} \tag{14}$$

where ε_t and η_t are standard normal and independent. We take $\phi = 0.6$, $\sigma_\eta^2 = (1-\phi^2)$, so that the marginal variance of the state x_t is $\sigma_x^2 = 1$. We simulate a single series of length $T = 200$, varying the measurement noise σ_ε^2 (but with fixed innovations ε_t) for the experiment. We take $N = 50$ and record the bias and the variance of the logarithm of the estimator of the likelihood for the two particle filter methods, SIR and the fully adapted particle filter (FAPF) of Pitt and Shephard (1999). The algorithm for the FAPF method is specified in Appendix A.2 and more details are provided in Pitt and Shephard (2001). In this case we can evaluate $p(y_{t+1}|x_t)$ explicitly and simulate from $p(x_{t+1}|x_t, y_{t+1})$, the two requirements for using the FAPF method. As the FAPF method essentially guides the particles using the future observation and integrates over the corresponding state when estimating the predictive density of the observations, the estimation of the likelihood should be much more efficient.

The bias and the variance are computed using 400 independent runs of both the SIR and FAPF filters. The likelihood for the data is given by the Kalman filter as the model is of simple state space form. Fig. 2 plots the variance and the bias in the log-likelihood estimator against σ_ε^{-1}. It is apparent that, as expected from part (ii) of Lemma 2, the bias downwards is about half the variance. It is also clear that as the measurement standard deviation σ_ε becomes smaller (so σ_ε^{-1} is larger) the standard SIR method does increasingly poorly whereas the performance of the FAPF actually improves. In all cases, it is found that the FAPF has smaller variance and bias. However, in examining the relative variances, given as the last row of Fig. 2, it is apparent that the FAPF is dramatically better for small σ_ε (large σ_ε^{-1}). The range of σ_ε chosen is not particularly extreme (no smaller than 1/9). We have found that this result applies to other models which may be fully adapted, see Sections 4.2 and 4.3, as the measurements becomes more informative. The variance, of course, directly translates into the necessary number of particles, as we need to take N such that the variance is around 0.85. Thus, Fig. 2 suggests that as σ_ε goes down towards about 1/9, the SIR method requires around a factor of 2000 more particles than the FAPF method.

For models where the measurements are informative but which cannot be fully adapted, the estimator resulting from applying the auxiliary particle filter, see Appendix A.2, can be used. There are several effective ways of constructing auxiliary proposals, see for example Pitt and Shephard (2001). For relatively uninformative measurement equations the gains over the standard SIR method may well be modest.

Fig. 3 displays the histogram (over 10,000 replications of the filter) of the log likelihood error $z = \log \widehat{p}_N(y|\theta) - \log p(y|\theta)$ and the fitted Gaussian distribution for the standard particle filter using the estimated mean and the estimated variance σ^2 (calculated using the 10,000 replications of z) only. We use $T = 50$ and 500 and varying N for $\sigma_\varepsilon = \sqrt{2}$. The asymptotic approximating density, which is $\mathcal{N}(-\sigma^2/2, \sigma^2)$, is also displayed using the estimated variance only. The boxes in Fig. 3 give the standard deviations. It is clear that even for small T, the asymptotic Gaussian approximation is good when σ is close to one for the log-likelihood estimator. As σ increases (N decreases) the approximation is a little worse. For large T, the approximation appears to be good even for small N. On the likelihood scale (left hand side), displaying $\exp(z) = \widehat{p}_N(y|\theta)/p(y|\theta)$, N needs to be very large for the density to be close to Gaussian. Of course our approximation, in Section 3.3, relies on the asymptotics for the log of the estimator $\log \widehat{p}_N(y|\theta; u)$. In particular we want this approximation to be good around the region of optimization for σ (close to one). We have found that the approximation holds up equally well for the other examples considered in this paper. In particular, it improves as T increases.

4.2. A mixture of autoregressive experts observed with noise model

We consider a two-component mixture of experts model observed with noise, where each expert is modeled as a first order autoregressive process. Section 5.3 motivates the model and applies it to GDP growth data. The model is given by

$$y_t = x_t + \sigma_\varepsilon \varepsilon_t \tag{15a}$$

$$x_t = c_{J_t} + \phi_{J_t} x_{t-1} + \tau_{J_t} \eta_t, \quad \text{for } J_t = 1, 2, \tag{15b}$$

$$\Pr(J_t = 1|x_{t-1}, x_{t-2}) = \frac{\exp(\xi_1 + \xi_2 x_{t-1} + \xi_3(x_{t-1} - x_{t-2}))}{1 + \exp(\xi_1 + \xi_2 x_{t-1} + \xi_3(x_{t-1} - x_{t-2}))}, \tag{15c}$$

where ε_t and η_t are standard normal and independent. The means of the first and second autoregressive experts are $\mu_1 = c_1(1-\phi_1)$ and $\mu_2 = c_1(1-\phi_2)$. To identify the two experts we assume that $\mu_1 < \mu_2$ so the first expert has a lower mean than the second expert. This is a state space model with the two dimensional state vector (x_t, x_{t-1}).

We generate a single series of length $T = 100$ from the model where the parameters are set as $\sigma_\varepsilon = 0.5$, $\phi_1 = 0.58$, $\phi_2 = 0.32$, $\log(\tau_1^2) = 0.54$, $\log(\tau_2^2) = 0.255$, $c_1 = -0.11$, $c_2 = 2.17$, $\xi_1 = -0.80$, $\xi_2 = -2.33$ and $\xi_3 = -1.53$. With the exception of σ_ε, they are the posterior means from the GDP growth example analyzed in Section 5.3. The parameter σ_ε is chosen to allow a moderate to high signal to noise ratio in the data.

We carry out a PMCMC scheme keeping the parameters fixed so that the Metropolis expression is given by (8). We record the proposed and accepted values of z which is the error in the estimated log likelihood. Table 1 summarizes the results of the simulation. From the theory of Section 3.3 we expect the mean of the proposed values of z arising from the PF to be around $-\sigma^2/2$ and the mean of the accepted values of z to be around $\sigma^2/2$, where σ^2 is the estimated variance of the draws from $g_N(z|\theta)$. The table shows that for the standard particle filter the results for the means and variances of $g_N(z|\theta)$ and $\pi_N(z|\theta)$ are close to the asymptotic (in N) theoretical results for $N \geq 400$ and that the standard deviation of $g_N(z|\theta)$ is close to 1 for $N = 400$. The table also shows that the empirical results for the fully adapted particle filter are close to the theoretical results for N as low as 25, and the standard particle filter needs more than 40 times the number of particles required by the fully adapted particle filter to achieve the same proposed variance as the fully adapted particle filter.

The left panel of Fig. 4 shows that the estimated and theoretical acceptance probabilities, given by $\Pr(A|\sigma)$ of Lemma 3(ii), for differing values of σ are close. The top four panels of Fig. 5 summarize the output when the PMCMC is run for two different values of the standard deviation (σ) of z, taking $N = 200$ and $N = 1400$, with the parameters fixed at their true values and

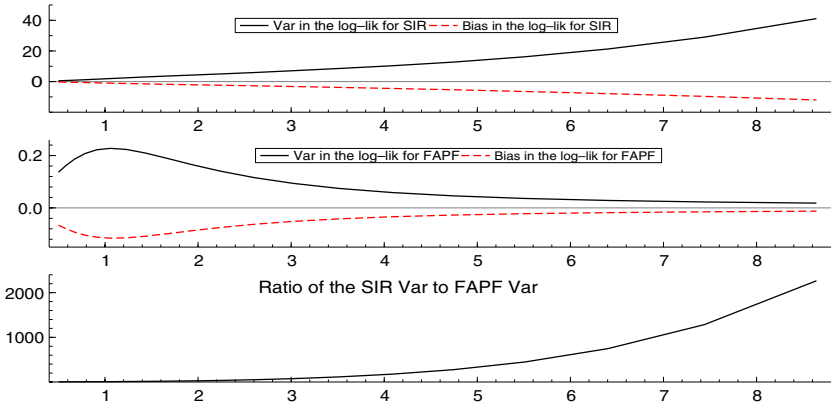


Fig. 2. AR(1) plus noise model. Number of replications is 400 and $N = 50$ for both filters. Length $T = 200$, $\phi = 0.6$, $\sigma_\eta^2 = (1-\phi^2)$. TOP: Bias and variance of SIR log-likelihood estimator against σ_ε^{-1}. MIDDLE: Bias and variance of FAPF log-likelihood estimator against σ_ε^{-1}. BOTTOM: Ratio of variance for SIR estimator to that of the FAPF estimator against σ_ε^{-1}.

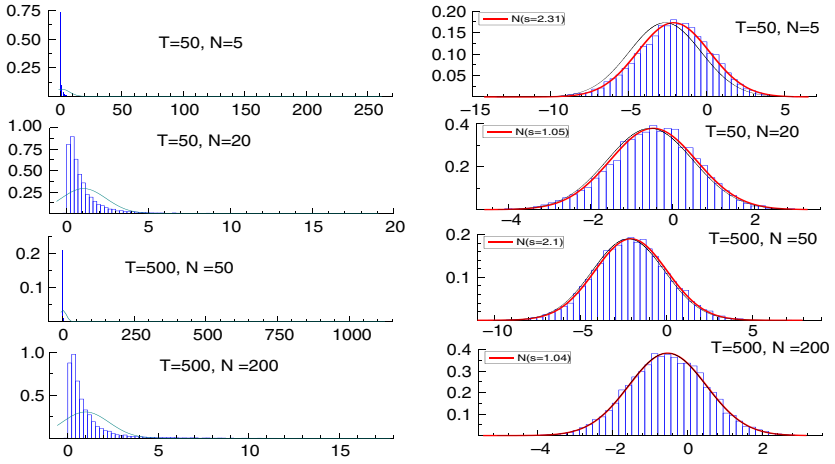


Fig. 3. AR(1) plus noise model with fixed parameters. Number of replications is 10,000. Displayed are the histograms and Gaussian approximations for the SIR likelihood estimator (divided by the likelihood) on the left and for the error in the log of the SIR likelihood estimator on the right. Both N and T vary as shown and $\phi = 0.6, \sigma_\eta^2 = (1-\phi^2)$, $\sigma_\varepsilon^2 = 2$. On the right, a Gaussian is fitted to the histogram (red/solid line) using the estimated mean and variance. On the right is the theoretical Gaussian density (black/dashed line) formed only from the estimated variance (mean $-\sigma^2/2$, variance σ^2). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

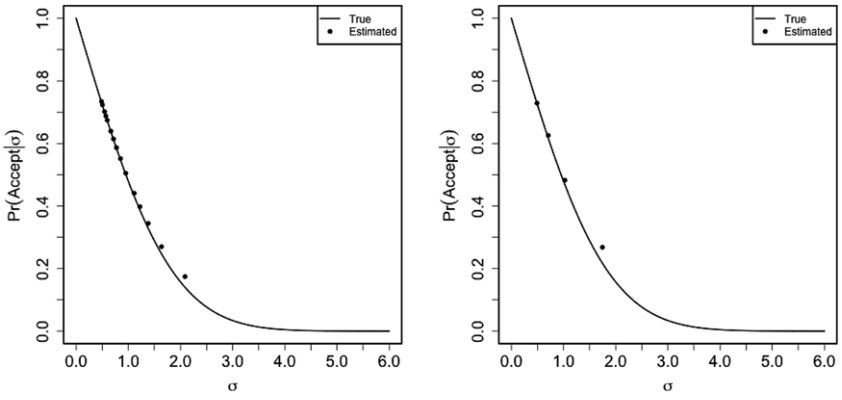


Fig. 4. Probability of acceptance in Metropolis–Hastings step against the standard deviation of the log-likelihood estimator. The theoretical probability of Lemma 3 is given by the solid line and the estimated probabilities are given by dots. For the estimated probabilities N is varied leading to the different values of σ. The left panel is for the mixture of experts AR(1) model with fixed parameters. The right panel is for the GARCH(1,1) model with noise with fixed parameters.

using the standard particle filter. The plots show that the empirical densities for $g_N(z|\sigma)$ and $\pi_N(z|\sigma)$ and the theoretical densities given by the asymptotic results in parts (iii) and (iv) of Lemma 2 are close, especially when σ is smaller than 1. The correlograms of z from $\pi_N(z|\theta)$ are displayed and decay in the manner the theory indicates.

The results are based on a sample of 100,000 (independent) draws from $g_N(z|\theta)$ and at least 200,000 draws from $\pi_N(z|\theta)$ using

Table 1
Mixture of experts plus noise model. The table shows the mean, variance and standard deviation of $g_N(z|\theta)$ and $\pi_N(z|\theta)$ for various values of the number of particles for both the standard particle filter and the fully adapted particle filter. The sample size is $T = 100$ and the parameters are fixed at their true values.

N	$g_N(z\|\theta)$			$\pi_N(z\|\theta)$		
	Mean	Var.	St.dev.	Mean	Var.	St.dev.
	Standard particle filter					
100	−1.9398	4.3531	2.0864	1.7775	3.2623	1.8062
200	−0.9231	1.9115	1.3826	0.8763	1.6889	1.2996
400	−0.4505	0.9064	0.9520	0.4420	0.8672	0.9312
1400	−0.1277	0.2546	0.5046	0.1251	0.2495	0.4995
	Fully adapted particle filter					
4	−1.0799	3.2170	1.7936	0.8132	1.3613	1.1667
8	−0.4507	1.0537	1.0265	0.4046	0.7398	0.8601
12	−0.2860	0.6206	0.7878	0.2683	0.5061	0.7114
25	−0.1279	0.2688	0.5185	0.1273	0.2471	0.4971

Table 2
GARCH model observed with noise. The table shows the mean, variance and standard deviation of $g_N(z|\theta)$ and $\pi_N(z|\theta)$ for various values of the number of particles for both the standard particle filter and the fully adapted particle filter. The sample size is $T = 526$ and the parameters are fixed at their posterior mean.

N	$g_N(z\|\theta)$			$\pi_N(z\|\theta)$		
	Mean	Var.	St.dev.	Mean	Var.	St.dev.
	Standard particle filter					
1,000	−1.4690	3.3408	1.8278	1.2254	2.2527	1.5009
2,500	−0.5692	1.2094	1.0997	0.5266	0.9729	0.9864
5,000	−0.2737	0.5729	0.7569	0.2611	0.4985	0.7061
10,000	−0.1350	0.2758	0.5252	0.1334	0.2566	0.5066
	Fully adapted particle filter					
50	−1.0230	2.3175	1.5223	1.1163	1.9663	1.4023
100	−0.4951	1.0973	1.0475	0.5234	0.9458	0.9725
250	−0.1926	0.4272	0.6536	0.2328	0.4197	0.6479
500	−0.1049	0.2118	0.4602	0.1024	0.2066	0.4546

the independent Metropolis Hastings algorithm for each number of particles.

4.3. GARCH model observed with noise

This section considers the GARCH(1,1) model observed with Gaussian noise which is a more flexible version of the basic GARCH(1,1) model. This is a simplified version of the factor GARCH model, with x_t the factor; see Fiorentini et al. (2004). Section 5.2 motivates this model, applies it to UK MSCI weekly returns and explains how to run a fully adapted particle filter for it. The model is described as

$$\begin{aligned} &y_t = x_t + \tau\varepsilon_t, \qquad x_t|\sigma_t^2 = \sigma_t\eta_t, \\ &\sigma_{t+1}^2 = \alpha + \beta x_t^2 + \gamma\sigma_t^2, \qquad x_0 \sim \mathcal{N}(0, \alpha/(1-\beta-\gamma)) \,. \end{aligned} \tag{16}$$

We use the estimated posterior means from the analysis in Section 5.2 as the fixed parameter values. These values are $\tau^2 = 0.00027$, $\alpha = 0.0000495$, $\beta = 0.89275$ and $\gamma = 0.03779$. A single dataset is generated from this model of length $T = 526$. Table 2 summarizes the results. The table shows that for the standard particle filter the means and variances of $g_N(z|\theta)$ and $\pi_N(z|\theta)$ are close to the asymptotic in N theoretical results only for $N \geq 2500$ and that the standard deviation of $g_N(z|\theta)$ is 1 for $N = 2500$. For the fully adapted particle filter the empirical results are close to the theoretical results only for $N \geq 250$, and the fully adapted particle filter needs about 1/25th of the number of particles to achieve the same variance of $g_N(z|\theta)$ as the standard particle filter.

The right panel of Fig. 4 compares the estimated and theoretical acceptance probability for differing values of σ. These results are based on a sample of 50,000 (independent) draws from the proposal and 50,000 draws from the posterior through the independent Metropolis–Hastings algorithm for each number of particles (as in (8)). The bottom panels in of Fig. 5 summarize the output when the PMCMC is run for two different values of the standard deviation (σ) of z, with the parameters fixed at their true values and using the standard particle filter. The plots show that the empirical densities for $g_N(z|\sigma)$ and $\pi_N(z|\sigma)$ and the theoretical densities given by the asymptotic results in parts (iii) and (iv) of Lemma 2 are close, especially when σ is smaller than 1. The correlograms for the accepted draws of z are displayed and decay

For all three models the theory appears to be remarkably close to what we observe in practice. This is particularly true for σ less than and around 1, the region where we hope our approximation is close as we optimize to achieve $\sigma = 0.92$. In this section we have kept the parameters fixed as we were concerned with the resulting behavior in the Markov chain for the error z in the log likelihood. In Section 5, we sample both the parameters and u and investigate performance as we vary N using both the standard and fully adapted particle filters.

5. Comparing the theory with empirical performance for a full PMCMC

The theory considers an idealized situation based on three assumptions. (i) the proposal is perfect in the sense that the proposal density for θ is its posterior and that the proposal is independent of previous iterates; (ii) the standard deviation of the estimated log likelihood error is kept constant by adjusting the number of particles for each θ; (iii) the error of the estimated log likelihood is assumed to be Gaussian as a function of the particles. However, when θ is also generated, neither the assumption of a perfect sampling scheme for the parameters nor the adjustment of the number of particles for θ is met in practice.

This section uses simulated and real data to compare the theoretical results with the performance of a full PMCMC. It also tries to separate out the effect on performance of the PMCMC due to using an estimated likelihood from the effect due to using an imperfect proposal. We consider the performance of both the standard particle filter and the fully adapted particle filter.

5.1. AR(1) plus noise model

We use data generated from the AR(1) plus noise model at (14) with $\sigma_\varepsilon = \sqrt{2}$. This model has two parameters (ϕ, σ_η^2). The prior distribution ϕ and σ_η^2 are respectively uniform on $(-1, 1)$ and an inverse gamma density with shape and scale parameters equal to 0.1. We use a single data set with $T = 500$ observations generated from the true model having $\phi = 0.6$ and $\sigma_\eta = \sqrt{1-\phi^2}$.

This model allows us to compare the results obtained by estimating the likelihood using the standard and fully adapted particle filters with the results obtained when the likelihood is evaluated using the Kalman filter. The unknown parameters are generated by two methods. The first is an independent Metropolis–Hastings scheme. The second is a random walk Metropolis scheme. Both methods are described in more detail below. The small number of parameters means that we can construct an independent Metropolis–Hastings scheme for the parameters that has a high acceptance rate when the exact likelihood is evaluated by the Kalman filter, and makes it possible to separate out the effect on the acceptance rates, inefficiencies and computing times of the parameter generating part of the PMCMC from that of using the particle filters to estimate the likelihood.

We first ran 100,000 replications using the fixed true parameters of both the standard SIR and the FAPF to compute the mean, variance and standard deviation of $g_N(z|\theta)$ and $\pi_N(z|\theta)$ for differ-

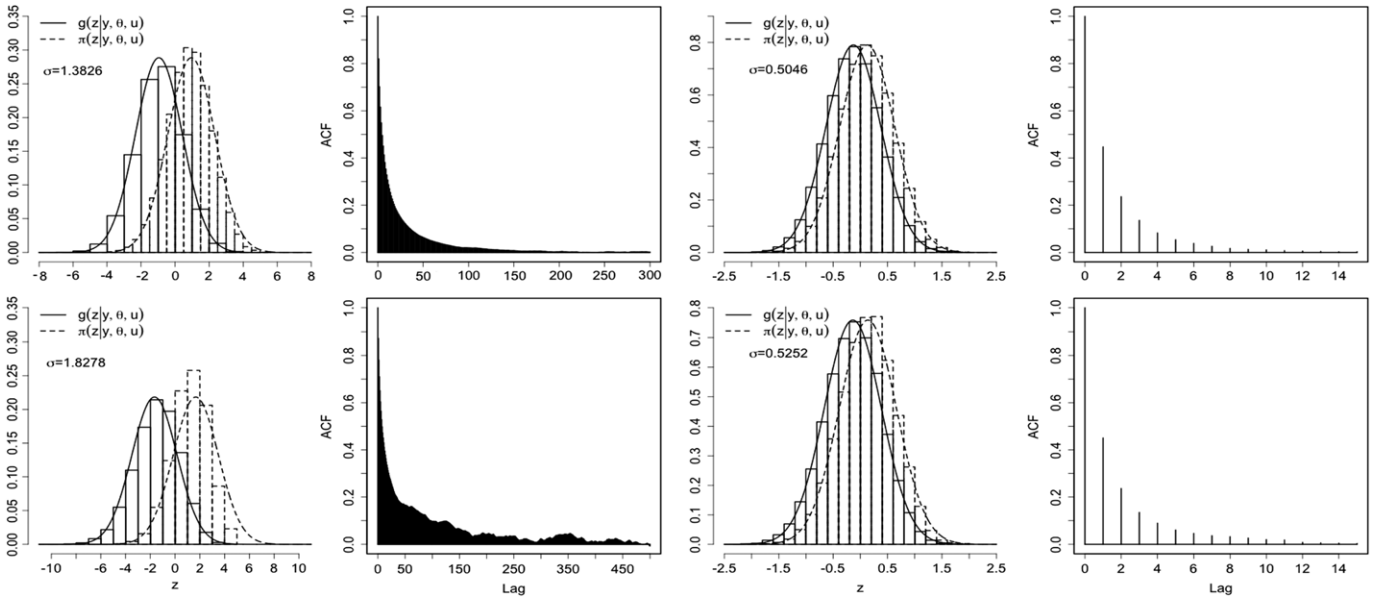


Fig. 5. MCMC output using the standard particle filter for fixed parameters and for two values of σ (and the number of particles N.) The top four panels are for the mixture of AR(1) experts model, with $\sigma = 1.3826$ for the first and second panels from the left and $\sigma = 0.5046$ for the third and fourth panels. The bottom four panels are for the GARCH model observed with noise, with $\sigma = 1.8278$ for the first and second panels from the left and $\sigma = 0.5252$ for the third and fourth panels. In each row, the first and third panels from the left are the theoretical densities $g_N(z|\theta)$ (solid line) and $\pi_N(z|\theta)$ (dotted line) together with the histograms of the draws for the error is the log-likelihood z. The second and fourth panels in each row are the empirical autocorrelations functions for the two values of σ.

Table 3
AR(1) model observed with noise. The table shows the mean, variance and standard deviation of $g_N(z|\theta)$ and $\pi_N(z|\theta)$ for various values of the number of particles for both the standard particle filter and the fully adapted particle filter. The artificial data set has $T = 500$ observations and the parameters are fixed at the true values.

N	$g_N(z\|\theta)$			$\pi_N(z\|\theta)$		
	Mean	Var.	St.dev.	Mean	Var.	St.dev.
	Standard particle filter					
70	−1.7526	3.5433	1.8824	1.6099	3.2008	1.7891
150	−0.8101	1.6450	1.2826	0.8196	1.6214	1.2734
225	−0.5451	1.0953	1.0466	0.5284	1.0687	1.0338
290	−0.4174	0.8421	0.9176	0.4270	0.8459	0.9197
500	−0.2418	0.4881	0.6986	0.2396	0.4863	0.6973
1000	−0.1206	0.2432	0.4931	0.1216	0.2422	0.4922
4000	−0.0302	0.0607	0.2463	0.0285	0.0604	0.2458
	Fully adapted particle filter					
12	−1.8514	3.7345	1.9325	1.7288	3.3914	1.8416
20	−1.1030	2.2326	1.4942	1.0784	2.0974	1.4483
40	−0.5516	1.1029	1.0502	0.5606	1.1161	1.0565
52	−0.4243	0.8501	0.9220	0.4267	0.8591	0.9269
100	−0.2213	0.4394	0.6629	0.2189	0.4387	0.6623
150	−0.1484	0.2954	0.5435	0.1437	0.2951	0.5432
700	−0.0315	0.0630	0.2510	0.0282	0.0624	0.2498

ent numbers of particles. Table 3 summarizes the results, which show that the optimal number of particles for the standard SIR filter is about 290 and for the FAPF it is about 52.

Next, we report the acceptance rates, inefficiencies (IF) and computing times (CT) when the two unknown parameters (ϕ, σ_η^2) are sampled from their posterior distributions. The acceptance rate of a sampling scheme is defined as the percentage of accepted draws; the inefficiency of the sampling scheme for a given parameter is defined as the variance of the parameter estimate divided by its variance if the sampling scheme generates independent iterates. We estimate the inefficiency factor, also known as the integrated autocorrelation time, for a given parameter as $IF = 1 + 2\sum_{j=1}^{L^\star} \hat{\rho}_j$, where $\hat{\rho}_j$ is the estimated autocorrelation of the parameter iterates at lag j, and L^* is that lag after which the estimated autocorrelations are randomly scattered about 0. This is the empirical estimator of (10). We define the computing time as $CT = N \times IF$ when the estimate of the likelihood is obtained by a particle filter.

The PMCMC results are shown in Fig. 6 and Table 4. Already, from Table 3, we would expect from theory that the optimal number of particles, N, should be around 290 and 52 for the standard particle filter and the fully adapted particle filter respectively. We first ran 100,000 iterations of the MCMC method using the true likelihood computed by the Kalman filter. We used both an adaptive independent Metropolis–Hastings (AIMH) algorithm, see Giordani and Kohn (2010), and an adaptive random walk method (ARWM), see Roberts and Rosenthal (2009). The acceptance rate and the inefficiencies were computed and are shown in the entries under Kalman filter at the top of Table 4 (for the two parameters ϕ and σ_η^2).

We then fixed the form of the proposal distribution, which is a mixture of normals, for IMH, and the form of the proposal used for the RWM and applied both forms of proposal where the likelihood is estimated using both the standard particle filter and the fully adapted particle filter. In all cases we ran the PMCMC scheme for 200,000 iterations. Table 4 summarizes the results, based on all draws, in terms of acceptance rates, inefficiencies and computing times for the two types of proposals, the two types of particle filter and for differing values of N.

The results are as expected since they show that the optimal numbers of particles for both particle filters are close to the number that gives the standard deviation of the error in the estimated log likelihood as 0.92. That is, the optimal number of particles for the standard particle filter is about 290 particles and 52 particles for the fully adapted particle filter.

Fig. 6 displays the results corresponding to Table 4 just for the IMH proposal. The top row in Fig. 6 plots the inefficiencies for the two parameters for the standard particle filter divided by the corresponding inefficiencies for the MCMC where the likelihood is evaluated exactly by the Kalman filter, which are both close to 1, as seen at the top of Table 4. We call this relative inefficiency RIF. The top row also plots the $RCT = N \times RIF$, which we call the relative computing time. The bottom row in Fig. 6 is similarly interpreted

Table 4
Acceptance rates, inefficiencies (IF) and computing time (CT) for (a single run of) the Gaussian AR(1) model observed with noise applied to an artificial data set using differing particle filters and number of particles. For the entries under Kalman filter we used the adaptive random walk and the adaptive independent MH algorithms as described in Section 5.2. For the entries under both the standard particle filter and the fully adapted particle filter, both random walk and independent Metropolis–Hastings proposals were fixed for all cases based on a previous run of their respective adaptive Metropolis–Hastings counterparts using the exact likelihood by the Kalman filter. We define the computing time CT $= N \times$ IF/1000 for the particle filters.

Algorithm	Number of particles N	Accept. rate	Inefficiency IF		Computing time CT	
			σ_η^2	ϕ	σ_η^2	ϕ
	Kalman filter					
ARWM	–	34.32	8.50	8.52	–	–
AIMH	–	94.07	1.24	1.16	–	–
	Standard particle filter					
	70	9.81	126.93	82.90	8.885	5.803
	150	18.24	24.66	21.57	3.700	3.235
	225	22.16	16.05	15.73	3.611	3.540
RWM	290	23.78	12.60	13.55	3.654	3.928
	500	27.38	10.27	10.45	5.137	5.227
	1000	30.58	9.38	9.17	9.375	9.168
	4000	32.95	8.70	8.57	34.788	34.289
	70	19.35	42.63	31.81	2.984	2.227
	150	37.16	11.00	9.37	1.650	1.405
	225	46.64	6.01	5.67	1.351	1.275
IMH	290	52.04	4.10	4.05	1.187	1.174
	500	61.81	2.95	2.86	1.476	1.432
	1000	71.74	2.23	2.09	2.232	2.086
	4000	84.46	1.47	1.42	5.873	5.678
	Fully adapted particle filter					
	12	7.40	157.03	125.49	1.884	1.506
	20	12.21	175.44	74.50	3.508	1.490
	40	19.07	21.22	22.24	0.848	0.889
RWM	52	21.48	16.36	18.03	0.851	0.937
	100	26.58	12.06	11.87	1.206	1.187
	150	28.46	10.10	10.27	1.516	1.540
	700	32.84	8.57	8.87	6.002	6.211
	12	13.26	89.22	83.44	1.070	1.001
	20	23.84	21.66	25.18	0.433	0.503
	40	38.98	8.48	8.93	0.3394	0.357
IMH	52	45.27	5.96	5.76	0.3100	0.3000
	100	58.79	3.14	3.09	0.3136	0.3091
	150	65.40	2.61	2.53	0.3917	0.3793
	700	82.46	1.57	1.56	1.098	1.089

for the fully adapted particle filter. Rather than plotting *RCT* and *RIF* against N, we equivalently display them against σ, the standard deviation of the log-likelihood evaluated at the fixed parameter values used for Table 3. We are altering the values of N to get the different values of σ on the horizontal axis providing a comparison with the theoretical results of Fig. 1.

Crucially, in Fig. 6 the inefficiencies *RIF* rise with σ as we would expect. The computing time is a convex function of σ, as expected from Fig. 1, with the minimum achieved at, or very close, to $\sigma = 0.92$. The computing time, given as *RCT* does not greatly vary over the range $\sigma = 0.5$ to $\sigma = 1.5$ as we would expect. The FAPF performs better than the SIR method, as the *RCT* is around 300 at the optimum for the FAPF rather than about 1000 at the optimum for the SIR method.

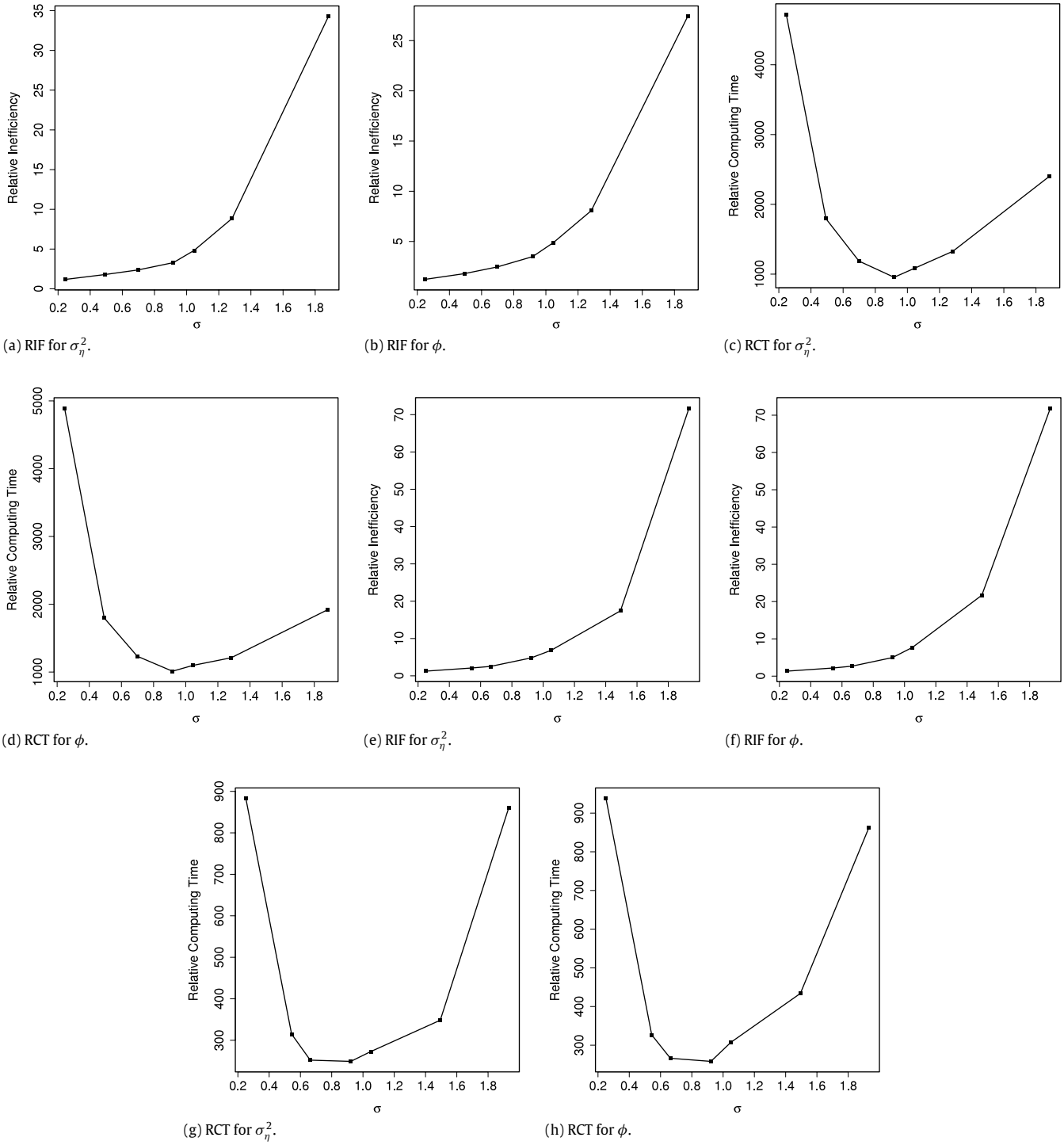
For the FAPF, the corresponding figures are 300 at the optimum σ and 900 at the boundaries of the range of σ. That means that for the standard SIR, the RCT is five times as large on the boundaries as it is at the optimum and is also three to five times larger than the relative computing time of the FAPF.

5.2. GARCH model observed with noise

The GARCH(1,1) model is used extensively to model financial returns (e.g. Bollerslev et al., 1994). In this section we consider the GARCH(1,1) model observed with Gaussian noise, which is described by Eq. (16). Malik and Pitt (2011) show that the model can be reparameterized as a conditional Markov chain in σ_t^2, where it is easy to generate from $p(\sigma_t^2|\sigma_{t-1}^2; y_t)$. We can obtain the fully adapted version of the particle filter for this model because the the measurement density is $y_t|\sigma_t^2, \tau^2 \sim \mathcal{N}(0, \tau^2 + \sigma_t^2)$, with the factor x_t integrated out. Crucially, as the noise $\tau \to 0$, the standard particle filter becomes increasingly inefficient but the adapted form becomes increasingly efficient as $\mathrm{Var}(\sigma_t^2|\sigma_{t-1}^2; y_t) \to 0$. Similar remarks apply to other members of the GARCH family, e.g. an EGARCH process observed with noise.

The parameters of the GARCH(1,1) model with noise in Eq. (16) are required to satisfy the following constraints: $\tau^2 > 0, \alpha > 0, \beta > 0, \gamma > 0$ and $\beta + \gamma < 1$. To facilitate the sampling of the parameters we transform α, β and γ as follows. Let $\phi = \beta + \gamma$, $\mu = \alpha/(1-\phi)$ and $\lambda = \beta/\phi$. Now put $\theta_1 = \mathrm{logit}(\phi)$, $\theta_2 = \log(\mu)$ and $\theta_3 = \mathrm{logit}(\lambda)$ so that θ_1 to θ_3 are unconstrained. The parameter τ^2 is not transformed. The prior on τ^2 is a half normal with the corresponding normal having standard deviation 10, the prior for θ_1 is $\mathcal{N}(3, 1.5^2)$, the prior for θ_2 is $\mathcal{N}(-1, 4^2)$ and the prior for θ_3 is $\mathcal{N}(0, 5^2)$. These priors are mildly informative.

MSCI UK index returns. We model the weekly MSCI UK index returns from 6 January 2000 to 28 January 2010 corresponding to 526 weekly observations. Table 5 summarizes the results of a single long run for each combination of particle filter, number of particles and adaptive Metropolis–Hastings sampling scheme for the parameters $(\tau^2, \theta_1, \theta_2, \theta_3)$. Similar results were obtained for the original GARCH parameterization. The table shows that the fully adapted particle filter performs much better than the standard particle filter both in terms of IF and CT for both adaptive



(a) RIF for σ_η^2. (b) RIF for ϕ. (c) RCT for σ_η^2.

(d) RCT for ϕ. (e) RIF for σ_η^2. (f) RIF for ϕ.

(g) RCT for σ_η^2. (h) RCT for ϕ.

Fig. 6. Relative inefficiencies (RIF) and relative computing times (RCT) for the standard particle filter (top row) and the fully adapted particle filter (bottom row) applied to the AR(1) plus noise model. Full MCMC applied to a single simulated time series with $T = 500$, $\phi = 0.6$ and $\sigma_\eta = 0.8$. RIF is calculated as the estimated integrated autocorrelation time from PMCMC output divided by that from the MCMC with the likelihood calculated by the Kalman filter. RCT $= N \times$ RIF. The top row is for the standard particle filter and the bottom row is for the fully adapted filter, see Table 4 and Section 5.1 for details. An IMH sampling scheme is used to generate the parameters.

sampling schemes for the parameters. For example, a fully adapted particle filter with 50 particles using the adaptive independent Metropolis Hastings sampler is more efficient than a standard particle filter using adaptive independent Metropolis Hastings and 1000 particles.

Next we consider the optimal choice of N in terms of CT for the adaptive independent Metropolis–Hastings sampler. For the standard particle filter, theory combined with Table 2 suggest that the optimal N is a little above $N = 2500$. Similarly, the optimal N for the fully adapted particle filter is a little above $N = 100$. Table 5 confirms these results for the full PMCMC and it is apparent that too small or too large a value of N results in CT rising quite substantially due to many rejections in the former case and expensive computations in the latter

Table 6 summarizes the posterior distributions of the four parameters. The standard deviation of the noise is estimated to be sizable ($\sqrt{E(\tau^2)} = 0.016$ or 1.6%), which makes the model substantially different from a standard GARCH(1,1).

Table 5
Acceptance rates, inefficiencies and computing times for (a single run of) the Gaussian GARCH model observed with noise applied to the UK index return using SIR and the fully adapted particle filters, number of particles (N) and the adaptive independent Metropolis–Hastings algorithms.

N	Accept. rate	Inefficiency				Computing Time/1000			
		τ^2	θ_1	θ_2	θ_3	τ^2	θ_1	θ_2	θ_3
Standard particle filter									
500	8.63	110.07	169.76	231.45	255.31	55.04	84.88	115.73	127.66
1,000	16.43	42.94	48.54	42.17	42.85	42.94	48.54	42.17	42.85
2,500	26.10	11.39	14.40	11.89	13.51	28.48	36.01	29.74	33.77
5,000	33.13	7.37	9.26	7.82	10.27	36.85	46.31	39.11	51.33
10,000	37.02	5.72	8.56	5.58	7.40	57.18	85.60	55.75	74.00
Fully adapted particle filter									
25	9.44	421.11	207.13	128.80	132.27	10.53	5.18	3.22	3.31
50	17.97	31.70	59.45	36.98	47.60	1.58	2.97	1.85	2.38
100	27.24	12.70	17.68	14.03	15.09	1.27	1.77	1.40	1.51
250	36.72	6.46	9.29	7.15	7.34	1.61	2.32	1.79	1.83
500	39.90	4.92	5.09	4.94	5.06	2.46	2.55	2.47	2.53

Table 6
Summary of statistics of the posterior distribution of the parameters for the GARCH(1,1) model with noise fitted to the UK MSCI index returns ($T = 526$). We used a fully adapted PF with $N = 100$ particles.

Parameter	Mean	St.dev.
τ^2	0.0002700	0.0000462
α	0.0000495	0.0000289
β	0.8927539	0.0672126
γ	0.0377854	0.0412842

We coded most of the algorithms in MATLAB, with a small proportion of the code written using C/Mex files. We carried out the estimation on an SGI Altix XE320 with the analysis in Section 5 carried out as follows. The updating schedule of the adaptive independent Metropolis Hastings was at 100, 200, 500, 1000, 1500, 2000, 3000, 4000, 5000, 10,000, 15,000, 20,000, and 50,000 iterates.

5.3. A mixture of experts model of GDP growth observed with noise

Various nonlinear and non-Gaussian features of the business cycle have been noticed since at least Keynes (1936), who believed recessions to be more volatile than expansions, and Friedman (1964), who believed deep recessions to be followed by periods of fast growth. In more recent times several nonlinear models have been estimated on real GDP growth or, less frequently, on industrial production and unemployment, in particular Markov switching models (Hamilton, 1989), and various (smooth) threshold models, (e.g. Tong, 1990; Terasvirta, 1994). To facilitate inference, this literature follows standard econometric practice in assuming that GDP growth is measured accurately. However, this seems unlikely even if revised data are used. The revisions in real GDP data from first to final release are in fact so large that one can only suspect that the final data contain sizable measurement errors (Zellner, 1992).

When adding measurement errors to a regime-switching model, sampling all the states conditional on the parameters and vice versa is a viable option as efficient MCMC samplers exist for this type of model; see Giordani et al. (2007) and Pitt et al. (2010). The same is not true of threshold models, however, which would require slow and sometimes unreliable single move samplers (see Pitt et al., 2010).

This section elegantly solves the problem by using the particle filter to integrate out the states. It illustrates the flexibility and wide applicability of the approach that combines particle filtering with adaptive sampling. All that is necessary for model estimation and model comparison by marginal likelihood is to code up a particle filter to estimate the likelihood and to code up the prior on the parameters.

Table 7
Prior and posterior means and standard deviations for the mixture of AR(1) experts plus noise models. All priors are independent normals.

Param	Prior mean	Prior std	Post. mean	Post. std
$\ln \sigma^2$	0.85	0.2	0.65	0.21
c_1	0	1	−0.11	0.52
ϕ_1	0.5	0.2	0.58	0.18
c_2	2	1	2.17	0.44
ϕ_2	0.5	0.2	0.32	0.14
$\ln \tau_1^2$	0.85	0.2	1.08	0.27
$\ln \tau_2^2$	0.85	0.2	0.51	0.25
ξ_1	0	10	−0.80	1.62
ξ_2	0	10	−2.33	1.31
ξ_2	0	10	−1.53	1.37

We assume that real GDP growth is measured with an error, and that the unobserved underlying process is a mixture of experts (Jacobs et al., 1991). Mixture of experts models are related to smooth threshold models and neural networks, but with a probabilistic rather than deterministic mixing of the experts (or components or regimes). The model is given by the equations at (15) in Section 4.2. Each of the two experts is an AR(1) process (further lags were not needed in our application) with its own intercept, persistence and innovation variance. The first expert is identified as a low growth regime and the constraint $c_1(1-\phi_1)^{-1} < c_2(1-\phi_2)^{-1}$ is imposed by rejection sampling. The probability of the low growth regime is a logistic function of x_{t-1} and $x_{t-1} - x_{t-2}$.

Like most signal plus noise models, this model is fully adapted. In our application the adaptive IMH sampler performed well, quickly reaching an acceptance rate of over 50%.

Data, priors and inference. We model the seasonally adjusted US real log GDP annualized growth from 1984 quarter 2 to 2010 quarter 3 (Source: US Department of Commerce: Bureau of Economic Analysis, series GDPC96, last updated 2010-12-22). Each of the ten model parameters has an independent normal prior. The gating function parameters ξ_1, ξ_2, ξ_3 have dispersed priors. The central moments of the other parameters are calibrated on an AR(1) estimated by OLS, with the OLS error variance split equally between measurement error and transition error, except that $E(c_1) < E(c_2)$ to reflect a prior of low and high growth regimes. Prior and posterior means and standard deviations are summarized in Table 7. The following results stand out: (i) $\ln \sigma_\varepsilon^2$ is sizable ($\sqrt{E(\sigma_\varepsilon^2)} = 1.38$) (ii) x_t in the low growth regimes is more persistent and more volatile than in the high growth regime (iii) the probability of the low growth regime is a negative function of both x_{t-1} and $x_{t-1} - x_{t-2}$, as expected.

6. Discussion

Designing an PMCMC scheme involves two considerations. First, it is important to design a good proposal $q(\theta|\theta')$ to ensure that the acceptance probability is reasonably high when the likelihood is known, i.e. as $N \to \infty$, for the PF estimator. Second, it is crucial, as we have shown, to select the number of particles N appropriately. The choice of the optimal number of particles N in the PMCMC method depends on many factors including the time dimension T, the dimension of the state and the signal to noise ratio. Crucially, we have reduced the problem into a single scalar quantity which is the error in the log of the estimated likelihood (z), for which we know the limiting distribution, in N. This limiting distribution provides an extremely good approximation to the finite N distribution of z over the relevant range of the standard deviation σ.

In practice when the MCMC routine is running it is difficult to determine whether high rejections are a consequence of a bad proposal $q(\theta|\theta')$ or too few particles. Our approach allows N to be chosen quite separately and robustly so that for the hypothetical perfect proposal $q(\theta|\theta') = \pi(\theta)$ the acceptance rate would be around 50% (see Lemma 5). The form of the proposal $q(\theta|\theta')$ can then be determined and if the acceptance rate is very low it will be apparent that this is due to the proposal $q(\theta|\theta')$ rather than the choice of N. This separation is extremely useful in practice as it means there are two separate optimization considerations and two separate problems to resolve.

It is possible using our framework to explore the properties of more general schemes advocated in the PMCMC framework of Andrieu et al. (2010). In particular, in the particle marginal Metropolis–Hastings update of Section 4.4 of their paper, Andrieu et al. (2010) allow for Gibbs type updates of the smoothed path of the states. This is simply a by-product of the algorithm already employed in that having accepted a new value of θ, we then generate from the smoothed path of the state $\widehat{\pi}_N(x|y;\theta)$ implied by the sequential Monte Carlo (SMC) scheme described by them. This is a fairly straightforward scheme to implement and involves randomly choosing (from N) a particle x_T, arising from the filter associated with the new accepted value of θ, and then tracing the ancestry of this particle backwards through time obtaining $x_{T-1}, x_{T-2}, \ldots, x_1$. This gives a full draw from the SMC smoothing density $\widehat{\pi}_N(x|y;\theta)$ and is an invariant sample from the true smoothing density $\pi(x|y;\theta)$. Importantly, the acceptance criterion remains the same as this step is performed after the new value for θ is accepted. This additional SMC step can be useful for two principal reasons. Firstly, the smoothed path of the states (rather than just the filtered path) may be of interest. Secondly, having obtained the sample from the smoothed path we can perform an additional standard MCMC (Gibbs-type) step updating to a new value of θ by targeting $\pi(\theta|y;x)$. In many cases, this density is simple to work with allowing a full update of the parameters.

In the applications, the standard particle filter and the fully adapted particle filters have been considered. The better performance (in terms of reducing the variance or equivalently reducing N) for the fully adapted particle filters suggests that for models where full adaption is not possible, the auxiliary particle filter (APF) may prove to be successful. The performance of the APF depends on how good the state proposal scheme is. However, there are guidelines given on this in Pitt and Shephard (1999) and Pitt and Shephard (2001). Guidelines for the proposals used in the APF have also been considered by, for example, Smith and Santos (2006) in the context of volatility models and by Durham and Gallant (2002) for latent diffusion models. The Durham and Gallant (2002) auxiliary particle (pages 309–312) has the particular advantage that auxiliary particle filter scheme does not degenerate as the number of Euler discretization points becomes large in sharp contrast to the standard particle filter applied to such models. In particular, the variance of the resulting likelihood estimator will be constant as the number of discretization points $M \longrightarrow \infty$.

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Appendix

A.1. Proofs

This section establishes the results of Sections 3.2 and 3.3.

Proof of Lemma 1. We now write $p(u) = p_U(u)$, whenever we need to explicitly identify that we are dealing with the density of u, and recall that the scalar $z = \psi(u;\theta) = \log\{\widehat{p}_N(y|\theta,u)/p(y|\theta)\}$. To avoid ambiguity in this proof we denote the joint posterior density of u and θ as $\widetilde{\pi}_N(u,\theta)$, where this is still defined by the right side of (3). We denote the corresponding conditional density of u given θ as $\widetilde{\pi}_N(u|\theta)$. We will denote the corresponding joint density in the lower dimensional space of z and θ as $\pi_N(z,\theta)$, and the conditional density as $\pi_N(z|\theta)$. From (3), $\widetilde{\pi}_N(u|\theta) = \exp(\psi(u;\theta))p(u)$. Let z^a be an auxiliary term so that there is a one to one transformation from u to (z,z^a). Then we can write $u = u(z,z^a)$ and let $J(z,z^a) = |\det(\partial u(z,z^a)/\partial(z,z^a))|$ be the absolute value of the Jacobian of the transformation. The density of z given θ is $g_N(z|\theta)$ and is given by

$$g_N(z|\theta) = \int p_U(u(z,z^a))J(z,z^a)dz^a,$$

by integrating over the auxiliary variable z^a. Hence,

$$\begin{aligned}\pi_N(z|\theta) &= \int \widetilde{\pi}_N(u(z,z^a)|\theta)J(z,z^a)dz^a \\ &= \exp(z)\int p_U(u(z,z^a)|\theta)J(z,z^a)dz^a \\ &= \exp(z)g_N(z|\theta).\end{aligned}$$

This obtains Part (i) of the lemma. Part (ii) follows from Part (i). Part (iii) follows because $\widehat{p}_N(y|\theta,u)$ is unbiased. □

Proof of Lemma 2. Part (i) follows from part (ii). We obtain part (ii) by applying the second order delta method (see e.g. Billingsley, 1985, p. 368) to obtain Eq. (7). To show part (iii), let $a_N = \gamma(\theta)/\sqrt{N}$ and $b_N = a_N^2/2$ where we omit to show that both a_N and b_N also depend on θ. Then, $b_N/a_N = a_N/2$. For $Z_N \sim g_N(z|\theta)$, the moment generating function of $(Z_N + b_N)/a_N$ is $M_g(s/a_N)\exp(sa_N/2)$ which tends to $\exp(s^2/2)$ as $N \to \infty$ by the central limit theorem. This means that $M_g(s/a_N) \to \exp(s^2/2)$ because $\exp(sa_N/2) \to 1$. Now suppose that $Z_N \sim \pi_N(z|\theta)$. Then the moment generating function of $(Z_N - b_N)/a_N$ is

$$\begin{aligned}&\int \exp\left(\frac{s(z-b_N)}{a_N}\right)\exp(z)g_N(z|\theta)dz \\ &= M_g(s/a_N + 1)(\exp(-a_N s/2)) \longrightarrow \exp(s^2/2),\end{aligned}$$

because $a_N s/2 \to 0$ and $(s/a_N + 1)/(s/a_N) \to 1$ as $N \to \infty$. This completes part (iii) of the proof. □

Proof of Lemma 3. Part (i) is obtained from

$$\begin{aligned}\Pr(A|z',\sigma) &= \int \min\{1,\exp(z-z')\}g(z|\sigma)dz\\ &= \int_{z'}^{\infty}\frac{1}{\sigma}\phi\left(\frac{z}{\sigma}\right)dz + \exp(-z')\int_{-\infty}^{z'}\exp(z)\frac{1}{\sigma}\phi\left(\frac{z}{\sigma}\right)dz\\ &= \Phi\left(-\frac{z'}{\sigma}\right)+\exp\left(-z'+\frac{\sigma^2}{2}\right)\Phi\left(\frac{z'}{\sigma}-\sigma\right).\end{aligned}$$

We use the following known result to obtain Part (ii). Suppose that $Z_1\sim N(\mu_1,\sigma^2)$, $Z_2\sim N(\mu_2,\sigma^2)$ and Z_1, Z_2 are independent. Then

$$\begin{aligned}\Pr(Z_1+Z_2\le 0) &= \Phi\left(\frac{-\mu_1-\mu_2}{\sqrt{2}\sigma}\right)\\ &= \int \Pr(Z_1\le -z_2|z_2)p(z_2)dz_2\\ &= \int \Phi\left(\frac{-z_2-\mu_1}{\sigma}\right)\frac{1}{\sigma}\phi\left(\frac{z_2-\mu_2}{\sigma}\right)dz_2.\end{aligned}$$

From above,

$$\begin{aligned}\Pr(A|\sigma) &= \int \Phi\left(-\frac{z'}{\sigma}\right)\frac{1}{\sigma}\phi\left(\frac{z'-\sigma^2}{\sigma}\right)dz'\\ &\quad+\exp\left(\frac{\sigma^2}{2}\right)\int\exp(-z')\Phi\left(\frac{z'}{\sigma}-\sigma\right)\\ &\quad\times\frac{1}{\sigma}\phi\left(\frac{z'-\sigma^2}{\sigma}\right)dz'\\ &= P_1+P_2,\end{aligned}$$

where

$$P_1=\Phi\left(-\frac{\sigma}{\sqrt{2}}\right)\quad\text{and}$$

$$P_2=\int\Phi\left(\frac{z'}{\sigma}-\sigma\right)\phi\left(\frac{z'}{\sigma}\right)dz'=\Phi\left(-\frac{\sigma}{\sqrt{2}}\right).\quad\square$$

Proof Lemma 4. We need to calculate the integrated autocorrelation time for $\zeta_j=h(\theta_j)$ for a given function $h(.)$. The Markov chain is actually on the joint space $\{\theta_j,z_j\}$ where the acceptance probability is given by (8) and only depends upon the current value z_j and the proposed value z. Although the chain is on this joint space, we really only need to be concerned with the marginal chain $\{z_j\}$ as, under our assumptions, this is generated independently from θ and the acceptance probability in the Metropolis expression, at Eq. (8), only depends upon the current and proposed values of z.

Without any loss of generality we will consider the beginning of the chain as $\{\theta_1,z_1\}$. We shall assume that the invariant distribution, i.e. posterior distribution, has been reached so that (θ_1,z_1) are jointly distributed according to

$$\pi(\theta,z)=\pi(\theta)\pi(z|\sigma)=\pi(\theta)g(z|\sigma)e^z,$$

where we write $\pi_N(z|\theta)$ and $g_N(z|\theta)$ as $\pi(z|\sigma)$ and $g(z|\sigma)$. Let $J=0$ if there is no jump in the θ iterates in the period 1 to $j+1$, with $J=1$ otherwise (at least one jump). Let $\overline{p}(z_1,\sigma)=1-\Pr(A|z_1,\sigma)$, where $\Pr(A|z_1,\sigma)$ is given in Lemma 3. In this context no jump means that all the Metropolis proposals are rejected and so the value of θ and z will remain the same as θ_1 and z_1. Then the probability of no jump over the interval is $\Pr(J=0|z_1,\sigma)=\overline{p}(z_1,\sigma)^j$, the probability of j successive rejections. The probability of at least one acceptance in the Metropolis scheme is $\Pr(J=1|z_1,\sigma)=1-\overline{p}(z_1,\sigma)^j$. Then

$$\begin{aligned}E(\zeta_{j+1}\zeta_1|z_1,\sigma) &= E(\zeta_{j+1}\zeta_1|z_1,\sigma,J=0)\Pr(J=0|z_1,\sigma)\\ &\quad+E(\zeta_{j+1}\zeta_1|z_1,\sigma,J=1)\Pr(J=1|z_1,\sigma).\end{aligned}$$

We note that ζ_{j+1} and ζ_1 are independent if $J=1$ as the proposal for θ is the true marginal posterior $\pi(\theta)$ and the proposal for Z is independent of θ. As a consequence,

$$E(\zeta_{j+1}\zeta_1|z_1,\sigma,J=1)=E_\pi(\zeta)^2=E_\pi[h(\theta)]^2.$$

Similarly, if there is no jump,

$$E(\zeta_{j+1}\zeta_1|z_1,\sigma,J=0)=E_\pi(\zeta^2)=E_\pi[h(\theta)^2].$$

So

$$E(\zeta_{j+1}\zeta_1|z_1,\sigma)=E_\pi(\zeta^2)\overline{p}(z_1,\sigma)^j+E_\pi(\zeta)^2\{1-\overline{p}(z_1,\sigma)^j\},$$

and

$$\begin{aligned}\mathrm{Cov}(\zeta_{j+1},\zeta_1|z_1,\sigma) &= \mathrm{E}(\zeta_{j+1}\zeta_1|z_1,\sigma)-\mathrm{E}(\zeta_{j+1}|z_1,\sigma)\mathrm{E}(\zeta_1|z_1,\sigma)\\ &= E_\pi(\zeta)^2(1-\overline{p}(z_1,\sigma))^j\\ &\quad+E_\pi(\zeta^2)\overline{p}(z_1,\sigma)^j-E_\pi(\zeta)^2\\ &= \{E_\pi(\zeta^2)-E_\pi(\zeta)^2\}\overline{p}(z_1,\sigma)^j\\ &= \mathrm{Var}_\pi(\zeta)\overline{p}(z_1,\sigma)^j,\end{aligned}$$

where the subscripts π indicate expectations and variances under $\pi(\theta)$, the true posterior for θ is the true marginal posterior. Then,

$$\begin{aligned}\mathrm{Cov}(\zeta_{j+1},\zeta_1|\sigma) &= E_{\pi(z_1|\sigma)}[\mathrm{Cov}(\zeta_{j+1},\zeta_1|z_1,\sigma)]\\ &= \mathrm{Var}(\zeta)E_{\pi(z|\sigma)}[\overline{p}(z,\sigma)^j].\end{aligned}$$

Let $\rho_j(\sigma)=\mathrm{Corr}(\zeta_{j+1}\zeta_1|\sigma)=E_{\pi(z|\sigma)}[\overline{p}(z,\sigma)^j]$. Then,

$$\begin{aligned}IF(\sigma) &= 1+2\sum_{j=1}^{\infty}\rho_j(\sigma)=1+2\sum_{j=1}^{\infty}E_{\pi(z|\sigma)}[\overline{p}(z,\sigma)^j]\\ &= E_{\pi(z|\sigma)}\left(1+2\sum_{j=1}^{\infty}\overline{p}(z,\sigma)^j\right),\end{aligned}$$

so that

$$IF(\sigma)=\int\frac{1+\overline{p}(z,\sigma)}{1-\overline{p}(z,\sigma)}\pi(z|\sigma)dz.$$

From Part (i) of Lemma 3

$$\begin{aligned}\overline{p}(z,\sigma) &= 1-\Phi\left(-\frac{z}{\sigma}\right)-\exp\left(-z+\frac{\sigma^2}{2}\right)\Phi\left(\frac{z}{\sigma}-\sigma\right)\\ &= \Phi(w+\sigma)-\exp\left(-\sigma w-\frac{1}{2}\sigma^2\right)\Phi(w)\\ &= p^*(w,\sigma)\end{aligned}$$

where $w=(z-\sigma^2)/\sigma$. Eq. (11) follows. $\square$

Proof of Lemma 5. Minimizing $\mathrm{CT}(\sigma)$ reduces to solving the first order condition for σ, $\frac{\partial\mathrm{IF}(\sigma)}{\partial\sigma}=\frac{2}{\sigma}\mathrm{IF}(\sigma)$. This can be solved numerically in a straightforward way to give $\sigma=0.92$. The rest of part (i) follows. Let $p^*(w,\sigma)$ be defined as in Lemma 4 and put $G(w,\sigma)=(1+p^*(w,\sigma))/(1-p^*(w,\sigma))$. Then,

$$\begin{aligned}G(w,\sigma) &= \frac{1+\Phi(w+\sigma)-\exp\left(-w\sigma-\sigma^2/2\right)\Phi(w)}{\Phi(-w-\sigma)+\exp\left(-w\sigma-\sigma^2/2\right)\Phi(w)}\\ &= \frac{\{1+\Phi(w+\sigma)\}\exp\left(w\sigma+\sigma^2/2\right)-\Phi(w)}{\exp\left(w\sigma+\sigma^2/2\right)\Phi(-w-\sigma)+\Phi(w)}.\end{aligned}$$

Next we use the inequality that for $x>0$, $\Phi(-x)<\phi(x)/x$ to show that for $\sigma+w>0$,

$$\begin{aligned}0 &< \exp\left(w\sigma+\sigma^2/2\right)\Phi(-w-\sigma)\\ &< \exp\left(w\sigma+\sigma^2/2\right)\phi(w+\sigma)/(w+\sigma)\\ &= \phi(w)/(w+\sigma)=O(\sigma^{-1})\end{aligned}$$

$$2>1+\Phi(w+\sigma)>2-\frac{\phi(-w-\sigma)}{w+\sigma}=2-O(\sigma^{-1}),$$

where $O(\sigma^{-1})$ means that $\sigma O(\sigma^{-1})$ is bounded as $\sigma \to \infty$. Hence,

$$G(w,\sigma) = \frac{(2-O(\sigma^{-1}))\exp\left(w\sigma+\sigma^2/2\right)-\Phi(w)}{O(\sigma^{-1})+\Phi(w)} = \frac{2\exp\left(w\sigma+\sigma^2/2\right)}{\Phi(w)} - 1 - O(\sigma^{-1}).$$

Hence,

$$\begin{aligned}\mathrm{IF}(\sigma) &= \int G(w,\sigma)\phi(w)dw \\ &= \int \frac{2\exp\left(w\sigma+\sigma^2/2\right)}{\Phi(w)}\phi(w)\,dw - 1 - O(\sigma^{-1}) \\ &= 2\exp\left(\frac{\sigma^2}{2}\right)\int \exp(w\sigma)\frac{\phi(w)}{\Phi(w)}dw - 1 - O(\sigma^{-1}) \\ &= 2\exp\left(\sigma^2\right)\int \frac{\phi(w-\sigma)}{\Phi(w)}dw - 1 - O(\sigma^{-1}) \\ &= 2\exp\left(\sigma^2\right) - 1 - O(\sigma^{-1}). \quad \square\end{aligned}$$

A.2. General ASIR particle filter

The auxiliary SIR (ASIR) filter of Pitt and Shephard (1999) may be thought of as a generalization of the SIR method of Gordon et al. (1993). We therefore focus on this more general approach. To simplify notation in this section, we omit to show dependence on the unknown parameter vector θ. The densities $g(y_{t+1}|x_t)$ and $g(x_{t+1}|x_t;y_{t+1})$ in Algorithm 1 are approximations to $p(y_{t+1}|x_t)$ and $p(x_{t+1}|x_t;y_{t+1})$ respectively such that we can evaluate $g(y_{t+1}|x_t)$ and generate from $g(x_{t+1}|x_t;y_{t+1})$. Their choice is discussed more fully below. It should be noted that $g(y_{t+1}|x_t)$ can be specified in unnormalized form for the algorithm.

The following algorithm describes the one time step ASIR update and is initialized with samples $x_0^k \sim p(x_0)$ with mass $1/N$ for $k = 1,\ldots,N$.

Algorithm 1. Given samples $x_t^k \sim p(x_t|y_{1:t})$ with mass π_t^k for $k = 1,\ldots,N$.

For $t = 0,\ldots,T-1$:

1. For $k = 1 : N$, compute $\omega_{t|t+1}^k = g(y_{t+1}|x_t^k)\pi_t^k$, $\pi_{t|t+1}^k = \frac{\omega_{t|t+1}^k}{\sum_{i=1}^N \omega_{t|t+1}^i}$.
2. For $k = 1 : N$, sample $\widetilde{x}_t^k \sim \sum_{i=1}^N \pi_{t|t+1}^i \delta(x_t - x_t^i)$.
3. For $k = 1 : N$, sample $x_{t+1}^k \sim g(x_{t+1}|\widetilde{x}_t^k;y_{t+1})$.
4. For $k = 1 : N$, compute

$$\omega_{t+1}^k = \frac{p(y_{t+1}|x_{t+1}^k)p(x_{t+1}^k|\widetilde{x}_t^k)}{g(y_{t+1}|\widetilde{x}_t^k)g(x_{t+1}^k|\widetilde{x}_t^k;y_{t+1})}, \qquad \pi_{t+1}^k = \frac{\omega_{t+1}^k}{\sum_{i=1}^N \omega_{t+1}^i}.$$

To motivate Algorithm 1, we note that the product density $p(y_{t+1}|x_{t+1})p(x_{t+1}|x_t)$ appears in its derivation and can be written as,

$$p(y_{t+1}|x_{t+1})p(x_{t+1}|x_t) = p(y_{t+1}|x_t)p(x_{t+1}|x_t;y_{t+1}),$$

where

$$p(y_{t+1}|x_t) = \int p(y_{t+1}|x_{t+1})p(x_{t+1}|x_t)dx_{t+1},$$

$$p(x_{t+1}|x_t;y_{t+1}) = p(y_{t+1}|x_{t+1})p(x_{t+1}|x_t)/p(y_{t+1}|x_t).$$

If we can evaluate $p(y_{t+1}|x_t)$ and generate from $p(x_{t+1}|x_t;y_{t+1})$, then we can take $g(y_{t+1}|x_t) = p(y_{t+1}|x_t)$ and $g(x_{t+1}|x_t;y_{t+1}) = p(x_{t+1}|x_t;y_{t+1})$ in 1, which Pitt and Shephard (2001) call the fully adapted form of the auxiliary particle filter. In this case Step 4 becomes redundant as $\omega_{t+1}^k = 1$, $(\pi_{t+1}^k = 1/N)$ and the method reduces to what Pitt and Shephard (2001) call the fully adapted algorithm. The fully adapted method is the most efficient in estimating the likelihood and is generally the optimal filter a single time step ahead.

Full adaptation is possible whenever $p(x_{t+1}|x_t)$ is conjugate in x_{t+1} to $p(y_{t+1}|x_{t+1})$. This occurs for example when the observation equation is Gaussian with $p(y_t|x_t) \sim N(H_t x_t, V_t)$ and the state transition equation is of the form $p(x_{t+1}|x_t) \sim \mathcal{N}(\mu(x_t), \Sigma(x_t))$.

The SIR method of Gordon et al. (1993) is a special case of Algorithm 1 when we can generate from $p(x_{t+1}|x_t)$, and we take $g(x_{t+1}|x_t) = p(x_{t+1}|x_t)$ and $g(y_{t+1}|x_t) = 1$. In this case, Step 1 above leaves the weights unchanged (as $\pi_{t|t+1}^k = \pi_t^k$).

In general, the goal of the auxiliary particle filter is to get as close to full adaption as possible, when full adaption is not analytically possible. This is achieved by making $g(y_{t+1}|x_t)$ as close to $p(y_{t+1}|x_t)$ as a function of x_t as possible (up to a constant of proportionality) and the density $g(x_{t+1}|x_t;y_{t+1})$ as close to $p(x_{t+1}|x_t;y_{t+1})$ as possible. Various procedures are available for doing this; see for example, Pitt and Shephard (2001) and Smith and Santos (2006).

The ASIR estimator of $p(y_t|y_{1:t-1})$ that we propose is

$$\widehat{p}_N(y_t|y_{1:t-1}) = \left\{\sum_{k=1}^N \frac{\omega_t^k}{N}\right\}\left\{\sum_{k=1}^N \omega_{t-1|t}^k\right\}, \tag{17}$$

where, in this section and the next, $\widehat{p}_N(y_t|y_{1:t-1})$ and $\widehat{p}_N(y_{1:t})$ mean $\widehat{p}_N(y_t|y_{1:t-1},u,\theta)$ and $\widehat{p}_N(y_{1:t}|u,\theta)$. The two sets of weights ω_t^k and $\omega_{t-1|t}^k$ are defined above and calculated as part of the ASIR Algorithm 1. For full adaption, $\omega_t^k = 1$ and $\omega_{t-1|t}^k = p(y_t|x_{t-1}^k)/N$ and the first summation in (17) disappears. For the SIR method, $\omega_t^k = p(y_t|x_t^k)$ and $\omega_{t-1|t}^k = \pi_{t-1}^k$ and the second summation in (17) disappears. The derivation of this estimator is given below.

The ASIR algorithm (Algorithm 1) is a flexible particle filter approach when combined with stratification. Theorem 1 in Appendix A.3 establishes that this algorithm together with the estimator of (17) is unbiased. This is important as it enables very efficient likelihood estimators from the ASIR method to be used within an MCMC algorithm. Our examples use the standard particle filter and the fully adapted particle filter.

We now define some terms that are used in Algorithm 1 and that will be useful for the proof of Theorem 1 and the derivation of the $\widehat{p}_N(y_t|y_{1:t-1})$.

$$\widehat{p}_N(x_t|y_{1:t}) = \sum_{k=1}^N \pi_t^k \delta(x_t - x_t^k), \quad \text{where } \pi_t^k \text{ is given in Step (4).}$$

$$\widehat{g}_N(x_t|y_{1:t+1}) = \sum_{k=1}^N \pi_{t|t+1}^k \delta(x_t - x_t^k), \quad \text{where } x_t^k \sim \widehat{p}_N(x_t|y_{1:t}) \tag{18}$$

and $\pi_{t|t+1}^k$ is defined in Step (1) of the algorithm.

$$\widehat{g}_N(x_t|y_{1:t}) = \int g(x_t|\widetilde{x}_{t-1};y_t)\widehat{g}_N(\widetilde{x}_{t-1}|y_{1:t})d\widetilde{x}_{t-1}, \tag{19}$$

$$\omega_{t|t+1}(x_t) = g(y_{t+1}|x_t)\pi_t, \omega_{t+1}(x_{t+1};x_t) = \frac{p(y_{t+1}|x_{t+1})p(x_{t+1}|x_t)}{g(y_{t+1}|x_t)g(x_{t+1}|x_t,y_{t+1})}.$$

The term $\widehat{p}_N(x_t|y_{1:t})$ is the empirical filtering density arising from Step 4 of Algorithm 1. The second term $\widehat{g}_N(x_t|y_{1:t+1})$, is the empirical "look ahead" approximation drawn in Step 2. The expression $\widehat{g}_N(x_t|y_{1:t})$ is the filtering approximation which we draw from in Step 3 (integrating over Step 2). Furthermore, we

have that in Algorithm 1, $\omega^k_{t|t+1} = \omega_{t|t+1}(x^k_t)\pi^k_t$ and $\omega^k_{t+1} = \omega_{t+1}(x^k_{t+1}; \widetilde{x}^k_t)$.

We now give a derivation of $\widehat{p}_N(y_t|y_{1:t-1})$ at (17). We note that

$$p(y_t|y_{1:t-1}) \simeq \int\int p(y_t|x_t)p(x_t|x_{t-1})\widehat{p}_N(x_{t-1}|y_{1:t-1})dx_t dx_{t-1}$$
$$= \left\{\sum_{k=1}^N \omega^k_{t-1|t}\right\} \int \omega_t(x_t; x_{t-1})\widehat{g}_N(x_t|y_{1:t})dx_t$$

which leads directly to $\widehat{p}_N(y_t|y_{1:t-1})$.

A.3. Proof that the ASIR likelihood is unbiased

Theorem 1. *The ASIR likelihood*

$$\widehat{p}_N(y_{1:t}) = \widehat{p}_N(y_1)\prod_{t=2}^T \widehat{p}_N(y_t|y_{1:t-1}),$$

is unbiased in the sense that $E(\widehat{p}_N(y_{1:t})) = p(y_{1:t})$.

Del Moral (2004) (Section 7.4.2, Proposition 7.4.1) proves the theorem by showing that the difference of the measure on the states induced by the particle filter and that of the limiting Feyman–Kac measure is a martingale. This appendix proves Theorem 1 using an iterated expectations argument on the estimated likelihood. We believe that our proof which deals specifically with the unbiasedness of the estimated likelihood is simpler and more direct which makes the proof of this fundamental result accessible to a much wider range of readers.

Let us define $\mathcal{A}_t = \{x^k_t; \pi^k_t\}$ as the swarm of particles, for $k = 1, \ldots, N$, at time t. So the particles x^k_t with associated probability π^k_t represent the filtering density $p(x_t|y_{1:t})$ for time t.

Lemma 6.

$$E[\widehat{p}_N(y_t|y_{1:t-1})|\mathcal{A}_{t-1}] = \sum_{k=1}^N p(y_t|x^k_{t-1})\pi^k_{t-1}.$$

Proof.

$$E[\widehat{p}_N(y_t|y_{1:t-1})|\mathcal{A}_{t-1}]$$
$$= E\left[\sum_{k=1}^N \frac{\omega_t(x^k_t; \widetilde{x}^k_{t-1})}{N}|\mathcal{A}_{t-1}\right]\left\{\sum_{j=1}^N \omega^j_{t-1|t}\right\},$$

as the weights $\omega^j_{t-1|t}$ are known given $\mathcal{A}_{t-1}$. So

$$E[\widehat{p}_N(y_t|y_{1:t-1})|\mathcal{A}_{t-1}]$$
$$= \int \omega_t(x_t; \widetilde{x}_{t-1})g(x_t|\widetilde{x}_{t-1}; y_t)\widehat{g}_N(\widetilde{x}_{t-1}|y_{1:t})dx_t d\widetilde{x}_{t-1}\left\{\sum_{j=1}^N \omega^j_{t-1|t}\right\},$$

using the terms (18) and (19),

$$= \int \sum_{k=1}^N \omega_t(x_t; x^k_{t-1})g(x_t|x^k_{t-1}; y_t)\frac{\omega_{t-1|t}(x^k_{t-1})}{\left(\sum_{j=1}^N \omega_{t-1|t}(x^j_{t-1})\right)}dx_t$$
$$\times \left\{\sum_{j=1}^N \omega^j_{t-1|t}\right\}$$
$$= \int \sum_{k=1}^N \omega_t(x_t; x^k_{t-1})g(x_t|x^k_{t-1}; y_t)\omega_{t-1|t}(x^k_{t-1})dx_t$$
$$= \sum_{k=1}^N \int \frac{p(y_t|x_t)p(x_t|x^k_{t-1})}{g(y_t|x^k_{t-1})g(x_t|x^k_{t-1}; y_t)}g(x_t|x^k_{t-1}; y_t)g(y_t|x^k_{t-1})\pi^k_{t-1}dx_t.$$

So

$$E[\widehat{p}_N(y_t|y_{1:t-1})|\mathcal{A}_{t-1}] = \sum_{k=1}^N \pi^k_{t-1}\int p(y_t|x_t)p(x_t|x^k_{t-1})dx_t$$
$$= \sum_{k=1}^N p(y_t|x^k_{t-1})\pi^k_{t-1}. \quad \square$$

Lemma 7.

$$E[\widehat{p}_N(y_{t-h:t}|y_{1:t-h-1})|\mathcal{A}_{t-h-1}] = \sum_{k=1}^N p(y_{t-h:t}|x^k_{t-h-1})\pi^k_{t-h-1}. \quad (20)$$

Proof. The proof is by induction.

Part A. This is true for $h = 0$ by Lemma 6.

Part B. We shall assume that (20) holds for h and show it then holds for $h + 1$.

For the case $h + 1$ the left hand side of (20) is given by,

$$E[\widehat{p}_N(y_{t-h-1:t}|y_{1:t-h-2})|\mathcal{A}_{t-h-2}]$$
$$= E[E[\widehat{p}_N(y_{t-h:t}|y_{1:t-h-1})|\mathcal{A}_{t-h-1}]\widehat{p}_N(y_{t-h-1}|y_{1:t-h-2})|\mathcal{A}_{t-h-2}]$$
$$= E\left[\sum_{k=1}^N p(y_{t-h:t}|x^k_{t-h-1})\pi^k_{t-h-1}\right.$$
$$\left.\times \sum_{i=1}^N \frac{\omega^i_{t-h-1}}{N}\sum_{j=1}^N \omega^j_{t-h-2|t-h-1}|\mathcal{A}_{t-h-2}\right],$$

using (20) for the case h and (17). So, recalling the definition of π_t in Step (4) of the algorithm,

$$E[\widehat{p}_N(y_{t-h-1:t}|y_{1:t-h-2})|\mathcal{A}_{t-h-2}]$$
$$= E\left[\left\{\sum_{k=1}^N p(y_{t-h:t}|x^k_{t-h-1})\frac{\omega^k_{t-h-1}}{\sum_{i=1}^N \omega^i_{t-h-1}}\right\}\right.$$
$$\left.\times \left\{\sum_{i=1}^N \frac{\omega^i_{t-h-1}}{N}\right\}\Bigg|\mathcal{A}_{t-h-2}\right]\left\{\sum_{j=1}^N \omega^j_{t-h-2|t-h-1}\right\},$$

due to the weights $\omega^j_{t-h-2|t-h-1}$ being known given $\mathcal{A}_{t-h-2}$,

$$= E\left[\frac{1}{N}\sum_{k=1}^N p(y_{t-h:t}|x^k_{t-h-1})\omega^k_{t-h-1}|\mathcal{A}_{t-h-2}\right]\left\{\sum_{j=1}^N \omega^j_{t-h-2|t-h-1}\right\}$$
$$= \left\{\sum_{j=1}^N \omega^j_{t-h-2|t-h-1}\right\}\int p(y_{t-h:t}|x_{t-h-1})\omega_{t-h-1}(x_{t-h-1}; \widetilde{x}_{t-h-2})$$
$$\times g(x_{t-h-1}|\widetilde{x}_{t-h-2}; y_{t-h-1})\widehat{g}_N(\widetilde{x}_{t-h-2}|y_{1:t-h-1})dx_{t-h-1}d\widetilde{x}_{t-h-2},$$

using the terms (18) and (19),

$$= \left\{\sum_{j=1}^N \omega^j_{t-h-2|t-h-1}\right\}\int \sum_{k=1}^N p(y_{t-h:t}|x_{t-h-1})$$
$$\times \omega_{t-h-1}(x_{t-h-1}; x^k_{t-h-2})g(x_{t-h-1}|x^k_{t-h-2}; y_{t-1})$$
$$\times \frac{g(y_{t-h-1}|x^k_{t-h-2})\pi^k_{t-h-2}}{\sum_{j=1}^N \omega^j_{t-h-2|t-h-1}}dx_{t-h-1}$$
$$= \sum_{k=1}^N \pi^k_{t-h-2}\int p(y_{t-h:t}|x_{t-h-1})\omega_{t-h-1}(x_{t-h-1}; x^k_{t-h-2})$$
$$\times g(x_{t-h-1}|x^k_{t-h-2}; y_{t-1})g(y_{t-h-1}|x^k_{t-h-2})dx_{t-h-1},$$

using the definition of ω_{t-h-1} (see Step 4 of Algorithm 1),

$$= \sum_{k=1}^{N} \pi_{t-h-2}^{k} \int p(y_{t-h:t}|x_{t-h-1}) p(y_{t-h-1}|x_{t-h-1}) \times p(x_{t-h-1}|x_{t-h-2}^{k}) dx_{t-h-1}$$

$$= \sum_{k=1}^{N} p(y_{t-h-1:t}|x_{t-h-2}^{k}) \pi_{t-h-2}^{k} \quad \text{as required.} \quad \square$$

Proof of Theorem 1. As a consequence we have from lemma that, with $h = t - 2$,

$$E[\widehat{p}_N(y_{1:t})|\mathcal{A}_0] = \sum_{k=1}^{N} p(y_{1:t}|x_0^k)\pi_0^k,$$

where $x_0^k \sim p(x_0)$ and $\pi_0^k = 1/N$,

$$E\left[\sum_{k=1}^{N} p(y_{1:t}|x_0^k)\pi_0^k\right] = \int p(y_{1:t}|x_0)p(x_0)dx_0 = p(y_{1:t}). \quad \square$$

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