FINITE ELEMENT INTERPOLATION OF NONSMOOTH FUNCTIONS SATISFYING BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we propose a modified Lagrange type interpolation operator to approximate functions in Sobolev spaces by continuous piecewise polynomials. In order to define interpolators for "rough" functions and to preserve piecewise polynomial boundary conditions, the approximated functions are averaged appropriately either on d- or $(d-1)$-simplices to generate nodal values for the interpolation operator. This combination of averaging and interpolation is shown to be a projection, and optimal error estimates are proved for the projection error.

1. INTRODUCTION

The approximation of functions in Sobolev spaces by functions in finite element spaces has been well studied (cf. [3] and the references therein). One approach is to show the approximability of the nodal, finite element interpolant of a function. However, the nodal value of the function may not be well defined if the function under consideration is too "rough". For example, functions in the Sobolev space H^1 have no pointwise value in two or more dimensions. In [4], Clément defined an optimal-order interpolation operator using local averaging (regularizing) to define nodal values for functions even in L^1. However, the interpolator does not preserve homogeneous boundary conditions naturally. By setting boundary nodal values to zero, the modified interpolation operator posed in [4] can solve this problem, but this approach cannot be easily generalized to nonhomogeneous boundary data.

In this note, we propose another local averaging interpolation operator. We restrict our attention to simplicial finite elements, although this restriction is not essential. However, we consider general, d-dimensional meshes, $d \geq 2$; we assume only that the meshes are nondegenerate, i.e., we do *not* assume that they are quasi-uniform. Averaging is done either on a d-simplex or on a face ($(d-1)$-simplex) of some d-simplex. This interpolation operator preserves homogeneous boundary conditions naturally. The key point is that this operator averages the interpolated functions on a subset (a $(d-1)$-simplex) of the boundary of the domain for each boundary nodal value, and it is of optimal order in approximation. We note that the interpolator is not defined for as broad a

class of functions as that of [4], but it is defined on functions smooth enough to have well-defined boundary values. A special case of this operator, interpolating $\overset{\circ}{H}{}^1$ functions by piecewise linear functions, was considered in [8]. We can also define an analogous interpolator for L^1 functions, using the techniques introduced here, that has properties similar to those of [4]. However, such an interpolator cannot be used to satisfy boundary conditions (such conditions are not well defined for L^1 functions in any case).

2. Interpolation

Let $\Omega \subset \mathscr{E}^d$ be a connected, open, bounded, d-dimensional domain with polyhedral boundary. We assume that $\partial\Omega$ is Lipschitz continuous for simplicity, although many of the results presented can be extended easily to domains with simple slits. Let $C^\infty(\Omega)$ be functions defined on Ω and having continuous derivatives of any order, and let $\overset{\circ}{C}{}^\infty(\Omega)$ be functions in $C^\infty(\Omega)$ with compact support in Ω. We denote by $L_p(\Omega)$ the completion of the functions in $C^\infty(\Omega)$ such that $\|f\|_{L_p(\Omega)} := (\int_\Omega |f(x)|^p dx)^{1/p}$ is finite, by $W_p^l(\Omega)$ the completion of the functions in $C^\infty(\Omega)$ such that $\|f\|_{W_p^l(\Omega)} := (\sum_{|\alpha|\le l} \|D^\alpha f\|_{L_p(\Omega)}^p)^{1/p}$ is finite, and by $\overset{\circ}{W}{}_p^l(\Omega)$ the completion of the functions in $\overset{\circ}{C}{}^\infty(\Omega)$ such that $\|f\|_{W_p^l(\Omega)}$ is finite. Similarly, we define seminorms, $|f|_{W_p^l(\Omega)} := (\sum_{|\alpha|=l} \|D^\alpha f\|_{L_p(\Omega)}^p)^{1/p}$. Here, $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index (each α_i is a nonnegative integer) with $|\alpha| := \sum_{i=1}^d \alpha_i$ and $D^\alpha := (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_d)^{\alpha_d}$. We denote $W_2^l(\Omega)$ and $\overset{\circ}{W}{}_2^l(\Omega)$ by $H^l(\Omega)$ and $\overset{\circ}{H}{}^l(\Omega)$, respectively, as usual. For the definition of $W_p^l(\Omega)$ for fractional-order l, see [1].

Let $\mathscr{T}_h$ be a simplicial subdivision of Ω with maximum mesh size

$$h := \max_{K \in \mathscr{T}_h} \operatorname{diam}(K)$$

that is *nondegenerate*:

$$(2.1) \qquad \max_{K \in \mathscr{T}_h} \frac{\operatorname{diam}(K)}{\rho_K} \le \gamma_0$$

with the constant γ_0 independent of h. Note that we do *not* assume that all simplices, K, are of comparable size (that is, the mesh need not be quasi-uniform). Here, K denotes a d-simplex, $\operatorname{diam}(K)$ denotes the diameter of K and ρ_K denotes the radius of the largest closed ball contained in $\overline{K}$. For simplicity, we make the assumption that the spatial variable has been normalized so that $\operatorname{diam}(\Omega) = 1$. We consider a finite element space V_h consisting of continuous piecewise polynomials

$$(2.2) \qquad V_h := \{ v \in C(\Omega) \mid v|_K \in P_K = \mathscr{P}_r^d \ \forall K \in \mathscr{T}_h\}$$

and the subspace $\overset{\circ}{V}_h$ consisting of continuous piecewise polynomials with homogeneous Dirichlet boundary condition:

(2.3) $$\overset{\circ}{V}_h := \{ v \in V_h \mid v|_{\partial\Omega} = 0 \}.$$

Here, $\mathscr{P}_r^d$ is the space consisting of polynomials of degree r or less in d variables. For simplicity, we made the assumption $\mathscr{P}_r^d = P_K$ although the theorems in this paper can be extended to cover cases where $\mathscr{P}_r^d \subset P_K$. By (2.1), we have a family of Lagrange finite elements,

$$(K, P_K, \Sigma_K), \qquad K \in \mathscr{T}_h,$$

which are all affine equivalent to a single reference finite element $(\hat{K}, \mathscr{P}_r^d, \hat{\Sigma})$ (cf. [3]). Here, $\hat{\Sigma}$ denotes the usual nodal variables for Lagrange interpolation consisting of point evaluations at appropriate points ("nodes") in $\hat{K}$, and Σ_K denotes evaluations at points in K that are the affine image of those for $\hat{\Sigma}$. For simplicity, we assume that $\hat{K}$ is a regular (equilateral) simplex having all edges of length one. We note that the theorems given in this paper also cover cases when the finite element spaces are of the form $\{ v \in V_h \mid v|_\Gamma = 0 \}$ for some $\Gamma \subset \partial\Omega$, provided that the triangulation matches Γ appropriately.

To define an interpolation operator on $W_p^l(\Omega)$, we make use of the nodal points, Σ_K, of the standard nodal interpolation operator. Let $\mathscr{N}_h = \{a_i\}_{i=1}^N$ be the set of all interpolation nodes of $\mathscr{T}_h$ and $\{\phi_i\}_{i=1}^N$ be the corresponding nodal basis of V_h. We choose, for any node a_i, either a d-simplex or a $(d-1)$-simplex, σ_i, according to the type of the node, a_i, as follows. If a_i is an interior point of some d-simplex, $K \in \mathscr{T}_h$, we let

(2.4) $$\sigma_i = K.$$

If a_i is an interior point of some face (which is a $(d-1)$-simplex), K', of a d-simplex, K, we let

(2.5) $$\sigma_i = K'.$$

For the rest of the $a_i \in \mathscr{N}_h$, which must be on some $(d-2)$-simplex, there is considerable freedom in picking σ_i. We may pick any $(d-1)$-simplex, K', such that $a_i \in \overline{K'}$, subject only to the restriction

(2.6) $$K' \subset \partial\Omega \quad \text{if } a_i \in \partial\Omega,$$

and we set $\sigma_i = K'$. We can see that the choice of σ_i in (2.6) is not unique. The restriction $K' \subset \partial\Omega$ in (2.6) for $a_i \in \partial\Omega$ is made for the purpose of preserving homogeneous boundary conditions.

For each face $(d-1)$-simplex, K', of K, there is a natural restriction of (K, P_K, Σ_K) that defines a finite element:

(2.7) $$(K', P_{K'}, \Sigma_{K'}) = (K, P_K, \Sigma_K)|_{K'}.$$

Here, $\Sigma_{K'}$ consists of point evaluations at the points from Σ_K that lie on the face K', and $P_{K'} = \mathscr{P}_r^{d-1}$. For example, the restriction to one dimension of a linear, triangular, finite element would be a linear, line segment, finite element. Since all (K, P_K, Σ_K) are affine equivalent to a single reference element, all $(K', P_{K'}, \Sigma_{K'})$ are affine equivalent to a single, $(d-1)$-dimensional reference element. Further, the nonzero functions that are restrictions of the nodal basis functions of P_K on K' comprise the nodal basis for $P_{K'}$. Moreover, (2.1) implies that the set of all face $(d-1)$-simplices,

$$(2.8) \qquad \{K' \mid K' \text{ is a face } ((d-1)\text{-simplex}) \text{ of } K, \text{ for some } K \in \mathscr{T}_h\},$$

is a nondegenerate family of $(d-1)$-simplices. We remark that the converse of the last statement is false. For example, a sequence of tetrahedra having nondegenerate faces (with each face approaching a unit right triangle in the limit) can degenerate into a plane; the limit is a unit square with two diagonals.

Let us denote by n_1 the dimension of $\mathscr{P}_r^d$ and by $n_0(\sigma_i)$ the dimension of $\mathscr{P}_r^{\dim \sigma_i}$, i.e., either the dimension of $\mathscr{P}_r^{d-1}$ or n_1. Let $a_{i,1} = a_i$, and let $\{a_{i,j}\}_{j=1}^{n_0}$ be the set of nodal points in σ_i. Here, σ_i is defined in (2.4)–(2.6) associated with the node a_i. For the nodal basis $\{\phi_{i,j}\}_{j=1}^{n_0}$ for σ_i, we have an $L^2(\sigma_i)$-dual basis $\{\psi_{i,j}\}$:

$$(2.9) \qquad \int_{\sigma_i} \psi_{i,j}(x)\phi_{i,k}(x)\,dx = \delta_{jk}, \qquad j, k = 1, 2, \ldots, n_0,$$

where δ_{jk} is the Kronecker delta. For simplicity, we let

$$(2.10) \qquad \qquad \psi_i = \psi_{i,1} \quad \forall a_i \in \mathscr{N}_i.$$

Therefore, we have

$$(2.11) \qquad \int_{\sigma_i} \psi_i(x)\phi_j(x)\,dx = \delta_{ij}, \qquad i, j = 1, 2, \ldots, N,$$

where ϕ_j is any nodal basis function of V_h.

We define an interpolation operator,

$$(2.12) \qquad \qquad \Pi : \quad W_p^l(\Omega) \to V_h(\Omega),$$

by

$$(2.13) \qquad \qquad \Pi v(x) = \sum_{i=1}^{N} \phi_i(x) \int_{\sigma_i} \psi_i(\xi)v(\xi)\,d\xi,$$

where

$$(2.14) \qquad \qquad l \geq 1 \quad \text{if} \quad p = 1 \quad \text{and} \quad l > 1/p \quad \text{otherwise}.$$

Here, Π depends on the choices of σ_i in (2.4)–(2.6), but we use the notation Π instead of $\Pi_{\{\sigma_i\}}$ for simplicity.

The condition (2.14) guarantees that the nodal values, $\{\Pi v(a_i)\}$, are well defined owing to the trace theorem (cf. [6] or [1]):

$$(2.15) \quad W_p^l(\Omega) \subset L^1(\sigma_i), \quad \text{i.e.,} \quad \|v\|_{L^1(\sigma_i)} \leq C(\Omega, \sigma_i)\|v\|_{W_p^l(\Omega)},$$
$$\forall v \in W_p^l(\Omega), \quad i = 1, \ldots, N.$$

Further, the condition (2.14) guarantees the validity of the homogeneous boundary condition:

$$(2.16) \quad \forall v \in \overset{\circ}{W}{}_p^l(\Omega), \quad v|_{\partial\Omega} = 0 \quad \text{in } L^1(\partial\Omega), \quad \text{i.e.,} \quad \|v\|_{L^1(\partial\Omega)} = 0.$$

By the choice of σ_i in (2.5), (2.6) we have, from (2.13) and (2.16), that

$$v \in \overset{\circ}{W}{}_p^l(\Omega) \implies \Pi v(a_i) = 0 \quad \forall a_i \in \partial\Omega,$$

and therefore that Π preserves the homogeneous boundary condition:

$$(2.17) \quad \Pi : \overset{\circ}{W}{}_p^l(\Omega) \to \overset{\circ}{V}_h.$$

Since (2.11) implies that

$$\int_{\sigma_i} \phi_i(x)v(x)\,dx = v(a_i) \quad \forall v \in V_h,$$

we conclude that Π is a projection,

$$(2.18) \quad \Pi v = v \quad \forall v \in V_h.$$

We summarize the above results in the following theorem.

Theorem 2.1. *Let l and p satisfy (2.14), and let $\mathcal{T}_h$ satisfy (2.1). Then the operator Π, defined in (2.13), is a projection from $W_p^l(\Omega)$ to V_h, defined in (2.2), with the property that $\overset{\circ}{W}{}_p^l(\Omega)$ is mapped to $\overset{\circ}{V}_h$, defined in (2.3).*

3. STABILITY

In the rest of this paper, we will use "$\lesssim \ldots$" and "$\simeq \ldots$" to denote "$\leq C \ldots$" and "$= C \ldots$", respectively, with the constant C independent of the mesh $\mathcal{T}_h$ and independent of the functions under consideration.

Let $\{\sigma_i\}$ be a set of simplices that are either d- or $(d-1)$-simplices satisfying (2.1). Let $\hat{\sigma}$ be either the d- or the $(d-1)$-dimensional reference simplex, let $\{\hat{\phi}_j\}$ be a nodal basis for $\hat{\sigma}$ and let $\{\hat{\psi}_j\}$ be the dual basis with respect to the inner product for $L^2(\hat{\sigma})$. Suppose the affine mapping

$$(3.1) \quad F(\hat{x}) = B\hat{x} + x_0$$

maps $\hat{\sigma}$ one-to-one and onto some $\sigma \in \{\sigma_i\}$; we then have (cf. [3])

$$\|B\| \leq \frac{h_\sigma}{\hat{\rho}} \lesssim h_\sigma \,,$$

(3.2) $$\|B^{-1}\| \leq \frac{\hat{h}_\sigma}{\rho_\sigma} \lesssim h_\sigma^{-1} \,,$$

$$\det(B) = \frac{\operatorname{meas}(\hat{\sigma})}{\operatorname{meas}(\sigma)} \lesssim \begin{cases} h_\sigma^d & \text{if } \dim(\sigma) = d \,, \\ h_\sigma^{d-1} & \text{if } \dim(\sigma) = d - 1 \,. \end{cases}$$

Here $h_\sigma = \operatorname{diam}(\sigma)$ and $\rho_\sigma = \operatorname{diam}$ (the largest closed ball contained in $\overline{\sigma}$), with $\hat{h}$ and $\hat{\rho}$ defined correspondingly. Let $\{\bar{\phi}_j\}$ be the nodal basis, and $\{\bar{\psi}_j\}$ the dual basis to $\{\bar{\phi}_j\}$ on σ:

$$\int_\sigma \bar{\psi}_j(x) \bar{\phi}_k(x) \, dx = \delta_{jk} \quad \forall j, \ k \,.$$

By the affine mapping defined in (3.1), it follows that

$$\int_{\hat{\sigma}} \bar{\psi}_j(B\hat{x} + \hat{x}_0) \bar{\phi}_k(B\hat{x} + \hat{x}_0) \det(B) \, d\hat{x} = \delta_{jk} \quad \forall j, \ k \,.$$

By the uniqueness of the dual basis and by noting $\hat{\phi}_j(\hat{x}) = \bar{\phi}_j(F(\hat{x}))$, we see that

(3.3) $$\hat{\psi}_j = \det(B) \bar{\psi}_j \quad \forall j \,.$$

By the regularity of the family $\{\sigma_i\}$ (both d- and $(d-1)$-simplices), we get the following lemma by combining (3.3) and (3.2).

Lemma 3.1. *For any node $a_i \in \mathcal{N}_h$,*

(3.4) $$\|\psi_i\|_{L^\infty(\sigma_i)} \lesssim h_K^{-\dim(\sigma_i)} \,,$$

where σ_i is the d- or $(d-1)$-simplex associated with a_i defined in (2.4)–(2.6), $K \in \mathcal{T}_h$ is such that $\sigma_i \subset \overline{K}$, and ψ_i is defined in (2.10). $\square$

Let $\sigma_i = K'$ be a $(d-1)$-simplex that is the face of a d-simplex, K, and let F defined in (3.1) map the reference $\hat{K}$ onto K such that $F(\hat{K}') = K'$ and $\hat{K}'$ is the $(d-1)$-dimensional reference simplex. Here we assume that $\hat{K}'$ is a face simplex of $\hat{K}$. Let us suppose that coordinates are chosen so that we can write

(3.5) $$F|_{\hat{K}'}(\hat{x}_{d-1}, 0) = (B_{d-1}\hat{x}_{d-1}, 0) + x_0$$

(in particular, $\hat{K}'$ lies in the plane $x_d = 0$). It follows from the trace theorem

and (3.2) that

$$\|v\|_{L^1(K')} \le |\det(B_{d-1})| \, \|\hat{v}\|_{L^1(\hat{K}')} \lesssim |\det(B_{d-1})| \, \|\hat{v}\|_{W_p^l(\hat{K})}$$

(3.6)
$$\lesssim h_K^{d-1} \|\hat{v}\|_{W_p^l(\hat{K})} \lesssim h_K^{d-1} \sum_{k=0}^{l} \|B\|^k |\det(B)|^{-1/p} |v|_{W_p^k(K)}$$

$$\lesssim \sum_{k=0}^{l} h_K^{d-1+k-d/p} |v|_{W_p^k(K)}.$$

Correspondingly, we have by Hölder's inequality that

(3.7)
$$\|v\|_{L^1(K)} \lesssim h_K^{d-d/p} |v|_{W_p^0(K)}.$$

Theorem 3.1. *Let* $v \in W_p^l(\Omega)$, *let* l *and* p *satisfy* (2.14), *let* $\mathcal{T}_h$ *satisfy* (2.1) *and let* $K \in \mathcal{T}_h$. *Then*

(3.8)
$$\|\Pi v\|_{W_q^m(K)} \lesssim \sum_{k=0}^{l} h_K^{k-m+\frac{d}{q}-\frac{d}{p}} |v|_{W_p^k(S_K)},$$

where

(3.9)
$$S_K = \text{interior}\left(\bigcup \{\overline{K_i} \mid \overline{K_i} \cap \overline{K} \ne \varnothing, \quad K_i \in \mathcal{T}_h\}\right)$$

and Π *is defined in* (2.13). *In* (3.8), $1 \le q \le \infty$ *and* m *is any nonnegative integer.*

Proof. Let F, defined in (3.1), map the reference simplex $\hat{K}$ onto K. We have, for any $v \in W_q^m(K)$,

$$|v|_{W_q^m(K)} \lesssim \|B^{-1}\|^m |\det(B)|^{-\frac{1}{q}} |\hat{v}|_{W_q^m(\hat{K})} \lesssim h_K^{-m+\frac{d}{q}} |\hat{v}|_{W_q^m(\hat{K})},$$

where $\hat{v} = v(F(\hat{x}))$. Further,

$$\|v\|_{W_q^m(K)} \lesssim h_K^{-m+\frac{d}{q}} \|\hat{v}\|_{W_q^m(\hat{K})},$$

since $h \le \text{diam}(\Omega)$. Without loss of generality, we may assume that the nodal points, $\{a_i \mid 1 \le i \le n_1\}$, comprise the nodal points for Σ_K. It follows then from (3.6)–(3.7) that, for any $v \in W_p^l(\Omega)$,

$$\|\Pi v\|_{W_q^m(K)} \le \sum_{i=1}^{n_1} |\Pi v(a_i)| \, \|\phi_i\|_{W_q^m(K)} \lesssim h_K^{-m+\frac{d}{q}} \max_{1 \le i \le n_1} \|\hat{\phi}_i\|_{W_q^m(\hat{K})} \sum_{i=1}^{n_1} |\Pi v(a_i)|$$

$$\lesssim h_K^{-m+\frac{d}{q}} \sum_{i=1}^{n_1} \left| \int_{\sigma_i} \psi_i(x) v(x) \, dx \right| \lesssim h_K^{-m+\frac{d}{q}} \sum_{i=1}^{n_1} \|\psi_i\|_{L^\infty(\sigma_i)} \|v\|_{L^1(\sigma_i)}$$

$$\lesssim h_K^{-m+\frac{d}{q}} \sum_{i=1}^{n_1} h_K^{-\dim(\sigma_i)} \|v\|_{L^1(\sigma_i)} \lesssim h_K^{-m+\frac{d}{q}} \sum_{i=1}^{n_1} \sum_{k=0}^{l} h_K^{k-d/p} |v|_{W_p^k(K_i)}$$

$$\lesssim \sum_{k=0}^{l} h_K^{-m+\frac{d}{q}+k-\frac{d}{p}} |v|_{W_p^k(S_K)},$$

where K_i is a neighboring d-simplex such that $\sigma_i \subset \overline{K}_i$ and S_K is defined in (3.9). Here, Lemma 3.1 was used to bound $\|\psi_i\|_{L^\infty(\sigma_i)}$, and we used the fact that (2.1) implies that K, σ_i, and K_i are all of comparable size, i.e., that a nondegenerate mesh is locally quasi-uniform in two or more dimensions. □

4. APPROXIMABILITY

We are going to consider $\|v - \Pi v\|_S$ for $v \in W_p^l(\Omega)$ and for various Sobolev norms, $\|\cdot\|_S$. For any $K \in \mathscr{T}_h$ and any polynomial $p \in \mathscr{P}_r^d$, we have by (2.18) and Theorem 3.1 that

$$\|v - \Pi v\|_{W_p^m(K)} \le \|v - p\|_{W_p^m(K)} + \|\Pi(p - v)\|_{W_p^m(K)}$$

(4.1)

$$\lesssim \sum_{k=0}^m h_K^{k-m} \|v - p\|_{W_p^k(S_K)}.$$

By (2.1), S_K is the finite union of domains each of which is star-shaped with respect to a ball of radius ρ times the diameter of S_K, with ρ depending only on γ_0. Thus, we can apply the Bramble–Hilbert lemma in the form developed in [5, p. 458] to the right-hand side of (4.1) to obtain, for all $\forall v \in W_p^l(S_K)$,

$$(4.2) \quad \inf_{p \in \mathscr{P}_r^d} \|v - p\|_{W_p^k(S_K)} \le C(d, r, \gamma_0) h_K^{l-k} |v|_{W_p^l(S_K)}, \qquad 0 \le k \le l \le r + 1.$$

In particular, we let the domains, D_j, in Theorem 7.1 of [5] be the interior of the closure of the union of pairs of simplices that share a common face. As a consequence of (2.1), such domains are each star-shaped with respect to a ball of radius ρ times the diameter of S_K, where ρ depends only on γ_0. Because of our assumption about the regularity of $\partial\Omega$, S_K is connected. In view of Remark 7.2 of [5], (4.2) holds with a constant depending only on d, r, γ_0 because the intersection of a pair of domains, D_j, contains one of the simplices that make up S_K and hence it contains a ball of radius $\tilde{\gamma}_0$ times the diameter of S_K, where $\tilde{\gamma}_0$ depends only on γ_0 in (2.1).

Combining (4.1) and (4.2) yields

$$(4.3) \qquad \|v - \Pi v\|_{W_p^m(K)} \le C(d, r, \gamma_0) h_K^{l-m} |v|_{W_p^l(S_K)}.$$

Since

$$\sup_{\check{K} \in \mathscr{T}_h} \{\operatorname{card}\{K \in \mathscr{T}_h \mid \check{K} \cap S_K \ne \varnothing\}\}$$

is a constant depending only on γ_0 in (2.1), we easily deduce our main theorem.

Theorem 4.1. *Let $v \in W_p^l(\Omega)$, let l and p satisfy (2.14), and let $\mathscr{T}_h$ satisfy (2.1). Then*

$$(4.4) \quad \left(\sum_{K \in \mathscr{T}_h} h_K^{p(m-l)} \|v - \Pi v\|_{W_p^m(K)}^p \right)^{1/p} \lesssim \|v\|_{W_p^l(\Omega)}, \qquad 0 \le m \le l \le r + 1,$$

where Π is defined in (2.13).

Letting $m = l$ and applying the triangle inequality, the following corollary is derived.

Corollary 4.1. *Let $v \in W_p^l(\Omega)$, let l and p satisfy (2.14), and let $\mathscr{T}_h$ satisfy (2.1). Then*

$$(4.5) \qquad \left(\sum_{K \in \mathscr{T}_h} \|\Pi v\|_{W_p^l(K)}^p \right)^{1/p} \lesssim \|v\|_{W_p^l(\Omega)},$$

where Π is defined in (2.13). $\square$

Recalling that $h = \max_{K \in \mathscr{T}_h} \text{diam}(K)$, the statment of Theorem 4.1 can be simplified as follows:

$$(4.6) \qquad \left(\sum_{K \in \mathscr{T}_h} \|v - \Pi v\|_{W_p^m(K)}^p \right)^{1/p} \lesssim h^{l-m} \|v\|_{W_p^l(\Omega)}, \qquad 0 \le m \le l \le r+1.$$

Note that the results obtained so far easily extend to domains with simple slits. In fact, the critical condition is that the set S_K in (3.9) be connected, and this can be arranged in such a case. We also note that if one is not interested in preserving boundary conditions, the techniques used here can also be used to define interpolators with optimal-order approximation properties for functions only in $L^1(\Omega)$. In such a case, the σ_i's are all chosen to be d-dimensional simplices containing the corresponding points, a_i. That is, the nodal values for all nodal points, a_i, are defined in the way that the nodal values for nodal points in the *interior* of each simplex were defined, cf. (2.4). In this way, one obtains an interpolator with properties similar to those of [4].

5. Applications

One way that the projection operator (2.13) can be used is to treat the Dirichlet problem for an elliptic boundary value problem,

$$(5.1) \qquad -\sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(\alpha_{ij} \frac{\partial u}{\partial x_i} \right) = 0 \quad \text{in} \quad \Omega,$$

$$u = g \quad \text{on} \quad \partial\Omega.$$

Here we assume the α_{ij} are bounded, measurable functions on Ω which form a uniformly positive definite matrix a.e. on Ω. Assuming g to be defined on all of Ω, and smooth enough for an interpolant, Πg, to be defined, then it is natural to define an approximant to u by seeking u_h in the space

$$V_h^g := \{v \in V_h \mid v = \Pi g \text{ on } \partial\Omega\} = \{v \in V_h \mid v - \Pi g \in \mathring{V}_h\}$$

such that

$$(5.2) \qquad a(u_h, v) := \int_\Omega \sum_{i,j=1}^{d} \alpha_{ij} \frac{\partial u_h}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx = 0 \quad \forall v \in \overset{\circ}{V}_h.$$

Note that u_h depends only on $\Pi g|_{\partial\Omega}$. If Π is the interpolator defined in (2.13), $\Pi g|_{\partial\Omega}$ depends only on $g|_{\partial\Omega}$. In particular, u furnishes an extension to $g|_{\partial\Omega}$, so we may think of u_h being defined using the space

$$V_h^g = \{v \in V_h \mid v - \Pi u \in \overset{\circ}{V}_h\},$$

since $\Pi g|_{\partial\Omega} = \Pi u|_{\partial\Omega}$. Note that not all interpolators, e.g. [4], would necessarily have this property.

We can derive error estimates using standard techniques in the natural energy norm

$$(5.3) \qquad \|v\|_E^2 := a(v, v).$$

Since $a(u_h, v) = a(u, v) = 0$ for $v \in \overset{\circ}{V}_h$, we have

$$\|\Pi u - u_h\|_E^2 = a(\Pi u - u_h, \Pi u - u_h) = a(\Pi u, \Pi u - u_h) = a(\Pi u - u, \Pi u - u_h).$$

Applying Schwarz's inequality, we find

$$\|\Pi u - u_h\|_E \leq \|\Pi u - u\|_E.$$

From the triangle inequality we thus have

$$\|u - u_h\|_E \leq 2\|u - \Pi u\|_E.$$

The error estimate (4.6) therefore implies that

$$(5.4) \qquad \|u - u_h\|_E \lesssim \max\{\|\alpha_{ij}\|_{L^\infty(\Omega)} \mid i, j = 1, \ldots, d\} h^{l-1} \|u\|_{H^l(\Omega)},$$
$$1 \leq l \leq r + 1.$$

For estimates in other Sobolev norms, see [7]. We note that Πg can be defined when standard (pointwise) interpolants may not be defined, e.g., for unbounded or discontinuous g. Moreover, Πg can be evaluated locally, so that the implementation of such a method is straightforward. Our techniques can be viewed as providing a systematic way of "averaging" the boundary data in such cases.

Another application of the interpolator defined in (2.13) is to extension of boundary data. Let us assume that $d = 2$ for simplicity. One can define Sobolev spaces $W_p^s(\partial\Omega)$ for real values of $s \in [0, 1]$ provided that $\partial\Omega$ is Lipschitz continuous, as we are assuming. Furthermore, there is a continuous extension operator, $E : W_p^{1-1/p}(\partial\Omega) \longrightarrow W_p^1(\Omega)$ for $1 \leq p < \infty$ (cf. [2] and references therein). Composing with Π, we obtain a map, $E_h = \Pi E : W_p^{1-1/p}(\partial\Omega) \longrightarrow V_h$, having the property that

$$(5.5) \qquad \|E_h v\|_{W_p^1(\Omega)} \lesssim \|v\|_{W_p^{1-1/p}(\partial\Omega)}.$$

Moreover, let Γ be any subset of $\partial\Omega$, and define

$$(5.6) \qquad\qquad \Gamma_h = \bigcup \{\sigma_i \mid \sigma_i \subset \Gamma\},$$

where σ_i is defined in (2.4)–(2.6). If v is a piecewise polynomial on Γ of degree r, then (2.11) implies that $E_h v = v$ on Γ_h. Thus we have an extension operator that maps piecewise polynomials on (parts of) the boundary to piecewise polynomials on Ω, that is bounded as a map from $W_p^{1-1/p}(\partial\Omega)$ to $W_p^1(\Omega)$.

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