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An improved semianalytical method for sensitivity analysis

S. Oral

Mechanical Engineering Department, Middle East Technical University, Ankara 06531, Turkey

Abstract The semi-analytical method is conveniently used to obtain design sensitivities. However, it may have serious accuracy problems in shape design. In this study, an improved semianalytical method is presented for the accurate computation of shape design sensitivities. The method is based on approximating the flexibility matrix by means of von Neumann series. In numerical examples, two cases for which the standard semianalytical method fails are considered. It is demonstrated that the sensitivities can be obtained very accurately by the improved method proposed.

1 Introduction

The accurate evaluation of the sensitivities of the system response to design variables is essential for an efficient optimization procedure. The overall finite difference technique provides accurate sensitivities, however it might be too slow for a large number of design variables. On the other hand, the analytical evaluation of the sensitivities may become a prohibitively difficult task if the system response is a very complicated function of the design variables. Hence, the semi-analytical method (SA) is generally the most convenient tool for complex problems such as shape design. However, it has been reported that severe inaccuracies may occur in the computation of shape sensitivities if this method is applied to structures which are built up by beam, plate or shell elements (Barthelemy and Haftka 1988; Olhoff and Rasmussen 1991). A few attempts have been made to improve the accuracy. Olhoff and Rasmussen (1990) developed correction factors to eliminate the errors in the method. Cheng *et al.* (1990) employed second-order information as a remedy. Cheng and Olhoff (1991) used rigid body displacements to detect the errors and correct the sensitivities. Mlejnek (1992) improved the method by the natural FEM approach which yields a nondefective incremental stiffness.

In this paper, the improved semi-analytical method (ISA) proposed by Hörnlein (1992) is applied to shape design sensitivity analysis. In this method, a series expansion is obtained for the sensitivities that converge to an exact value. The method is based on approximating the system flexibility by means of von Neumann series. The accuracy of the method is illustrated by numerical examples.

2 The method

Consider the static equilibrium equations

$$\mathbf{K}\mathbf{u} = \mathbf{f}\,, \tag{1}$$

where the stiffness matrix $\mathbf{K}$ and the displacement vector $\mathbf{u}$ are functions of design variables x_i, $i = 1, \ldots, n$. The load vector $\mathbf{f}$ is taken to be constant. Then the derivative of displacements with respect to the i-th design variable is

$$\frac{\partial \mathbf{u}}{\partial x_i} = -\mathbf{K}^{-1}\frac{\partial \mathbf{K}}{\partial x_i}\mathbf{u}\,. \tag{2}$$

For an increment $\Delta x_i = \varepsilon x_i$ in the design variable, ε being a small disturbance, the stiffness matrix becomes

$$\mathbf{K}^{(i)} = \mathbf{K} + \Delta\mathbf{K}^{(i)}\,, \tag{3}$$

and the displacements can be calculated as

$$\mathbf{u}^{(i)} = {\mathbf{K}^{(i)}}^{-1}\mathbf{f}\,, \tag{4}$$

where ${\mathbf{K}^{(i)}}^{-1}$ can be expressed as

$${\mathbf{K}^{(i)}}^{-1} = [\mathbf{K} + \Delta\mathbf{K}^{(i)}]^{-1} = [\mathbf{I} + \mathbf{K}^{-1}\Delta\mathbf{K}^{(i)}]^{-1}\mathbf{K}^{-1}\,, \tag{5}$$

where $\mathbf{I}$ is an identity matrix.

For a matrix $\mathbf{M}$ whose spectral radius satisfies the condition

$$\rho(\mathbf{M}) = \max_j |\lambda_j| < 1\,, \tag{6}$$

von Neumann series gives

$$(\mathbf{I} + \mathbf{M})^{-1} = \sum_{k=1}^{\infty}(-1)^k\mathbf{M}^k\,. \tag{7}$$

Hence, one can express ${\mathbf{K}^{(i)}}^{-1}$ as

$${\mathbf{K}^{(i)}}^{-1} = \sum_{k=1}^{\infty}(-1)^k[\mathbf{K}^{-1}\Delta\mathbf{K}^{(i)}]^k\mathbf{K}^{-1}\,, \tag{8}$$

and these series will converge to the exact value provided that $\rho(\mathbf{K}^{-1}\Delta\mathbf{K}^{(i)}) < 1$. Using (1) and (8), (4) can be written as

$$\mathbf{u}^{(i)} = {\mathbf{K}^{(i)}}^{-1}\mathbf{f} = \sum_{k=0}^{\infty}(-1)^k[\mathbf{K}^{-1}\Delta\mathbf{K}^{(i)}]^k\mathbf{u}\,. \tag{9}$$

Then the sensitivity $\partial\mathbf{u}/\partial x_i$ can be approximated as

$$\frac{\partial \mathbf{u}}{\partial x_i} \cong \frac{\mathbf{u}^{(i)} - \mathbf{u}}{\Delta x_i} = \frac{1}{\Delta x_i}\sum_{k=1}^{\infty}(-1)^k[\mathbf{K}^{-1}\Delta\mathbf{K}^{(i)}]^k\mathbf{u}\,. \tag{10}$$

It is seen that the first term of the above series gives the SA sensitivity as

$$\frac{\partial \mathbf{u}}{\partial x_i} \cong \frac{\Delta\mathbf{u}}{\Delta x_i} = -\mathbf{K}^{-1}\frac{\Delta\mathbf{K}}{\Delta x_i}\mathbf{u}\,. \tag{11}$$

This shows the deficiency of SA, since accuracy cannot be expected by taking only the first term of a series. Denoting $[\mathbf{K}^{-1}\Delta\mathbf{K}^{(i)}]^k\mathbf{u}$ as $\eta_k^{(i)}$, one obtains

$$\frac{\partial \mathbf{u}}{\partial x_i} \cong \frac{1}{\Delta x_i}[-\eta_1^{(i)} + \eta_2^{(i)} - \eta_3^{(i)} + \ldots]\,. \tag{12}$$

Note that for each additional term in this series only one back-substitution is needed as

$$\mathbf{LDL}^{\mathrm{T}}\eta_k^{(i)} = \Delta\mathbf{K}^{(i)}\eta_{k-1}^{(i)}, \tag{13}$$

where $\eta_0^{(i)} = \mathbf{u}$ and $\mathbf{LDL}^{\mathrm{T}} = \mathbf{K}$ is the Crout decomposition available from the analysis. Hence, accurate sensitivities can be obtained cost-effectively by the present method provided that the perturbation ε in determining Δx_i is chosen small enough to satisfy the condition on the spectral radius.

3 Numerical studies

In this section two problems for which SA (i.e. $k = 1$ in ISA) fails are considered. It is demonstrated that almost exact sensitivities can be obtained by the proposed method except for rounding-off errors.

3.1 Beam in pure bending

The geometry of the beam is shown in Fig. 1. The beam consists of m number of elements of equal length 1. It is fixed at $x = 0$ and loaded by a tip moment $M = 1/m$ at the free end P where $x = L = m$. The rigidity $EI = 1$ throughout. The sensitivity of the transverse displacement at the free end, v_P, with respect to the total length L can be determined analytically as

$$\frac{\partial v_P}{\partial L} = \frac{\partial}{\partial L}\left(\frac{1}{2}\frac{ML^2}{EI}\right) = 1,$$

and is independent of the number of elements m. If the sensitivity is computed by SA with a perturbation $\varepsilon = 0.0001$, then the error increases rapidly with m and it becomes as high as 250% when $m = 100$. If, however, the present method is used, the sensitivities obtained are in excellent agreement with the analytical results even for $k = 2$. Mlejnek (1992) also obtained accurate results in this problem by the natural FEM approach. The results obtained by the present method for various values of k are tabulated in Table 1 and are compared with Mlejnek's solution.

Table 1. Sensitivity of tip deflection with respect to beam length

	$\partial v_E/\partial L$	Error %
SA	-1.49935	-249.935
ISA, $k = 1$	-1.49935	-249.935
ISA, $k = 2$	0.99977	-0.023
ISA, $k = 3$	0.99883	-0.117
ISA, $k = 4$	1.00005	0.005
ISA, $k = 5$	1.00005	0.005
Natural FEM (Mlejnek 1992)	0.99990	-0.010
Cartesian FEM (Mlejnek 1992)	0.99980	-0.020
Analytical	1.0	

3.2 Plate under uniform load

A square plate with width $a = 1$, thickness $h = 0.02$, elasticity modulus $E = 10920$ and Poisson's ratio $\nu = 0.3$ is considered, as shown in Fig. 2. The plate is simply supported along all edges and is subjected to a uniform transverse load of intensity $q = 1$. The entire plate is modelled by 16 rectangular Kirchhoff elements. The width a is taken as the design variable. In this problem the sensitivity of the transverse deflections with respect to the design variable a can easily be determined analytically, e.g. for the centre deflection w_c, the analytical value of the sensitivity is

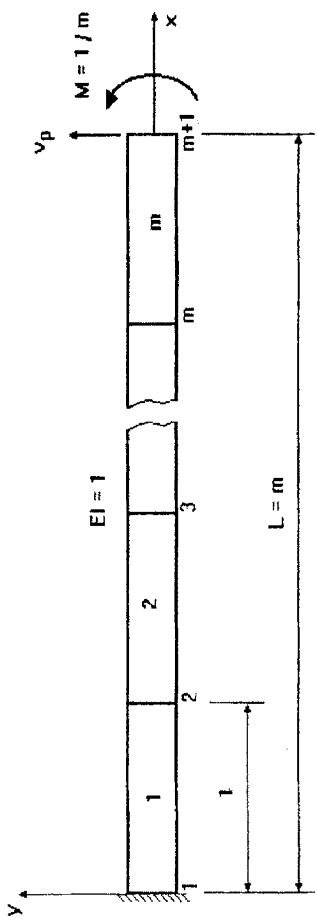

Fig. 1. Cantilever beam under end moment

$$\frac{\partial w_c}{\partial a} = \frac{4w_c}{a} = 2.031\,.$$

SA gives very inaccurate results with a perturbation $\varepsilon = 0.01$. However, almost exact results are obtained by ISA even with few terms in the series. The sensitivities computed for various points in the plate are tabulated in Table 2 and are compared with the results of Yamazaki and Vanderplaats (1993), and the analytical values.

4 Conclusions

An improved method has been presented for the semi-analytical computation of sensitivities. The method gives a series expansion for the sensitivities. The first term of the series corresponds to the standard semi-analytical method. Only a few terms need to be considered for the accurate computation of the sensitivities. The method can readily be implemented in the existing finite element packages without requiring special element formulations, correction factors or second-order information. The only modification to be done is existing conventional SA algorithms is to implement a recursive call of the equation solver (13). Very accurate results have been obtained in two shape design problems for which

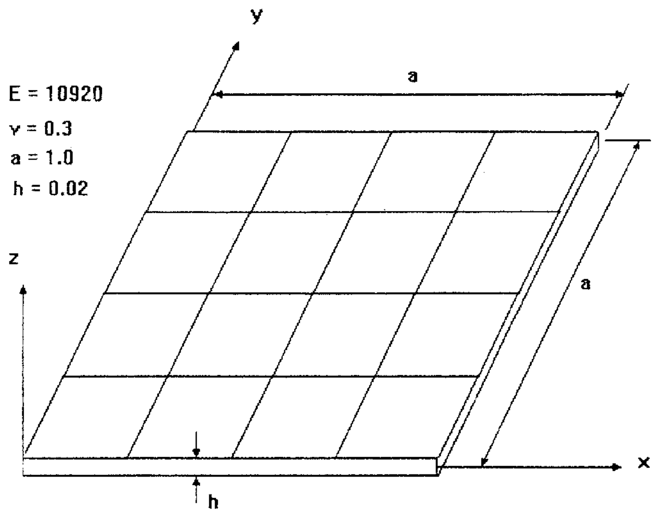

Fig. 2. Simply supported square plate under uniform load

Table 2. Sensitivity of deflections with respect to plate width, $\partial w/\partial a$

x/a	0.125	0.250	0.375	0.500
SA	0.4803	0.8673	1.110	1.193
ISA, $k=1$	0.4803	0.8673	1.110	1.193
ISA, $k=2$	0.7919	1.443	1.861	2.003
ISA, $k=3$	0.7985	1.455	1.877	2.020
ISA, $k=4$	0.8010	1.460	1.883	2.027
ISA, $k=5$	0.8010	1.460	1.883	2.027
FEM (Yamazaki and Vanderplaats 1993)	0.9933	1.742	1.965	2.041
FDM (Yamazaki and Vanderplaats 1993)	0.9934	1.742	1.965	2.041
Analytical	0.8116	1.469	1.888	2.031

the standard semi-analytical approach fails.

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