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A SIMPLE ERROR ESTIMATOR AND ADAPTIVE PROCEDURE FOR PRACTICAL ENGINEERING ANALYSIS

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SUMMARY

A new error estimator is presented which is not only reasonably accurate but whose evaluation is computationally so simple that it can be readily implemented in existing finite element codes.

The estimator allows the global energy norm error to be well estimated and also gives a good evaluation of local errors. It can thus be combined with a full adaptive process of refinement or, more simply, provide guidance for mesh redesign which allows the user to obtain a desired accuracy with one or two trials.

When combined with an automatic mesh generator a very efficient guidance process to analysis is available.

Estimates other than the energy norm have successfully been applied giving, for instance, a predetermined accuracy of stresses.

INTRODUCTION

The subject of error estimates for finite element solutions and a consequent adaptive analysis, in which the approximation is successively refined to reach predetermined standards of accuracy, is central to the effective use of finite element codes for practical, engineering, analysis. Much of the pioneering mathematical work[1–9] is being today 'translated' to engineering usage. A recent survey of such work shows the principal directions of this effort.[10] However, many difficulties remain and these are responsible for the fact that to this date almost no commercially available codes include provisions for error estimating or adaptivity. The two main problems are

(1) the cost of computations associated with error estimation and the difficulty of implementing such computations into an existing code structure
(2) the virtual impossibility of embracing a fully adaptive structure into an existing code structure.

This paper concentrates on a new process for providing a simple error estimator which overcomes problem (1). By coupling the error estimator with an automatic mesh generator (such as is already featured in many 'practical' codes) many of the difficulties of problem (2) can be bypassed and a fully or partially automated process can be made available within an existing code structure.

Two main directions of refining a finite element solution exist. The first is the simple reduction of the subdivision size (h-refinement), which is a natural way for most engineers.

The second refinement process increases the order of polynomial trial function approximation in a predefined element subdivision (p-refinement).

This last procedure has particular advantages in many (elliptic) situations when combined with a *hierarchic formulation*. We have discussed these in detail elsewhere[4–9] showing the relative ease of obtaining error estimates, fast convergence and possibilities of iterative solutions of refinement

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equations. However, the efficiency here involves generally abandoning a standard finite element structure. For this reason in the present paper we shall concentrate on an efficient *h*-refinement process.

THE MODEL PROBLEM

We shall focus our attention here on the solution of a linear elastic problem which may be posed by the differential equilibrium equation

$$\mathbf{Lu} - \mathbf{q} \equiv \mathbf{S}^{\mathrm{T}}\mathbf{DSu} - \mathbf{q} = \mathbf{0} \tag{1}$$

in a domain Ω, with prescribed displacement

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on the boundary} \quad \Gamma_u \tag{2a}$$

and prescribed traction

$$\mathbf{GDSu} = \bar{\mathbf{t}} \quad \text{on the boundary} \quad \Gamma_t \tag{2b}$$

with

$$\Gamma = \Gamma_u \bigcup \Gamma_t$$

In the above, the matrix operator $\mathbf{S}$ defines the strain $\boldsymbol{\varepsilon}$ as

$$\boldsymbol{\varepsilon} = \mathbf{Su} \tag{3}$$

and the elasticity matrix $\mathbf{D}$ defines the stresses as

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \tag{4}$$

In a finite element approximation with

$$\mathbf{u} \approx \hat{\mathbf{u}} = \mathbf{N}\bar{\mathbf{u}} \tag{5}$$

we obtain the approximating equations by a standard Galerkin process (or equivalently by minimizing the potential energy) to obtain[11]

$$\mathbf{Ku} - \mathbf{f} = \mathbf{0} \tag{6}$$

where

$$\mathbf{K} = \int_\Omega (\mathbf{SN})^{\mathrm{T}}\mathbf{D}(\mathbf{SN})\,\mathrm{d}\Omega$$

$$\mathbf{f} = \int_\Omega \mathbf{N}^{\mathrm{T}}\mathbf{q}\,\mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^{\mathrm{T}}\bar{\mathbf{t}}\,\mathrm{d}\Gamma$$

and the stresses are calculated as

$$\hat{\boldsymbol{\sigma}} = (\mathbf{DSN})\bar{\mathbf{u}} \tag{7}$$

The approximate solution $\hat{\mathbf{u}}, \hat{\boldsymbol{\sigma}}$ differs from the exact values $\mathbf{u}, \boldsymbol{\sigma}$ and the difference is the error. Thus, for displacements

$$\mathbf{e} = \mathbf{u} - \hat{\mathbf{u}} \tag{8a}$$

for stresses

$$\mathbf{e}_\sigma = \boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}} \tag{8b}$$

etc.

ERROR NORMS AND ERROR MEASURES

A pointwise definition of errors, as given in equations (8a) and (8b) is generally difficult to specify, and various integral measures are more conveniently adopted. One of the most common of such measures is the 'energy norm' written for a general problem as

$$\|\mathbf{e}\| = \left(\int_\Omega \mathbf{e}^{\mathrm{T}} \mathbf{L} \mathbf{e} \, \mathrm{d}\Omega \right)^{1/2} \tag{9}$$

or in the specific case of elasticity as

$$\|\mathbf{e}\| = \left(\int_\Omega (\mathbf{S}\mathbf{e})^{\mathrm{T}} \mathbf{D} (\mathbf{S}\mathbf{e}) \, \mathrm{d}\Omega \right)^{1/2} = \left(\int_\Omega (\mathbf{e}_\varepsilon^{\mathrm{T}}) \mathbf{D} (\mathbf{e}_\varepsilon) \, \mathrm{d}\Omega \right)^{1/2}$$

$$= \left(\int_\Omega (\mathbf{e}_\sigma^{\mathrm{T}}) \mathbf{D}^{-1} (\mathbf{e}_\sigma) \mathrm{d}\Omega \right)^{1/2} \tag{10}$$

A more direct measure is the so called L_2 norm, which can be associated with the errors in any quantity. Thus for the displacement **u** the L_2 norm of the error **e** is

$$\|\mathbf{e}\|_{L_2} = \left(\int_\Omega \mathbf{e}^{\mathrm{T}} \mathbf{e} \, \mathrm{d}\Omega \right)^{1/2} \tag{11}$$

and for stresses

$$\|\mathbf{e}_\sigma\|_{L_2} = \left(\int_\Omega (\mathbf{e}_\sigma)^{\mathrm{T}} (\mathbf{e}_\sigma) \, \mathrm{d}\Omega \right)^{1/2} \tag{12}$$

the latter expression differing from the energy norm only by the weighting **D**.

Although all the norms written above are defined on the whole domain, we note that the square of each can be obtained by summing element contributions. Thus

$$\|\mathbf{e}\|^2 = \sum_{i=1}^{m} \|\mathbf{e}\|_i^2 \tag{13}$$

where i represents an element contribution and m the total element number. Indeed for an 'optimal' mesh we shall generally try to make the contributions to this square of the norm equal for all elements.[12,13]

Although the absolute value of the energy (or indeed the L_2) norm has little physical meaning, the relative percentage error, e.g.

$$\eta = \frac{\|\mathbf{e}\|}{\|\mathbf{u}\|} \times 100 \, \text{per cent} \tag{14}$$

is more easily interpreted. If the last of the expressions (10) is examined it is evident that equation (14) represents a weighted R.M.S. (root mean square) percentage error in the stresses.

The percentage error η can be determined for the whole domain or for element subdomains. The local definition is obviously more meaningful.

We can define η_L similarly to equation (14), but with reference to the L_2 norm and, as will be shown later, similar results are obtainable.

A very relevant but seldom used measure defines the R.M.S. *stress error in absolute terms.* Using equation (12) we note that this can be defined in the domain Ω as

$$\Delta\sigma = \left(\frac{\|\mathbf{e}_\sigma\|^2_{L_2}}{\Omega}\right)^{1/2} \tag{15}$$

and indeed similarly in element subdomains. We have, in some of the examples given later, used this measure in the element subdomains as its interpretation in engineering terms is most evident.

THE ERROR ESTIMATE

In elasticity (or indeed other) problems embraced by equation (1) C_0 continuity is assumed in the trial functions of equation (5), resulting in a discontinuous approximation of $\hat{\boldsymbol{\sigma}}$. In Figure 1 we illustrate a one-dimensional linear approximation for $\hat{u}$ and the corresponding stress $\hat{\sigma}$. To obtain acceptable results for stresses resort is generally made to a nodal averaging or projection process in which it is assumed that the stress $\boldsymbol{\sigma}^*$ is interpolated by the same function as the displacement, i.e.

$$\boldsymbol{\sigma}^* = \mathbf{N}\bar{\boldsymbol{\sigma}}^* \tag{16}$$

and

$$\int_\Omega \mathbf{N}^{\mathrm{T}}(\boldsymbol{\sigma}^* - \hat{\boldsymbol{\sigma}})\,\mathrm{d}\Omega = 0 \tag{17}$$

On substitution of equations (16) and (17) this yields

$$\bar{\boldsymbol{\sigma}}^* = \mathbf{A}^{-1}\int_\Omega \mathbf{N}^{\mathrm{T}}\mathbf{DSN}\,\mathrm{d}\Omega\,\bar{\mathbf{u}} \tag{18}$$

where

$$\mathbf{A} = \int_\Omega \mathbf{N}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}\Omega$$

and again in Figure 1 we show such smoothed stresses.

At this stage we should note that the calculation of such stress projections is very simple if a 'lumped' form of the matrix $\mathbf{A}$ is used—and in particular if both integrals of (18) are obtained by nodal quadrature.[14,15]

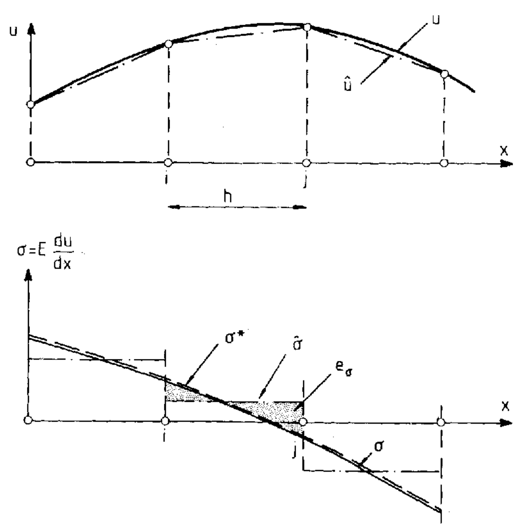

Figure 1. The nature of approximation to the error in stress (e_σ) for a one-dimensional problem and linear shape functions

It is intuitively 'obvious' that $\boldsymbol{\sigma}^*$ is in fact a better approximation than $\hat{\boldsymbol{\sigma}}$ and we shall use it to estimate the error $\mathbf{e}_\sigma$ (namely equation (8b)), i.e. put

$$\mathbf{e}_\sigma \approx \boldsymbol{\sigma}^* - \hat{\boldsymbol{\sigma}} \tag{19}$$

to evaluate the various error norms.

This process is easily implemented into any code as these generally already evaluate the appropriate $\boldsymbol{\sigma}^*$ and $\hat{\boldsymbol{\sigma}}$ quantities.

In Figure 1 we show the value of e_σ corresponding to a linear interpolation of u.

In the case illustrated it is easy to show that in fact the approximation of equation (19) is a good one and convergent to the exact error. The proof is simple:

(a) It is well known that the approximation of equation (6) yields exact nodal values of u in the one-dimensional case.[16]

(b) The projection evaluates an approximation to the first derivative at nodes which is one order higher than the linear approximation (provided that the modulus E, the one-dimensional equivalent of $\mathbf{D}$, is constant).

Thus expression (19) is true at least to $O(h)$ and the estimate converges here to the correct error.

Extension of this proof to two and three dimensions follows similar reasoning and indeed the procedure is valid not only for linear but also for higher order approximating shape functions $\mathbf{N}$. We shall show its effectiveness in later examples.

Most of the previously used error estimates involve in the energy norm the evaluation of integrals including interelement stress discontinuities. For instance the estimators derived in References 1–9 for bilinear elements ($p = 1$) are of the form

$$\varepsilon^2 = \sum_{i=1}^{m} \varepsilon_i^2 = \sum_{i=1}^{m} \left(C_1 \int_{\Omega_i} r^2 \, \mathrm{d}\Omega + C_2 \int_{I_i} J^2 \, \mathrm{d}I \right) \tag{20}$$

with I_i being the element interface, r the residual of the equation (1), J the inter-element traction jump, and

$$C_1 = \frac{h_i^2}{24K}, \quad C_2 = \frac{h_i}{24K}$$

($K = E/(1 - \nu)$ for plane stress elasticity problems with linear shape functions[6,7]).

Such estimators are more complex to evaluate than the present one, as they require explicit determination of the interelement traction jumps and an integration along element interfaces. However, the present simple form is approximatly equivalent to these earlier forms. Indeed for the bilinear element this equivalence has been demonstrated by Rank,[17] and an outline of his proof is given in the Appendix.

The effectivity index of any error estimator is the ratio of the predicted to correct error values. Thus

$$\theta = \frac{{}^0\|\mathbf{e}\|}{\|\mathbf{e}\|} \tag{21}$$

where the superscript 0 refers to predicted values.

Predicted values of percentage error are obtained as

$${}^0\eta = ({}^0\|\mathbf{e}\|^2/(\|\hat{\mathbf{u}}\|^2 + {}^0\|\mathbf{e}\|^2))^{1/2} \tag{22}$$

Finally an empirical correction multiplying factor to our estimates has been found useful. This depends on the elements used and is 1·1 for bilinear elements, 1·3 for linear triangles, 1·6 for biquadratic 9-node isoparametric elements and 1·4 for quadratic triangles.

In the subsequent examples we show both corrected and uncorrected values with θ^* and η^* referring to the latter.

The new error estimator was first presented in Reference 18 by the authors.

REFINEMENT STRATEGY

The refinement strategy will of course depend on the nature of the criteria on accuracy which we wish to satisfy.

A very common requirement is to specify the achievement of a certain minimum percentage error in the energy (or L_2) norm.

We thus require that after the final analysis is completed the condition that

$$\eta \leqslant \bar{\eta} \tag{23}$$

be satisfied for the whole domain, where $\bar{\eta}$ is the maximum permissible error.

If we assume that the error is equally distributed between elements the requirement (23) can be translated into our placing a limit on the error in each element. Thus for each element we require that, given that m is the total number of elements,

$$\|\mathbf{e}\|_i \leqslant \bar{\eta}[(\|\hat{\mathbf{u}}\|^2 + \|\mathbf{e}\|^2)/m]^{1/2} = \bar{e}_m \tag{24}$$

As the error is in fact computed for each element we can readily check where refinement is necessary. Indeed the ratio

$$\xi_i = \frac{\|\mathbf{e}\|_i}{\bar{e}_m} > 1 \tag{25}$$

defines the elements to be refined, and its value can, assuming a certain rate of convergence, decide the degree of subdivision or element size needed. For instance if the current element size is h_i, and we assume the rate of convergence of the error to be $O(h^p)$ (in the area covered by the element) then the predicted element size should be†

$$h = h_i/\xi_i^{1/p}, \quad \text{for} \quad \xi_i > 1 \tag{26}$$

Generally we shall assume that p is equal to the polynomial order of the trial function used, though of course this simple convergence rule is not valid near singularities.

The predicted element size distribution can be achieved in any manner practicable. In some of the examples we shall simply use a successive subdivision; in others a completely new mesh is generated using the requirement (26) and it will be seen that this can lead to the achievement of the final solution in a single trial.

The criterion of achieving a globally acceptable energy norm percentage error gives little direct information about the stresses.

A very logical criterion is to insist that within each element the stress deviation evaluated locally by the expression (15) should be below a certain minimum.

This minimum can be selected *a priori* as a fraction say of maximum permissible stress or it can be fixed as a percentage of the mean $\boldsymbol{\sigma}$ value in the whole problem. Whichever is applicable the requirement is similar to that of equation (24) with

$$\Delta\sigma_i \leqslant \Delta\sigma_{\text{permissible}} \tag{27}$$

† Of course the expression (26) can be used with $\xi_i < 1$ indicating that in some areas a coarser subdivision is permissible.

Again if we specify ξ_i as

$$\xi_i = \frac{\Delta\sigma_i}{\Delta\bar{\sigma}} \tag{28}$$

the expression (26) can be used to guide the refinement. However, the satisfaction of such a local criterion will be more demanding, as we shall show later.

EXAMPLES

The procedures for evaluating the errors and comparison with 'exact' error values are illustrated in a series of examples. In these the 'exact' solution is provided either analytically or by using an extremely fine subdivision.

Bilinear quadrilaterals (4-node isoparametric elements) are used in examples presented in Figures 2–5; *Biquadratic quadrilaterals* (9-node isoparametric elements) are used in Figures 6–9; and, finally, *linear and quadratic triangles* (3- and 6-node elements) are used in Figures 10–18.

In all these examples the global effectivity indices are given for various subdivisions as well as local ones in two cases. The reader can observe the excellent prediction of error and convergence of the effectivity indices to unity with refinement.

In all the examples using quadrilaterals the refinement strategy allowed only a halving of the mesh at each step to avoid excessive constraints.

This type of strategy leads to a good refinement but is not economical.

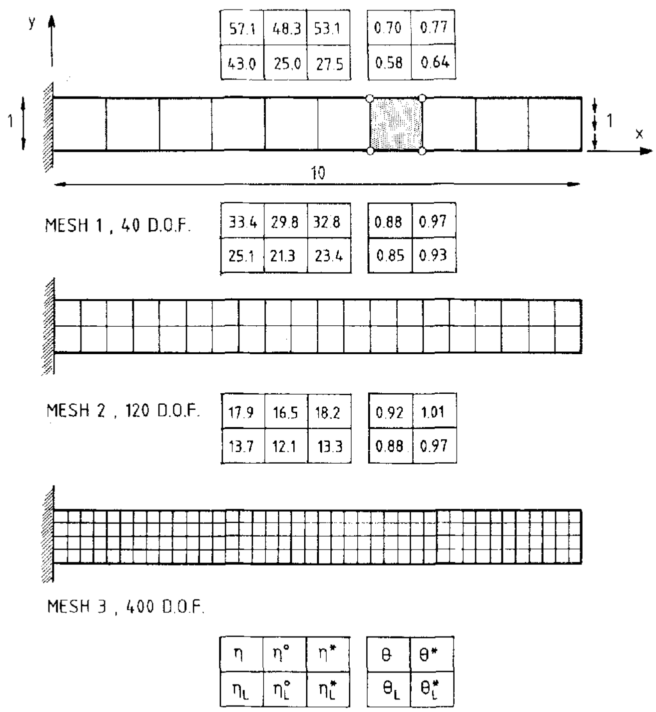

Figure 2. Bilinear elements: cantilever beam—plane stress. $E = 10^5$, $\nu = 0{\cdot}3$. Analysis and error estimates for uniform subdivision. η: actual percentage error in energy norm. η^0: predicted percentage error in energy norm. η^*: predicted percentage error using corrective factor. θ effectivity index. θ^* effectivity index using corrective factor. Suffix 'L' indicates use of the L_2 norm

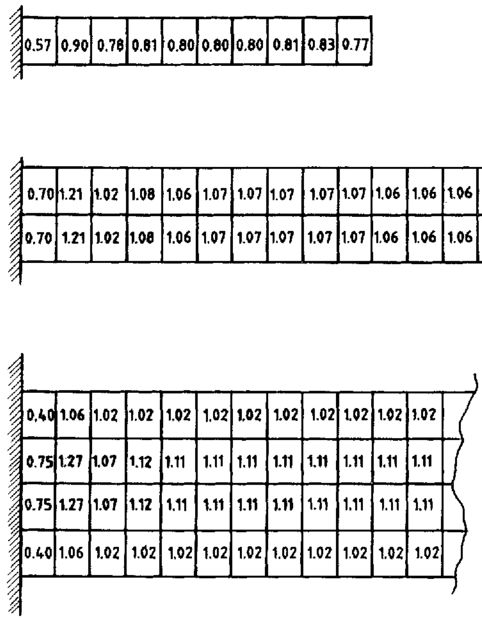

Figure 3. Local effectivity indices for problem 1 Figure 2 (meshes 1–3) [θ^* energy norm] (in the L_2 norm the results are very similar)

In the examples in which triangular elements are used we proceeded to impose directly the element spacing required by equation (26), having access to an excellent new mesh generator recently developed.[19]

Figures 10–16 show attempts to achieve a final accuracy of 5 per cent in one step of refinement. Though this target is not completely reached, it is reasonably approximated.

Figure 17 shows the distribution of $\Delta\sigma_i$ for the problem of Figure 12 (mesh 1) as well as the maximum value of ξ. Figure 18 shows the refinement now using the $\Delta\sigma$ criterion, there the max ξ has now been reduced to values below 3.

CONCLUDING REMARKS

The error estimator introduced in this paper and the simple refinement strategy suggested are applicable to problems other than stress analysis alone. The method can be used for almost *any* linear finite element discretization and currently work is in progress in developing a non-linear methodology.

The original applications of the adaptive process here suggested were made in the context of optimization, where it is logical to impose a tolerance in such quantities as stresses which govern the design. Clearly the main objective of adding a certain degree of measured confidence to results of finite element analysis is now being made easier.

To achieve a given accuracy with the least effort, mesh generation procedures are essential. Indeed these should be capable of designing a mesh from the specification of element size distribution. Such a mesh generator can be devised for triangles in two dimensions, as shown

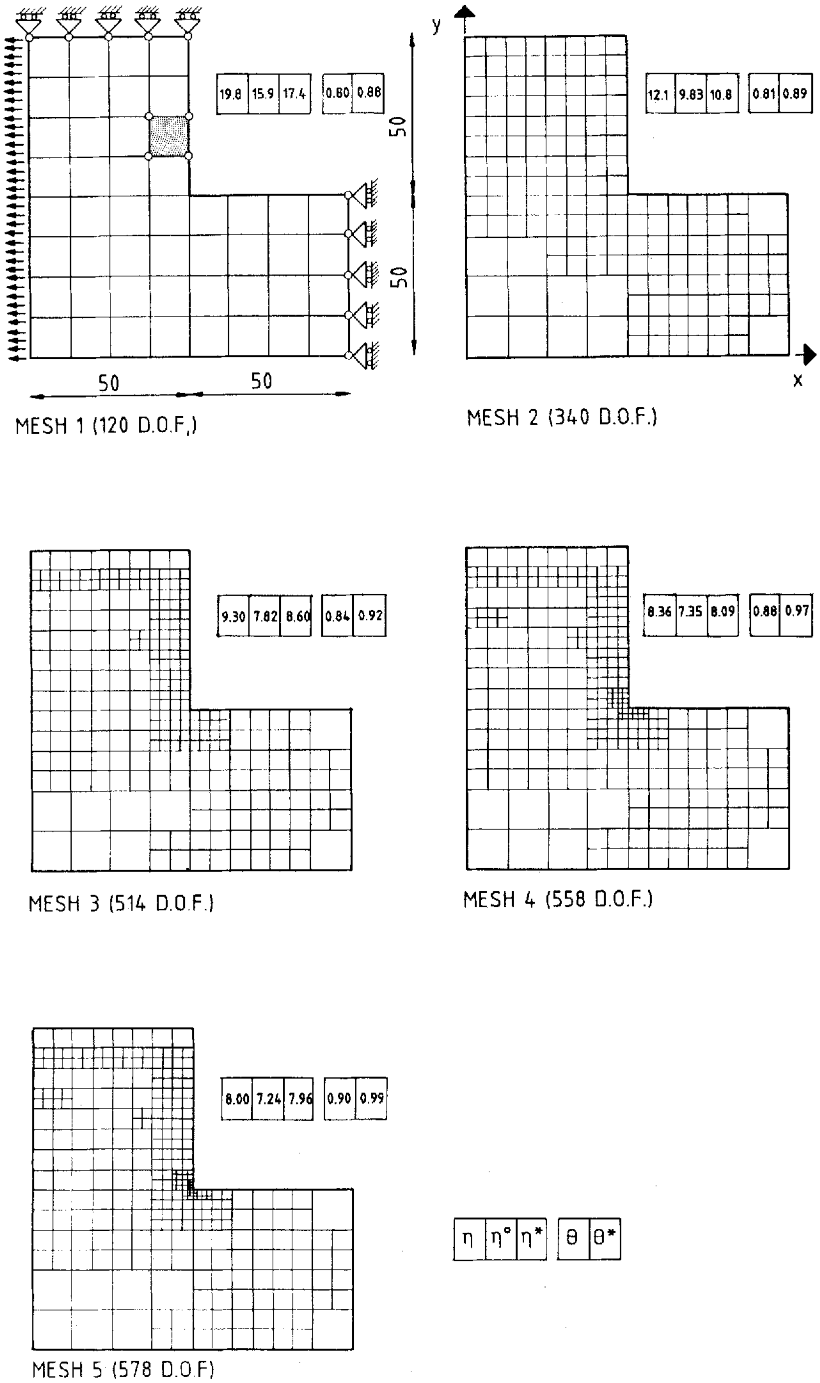

Figure 4. Bilinear elements: an L-shaped region in plane stress—sequences of mesh refinement

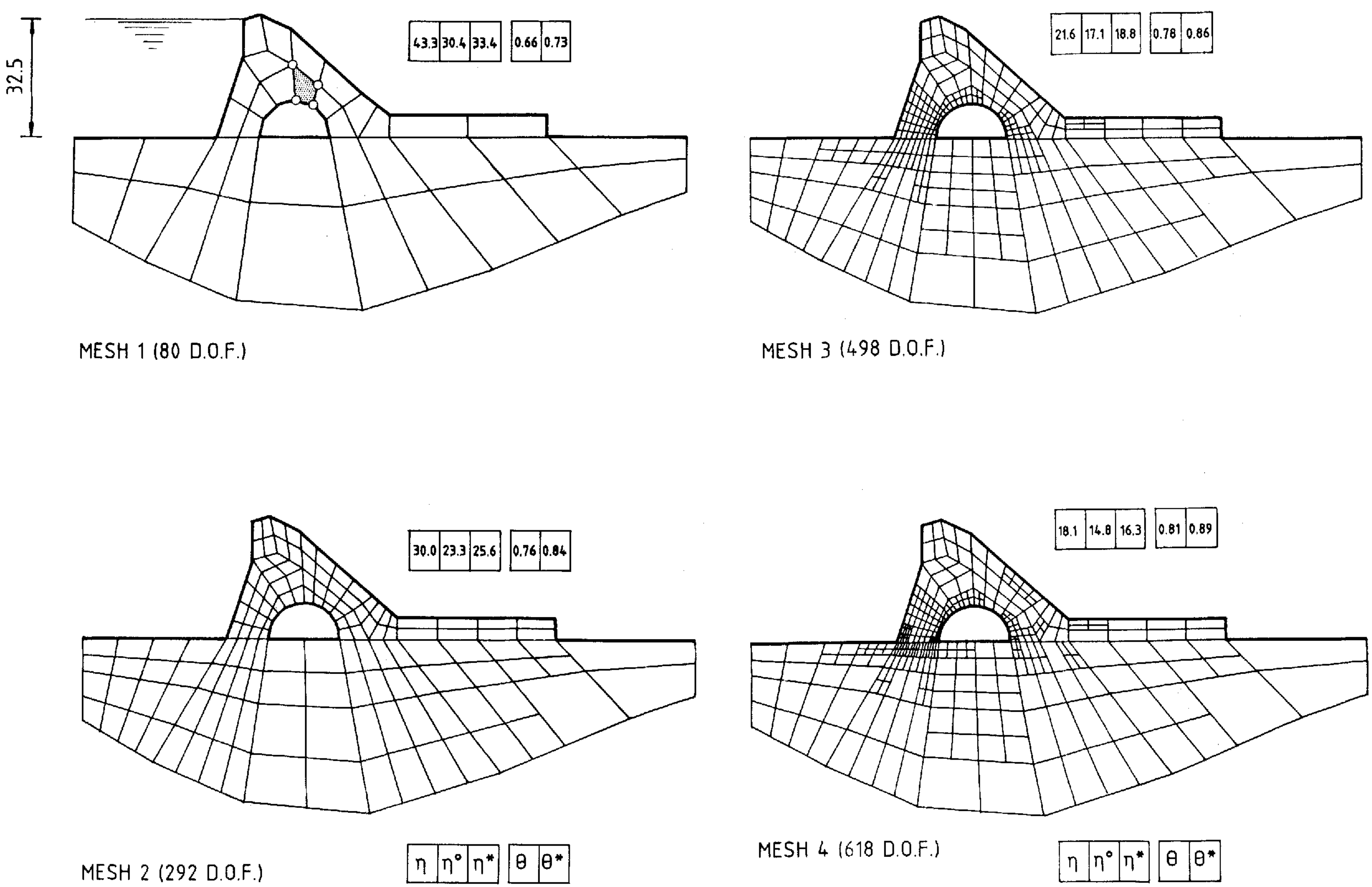

Figure 5. Bilinear elements: plane strain analysis of a dam with perforation; water loading only

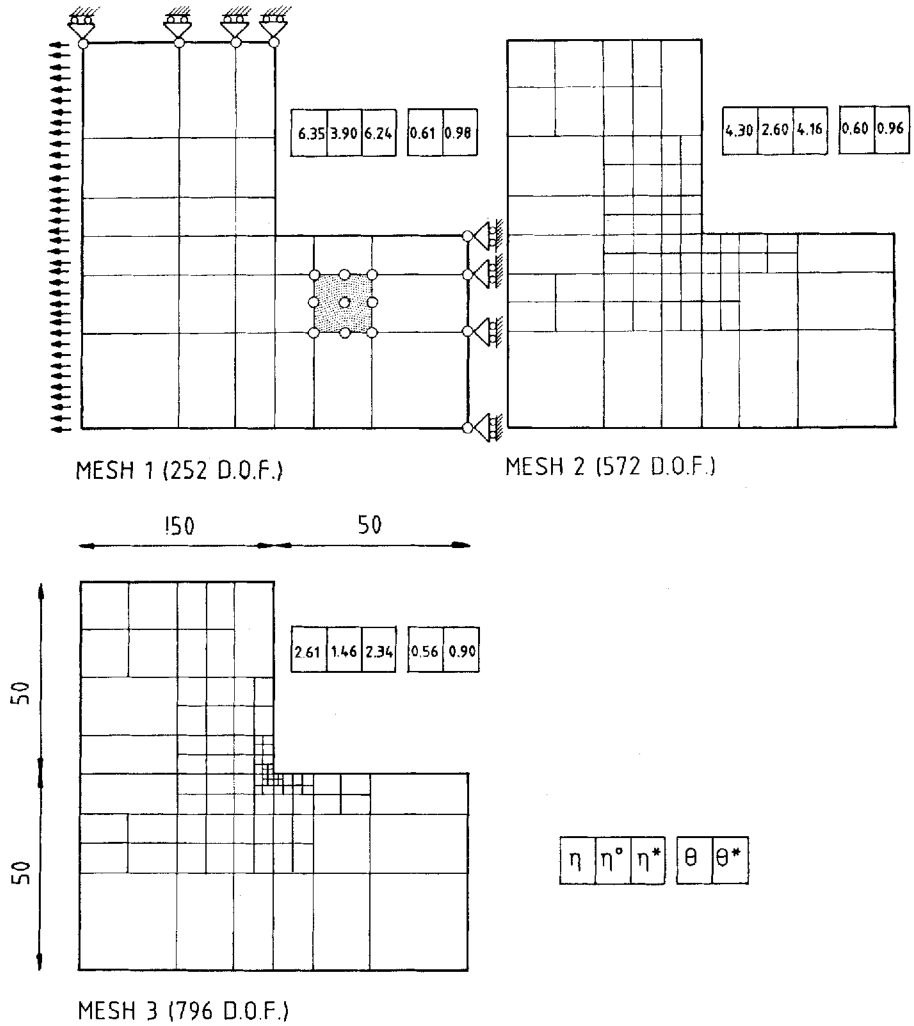

Figure 6. Biquadratic elements (9 node): an L-shaped region in plane stress—sequences of mesh refinement

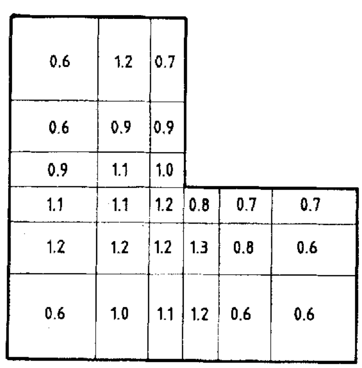

Figure 7. Local effectivity indices: problem of Figure 6 (mesh 1) [θ^*]

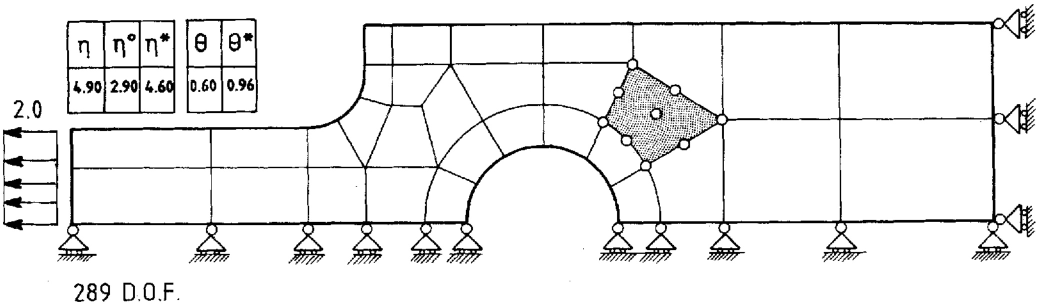

Figure 8. Biquadratic elements: perforated tension bar in plane stress

in Reference 19, and generalization to three dimensions following similar lines is in progress. However, such a mesh generator for quadrilateral (or 'brick') type elements does not exist—and would appear to be difficult to construct. For this reason we feel that the use of triangles (or tetrahedra) with either linear or quadratic basis functions may become more popular in the future.

A final remark concerns the meaning of the error estimates. These always concern the actual stresses computed by such expressions as equation (7) though in practical codes the results which are generally given in the output are those of stresses to which some form of averaging or projection was applied. The error computed with such stresses is invariably lower and in this sense the error estimates given here are on the safe side. (This remark of course applies with equal validity to all the other estimator forms quoted in References 1–10)

Below we show, for the example of Figure 2, the ratio of the true error in energy evaluated on the basis of $\mathbf{e}_\sigma = \boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}$ and that derived on the basis of smoothed (projected) stresses $\mathbf{e}_\sigma^* = \boldsymbol{\sigma} - \boldsymbol{\sigma}^*$. Thus, defining

$$\alpha = \frac{\|\mathbf{e}\|}{\|\mathbf{e}^*\|}$$

we have

$$\text{mesh } 1(10\,\text{elements})\colon \alpha = 1{\cdot}4$$
$$\text{mesh } 2(40\,\text{elements})\colon \alpha = 2{\cdot}1$$
$$\text{mesh } 3(160\,\text{elements})\colon \alpha = 2{\cdot}5$$

If our concern is mainly with the stresses it is evident that these are, if averaged, estimated with much better precision than indicated by error estimators (with the factor being close to two on the error). However this remark does not apply to the energy norm evaluated from displacements directly.

The current work shows that error estimates of the form derived here are equally useful in non-linear applications and, indeed, similar procedures applied to hyperbolic problems have been shown to be very effective.[19] In the latter problem a directionally orientated mesh refinement was effectively used and extension of such refinement to problems of structural mechanics is in progress.

ACKNOWLEDGEMENT

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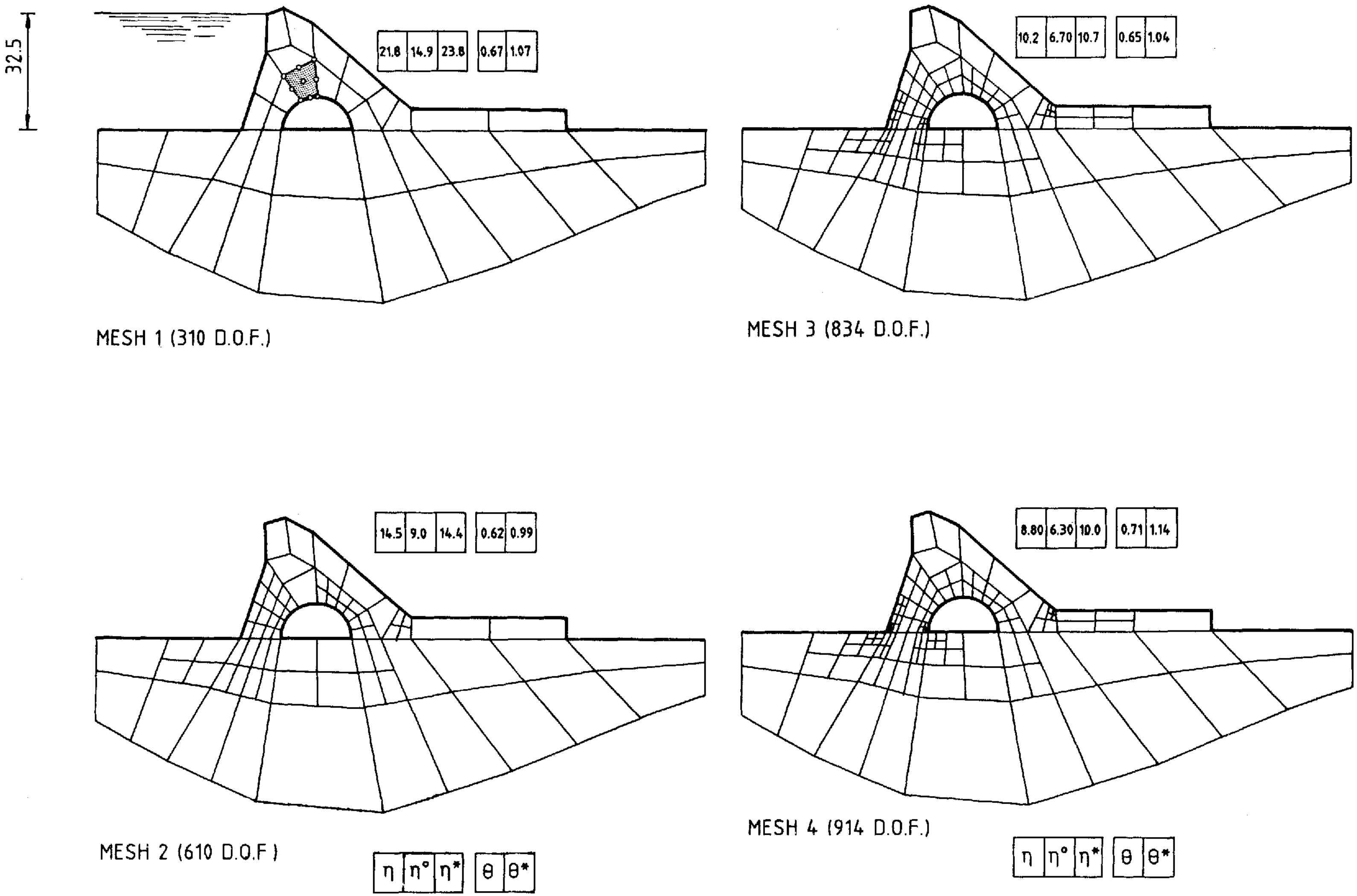

Figure 9. Biquadratic elements: plane strain analysis of a dam with perforation; water loading only

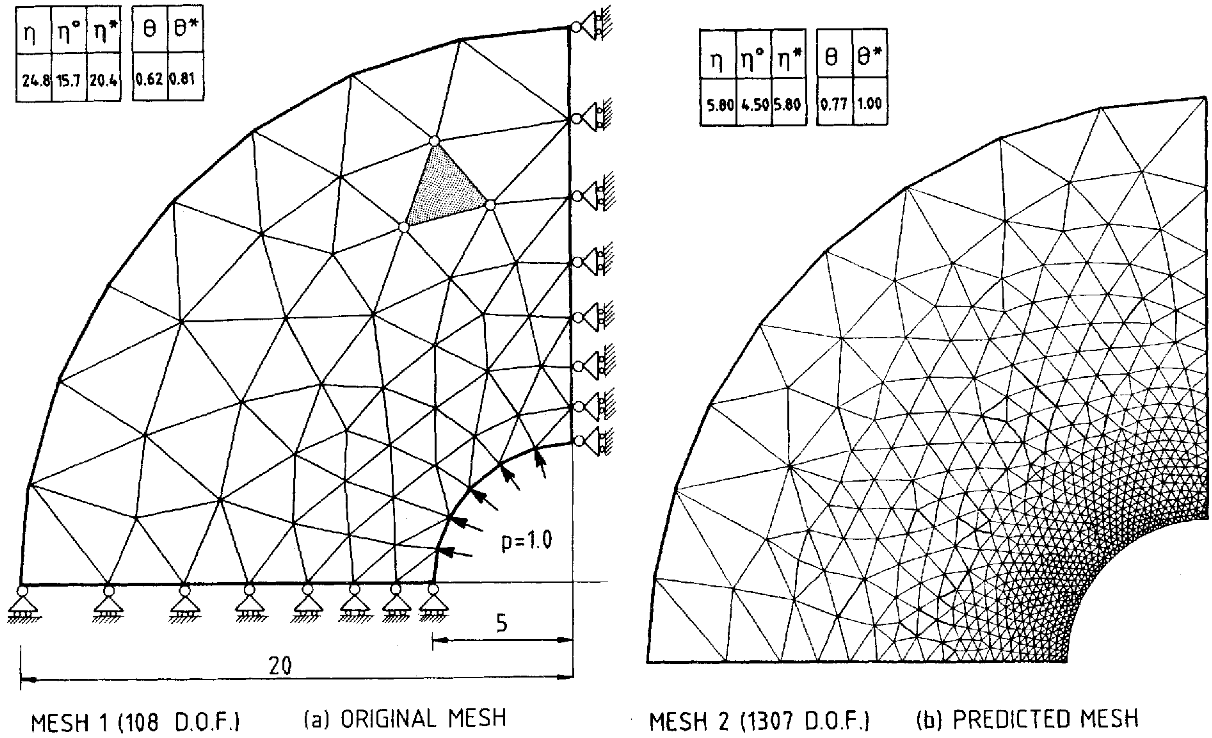

Figure 10. Linear triangle—automatic mesh generation to achieve 5 per cent accuracy: thick circular cylinder under internal pressure (plane strain)

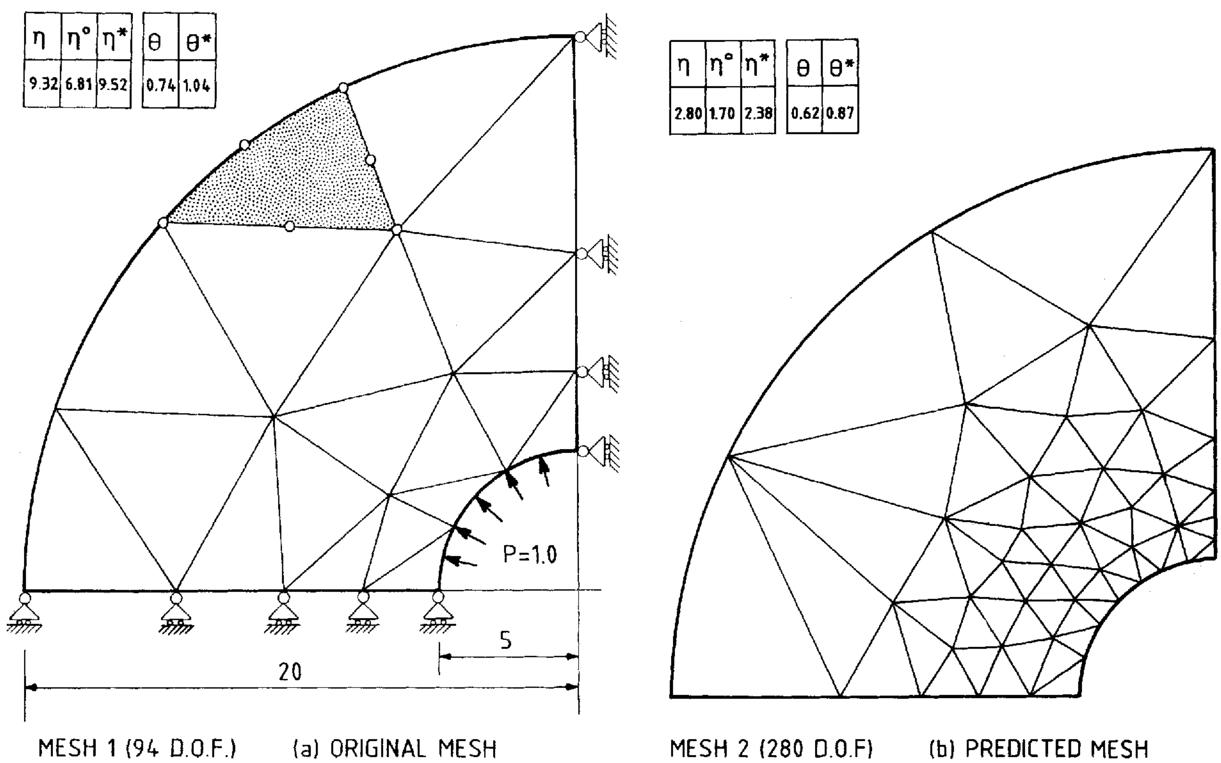

Figure 11. Quadratic triangle—automatic mesh generation to achieve 5 per cent accuracy: thick circular cylinder under internal pressure (plane strain)

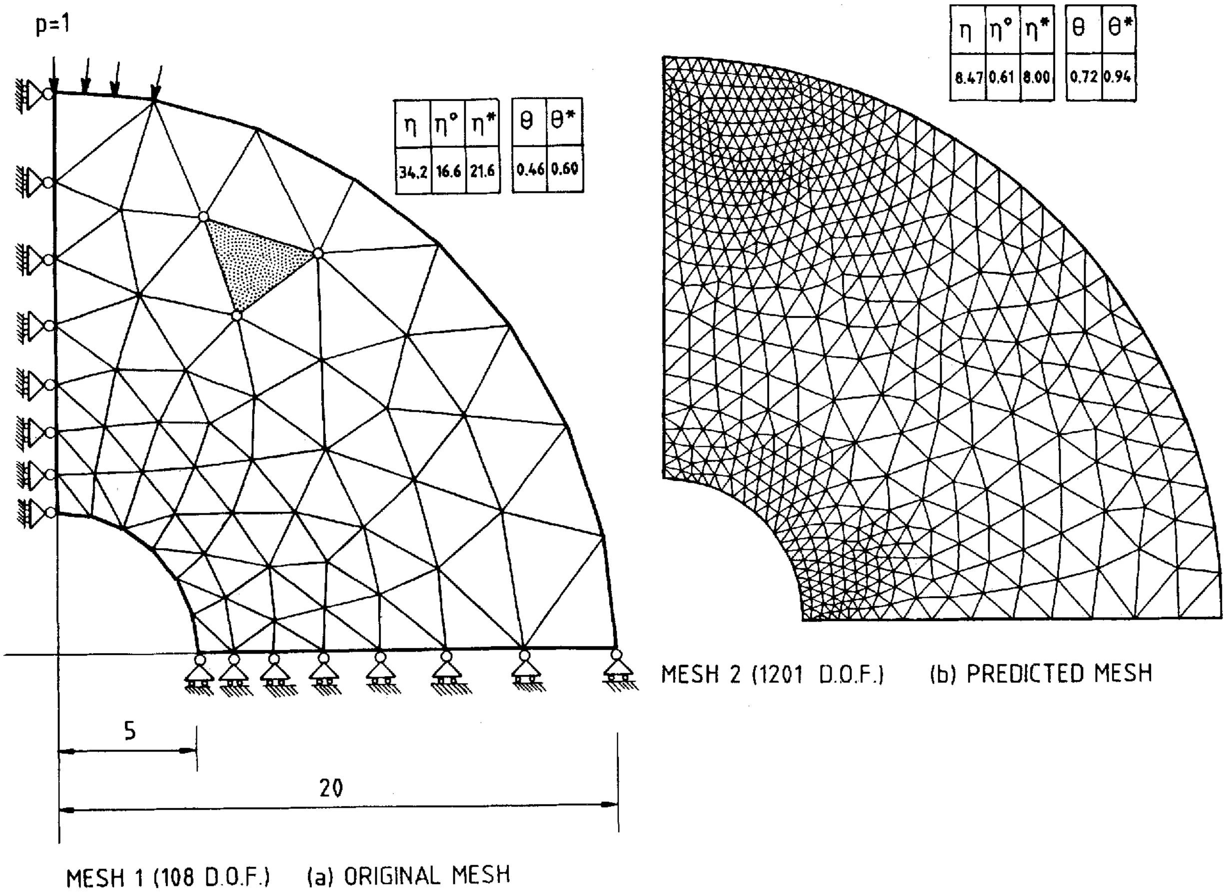

Figure 12. Linear triangle—automatic mesh generation to achieve 5 per cent accuracy: thick cylinder under diametral load

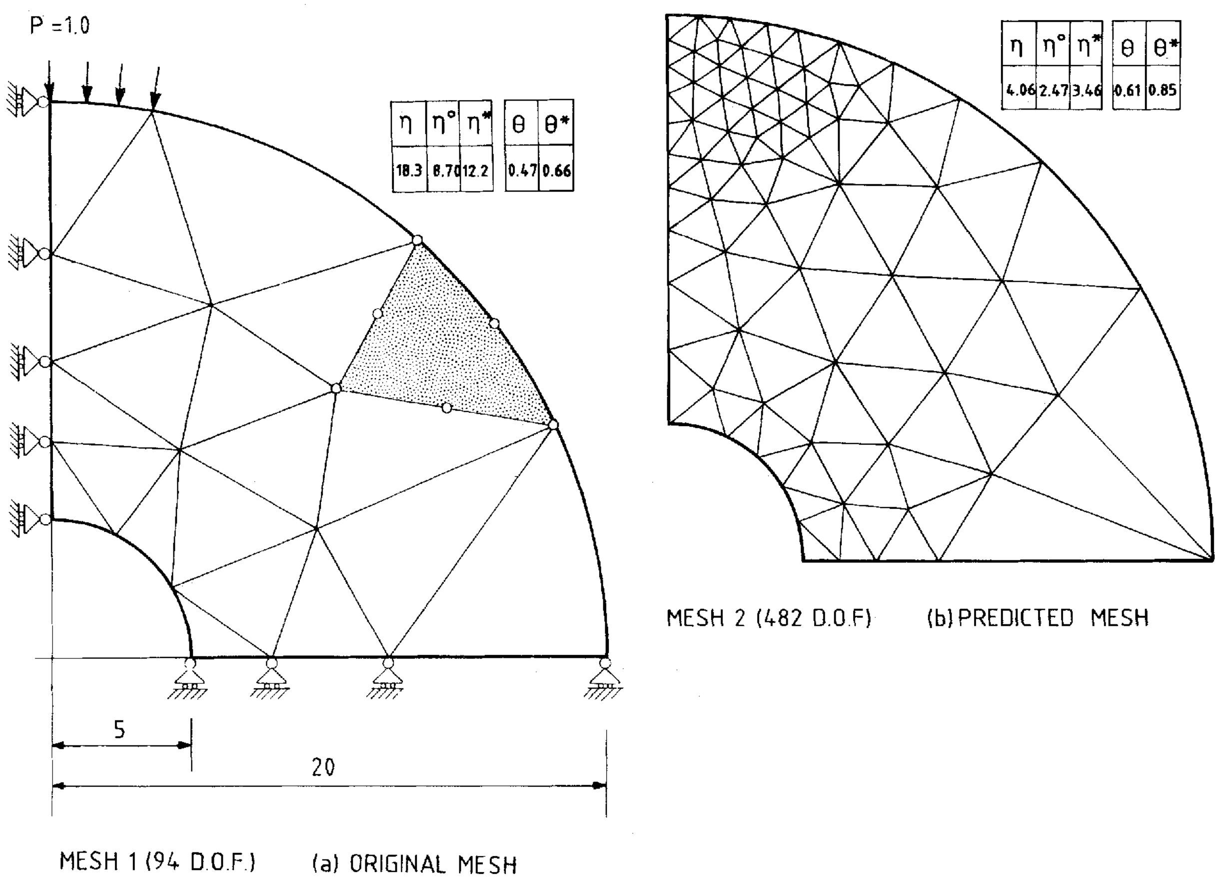

Figure 13. Quadratic triangle—automatic mesh generation to achieve 5 per cent accuracy: thick cylinder under diametral load

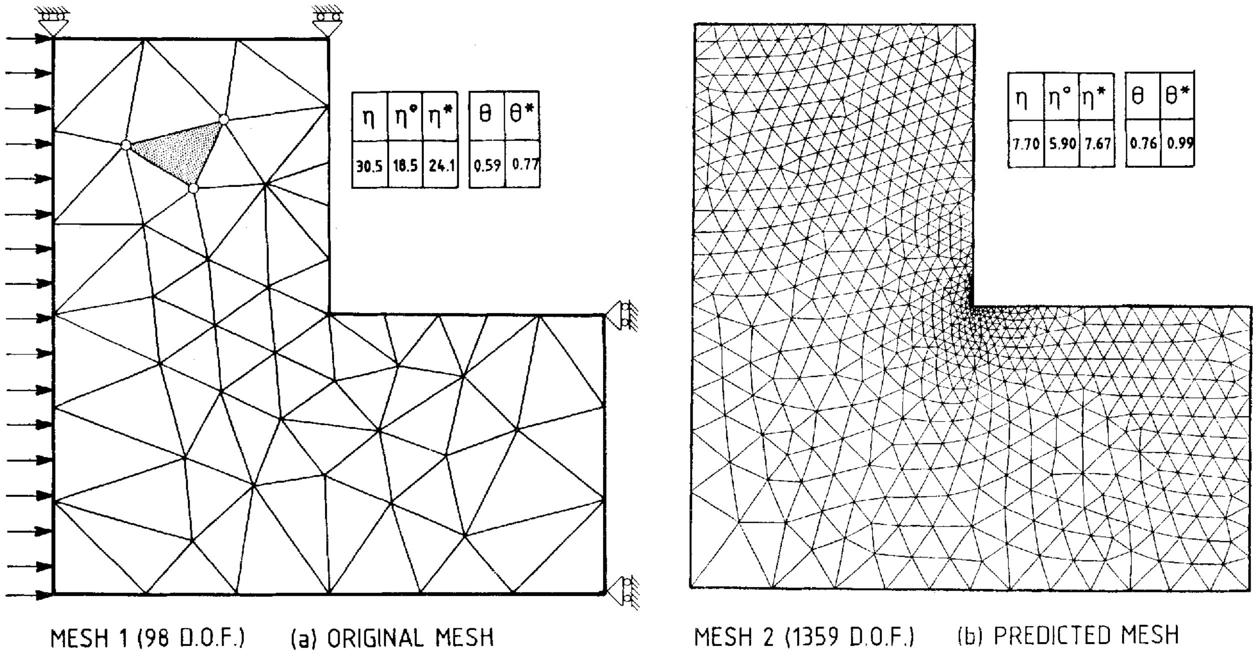

Figure 14. Linear triangle—automatic mesh generation to achieve 5 per cent accuracy: L-shaped region in plane stress

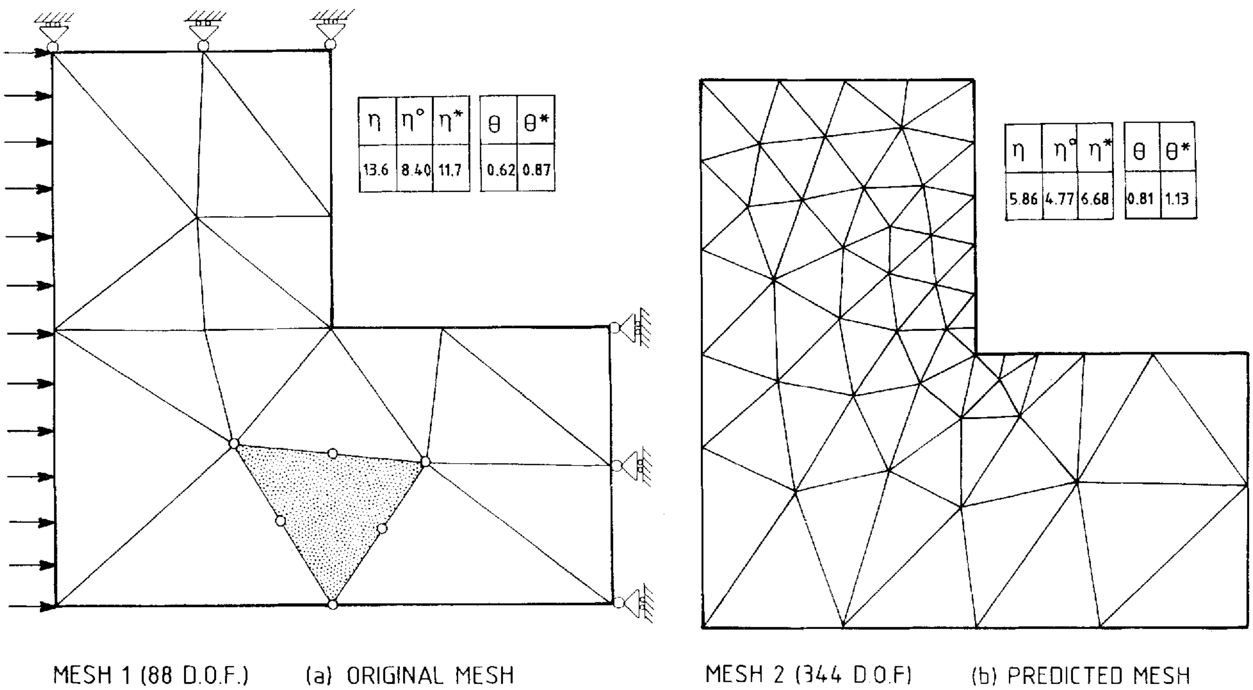

Figure 15. Quadratic triangle—automatic mesh generation to achieve 5 per cent accuracy; L-shaped region in plane stress

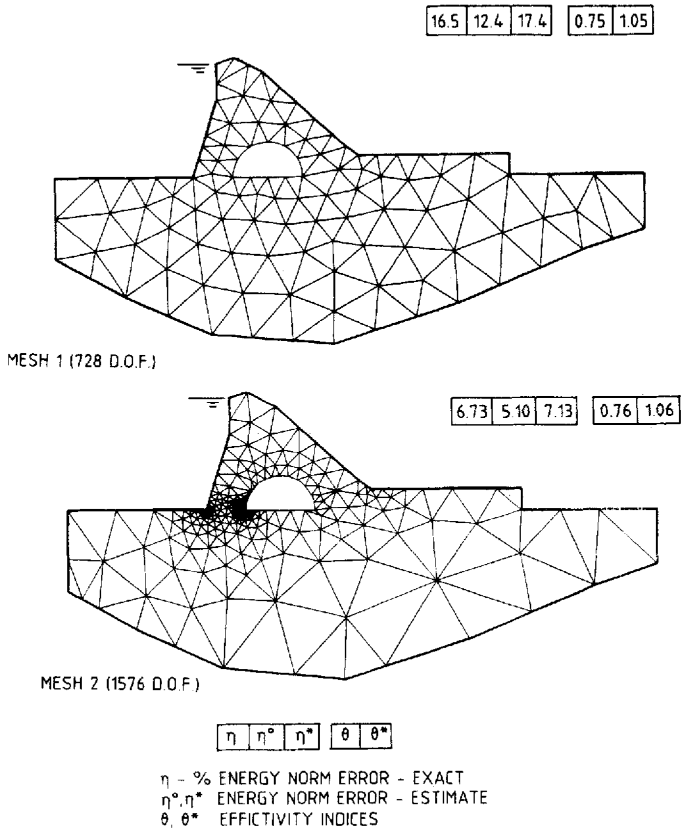

Figure 16. Quadratic triangle—automatic mesh generation to achieve 5 per cent accuracy plane strain analysis of a dam with perforation water loading only

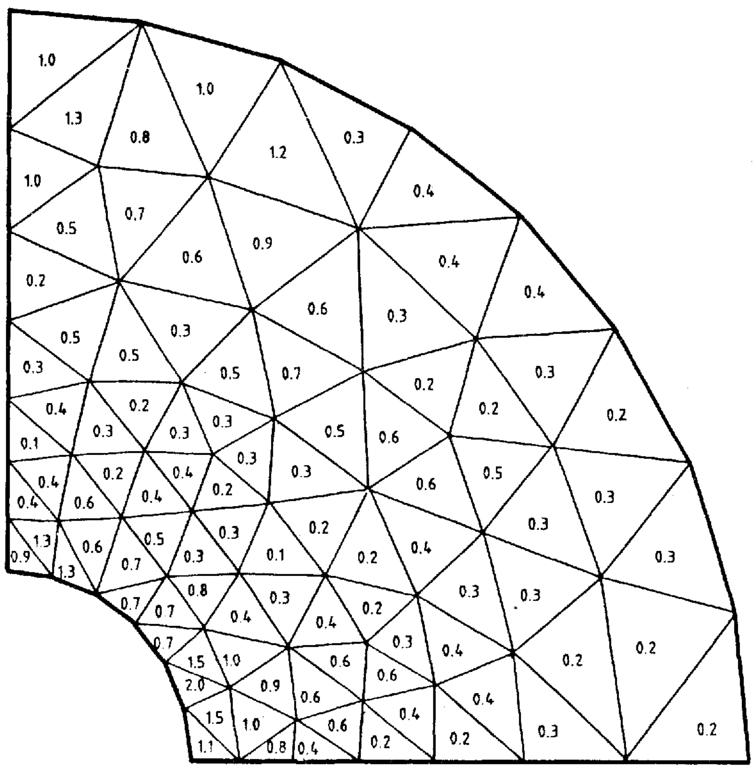

Figure 17. The distribution of $\Delta\sigma_i$ of Figure 11. The numbers above are $\Delta\sigma_i \times 10$: $\Delta\sigma_{\text{permissible}} = 0{\cdot}013$ (5 per cent R.M.S. σ), max. $\Delta\sigma_i = 0{\cdot}2$ max. $\xi_i = 15{\cdot}3$

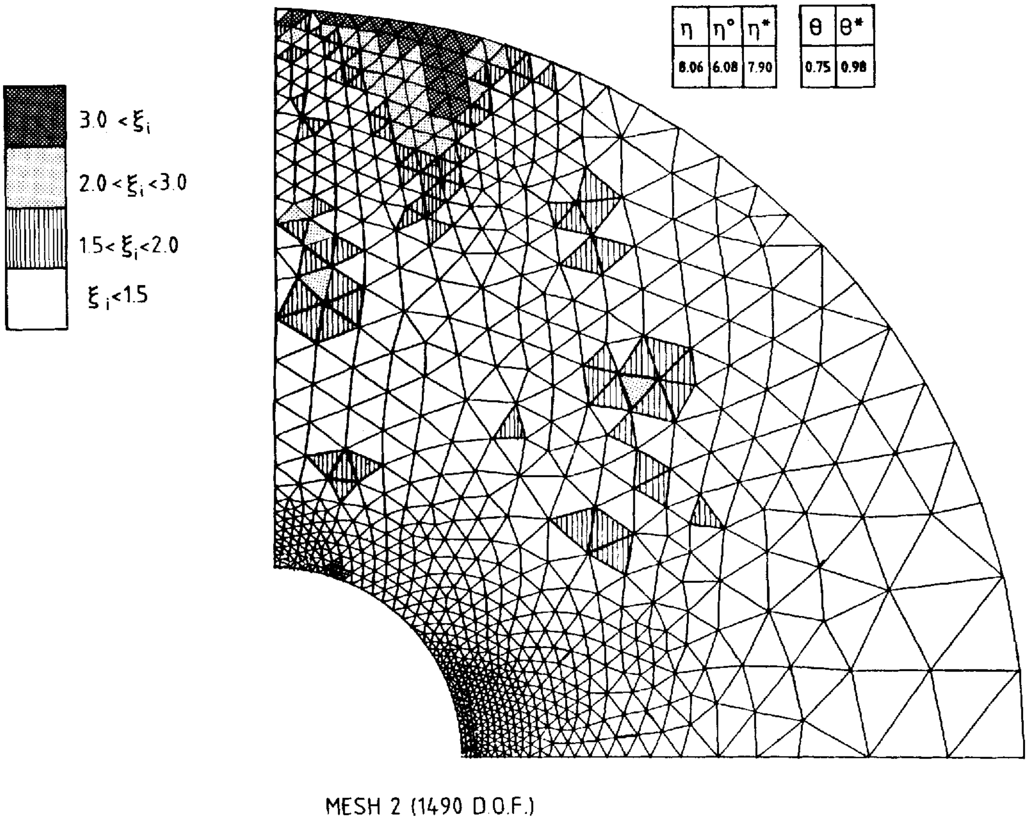

Figure 18. Predicted mesh for Figure 13, using the $\Delta\sigma$ criterion of refinement. Max. $\Delta\sigma_i = 0{\cdot}127$, max. $\xi_i = 6{\cdot}0$; 11·0 per cent of all elements still show $\xi_i > 2{\cdot}0$ ($\xi_i = \Delta\sigma_i / \Delta\sigma_{\text{permissible}}$)

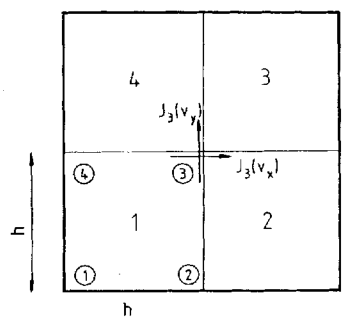

Figure 19. A typical mesh

APPENDIX

Instead of presenting the results here in the general form, we shall discuss the equivalence of the error estimators by means of the model equation

$$-K\Delta u = f, \quad \text{in} \quad \Omega \tag{29}$$

$$u = 0, \quad \text{on} \quad \Gamma_u \tag{30a}$$

$$K\frac{\partial u}{\partial n} = \bar{t}, \quad \text{on} \quad \Gamma_t \tag{30b}$$

with

$$\mathbf{v} = (v_x, v_y)^{\mathrm{T}} = -K \quad \text{grad} \quad u \tag{31}$$

and constant f and K values.

We shall consider bilinear finite element approximation and, for simplicity, a square mesh with side length h of each element (Figure 19).

The error estimator (20) in element i is now of the form

$$\varepsilon_i^2 = \frac{h}{24\,K}\int_I [J^2(v_x) + J^2(v_y)]\,\mathrm{d}I \tag{32}$$

as r here turns out to be zero, because of the use of bilinear functions.

The error estimator (10) with (19) in element i has the form

$$\|\mathbf{e}\|_i^2 = \frac{1}{K}\int_{\Omega_i} (\mathbf{v}^* - \hat{\mathbf{v}})^{\mathrm{T}}(\mathbf{v}^* - \hat{\mathbf{v}})\,\mathrm{d}\Omega \tag{33}$$

where

$$\mathbf{v}^* = \mathbf{N}\bar{\mathbf{v}}^* \tag{34}$$

The smoothed nodal values $\bar{\mathbf{v}}^*$ are the averaged nodal values of $\hat{\mathbf{v}}$. Considering the continuity of $\hat{v}_x$ and $\hat{v}_y$, we see immediately that, for example, at node 3 of element 1 of Figure 19

$$(v_x^* - \hat{v}_x)_3 = \tfrac{1}{2}J_3(v_x) \tag{35a}$$

and

$$(v_y^* - \hat{v}_y)_3 = \tfrac{1}{2}J_3(v_y) \tag{35b}$$

This shows that the difference of $\mathbf{v}^*$ and $\hat{\mathbf{v}}$ can be represented by the jumps. Thus, by taking into account the appropriate signs of the jumps at nodes 1 to 4 of the element, equation (33) can be rewritten as

$$\|\mathbf{e}\|_i^2 = \frac{h^2}{16\,K}(\mathbf{J}_x^{\mathrm{T}}\mathbf{A}\mathbf{J}_x + \mathbf{J}_y^{\mathrm{T}}\mathbf{A}\mathbf{J}_y) \tag{36}$$

where

$$\mathbf{J}_x = (-J_1(v_x), \quad J_2(v_x), \quad J_3(v_x), \quad -J_4(v_x))^{\mathrm{T}}$$

$$\mathbf{J}_y = (-J_1(v_y), \quad -J_2(v_y), \quad J_3(v_y), \quad J_4(v_y))^{\mathrm{T}}$$

$$\mathbf{A} = \int_{\Omega_i} \mathbf{N}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}\Omega$$

and Ω_i is a square element of side length 2.

Evaluating (36) and noting that

$$(J_1 - J_2 - J_3 + J_4)^2 \geqslant 0, \quad (J_1 + J_2 - J_3 - J_4)^2 \geqslant 0$$

and

$$|2J_jJ_k| \leqslant J_j^2 + J_k^2, \quad j,k = 1,2,3,4$$

we finally obtain

$$\frac{h^2}{144\,K}\sum_{j=1}^{4}[J_j^2(v_x) + J_j^2(v_y)] \leqslant \|\mathbf{e}\|_i^2 \leqslant \frac{h^2}{16\,K}\sum_{j=1}^{4}[J_j^2(v_x) + J_j^2(v_y)] \tag{37}$$

Noting that jumps are linear functions along the element edges, the numerical approximation of equation (32) yields the following expression:

$$\varepsilon_i^2 = \frac{h^2}{48\,K} \sum_{j=1}^{4} [J_j^2(v_x) + J_j^2(v_y)] \tag{38}$$

Comparing (38) and (37) we have

$$\tfrac{1}{3}\varepsilon_i^2 \leqslant \|\mathbf{e}\|_i^2 \leqslant 3\varepsilon_i^2 \tag{39}$$

therefore

$$\frac{1}{\sqrt{3}}\varepsilon \leqslant \|\mathbf{e}\| \leqslant \sqrt{3}\varepsilon \tag{40}$$

and the equivalence of the two error estimator holds.

If **A** is 'lumped' or diagonalized in the evaluation of (36), we have

$$\|\mathbf{e}\|_i^2 = \frac{h^2}{16\,K} \sum_{j=1}^{4} [J_j^2(v_x) + J_j^2(v_y)] \tag{41}$$

With this assumption we have simply

$$\|\mathbf{e}\| = \sqrt{3}\varepsilon \tag{42}$$

This perhaps gives an indication why some upward correction of the estimators may be needed. The above discussion has shown the equivalence of error estimators (20) and (10) for a two-dimensional square mesh. For general bilinear elements the equivalence of the error estimators follows from the fact that in this case ε_i and $\|\mathbf{e}\|_i$ differ from (32) and (33) only by constants which depend on the Jacobian of the mapping. As long as the Jacobian is bounded, i.e. there exist two constants $C_1, C_2 \geqslant C > 0$, such that

$$C_1 h^2 \leqslant |\det J| \leqslant C_2 h^2$$

then, the error estimators are again equivalent.

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