The Finite Element Method for Elliptic Problems

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The Finite Element Method for Elliptic Problems

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This SIAM edition is an unabridged republication of the work first published by North-Holland, Amsterdam, New York, Oxford, 1978.

10 9 8 7 6 5 4 3

Library of Congress Cataloging-in-Publication Data

Ciarlet, Philippe G.

The finite element method for elliptic problems / Philippe G. Ciarlet.

p. cm. -- (Classics in applied mathematics ; 40)

Includes bibliographical references and index.

ISBN 978-0-898715-14-9 (pbk.)

1. Differential equations, Elliptic--Numerical solutions. 2. Boundary value problems--Numerical solutions. 3. Finite element method. I. Title. II. Series.

QA377 .C53 2002

515'.353--dc21

2002019515

To Monique

TABLE OF CONTENTS

PREFACE TO THE CLASSICS EDITION

Although almost 25 years have elapsed since the manuscript of this book was completed, it is somewhat comforting to see that the content of Chapters 1 to 6, which together could be summarized under the title "The Basic Error Estimates for Elliptic Problems," is still essentially up-to-date. More specifically, the topics covered in these chapters are the following:

- description and mathematical analysis of various problems found in linearized elasticity, such as the membrane and plate equations, the equations of three-dimensional elasticity, and the obstacle problem;
- description of conforming finite elements used for approximating second-order and fourth-order problems, including composite and singular elements;
- derivation of the fundamental error estimates, including those in maximum norm, for conforming finite element methods applied to second-order problems;
- derivation of error estimates for the obstacle problem;
- description of finite element methods with numerical integration for second-order problems and derivation of the corresponding error estimates;
- description of nonconforming finite element methods for second-order and fourth-order problems and derivation of the corresponding error estimates;
- description of the combined use of isoparametric finite elements and isoparametric numerical integration for second-order problems posed over domains with curved boundaries and derivation of the corresponding error estimates;
- derivation of the error estimates for polynomial, composite, and singular finite elements used for solving fourth-order problems.

Otherwise, the topics considered in Chapters 7 and 8 have since undergone considerable progress. Additionally, new topics have emerged that often address the essential issue of the actual implementation of the finite element method. The interested reader may thus wish to consult the following more recent books, the list of which is by no means intended to be exhaustive:

- for further types of error estimates, a posteriori error estimates, locking phenomena, and numerical implementation: Brenner and Scott (1994), Wahlbin (1991, 1995), Lucquin and Pironneau (1998), Apel (1999), Ainsworth and Oden (2000), Bramble and Zhang (2000), Frey and George (2000), Zienkiewicz and Taylor (2000), Babuska and Strouboulis (2001), Braess (2001);
- for mixed and hybrid finite element methods: Girault and Raviart (1986), Brezzi and Fortin (1991), Robert and Thomas (1991);
- for finite element approximations of eigenvalue problems: Babuska and Osborn (1991);
- for finite element approximations of variational inequalities: Glowinski (1984);
- for finite element approximations of shell problems: Bernadou (1995), Bathe (1996);
- for finite element approximations of time-dependent problems: Raviart and Thomas (1983), Thomée (1984), Hughes (1987), Fujita and Suzuki (1991).

Last but not least, it is my pleasure to express my sincere thanks to Sara J. Triller, Arjen Sevenster, and Gilbert Strang, whose friendly cooperation made this reprinting possible.

Philippe G. Ciarlet
October 2001

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PREFACE

The *objective* of this book is to analyze within reasonable limits (it is not a treatise) the basic mathematical aspects of the finite element method. The book should also serve as an introduction to current research on this subject.

On the one hand, it is also intended to be a working textbook for advanced courses in Numerical Analysis, as typically taught in graduate courses in American and French universities. For example, it is the author's experience that a one-semester course (on a three-hour per week basis) can be taught from Chapters 1, 2 and 3 (with the exception of Section 3.3), while another one-semester course can be taught from Chapters 4 and 6.

On the other hand, it is hoped that this book will prove to be useful for researchers interested in advanced aspects of the numerical analysis of the finite element method. In this respect, Section 3.3, Chapters 5, 7 and 8, and the sections on "Additional Bibliography and Comments" should provide many suggestions for conducting seminars.

Although the emphasis is mathematical, it is one of the author's wishes that some parts of the book will be of some value to engineers, whose familiar objects are perhaps seen from a different viewpoint. Indeed, in the selection of topics, we have been careful in considering only actual problems and we have likewise restricted ourselves to finite element methods which are actually used in contemporary engineering applications.

The *prerequisites* consist essentially in a good knowledge of Analysis and Functional Analysis, notably: Hilbert spaces, Sobolev spaces, and Differential Calculus in normed vector spaces. Apart from these preliminaries and some results on elliptic boundary value problems (regularity properties of the solutions, for example), the book is mathematically self-contained.

The *main topics covered* are the following:

Description and mathematical analysis of linear second- and fourth-order boundary value problems which are typically found in elasticity theory: System of equations of two-dimensional and three-dimensional elasticity, problems in the theory of membranes, thin plates, arches, thin shells (Chapters 1 and 8).

Description and mathematical analysis of some nonlinear second-order boundary value problems, such as the obstacle problem (and more generally problems modeled by variational inequalities), the minimal surface problem, problems of monotone type (Chapter 5).

Description of conforming finite element methods for solving second-order or fourth-order problems (Chapter 2).

Analysis of the convergence properties of such methods for second-order problems, including the uniform convergence (Chapter 3), and fourth-order problems (Section 6.1).

Description and convergence analysis of finite element methods with numerical integration (Section 4.1).

Description and convergence analysis of nonconforming finite element methods for second-order problems (Section 4.2) and fourth-order problems (Section 6.2).

Description and interpolation theory for isoparametric finite elements (Section 4.3).

Description and convergence analysis of the combined use of isoparametric finite elements and numerical integration for solving second-order problems over domains with curved boundaries (Section 4.4).

Convergence analysis of finite element approximations of some nonlinear problems (Chapter 5).

Description and convergence analysis of a mixed finite element method for solving the biharmonic problem, with an emphasis on duality theory, especially as regards the solution of the associated discrete problem (Chapter 7).

Description and convergence analysis of finite element methods for arches and shells, including an analysis of the approximation of the geometry by curved and flat elements (Chapter 8).

For more detailed information, the reader should consult the Introductions of the Chapters.

It is also appropriate to comment on some of the *omitted topics*. As suggested by the title, we have restricted ourselves to elliptic problems, and this restriction is obviously responsible for the omission of finite element methods for time-dependent problems, a subject which would require another volume. In fact, for such problems, the content of this book should amply suffice for those aspects of the theory which are directly related to the finite element method. The additional analysis, due to the change in the nature of the partial differential equation, requires functional analytic tools of a different nature.

The main omissions within the realm of elliptic boundary value problems concern the so-called hybrid and equilibrium finite element methods, and also mixed methods other than that described in Chapter 7. There are basically two reasons behind these omissions: First, the basic theory for such methods was not yet in a final form by the time this book was completed. Secondly, these methods form such wide and expanding a topic that their inclusion would have required several additional chapters. Other notable omissions are finite element methods for approximating the solution of particular problems, such as problems on unbounded domains, Stokes and Navier–Stokes problems and eigenvalue problems.

Nevertheless, introductions to, and references for, the topics mentioned in the above paragraph are given in the sections titled "Additional Bibliography and Comments".

As a rule, all topics which would have required further analytic tools (such as nonintegral Sobolev spaces for instance) have been deliberately omitted.

Many results are left as exercises, which is not to say that they should be systematically considered less important than those proved in the text (their inclusion in the text would have meant a much longer book).

The book comprises *eight chapters*. Chapter n, $1 \leq n \leq 8$, contains an *introduction*, several *sections* numbered Section $n.1$, Section $n.2$, etc..., and a section "Bibliography and Comments", sometimes followed by a section "Additional Bibliography and Comments". Theorems, remarks, formulas, figures, and exercises, found in each section are numbered with a three-number system. Thus the second theorem of Section 3.2 is "Theorem 3.3.3", the fourth remark in Section 4.4 is "Remark 4.4.4", the twelfth formula of Section 8.3 is numbered (8.3.12) etc.... The end of a theorem or of a remark is indicated by the symbol □.

Since the sections (which correspond to a logical subdivision of the text) may vary considerably in length, unnumbered subtitles have been added in each section to help the reader (they appear in the table of contents).

The *theorems* are intended to represent important results. Their number have been kept to a minimum, and there are no lemmas, propositions, or corollaries. This is why the proofs of the theorems are sometimes fairly long. In principle, one can skip the *remarks* during a first reading. When a term is defined, it is set in italics. Terms which are only given a loose or intuitive meaning are put between quotation marks. There are very few references in the body of the text. All relevant bibliographical material is instead indicated in the sections "Bibliography and Comments" and "Additional Bibliography and Comments".

Underlying the writing of this book, there has been a deliberate attempt to put an emphasis on *pedagogy*. In particular:

All pertinent prerequisite material is clearly delineated and kept to a minimum. It is introduced only when needed.

Complete proofs are generally given. However, some technical results or proofs which resemble previous proofs are occasionally left to the reader.

The chapters are written in such a way that it should not prove too hard for a reader already reasonably familiar with the finite element method to read a given chapter almost independently of the previous chapters. Of course, this is at the expense of some redundancies, which are purposefully included. For the same reason, the index, the glossary of symbols and the interdependence table should be useful.

It is in particular with an eye towards classroom use and self-study that *exercises* of varying difficulty are included at the end of the sections. Some exercises are easy and are simply intended to help the reader in getting a better understanding of the text. More challenging problems (which are generally provided with hints and/or references) often concern significant extensions of the material of the text (they generally comprise several questions, numbered (i), (ii), . . .).

In most sections, a significant amount of material (generally at the beginning) is devoted to the introductive and descriptive aspects of the topic under consideration.

Many *figures* are included, which hopefully will help the reader. Indeed, it is the author's opinion that one of the most fascinating aspects of the finite element method is that it entails a rehabilitation of old-fashioned "classical" geometry (considered as completely obsolete, it has almost disappeared in the curriculae of French secondary schools).

There was no systematic attempt to compile an exhaustive *bibliography*. In particular, most references before 1970 and/or from the engineering literature and/or from Eastern Europe are not quoted. The interested reader is referred to the bibliography of Whiteman (1975). An effort was made, however, to include the *most recent* references (published or unpublished) of which the author was aware, as of October, 1976.

In attributing proper names to some finite elements and theorems, we have generally simply followed the common usages in French universities, and we hope that these choices will not stir up controversies. Our purpose was not to take issues but rather to give due credit to some of those who are clearly responsible for the invention, or the mathematical justification of, some aspects of the finite element method.

For providing a very stimulating and challenging scientific atmosphere, I wish to thank all my colleagues of the Laboratoire d'Analyse Numérique at the Université Pierre et Marie Curie, particularly Pierre-Arnaud Raviart and Roland Glowinski. Above all, it is my pleasure to express my very deep gratitude to Jacques-Louis Lions, who is responsible for the creation of this atmosphere, and to whom I personally owe so much.

For their respective invitations to Bangalore and Montréal, I express my sincere gratitude to Professor K.G. Ramanathan and to Professor A. Daigneault. Indeed, this book is an outgrowth of Lectures which I was privileged to give in Bangalore as part of the "Applied Mathematics Programme" of the Tata Institute of Fundamental Research, Bombay, and at the University of Montréal, as part of the "Séminaire de Mathématiques Supérieures".

For various improvements, such as shorter proofs and better exposition at various places, I am especially indebted to J. Tinsley Oden, Vidar Thomée, Annie Puech-Raoult and Michel Bernadou, who have been kind enough to entirely read the manuscript.

For kindly providing me with computer graphics and drawings of actual triangulations, I am indebted to Professors J. H. Argyris, C. Felippa, R. Glowinski and O. C. Zienkiewicz, and to the Publishers who authorized the reprinting of these figures.

For their understanding and kind assistance as regards the material realization of this book, sincere thanks are due to Mrs. Damperat, Mrs. Theis and Mr. Riguet.

For their expert, diligent, and especially fast, typing of the entire manuscript, I very sincerely thank Mrs. Bugler and Mrs. Guille.

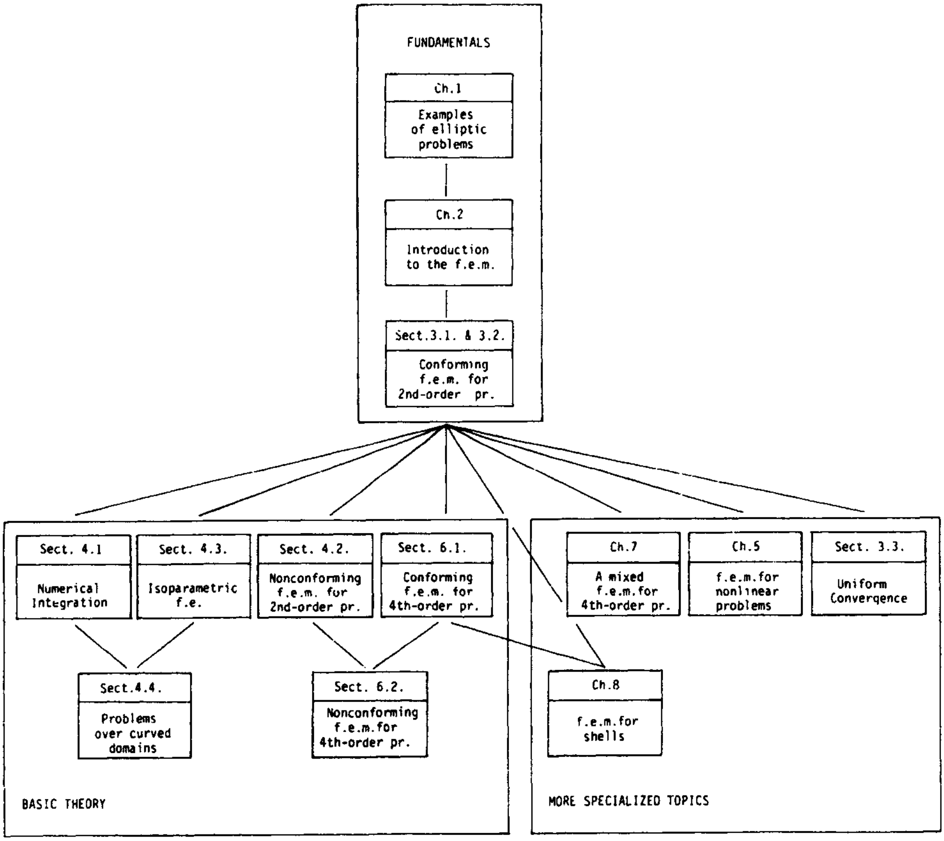
For a considerable help in proofreading and in the general elaboration of the manuscript, and for a permanent comprehension in spite of a finite, but

large, number of lost week-ends and holidays, I deeply thank the one to whom this book is dedicated.

The author welcomes in advance all comments, suggestions, criticisms, etc.

December 1976 Philippe G. Ciarlet

GENERAL PLAN AND INTERDEPENDENCE TABLE



"A mathematician's nightmare is a sequence n_ϵ that tends to 0 as ϵ becomes infinite."

Paul R. HALMOS: *How to Write Mathematics*, A.M.S., 1973.