CHAPTER 8

# FINITE ELEMENT METHODS FOR SHELLS

## Introduction

In Section 8.1, we give a description of a model of the shell problem, known as *Koiter's model.* The fact that a shell is a body with "small"' thickness makes it possible to use as the only unknown the displacement of the middle surface of the shell and, consequently, there are only two independent variables, namely the curvilinear coordinates of the middle surface of the shell.

Restricting ourselves to the linear theory, it can be shown that *the corresponding strain energy is elliptic.* However, since the proof of this fact is lengthy, we content ourselves to show the ellipticity of the strain energy of an arch (Theorem 8.1.2). This simplificationn is justified, inasmuch as the arch problem is a "model problem" for the shell problem.

In the following two sections, we examine the approximation of such problems by finite element methods. There are essentially five types of such approximations:

(i) The shell is considered as a *three-dimensional body* and, accordingly, three-dimensional isoparametric finite elements are used (therefore, the numerical analysis of such methods is known). Let us add that, in the engineers' experience, this method seems in some cases to be competitive with methods which are specifically based on a two-dimensional model.

(ii) The reduction of a three-dimensional model to a two-dimensional model is performed *not on the continuous problem*, but on the finite element model itself. The principle of this method is very attractive, but little seems to be known as regards its analysis.

(iii) The first example of a finite element method which uses only the two-dimensional model is the "ideal" one (and again its convergence analysis is known): In some instances, the strain energy (which involves partial derivatives of the mapping which defines the middle surface of the shell) and the potential energy of the exterior forces can be *exactly reproduced* in the finite element spaces. This happens in special cases where all coefficients in the energy of the shell are constant functions, such as when the shell is a portion of a right circular cylinder. Incidentally, in this case one may consider that the functions in the finite element spaces are piecewise polynomials expressed in terms of the curvilinear coordinates along the middle surface of the shell.

(iv) In general, the geometry of the shell has to be approximated. This approximation results in an approximate shell or, equivalently, in an approximate energy. In Section 8.2, this type of method is analyzed and, in so doing, we are led to the definition of *conforming finite element methods for shells*. A general convergence result is proved (Theorem 8.2.4), which depends upon a careful comparison between the exact and the approximate strain energies (Theorem 8.2.1).

(v) The last category of finite element methods for shells consists in approximating the geometry in too crude a manner, so that the method is no longer conforming. Following a recent work of C. Johnson, we present in Section 8.3 the corresponding analysis in the case of a circular arch. Here the arch is approximated by straight segments and, consequently, the strain energy is written as a sum of strain energies of "elementary" straight beams. It is proved that such a method is convergent, provided the functions in the finite element spaces satisfy appropriate *compatibility relations*, which essentially compensate for the inadequate approximation of the geometry.

## 8.1. The shell problem

### *Geometrical preliminaries. Koiter's model*

Let $\Omega$ be a bounded subset in a plane $\mathscr{E}^2$, with boundary $\Gamma$. Then a *shell* $\mathscr{S}$ is the image of the set $\bar{\Omega}$ by a mapping $\varphi$: $\bar{\Omega} \subset \mathscr{E}^2 \to \mathscr{E}^3$, where $\mathscr{E}^3$ is the usual Euclidean space. In fact, the surface $\mathscr{S}$ is the *middle surface* of the shell, but since we are only considering "thin" shells, *we shall constantly identify the shell with its middle surface.* The data $\Gamma$ and $\varphi$ are assumed to be sufficiently smooth for all subsequent purposes.

We will denote by $(\boldsymbol{a}, \boldsymbol{b}) \to \boldsymbol{a} \cdot \boldsymbol{b}$, $\|.\|$, and $\boldsymbol{e}^i$, $1 \leqslant i \leqslant 3$, the Euclidean scalar product, Euclidean norm, and an orthonormal basis of the space $\mathscr{E}^3$, respectively.

We shall assume that all points of the shell $\mathscr{S} = \varphi(\bar{\Omega})$ are *regular*, in the sense that the two vectors

$$\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\varphi}, \quad \alpha = 1, 2, \tag{8.1.1}$$

are linearly independent, for all points $\xi = (\xi^1, \xi^2) \in \bar{\Omega}$.

As a rule, we shall use Greek letters: $\alpha, \beta, \tau, \ldots,$ for indices which take their values in the set $\{1, 2\}$, while Latin letters: $i, j, k, \ldots,$ will be used for indices which take their values in the set $\{1, 2, 3\}$. For these indices, we shall use Einstein's convention for summation. Finally, the usual symbols, such as $\partial_\alpha$, $\partial_{\alpha\beta}$, etc..., shall be used also for partial derivatives of *vector-valued* functions of the form $\boldsymbol{\theta} = \theta_i \boldsymbol{e}^i : \bar{\Omega} \subset \mathscr{E}^2 \to \mathscr{E}^3$. Thus, for instance, one has $\partial_\alpha \boldsymbol{\theta} = \partial_\alpha \theta_i \boldsymbol{e}^i$, $\partial_{\alpha\beta} \boldsymbol{\theta} = \partial_{\alpha\beta} \theta_i \boldsymbol{e}^i$, etc...

The vectors $\boldsymbol{a}_\alpha$ are tangent to the *curvilinear coordinate lines* $\varphi(\xi^\beta =$ constant$)$, $\beta \neq \alpha$, and they define the tangent plane at the point $\varphi(\xi)$. We introduce the vector (Fig. 8.1.1)

$$\boldsymbol{a}_3 = \boldsymbol{a}^3 = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{\|\boldsymbol{a}_1 \times \boldsymbol{a}_2\|}. \tag{8.1.2}$$

The *first fundamental form* $(a_{\alpha\beta})$ of the surface is defined by

$$a_{\alpha\beta} = a_{\beta\alpha} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta = \partial_\alpha \boldsymbol{\varphi} \cdot \partial_\beta \boldsymbol{\varphi}. \tag{8.1.3}$$

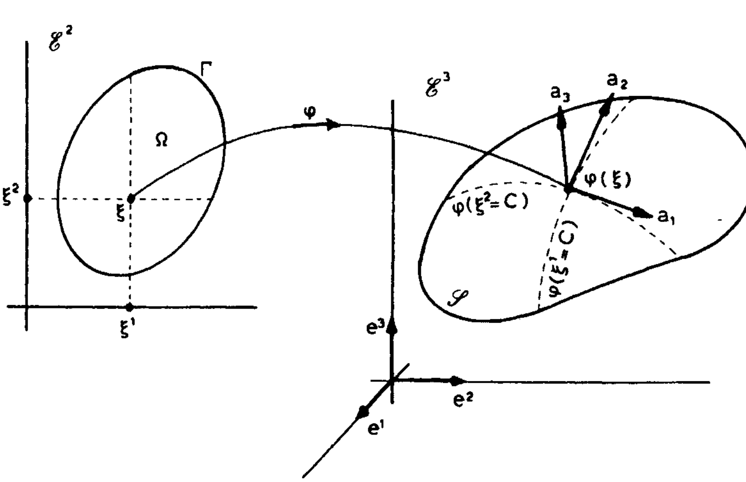


Fig. 8.1.1

With the *covariant basis* $(\boldsymbol{a}_\alpha)$ is associated (Fig. 8.1.2) the *contravariant basis* $(\boldsymbol{a}^\alpha)$ of the tangent plane, which is defined through the relations

$$\boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta^\alpha_\beta \quad \text{(no summation if } \alpha = \beta), \tag{8.1.4}$$

where $\delta^\alpha_\beta$ is the Kronecker symbol. We then have

$$\boldsymbol{a}_\alpha = a_{\alpha\beta}\boldsymbol{a}^\beta, \quad \boldsymbol{a}^\alpha = a^{\alpha\beta}\boldsymbol{a}_\beta, \quad a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta = a^{\beta\alpha}, \tag{8.1.5}$$

where the matrix $(a^{\alpha\beta})$ is the inverse of the matrix $(a_{\alpha\beta})$, which is always invertible since all points are regular, by assumption.

We recall that the *area measure* $\mathrm{d}S$ along the surface $\mathcal{S}$ is given by

$$\mathrm{d}S = \sqrt{a}\,\mathrm{d}\xi, \tag{8.1.6}$$

where

$$a = \det(a_{\alpha\beta}) = a_{11}a_{22} - (a_{12})^2. \tag{8.1.7}$$

We now come to the shell model, which is another example of a familiar problem: The solution $\boldsymbol{u}$, which will be defined below, minimizes the *shell energy*

$$J(\boldsymbol{v}) = \tfrac{1}{2}a(\boldsymbol{v}, \boldsymbol{v}) - f(\boldsymbol{v}), \tag{8.1.8}$$

when the functions vary over an appropriate space $\boldsymbol{V}$. We shall therefore successively define the bilinear form $a(.,.)$, the linear form $f$, and the space $\boldsymbol{V}$.

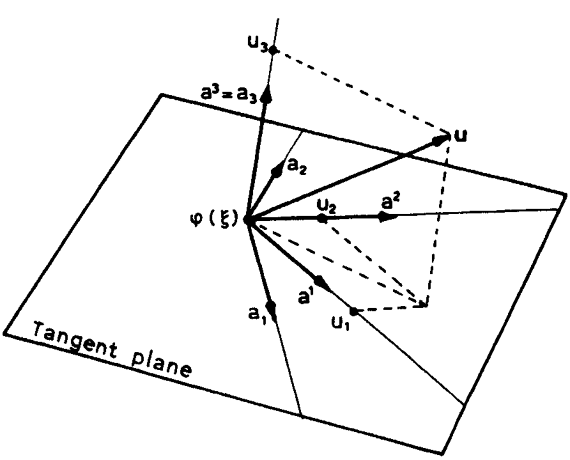


Fig. 8.1.2

*The unknowns are the three functions*

$$u_i: \xi \in \bar{\Omega} \to u_i(\xi) \in \mathbf{R},$$

which represent the *covariant components* of the displacement $\boldsymbol{u} = \boldsymbol{u}(\xi)$ of the point $\boldsymbol{\varphi}(\xi)$. In other words (Fig. 8.1.2), we have

$$\boldsymbol{u} = u_i \boldsymbol{a}^i. \tag{8.1.9}$$

Of course, it should be remembered that *the vectors $\boldsymbol{a}^i$ are also functions of $\xi \in \Omega$.*

The *strain energy* $\frac{1}{2}a(\boldsymbol{v}, \boldsymbol{v})$ *of the shell* is a surface integral:

$$\frac{1}{2}a(\boldsymbol{v}, \boldsymbol{v}) = \int_{\mathcal{S}} A(\cdots)\,\mathrm{d}S = \int_{\Omega} A(\cdots)\sqrt{a}\,\mathrm{d}\xi. \tag{8.1.10}$$

The function $A(\cdots)$ is given by (cf. KOITER (1970), eq. (3.16)):

$$A(\cdots) = \frac{Ee}{2(1-\sigma^2)}\{(1-\sigma)\gamma^\alpha_\beta\gamma^\beta_\alpha + \sigma\gamma^\alpha_\alpha\gamma^\beta_\beta\}$$
$$+ \frac{Ee^3}{24(1-\sigma^2)}\{(1-\sigma)\bar{\rho}^\alpha_\beta\bar{\rho}^\beta_\alpha + \sigma\bar{\rho}^\alpha_\alpha\bar{\rho}^\beta_\beta\}, \tag{8.1.11}$$

where $e$ is the thickness of the shell, $E$ is its Young modulus, $\sigma$ is its Poisson coefficient, and the mixed tensors $(\gamma^\alpha_\beta)$ and $(\bar{\rho}^\alpha_\beta)$ are obtained from the doubly covariant *strain tensor* $(\gamma_{\alpha\beta})$ and *change of curvature tensor* $(\bar{\rho}_{\alpha\beta})$ through the tensorial operations

$$\gamma^\alpha_\beta = a^{\alpha\nu}\gamma_{\nu\beta}, \quad \bar{\rho}^\alpha_\beta = a^{\alpha\nu}\bar{\rho}_{\nu\beta}. \tag{8.1.12}$$

The tensors $(\gamma_{\alpha\beta})$ and $(\bar{\rho}_{\alpha\beta})$ are given by

$$\gamma_{\alpha\beta} = \gamma_{\beta\alpha} = \tfrac{1}{2}(v_{\alpha|\beta} + v_{\beta|\alpha}) - b_{\alpha\beta}v_3, \tag{8.1.13}$$

$$\bar{\rho}_{\alpha\beta} = \bar{\rho}_{\beta\alpha} = v_{3|\alpha\beta} - c_{\alpha\beta}v_3 + b^\lambda_\alpha v_{\lambda|\beta} + b^\lambda_\beta v_{\lambda|\alpha} + b^\lambda_{\alpha|\beta}v_\lambda, \tag{8.1.14}$$

where the various symbols occuring in these expressions will now be defined.

The *second fundamental form* $(b_{\alpha\beta})$ of the surface is given by

$$b_{\alpha\beta} = b_{\beta\alpha} = -\boldsymbol{a}_\alpha \cdot \partial_\beta \boldsymbol{a}_3 = \boldsymbol{a}_3 \cdot \partial_\beta \boldsymbol{a}_\alpha = \tfrac{1}{2}\boldsymbol{a}_3 \cdot (\partial_\beta \boldsymbol{a}_\alpha + \partial_\alpha \boldsymbol{a}_\beta), \tag{8.1.15}$$

from which the *third fundamental form* $(c_{\alpha\beta})$ of the surface is derived by letting

$$c_{\alpha\beta} = c_{\beta\alpha} = b^\lambda_\alpha b_{\lambda\beta}, \quad \text{where} \quad b^\lambda_\alpha = a^{\lambda\beta}b_{\alpha\beta}. \tag{8.1.16}$$

Then the *Christoffel symbols* $(\Gamma^\alpha_{\beta\gamma})$ of the surface are defined by the formulas

$$\Gamma^{\alpha}_{\beta\gamma} = \boldsymbol{a}^{\alpha} \cdot \partial_{\beta}\boldsymbol{a}_{\gamma} = a^{\alpha\nu}\boldsymbol{a}_{\nu} \cdot \partial_{\beta}\boldsymbol{a}_{\gamma} = a^{\alpha\nu}\partial_{\nu}\boldsymbol{\varphi} \cdot \partial_{\gamma\beta}\boldsymbol{\varphi}. \tag{8.1.17}$$

These functions are symmetric with respect to the lower indices in the sense that

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}, \tag{8.1.18}$$

and they satisfy

$$\Gamma^{\alpha}_{\beta\gamma} = a^{\alpha\lambda}\Gamma_{\lambda\beta\gamma}, \quad \text{with} \quad \Gamma_{\alpha\beta\gamma} = \tfrac{1}{2}(\partial_{\gamma}a_{\alpha\beta} + \partial_{\beta}a_{\alpha\gamma} - \partial_{\alpha}a_{\beta\gamma}). \tag{8.1.19}$$

Then the *covariant derivatives* $v_{\alpha|\beta}$, $v_{3|\alpha\beta}$, $b^{\delta}_{\alpha|\beta}$ are given by

$$v_{\alpha|\beta} = \partial_{\beta}v_{\alpha} - \Gamma^{\gamma}_{\alpha\beta}v_{\gamma}, \tag{8.1.20}$$

$$v_{3|\alpha\beta} = \partial_{\alpha\beta}v_3 - \Gamma^{\gamma}_{\alpha\beta}\partial_{\gamma}v_3, \tag{8.1.21}$$

$$b^{\delta}_{\alpha|\beta} = \partial_{\beta}b^{\delta}_{\alpha} + \Gamma^{\delta}_{\beta\lambda}b^{\lambda}_{\alpha} - \Gamma^{\lambda}_{\alpha\beta}b^{\delta}_{\lambda}. \tag{8.1.22}$$

Using the Mainardi–Codazzi and Ricci identities, one can show the equalities

$$b^{\delta}_{\alpha|\beta} = b^{\delta}_{\beta|\alpha}, \tag{8.1.23}$$

which in turn imply the symmetry of the tensor $\bar{\rho}_{\alpha\beta}$ of (8.1.14).

From all the previous formulas, it follows that in the integrand (8.1.11) appearing in the strain energy (8.1.10) of the shell, one finds the three functions $v_i$ and some of their partial derivatives, which we shall sometimes record as the following twelve functions $V_I$, $1 \leq I \leq 12$:

$$(V_I)_{I=1}^{12} = \begin{cases} v_1, \partial_1 v_1, \partial_2 v_1, \\ v_2, \partial_1 v_2, \partial_2 v_2, \\ v_3, \partial_1 v_3, \partial_2 v_3, \partial_{11}v_3, \partial_{12}v_3, \partial_{22}v_3, \end{cases} \tag{8.1.24}$$

while the notation $\boldsymbol{v}$ is reserved for the triple $(v_1, v_2, v_3)$.

Associating as in (8.1.24) twelve functions $U_I$, $1 \leq I \leq 12$, with another generic function $\boldsymbol{u} = (u_1, u_2, u_3)$, we are able to state the main properties of the bilinear form (8.1.10). The proof, which is a matter of lengthy verifications, is left as a problem (Exercise 8.1.1).

**Theorem 8.1.1.** *The bilinear form which occurs in the definition of the strain energy of the shell is of the following form:*

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} \sum_{I,J=1}^{12} A_{IJ}(\xi)U_I V_J \, \mathrm{d}\xi. \tag{8.1.25}$$

*Denoting by $\varphi_i$ the components of the mapping $\varphi = \varphi_i e^i$, we have, for each $(I, J)$:*

$$A_{IJ}(\xi) = f_{IJ}(\partial_\alpha \varphi_i(\xi), \partial_{\alpha\beta}\varphi_i(\xi), \partial_{\alpha\beta\gamma}\varphi_i(\xi)), \tag{8.1.26}$$

*where the function $f_{IJ}$ is a quotient between a polynomial in its arguments and a denominator which is an integer power of the expression*

$$\sqrt{a} = \sqrt{\det(a_{\alpha\beta})} = \left(\left(\sum_{i=1}^{3} (\partial_1\varphi_i)^2\right)\left(\sum_{j=1}^{3} (\partial_2\varphi_j)^2\right) - \left(\sum_{i=1}^{3} \partial_1\varphi_i\partial_2\varphi_i\right)^2\right)^{1/2}. \tag{8.1.27}$$

*The bilinear form is symmetric in its arguments $u$ and $v$ and, finally, it is defined and continuous over the space $H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega)$.* □

The *potential energy of the exterior forces* is another surface integral, of the form

$$f(v) = \int_{\mathcal{S}} f \cdot v \,\mathrm{d}S = \int_\Omega f^i v_i \sqrt{a}\,\mathrm{d}\xi, \tag{8.1.28}$$

where the functions $f^i$ represent the contravariant components, i.e., over the basis $(a_i)$, of the reduced density per unit surface of the exterior forces. Clearly, such a linear form is also continuous over the space $H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega)$.

*Existence of a solution. Proof for the arch problem*

Let then $V$ be a space such as

$$V = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^2(\Omega), \quad \text{or}$$
$$V = H_0^1(\Omega) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)), \tag{8.1.29}$$

which corresponds to the case of a *clamped shell*, and a *simply supported shell*, respectively. Then the problem of showing the existence of a displacement $u \in V$ such that $J(u) = \inf_{v \in V} J(v)$, or equivalently such that $a(u, v) = f(v)$ for all $v \in V$, with $J$, $a(.,.)$ and $f(.)$ as in (8.1.8), (8.1.25) and (8.1.28) respectively, reduces to the problem of showing the $V$-ellipticity of the bilinear form.

This is done in BERNADOU & CIARLET (1976), under the assumption that the mapping $\varphi$ is of class $\mathscr{C}^3(\bar{\Omega})$. Rather than giving the lengthy and fairly intricate proof here, we shall instead focus our attention on a simpler problem, which nevertheless displays all the essential features of the general shell problem: the *arch problem*, where a single variable (instead of two) is needed.

The arch $\mathscr{A}$ is assumed to be in a plane just as the forces which act on it. Then following Fig. 8.1.3, which should be self explanatory as regards the various notations introduced, the *energy of the arch* $\mathscr{A}$ has the following form:

$$J(v)=\frac{1}{2}\int_I\left\{EA\left(v_1'-\frac{v_2}{R}\right)^2+EI\left(\left(v_2'+\frac{v_1}{R}\right)'\right)^2\right\}\mathrm{d}s-\int_I f\cdot v\,\mathrm{d}s. \tag{8.1.30}$$

In this expression, *the parameter $s$ is the curvilinear abcissa along the arch* and thus the vector $a^1=\varphi'$ is a unit vector, the functions $v_1$ and $v_2\colon I\to\mathbf{R}$ are the tangential and normal components of the admissible displacements $v=v_1a^1+v_2a^2$, the function $R\colon \bar{I}\to\mathbf{R}$ is the (algebraically counted) radius of curvature, so that the function $1/R\colon\bar{I}\to\mathbf{R}$ is the *curvature* of the arch. Finally, the constant $E$ is the Young modulus of the material of which the arch is composed, the constant $A$ is the area of a cross-section of the arch and the constant $I$ is the moment of inertia of a cross-section of the arch. Since these three constants are strictly

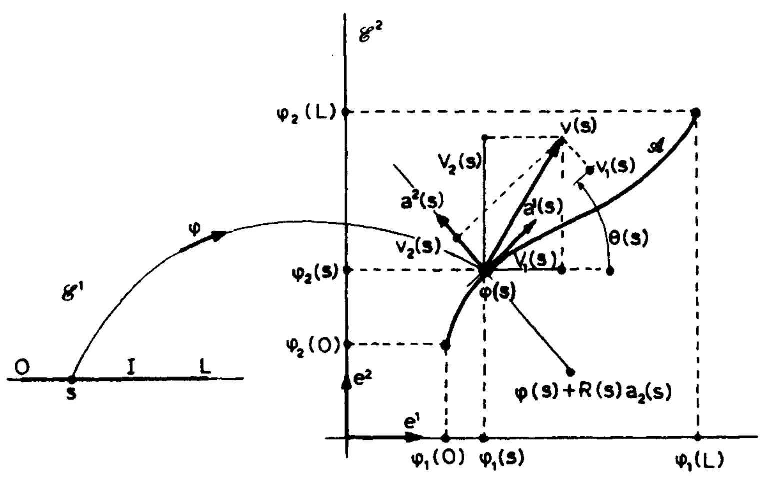


Fig. 8.1.3

positive, there is no loss of generality in assuming that $EA = EI = 1$, as will be henceforth assumed.

In the next theorem, the proof of the ellipticity of the bilinear form is given in a form which is similar to that given in BERNADOU & CIARLET (1976) for a shell.

**Theorem 8.1.2.** *If the function* $1/R$ *is continuously differentiable over the interval* $\bar{I}$, *the bilinear form defined by*

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_I \left\{ \left(u_1' - \frac{u_2}{R}\right)\left(v_1' - \frac{v_2}{R}\right) + \left(u_2' + \frac{u_1}{R}\right)'\left(v_2' + \frac{v_1}{R}\right)' \right\} \mathrm{d}s \tag{8.1.31}$$

*is* $(H_0^1(I) \times (H^2(I) \cap H_0^1(I)))$*-elliptic, and thus, it is a fortiori* $(H_0^1(I) \times H_0^2(I))$*-elliptic.*

**Proof.** We shall equip the space

$$V = H_0^1(I) \times (H^2(I) \cap H_0^1(I)) \tag{8.1.32}$$

with the norm

$$|v| = (|v_1|_{1,I}^2 + |v_2|_{2,I}^2)^{1/2} \tag{8.1.33}$$

(it is easily verified that over the space $H^2(I) \cap H_0^1(I)$, the semi-norm $|\cdot|_{2,I}$ is a norm, equivalent to the norm $\|\cdot\|_{2,I}$).

The proof consists of three steps.

(i) *There exist a constant* $\lambda > 0$ *and a constant* $\mu$ *such that*

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq \lambda |\boldsymbol{v}|^2 + \mu(|v_1|_{0,I}^2 + \|v_2\|_{1,I}^2),$$
$$\text{for all } \boldsymbol{v} = (v_1, v_2) \in H^1(I) \times H^2(I). \tag{8.1.34}$$

Let $\beta = |1/R|_{0,\infty,I}$. For all $\epsilon > 0$, we have

$$\int_I \left(v_1' - \frac{v_2}{R}\right)^2 \mathrm{d}s \geq |v_1|_{1,I}^2 - 2\beta|v_1|_{1,I}|v_2|_{0,I}$$
$$\geq (1 - \epsilon\beta)|v_1|_{1,I}^2 - \frac{\beta}{\epsilon}|v_2|_{0,I}^2,$$

and thus if we choose $\epsilon \in ]0, 1/\beta[$, we have found a constant $\lambda_1 > 0$ and a constant $\mu_1$ such that

$$\int_I \left(v_1' - \frac{v_2}{R}\right)^2 \mathrm{d}s \geq \lambda_1|v_1|_{1,I}^2 + \mu_1|v_2|_{0,I}^2. \tag{8.1.35}$$

Likewise, there exist a constant $\lambda_2 > 0$ and a constant $\mu_2'$ such that

$$\int_I \left(\left(v_2' + \frac{v_1}{R}\right)'\right)^2 \mathrm{d}s \geq \lambda_2 |v_2|_{2,I}^2 + \mu_2' \left|\left(\frac{v_1}{R}\right)'\right|_{0,I}^2.$$

Since

$$\left|\left(\frac{v_1}{R}\right)'\right|_{0,I} \leq \beta |v_1|_{1,I} + \beta^2 |R'|_{0,\infty,I} |v_1|_{0,I},$$

we have found a constant $\lambda_2 > 0$ and two constants $\mu_2$ and $\nu_2$ such that

$$\forall v_1 \in H^1(I), \quad \forall v_2 \in H^2(I),$$

$$\int_I \left(\left(v_2' + \frac{v_1}{R}\right)'\right)^2 \mathrm{d}s \geq \lambda_2 |v_2|_{2,I}^2 + \mu_2 |v_1|_{1,I}^2 + \nu_2 |v_1|_{0,I}^2. \tag{8.1.36}$$

If the constant $\mu_2$ is positive, then inequality (8.1.34) is a direct consequence of inequalities (8.1.35) and (8.1.36). If $\mu_2 < 0$, let $\epsilon$ be so chosen that $0 < \epsilon < \min\{\lambda_1/|\mu_2|, 1\}$. Then

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq \int_I \left(v_1' - \frac{v_2}{R}\right)^2 \mathrm{d}s + \epsilon \int_I \left(\left(v_2' + \frac{v_1}{R}\right)'\right)^2 \mathrm{d}s$$

$$\geq (\lambda_1 - \epsilon|\mu_2|)|v_1|_{1,I}^2 - \epsilon|\nu_2|\,|v_1|_{0,I}^2 + \mu_1 |v_2|_{0,I}^2 + \epsilon\lambda_2 |v_2|_{2,I}^2,$$

and inequality (8.1.34) is proved in all cases.

(ii) *The mapping*

$$\boldsymbol{v} \to \sqrt{a(\boldsymbol{v}, \boldsymbol{v})}$$

*is a norm over the space V.* Clearly it is a semi-norm, so it remains to prove that $a(\boldsymbol{v}, \boldsymbol{v}) = 0$ implies $\boldsymbol{v} = \boldsymbol{0}$, i.e., we face the problem of solving, *in the sense of distributions*, the coupled system of differential equations:

$$\begin{cases} v_1' - \dfrac{v_2}{R} = 0 & \text{on} \quad I, \\ \left(v_2' + \dfrac{v_1}{R}\right)' = 0 & \text{on} \quad I, \end{cases} \tag{8.1.37}$$

along with the boundary conditions:

$$v_1(0) = v_2(0) = v_1(L) = v_2(L) = 0. \tag{8.1.38}$$

As suggested by the geometry of the problem (Fig. 8.1.3), let us introduce the angle $\theta$ between the vectors $\boldsymbol{e}^1$ and $\boldsymbol{a}^1$, so that the following relations hold:

$$\theta' = \frac{1}{R}, \tag{8.1.39}$$

$$\varphi_1' = \cos\theta, \quad \varphi_2' = \sin\theta. \tag{8.1.40}$$

We also introduce the Cartesian components $V_1$ and $V_2$ of the displacement. From the relations

$$\forall s \in I, \quad \boldsymbol{v}(s) = v_1(s)\boldsymbol{a}^1(s) + v_2(s)\boldsymbol{a}^2(s) = V_1(s)\boldsymbol{e}^1 + V_2(s)\boldsymbol{e}^2,$$

we deduce that

$$V_1 = v_1 \cos\theta - v_2 \sin\theta,$$

$$V_2 = v_1 \sin\theta + v_2 \cos\theta.$$

Since the functions $v_1$ and $v_2$ are both in the space $H^1(I)$ and since the function $\theta$ is in the space $\mathscr{C}^1(\bar{I})$, both functions $V_1$ and $V_2$ are in the space $H^1(I)$, and

$$\begin{cases} V_1' = (v_1' - v_2\theta')\cos\theta - (v_2' + v_1\theta')\sin\theta, \\ V_2' = (v_2' + v_1\theta')\cos\theta + (v_1' - v_2\theta')\sin\theta. \end{cases} \tag{8.1.41}$$

The second differential equation of (8.1.37) implies the existence of a constant $a$ such that

$$v_2' + \frac{v_1}{R} = a, \tag{8.1.42}$$

so that we obtain, upon combining relations (8.1.37), (8.1.39), (8.1.40), (8.1.41) and (8.1.42):

$$V_1' + a\varphi_2' = 0,$$

$$V_2' - a\varphi_1' = 0.$$

Therefore, there exist constants $b_1$ and $b_2$ such that

$$V_1 = -a\varphi_2 + b_1,$$

$$V_2 = a\varphi_1 + b_2.$$

Let then $\boldsymbol{e}^3 = \boldsymbol{e}^1 \times \boldsymbol{e}^2$. We have proved that *the general solution of the differential system* (8.1.37) *is of the form*

$$\boldsymbol{v} = \boldsymbol{a} \times \boldsymbol{\varphi} + \boldsymbol{b}, \tag{8.1.43}$$

*where the constant vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *are given by*

$$\boldsymbol{a} = a\boldsymbol{e}^3, \quad \boldsymbol{b} = b_1\boldsymbol{e}^1 + b_2\boldsymbol{e}^2 \tag{8.1.44}$$

(see Remark 8.1.1 for the interpretation of such a solution).

Finally, it is an easy matter to show that any solution of the form (8.1.43) necessarily vanishes when it is subjected to the boundary conditions (8.1.38), since $\varphi(0) \neq \varphi(L)$ by assumption.

(iii) *Using steps* (i) *and* (ii), *we are in a position to show the V-ellipticity of the bilinear form.* If it were not $V$-elliptic, there would exist a sequence $\boldsymbol{v}^k = (v_1^k, v_2^k) \in V$ such that

$$\lim_{k\to\infty} a(\boldsymbol{v}^k, \boldsymbol{v}^k) = 0, \quad |\boldsymbol{v}^k| = 1 \quad \text{for all } k. \tag{8.1.45}$$

Since the sequence $(v_1^k)$ is bounded in $H_0^1(I)$, there exists a subsequence, which we shall still denote by $(v_1^k)$ for convenience, which converges weakly in $H_0^1(I)$ and converges strongly in $L^2(I)$ to the same limit $v_1$.

Likewise, the boundedness of the sequence $(v_2^k)$ in $H^2(I) \cap H_0^1(I)$ implies that there exists a subsequence, still denoted by $(v_2^k)$, which converges weakly in $H^2(I) \cap H_0^1(I)$ and converges strongly in $H^1(I)$ to the same limit $v_2$.

The function $\boldsymbol{v} \in V \to a(\boldsymbol{v}, \boldsymbol{v})$ is continuous for the strong topology of the space $V$ and it is a convex function (it is even a strictly convex function since its second derivative is positive definite, as was shown in step (ii)). Therefore, it is weakly lower semi-continuous. As a consequence, we have, setting $\boldsymbol{v} = (v_1, v_2)$,

$$a(\boldsymbol{v}, \boldsymbol{v}) \leq \liminf_{k\to\infty} a(\boldsymbol{v}^k, \boldsymbol{v}^k) = \lim_{k\to\infty} a(\boldsymbol{v}^k, \boldsymbol{v}^k) = 0,$$

and thus $\boldsymbol{v} = \boldsymbol{0}$ by step (ii). By step (i), we have for all $k$,

$$a(\boldsymbol{v}^k, \boldsymbol{v}^k) \geq \lambda |\boldsymbol{v}^k|^2 + \mu(|v_1^k|_{0,I}^2 + \|v_2^k\|_{1,I}^2)$$

(cf. (8.1.34)). We have therefore reached a contradiction, since

$$\lim_{k\to\infty} a(\boldsymbol{v}^k, \boldsymbol{v}^k) = 0 \quad \text{(cf. (8.1.45))},$$

$$\lim_{k\to\infty} \{\lambda |\boldsymbol{v}^k|^2 + \mu(|v_1^k|_{0,I}^2 + \|v_2^k\|_{1,I}^2)\} = \lambda > 0. \qquad \square$$

**Remark 8.1.1.** By step (ii) in the preceding proof, any displacement $\boldsymbol{v} \in H^1(\Omega) \times H^2(\Omega)$ which satisfies $a(\boldsymbol{v}, \boldsymbol{v}) = 0$ is of the form (8.1.43), i.e., it corresponds to a *rigid body motion* in the plane of the arch. This condition is an instance of the *rigid displacement condition* that a mathematical model for an elastic system should be such that *the vanishing of the strain energy corresponds to rigid body motions* (similar conclusions hold for the system of linear elasticity; cf. Exercise 1.2.4).

In addition, this interpretation provides an approach for integrating in a simple way the differential system (8.1.37) by suggesting the introduction of the functions $\boldsymbol{\theta}$, $V_1$ and $V_2$. □

*Exercises*

***8.1.1.*** Prove all the statements of Theorem 8.1.1.

***8.1.2.*** Let $\bar{\Omega} = [\alpha, \beta] \times [-H, +H]$ and

$$\varphi(\xi) = R \cos \xi^1 \boldsymbol{e}^1 + R \sin \xi^1 \boldsymbol{e}^2 + \xi^2 \boldsymbol{e}^3$$

so that the shell $\mathscr{S} = \varphi(\bar{\Omega})$ is a portion of a right circular cylinder (Fig. 8.1.4).

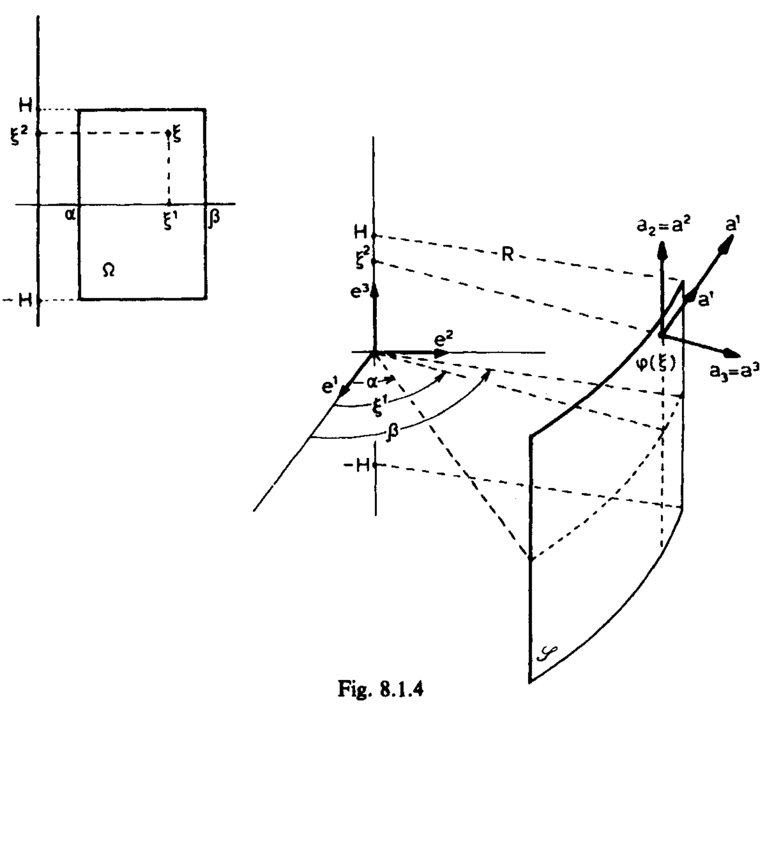


Fig. 8.1.4

Show that the energy of the shell $\mathcal{S}$ has the following expression:

$$J(\boldsymbol{v}) = \frac{Ee}{2(1-\sigma^2)} \int_\alpha^\beta \int_{-H}^{+H} \Big\{(1-\sigma)\Big(\frac{1}{R^4}(\partial_1 v_1 + Rv_3)^2 + \frac{1}{2R^2}(\partial_2 v_1 + \partial_1 v_2)^2 + (\partial_2 v_2)^2\Big) + \sigma\Big(\frac{1}{R^2}(\partial_1 v_1 + Rv_3)^2 + (\partial_2 v_2)^2\Big)\Big\} R \, \mathrm{d}\xi^1 \, \mathrm{d}\xi^2 + \frac{Ee^3}{24(1-\sigma^2)} \int_\alpha^\beta \int_{-H}^{+H} \Big\{(1-\sigma) \times \Big(\frac{1}{R^4}\Big(\partial_{11} v_3 - v_3 - \frac{2}{R}\partial_1 v_1\Big)^2 + \frac{2}{R^2}\Big(\partial_{12} v_3 - \frac{1}{R}\partial_2 v_1\Big)^2 + (\partial_{22} v_3)^2\Big) + \sigma\Big(\frac{1}{R^2}\Big(\partial_{11} v_3 - v_3 - \frac{2}{R}\partial_1 v_1\Big) + \partial_{22} v_3\Big)^2\Big\} R \, \mathrm{d}\xi^1 \, \mathrm{d}\xi^2 - \int_\alpha^\beta \int_{-H}^{+H} \{f^1 v_1 + f^2 v_2 + f^3 v_3\} R \, \mathrm{d}\xi^1 \, \mathrm{d}\xi^2.$$

***8.1.3.*** Let $\bar{\Omega}$ be a rectangle with sides parallel to the coordinate axes and let $\varphi(\xi) = \xi^1 \boldsymbol{e}^1 + \xi^2 \boldsymbol{e}^2$, i.e., the shell is a plate. Is the energy in this case identical to the energy of a plate as given in (1.2.46)?

***8.1.4.*** In the case of a *clamped circular arch* ($R$ = constant), one can give another proof of Theorem 8.1.2, along the following lines, suggested by C. Johnson.

(i) For any $\boldsymbol{v} = (v_1, v_2)$ in the space $V = H_0^1(I) \times H_0^2(I)$, let

$$g_1 = v_1' - \frac{v_2}{R}, \quad g_2 = v_2'' + \frac{v_1'}{R}, \quad F = g_2 - \frac{g_1}{R},$$

and show that

$$\forall s \in I, \quad v_2(s) = \int_0^s R \sin\Big(\frac{1}{R}(s-t)\Big) F(t) \, \mathrm{d}t.$$

(ii) For all $\boldsymbol{v} \in V$, deduce from (i) that $|v_2|_{0,I}^2 \leq c_1 a(\boldsymbol{v}, \boldsymbol{v})$, and then that $|v_2|_{2,I}^2 \leq c_2 a(\boldsymbol{v}, \boldsymbol{v})$ and finally that $|v_1|_{1,I}^2 \leq c_3 a(\boldsymbol{v}, \boldsymbol{v})$, for some constants $c_1$, $c_2$ and $c_3$ independent of $\boldsymbol{v} \in V$.

***8.1.5.*** Consider a *circular arch*, i.e., for which the radius of curvature is a constant. Assuming the solution $\boldsymbol{u} = (u_1, u_2)$ of the associated variational equations is smooth enough, derive the associated system of two differential equations and the boundary conditions corresponding to the choices $V = H_0^1(I) \times H_0^2(I)$ and $V = H_0^1(I) \times (H^2(I) \cap H_0^1(I))$.

## 8.2. Conforming methods

*The discrete problem. Approximation of the geometry. Approximation of the displacement*

We shall assume throughout this section that *the set* $\bar{\Omega}$ *is a polygon* (cf. Remark 8.2.1). Thus we may cover the set $\bar{\Omega}$ by triangulations $\mathcal{T}_h$ in such a way that $\bar{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$, the sets $K$ being the finite elements of the triangulation. *With such a triangulation are associated three finite element spaces* $\Phi_h$, $V_h$, $W_h$, whose specific properties will be subsequently described (actually, the present analysis immediately extends to the case where the spaces $\Phi_h$, $V_h$, $W_h$ would be associated with different triangulations, an unrealistic case from a practical viewpoint, however).

*The discrete problem then requires two approximations.*

(i) *Approximation of the geometry of the surface:*
If $\Theta_h$ denotes the $\Phi_h$-interpolation operator, then with the given mapping $\varphi = \varphi_i e^i$ is associated the *approximate mapping*

$$\varphi_h = \varphi_{ih} e^i, \quad \text{with} \quad \varphi_{ih} = \Theta_h \varphi_i, \quad 1 \le i \le 3. \tag{8.2.1}$$

Notice that if the finite elements of the space $\Phi_h$ are not of class $\mathscr{C}^0$, then the mapping $\varphi_h$ is *a priori* defined only on the union $\bigcup_{K\in\mathcal{T}_h} \mathring{K}$ of the interiors of the finite elements.

(ii) *Approximation of the components of the displacement:* The approximations $u_{\alpha h}$ of the components $u_\alpha$, $\alpha = 1, 2$, belong to the space $V_h$, while the approximation $u_{3h}$ of the component $u_3$ belongs to the space $W_h$. Therefore the discrete solution $\boldsymbol{u}_h = (u_{1h}, u_{2h}, u_{3h})$ is in the space

$$\boldsymbol{V}_h = V_h \times V_h \times W_h. \tag{8.2.2}$$

The *discrete problem* is then defined as follows:
The discrete solution $\boldsymbol{u}_h \in \boldsymbol{V}_h$ is such that

$$\forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \quad a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = f_h(\boldsymbol{v}_h), \tag{8.2.3}$$

where the *approximate bilinear form* $a_h(.,.)$ is given by

$$a_h(\boldsymbol{u}, \boldsymbol{v}) = \sum_{K\in\mathcal{T}_h} \int_{\mathring{K}} \sum_{I,J=1}^{12} A_{IJh}(\xi) U_I V_J \, \mathrm{d}\xi, \tag{8.2.4}$$

with

$$A_{IJh}(\xi) = f_{IJ}(\partial_\alpha \varphi_{ih}(\xi), \partial_{\alpha\beta}\varphi_{ih}(\xi), \partial_{\alpha\beta\gamma}\varphi_{ih}(\xi)), \tag{8.2.5}$$

*the functions $f_{IJ}$ being the same as in Theorem* 8.1.1. In other words, *the coefficients $A_{IJh}$ are expressed in terms of the partial derivatives of the approximate mapping $\varphi_h$ exactly as the coefficients $A_{IJ}$ are expressed in terms of the same partial derivatives of the mapping $\varphi$.*

In the same fashion, the *approximate linear form $f_h$* is given by

$$f_h(v) = \sum_{K \in \mathcal{T}_h} \int_{\hat{K}} f \cdot v \sqrt{a_h} \, \mathrm{d}\xi, \tag{8.2.6}$$

where (compare with (8.1.27))

$$\sqrt{a_h} = \left( \left( \sum_{i=1}^{3} (\partial_1 \varphi_{ih})^2 \right) \left( \sum_{j=1}^{3} (\partial_2 \varphi_{jh})^2 \right) - \left( \sum_{i=1}^{3} \partial_1 \varphi_{ih} \partial_2 \varphi_{ih} \right)^2 \right)^{1/2}. \tag{8.2.7}$$

Notice that the replacement of the functions $A_{IJ}$ by the functions $A_{IJh}$ amounts to replacing each covariant derivative with respect to the surface $\mathcal{S}$ by the analogous covariant derivative with respect to the *approximate surface*

$$\mathcal{S}_h = \varphi_h(\bar{\Omega}). \tag{8.2.8}$$

Therefore *the approximate energy*

$$J_h(v) = \tfrac{1}{2} a_h(v, v) - f_h(v) \tag{8.2.9}$$

*may be viewed either as an approximation of the energy of the shell $\mathcal{S}$* or *as the exact energy of the approximate shell $\mathcal{S}_h$.*

**Remark 8.2.1.** If the boundary $\Gamma$ is *curved*, then a *third approximation* has to be taken into account, and similarly, a *fourth approximation has* to be considered in case *numerical quadrature schemes* are used for computing the coefficients of the resulting linear system. Taking these approximations into account requires an extension of the analysis made in Chapter 4. See BERNADOU (1976). □

*Finite element methods conforming for the displacements*

For the sake of definiteness, we shall assume in the sequel that we are considering the case of a *clamped shell*, i.e., the space $V$ is given by

$$V = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^2(\Omega), \tag{8.2.10}$$

but it should be clear that the subsequent analysis extends readily to other situations, such as that of a simply supported shell, etc...

We shall say that the discrete problem is *conforming for the dis-*

*placements* if the inclusions

$$V_h \subset H_0^1(\Omega), \quad W_h \subset H_0^2(\Omega), \quad \text{i.e.,}\ V_h \subset V,$$

hold (after we have established sufficient conditions for convergence in Theorem 8.2.4, we will also define what may be understood by a discrete problem which is "conforming for the geometry").

As regards the construction of the spaces $\Phi_h$, $V_h$ and $W_h$, let us consider one *example*, as described in ARGYRIS & LOCHNER (1972), ARGYRIS, HAASE & MALEJANNAKIS (1973). Let $K$ be any triangle of the triangulation $\mathcal{T}_h$, with vertices $a_i$ and mid-points $b_i$ along the sides, and where the vectors $\nu_i$ denote the heights of the triangle (Fig. 8.2.1).

Then if the mapping $\varphi: \bar{\Omega} \to \mathscr{E}^3$ is of class $\mathscr{C}^2$, there exists for each triangle $K \in \mathcal{T}_h$ a unique mapping $F_K: K \to \mathscr{E}^3$ such that:

$$
\begin{aligned}
&F_K \in (P_5(K))^3,\\
&F_K(a_i) = \varphi(a_i), \quad 1 \le i \le 3,\\
&DF_K(a_i)(a_{i-1}-a_i) = D\varphi(a_i)(a_{i-1}-a_i), \quad 1 \le i \le 3,\\
&DF_K(a_i)(a_{i+1}-a_i) = D\varphi(a_i)(a_{i+1}-a_i), \quad 1 \le i \le 3,\\
&DF_K(b_i)\nu_i = D\varphi(b_i)\nu_i, \quad 1 \le i \le 3,\\
&D^2F_K(a_i)(a_{j+1}-a_j)^2 = D^2\varphi(a_i)(a_{j+1}-a_j)^2, \quad 1 \le i, j \le 3,
\end{aligned}
$$

where the indices $i$ and $j$ are counted modulo 3, if necessary. We recognize here the *Argyris triangle*, which was introduced in Section 2.2, and whose interpolation properties were analyzed in Section 6.1.

We then choose the approximate mapping $\varphi_h: \bar{\Omega} \to \mathscr{E}^3$ so that, for each

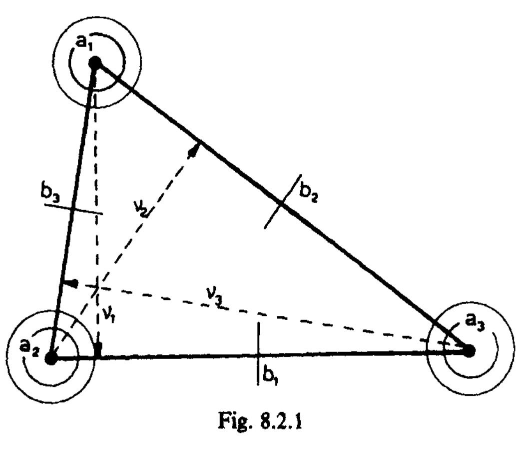


Fig. 8.2.1

$K \in \mathcal{T}_h$, its restriction to the set $K$ coincides with the mapping $F_K$. Once the space $\Phi_h$ is defined in this fashion, we let $V_h$, and $W_h$, be the subspaces of $\Phi_h$ whose functions $v_h$ and $w_h$ satisfy the boundary condition $v_h = 0$ on $\Gamma$, and the boundary condition $w_h = \partial_\nu w_h = 0$ on $\Gamma$, respectively, i.e., with the notations of Chapter 2, we let $V_h = \Phi_{0h}$ and $W_h = \Phi_{00h}$. Since the inclusion $\Phi_h \subset \mathscr{C}^1(\bar{\Omega})$ holds, the inclusions $V_h \subset H_0^1(\Omega)$ and $W_h \subset H_0^2(\Omega)$ surely hold, and this method is therefore conforming for the displacements.

Actually, one has even "too much", in that the inclusion $V_h \subset H_0^1(\Omega) \cap H^2(\Omega)$ holds. However, it is clear that using basically a single finite element space has obvious advantages in terms of the actual numerical implementation of the method. This approach is also similar to that of DUPUIS & GOËL (1970) and DUPUIS (1971), who approximate the *Cartesian* components of the displacement $\boldsymbol{u}$, i.e., over the basis $(\boldsymbol{e}^i)$, so that all three components are in $H^2(\Omega)$, in general.

Let us next return to the general discussion. As far as the error analysis is concerned, it is clear that the ideal situation would correspond to the equality $\varphi = \varphi_h$ which implies the equalities $a_h(.,.) = a(.,.)$ and $f_h(.) = f(.)$. However it is equally clear that this is an exceptional situation. For instance, if we use the Argyris triangle, this would happen only if the restrictions $\varphi_{|K}$ belong to the spaces $(P_5(K))^3$ for all triangles $K \in \mathcal{T}_h$.

Nevertheless, we wish to emphasize the fact that *there are instances where this general approach would yield $\varphi \neq \varphi_h$, while the most straightforward approach yields the equalities $a_h(.,.) = a(.,.)$ and $f_h(.) = f(.)$*. To make this point clear, let us consider the case where the surface $\mathcal{S}$ is a portion of a right circular cylinder, whose energy was given in Exercise 8.1.2. In this case, the energy is expressed uniquely in terms of the functions $v_i$ and their derivatives since the functions $\varphi_i$ and their derivatives appear only as constants. Therefore the obvious discretization of this problem consists in minimizing the *same* energy over the space $V_h$. In this fashion, there is no approximation of the geometry so that one may consider that the approximated displacement are *piecewise polynomials in the curvilinear coordinates which define the surface $\mathcal{S}$.*

Let us assume on the other hand that we had applied the general approach to this particular case. Since the mapping $\varphi$ is given by

$$\varphi(\xi) = R \cos \xi^1 \boldsymbol{e}^1 + R \sin \xi^1 \boldsymbol{e}^2 + \xi^2 \boldsymbol{e}^3 = \varphi_i(\xi)\boldsymbol{e}^i,$$

for $\xi = (\xi^1, \xi^2) \in [\alpha, \beta] \times [-H, +H]$, any standard finite element space $\Phi_h$

(whose functions are essentially piecewise polynomials) would *not* contain the two functions $\varphi_1$ and $\varphi_2$, and thus, this approach would necessarily require an approximation of the energy.

Likewise, between several available mappings $\varphi$ for a given shell $\mathcal{S}$, one should choose the "simplest" one. To illustrate this point, let us consider again the case of a portion of a right circular cylinder. With the same notations as in Fig. 8.1.4, assume that $0<\alpha<\beta<\pi$, so that another possible mapping $\varphi^*$ is given by

$$\varphi^*(\eta)=\eta^1 e^1+\sqrt{R^2-(\eta^1)^2}e^2+\eta^2 e^3=\varphi_i^*(\eta)e^i,$$

for $\eta=(\eta^1,\eta^2)\in[R\cos\alpha, R\cos\beta]\times[-H,+H]$. Then, had we chosen this mapping, some partial derivatives of the function $\varphi_2^*$ would have resulted in non polynomial functions $A_{\beta\lambda}^*(\eta)$, and therefore the energy could not have been exactly reproduced in the subspace $V_h$.

### *Consistency error estimates*

Let us turn to the estimation of the error. In the rest of this section, we shall assume that we are given *three families of finite element spaces $\Phi_h$, $V_h$ and $W_h$*. In order to avoid lengthy statements of theorems, we shall assume throughout this section that, whenever they are needed, hypotheses (H1), (H2), (H3) (cf. Section 3.2) or (H1*), (H2*) (cf. Section 6.1) are satisfied by any one of the above family of finite element spaces. However, we shall record the basic inclusions which govern the orders of convergence, as in (8.2.13), (8.2.25) through (8.2.27).

We denote by $P_K$, $P_K'$, $P_K''$, the spaces spanned by the restrictions to a given finite element $K$ of the functions in the space $\Phi_h$, $V_h$, $W_h$ respectively.

The error analysis depends essentially upon estimates of the expressions $|a(u,v)-a_h(u,v)|$ and $|f(v)-f_h(v)|$, which are derived in the following theorem.

We shall use the product norm

$$v\in H^1(\Omega)\times H^1(\Omega)\times H^2(\Omega)\to\|v\|=(\|v_1\|_{1,\Omega}^2+\|v_2\|_{1,\Omega}^2+\|v_3\|_{2,\Omega}^2)^{1/2}, \tag{8.2.11}$$

which, over the space $V$ of (8.2.10) is equivalent to the semi-norm

$$v\to|v|=(|v_1|_{1,\Omega}^2+|v_2|_{1,\Omega}^2+|v_3|_{2,\Omega}^2)^{1/2}. \tag{8.2.12}$$

**Theorem 8.2.1.** *We assume that the spaces $\Phi_h$ are such that the in-*

*clusions*

$$\forall K \in \bigcup_h \mathcal{T}_h, \quad P_m(K) \subset P_K \subset \mathscr{C}^3(K) \tag{8.2.13}$$

*hold, for a given integer* $m \geqslant 3$. *Then if* $h$ *is sufficiently small, the approximate bilinear form of* (8.2.4) *is also defined over the space* $H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega)$ *and there exists a constant* $C$ *independent of* $h$ *such that, for all* $\boldsymbol{u}, \boldsymbol{v} \in H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega)$,

$$|a(\boldsymbol{u}, \boldsymbol{v}) - a_h(\boldsymbol{u}, \boldsymbol{v})| \leq C\|\boldsymbol{u}\|\,\|\boldsymbol{v}\| h^{m-2}. \tag{8.2.14}$$

*Similarly, if* $h$ *is sufficiently small, there exists a constant* $C$ *independent of* $h$ *such that*,

$$|f(\boldsymbol{v}) - f_h(\boldsymbol{v})| \leq C|\boldsymbol{f}|_{0,\Omega}\|\boldsymbol{v}\| h^m, \tag{8.2.15}$$

*where* $f_h$ *denotes the approximate linear form of* (8.2.6), *and*

$$|\boldsymbol{f}|_{0,\Omega} = \Big(\sum_{i=1}^{3} |f^i|_{0,\Omega}^2\Big)^{1/2}.$$

**Proof.** In view of the assumption (8.2.13) made upon the spaces $\Phi_h$, there exists a constant $C$ independent of $h$ such that

$$\begin{cases} \sup_{\xi \in \cup \hat{K}} |\partial_\alpha \varphi_i(\xi) - \partial_\alpha \varphi_{ih}(\xi)| \leq C h^m |\varphi_i|_{m+1,\infty,\hat{\Omega}}, \\ \sup_{\xi \in \cup \hat{K}} |\partial_{\alpha\beta} \varphi_i(\xi) - \partial_{\alpha\beta} \varphi_{ih}(\xi)| \leq C h^{m-1} |\varphi_i|_{m+1,\infty,\hat{\Omega}}, \\ \sup_{\xi \in \cup \hat{K}} |\partial_{\alpha\beta\gamma} \varphi_i(\xi) - \partial_{\alpha\beta\gamma} \varphi_{ih}(\xi)| \leq C h^{m-2} |\varphi_i|_{m+1,\infty,\hat{\Omega}}, \end{cases} \tag{8.2.16}$$

for all $i \in \{1, 2, 3\}$, and for all $\alpha, \beta, \gamma \in \{1, 2\}$.

Let $\psi_{i\alpha}$, $i \in \{1, 2, 3\}$, $\alpha \in \{1, 2\}$, be real-valued functions defined over the union $\bigcup \hat{K}$. With these functions, we associate the function

$$a(\psi_{i\alpha}) = \Big(\sum_{i=1}^{3} (\psi_{i1})^2\Big)\Big(\sum_{j=1}^{3} (\psi_{j2})^2\Big) - \Big(\sum_{i=1}^{3} \psi_{i1}\psi_{i2}\Big)^2,$$

i.e., the function constructed from the function $\psi_{i\alpha}$ exactly as the function $a$, and the approximate function $a_h$, are constructed from the functions $\partial_\alpha\varphi_i$, and $\partial_\alpha\varphi_{ih}$, respectively (cf. (8.1.27) and (8.2.7)). We then claim that there exist two constants $\delta$ and $a_0 > 0$ such that, for all functions $\psi_{i\alpha}$ which satisfy the uniform bound

$$\sup_{\xi \in \cup \hat{K}} |\partial_\alpha \varphi_i(\xi) - \psi_{i\alpha}(\xi)| \leq \delta,$$

then

$$\inf_{\xi \in \cup \hat{K}} a(\psi_{i\alpha}(\xi)) \geq a_0 > 0. \tag{8.2.17}$$

To see this, we remark that $a(\psi_{i\alpha})$ is the square of the norm of the vector $(\psi_{i1}e^i) \times (\psi_{i2}e^i)$. Since the norms of the vectors $a_\alpha(\xi) = \partial_\alpha\varphi_i(\xi)e^i$ are bounded below by a strictly positive constant independent of the point $\xi \in \bar{\Omega}$, this property is also true of all corresponding vectors $\psi_{i\alpha}e^i$ for a sufficiently small quantity $\sup_{\xi \in \cup \hat{K}} |\partial_\alpha\varphi_i(\xi) - \psi_{i\alpha}(\xi)|$. Likewise, since the cosine of the angle between the two vectors $a_\alpha(\xi)$ is bounded away from 1, independently of $\xi \in \bar{\Omega}$, we deduce that, for a sufficiently small quantity $\sup_{\xi \in \cup \hat{K}} |\partial_\alpha\varphi_i(\xi) - \psi_{i\alpha}(\xi)|$, the cosine of the angle between all corresponding vectors $\psi_{i\alpha}e^i$ has the same property. Thus the modulus of their vector product is certainly bounded below by a strictly positive constant independent of the point $\xi \in \bar{\Omega}$.

Let then $h_0$ be such that

$$\forall h \leq h_0, \quad \sup_{\xi \in \cup \hat{K}} |\partial_\alpha\varphi_i(\xi) - \partial_\alpha\varphi_{ih}(\xi)| \leq \delta, \tag{8.2.18}$$

which is certainly possible, in view of the first of the uniform bounds given in (8.2.16). Since the only denominators which may occur in the functions $A_{IJ}$ (resp. the functions $A_{IJh}$) are integer powers of the function $\sqrt{a}$ (resp. the function $\sqrt{a_h}$), as was stated in Theorem 8.1.1, and since these same functions are otherwise regular, we deduce that the approximate bilinear form $a_h(.,.)$ is well defined over the space $V$ for all $h \leq h_0$. To compare it with the bilinear form $a(.,.)$, we observe that

$$\forall u, v \in V, \quad |a(u, v) - a_h(u, v)| \leq \gamma_h \|u\| \|v\|, \tag{8.2.19}$$

where

$$\gamma_h = \sum_{I,J=1}^{12} \sup_{\xi \in \cup \hat{K}} |A_{IJ}(\xi) - A_{IJh}(\xi)|.$$

Using again Theorem 8.1.1, and the definition (8.2.5) of the functions $A_{IJh}$, we obtain

$$A_{IJ}(\xi) - A_{IJh}(\xi) = f_{IJ}(\partial_\alpha\varphi_i(\xi), \partial_{\alpha\beta}\varphi_i(\xi), \partial_{\alpha\beta\gamma}\varphi_i(\xi)) - f_{IJ}(\partial_\alpha\varphi_{ih}(\xi), \partial_{\alpha\beta}\varphi_{ih}(\xi), \partial_{\alpha\beta\gamma}\varphi_{ih}(\xi)).$$

Since all points $(\psi_{i\alpha}(\xi), \psi_{i\alpha\beta}(\xi), \psi_{i\alpha\beta\gamma}(\xi))$ of the segments joining the points $(\partial_\alpha\varphi_i(\xi), \partial_{\alpha\beta}\varphi_i(\xi), \partial_{\alpha\beta\gamma}\varphi_i(\xi))$ and $(\partial_\alpha\varphi_{ih}(\xi), \partial_{\alpha\beta}\varphi_{ih}(\xi), \partial_{\alpha\beta\gamma}\varphi_{ih}(\xi))$ are such that

$$\forall \xi \in \bigcup \mathring{K}, \quad |\psi_{i\alpha}(\xi) - \partial_\alpha\varphi_i(\xi)| \leq |\partial_\alpha\varphi_{ih}(\xi) - \partial_\alpha\varphi_i(\xi)| \leq \delta$$

for all $h \leq h_0$ by (8.2.18), it follows that the functions $f_{IJ}$ are continuously differentiable along these segments for all $\xi$. Since, in addition, all these points are in a compact subset of $\mathbf{R}^{27}$ of the form (cf. (8.2.17))

$$\{(X_{i\alpha}, X_{i\alpha\beta}, X_{i\alpha\beta\gamma}) \in \mathbf{R}^{27};\ |X_{i\alpha}|, |X_{i\alpha\beta}|, |X_{i\alpha\beta\gamma}| \leq \rho,\ a(X_{i\alpha}) \geq a_0\},$$

it follows that, along these segments, all partial derivatives of the first order of the functions $f_{IJ}$ (with respect to the arguments $X_{i\alpha}$, $X_{i\alpha\beta}$, $X_{i\alpha\beta\gamma}$) are bounded above in the norm $|.|_{0,\infty,\cup\mathring{K}}$ by some constants $M_{i\alpha}$, $M_{i\alpha\beta}$ and $M_{i\alpha\beta\gamma}$, respectively. Therefore, an application of Taylor's formula yields:

$$\begin{aligned}\gamma_h &= \sum_{I,J=1}^{12} \sup_{\xi\in\cup\mathring{K}} |A_{IJ}(\xi) - A_{IJh}(\xi)| \\ &\leq 144\Big(\sum_{i,\alpha} M_{i\alpha} \sup_{\xi\in\cup\mathring{K}} |\partial_\alpha\varphi_i(\xi) - \partial_\alpha\varphi_{ih}(\xi)| \\ &\quad + \sum_{i,\alpha\beta} M_{i\alpha\beta} \sup_{\xi\in\cup\mathring{K}} |\partial_{\alpha\beta}\varphi_i(\xi) - \partial_{\alpha\beta}\varphi_{ih}(\xi)| \\ &\quad + \sum_{i,\alpha\beta\gamma} M_{i\alpha\beta\gamma} \sup_{\xi\in\cup\mathring{K}} |\partial_{\alpha\beta\gamma}\varphi_i(\xi) - \partial_{\alpha\beta\gamma}\varphi_{ih}(\xi)|\Big),\end{aligned} \tag{8.2.20}$$

and the conclusion follows by combining inequalities (8.2.16), (8.2.19) and (8.2.20).

The difference $|f(\boldsymbol{v}) - f_h(\boldsymbol{v})|$ is studied analogously. Since partial derivatives of the first order only (of the functions $\varphi_i$) appear in the surface element $\mathrm{d}S = \sqrt{a}\,\mathrm{d}\xi$, we are in this case led to the exponent $m$. □

As was indicated in the previous section, the bilinear form $a(.,.)$ is $V$-elliptic, i.e., there exists a constant $\alpha > 0$ such that for all $\boldsymbol{v} = (v_1, v_2, v_3) \in V = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^2(\Omega)$, one has

$$\alpha |\boldsymbol{v}|^2 \leq a(\boldsymbol{v}, \boldsymbol{v}), \tag{8.2.21}$$

where the norm $|.|$ has been defined in (8.2.12).

**Theorem 8.2.2.** *Assume that the discrete problems are conforming for* *the displacements, and that the spaces $\Phi_h$ are such that the inclusions* (8.2.13) *are satisfied for all $K \in \bigcup_h \mathcal{T}_h$.*

*Then, if $h$ is sufficiently small, the approximate bilinear form $a_h(.,.)$ is $V_h$-elliptic and therefore, the discrete problem has a unique solution.*

*The bilinear forms $a_h(.,.)$ are also $V$-elliptic and continuous over the space $V$, uniformly with respect to $h$, in that there exist two constants $\tilde{\alpha} > 0$ and $\tilde{M}$ such that for all $h$ sufficiently small,*

$$\forall v \in V, \quad \tilde{\alpha}\|v\|^2 \leq a_h(v, v), \tag{8.2.22}$$

$$\forall u, v \in V, \quad |a_h(u, v)| \leq \tilde{M}\|u\|\,\|v\|. \tag{8.2.23}$$

**Proof.** Let $C$ be the constant appearing in inequality (8.2.14). Then using inequality (8.2.21), we find that, for all $v \in V$,

$$\begin{aligned} a_h(v, v) &= a(v, v) + (a_h(v, v) - a(v, v)) \\ &\geq \alpha |v|^2 - Ch^{m-2}\|v\|^2, \end{aligned}$$

and thus there exists a constant $\tilde{\alpha} > 0$ such that inequalities (8.2.22) hold, provided $h$ is sufficiently small. Likewise, the bilinear form $a(.,.)$ being continuous, there exists a constant $M$ such that, for all $u, v \in V$,

$$\begin{aligned} |a_h(u, v)| &= |a(u, v) + (a_h(u, v) - a(u, v))| \\ &\leq (M + Ch^{m-2})\|u\|\,\|v\|, \end{aligned}$$

which proves the validity of inequality (8.2.23). □

*Abstract error estimate*

Thus we have another instance of a family of discrete problems for which the associated bilinear forms are *uniformly $V_h$-elliptic.* With this property as our main assumption, we first derive an abstract upper bound for the error. As usual, *consistency conditions* can be derived from inequality (8.2.24) below.

**Theorem 8.2.3.** *Given a family of discrete problems conforming for the displacements, for which the inequalities* (8.2.22) *and* (8.2.23) *hold for all $h$, there exists a constant $C$ independent of $h$ such that*

$$\begin{aligned} \|u - u_h\| \leq C\Big(&\inf_{v_h \in V_h} \|u - v_h\| + \sup_{w_h \in V_h} \frac{|a(u, w_h) - a_h(u, w_h)|}{\|w_h\|} \\ &+ \sup_{w_h \in V_h} \frac{|f(w_h) - f_h(w_h)|}{\|w_h\|}\Big). \end{aligned} \tag{8.2.24}$$

**Proof.** Let $v_h$ be an arbitrary element in the space $V_h$. We may write

$$\begin{aligned}\tilde{\alpha}\|v_h - u_h\|^2 &\leq a_h(v_h - u_h, v_h - u_h)\\ &= a_h(v_h - u, v_h - u_h) + \{a_h(u, v_h - u_h) - a(u, v_h - u_h)\}\\ &\quad + \{f(v_h - u_h) - f_h(v_h - u_h)\},\end{aligned}$$

from which we deduce

$$\begin{aligned}\tilde{\alpha}\|v_h - u_h\| &\leq \tilde{M}\|u - v_h\| + \frac{|a_h(u, v_h - u_h) - a(u, v_h - u_h)|}{\|v_h - u_h\|}\\ &\quad + \frac{|f(v_h - u_h) - f_h(v_h - u_h)|}{\|v_h - u_h\|}\\ &\leq \tilde{M}\|u - v_h\| - \sup_{w_h \in V_h} \frac{|a_h(u, w_h) - a(u, w_h)|}{\|w_h\|}\\ &\quad + \sup_{w_h \in V_h} \frac{|f(w_h) - f_h(w_h)|}{\|w_h\|},\end{aligned}$$

and the conclusion follows by combining the above inequality with the triangular inequality

$$\|u - u_h\| \leq \|u - v_h\| + \|v_h - u_h\|. \qquad \square$$

*Estimate of the error* $\left(\sum_{\alpha=1}^{2} \|u_\alpha - u_{\alpha h}\|_{1,\Omega}^2 + \|u_3 - u_{3h}\|_{2,\Omega}^2\right)^{1/2}$

We are now in a position to obtain sufficient conditions for convergence (to shorten the statement of the next theorem, it is to be implicitly understood that possible additional hypotheses upon the integers $k$ and $l$ may be needed so as to insure that the $V_h$-interpolation operator, or the $W_h$-interpolation operator, are well defined).

**Theorem 8.2.4.** *Assume that the discrete problems are conforming for the displacements and that the spaces $\Phi_h$, $V_h$ and $W_h$ are such that, for all $\mathcal{T}_h$ and all $K \in \mathcal{T}_h$,*

$$P_m(K) \subset P_K \subset \mathscr{C}^3(K) \tag{8.2.25}$$

*for some integer $m \geq 3$,*

$$P_k(K) \subset P_K' \subset H^1(K) \tag{8.2.26}$$

*for some integer $k \geqslant 1$,*

$$P_l(K) \subset P_K'' \subset H^2(K) \tag{8.2.27}$$

*for some integer $l \geqslant 2$, respectively.*

*Then if the solution $u = (u_1, u_2, u_3)$ belongs to the space*

$$H^{k+1}(\Omega) \times H^{k+1}(\Omega) \times H^{l+1}(\Omega), \tag{8.2.28}$$

*there exists a constant $C$ independent of $h$ such that*

$$\|u - u_h\| \leqslant Ch^{\min\{k,l-1,m-2\}}. \tag{8.2.29}$$

**Proof.** One has

$$\inf_{v_h \in V_h} \|u - v_h\| \leqslant \|u - \Pi_h u\| = (\|u_1 - \Pi_h u_1\|_{1,\Omega}^2 + \|u_2 - \Pi_h u_2\|_{1,\Omega}^2 + \|u_3 - \Lambda_h u_3\|_{2,\Omega}^2)^{1/2},$$

where $\Pi_h u = (\Pi_h u_1, \Pi_h u_2, \Lambda_h u_3)$ is the $V_h$-interpolant of the solution $u$. Since it follows that $\Pi_h u_\alpha$ and $\Lambda_h u_3$ are the $V_h$-interpolants of the function $u_\alpha$ and the $W_h$-interpolant of the function $u_3$, respectively, an application of the standard error estimates shows that

$$\inf_{v_h \in V_h} \|u - v_h\| \leqslant C\{(|u_1|_{k+1,\Omega} + |u_2|_{k+1,\Omega})h^k + |u_3|_{l+1,\Omega} h^{l-1})\},$$

for some constant $C$ independent of $h$.

From inequalities (8.2.14) and (8.2.15) of Theorem 8.2.1, we derive the *consistency error estimates*:

$$\sup_{w_h \in V_h} \frac{|a(u, w_h) - a_h(u, w_h)|}{\|w_h\|} \leqslant C\|u\|h^{m-2},$$

$$\sup_{w_h \in V_h} \frac{|f(w_h) - f_h(w_h)|}{\|w_h\|} \leqslant C|f|_{0,\Omega} h^m,$$

and the conclusion follows by combining the last three inequalities and inequality (8.2.24) of Theorem 8.2.3. □

For instance, this result shows that the Argyris triangle yields an $O(h^3)$ convergence since it corresponds to the values $k = l = m = 5$. This is to be compared with the $O(h^4)$ convergence which it yields for plates: *the decrease of one in the order of convergence is due to the approximation of the geometry.*

**Remark 8.2.2.** In some shell models, partial derivatives of orders only 1 and 2 of the mapping $\tilde{\varphi}$ appear in the functions $A_{IJ}$. For such models, the analogues of Theorems 8.2.1 and 8.2.4 hold with the exponent $(m-1)$ instead of $(m-2)$. □

*Finite element methods conforming for the geometry*

In view of Theorem 8.2.4, we shall say that a finite element method is *conforming for the geometry if the inclusions*

$$\forall K \in \mathcal{T}_h, \quad P_m(K) \subset P_K \subset \mathscr{C}^3(K)$$

*hold for some integer* $m \geq 3$, so that we may obtain convergence with these sole conditions (as regards the geometry).

In this definition, it is unexpected that *no continuity is required across adjacent finite elements for the functions in the space* $\Phi_h$, and this is a conclusion which differs from the requirement, usually found in the engineering literature, that the inclusion $\Phi_h \subset \mathscr{C}^1(\bar{\Omega})$ should hold. We believe that the origin of this difference is that there are essentially two points of view:

Either one can argue in terms of the approximate surface $\mathscr{S}_h = \varphi_h(\bar{\Omega})$ (cf. (8.2.8)) and, for physical reasons, this imposes some regularity requirements (such as $\mathscr{C}^1$-continuity) on the mapping $\varphi_h$. In this interpretation, one may think of the discrete solution $u_h(\xi)$ as a displacement attached to the point $\varphi_h(\xi)$. Or one can consider that the main objective is to get a good approximation of the bilinear form $a(.,.)$: From this point of view, the proof of Theorem 8.2.1 shows that possible discontinuities of the approximate mapping $\varphi_h$ along sides common to adjacent finite elements are irrelevant. What matters is only that sufficiently good "local" uniform approximations of the coefficients $A_{IJ}$ can be obtained and this is exactly a consequence of the definition of a conforming method for the geometry. Let us add that in this second interpretation, which is chosen here, we think of the three functions $u_{ih}$ *defined over the set* $\bar{\Omega}$ as approximations of the three functions $u_i$ *also defined over the set* $\bar{\Omega}$, even if these functions, by means of the coordinate system $(\varphi(\xi), a^i(\xi))$, allow to derive the displacement of the point $\varphi(\xi)$.

*Conforming finite element methods for shells*

In light of the preceding analysis, we shall say that *a finite element method for solving the shell problem is conforming* if it is both conforming for the displacement and conforming for the geometry, in the senses understood in this section. Consequently, *a finite element method for solving the shell problem will be called nonconforming* if it is not conforming in the previous sense.

## 8.3. A nonconforming method for the arch problem

*The circular arch problem*

Our purpose is to analyze a *nonconforming method* for solving the simplest problem similar to the shell problem: *the circular arch problem.*

We consider a circular arch of radius $R$ (Fig. 8.3.1) and, for definiteness, we shall assume that the arch is clamped. Setting the physical constants EA and EI equal to one in the energy (8.1.30), the variational problem corresponds to the following data (notice the change of sign

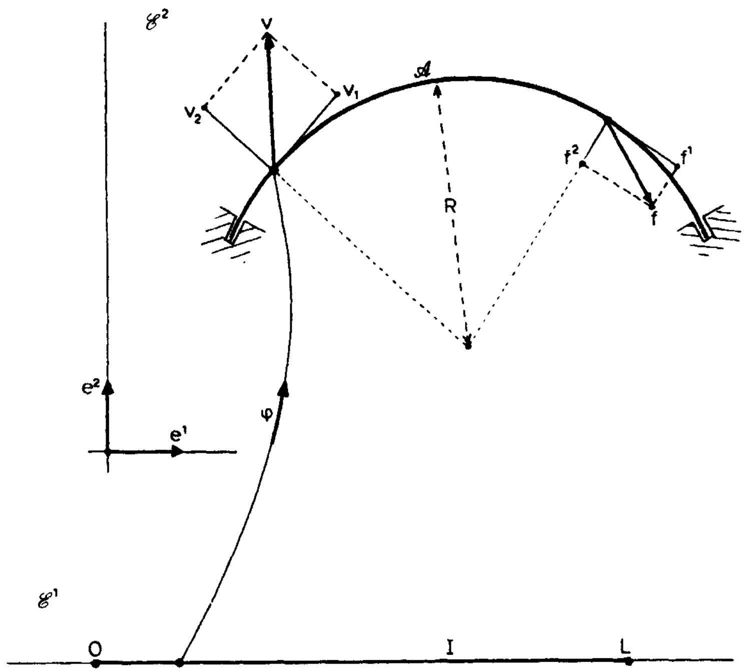


Fig. 8.3.1

because $R$ is a positive constant in the present case):

$$\begin{cases} V = H_0^1(I) \times H_0^2(I), \\ a(\boldsymbol{u}, \boldsymbol{v}) = \int_I \left\{ \left(u_1' + \frac{u_2}{R}\right)\left(v_1' + \frac{v_2}{R}\right) + \left(u_2'' - \frac{u_1'}{R}\right)\left(v_2'' - \frac{v_1'}{R}\right) \right\} \mathrm{d}s, \\ f(\boldsymbol{v}) = \int_I \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}s = \int_I (f^1 v_1 + f^2 v_2) \, \mathrm{d}s, \end{cases} \tag{8.3.1}$$

where $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{v} = (v_1, v_2)$ are functions of the curvilinear abscissa $s \in \bar{I} = [0, L]$. We recall that in Theorem 8.1.2, we have proved the $V$-ellipticity of bilinear forms which contain that of (8.3.1) as a special case.

*A natural finite element approximation*

***Let us first review what the most straightforward finite element method would be for solving this problem.*** Since all coefficients appearing in the bilinear form are constant, it is not necessary to approximate the geometry as we pointed out in Section 8.2. In other words, the discrete problem consists in letting $a_h(.,.) = a(.,.)$ and $f_h(.) = f(.)$, and then in looking for a discrete solution $\boldsymbol{u}_h = (u_{1h}, u_{2h}) \in \boldsymbol{V}_h = V_h \times W_h$, where $V_h$ and $W_h$ are subspaces of $H_0^1(I)$ and $H_0^2(I)$, respectively.

The simplest choices that can be made for these spaces are the following: Let

$$\bar{I} = [0, L] = \bigcup_{i=1}^{M} I_i, \quad \text{with} \quad I_i = [s_{i-1}, s_i],\ s_i = ih, \quad 1 \leq i \leq M, \tag{8.3.2}$$

be a *uniform partition* of the interval $\bar{I}$ associated with a mesh size $h = L/M$, $M$ being a strictly positive integer.

We let $V_h$ be the space of functions $v_{1h} \in \mathscr{C}^0(\bar{I})$ for which the restrictions $v_{1h}|_{I_i}$ span the space $P_1(I_i)$, $1 \leq i \leq M$, and which satisfy the boundary conditions $v_{1h}(0) = v_{2h}(0) = 0$, and we let $W_h$ be the space of functions $v_{2h} \in \mathscr{C}^1(\bar{I})$ for which the restrictions $v_{2h}|_{I_i}$ span the space $P_3(I_i)$, $1 \leq i \leq M$, and which satisfy the boundary conditions $v_{2h}(0) = v_{2h}'(0) = v_{2h}(L) = v_{2h}'(L) = 0$. Therefore, the degrees of freedom of the space $V_h$ are $v_{1h}(s_i)$, $1 \leq i \leq M-1$, and the degrees of freedom of the space $W_h$ are $v_{2h}(s_i)$, $v_{2h}'(s_i)$, $1 \leq i \leq M-1$.

The discrete solution $\boldsymbol{u}_h$ satisfies the equations

$$\forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \quad a(\boldsymbol{u}_h, \boldsymbol{v}_h) = f(\boldsymbol{v}_h), \tag{8.3.3}$$

and it is straightforward to show that if the solution $\boldsymbol{u} = (u_1, u_2)$ belongs to the space $V \cap (H^2(I) \times H^3(I))$, there exists a constant $C$ independent of $h$ such that

$$|\boldsymbol{u} - \boldsymbol{u}_h| \leq Ch(|u_1|_{2,I}^2 + |u_2|_{3,I}^2)^{1/2}, \tag{8.3.4}$$

where

$$\boldsymbol{v} = (v_1, v_2) \in V \to |\boldsymbol{v}| = (|v_1|_{1,I}^2 + |v_2|_{2,I}^2)^{1/2} \tag{8.3.5}$$

is a norm on the space $V$. We shall occasionally use the equivalent norm

$$\boldsymbol{v} = (v_1, v_2) \to \|\boldsymbol{v}\| = (\|v_1\|_{1,I}^2 + \|v_2\|_{2,I}^2)^{1/2}. \tag{8.3.6}$$

*Finite element methods conforming for the geometry*

Let us henceforth forget that we need not approximate the bilinear form of (8.3.1). If we follow the analysis made in Section 8.2, we are led to approximate the mapping $\varphi$ by a mapping $\varphi_h$ whose components lie in a finite element space $\Phi_h$ associated with the partition (8.3.2) of the interval $[0, L]$. We let $P_{I_i}$ denote the spaces spanned by the restrictions to the sets $I_i$, $1 \leq i \leq M$, of the functions in the space $\Phi_h$.

Since the third derivative of the mapping $\varphi$ does not appear in the bilinear form (to see this, it suffices to choose any parametrization of the arch in which the derivative of the mapping are not constant), a method is *conforming for the geometry* provided the inclusions $P_2(I_i) \subset P_{I_i}$, $1 \leq i \leq M$, hold.

*A finite element method which is not conforming for the geometry. Definition of the discrete problem*

We shall now analyze a method for which the spaces $P_{I_i}$ coincide with the spaces $P_1(I_i)$, $1 \leq i \leq M$, and which consequently is *not* conforming for the geometry. More precisely, let $\Phi_h$ denote the space of functions which are affine on each interval $I_i$, $1 \leq i \leq M$, and continuous over the interval $\bar{I}$. Then the approximate arch is defined by

$$\mathcal{A}_h = \varphi_h(\bar{I}), \tag{8.3.7}$$

where

$$\varphi_h = \varphi_{1h}e^1 + \varphi_{2h}e^2 \quad \text{with} \quad \varphi_{ih} \in \Phi_h, \quad i = 1, 2, \tag{8.3.8}$$

and the mapping $\varphi_h$ is uniquely determined by the interpolation conditions

$$\varphi_h(s_i) = \varphi(s_i), \quad 0 \leq i \leq M. \tag{8.3.9}$$

Since the second derivative of the approximate mapping $\varphi_h$ vanishes on each interval $\mathring{I}_i$, the corresponding "approximate" radius of curvature is infinite. Therefore, following the approach of Section 8.2 (cf. equations (8.2.4) and (8.2.5)) and the expression of the bilinear form as given in (8.3.1), we are led to the following approximate bilinear form:

$$a_h^*(u_h^*, v_h^*) = \sum_{i=1}^{M} \int_{I_i} (u_{1h}^{*\prime}(s) v_{1h}^{*\prime}(s) + \\ + u_{2h}^{*\prime\prime}(s) v_{2h}^{*\prime\prime}(s)) \sqrt{(\varphi_{1h}'(s))^2 + (\varphi_{2h}'(s))^2}\, \mathrm{d}s, \tag{8.3.10}$$

for functions $u_{1h}^*$, and $u_{2h}^*$, belonging to an appropriate finite element space contained in the space $\Pi_{i=1}^M H^1(I_i)$, and in the space $\Pi_{i=1}^M H^2(I_i)$, respectively.

The element of arc length along the approximate arch $\mathcal{A}_h$ is given by

$$\mathrm{d}\tilde{s} = \sqrt{(\varphi_{1h}')^2 + (\varphi_{2h}')^2}\, \mathrm{d}s = \frac{\sin \theta_h}{\theta_h} \mathrm{d}s, \tag{8.3.11}$$

where the angle $\theta_h$ is such that (Fig. 8.3.2)

$$h = 2R\theta_h. \tag{8.3.12}$$

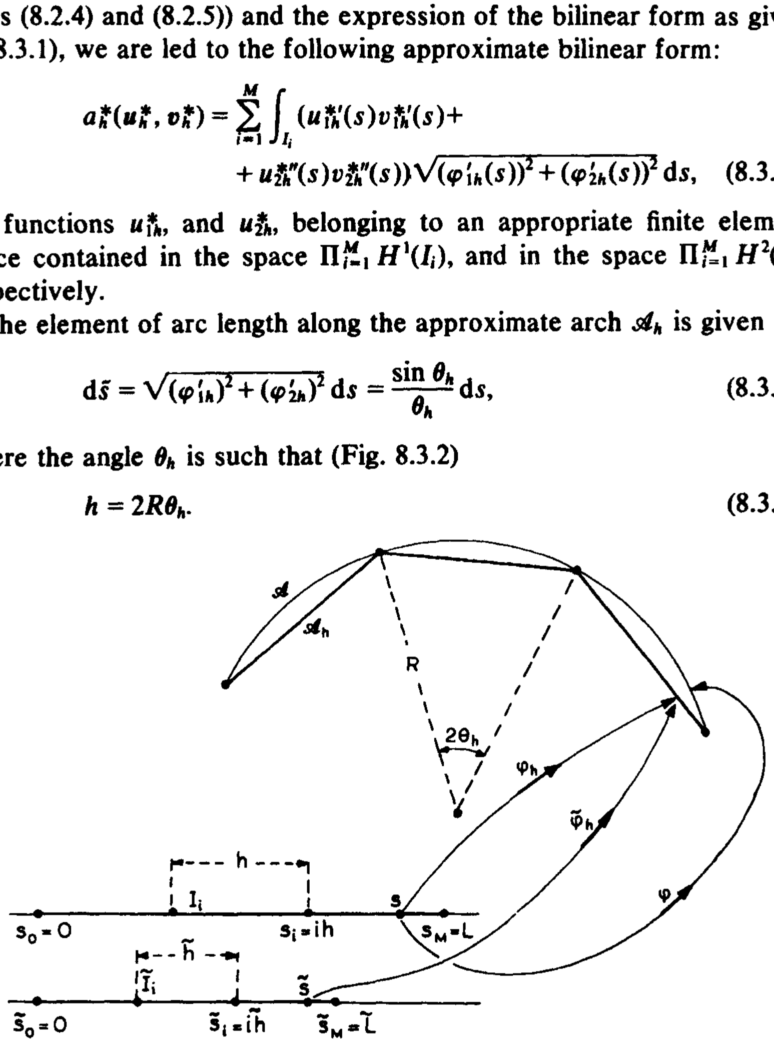


Fig. 8.3.2

Usually, the discrete problem is rather defined in terms of the abcissa $\tilde{s}$ along the approximate arch $\mathscr{A}_h$, which is given by

$$\tilde{s} = \frac{\sin\theta_h}{\theta_h} s, \quad 0 \le s \le L. \tag{8.3.13}$$

This being the case, we can also write (Fig. 8.3.2)

$$\mathscr{A}_h = \tilde{\varphi}_h(\bar{\tilde{I}}), \tag{8.3.14}$$

with

$$\bar{\tilde{I}} = [0, \tilde{L}], \quad \tilde{L} = \frac{\sin\theta_h}{\theta_h} L, \tag{8.3.15}$$

$$\tilde{\varphi}_h(\tilde{s}) = \varphi_h(s) \quad \text{for all } \tilde{s} = \frac{\sin\theta_h}{\theta_h} s, \quad 0 \le s \le L, \tag{8.3.16}$$

and we shall associate the uniform partition

$$\bar{\tilde{I}} = [0, \tilde{L}] = \bigcup_{i=1}^{M} \tilde{I}_i, \quad \text{with} \quad \tilde{I}_i = [\tilde{s}_{i-1}, \tilde{s}_i], \quad \tilde{s}_i = i\tilde{h},$$

$$1 \le i \le M, \tilde{h} = \frac{\sin\theta_h}{\theta_h} h, \tag{8.3.17}$$

with the partition of (8.3.2). Notice that

$$\tilde{h} = 2R\sin\theta_h. \tag{8.3.18}$$

Thus, rather than looking for the functions $u_{1h}^*, u_{2h}^*$: $s \in \bar{I} \to \mathbf{R}$, we shall look instead for functions $\tilde{u}_{1h}, \tilde{u}_{2h}$: $\tilde{s} \in \bar{\tilde{I}} \to \mathbf{R}$. To get the simplest correspondences, it suffices to let

$$\tilde{u}_{1h}(\tilde{s}) = \frac{\tilde{h}}{h} u_{1h}^*(s), \quad \tilde{u}_{2h}(\tilde{s}) = \left(\frac{\tilde{h}}{h}\right)^2 u_{2h}^*(s), \tag{8.3.19}$$

so that the strain energy (8.3.10) of the approximate arch takes the form

$$\tilde{a}_h(\tilde{u}_h, \tilde{v}_h) = a_h^*(u_h^*, v_h^*) = \sum_{i=1}^{M} \int_{\tilde{I}_i} (\tilde{u}'_{1h}(\tilde{s})\tilde{v}'_{1h}(\tilde{s}) + \tilde{u}''_{2h}(\tilde{s})\tilde{v}''_{2h}(\tilde{s}))\,\mathrm{d}\tilde{s}, \tag{8.3.20}$$

i.e., *it is written as a sum of strain energies of "elementary" straight beams* $\varphi_h(I_i) = \tilde{\varphi}_h(\tilde{I}_i)$, $1 \le i \le M$, *which is indeed the main feature of such a method.*

We are therefore led to look for a discrete solution $\tilde{u}_h = (\tilde{u}_{1h}, \tilde{u}_{2h})$ in a space $\tilde{V}_h$ whose elements $\tilde{v}_h = (\tilde{v}_{1h}, \tilde{v}_{2h})$ are such that $\tilde{v}_{1h} \in \Pi_{i=1}^M H^1(\tilde{I}_i)$ and $\tilde{v}_{2h} \in \Pi_{i=1}^M H^2(\tilde{I}_i)$.

In view of the definitions of the spaces $V_h$ and $W_h$, we shall assume for definiteness (leaving out for the time being the boundary conditions) that the restrictions $\tilde{v}_{1h|\tilde{I}_i}$ span the space $P_1(\tilde{I}_i)$, and that the restrictions $\tilde{v}_{2h|\tilde{I}_i}$ span the space $P_3(\tilde{I}_i)$, $1 \leq i \leq M$. In this fashion, a function $\tilde{v}_h = (\tilde{v}_{1h}, \tilde{v}_{2h}) \in \tilde{V}_h$ is specified by the parameters

$$\tilde{v}_{1h}(\tilde{s}_{i-1}^+), \quad \tilde{v}_{1h}(\tilde{s}_i^-), \quad 1 \leq i \leq M,$$

and

$$\tilde{v}_{2h}(\tilde{s}_{i-1}^+), \quad \tilde{v}_{2h}'(\tilde{s}_{i-1}^+), \quad \tilde{v}_{2h}(\tilde{s}_i^-), \quad \tilde{v}_{2h}'(\tilde{s}_i^-), \quad 1 \leq i \leq M.$$

To find the compatibility relations between these parameters, it suffices to express that they correspond to a well-defined *displacement*

$$\boldsymbol{v}_h(s_i) = v_{1h}(s_i)\boldsymbol{a}_1(s_i) + v_{2h}(s_i)\boldsymbol{a}_2(s_i) \tag{8.3.21}$$

(Fig. 8.3.3) of the point $\varphi(s_i)$, and to a well-defined *rotation*

$$\omega_h(s_i) = v_{2h}'(s_i) - \frac{1}{R} v_{1h}(s_i) \tag{8.3.22}$$

of the same point $\varphi(s_i)$, for all $i = 0, 1, \ldots, M$.

With the self-explanatory notation of Fig. 8.3.3, we obtain

$$\begin{cases} \tilde{\boldsymbol{v}}_h(\tilde{s}_i^-) = \tilde{v}_{1h}(\tilde{s}_i^-)\boldsymbol{a}_1(\tilde{s}_i^-) + \tilde{v}_{2h}(\tilde{s}_i^-)\boldsymbol{a}_2(\tilde{s}_i^-), \quad 1 \leq i \leq M, \\ \tilde{\boldsymbol{v}}_h(\tilde{s}_i^+) = \tilde{v}_{1h}(\tilde{s}_i^+)\boldsymbol{a}_1(\tilde{s}_i^+) + \tilde{v}_{2h}(\tilde{s}_i^+)\boldsymbol{a}_2(\tilde{s}_i^+), \quad 0 \leq i \leq M-1, \\ \tilde{\omega}_h(\tilde{s}_i^-) = \tilde{v}_{2h}'(\tilde{s}_i^-), \quad 1 \leq i \leq M, \quad \tilde{\omega}_h(\tilde{s}_i^+) = \tilde{v}_{2h}'(\tilde{s}_i^+), \\ \hfill 0 \leq i \leq M-1, \end{cases} \tag{8.3.23}$$

and thus we must have

$$\begin{cases} \tilde{v}_{1h}(\tilde{s}_i^-) = \cos\theta_h v_{1h}(s_i) + \sin\theta_h v_{2h}(s_i), \\ \tilde{v}_{2h}(\tilde{s}_i^-) = -\sin\theta_h v_{1h}(s_i) + \cos\theta_h v_{2h}(s_i), \\ \tilde{v}_{1h}(\tilde{s}_i^+) = \cos\theta_h v_{1h}(s_i) - \sin\theta_h v_{2h}(s_i), \\ \tilde{v}_{2h}(\tilde{s}_i^+) = \sin\theta_h v_{1h}(s_i) + \cos\theta_h v_{2h}(s_i), \\ \tilde{v}_{2h}'(\tilde{s}_i^-) = \tilde{v}_{2h}'(\tilde{s}_i^+) = v_{2h}'(s_i) - \frac{1}{R} v_{1h}(s_i). \end{cases} \tag{8.3.24}$$

Using relations (8.3.21), (8.3.22), (8.3.23) and (8.3.24), we deduce the

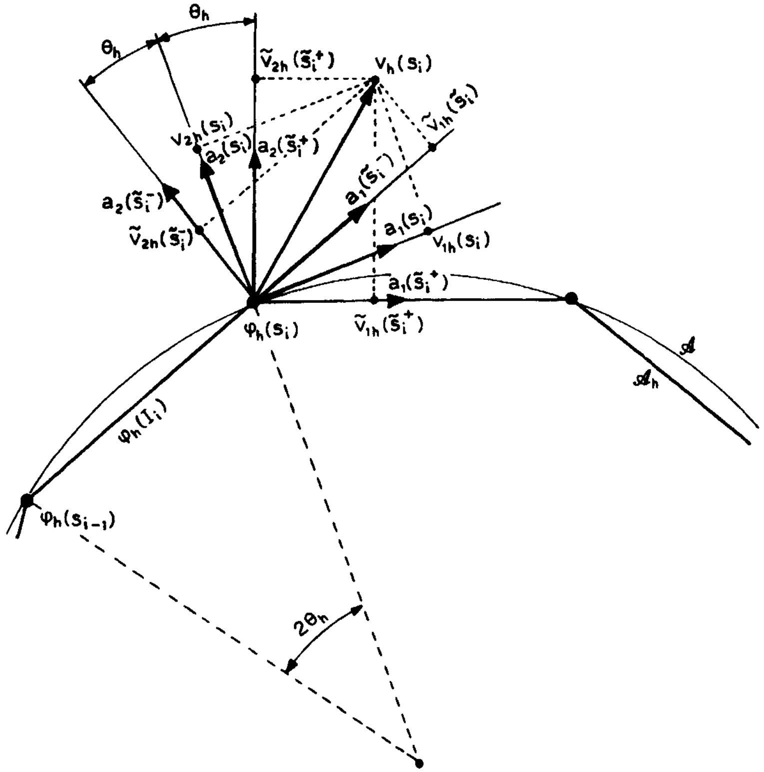


Fig. 8.3.3

*compatibility relations* which the functions $\tilde{v}_{1h}$ and $\tilde{v}_{2h}$ should satisfy:

$$\left\{\begin{aligned} &\cos\theta_h\tilde{v}_{1h}(\tilde{s}_i^-)-\sin\theta_h\tilde{v}_{2h}(\tilde{s}_i^-)=\cos\theta_h\tilde{v}_{1h}(\tilde{s}_i^+)+\sin\theta_h\tilde{v}_{2h}(\tilde{s}_i^+)\\ &\qquad\qquad\qquad\qquad\qquad(=v_{1h}(s_i)),\\ &\sin\theta_h\tilde{v}_{1h}(\tilde{s}_i^-)+\cos\theta_h\tilde{v}_{2h}(\tilde{s}_i^-)=-\sin\theta_h\tilde{v}_{1h}(\tilde{s}_i^+)+\cos\theta_h\tilde{v}_{2h}(\tilde{s}_i^+)\\ &\qquad\qquad\qquad\qquad\qquad(=v_{2h}(s_i)),\\ &\tilde{v}_{2h}'(\tilde{s}_i^-)=\tilde{v}_{2h}'(\tilde{s}_i^+)\quad\left(=v_{2h}'(s_i)-\frac{1}{R}v_{1h}(s_i)\right).\end{aligned}\right. \tag{8.3.25}$$

Thus at each point $s_i$, $0\leqslant i\leqslant M$, one may consider that the *independent parameters* are the values $v_{1h}(s_i)$, $v_{2h}(s_i)$, $v_{2h}'(s_i)$, from which the parameters $\tilde{v}_{1h}(\tilde{s}_i^-)$, $\tilde{v}_{1h}(\tilde{s}_i^+)$, $\tilde{v}_{2h}(\tilde{s}_i^-)$, $\tilde{v}_{2h}(\tilde{s}_i^+)$, $\tilde{v}_{2h}'(\tilde{s}_i^-)$, $\tilde{v}_{2h}'(\tilde{s}_i^+)$ are derived through relations (8.3.24).

Finally, the following boundary conditions will be included in the definition of the spaces $\tilde{V}_h$:

$$\tilde{v}_{1h}(0^+) = \tilde{v}_{2h}(0^+) = \tilde{v}'_{2h}(0^+) = \tilde{v}_{1h}(L^-) = \tilde{v}_{2h}(L^-) = \tilde{v}'_{2h}(L^-) = 0, \tag{8.3.26}$$

or equivalently, using relations (8.3.25),

$$v_{1h}(0) = v_{2h}(0) = v'_{2h}(0) = v_{1h}(L) = v_{2h}(L) = v'_{2h}(L) = 0. \tag{8.3.27}$$

To sum up, *the space $\tilde{V}_h$ is completely defined*: it is composed of pairs $(\tilde{v}_{1h}, \tilde{v}_{2h})$

(i) whose restrictions $v_{1h|\tilde{I}_i}$ and $v_{2h|\tilde{I}_i}$ span the space $P_1(\tilde{I}_i)$ and the space $P_3(\tilde{I}_i)$, respectively,·

(ii) which satisfy the compatibility relations (8.3.25), and

(iii) which satisfy the boundary conditions (8.3.26).

By relations (8.3.24)–(8.3.25) and (8.3.26)–(8.3.27), there exists a bijection

$$(\tilde{v}_{1h}, \tilde{v}_{2h}) \in \tilde{V}_h \to (v_{1h}, v_{2h}) \in \boldsymbol{V}_h = V_h \times W_h, \tag{8.3.28}$$

where $\boldsymbol{V}_h = V_h \times W_h$ is the "conforming" subspace introduced at the beginning of this section.

The *discrete problem* then consists in finding an element $\tilde{u}_h \in \tilde{V}_h$ such that (cf. (8.3.20))

$$\forall \tilde{v}_h \in \tilde{V}_h, \quad \tilde{a}_h(\tilde{u}_h, \tilde{v}_h) = h \sum_{i=1}^{M-1} (f \cdot v_h)(s_i), \tag{8.3.29}$$

or, more explicitly, such that

$$\sum_{i=1}^{M} \int_{\tilde{I}_i} (\tilde{u}'_{1h}(\tilde{s})\tilde{v}'_{1h}(\tilde{s}) + \tilde{u}''_{2h}(\tilde{s})\tilde{v}''_{2h}(\tilde{s}))\, d\tilde{s} = \\ = h \sum_{i=1}^{M-1} \{f^1(s_i)v_{1h}(s_i) + f^2(s_i)v_{2h}(s_i)\}, \tag{8.3.30}$$

for all $\tilde{v}_h \in \tilde{V}_h$, where the functions $\tilde{v}_h \in \tilde{V}_h$ and $v_h \in \boldsymbol{V}_h$ are in the correspondence (8.3.28).

**Remark 8.3.1.** In principle, the approximate linear form should also be given as a sum of integrals over the intervals $\tilde{I}_i$. However, the right-hand side of the discrete variational problem is usually defined as in (8.3.30) in the engineering literature, where it is considered that the applied force is approximated by a sum of concentrated forces, an obvious simplification for computational purposes. In addition, this simplification is theoretically justified, since it does not decrease the order of convergence, as we shall see. □

Notice that the discrete problem (8.3.29) can also be written as a problem posed *over the space* $V_h$: To find $u_h \in V_h$ such that

$$\forall v_h \in V_h, \quad a_h(u_h, v_h) = f_h(v_h), \tag{8.3.31}$$

where, *by definition*, the approximate bilinear form $a_h$ is given by

$$a_h(u_h, v_h) = \tilde{a}_h(\tilde{u}_h, \tilde{v}_h), \tag{8.3.32}$$

for all $u_h, v_h \in V_h$ and $\tilde{u}_h, \tilde{v}_h \in \tilde{V}_h$ in the correspondence (8.3.28), and the approximate linear form is given by

$$f_h(v_h) = h \sum_{i=1}^{M-1} (f \cdot v_h)(s_i). \tag{8.3.33}$$

This is why our first task (Theorem 8.3.1) will be to explicitly compute the bilinear form $a_h(.\,,.)$. Since the space $V_h$ is a subspace of the space $V$, we are exactly in the same abstract setting as we were when we studied the effect of numerical integration in Section 4.1. Accordingly, our objective is to be in a positition to apply the abstract error bound of Theorem 4.1.1. Therefore, we shall successively evaluate the quantities $|a_h(v_h, w_h) - a(v_h, w_h)|$ for $v_h, w_h \in V_h$ (Theorem 8.3.2) and $|f_h(w_h) - f(w_h)|$ for $w_h \in V_h$ (Theorem 8.3.3), before we combine them in our final result (Theorem 8.3.4).

**Theorem 8.3.1.** *Let* $\tilde{v}_h = (\tilde{v}_{1h}, \tilde{v}_{2h}) \in \tilde{V}_h$ *and* $v_h = (v_{1h}, v_{2h}) \in V_h$ *be in the correspondence* (8.3.28). *Then we have*

$$\tilde{a}_h(\tilde{v}_h, \tilde{v}_h) = a_h(v_h, v_h), \tag{8.3.34}$$

*where*

$$\begin{aligned} a_h(v_h, v_h) = &\sum_{i=1}^{M} \int_{I_i} \left(\frac{h}{\tilde{h}} \cos \theta_h v_{1h}'(s) + \frac{v_{2h}(s_{i-1}) + v_{2h}(s_i)}{2R}\right)^2 \frac{\tilde{h}}{h}\, \mathrm{d}s \\ &+ \sum_{i=1}^{M} \int_{I_i} \left(\left(\frac{h}{\tilde{h}}\right)^2 \cos \theta_h v_{2h}''(s) - \frac{1}{R}\frac{h}{\tilde{h}}\, v_{1h}'(s)\right. \\ &+ \left\{\frac{1}{\tilde{h}}\left(1 - \cos \theta_h \frac{h}{\tilde{h}}\right)\right\}\left\{v_{2h}'(s_{i-1})\left(6\frac{(s - s_{i-1})}{h} - 4\right)\right. \\ &\left.\left. + v_{2h}'(s_i)\left(6\frac{(s - s_{i-1})}{h} - 2\right)\right\}\right) \frac{\tilde{h}}{h}\, \mathrm{d}s. \end{aligned} \tag{8.3.35}$$

**Proof.** Recall that (cf. (8.3.20))

$$\tilde{a}_h(\tilde{v}_h, \tilde{v}_h) = \sum_{i=1}^{M} \int_{\tilde{I}_i} (\tilde{v}'_{1h}(\tilde{s}))^2 \,\mathrm{d}\tilde{s} + \sum_{i=1}^{M} \int_{\tilde{I}_i} (\tilde{v}''_{2h}(\tilde{s}))^2 \,\mathrm{d}\tilde{s}.$$

To prove equality (8.3.35) we shall in fact prove more: With obvious notations, equality (8.3.35) can be written as

$$\sum_{i=1}^{M} \int_{\tilde{I}_i} \{(\tilde{\varphi}_i^1(\tilde{s}))^2 + (\tilde{\varphi}_i^2(\tilde{s}))^2\} \,\mathrm{d}\tilde{s} = \sum_{i=1}^{M} \int_{I_i} \{(\theta_i^1(s))^2 + (\theta_i^2(s))^2\} \frac{\tilde{h}}{h} \,\mathrm{d}s,$$

and we have

$$\sum_{i=1}^{M} \int_{\tilde{I}_i} \{(\tilde{\varphi}_i^1(\tilde{s}))^2 + (\tilde{\varphi}_i^2(\tilde{s}))^2\} \,\mathrm{d}\tilde{s} =$$

$$= \sum_{i=1}^{M} \int_{I_i} \left\{ \left(\tilde{\varphi}_i^1\left(\frac{\tilde{h}}{h} s\right)\right)^2 + \left(\tilde{\varphi}_i^2\left(\frac{\tilde{h}}{h} s\right)\right)^2 \right\} \frac{\tilde{h}}{h} \,\mathrm{d}s.$$

Then we shall derive the stronger equalities

$$\tilde{\varphi}_i^1\left(\frac{\tilde{h}}{h} s\right) = \theta_i^1(s) \text{ and } \tilde{\varphi}_i^2\left(\frac{\tilde{h}}{h} s\right) = \theta_i^2(s) \quad \text{for} \quad s \in I_i, \quad 1 \leq i \leq M.$$

Since, on each interval $\tilde{I}_i$, $\tilde{v}_{1h}$ is a polynomial of degree one, we can write, for all $\tilde{s} \in \tilde{I}_i$,

$$\begin{aligned}\tilde{v}'_{1h}(\tilde{s}) &= \frac{\tilde{v}_{1h}(\tilde{s}_i^-) - \tilde{v}_{1h}(\tilde{s}_{i-1}^+)}{\tilde{h}} \\ &= \frac{1}{\tilde{h}} \{\cos \theta_h (v_{1h}(s_i) - v_{1h}(s_{i-1})) + \sin \theta_h (v_{2h}(s_i) + v_{2h}(s_{i-1}))\},\end{aligned}$$

by (8.3.24), and thus, since $v_{1h}$ is also a polynomial of degree one in the variable $s$ over $I_i$,

$$\tilde{v}'_{1h}(\tilde{s}) = \frac{h}{\tilde{h}} \cos \theta_h v'_{1h}(s) + \frac{v_{2h}(s_{i-1}) + v_{2h}(s_i)}{2R},$$

where we have also used the relation $2R \sin \theta_h = \tilde{h}$ ((cf. (8.3.18)). Therefore, the first equality is proved.

Since on each interval $\tilde{I}_i$, $\tilde{v}_{2h}$ is a polynomial of degree three, we can write, for all $\tilde{s} \in \tilde{I}_i$,

$$\begin{aligned}\tilde{v}''_{2h}(\tilde{s}) = {} & \tilde{v}_{2h}(\tilde{s}_{i-1}^+)\left(\frac{12(\tilde{s} - \tilde{s}_i) + 6\tilde{h}}{\tilde{h}^3}\right) + \tilde{v}_{2h}(\tilde{s}_i^-)\left(\frac{-12(\tilde{s} - \tilde{s}_{i-1}) + 6\tilde{h}}{\tilde{h}^3}\right) \\ & + \tilde{v}'_{2h}(\tilde{s}_{i-1}^+)\left(\frac{6(\tilde{s} - \tilde{s}_i) + 2\tilde{h}}{\tilde{h}^2}\right) + \tilde{v}'_{2h}(s_i^-)\left(\frac{6(\tilde{s} - \tilde{s}_{i-1}) - 2\tilde{h}}{\tilde{h}^2}\right).\end{aligned}$$

Using relations (8.3.24) and equalities $\tilde{s} = (\tilde{h}/h)s$, $\tilde{h} = 2R\sin\theta_h$ (cf. (8.3.13) and (8.3.18)), we obtain

$$
\begin{aligned}
\tilde{v}''_{2h}(\tilde{s}) = {} & \left(\frac{h}{\tilde{h}}\right)^2 \left\{ \left(\sin\theta_h v_{1h}(s_{i-1}) + \cos\theta_h v_{2h}(s_{i-1})\right)\left(\frac{12(s-s_i)+6h}{h^3}\right) \right. \\
& \left. + \left(-\sin\theta_h v_{1h}(s_i) + \cos\theta_h v_{2h}(s_i)\right)\left(\frac{-12(s-s_{i-1})+6h}{h^3}\right) \right\} \\
& + \left(\frac{h}{\tilde{h}}\right) \left\{ \left(v'_{2h}(s_{i-1}) - \frac{1}{R} v_{1h}(s_{i-1})\right)\left(\frac{6(s-s_i)+2h}{h^2}\right) \right. \\
& \left. + \left(v'_{2h}(s_i) - \frac{1}{R} v_{1h}(s_i)\right)\left(\frac{6(s-s_{i-1})-2h}{h^2}\right) \right\} \\
= {} & \left(\frac{h}{\tilde{h}}\right)^2 \cos\theta_h \left\{ v_{2h}(s_{i-1})\left(\frac{12(s-s_i)+6h}{h^3}\right) \right. \\
& + v_{2h}(s_i)\left(\frac{-12(s-s_{i-1})+6h}{h^3}\right) + v'_{2h}(s_{i-1})\left(\frac{6(s-s_i)+2h}{h^2}\right) \\
& \left. + v'_{2h}(s_i)\left(\frac{6(s-s_{i-1})-2h}{h^2}\right) \right\} + \frac{h}{\tilde{h}}\left(1 - \cos\theta_h \frac{h}{\tilde{h}}\right) \\
& \times \left\{ v'_{2h}(s_{i-1})\left(\frac{6(s-s_i)+2h}{h^2}\right) + v'_{2h}(s_i)\left(\frac{6(s-s_{i-1})-2h}{h^2}\right) \right\} \\
& + \frac{1}{R}\frac{h}{\tilde{h}}\left\{\frac{v_{1h}(s_{i-1}) - v_{1h}(s_i)}{h}\right\} = \left(\frac{h}{\tilde{h}}\right)^2 \cos\theta_h v''_{2h}(s) \\
& + \frac{h}{\tilde{h}}\left(1 - \cos\theta_h \frac{h}{\tilde{h}}\right)\left\{ v'_{2h}(s_{i-1})\left(\frac{6(s-s_i)+2h}{h^2}\right) \right. \\
& \left. + v'_{2h}(s_i)\left(\frac{6(s-s_{i-1})-2h}{h^2}\right) \right\} - \frac{1}{R}\frac{h}{\tilde{h}} v'_{1h}(s),
\end{aligned}
$$

where we have taken into account the fact that $v_{1h}$, and $v_{2h}$, are polynomials of degree one and three, respectively, in the variable $s$ on each interval $I_i$. Thus the second equality is proved. □

*Consistency error estimates*

When this is not explicitly stated, it is understood in the remainder of this section that the letter $C$ stands for any constant independent of $h$.

**Theorem 8.3.2.** *There exists a constant $C$ independent of $h$ such that,* *for all* $v_h \in V_h$, $w_h \in V_h$,

$$|a_h(v_h, w_h) - a(v_h, w_h)| \leq Ch\|v_h\|\,\|w_h\|, \tag{8.3.36}$$

*where* $\|.\|$ *is the norm defined in* (8.3.6).

*Consequently, for* $h$ *sufficiently small, the approximate bilinear forms* $a_h(.,.)$ *are uniformly* $V_h$*-elliptic.*

**Proof.** With self-explanatory notation, the bilinear forms $a(.,.)$ and $a_h(.,.)$ are of the following form, for all $v, w \in V_h$ (cf. (8.3.1) and (8.3.35)):

$$a(v, w) = \sum_{i=1}^{M} \int_{I_i} AvAw \, \mathrm{d}s + \sum_{i=1}^{M} \int_{I_i} BvBw \, \mathrm{d}s,$$

$$a_h(v, w) = \sum_{i=1}^{M} \int_{I_i} A_hvA_hw \, \mathrm{d}s + \sum_{i=1}^{M} \int_{I_i} B_hvB_hw \, \mathrm{d}s$$

(notice that since the bilinear form $a_h(.,.)$ is symmetric, it sufficed to compute it on the diagonal $v = w$, as we did in Theorem 8.3.1). We shall use the inequalities

$$\left|\int_{I_i} AvAw \, \mathrm{d}s - \int_{I_i} A_hvA_hw \, \mathrm{d}s\right| \leq |Av - A_hv|_{0,I_i}|Aw - A_hw|_{0,I_i} + \\ + |Av|_{0,I_i}|Aw - A_hw|_{0,I_i} + |Aw|_{0,I_i}|Av - A_hv|_{0,I_i}, \tag{8.3.37}$$

and similar inequalities for the other integrals. Since

$$Av = v_1' + \frac{1}{R} v_2, \quad A_hv = \sqrt{\frac{\tilde{h}}{h}}\left(\frac{h}{\tilde{h}} \cos \theta_h v_1' + \frac{1}{R} \frac{v_2(s_{i-1}) + v_2(s_i)}{2}\right),$$

we deduce

$$Av - A_hv = \left(1 - \sqrt{\frac{h}{\tilde{h}}} \cos \theta_h\right) v_1' + \frac{1}{R}\sqrt{\frac{\tilde{h}}{h}}\left(v_2 - \frac{v_2(s_{i-1}) + v_2(s_i)}{2}\right) \\ + \frac{1}{R}\left(1 - \sqrt{\frac{\tilde{h}}{h}}\right) v_2,$$

and consequently,

$$|Av - A_hv|_{0,I_i} \leq \left|1 - \sqrt{\frac{h}{\tilde{h}}} \cos \theta_h\right| |v_1|_{1,I_i} \\ + \frac{1}{R}\sqrt{\frac{\tilde{h}}{h}}\left|v_2 - \frac{v_2(s_{i-1}) + v_2(s_i)}{2}\right|_{0,I_i} \\ + \frac{1}{R}\left|1 - \sqrt{\frac{\tilde{h}}{h}}\right| |v_{22}|_{0,I_i}.$$

On the one hand, using (8.3.12) and (8.3.17), we obtain

$$\left|1-\sqrt{\frac{h}{\tilde h}}\cos\theta_h\right| = \mathrm{O}(h^2),\quad \sqrt{\frac{\tilde h}{h}} = 1+\mathrm{O}(h^2),\quad \left|1-\sqrt{\frac{\tilde h}{h}}\right| = \mathrm{O}(h^2),$$

and, on the other hand, there exists some constant $C$ such that

$$\forall v_2 \in H^1(I_i),\quad \left|v_2 - \frac{v_2(s_{i-1})+v_2(s_i)}{2}\right|_{0,I_i} \le Ch|v_2|_{1,I_i},$$

since the mapping

$$\Pi : v_2 \in H^1(I_i) \to \frac{v_2(s_{i-1})+v_2(s_i)}{2} \in L^2(I_i)$$

preserves polynomials of degree zero (Theorem 3.1.4). Since

$$|Av|_{0,I_i} \le |v_1|_{1,I_i} + \frac{1}{R}|v_2|_{0,I_i},$$

we eventually find, upon combining the above inequalities in inequality (8.3.37), that

$$\forall v, w \in V_h,\quad \left|\int_{I_i} AvAw\,\mathrm{d}s - \int_{I_i} A_hvA_hw\,\mathrm{d}s\right| \le Ch\|v\|_{I_i}\|w\|_{I_i},$$

where

$$\|v\|_{I_i} = (\|v_1\|^2_{1,I_i} + \|v_2\|^2_{2,I_i})^{1/2},$$

and therefore,

$$\left|\sum_{i=1}^{M}\int_{I_i} AvAw\,\mathrm{d}s - \sum_{i=1}^{M}\int_{I_i} A_hvA_hw\,\mathrm{d}s\right| \le Ch\|v\|\,\|w\|.$$

It remains to consider the analogous expression, where

$$Bv = v_2'' - \frac{1}{R}v_1',$$

$$B_hv = \left(\frac{h}{\tilde h}\right)^{3/2}\cos\theta_h v_2'' - \frac{1}{R}\sqrt{\frac{h}{\tilde h}}\,v_1' +$$

$$+\frac{1}{\sqrt{h\tilde h}}\left(1-\cos\theta_h\frac{h}{\tilde h}\right)\left\{v_2'(s_{i-1})\left(6\frac{(s-s_{i-1})}{h}-4\right)\right.$$

$$\left.+\,v_2'(s_i)\left(6\frac{(s-s_{i-1})}{h}-2\right)\right\},$$

so that

$$Bv - B_h v = \left(1 - \left(\frac{h}{\bar{h}}\right)^{3/2} \cos \theta_h\right) v_2'' - \frac{1}{R}\left(1 - \sqrt{\frac{h}{\bar{h}}}\right) v_1' -$$
$$- \frac{1}{\sqrt{h\bar{h}}}\left(1 - \cos \theta_h \frac{h}{\bar{h}}\right)\left\{ v_2'(s_{i-1})\left(6 \frac{(s - s_{i-1})}{h} - 4\right)\right.$$
$$\left. + v_2'(s_i)\left(\frac{6(s - s_{i-1})}{h} - 2\right)\right\}.$$

Using (8.3.12) and (8.3.17), it is first established that

$$\left|1 - \left(\frac{h}{\bar{h}}\right)^{3/2} \cos \theta_h\right| = O(h^2), \quad \left|1 - \sqrt{\frac{h}{\bar{h}}}\right| = O(h^2),$$
$$\frac{1}{\sqrt{h\bar{h}}}\left(1 - \cos \theta_h \frac{h}{\bar{h}}\right) = O(h).$$

Next, one has

$$\left|\left(6 \frac{(\cdot - s_{i-1})}{h} - 4\right)\right|_{0,I_i} = O(h^{1/2}), \quad \left|\left(6 \frac{(\cdot - s_{i-1})}{h} - 2\right)\right|_{0,I_i} = O(h^{1/2}),$$

and, finally, there exists a constant $C$ such that

$$\forall v_2 \in H^2(I_i), \quad |v_2|_{1,\infty,I_i} \leq Ch^{-1/2} \|v_2\|_{2,I_i}.$$

Combining the above relations with an inequality similar to inequality (8.3.37), we find that

$$\forall v, w \in V_h, \quad \left|\int_{I_i} BvBw \, ds - \int_{I_i} B_h v B_h w \, ds\right| \leq Ch\|v\|_{I_i}\|w\|_{I_i},$$

and inequality (8.3.36) is proved.

The uniform $V_h$-ellipticity of the approximate bilinear forms is proved as in Theorem 8.2.2. □

Concerning the approximation of the linear form, we have the following result, whose proof is left to the reader (Exercise 8.3.1):

**Theorem 8.3.3.** *Assume that the functions $f^1$ and $f^2$ are Lipschitz-continuous on the interval $\bar{I}$. Then there exists a constant $C$ independent of $h$ (but dependent on the functions $f^1, f^2$) such that, for all $w_h \in V_h$,*

$$|f(w_h) - f_h(w_h)| \leq Ch\|w_h\|. \qquad \square \quad (8.3.38)$$

*Estimate of the error* $(|u_1 - u_{1h}|_{1,I}^2 + |u_2 - u_{2h}|_{2,I}^2)^{1/2}$

We are now in a position to prove the main result of this section.

**Theorem 8.3.4.** *Assume that the solution* $\boldsymbol{u} = (u_1, u_2)$ *belongs to the space* $V \cap (H^2(I) \times H^3(I))$, *and that the functions* $f^1$ *and* $f^2$ *are Lipschitz-continuous on the interval* $I$. *Then there exists a constant* $C$ *independent of* $h$ *(but dependent on the solution* $\boldsymbol{u}$*) such that*

$$|\boldsymbol{u} - \boldsymbol{u}_h| = (|u_1 - u_{1h}|_{1,I}^2 + |u_2 - u_{2h}|_{2,I}^2)^{1/2} \leq Ch, \tag{8.3.39}$$

where $\boldsymbol{u}_h$ is the solution of the discrete problem (8.3.31).

**Proof.** Since the approximate bilinear forms are uniformly $V_h$-elliptic for $h$ sufficiently small (Theorem 8.3.2), we may apply Theorem 4.1.1: If we let $\Pi_h \boldsymbol{u} = (\Pi_h u_1, \Lambda_h u_2)$ denote the $V_h$-interpolant of the solution $\boldsymbol{u}$, we have

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| \leq C\Bigg(\|\boldsymbol{u} - \Pi_h \boldsymbol{u}\| + \sup_{\boldsymbol{w}_h \in V_h} \frac{|a(\Pi_h \boldsymbol{u}, \boldsymbol{w}_h) - a_h(\Pi_h \boldsymbol{u}, \boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|} + \\ + \sup_{\boldsymbol{w}_h \in V_h} \frac{|f(\boldsymbol{w}_h) - f_h(\boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|}\Bigg).$$

Using the regularity assumption on the solution $\boldsymbol{u}$, we obtain,

$$\|\boldsymbol{u} - \Pi_h \boldsymbol{u}\| \leq Ch(|u_1|_{2,I}^2 + |u_2|_{3,I}^2)^{1/2}.$$

Next, using Theorem 8.3.2, we find the *consistency error estimate*

$$\sup_{\boldsymbol{w}_h \in V_h} \frac{|a(\Pi_h \boldsymbol{u}, \boldsymbol{w}_h) - a_h(\Pi_h \boldsymbol{u}, \boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|} \leq Ch\|\Pi_h \boldsymbol{u}\| \leq Ch\|\boldsymbol{u}\|,$$

where for the second inequality, we have used the fact that the operators $\Pi_h$ and $\Lambda_h$ preserve polynomials of degree zero and one, respectively.

Finally, we have by Theorem 8.3.3,

$$\sup_{\boldsymbol{w}_h \in V_h} \frac{|f(\boldsymbol{w}_h) - f_h(\boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|} \leq Ch,$$

and the proof is complete. □

We are now in a position to state the main conclusion of the present analysis: In order to compensate the discrepancies between the strain energies of the continuous and of the discrete problem, the functions in the spaces $\tilde{V}_h$ and $\tilde{W}_h$ must satisfy the compatibility relations (8.3.25) across the mesh points $\tilde{s}_i$. As a consequence, the inclusions $\tilde{V}_h \subset H^1(\tilde{I})$ and $\tilde{W}_h \subset H^2(\tilde{I})$ no longer hold.

In other words, *the fact that the method is not conforming for the geometry implies that it is not conforming also for the displacements.*

*Exercise*

*8.3.1.* Prove Theorem 8.3.3. Recall (cf. (8.3.1) and (8.3.33)) that for all $w \in V_h$, we have

$$f(w) - f_h(w) = \int_0^L (f^1(s)w_1(s) + f^2(s)w_2(s))\, ds - h \sum_{i=1}^{M-1} (f^1(s_i)w_1(s_i) + f^2(s_i)w_2(s_i)).$$

**Bibliography and comments**

*8.1.* There exist numerous references for the shell problem. The description of Koiter's model (including the nonlinear case) is found at various stages of its development in KOITER (1966, 1970), KOITER & SIMMONDS (1972). This model is based on certain physical hypotheses (essentially about the stress distribution across the thickness of the shell), which JOHN (1965) has theoretically justified.

A different model has been proposed by NAGHDI (1963, 1972). A simplified theory of the so-called "shallow" shells is presented in TIMOSHENKO & WOINOWSKY-KRIEGER (1959) and WASHIZU (1968).

For references in tensor calculus and differential geometry, the reader may consult GOUYON (1963), LELONG-FERRAND (1963), LICHNÉROWICZ (1967), VALIRON (1950, Chapters 12, 13, 14).

The ellipticity of Koiter's model in the linear case is proved in BERNADOU & CIARLET (1976), where the proof follows basically a method set up in CIARLET (1976a) for circular arches. Following CIARLET (1976c), we have presented here the extension of this method to the case of an arch of varying curvature. For the expression of the corresponding elastic energy, see for example MOAN (1974).

As regards the question of ellipticity for various shell models, we mention the works of ROUGÉE (1969) for "cylindrical" shells, COUTRIS (1973) for Naghdi's model, GORDEZIANI (1974) for the model of VEKUA (1965), SHOIKET (1974) for the model of NOVOZHILOV (1970). For cylindrical shells, see KOLAKOWSKI & DRYJA (1974), MIYOSHI (1973b).

*8.2.* The content of this section is essentially based on CIARLET (1976b). Conforming finite element methods of the type considered here are described in ARGYRIS & LOCHNER (1972), ARGYRIS, HAASE & MALEJANNAKIS (1973). Related methods are discussed in DUPUIS & GÖEL (1970b), DUPUIS (1971). These are only a few among the many papers which are concerned with the description of the application of finite element methods to shells and the various computational problems attached with them. In this direction, let us quote FRIED (1971b), GALLAGHER (1973). In the case of large deflections, see BATOZ, CHATTOPADHYAY & DHATT (1976), MATSUI & MATSUOKA (1976).

By contrast, there are very few papers that deal with the numerical analysis of such methods. MIYOSHI (1973b) has analyzed the convergence of a mixed finite element method for cylindrical shells and GELLERT & LAURSEN (1976) study a mixed method for arches. KIKUCHI & ANDO (1972d, 1973b) have described the application of a simplified hybrid method to shallow shells. A hybrid method for shells is also considered by STEPHAN & WEISSGERBER (1976). MOAN (1974) has examined the asymptotic rate of energy convergence for arches. KIKUCHI (1975a) has proposed a simplified method for thin "shallow" shells. Let us also mention the analysis of CLÉMENT & DESCLOUX (1972) regarding the validity of the rigid displacement condition (Remark 8.1.1) for the discrete problem.

The effect of curved boundaries and numerical integration (Remark 8.2.1) is analyzed in BERNADOU (1976) along the lines of the present treatment.

As was pointed out in the introduction, an elegant way of approximating shell problems consists in using finite elements directly derived from three-dimensional finite elements by reducing their thickness. See AHMAD, IRONS & ZIENKIEWICZ (1970), ZIENKIEWICZ, TAYLOR & TOO (1971). The corresponding numerical analysis is yet to be done. A related, and challenging, problem is to describe and analyze the "intermediate" finite elements which should be used at the junctions between two-dimensional or three-dimensional portions of a single mechanical structure.

*8.3.* The content of this section is based on a recent paper by JOHNSON

(1975), which we have presented along the lines of Section 8.2 (the expressions (8.3.22) of the rotations at the points $\varphi(s_i)$ are justified for example in NOVOZHILOV (1970)). For a related analysis, see KIKUCHI (1975b, 1976a).

It is an open problem to extend this type of analysis to general shell problems. In this direction, DAWE (1972) considers a method which uses "flat" elements, i.e., for which one has $\varphi_{h|K} \in (P_1(K))^3$ for all $K \in \mathcal{T}_h$. More general, but still nonconforming, elements are also in use. See for example IRONS (1974a).