CHAPTER 1

ELLIPTIC BOUNDARY VALUE PROBLEMS

Introduction

Many problems in elasticity are mathematically represented by the following minimization problem: The unknown u, which is the *displacement* of a *mechanical system*, satisfies

$$u \in U \quad \text{and} \quad J(u) = \inf_{v \in U} J(v),$$

where the set U of *admissible displacements* is a closed convex subset of a Hilbert space V, and the *energy* J of the system takes the form

$$J(v) = \tfrac{1}{2}a(v, v) - f(v),$$

where $a(\cdot, \cdot)$ is a symmetric bilinear form and f is a linear form, both defined and continuous over the space V. In Section 1.1, we first prove a general existence result (Theorem 1.1.1), the main assumptions being the *completeness* of the space V and the V-*ellipticity* of the bilinear form. We also describe other formulations of the same problem (Theorem 1.1.2), known as its *variational formulations*, which, in the absence of the assumption of symmetry for the bilinear form, make up *variational problems* on their own. For such problems, we give an existence theorem when $U = V$ (Theorem 1.1.3), which is the well-known *Lax–Milgram lemma.*

All these problems are called *abstract problems* inasmuch as they represent an "abstract" formulation which is common to many examples, such as those which are examined in Section 1.2.

From the analysis made in Section 1.1, a candidate for the space V must have the following properties: It must be complete on the one hand, and it must be such that the expression $J(v)$ is well-defined for all functions $v \in V$ on the other hand (V is a "space of finite energy"). The *Sobolev spaces* fulfill these requirements. After briefly mentioning some of their properties (other properties will be introduced in later sections, as needed), we examine in Section 1.2 specific examples of the abstract problems of Section 1.1, such as the *membrane problem*, the *clamped plate problem*, and the *system of equations of linear elasticity*, which is by far the most significant example. Indeed, even though throughout this book we will often find it convenient to work with the simpler looking problems described at the beginning of Section 1.2, it must not be forgotten that these are essentially convenient *model problems* for the system of linear elasticity.

Using various *Green's formulas* in Sobolev spaces, we show that when solving these problems, one solves, at least *formally*, elliptic boundary value problems of the second and fourth order posed in the classical way.

1.1. Abstract problems

The symmetric case. Variational inequalities

All functions and vector spaces considered in this book are real.

Let there be given a normed vector space V with norm $\|\cdot\|$, a *continuous* bilinear form $a(\cdot,\cdot): V \times V \to \mathbf{R}$, a *continuous* linear form $f: V \to \mathbf{R}$ and a non empty subset U of the space V. With these data we associate an *abstract minimization problem*: Find an element u such that

$$u \in U \quad \text{and} \quad J(u) = \inf_{v \in U} J(v), \tag{1.1.1}$$

where the functional $J: V \to \mathbf{R}$ is defined by

$$J: v \in V \to J(v) = \tfrac{1}{2}a(v, v) - f(v). \tag{1.1.2}$$

As regards existence and uniqueness properties of the solution of this problem, the following result is essential.

Theorem 1.1.1. *Assume in addition that*

(i) *the space V is complete,*
(ii) *U is a closed convex subset of V,*
(iii) *the bilinear form $a(\cdot,\cdot)$ is symmetric and V-elliptic, in the sense that*

$$\exists \alpha > 0, \quad \forall v \in V, \quad \alpha\|v\|^2 \leq a(v, v). \tag{1.1.3}$$

Then the abstract minimization problem (1.1.1) *has one and only one solution.*

Proof. The bilinear form $a(\cdot,\cdot)$ is an inner product over the space V, and the associated norm is equivalent to the given norm $\|\cdot\|$. Thus the space V is a Hilbert space when it is equipped with this inner product. By the Riesz representation theorem, there exists an element $\sigma f \in V$ such that

$$\forall v \in V, \quad f(v) = a(\sigma f, v),$$

so that, taking into account the symmetry of the bilinear form, we may rewrite the functional as

$$J(v) = \tfrac{1}{2}a(v, v) - a(\sigma f, v) = \tfrac{1}{2}a(v - \sigma f, v - \sigma f) - \tfrac{1}{2}a(\sigma f, \sigma f).$$

Hence solving the abstract minimization problem amounts to minimizing the distance between the element σf and the set U, with respect to the norm $\sqrt{a(\cdot,\cdot)}$. Consequently, the solution is simply the projection of the element σf onto the set U, with respect to the inner product $a(\cdot,\cdot)$. By the projection theorem, such a projection exists and is unique, since U is a closed convex subset of the space V. □

Next, we give equivalent formulations of this problem.

Theorem 1.1.2. *An element u is the solution of the abstract minimization problem* (1.1.1) *if and only if it satisfies the relations*

$$u \in U \quad \textit{and} \quad \forall v \in U, \quad a(u, v-u) \geq f(v-u), \tag{1.1.4}$$

in the general case, or

$$u \in U \quad \textit{and} \quad \begin{cases} \forall v \in U, & a(u, v) \geq f(v), \\ & a(u, u) = f(u), \end{cases} \tag{1.1.5}$$

if U is a closed convex cone with vertex 0, *or*

$$u \in U \quad \textit{and} \quad \forall v \in U, \quad a(u, v) = f(v), \tag{1.1.6}$$

if U is a closed subspace.

Proof. The projection u is completely characterized by the relations

$$u \in U \quad \text{and} \quad \forall v \in U, \quad a(\sigma f - u, v - u) \leq 0, \tag{1.1.7}$$

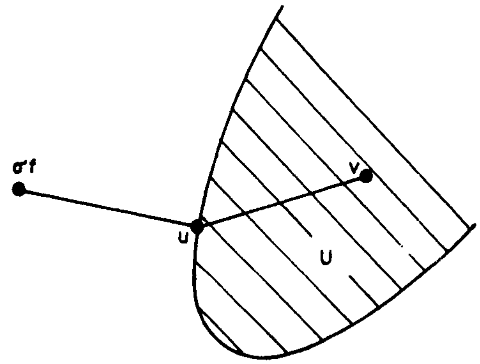

Fig. 1.1.1

the geometrical interpretation of the last inequalities being that the angle between the vectors $(\sigma f - u)$ and $(v - u)$ is obtuse (Fig. 1.1.1) for all $v \in U$. These inequalities may be written as

$$\forall v \in U, \quad a(u, v-u) \geqslant a(\sigma f, v-u) = f(v-u),$$

which proves relations (1.1.4).

Assume next U is a closed convex cone with vertex 0. Then the point $(u+v)$ belongs to the set U whenever the point v belongs to the set U (Fig. 1.1.2).

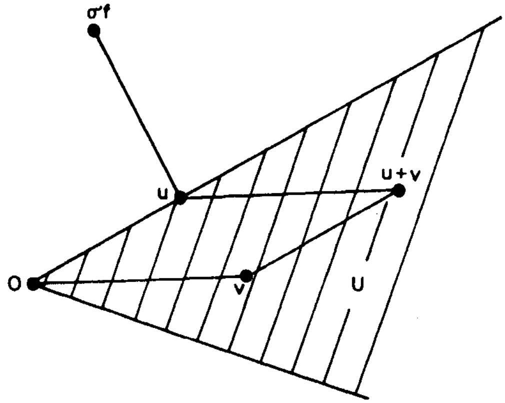

Fig. 1.1.2

Therefore, upon replacing v by $(u + v)$ in inequalities (1.1.4), we obtain the inequalities

$$\forall v \in U, \quad a(u, v) \geqslant f(v),$$

so that, in particular, $a(u, u) \geqslant f(u)$. Letting $v = 0$ in (1.1.4), we obtain $a(u, u) \leqslant f(u)$, and thus relations (1.1.5) are proved. The converse is clear.

If U is a subspace (Fig. 1.1.3), then inequalities (1.1.5) written with v and $-v$ yield $a(u, v) \geqslant f(v)$ and $a(u, v) \leqslant f(v)$ for all $v \in U$, from which relations (1.1.6) follow. Again the converse is clear. □

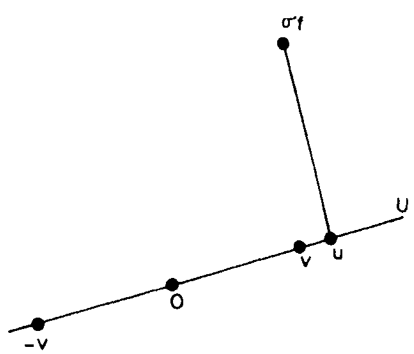

Fig. 1.1.3

The characterizations (1.1.4), (1.1.5) and (1.1.6) are called *variational formulations* of the original minimization problem, the equations (1.1.6) are called *variational equations*, and the inequalities of (1.1.4) and (1.1.5) are referred to as *variational inequalities*. The terminology "variational" will be justified in Remark 1.1.2.

Remark 1.1.1. Since the projection mapping is linear if and only if the subset U is a subspace, it follows that *problems associated with variational inequalities are generally non linear*, the linearity or non linearity being that of the mapping $f \in V' \to u \in V$, where V' is the dual space of V, all other data being fixed. One should not forget, however, that if the resulting problem is linear when one minimizes over a subspace this is also because the functional is *quadratic* i.e., it is of the form (1.1.2). The minimization of more general functionals over a subspace would correspond to nonlinear problems (cf. Section 5.3). □

Remark 1.1.2. *The variational formulations of Theorem* 1.1.2 *may be also interpreted from the point of view of Differential Calculus*, as follows. We first observe that the functional J is differentiable at every point $u \in V$, its (Fréchet) derivative $J'(u) \in V'$ being such that

$$\forall v \in V, \quad J'(u)v = a(u, v) - f(v). \tag{1.1.8}$$

Let then u be the solution of the minimization problem (1.1.1), and let $v = u + w$ be any point of the convex set U. Since the points $(u + \theta w)$ belong to the set U for all $\theta \in [0, 1]$ (Fig. 1.1.4), we have, by definition of

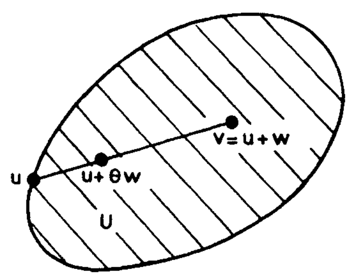

Fig. 1.1.4

the derivative $J'(u)$,

$$0 \leqslant J(u + \theta w) - J(u) = \theta J'(u)w + \theta \|w\| \epsilon(\theta)$$

for all $\theta \in [0, 1]$, with $\lim_{\theta \to 0} \epsilon(\theta) = 0$. As a consequence, we necessarily have

$$J'(u)w \geqslant 0, \tag{1.1.9}$$

since otherwise the difference $J(u + \theta w) - J(u)$ would be strictly negative for θ small enough. Using (1.1.8), inequality (1.1.9) may be rewritten as

$$J'(u)w = J'(u)(v - u) = a(u, v - u) - f(v - u) \geqslant 0,$$

which is precisely (1.1.4). Conversely, assume we have found an element $u \in U$ such that

$$\forall v \in U, \quad J'(u)(v - u) \geqslant 0. \tag{1.1.10}$$

The second derivative $J''(u) \in \mathscr{L}_2(V; \mathbf{R})$ of the functional J is independent of $u \in V$ and it is given by

$$\forall v, w \in V, \quad J''(u)(v, w) = a(v, w). \tag{1.1.11}$$

Therefore, an application of Taylor's formula for any point $v = u + w$ belonging to the set U yields

$$J(u + w) - J(u) = J'(u)(w) + \frac{1}{2}a(w, w) \geq \frac{\alpha}{2}\|w\|^2, \tag{1.1.12}$$

which shows that u is a solution of problem (1.1.1). We have $J(v) - J(u) > 0$ unless $v = u$ so that we see once again the solution is unique.

Arguing as in the proof of Theorem 1.1.2, it is an easy matter to verify that inequalities (1.1.10) are equivalent to the relations

$$\forall v \in U, \quad J'(u)v \geq 0 \quad \text{and} \quad J'(u)u = 0, \tag{1.1.13}$$

when U is a convex cone with vertex 0, alternately,

$$\forall v \in U, \quad J'(u)v = 0, \tag{1.1.14}$$

when U is a subspace. Notice that relations (1.1.13) coincide with relations (1.1.5), while (1.1.14) coincide with (1.1.6).

When $U = V$, relations (1.1.14) reduce to the familiar condition that the *first variation* of the functional J, i.e., the first order term $J'(u)w$ in the Taylor expansion (1.1.12), vanishes for all $w \in V$ when the point u is a minimum of the function $J: V \to \mathbf{R}$, this condition being also sufficient if the function J is convex, as is the case here. Therefore the various relations (1.1.4), (1.1.5) and (1.1.6), through the equivalent relations (1.1.10), (1.1.13) and (1.1.14), appear as generalizations of the previous condition, the expression $a(u, v - u) - f(v - u) = J'(u)(v - u)$, $v \in U$, playing in the present situation the role of the first variation of the functional J *relative to the convex set* U. It is in this sense that the formulations of Theorem 1.1.2 are called "variational". □

The nonsymmetric case. The Lax–Milgram lemma

Without making explicit reference to the functional J, we now define an *abstract variational problem*: Find an element u such that

$$u \in U \quad \text{and} \quad \forall v \in U, \quad a(u, v - u) \geq f(v - u), \tag{1.1.15}$$

or, find an element u such that

$$u \in U \quad \text{and} \quad \begin{cases} \forall v \in U, & a(u,v) \geq f(v), \\ a(u,u) = f(u), \end{cases} \tag{1.1.16}$$

if U is a cone with vertex 0, or, finally, find an element u such that

$$u \in U \quad \text{and} \quad \forall v \in V, \quad a(u,v) = f(v), \tag{1.1.17}$$

if U is a subspace. By Theorem 1.1.1, each such problem has one and only one solution if the space V is complete, the subset U of V is closed and convex, and the bilinear form is V-elliptic, continuous, and symmetric. If the assumption of symmetry of the bilinear form is dropped, the above variational problem still has one and only one solution (LIONS & STAMPACCHIA (1967)) if the space V is a Hilbert space, but there is no longer an associated minimization problem. Here we shall confine ourselves to the case where $U = V$.

Theorem 1.1.3 (*Lax–Milgram lemma*). *Let V be a Hilbert space, let $a(\cdot,\cdot)$: $V \times V \to \mathbf{R}$ be a continuous V-elliptic bilinear form, and let f: $V \to \mathbf{R}$ be a continuous linear form.*

Then the abstract variational problem: Find an element u such that

$$u \in V \quad \text{and} \quad \forall v \in V, \quad a(u,v) = f(v), \tag{1.1.18}$$

has one and only one solution.

Proof. Let M be a constant such that

$$\forall u, v \in V, \quad |a(u,v)| \leq M\|u\|\,\|v\|. \tag{1.1.19}$$

For each $u \in V$, the linear form $v \in V \to a(u,v)$ is continuous and thus there exists a unique element $Au \in V'$ (V' is the dual space of V) such that

$$\forall v \in V, \quad a(u,v) = Au(v). \tag{1.1.20}$$

Denoting by $\|\cdot\|^*$ the norm in the space V', we have

$$\|Au\|^* = \sup_{v \in V} \frac{|Au(v)|}{\|v\|} \leq M\|u\|.$$

Consequently, the linear mapping A: $V \to V'$ is continuous, with

$$\|A\|_{\mathscr{L}(V;\,V')} \leq M. \tag{1.1.21}$$

Let τ: $V' \to V$ denote the Riesz mapping which is such that, by definition,

$$\forall f \in V', \quad \forall v \in V, \quad f(v) = ((\tau f, v)), \tag{1.1.22}$$

$((\cdot,\cdot))$ denoting the inner product in the space V. Then solving the variational problem (1.1.18) is equivalent to solving the equation $\tau A u = \tau f$. We will show that this equation has one and only one solution by showing that, for appropriate values of a parameter $\rho > 0$, the affine mapping

$$v \in V \to v - \rho(\tau A v - \tau f) \in V \tag{1.1.23}$$

is a contraction. To see this, we observe that

$$\begin{aligned}\|v - \rho\tau A v\|^2 &= \|v\|^2 - 2\rho((\tau A v, v)) + \rho^2\|\tau A v\|^2\\ &\leq (1 - 2\rho\alpha + \rho^2 M^2)\|v\|^2,\end{aligned}$$

since, using inequalities (1.1.3) and (1.1.21),

$$\begin{aligned}((\tau A v, v)) &= A v(v) = a(v, v) \geq \alpha\|v\|^2,\\ \|\tau A v\| &= \|A v\|^* \leq \|A\|\,\|v\| \leq M\|v\|.\end{aligned}$$

Therefore the mapping defined in (1.1.23) is a contraction whenever the number ρ belongs to the interval $]0, 2\alpha/M^2[$ and the proof is complete. □

Remark 1.1.3. It follows from the previous proof that the mapping $A: V \to V'$ is onto. Since

$$\alpha\|u\|^2 = a(u, u) = f(u) \leq \|f\|^*\|u\|,$$

the mapping A has a continuous inverse A^{-1}, with

$$\|A^{-1}\|_{\mathscr{L}(V';\,V)} \leq \frac{1}{\alpha}.$$

Therefore the variational problem (1.1.18) is *well-posed* in the sense that its solution *exists, is unique*, and *depends continuously on the data f* (all other data being fixed). □

Exercises

1.1.1. Show that if u_i, $i = 1, 2$, are the solutions of minimization problems (1.1.1) corresponding to linear form $f_i \in V'$, $i = 1, 2$, then

$$\|u_1 - u_2\| \leq \frac{1}{\alpha}\|f_1 - f_2\|^*.$$

(i) Give a proof which uses the norm reducing property of the projection operator.

(ii) Give another proof which also applies to the variational problem (1.1.15).

1.1.2. The purpose of this exercise is to give an alternate proof of the Lax–Milgram lemma (Theorem 1.1.3). As in the proof given in the text, one first establishes that the mapping $\mathcal{A} = \tau \cdot A: V \to V$ is continuous with $\|\mathcal{A}\| \leq M$, and that $\alpha\|v\| \leq \|\mathcal{A}v\|$ for all $v \in V$. It remains to show that $\mathcal{A}(V) = V$.

(i) Show that $\mathcal{A}(V)$ is a closed subspace of V.

(ii) Show that the orthogonal complement of $\mathcal{A}(V)$ in the space V is reduced to $\{0\}$.

1.2. Examples of elliptic boundary value problems

The Sobolev spaces $H^m(\Omega)$. Green's formulas

Let us first briefly recall some results from Differential Calculus. Let there be given two normed vector spaces X and Y and a function $v: A \to Y$, where A is a subset of X. If the function is k times differentiable at a point $a \in A$, we shall denote $D^k v(a)$, or simply $Dv(a)$ if $k = 1$, its k-th (*Fréchet*) *derivative*. It is a symmetric element of the space $\mathcal{L}_k(X; Y)$, whose norm is given by

$$\|D^k v(a)\| = \sup_{\substack{\|h_i\| \leq 1 \\ 1 \leq i \leq k}} \|D^k v(a)(h_1, h_2, \ldots, h_k)\|.$$

We shall also use the alternate notations $Dv(a) = v'(a)$ and $D^2 v(a) = v''(a)$.

In the special case where $X = \mathbf{R}^n$ and $Y = \mathbf{R}$, let e_i, $1 \leq i \leq n$, denote the canonical basis vectors of $\mathbf{R}^n$. Then the usual partial derivatives will be denoted by, and are given by, the following:

$$\partial_i v(a) = Dv(a)e_i,$$
$$\partial_{ij} v(a) = D^2 v(a)(e_i, e_j),$$
$$\partial_{ijk} v(a) = D^3 v(a)(e_i, e_j, e_k), \text{ etc.} \ldots$$

Occasionally, we shall use the notation $\nabla v(a)$, or grad $v(a)$, to denote the *gradient* of the function v at the point a, i.e., the vector in $\mathbf{R}^n$ whose components are the partial derivatives $\partial_i v(a)$, $1 \leq i \leq n$.

We shall also use the *multi-index notation*: Given a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{N}^n$, we let $|\alpha| = \sum_{i=1}^n \alpha_i$. Then the partial derivative $\partial^\alpha v(a)$ is the result of the application of the $|\alpha|$-th derivative $D^{|\alpha|}v(a)$ to any $|\alpha|$-vector of $(\mathbf{R}^n)^{|\alpha|}$ where each vector e_i occurs α_i times, $1 \leqslant i \leqslant n$. For instance, if $n = 3$, we have $\partial_1 v(a) = \partial^{(1,0,0)}v(a)$, $\partial_{123}v(a) = \partial^{(1,1,1)}v(a)$, $\partial_{111}v(a) = \partial^{(3,0,0)}v(a)$, etc...

There exist constants $C(m, n)$ such that for any partial derivative $\partial^\alpha v(a)$ with $|\alpha| = m$ and any function v,

$$|\partial^\alpha v(a)| \leqslant \|D^m v(a)\| \leqslant C(m, n) \max_{|\alpha|=m} |\partial^\alpha v(a)|,$$

where it is understood that the space $\mathbf{R}^n$ is equipped with the Euclidean norm.

As a rule, we shall represent by symbols such as $D^k v$, v'', $\partial_i v$, $\partial^\alpha v$, etc..., the *functions* associated with any derivative or partial derivative.

When $h_1 = h_2 = \cdots = h_k = h$, we shall simply write

$$D^k v(a)(h_1, h_2, \ldots, h_k) = D^k v(a)h^k.$$

Thus, given a real-valued function v, *Taylor's formula of order* k is written as

$$v(a+h) = v(a) + \sum_{l=1}^{k} \frac{1}{l!} D^l v(a)h^l + \frac{1}{(k+1)!} D^{k+1} v(a+\theta h)h^{k+1},$$

for some $\theta \in]0, 1[$ (whenever such a formula applies).

Given a bounded open subset Ω in $\mathbf{R}^n$, the space $\mathcal{D}(\Omega)$ consists of all indefinitely differentiable functions $v: \Omega \to \mathbf{R}$ with compact support.

For each integer $m \geqslant 0$, the *Sobolev space* $H^m(\Omega)$ consists of those functions $v \in L^2(\Omega)$ for which all partial derivatives $\partial^\alpha v$ (in the distribution sense), with $|\alpha| \leqslant m$, belong to the space $L^2(\Omega)$, i.e., for each multi-index α with $|\alpha| \leqslant m$, there exists a function $\partial^\alpha v \in L^2(\Omega)$ which satisfies

$$\forall \phi \in \mathcal{D}(\Omega), \quad \int_\Omega \partial^\alpha v\, \phi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v \partial^\alpha \phi \,\mathrm{d}\chi. \tag{1.2.1}$$

Equipped with the norm

$$\|v\|_{m,\Omega} = \Big(\sum_{|\alpha|\leqslant m} \int_\Omega |\partial^\alpha v|^2 \,\mathrm{d}x\Big)^{1/2},$$

the space $H^m(\Omega)$ *is a Hilbert space.* We shall also make frequent use of the semi-norm

$$|v|_{m,\Omega} = \Big(\sum_{|\alpha|= m} \int_\Omega |\partial^\alpha v|^2 \,\mathrm{d}x\Big)^{1/2}.$$

We define the *Sobolev space*

$$H_0^m(\Omega) = (\mathcal{D}(\Omega))^-,$$

the closure being understood in the sense of the norm $\|\cdot\|_{m,\Omega}$.

When the set Ω is bounded, there exists a constant $C(\Omega)$ such that

$$\forall v \in H_0^1(\Omega), \quad |v|_{0,\Omega} \leq C(\Omega)|v|_{1,\Omega}, \tag{1.2.2}$$

this inequality being known as the *Poincaré–Friedrichs inequality*.

Therefore, *when the set Ω is bounded, the semi-norm $|\cdot|_{m,\Omega}$ is a norm over the space $H_0^m(\Omega)$, equivalent to the norm $\|\cdot\|_{m,\Omega}$* (another way of reaching the same conclusion is indicated in the proof of Theorem 1.2.1 below).

The next definition will be sufficient for most subsequent purposes whenever some smoothness of the boundary is needed. It allows the consideration of all commonly encountered shapes without cusps. Following NEČAS (1967), we say that an open set Ω has a *Lipschitz-continuous boundary* Γ if the following conditions are fulfilled: There exist constants $\alpha > 0$ and $\beta > 0$, and a finite number of local coordinate systems and local maps a_r, $1 \leq r \leq R$, which are *Lipschitz-continuous on their respective domains of definitions* $\{\hat{x}^r \in \mathbf{R}^{n-1};\ |\hat{x}^r| \leq \alpha\}$, such that (Fig. 1.2.1):

$$\Gamma = \bigcup_{r=1}^{R} \{(x_1^r, \hat{x}^r);\ x_1^r = a_r(\hat{x}^r),\quad |\hat{x}^r| < \alpha\},$$

$$\{(x_1^r, \hat{x}^r);\quad a_r(\hat{x}^r) < x_1^r < a_r(\hat{x}^r) + \beta;\quad |\hat{x}^r| < \alpha\} \subset \Omega, \quad 1 \leq r \leq R,$$

$$\{(x_1^r, \hat{x}^r);\quad a_r(\hat{x}^r) - \beta < x_1^r < a_r(\hat{x}^r);\quad |\hat{x}^r| < \alpha\} \subset \complement\, \bar{\Omega}, \quad 1 \leq r \leq R,$$

where $\hat{x}^r = (x_2^r, \ldots, x_n^r)$, and $|\hat{x}^r| < \alpha$ stands for $|x_i^r| < \alpha$, $2 \leq i \leq n$. Notice in passing that *an open set with a Lipschitz-continuous boundary is bounded.*

Occasionally, we shall also need the following definitions: A boundary is *of class* $\mathscr{X}$ if the functions a_r: $|\hat{x}^r| \leq \alpha \to \mathbf{R}$ are of class $\mathscr{X}$ (such as $\mathscr{C}^m$ or $\mathscr{C}^{m,\alpha}$), and a boundary is said to be *sufficiently smooth* if it is of class $\mathscr{C}^m$, or $\mathscr{C}^{m,\alpha}$, for sufficient high values of m, or m and α (for a given problem).

In the remaining part of this section, it will be always understood that Ω is an open subset in $\mathbf{R}^n$ with a Lipschitz-continuous boundary. This being the case, a superficial measure, which we shall denote $d\gamma$, can be

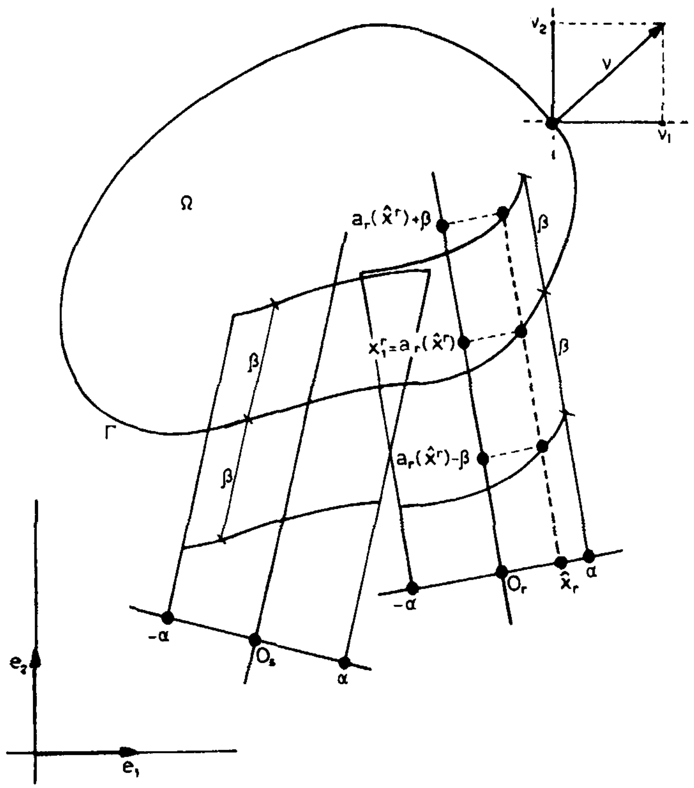

Fig. 1.2.1

defined along the boundary, so that it makes sense to consider the spaces $L^2(\Gamma)$, whose norm shall be denoted $\|\cdot\|_{L^2(\Gamma)}$.

Then it can be proved that there exists a constant $C(\Omega)$ such that

$$\forall v \in \mathscr{C}^\infty(\bar{\Omega}), \quad \|v\|_{L^2(\Gamma)} \le C(\Omega)\|v\|_{1,\Omega}. \tag{1.2.3}$$

Since in this case $(\mathscr{C}^\infty(\bar{\Omega}))^- = H^1(\Omega)$, the closure being understood in the sense of the norm $\|\cdot\|_{1,\Omega}$, there exists a continuous linear mapping tr: $v \in H^1(\Omega) \to \operatorname{tr} v \in L^2(\Gamma)$, which is called the *trace operator.* However when no confusion should arise, we shall simply write $\operatorname{tr} v = v$. The *following characterization holds*:

$$H_0^1(\Omega) = \{v \in H^1(\Omega);\quad v = 0 \quad \text{on} \quad \Gamma\}.$$

Since the unit outer normal $\nu = (\nu_1, \dots, \nu_n)$ (Fig. 1.2.1) exists almost everywhere along Γ, the (outer) *normal derivative operator*:

$$\partial_\nu = \sum_{i=1}^{n} \nu_i \partial_i$$

is defined almost everywhere along Γ for smooth functions. Extending its definition to $\partial_\nu = \sum_{i=1}^{n} \nu_i \operatorname{tr} \partial_i$ for functions in the space $H^2(\Omega)$, *the following characterization holds*:

$$H_0^2(\Omega) = \{v \in H^2(\Omega);\quad v = \partial_\nu v = 0 \quad \text{on} \quad \Gamma\}.$$

Given two functions $u, v \in H^1(\Omega)$, *the following fundamental Green's formula*

$$\int_\Omega u \partial_i v \, dx = -\int_\Omega \partial_i u \, v \, dx + \int_\Gamma u \, v \nu_i \, d\gamma \tag{1.2.4}$$

holds for any $i \in [1, n]$. From this formula, other *Green's formulas* may be easily deduced. For example, replacing u by $\partial_i u$ and taking the sum from 1 to n, we get

$$\int_\Omega \sum_{i=1}^{n} \partial_i u \partial_i v \, dx = -\int_\Omega \Delta u \, v \, dx + \int_\Gamma \partial_\nu u \, v \, d\gamma \tag{1.2.5}$$

for all $u \in H^2(\Omega)$, $v \in H^1(\Omega)$. As a consequence, we obtain by subtraction:

$$\int_\Omega (u \Delta v - \Delta u \, v) \, dx = \int_\Gamma (u \partial_\nu v - \partial_\nu u \, v) \, d\gamma \tag{1.2.6}$$

for all $u, v \in H^2(\Omega)$. Replacing u by Δu in formula (1.2.6), we obtain

$$\int_\Omega \Delta u \Delta v \, dx = \int_\Omega \Delta^2 u \, v \, dx - \int_\Gamma \partial_\nu \Delta u \, \nu \, d\gamma + \int_\Gamma \Delta u \partial_\nu n \, d\gamma \tag{1.2.7}$$

for all $u \in H^4(\Omega)$, $v \in H^2(\Omega)$. As another application of formula (1.2.4), let us prove the relation

$$\forall v \in H_0^2(\Omega), \quad |\Delta v|_{0,\Omega} = |v|_{2,\Omega}, \tag{1.2.8}$$

which implies that, *over the space* $H_0^2(\Omega)$, *the semi-norm* $v \to |\Delta v|_{0,\Omega}$ *is a norm, equivalent to the norm* $\|\cdot\|_{2,\Omega}$: We have, by definition,

$$|v|_{2,\Omega}^2 = \int_\Omega \left\{ \sum_i (\partial_{ii} v)^2 + \sum_{i \neq j} (\partial_{ij} v)^2 \right\} dx,$$

$$|\Delta v|_{0,\Omega}^2 = \int_\Omega \left\{ \sum_i (\partial_{ii} v)^2 + \sum_{i \neq j} \partial_{ii} v \partial_{jj} v \right\} dx.$$

Clearly, it suffices to prove relations (1.2.8) for all functions $v \in \mathcal{D}(\Omega)$. For these functions we have

$$\int_\Omega (\partial_{ij}v)^2 \, dx = -\int_\Omega \partial_i v \partial_{ijj} v \, dx = \int_\Omega \partial_{ii} v \partial_{jj} v \, dx,$$

as two applications of Green's formula (1.2.4) show, and thus (1.2.8) is proved.

For $n = 2$, let $\tau = (\tau_1, \tau_2)$ denote the unit tangential vector along the boundary Γ, oriented in the usual way. In addition to the normal derivative operator ∂_ν, we introduce the differential operators ∂_τ, $\partial_{\nu\tau}$, $\partial_{\tau\tau}$ defined by

$$\partial_\tau v(a) = Dv(a)\tau = \sum_{i=1}^{2} \tau_i \partial_i v(a),$$

$$\partial_{\nu\tau} v(a) = D^2 v(a)(\nu, \tau) = \sum_{i,j=1}^{2} \nu_i \tau_j \partial_{ij} v(a),$$

$$\partial_{\tau\tau} v(a) = D^2 v(a)(\tau, \tau) = \sum_{i,j=1}^{2} \tau_i \tau_j \partial_{ij} v(a).$$

Then we shall make use of the following Green's formula, whose proof is left as an exercise (Exercise 1.2.1):

$$\int_\Omega \{2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v - \partial_{22}u\partial_{11}v\} \, dx = \int_\Gamma \{-\partial_{\tau\tau}u\partial_\nu v + \partial_{\nu\tau}u\partial_\tau v\} \, d\gamma. \quad (1.2.9)$$

This relation holds for all functions $u \in H^3(\Omega)$, $v \in H^2(\Omega)$.

First examples of second-order boundary value problems

We next proceed to examine several examples of minimization and variational problems. According to the analysis made in Section 1.1, we need to specify for each example the space V, a subset U of the space V, a bilinear form $a(\cdot,\cdot)$: $V \times V \to \mathbf{R}$, and a linear form f: $V \to \mathbf{R}$. In fact, the examples given in this section correspond to the case where $U = V$, i.e., they all correspond to linear problems (Remark 1.1.1). A non linear problem is considered in Exercise 1.2.5, and another one will be considered in Section 5.1.

The *first example* corresponds to the following data:

$$\begin{cases} V = U = H_0^1(\Omega), \\ a(u, v) = \displaystyle\int_\Omega \Big(\sum_{i=1}^n \partial_i u \partial_i v + auv\Big)\,\mathrm{d}x, \\ f(v) = \displaystyle\int_\Omega fv\,\mathrm{d}x, \end{cases} \tag{1.2.10}$$

and to the following assumptions on the functions a and f:

$$a \in L^\infty(\Omega), \quad a \geq 0 \text{ a.e. } \quad \text{on} \quad \Omega, \quad f \in L^2(\Omega). \tag{1.2.11}$$

To begin with, it is clear that the symmetric bilinear form $a(\cdot,\cdot)$ is continuous since for all $u, v \in H^1(\Omega)$,

$$\begin{aligned} |a(u, v)| &\leq \sum_{i=1}^n |\partial_i u|_{0,\Omega} |\partial_i v|_{0,\Omega} + |a|_{0,\infty,\Omega} |u|_{0,\Omega} |v|_{0,\Omega} \\ &\leq \max\{1, |a|_{0,\infty,\Omega}\} \|u\|_{1,\Omega} \|v\|_{1,\Omega}, \end{aligned}$$

where $|\cdot|_{0,\Omega}$ and $|\cdot|_{0,\infty,\Omega}$ denote the norms of the space $L^2(\Omega)$ and $L^\infty(\Omega)$ respectively, and it is $H_0^1(\Omega)$-elliptic since, for all $v \in H^1(\Omega)$,

$$a(v, v) \geq \int_\Omega \sum_{i=1}^n (\partial_i v)^2\,\mathrm{d}x = |v|_{1,\Omega}^2$$

(the semi-norm $|\cdot|_{1,\Omega}$ is a norm over the space $H_0^1(\Omega)$, equivalent to the norm $\|\cdot\|_{1,\Omega}$). Next, the linear form f is continuous since for all $v \in H^1(\Omega)$,

$$|f(v)| \leq |f|_{0,\Omega} |v|_{0,\Omega} \leq |f|_{0,\Omega} \|v\|_{1,\Omega}.$$

Therefore, by Theorem 1.1.1, there exists a unique function $u \in H_0^1(\Omega)$ which minimizes the functional

$$J: v \to J(v) = \frac{1}{2} \int_\Omega \Big\{\sum_{i=1}^n (\partial_i v)^2 + av^2\Big\}\,\mathrm{d}x - \int_\Omega fv\,\mathrm{d}x \tag{1.2.12}$$

over the space $H_0^1(\Omega)$, or equivalently, by Theorem 1.1.2, which satisfies the variational equations

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \Big\{\sum_{i=1}^n \partial_i u \partial_i v + auv\Big\}\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x. \tag{1.2.13}$$

Using these equations, we proceed to show that *we are also solving a partial differential equation in the distributional sense.* More specifically, let $\mathcal{D}'(\Omega)$ denote the *space of distributions over the set* Ω, i.e., the dual space of the space $\mathcal{D}(\Omega)$ equipped with the Schwartz topology, and let $\langle\cdot,\cdot\rangle$ denote the duality pairing between the spaces $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$. *If g is a locally integrable function over Ω, we shall identify it with the distribution g: $\phi \in \mathcal{D}(\Omega) \to \int_\Omega g\phi \, dx$.*

Since the inclusion

$$\mathcal{D}(\Omega) \subset V = H_0^1(\Omega)$$

holds, the variational equations (1.2.13) are satisfied for all functions $v \in \mathcal{D}(\Omega)$. Therefore, by definition of the differentiation for distributions, we may write

$$\forall \phi \in \mathcal{D}(\Omega), \quad a(u, \phi) = \sum_{i=1}^{n} \langle \partial_i u, \partial_i \phi \rangle + \langle au, \phi \rangle = \langle -\Delta u + au, \phi \rangle.$$

Since $f(\phi) = \langle f, \phi \rangle$ for all $\phi \in \mathcal{D}(\Omega)$, it follows from the above relations that u is a solution of the partial differential equation $-\Delta u + au = f$ in $\mathcal{D}'(\Omega)$.

To sum up, the solution u of the minimization (or variational) problem associated with the data (1.2.10) is also a solution of the problem: *Find a distribution $u \in \mathcal{D}'(\Omega)$ such that*

$$u \in H_0^1(\Omega) \quad \textit{and} \quad -\Delta u + au = f \textit{ in } \mathcal{D}'(\Omega), \tag{1.2.14}$$

and conversely, if a distribution u satisfies (1.2.14), it is a solution of the original problem. To see this, we observe that the equalities

$$\forall \phi \in \mathcal{D}(\Omega), \quad a(u, \phi) = \langle -\Delta u + au, \phi \rangle = \langle f, \phi \rangle = f(\phi)$$

hold in fact for all functions $\phi \in H_0^1(\Omega)$ since $\mathcal{D}(\Omega)$ is a dense subspace of the space $H_0^1(\Omega)$.

Remembering that the functions in the space $H_0^1(\Omega)$ have a vanishing trace along Γ, we shall say that we have *formally solved* the *associated boundary value problem*

$$\begin{cases} -\Delta u + au = f & \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{1.2.15}$$

Problem (1.2.15) is called a *homogeneous Dirichlet problem* for the operator $u \to -\Delta u + au$, since it is formally posed exactly as in the classical sense where, typically, one would seek a solution in the space $\mathscr{C}^0(\bar{\Omega}) \cap \mathscr{C}^2(\Omega)$. Actually, when the data are sufficiently smooth, it can be proved (but this is non trivial) that the solution of (1.2.14) is also a solution of (1.2.15) in the classical sense. Nevertheless, one should keep in mind that, in general, nothing guarantees that the partial differential equation $-\Delta u + au = f$ in Ω can be given a sense otherwise than in the space $\mathcal{D}'(\Omega)$. Likewise the boundary condition $u = 0$ on Γ cannot be understood in general in other than the sense of a vanishing trace, or even in no sense at all if the set Ω were "only" supposed to be bounded.

With $a = 0$ and $n = 2$, the problem under analysis is called the *membrane problem*: It arises when one considers the problem of finding the *equilibrium position of an elastic membrane*, with tension τ, under the action of a "vertical" force, of density $F = \tau f$, and lying in the "horizontal" plane, of coordinates (x_1, x_2), when $f = 0$, as shown in Fig. 1.2.2 (where the vertical scale is considerably distorted if it were to correspond to an actual membrane). More general situations are considered in Exercise 1.2.2 and Section 5.1.

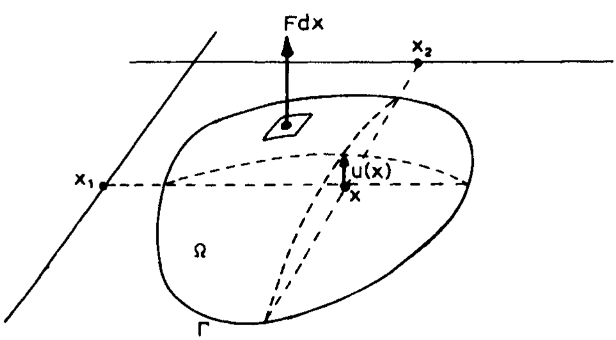

Fig. 1.2.2

The *second example* corresponds to the following data:

$$\begin{cases} V = U = H^1(\Omega), \\ a(u, v) = \displaystyle\int_\Omega \left(\sum_{i=1}^n \partial_i u \partial_i v + auv\right) \mathrm{d}x, \\ f(v) = \displaystyle\int_\Omega fv \,\mathrm{d}x + \int_\Gamma gv \,\mathrm{d}\gamma, \end{cases} \tag{1.2.16}$$

with the following assumptions on the functions a, f and g:

$$a \in L^\infty(\Omega), \quad a \geq a_0 > 0 \text{ a.e. on } \Omega, \quad f \in L^2(\Omega), \quad g \in L^2(\Gamma), \tag{1.2.17}$$

for some constant a_0.

The bilinear form is $H^1(\Omega)$-elliptic since $a(v, v) \geq \min\{1, a_0\}\|v\|_{1,\Omega}^2$ for all $v \in H^1(\Omega)$ (in Exercise 1.2.3, a case where $a_0 = 0$ is considered). The linear form $v \in H^1(\Omega) \to \int_\Gamma gv \,\mathrm{d}\gamma$ is continuous since by inequality (1.2.3),

$$\left| \int_\Gamma gv \,\mathrm{d}\gamma \right| \leq \|g\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} \leq C(\Omega) \|g\|_{L^2(\Gamma)} \|v\|_{1,\Omega}.$$

Therefore there exists a unique function $u \in H^1(\Omega)$ which minimizes the functional

$$J: v \to J(v) = \frac{1}{2} \int_\Omega \left\{ \sum_{i=1}^n (\partial_i v)^2 + av^2 \right\} \mathrm{d}x - \int_\Omega fv \,\mathrm{d}x - \int_\Gamma gv \,\mathrm{d}\gamma,$$

over the space $H^1(\Omega)$ or equivalently, such that

$$\forall v \in H^1(\Omega), \int_\Omega \left\{ \sum_{i=1}^n \partial_i u \partial_i v + auv \right\} \mathrm{d}x = \int_\Omega fv \,\mathrm{d}x + \int_\Gamma gv \,\mathrm{d}\gamma. \tag{1.2.18}$$

Because $\mathcal{D}(\Omega)$ is a subspace of the space $H^1(\Omega)$, an argument similar to the one used for the first example shows that u is also a solution of the partial differential equation $-\Delta u + au = f$ in $\mathcal{D}'(\Omega)$. Thus we have

$$\forall v \in H^1(\Omega), \int_\Omega (-\Delta u + au) v \,\mathrm{d}x = a(u, v) - \int_\Gamma gv \,\mathrm{d}\gamma.$$

To sum up, the solution u of the minimization (or variational) problem associated with the data (1.2.16) is also a solution of the problem: *Find a distribution $u \in \mathcal{D}'(\Omega)$ such that*

$$\begin{cases} u \in H^1(\Omega), \quad -\Delta u + au = f \quad \text{in} \quad \mathcal{D}'(\Omega), \\ \forall v \in H^1(\Omega), \quad \int_\Omega (-\Delta u + au) v \,\mathrm{d}x = a(u, v) - \int_\Gamma gv \,\mathrm{d}\gamma. \end{cases} \tag{1.2.19}$$

and, conversely, if a distribution u is a solution of problem (1.2.19), it is clearly a solution of the variational equations (1.2.18).

If we assume additional smoothness on the solution, *the second relations* (1.2.19) *can be interpreted as playing the role of boundary conditions*. If the solution u is in the space $H^2(\Omega)$, for example, an application of Green's formula (1.2.5) shows that, for all $v \in H^1(\Omega)$,

$$\begin{aligned} a(u, v) &= \int_\Omega (-\Delta u + au) v \,\mathrm{d}x + \int_\Gamma \partial_\nu u \, v \,\mathrm{d}\gamma, \\ &= \int_\Omega fv \,\mathrm{d}x + \int_\Gamma gv \,\mathrm{d}\gamma. \end{aligned} \tag{1.2.20}$$

Therefore the conjunction of relations (1.2.19) and (1.2.20) implies that

$$\forall v \in H^1(\Omega) \int_\Gamma \partial_\nu u \, v \, \mathrm{d}\gamma = \int_\Gamma gv \, \mathrm{d}\gamma. \tag{1.2.21}$$

From these, one deduces that $\partial_\nu u = g$ on Γ.

Consequently, we shall say that *we have formally solved* the *associated boundary value problem*:

$$\begin{cases} -\Delta u + au = f & \text{in} \quad \Omega, \\ \partial_\nu u = g & \text{on} \quad \Gamma, \end{cases} \tag{1.2.22}$$

which is called a *nonhomogeneous Neumann problem* if $g \neq 0$, or a *homogeneous Neumann problem* if $g = 0$, for the operator $u \to -\Delta u + au$.

Remark 1.2.1. Without using differentiation of distributions, Green's formula (1.2.5) gives another way to obtain the partial differential equation since

$$\forall \phi \in \mathcal{D}(\Omega), \quad a(u, \phi) = \int_\Omega (-\Delta u + au)\phi \, \mathrm{d}x = \langle -\Delta u + au, \phi \rangle.$$

Of course, this is not a coincidence: The definition of differentiation for distributions is precisely based upon the fundamental Green's formula (1.2.4). □

In the *third example*, we shall extend in three directions the previous analysis: First the associated partial differential equation will have non constant coefficients. Secondly, the bilinear form will not be necessarily symmetric so that Theorem 1.1.3 will be needed for the existence analysis, and thirdly, the space V will be "intermediate" between the spaces $H_0^1(\Omega)$ and $H^1(\Omega)$. The data are the following:

$$\begin{cases} V = U = \{v \in H^1(\Omega); \quad v = 0 \quad \text{on} \quad \Gamma_0\}, \\ a(u, v) = \displaystyle\int_\Omega \left\{ \sum_{i,j=1}^n a_{ij} \partial_i u \partial_j v + auv \right\} \mathrm{d}x, \\ f(v) = \displaystyle\int_\Omega fv \, \mathrm{d}x + \int_{\Gamma_1} gv \, \mathrm{d}\gamma, \end{cases} \tag{1.2.23}$$

where $\Gamma_0 = \Gamma - \Gamma_1$ is a $\mathrm{d}\gamma$-measurable subset of the boundary Γ with a strictly positive $\mathrm{d}\gamma$-measure, and the functions a_{ij}, a and f satisfy the following assumptions:

$$\begin{cases} a_{ij} \in L^\infty(\Omega), \quad 1 \leq i,j \leq n, \quad a \in L^\infty(\Omega), \quad a \geq 0 \text{ a.e. on } \Omega, \\ f \in L^2(\Omega), \quad g \in L^2(\Gamma_1), \end{cases} \tag{1.2.24}$$

$$\exists \beta > 0, \quad \forall \xi_i, \quad 1 \leq i \leq n, \quad \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \geq \beta \sum_{i=1}^n \xi_i^2 \text{ a.e. on } \Omega. \tag{1.2.25}$$

The V-ellipticity of the bilinear form will be a consequence of the following result.

Theorem 1.2.1. *Let Ω be a connected bounded open subset of* $\mathbf{R}^n$. *Then the space V defined in* (1.2.23) *is a closed subspace of $H^1(\Omega)$.*

If the $\mathrm{d}\gamma$*-measure of Γ_0 is strictly positive, the semi-norm $|\cdot|_{1,\Omega}$ is a norm over the space V, equivalent to the norm $\|\cdot\|_{1,\Omega}$.*

Proof. Let (v_k) be a sequence of functions in the space V which converges to an element $v \in H^1(\Omega)$. Since the sequence $(\mathrm{tr}\, v_k)$ converges to $\mathrm{tr}\, v$ in the space $L^2(\Gamma)$ (cf. inequalities (1.2.3)), it contains a subsequence which converges almost everywhere to $\mathrm{tr}\, v$ and thus $\mathrm{tr}\, v = 0$ a.e. on Γ_0. This implies that the function v belongs to the space V.

Next, let us show that $|\cdot|_{1,\Omega}$ is a norm over the space V. Let v be a function in the space V which satisfies $|v|_{1,\Omega} = 0$. Then it is a constant by virtue of the connectedness of the set Ω and, being as such a smooth function, its trace is the same constant. That this constant is zero follows from the fact that the trace vanishes on the set Γ_0, whose $\mathrm{d}\gamma$-measure is strictly positive.

Finally, assume that the two norms $|\cdot|_{1,\Omega}$ and $\|\cdot\|_{1,\Omega}$ are not equivalent over the space V. Then there exists a sequence (v_k) of functions $v_k \in V$ such that

$$\begin{cases} \forall k, \quad \|v_k\|_{1,\Omega} = 1, \\ \qquad \lim_{k \to \infty} |v_k|_{1,\Omega} = 0. \end{cases}$$

By *Rellich's theorem*, any bounded sequence in the space $H^1(\Omega)$ contains a subsequence which converges in $L^2(\Omega)$, so that there exists a sequence (v_l) of functions $v_l \in V$ which converges in the space $L^2(\Omega)$ and which is such that $\lim_{l\to\infty} |v_l|_{1,\Omega} = 0$. Thus the sequence (v_l) is a

Cauchy sequence in the complete space V and therefore it converges in the norm $\|\cdot\|_{1,\Omega}$ to an element $v \in V$.

Since $|v|_{1,\Omega} = \lim_{l\to\infty} |v_l|_{1,\Omega} = 0$, we deduce that $v = 0$, which is in contradiction with the equalities $\|v_k\|_{1,\Omega} = 1$ for all k. □

From this theorem, we infer that the bilinear form of (1.2.23) is V-elliptic since we have $a(v,v) \geq \beta |v|_{1,\Omega}^2$ for all $v \in H^1(\Omega)$, as an application of the inequalities of (1.2.24) and (1.2.25) shows.

By the Lax–Milgram lemma (Theorem 1.1.3), there exists a unique function $u \in V$ which satisfies the variational equations

$$\forall v \in V,$$
$$\int_\Omega \left\{ \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j v + auv \right\} \mathrm{d}x = \int_\Omega fv \,\mathrm{d}x + \int_{\Gamma_1} gv \,\mathrm{d}\gamma. \tag{1.2.26}$$

Referring once again to formula (1.2.4), we obtain another *Green's formula*:

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j v \,\mathrm{d}x = -\int_\Omega \sum_{i,j=1}^n \partial_j(a_{ij}\partial_i u) v \,\mathrm{d}x + \int_\Gamma \sum_{i,j=1}^n a_{ij}\partial_i u \, v\nu_j \,\mathrm{d}\gamma, \tag{1.2.27}$$

valid for all functions $u \in H^2(\Omega)$, $v \in H^1(\Omega)$, provided the functions a_{ij} are smooth enough so that the functions $a_{ij}\partial_i u$ belong to the space $H^1(\Omega)$ (for example, $a_{ij} \in \mathscr{C}^1(\bar{\Omega})$). Using (1.2.27), we conclude that *we have formally solved the boundary value problem*

$$\begin{cases} -\sum_{i,j=1}^n \partial_j(a_{ij}\partial_i u) + au = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma_0 \\ \sum_{i,j=1}^n a_{ij}\nu_j \partial_i u = g & \text{on} \quad \Gamma_1, \end{cases} \tag{1.2.28}$$

which is called a *homogeneous mixed problem* if $g = 0$, or a *nonhomogeneous mixed problem* if $g \neq 0$, for the operator

$$u \to -\sum_{i,j=1}^n \partial_j(a_{ij}\partial_i u) + au, \tag{1.2.29}$$

assuming in both cases the $\mathrm{d}\gamma$-measures of Γ_0 and Γ_1 are strictly positive. Notice that condition (1.2.25) is the classical *ellipticity con-*

dition for an operator such as that of (1.2.29). The operator

$$u \to \sum_{i,j=1}^{n} a_{ij}\nu_j \partial_i u$$

is called the *conormal derivative operator* associated with the operator of (1.2.29).

If $\Gamma = \Gamma_0$, or $\Gamma = \Gamma_1$, then we have formally solved a *homogeneous Dirichlet problem*, or a *homogeneous* or a *nonhomogeneous Neumann problem*, for the operator of (1.2.29) (in the second case, we would require an inequality such as $a \geq a_0 > 0$ a.e. on Ω to get existence).

The elasticity problem

We now come to the *fourth example* which is by far the most significant. Let Ω be a bounded open connected subset of $\mathbf{R}^3$ with a Lipschitz-continuous boundary. We define the space

$$\begin{aligned} \boldsymbol{V} = \boldsymbol{U} = \{\boldsymbol{v} = (v_1, v_2, v_3) \in (H^1(\Omega))^3;\\ v_i = 0 \quad \text{on} \quad \Gamma_0, \quad 1 \leq i \leq 3\}, \end{aligned} \tag{1.2.30}$$

where Γ_0 is a $\mathrm{d}\gamma$-measurable subset of Γ, with a strictly positive $\mathrm{d}\gamma$-measure. The space $\boldsymbol{V}$ is equipped with the product norm

$$\boldsymbol{v} = (v_1, v_2, v_3) \to \|\boldsymbol{v}\|_{1,\Omega} = \left(\sum_{i=1}^{3} \|v_i\|_{1,\Omega}^2\right)^{1/2}.$$

For any $\boldsymbol{v} = (v_1, v_2, v_3) \in (H^1(\Omega))^3$, we let

$$\epsilon_{ij}(\boldsymbol{v}) = \epsilon_{ji}(\boldsymbol{v}) = \tfrac{1}{2}(\partial_j v_i + \partial_i v_j), \quad 1 \leq i, j \leq 3, \tag{1.2.31}$$

and

$$\sigma_{ij}(\boldsymbol{v}) = \sigma_{ji}(\boldsymbol{v}) = \lambda\left(\sum_{k=1}^{3} \epsilon_{kk}(\boldsymbol{v})\right)\delta_{ij} + 2\mu\epsilon_{ij}(\boldsymbol{v}), \quad 1 \leq i, j \leq 3, \tag{1.2.32}$$

where δ_{ij} is the Kronecker's symbol, and λ and μ are two constants which are assumed to satisfy $\lambda > 0$, $\mu > 0$. We define the bilinear form

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &= \int_\Omega \sum_{i,j=1}^{3} \sigma_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v})\,\mathrm{d}x \\ &= \int_\Omega \left\{\lambda \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} + 2\mu \sum_{i,j=1}^{3} \epsilon_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v})\right\}\mathrm{d}x, \end{aligned} \tag{1.2.33}$$

and the linear form

$$f(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}\gamma = \int_\Omega \sum_{i=1}^3 f_i v_i\,\mathrm{d}x + \int_{\Gamma_1} \sum_{i=1}^3 g_i v_i\,\mathrm{d}\gamma, \tag{1.2.34}$$

where $\boldsymbol{f} = (f_1, f_2, f_3) \in (L^2(\Omega))^3$ and $\boldsymbol{g} = (g_1, g_2, g_3) \in (L^2(\Gamma_1))^3$, with $\Gamma_1 = \Gamma - \Gamma_0$ are given functions.

It is clear that these bilinear and linear forms are continuous over the space V. To prove the V-ellipticity of the bilinear form (see Exercise 1.2.4), one needs *Korn's inequality*: There exists a constant $C(\Omega)$ such that, for all $\boldsymbol{v} = (v_1, v_2, v_3) \in (H^1(\Omega))^3$,

$$\|\boldsymbol{v}\|_{1,\Omega} \leq C(\Omega)\Big(\sum_{i,j=1}^3 |\epsilon_{ij}(\boldsymbol{v})|^2_{0,\Omega} + \sum_{i=1}^3 |v_i|^2_{0,\Omega}\Big)^{1/2}. \tag{1.2.35}$$

This is a nontrivial inequality, whose proof may be found in DUVAUT & LIONS (1972, Chapter 3, §3.3), or in FICHERA (1972, Section 12). From it, one deduces that over the space V defined in (1.2.30) *the mapping*

$$\boldsymbol{v} = (v_1, v_2, v_3) \to |\boldsymbol{v}| = \Big(\sum_{i,j=1}^3 |\epsilon_{ij}(\boldsymbol{v})|^2_{0,\Omega}\Big)^{1/2}$$

is a norm, equivalent to the product norm, as long as the $\mathrm{d}\gamma$-measure of Γ_0 is strictly positive, which is the case here (again the reader is referred to Exercise 1.2.4).

The V-ellipticity is therefore a consequence of the inequalities $\lambda > 0$, $\mu > 0$, since by (1.2.33)

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq 2\mu|\boldsymbol{v}|^2.$$

We conclude that there exists a unique function $\boldsymbol{u} \in V$ which minimizes the functional

$$J(\boldsymbol{v}) = \frac{1}{2}\int_\Omega \Big\{\lambda(\operatorname{div}\boldsymbol{v})^2 + 2\mu \sum_{i,j=1}^3 (\epsilon_{ij}(\boldsymbol{v}))^2\Big\}\,\mathrm{d}x - \Big(\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}\gamma\Big) \tag{1.2.36}$$

over the space V, or equivalently, which is such that

$$\forall \boldsymbol{v} \in V, \quad \int_\Omega \sum_{i,j=1}^3 \sigma_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}\gamma. \tag{1.2.37}$$

Since relations (1.2.37) are satisfied by all functions $v \in (\mathcal{D}(\Omega))^3$, they could yield the associated partial differential equation. However, as was pointed out in Remark 1.2.1, it is equivalent to proceed through Green's formulas, which in addition have the advantage of yielding boundary conditions too.

Using Green's formula (1.2.4), we obtain, for all $u \in (H^2(\Omega))^3$ and all $v \in (H^1(\Omega))^3$:

$$\int_\Omega \sigma_{ij}(u)\partial_j v_i \, dx = -\int_\Omega (\partial_j \sigma_{ij}(u)) v_i \, dx + \int_\Gamma \sigma_{ij}(u) v_i \nu_j \, d\gamma,$$

so that, using definitions (1.2.31) and (1.2.32), we have proved that the following *Green's formula* holds:

$$\int_\Omega \sum_{i,j=1}^3 \sigma_{ij}(u)\epsilon_{ij}(v)\, dx = \int_\Omega \sum_{i=1}^3 \Big(-\sum_{j=1}^3 \partial_j\sigma_{ij}(u)\Big) v_i \, dx + \int_\Gamma \sum_{i=1}^3 \Big(\sum_{j=1}^3 \sigma_{ij}(u)\nu_j\Big) v_i \, d\gamma, \qquad (1.2.38)$$

for all functions $u \in (H^2(\Omega))^3$ and $v \in (H^1(\Omega))^3$.

Arguing as in the previous examples, we find that we are formally solving the equations

$$-\sum_{j=1}^3 \partial_j \sigma_{ij}(u) = f_i, \quad 1 \le i \le 3. \qquad (1.2.39)$$

It is customary to write these equations in vector form:

$$-\mu \, \Delta u - (\lambda + \mu) \operatorname{grad} \operatorname{div} u = f \quad \text{in} \quad \Omega,$$

which is derived from (1.2.39) simply by using relations (1.2.32).

Taking equations (1.2.39) into account, the variational equations (1.2.37) reduce to

$$\forall v \in V, \quad \int_{\Gamma_1} \sum_{i=1}^3 \Big(\sum_{j=1}^3 \sigma_{ij}(u)\nu_j\Big) v_i \, d\gamma = \int_{\Gamma_1} \sum_{i=1}^3 g_i v_i \, d\gamma,$$

since $v = 0$ on $\Gamma_0 = \Gamma - \Gamma_1$.

To sum up, *we have formally solved the following associated boundary value problem*:

$$\begin{cases} -\mu \, \Delta u - (\lambda + \mu) \operatorname{grad} \operatorname{div} u = f \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \Gamma_0, \\ \sum_{j=1}^3 \sigma_{ij}(u)\nu_j = g_i \quad \text{on} \quad \Gamma_1, \quad 1 \le i \le 3, \end{cases} \qquad (1.2.40)$$

which is known as the *system of equations of linear elasticity*. Let us mention that a completely analogous analysis holds in two dimensions, in which case the resulting problem is called the *system of equations of two-dimensional, or plane, elasticity*, the above one being also called by contrast the *system of three-dimensional elasticity*. Accordingly, the variational problem associated with the data (1.2.30), (1.2.33) and (1.2.34) is called the *(three- or two-dimensional) elasticity problem*.

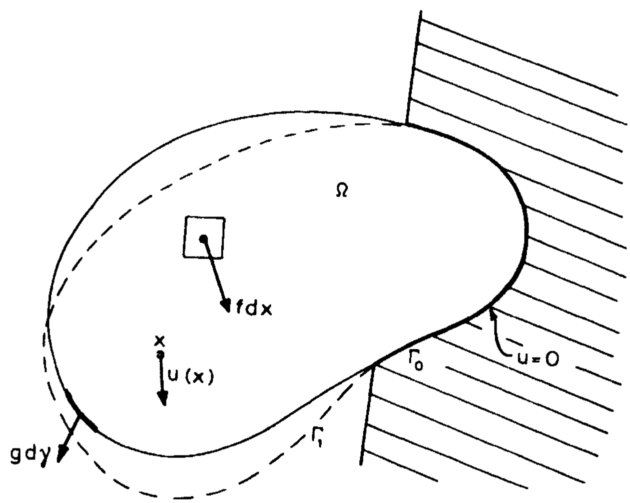

Fig. 1.2.3

Assuming "small" displacements and "small" strains, this system describes the state of a body (Fig. 1.2.3) which occupies the set $\bar{\Omega}$ in the absence of forces, u denoting the displacement of the points of $\bar{\Omega}$ under the influence of given forces (as usual, the scale for the displacements is distorted in the figure).

The body $\bar{\Omega}$ cannot move along Γ_0, and along Γ_1, surface forces of density g are given. In addition, a volumic force, of density f, is prescribed inside the body $\bar{\Omega}$.

Then we recognize in $(\epsilon_{ij}(u))$ the *strain tensor* while $(\sigma_{ij}(u))$ is the *stress tensor*, the relationship between the two being given by the linear equations (1.2.32) known in Elasticity as *Hooke's law* for isotropic bodies. The constants λ and μ are the *Lamé coefficients* of the material of which the body is composed.

The variational equations (1.2.37) represent the *principle of virtual work*, valid for all *kinematically admissible displacements* $\boldsymbol{v}$, i.e., which satisfy the boundary condition $\boldsymbol{v} = \boldsymbol{0}$ on Γ_0.

The functional J of (1.2.36) is the *total potential energy* of the body. It is the sum of the *strain energy*:

$$\frac{1}{2}\int \left\{\lambda(\operatorname{div} \boldsymbol{v})^2 + 2\mu \sum_{i,j=1}^{3} (\epsilon_{ij}(\boldsymbol{v}))^2\right\} \mathrm{d}x,$$

and of the *potential energy of the exterior forces*: $-(\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g} \cdot \boldsymbol{v}\, \mathrm{d}\gamma)$.

This example is probably the most crucial one, not only because it has obviously many applications, but essentially because its variational formulation, described here, is basically responsible for the invention of the finite element method by engineers.

Remark 1.2.2. It is interesting to notice that the strict positiveness of the $\mathrm{d}\gamma$-measure of Γ_0 has a physical interpretation: It is intuitively clear that in case the $\mathrm{d}\gamma$-measure Γ_0 would vanish, the body would be free and therefore there could not exist an equilibrium position in general. □

Remark 1.2.3. The *membrane problem*, which we have already described, the *plate problem*, which we shall soon describe in this section, and the *shell problem* (Section 8.1), *are derived from the elasticity problem*, through a process which can be briefly described as follows: Because such bodies have a "small" thickness, simplifying *a priori* assumptions can be made (such as linear variations of the stresses over the thickness) which, together with other assumptions (on the constitutive material in the case of membranes, or on the orthogonality of the exterior forces in the case of membranes and plates), allow one to integrate the energy (1.2.36) over the thickness. In this fashion, the problem is reduced to a problem in two variables, and only one function (the "vertical" displacement) in case of membranes and plates. All this is at the expense of a greater mathematical complexity in case of plates and shells however, as we shall see. □

Remark 1.2.4. Since problem (1.2.40) is called system of *linear* elasticity, the linearity being of course that of the mapping $(\boldsymbol{f}, \boldsymbol{g}) \to \boldsymbol{u}$, it is worth saying how this problem might become nonlinear. This may happen in three nonexclusive ways:

(i) Instead of minimizing the energy over the space V, we minimize it over a subset U which is not a subspace. This circumstance, which we already commented upon (Remark 1.1.1) is examined in Exercise 1.2.5 for a simpler model. Another example is treated in Section 5.1.

(ii) Instead of considering the "linearized" strain tensor (1.2.31), the "full" tensor is considered, i.e., we let

$$\epsilon_{ij}(\mathbf{v}) = \frac{1}{2}\Big(\partial_j v_i + \partial_i v_j + \sum_{k=1}^{3} \partial_i v_k \partial_j v_k\Big), \quad 1 \leq i, j \leq 3.$$

Actually, it suffices that for at least one pair (i, j), the above expression be considered. This is the case for instance of the *von Karmann's model of a clamped plate.*

(iii) The linear relation (1.2.32) between the strain tensor and the stress tensor is replaced by a nonlinear relation. □

Examples of fourth-order problems: The biharmonic problem, the plate problem

Whereas in the preceding examples the spaces V were contained in the space $H^1(\Omega)$, we consider in the last examples Sobolev spaces which involve second-order derivatives. We begin with the following data:

$$\begin{cases} V = U = H_0^2(\Omega), \\ a(u, v) = \displaystyle\int_\Omega \Delta u \,\Delta v \,\mathrm{d}x, \\ f(v) = \displaystyle\int_\Omega f v \,\mathrm{d}x, \quad f \in L^2(\Omega). \end{cases} \tag{1.2.41}$$

Since the mapping $v \to |\Delta v|_{0,\Omega}$ is a norm over the space $H_0^2(\Omega)$, as we showed in (1.2.8), the bilinear form is $H_0^2(\Omega)$-elliptic. Thus there exists a unique function $u \in H_0^2(\Omega)$ which minimizes the functional

$$J: v \to J(v) = \frac{1}{2}\int_\Omega |\Delta v|^2 \,\mathrm{d}x - \int_\Omega f v \,\mathrm{d}x \tag{1.2.42}$$

over the space $H_0^2(\Omega)$ or, equivalently, which satisfies the variational equations

$$\forall v \in H_0^2(\Omega), \quad \int_\Omega \Delta u \,\Delta v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x. \tag{1.2.43}$$

Using Green's formula (1.2.7):

$$\int_\Omega \Delta u\, \Delta v \,\mathrm{d}x = \int_\Omega \Delta^2 u\, v\,\mathrm{d}x - \int_\Gamma \partial_\nu \Delta u\, v\,\mathrm{d}\gamma + \int_\Gamma \Delta u \partial_\nu v\,\mathrm{d}\gamma,$$

we find that we have formally solved the following *homogeneous Dirichlet problem for the biharmonic operator* Δ^2:

$$\begin{aligned} &\Delta^2 u = f \quad \text{in} \quad \Omega,\\ &u = \partial_\nu u = 0 \quad \text{on} \quad \Gamma. \end{aligned} \tag{1.2.44}$$

We shall indicate a physical origin of this problem in the section "Additional Bibliography and Comments" of Chapter 4.

As our *last example*, we let, for $n = 2$,

$$\begin{cases} V = U = H_0^2(\Omega),\\ a(u, v) = \displaystyle\int_\Omega \{\Delta u\, \Delta v + (1-\sigma)(2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v \\ \qquad\qquad - \partial_{22}u\partial_{11}v)\}\,\mathrm{d}x \\ \qquad\quad = \displaystyle\int_\Omega \{\sigma\Delta u\, \Delta v + (1-\sigma)(\partial_{11}u\partial_{11}v + \partial_{22}u\partial_{22}v \\ \qquad\qquad + 2\partial_{12}u\partial_{12}v)\}\,\mathrm{d}x,\\ f(v) = \displaystyle\int_\Omega fv\,\mathrm{d}x, \quad f \in L^2(\Omega). \end{cases} \tag{1.2.45}$$

These data correspond to the variational formulation of the (*clamped*) *plate problem*, which concerns the equilibrium position of a plate of constant thickness e under the action of a transverse force, of density $F = (Ee^3/12(1-\sigma^2))f$ per unit area. The constants $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$ and $\sigma = \lambda/2(\lambda + \mu)$ are respectively the *Young's modulus* and the *Poisson's coefficient* of the plate, λ and μ being the Lamé's coefficients of the plate material. When $f = 0$, the plate is in the plane of coordinates (x_1, x_2) (Fig. 1.2.4). The condition $u \in H_0^2(\Omega)$ takes into account the fact that the plate is clamped (see the boundary conditions in (1.2.48) below).

As we pointed out in Remark 1.2.3, the expressions given in (1.2.45) for the bilinear form and the linear form are obtained upon integration over the thickness of the plate of the analogous quantities for the elasticity problem. This integration results in a simpler problem, in that there are now only two independent variables. However, this advantage is compensated by the fact that second partial derivatives are now present

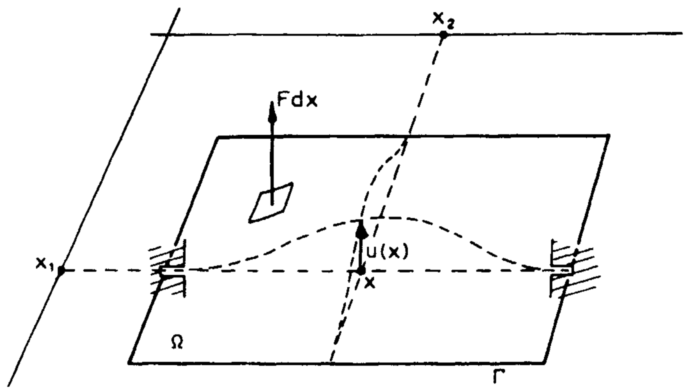

Fig. 1.2.4

in the bilinear form. This will result in a fourth-order partial differential equation. See (1.2.48).

The Poisson's coefficient σ satisfying the inequalities $0 < \sigma < \frac{1}{2}$, the bilinear form is $H_0^2(\Omega)$-elliptic, since we have

$$\forall v \in H^2(\Omega), \quad a(v, v) = \sigma|\Delta v|_{0,\Omega}^2 + (1-\sigma)|v|_{2,\Omega}^2.$$

Thus, there exists a unique function $u \in H_0^2(\Omega)$ which minimizes the *total potential energy of the plate*:

$$J(v) = \frac{1}{2}\int_\Omega \{|\Delta v|^2 + 2(1-\sigma)((\partial_{12}v)^2 - \partial_{11}v\partial_{22}v)\}\,\mathrm{d}x - \int_\Omega fv\,\mathrm{d}x, \tag{1.2.46}$$

over the space $H_0^2(\Omega)$ or, equivalently, which is solution of the variational equations

$$\forall v \in H_0^2(\Omega), \quad \int_\Omega \{\Delta u\,\Delta v + (1-\sigma)(2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v - \partial_{22}u\partial_{11}v)\}\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x. \tag{1.2.47}$$

Using Green's formulas (1.2.7) and (1.2.9):

$$\int_\Omega \Delta u\,\Delta v = \int_\Omega \Delta^2 u\; v\,\mathrm{d}x - \int_\Gamma \partial_\nu \Delta u\; v\,\mathrm{d}\gamma + \int_\Gamma \Delta u \partial_\nu v\,\mathrm{d}\gamma,$$

$$\int_\Omega \{2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v - \partial_{22}u\partial_{11}v\}\,\mathrm{d}x = \int_\Gamma \{-\partial_{\tau\tau}u\partial_\nu v + \partial_{\nu\tau}u\partial_\tau v\}\,\mathrm{d}\gamma,$$

we find that we have again solved, at least formally, the *homogeneous Dirichlet problem for the biharmonic operator* Δ^2:

$$\begin{cases} \Delta^2 u = f & \text{in} \quad \Omega, \\ u = \partial_\nu u = 0 & \text{on} \quad \Gamma. \end{cases} \tag{1.2.48}$$

Therefore, *in spite of a different bilinear form, we eventually find the same problem as in the previous example.* This is so because, in view of the second Green's formula which we used, the contribution of the integral

$$\int_\Omega (1-\sigma)\{2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v - \partial_{22}u\partial_{11}v\}\,dx$$

is zero when the functions v are in the space $\mathscr{D}(\Omega)$, and consequently in its closure $H_0^2(\Omega)$. Thus, the partial differential equation is still $\Delta^2 u = f$ in Ω. However different boundary conditions might result from another choice for the space V. See Exercise 1.2.7.

To distinguish the two problems, we shall refer to a fourth-order problem corresponding to the functional of (1.2.42) as a *biharmonic problem*, while we shall refer to a fourth-order problem corresponding to the functional of (1.2.46) as a *plate problem*.

In this section, we have examined various minimization or variational problems with each of which is associated a boundary value problem for which the partial differential operator is elliptic (incidentally, this correspondence is not one-to-one, as the last two examples show). This is why, by extension, these minimization or variational problems are themselves called *elliptic boundary value problems*. For the same reasons, such problems are said to be *second-order problems*, or *fourth-order problems*, when the associated partial differential equation is of order two or four, respectively.

Finally, one should recall that even though the association between the two formulations may be formal, it is possible to prove, *under appropriate smoothness assumptions on the data*, that a solution of any of the variational problems considered here is also a solution in the classical sense of the associated boundary value problem.

Remark 1.2.5. In this book, one could conceivably omit all reference to the associated classical boundary value problems, inasmuch as the finite element method is based only on the variational formulations. By contrast, finite difference methods are most often derived from the classical formulations. □

Exercises

1.2.1. Prove Green's formula (1.2.9). The reader should keep in mind that the derivative $\partial_{\tau\tau}v$ generally differs from the second derivative of the function v, considered as a function of the curvilinear abcissa along the boundary.)

1.2.2. Let the space $V = U$ and the bilinear form be as in (1.2.10), and let the linear form be defined by

$$f(v) = \int_\Omega fv \, dx - a(u_0, v),$$

where the functions f and a satisfy assumptions (1.2.11) and u_0 is a given function in the space $H^1(\Omega)$. Show that these data correspond to the formal solution of the *nonhomogeneous Dirichlet problem for the operator* $u \to -\Delta u + au$, i.e.,

$$\begin{cases} -\Delta u + au = f & \text{in} \quad \Omega, \\ u = u_0 & \text{on} \quad \Gamma. \end{cases}$$

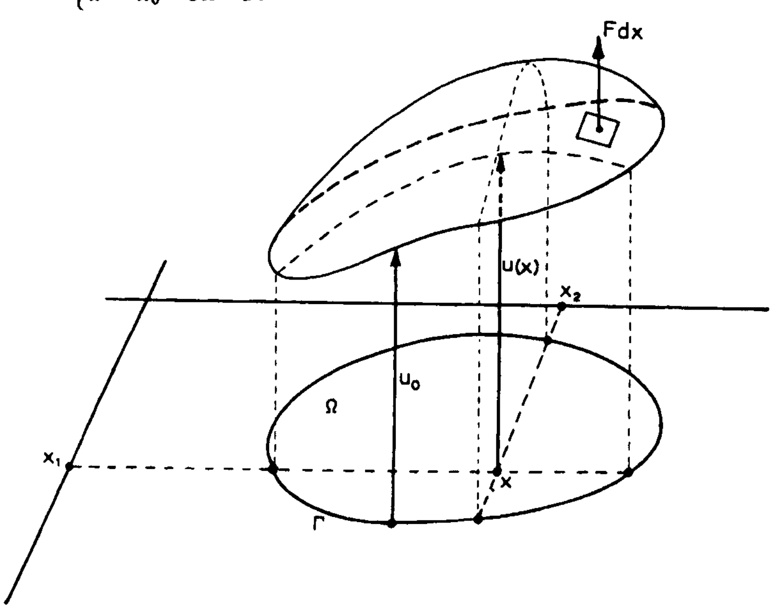

Fig. 1.2.5

Is it equivalent to minimizing the functional (1.2.12) over the subset

$$U = \{v \in H^1(\Omega);\quad (v - u_0) \in H_0^1(\Omega)\}$$

of the space $V = H^1(\Omega)$?

With $a = 0$ and $n = 2$, this is another *membrane problem*. See Fig. 1.2.5, which is self-explanatory.

1.2.3. Find a variational problem which amounts to solving the non-homogeneous Neumann problem for the operator $-\Delta$, i.e., problem (1.2.22) when the function a vanishes identically, and when the equality $\int_\Omega f\,\mathrm{d}x + \int_\Gamma g\,\mathrm{d}\gamma = 0$ holds. [Hint: Use the fact that over the quotient space $H^1(\Omega)/P_0(\Omega)$, $P_0(\Omega)$: space of constant functions over Ω, the semi-norm $|\cdot|_{1,\Omega}$ is a norm, equivalent to the quotient norm. See Theorem 3.1.1 for a proof.]

1.2.4. Let Ω be a connected open subset of $\mathbf{R}^n$, with $n = 2$ or 3, and let Γ_0 be a dγ-measurable subset of its boundary Γ, assumed to be Lipschitz-continuous. Let

$$\boldsymbol{V} = \{\boldsymbol{v} = (v_i) \in (H^1(\Omega))^n;\quad v_i = 0 \quad \text{on} \quad \Gamma_0,\quad 1 \leq i \leq n\}$$

(i) Show that $\boldsymbol{V}$ is a closed subspace of the space $(H^1(\Omega))^n$.

(ii) Show that the mapping

$$|\cdot|: \boldsymbol{v} \in \boldsymbol{V} \to |\boldsymbol{v}| = \left(\sum_{i,j=1}^n |\epsilon_{ij}(\boldsymbol{v})|_{0,\Omega}^2\right)^{1/2}$$

is a norm over the space $\boldsymbol{V}$, if the dγ-measure of Γ_0 is strictly positive. [Hint: Show that a function $\boldsymbol{v} \in (H^1(\Omega))^n$ which satisfies $|\boldsymbol{v}| = 0$ is of the form $\boldsymbol{v}(x) = \boldsymbol{a} \times x + \boldsymbol{b}$ for some constant vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e., the displacement $\boldsymbol{v}$ is a *rigid body motion*. Such a result is proved for example in HLAVÁČEK & NEČAS (1970, Lemma II.1). See also Section 8.1 for analogous ideas.]

(iii) Using Korn's inequality (1.2.35), show that the norm $|\cdot|$ is equivalent to the norm $\|\cdot\|_{1,\Omega}$. [Hint: Argue as in Theorem 1.2.1.]

1.2.5. Let

$$V = H^1(\Omega),\quad a(u, v) = \int_\Omega \left\{\sum_{i=1}^n \partial_i u \partial_i v + auv\right\} \mathrm{d}x,\quad f(v) = \int_\Omega fv\,\mathrm{d}x,$$

and let

$$U = \{v \in H^1(\Omega);\quad v \geq 0 \text{ a.e.} \quad \text{on} \quad \Gamma\}.$$

Show that U is a closed convex cone with vertex 0. Using characterizations (1.1.5) of Theorem 1.1.2 show that the associated variational problem amounts to formally solving the boundary value problem

$$\begin{cases} -\Delta u + au = f & \text{in} \quad \Omega, \\ u \geqslant 0, \quad \partial_\nu u \geqslant 0, \quad u\partial_\nu u = 0 & \text{on} \quad \Gamma. \end{cases}$$

This type of nonlinear problem is a model problem for *Signorini problems*, i.e., problems in elasticity for which the boundary conditions are *unilateral constraints* such as the above ones. For extensive discussions of such problems, see DUVAUT & LIONS (1972), FICHERA (1972).

1.2.6. Extend the analysis made for the data (1.2.23) to the case where the bilinear form is given by

$$a(u, v) = \int_\Omega \left\{ \sum_{i,j=1}^n a_{ij}\partial_i u \partial_j v + \sum_{i=1}^n a_i \partial_i u \, v + auv \right\} \mathrm{d}x,$$

the functions a_i being in the space $L^\infty(\Omega)$. In particular, find sufficient conditions for the V-ellipticity of the bilinear form.

1.2.7. Let the bilinear form and the linear form be as in (1.2.45), and let

$$V = U = H^2(\Omega) \cap H_0^1(\Omega) = \{v \in H^2(\Omega); \quad v = 0 \quad \text{on} \quad \Gamma\}.$$

This is a mathematical model for a *simply supported plate.* Using the fact that $v \to |\Delta v|_{0,\Omega}$ is again a norm over the space V, equivalent to the norm $\|\cdot\|_{2,\Omega}$, analyze the associated variational problem. What is the associated boundary value problem?

1.2.8. Let

$$\begin{cases} V = U = H^2(\Omega) \cap H_0^1(\Omega) = \{v \in H^2(\Omega); \quad v = 0 \quad \text{on} \quad \Gamma\}, \\ a(u, v) = \displaystyle\int_\Omega \Delta u \, \Delta v \, \mathrm{d}x, \\ f(v) = \displaystyle\int_\Omega fv \, \mathrm{d}x - \int_\Gamma \lambda \partial_\nu v \, \mathrm{d}\gamma, \quad f \in L^2(\Omega), \quad \lambda \in L^2(\Gamma). \end{cases}$$

Using as in Exercise 1.2.7 the fact that $v \to |\Delta v|_{0,\Omega}$ is a norm over the space V, equivalent to the norm $\|\cdot\|_{2,\Omega}$, analyze the associated variational problem. In particular, show that it can be decomposed into two second order problems. What is the associated boundary value problem? Does it share the same property?

Bibliography and comments

1.1. The original reference of the Lax–Milgram lemma is LAX & MILGRAM (1954). Our proof follows the method of LIONS & STAMPACCHIA (1967), where it is applied to the general variational problem (1.1.15), and where the case of semi-positive definite bilinear forms is also considered. STAMPACCHIA (1964) had the original proof. For constructive existence proofs and additional references, see also GLOWINSKI, LIONS & TRÉMOLIÈRES (1976a).

I. BABUSKA (BABUSKA & AZIZ (1972, Theorem 5.2.1)) has extended the Lax–Milgram lemma to the case of bilinear forms defined on a product of two distinct Hilbert spaces. This extension turns out to be a useful tool for the analysis of some finite element methods (BABUŠKA (1971b)).

1.2. For treatments of Differential Calculus with Fréchet derivatives, the reader may consult CARTAN (1967), DIEUDONNÉ (1967), SCHWARTZ (1967). For the theory of distributions and its applications to partial differential equations, see SCHWARTZ (1966). Other references are TRÈVES (1967), SHILOV (1968), VO-KHAC KHOAN (1972a, 1972b). The Sobolev spaces are extensively studied in LIONS (1962) and NEČAS (1967). See also ADAMS (1975). The original reference is SOBOLEV (1950).

Thorough treatments of the variational formulations of elliptic boundary value problems are given in LIONS (1962), AGMON (1965), NEČAS (1967), LIONS & MAGENES (1968), VO-KHAC KHOAN (1972b). Shorter accounts are given in AUBIN (1972), BABUŠKA & AZIZ (1972), ODEN & REDDY (1976a). More specialized treatments, particularly for nonlinear problems, are LADYŽENSKAJA & URAL'CEVA (1968), LIONS (1969), EKELAND & TÉMAM (1974). For regularity results, see GRISVARD (1976), KONDRAT'EV (1967).

For more classically oriented treatments, see for example BERS, JOHN & SCHECHTER (1964), COURANT & HILBERT (1953, 1962), MIRANDA (1970), STAKGOLD (1968).

As an introduction to classical elasticity theory, notably for the elasticity problem, the clamped plate problem, the membrane problem, see for example LANDAU & LIFSCHITZ (1967). For the variational formulations of problems in elasticity along the lines followed here, consult DUVAUT & LIONS (1972), FICHERA (1972), ODEN & REDDY (1976b).