Parameter estimation for multiscale diffusions with continuous and discrete moving averages

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1 Introduction

Let $\varepsilon > 0$ and let us consider the one-dimensional multiscale stochastic differential equation (SDE)

$$\mathrm{d}X_t^\varepsilon = -\alpha V_0'(X_t^\varepsilon)\,\mathrm{d}t - \frac{1}{\varepsilon}V_1'\left(\frac{X_t^\varepsilon}{\varepsilon}\right) + \sqrt{2\sigma}\,\mathrm{d}W_t, \tag{1.1}$$

where the drift coefficient α and the diffusion coefficient σ are positive real parameters, possibly unknown, and W_t is a standard one-dimensional Brownian motion. The functions $V_0, V_1 \colon \mathbb{R} \to \mathbb{R}$ are slow and fast potentials driving the dynamics of the solution X_t^ε. Theory of homogenization [1] guarantees the existence of an SDE of the form

$$\mathrm{d}X_t^0 = -AV_0'(X)\,\mathrm{d}t + \sqrt{2\Sigma}\,\mathrm{d}W_t, \tag{1.2}$$

where the fast dynamics have been eliminated, such that $X_t^\varepsilon \to X_t^0$ in law as random variables with values in $\mathcal{C}^0((0,T))$. The drift and diffusion coefficients of the homogenized dynamics A and Σ are given by $A = K\alpha$ and $\Sigma = K\sigma$, where K can be computed as introduce theory.

In order to estimate the drift coefficient, one considers the likelihood function

$$L_T(X_t) = \exp\left\{\int_0^T -AV_0'(X_t)\,\mathrm{d}X_t - \frac{1}{2}\int_0^T A^2V_0'(X_t)^2\,\mathrm{d}t\right\},$$

whose logarithm $\ell_T(X_t) = \log L_T(X_t)$ can be maximised thus giving the estimator

$$\widehat{A} = -\frac{\int_0^T V_0'(X_t)\,\mathrm{d}X_t}{\int_0^T V_0'(X_t)^2\,\mathrm{d}t}.$$

The diffusion coefficient can be computed as the quadratic variation of the path, i.e., given a sequence of partitions $\mathcal{P}_h = \{t_k\}_{k=0}^{N_h}$, of the interval $[0,T]$, where $h \coloneqq \sup_k(t_k - t_{k-1})$, we have

$$\Sigma = \frac{1}{2T}\lim_{h\to 0}\sum_{k=1}^{N_h}(X_{t_k}^0 - X_{t_{k-1}}^0)^2, \tag{1.3}$$

in probability and for all $T > 0$.

In a Bayesian setting, we can fix a prior Λ with density λ and the posterior is then given by

$$\mu_T(B) = \frac{\int_B L_T(A)\lambda(A)\,\mathrm{d}A}{\int_{\mathcal{A}} L_T(A)\lambda(A)\,\mathrm{d}A}.$$

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2 Point estimates from continuous data

In this section, we study the convergence with respect to the parameter ε of point estimates of the drift and the diffusion coefficients when the estimator is computed employing data coming from the multiscale model.

2.1 Drift coefficient

Let $X^\varepsilon \coloneqq (X_t^\varepsilon, 0 \le t \le T)$ be the solution of (1.1) and define $\mathcal{H}_\Delta(X^\varepsilon)$ as

$$\mathcal{H}_\Delta(X^\varepsilon)_t \coloneqq \begin{cases} X_0, & t = 0, \\ \frac{1}{t}\int_0^t X_s \,\mathrm{d}s, & 0 < t < \Delta, \\ \frac{1}{\Delta}\int_{t-\Delta}^t X_s \,\mathrm{d}s, & \Delta \le t \le T, \end{cases} \tag{2.1}$$

with $\Delta > 0$. Let us denote for ease of notation, $Z_t^\varepsilon \coloneqq \mathcal{H}_\Delta(X^\varepsilon)_t$. The maximum likelihood estimator of the drift coefficient is then

$$\widehat{A}_{T,\Delta}(Z_t^\varepsilon) = -\frac{\int_0^T V_0'(Z_t^\varepsilon)\,\mathrm{d}Z_t^\varepsilon}{\int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t}.$$

Let us remark that for $0 < t < \Delta$,

$$\mathrm{d}(tZ_t^\varepsilon) = X_t \,\mathrm{d}t,$$

which implies

$$\mathrm{d}Z_t^\varepsilon = \frac{1}{t}(X_t^\varepsilon - Z_t^\varepsilon)\,\mathrm{d}t.$$

For $\Delta \le t \le T$, instead

$$\mathrm{d}Z_t^\varepsilon = \frac{1}{\Delta}(X_t^\varepsilon - X_{t-\Delta}^\varepsilon)\,\mathrm{d}t.$$

We rewrite the estimator as

$$\widehat{A}_{T,\Delta}(Z_t^\varepsilon) = -\frac{\int_0^\Delta V_0'(Z_t^\varepsilon)\frac{1}{t}(X_t^\varepsilon - Z_t^\varepsilon)\,\mathrm{d}t}{\int_0^T V_0'(Z_t^\varepsilon)^2\,\mathrm{d}t} - \frac{\int_\Delta^T V_0'(Z_t^\varepsilon)(X_t^\varepsilon - X_{t-\Delta}^\varepsilon)\,\mathrm{d}t}{\Delta\int_0^T V_0'(Z_t^\varepsilon)^2\,\mathrm{d}t}.$$

The goal of this section is proving the following result.

Theorem 2.1. *Under assumption add assumptions, if there exists $\zeta \in (0,2)$ such that $\Delta = \varepsilon^\zeta$ and $\gamma > \zeta$ such that $T = \varepsilon^{-\gamma}$, it holds*

$$\lim_{\varepsilon\to 0} \widehat{A}_{T,\Delta}(Z_t^\varepsilon) = A, \quad \textit{in law.}$$

It is useful in the following to rewrite (1.1) as a system of two coupled SDEs. In particular, introducing the variable $Y_t^\varepsilon \coloneqq X_t^\varepsilon/\varepsilon$, one has

$$\begin{aligned} \mathrm{d}X_t^\varepsilon &= -\alpha V_0'(X_t^\varepsilon)\,\mathrm{d}t - \frac{1}{\varepsilon}V_1'(Y_t^\varepsilon) + \sqrt{2\sigma}\,\mathrm{d}W_t, \\ \mathrm{d}Y_t^\varepsilon &= -\frac{\alpha}{\varepsilon}V_0'(X_t^\varepsilon)\,\mathrm{d}t - \frac{1}{\varepsilon^2}V_1'(Y_t^\varepsilon) + \sqrt{\frac{2\sigma}{\varepsilon^2}}\,\mathrm{d}W_t. \end{aligned} \tag{2.2}$$

The analysis necessary to prove Theorem 2.1 is based on the expansion

$$\begin{aligned} X_t^\varepsilon - X_{t-\Delta}^\varepsilon = &- \alpha\int_{t-\Delta}^t V_0'(X_s^\varepsilon)\big(1+\Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}s \\ &+ \sqrt{2\sigma}\int_{t-\Delta}^t \big(1+\Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}W_s \\ &- \varepsilon\big(\Phi'(Y_t^\varepsilon) - \Phi'(Y_{t-\Delta}^\varepsilon)\big), \end{aligned} \tag{2.3}$$

for $t \ge \Delta$ (see [2, Equation (5.8)]). The following lemma ensures that the process Z_t^ε has bounded moments.

Lemma 2.1. *The process Z_t^ε has bounded moments of all order, i.e., for all $p \geq 1$ and $t \geq 0$ it holds*

$$\mathbb{E}^{\mu^\varepsilon} |Z_t^\varepsilon|^p \leq C,$$

for $C > 0$ a constant uniform in $\varepsilon \to 0$.

Proof. The process X_t^ε has bounded moments (see [2, Corollary 5.4]), which implies the desired result with an application of the Hölder inequality. In fact, for $0 < t < \Delta$,

$$\begin{aligned}\mathbb{E}^{\mu^\varepsilon} |Z_t^\varepsilon|^p &\leq \frac{t^{p-1}}{t^p} \int_0^t \mathbb{E}^{\mu^\varepsilon} |X_s^\varepsilon|^p \,\mathrm{d}s \\ &\leq t^{-1} \int_0^t C \,\mathrm{d}s = C.\end{aligned}$$

For $\Delta \leq t \leq T$ the procedure is analogue. $\square$

In the following lemma the difference between the processes X_t^ε and Z_t^ε is bounded.

Lemma 2.2. *Under assumptions add assumptions*

$$\mathbb{E}^{\mu^\varepsilon} |X_t^\varepsilon - Z_t^\varepsilon|^p \leq C(\Delta^p + \Delta^{p/2} + \varepsilon^p),$$

where $C > 0$ is a constant independent of Δ and ε.

Proof. By definition of Z_t^ε for $\Delta \leq t \leq T$ and applying the Hölder inequality we have

$$\begin{aligned}\mathbb{E}^{\mu^\varepsilon} |X_t^\varepsilon - Z_t^\varepsilon|^p &= \Delta^{-p}\, \mathbb{E}^{\mu^\varepsilon} \left| \int_{t-\Delta}^t (X_t^\varepsilon - X_s^\varepsilon) \,\mathrm{d}s \right|^p \\ &\leq \Delta^{-1} \int_{t-\Delta}^t \mathbb{E}^{\mu^\varepsilon} |X_t^\varepsilon - X_s^\varepsilon|^p \,\mathrm{d}s\end{aligned}$$

We can now apply [2, Lemma 6.1] to the integrand to obtain

$$\mathbb{E}^{\mu^\varepsilon} |X_t^\varepsilon - Z_t^\varepsilon|^p \leq C\Delta^{-1} \int_{t-\Delta}^t (\Delta^p + \Delta^{p/2} + \varepsilon^p) \,\mathrm{d}s,$$

which implies the desired result. The case $0 < t \leq T$ can be proved analogously. $\square$

Lemma 2.3 (See [2, Proposition 5.8])**.** *Under assumptions add assumptions, it holds in law*

$$\alpha \int_{t-\Delta}^t V_0'(X_s^\varepsilon)\big(1 + \Phi'(Y_s^\varepsilon)\big) \,\mathrm{d}s = A\Delta V_0'(Z_t^\varepsilon) + R(\varepsilon, \Delta),$$

where for every $p > 0$ and if Δ and ε are sufficiently small, then

$$\Big(\mathbb{E}^{\mu^\varepsilon} |R(\varepsilon, \Delta)|^p\Big)^{1/p} \leq C(\varepsilon^2 + \Delta^{1/2}\varepsilon + \Delta^{3/2}),$$

where $C > 0$ is independent of ε and Δ.

Proof. Let us denote $\Psi(t) \coloneqq 1 + \Phi'(Y_t^\varepsilon)$. Then

$$\begin{aligned}\mathbb{E}^{\mu^\varepsilon} |R(\varepsilon, \Delta)|^p &= \mathbb{E}^{\mu^\varepsilon} \left| \int_{t-\Delta}^t \alpha V_0'(X_s^\varepsilon)\Psi(s) \,\mathrm{d}s - \Delta A V_0'(Z_t^\varepsilon) \right|^p \\ &\leq C\, \mathbb{E}^{\mu^\varepsilon} \left| V_0'(Z_t^\varepsilon) \int_{t-\Delta}^t (\alpha\Psi(s) - A) \,\mathrm{d}s \right|^p \\ &\quad + C\, \mathbb{E}^{\mu^\varepsilon} \left| \int_{t-\Delta}^t \alpha \left(V_0'(X_t^\varepsilon) - V_0'(Z_t^\varepsilon)\right) \Psi(s) \,\mathrm{d}s \right|^p.\end{aligned}$$

The result is then obtained following the proof of [2, Proposition 5.8] and replacing [2, Lemma 6.1] with Lemma 2.2, and [2, Corollary 4.1] with Lemma 2.1. $\square$

We can now prove Theorem 2.1.

Proof of Theorem 2.1. Consider the decomposition (2.3). Denoting

$$J_t := \sqrt{2\sigma} \int_{t-\Delta}^t \left(1 + \Phi'(Y_s^\varepsilon)\right) \mathrm{d}W_s,$$

we have due to Lemma 2.3 the equality in law

$$X_t^\varepsilon - X_{t-\Delta}^\varepsilon = -A\Delta V'(Z_t^\varepsilon) + J_t^\Delta + R(\varepsilon, \Delta),$$

where, since $\zeta \in (0,1)$, we have

$$\left(\mathbb{E}^{\mu^\varepsilon} \left|R(\varepsilon, \Delta)\right|^p\right)^{1/p} \leq C(\varepsilon^2 + \varepsilon^{3\zeta/2})$$

Therefore, we have that the estimator satisfies

$$\begin{aligned} \widehat{A}_{T,\Delta}(Z_t^\varepsilon) &= A - A\frac{\int_0^\Delta V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t}{\int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t} - \frac{\int_0^\Delta V_0'(Z_t^\varepsilon)\frac{1}{t}(X_t^\varepsilon - Z_t^\varepsilon) \,\mathrm{d}t}{\int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t} \\ &\quad - \frac{\int_\Delta^T V_0'(Z_t^\varepsilon) J_t \,\mathrm{d}t}{\Delta \int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t} - \frac{R(\varepsilon,\Delta)\int_\Delta^T V_0'(Z_t^\varepsilon) \,\mathrm{d}t}{\Delta \int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t} \\ &=: A - I_1 - I_2 - I_3 - I_4, \end{aligned} \tag{2.4}$$

in law. Let us analyse the terms I_i, $i = 1, \ldots, 4$ separately. Let us consider I_1 and multiply both the numerator and the denominator by $1/T$. Due to assumption add assumption and Lemma 2.1, we have

$$\frac{A}{T} \mathbb{E}^{\mu^\varepsilon} \left| \int_0^\Delta V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t \right| \leq C\varepsilon^{\gamma+\zeta},$$

for a constant $C > 0$ independent of Δ and ε. Hence the numerator vanishes in L^1 and thus in law for $\varepsilon \to 0$. We split the denominator as

$$\frac{1}{T}\int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t = \frac{1}{T}\int_0^T V_0'(X_t^\varepsilon)^2 \,\mathrm{d}t + \frac{1}{T}\int_0^T \left(V_0'(Z_t^\varepsilon)^2 - V_0'(X_t^\varepsilon)^2\right) \,\mathrm{d}t$$

For the first term, we have by the ergodic theorem

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T V_0'(X_t^\varepsilon)^2 \,\mathrm{d}t = \mathbb{E}^{\mu^\varepsilon} \left|V_0'\right|^2, \quad \text{a.s.}$$

For the second term, we have applying Cauchy–Schwarz's inequality and due to assumption add assumption and Lemma 2.2

$$\begin{aligned} \frac{1}{T}\mathbb{E}^{\mu^\varepsilon} \left| \int_0^T \left(V_0'(Z_t^\varepsilon)^2 - V_0'(X_t^\varepsilon)^2\right) \,\mathrm{d}t \right| &\leq \frac{C}{T}\int_0^T \left(\mathbb{E}^{\mu^\varepsilon} \left|V_0'(Z_t^\varepsilon) - V_0'(X_t^\varepsilon)\right|^2\right)^{1/2} \,\mathrm{d}t \\ &\leq C\left(\Delta + \Delta^{1/2} + \varepsilon\right), \end{aligned}$$

which implies that the denominator tends to a finite value in probability for $\varepsilon \to 0$. Therefore, by Slutsky's theorem,

$$\lim_{\varepsilon\to 0} I_1 = 0, \quad \text{in law.}$$

Let us now consider I_2 and multiply numerator and denominator by $1/T$. The denominator is the same as I_1, and therefore does not need to be treated further. The numerator can be bounded in L^1 as

$$\frac{1}{T}\mathbb{E}^{\mu^\varepsilon} \left| \int_0^\Delta V_0'(Z_t^\varepsilon)\frac{1}{t}(X_t^\varepsilon - Z_t^\varepsilon) \,\mathrm{d}t \right| \leq \frac{C}{\Delta T}\int_0^\Delta \frac{\Delta}{t}\mathbb{E}^{\mu^\varepsilon} \left|X_t^\varepsilon - Z_t^\varepsilon\right| \,\mathrm{d}t,$$

which, since $Z_0^\varepsilon = X_0^\varepsilon$, vanishes for $\varepsilon \to 0$. Hence, an application of Slutsky's theorem yields

$$\lim_{\varepsilon\to 0} I_2 = 0, \quad \text{in law.}$$

We consider now I_3, which can be rewritten as

$$\begin{aligned} I_3 &= \frac{1}{\sqrt{T\Delta}} \frac{\frac{1}{\sqrt{T\Delta}} \int_\Delta^T V_0'(Z_t^\varepsilon) J_t \,\mathrm{d}t}{\frac{1}{T}\int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t} \\ &= \varepsilon^{(\gamma-\zeta)/2} \frac{\frac{1}{\sqrt{T\Delta}} \int_\Delta^T V_0'(Z_t^\varepsilon) J_t \,\mathrm{d}t}{\frac{1}{T}\int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t} \end{aligned}$$

Let us remark that J_t is a martingale and that by Itô isometry

$$\mathbb{E}^{\mu^\varepsilon} |J_\Delta|^2 = 2\Sigma\Delta,$$

Therefore, we can apply the central limit theorem for martingales to the numerator and obtain the equality in law

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{\sqrt{T\Delta}} \int_\Delta^T V_0'(Z_t^\varepsilon) J_t \,\mathrm{d}t &= \frac{1}{\sqrt{\Delta}} \mathcal{N}\left(0, \mathbb{E}^{\mu^\varepsilon}\left(\left|V_0'(X_0^\varepsilon)\right|^2 |J_\Delta|^2\right)\right) \\ &= C\mathcal{N}(0,1). \end{aligned}$$

The denominator is the same as in I_2 and I_3 and tends in probability to a finite value. Hence, since by hypothesis $\gamma > \zeta$, we have

$$\lim_{\varepsilon\to 0} I_3 = 0, \quad \text{in law.}$$

For the last term I_4, we have

$$I_4 = \frac{\varepsilon^{\gamma-\zeta} R(\varepsilon,\Delta) \int_\Delta^T V_0'(Z_t^\varepsilon)\,\mathrm{d}t}{\frac{1}{T}\int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t}.$$

For the numerator, we have by the Cauchy–Schwarz inequality and due to Lemma 2.3

$$\begin{aligned} \varepsilon^{\gamma-\zeta}\, \mathbb{E}^{\mu^\varepsilon} \left| R(\varepsilon,\Delta) \int_\Delta^T V_0'(Z_t^\varepsilon)\,\mathrm{d}t \right| &\le \varepsilon^{\gamma-\zeta} \left(\mathbb{E}^{\mu^\varepsilon} |R(\varepsilon,\Delta)|^2\right)^{1/2} \left(\mathbb{E}^{\mu^\varepsilon} \left|\int_\Delta^T V_0'(Z_t^\varepsilon)\,\mathrm{d}t\right|^2\right)^{1/2} \\ &\le C\varepsilon^{\gamma-\zeta}(\varepsilon^2 + \varepsilon^{3\zeta/2})\varepsilon^{-\gamma} \\ &\le C\left(\varepsilon^{2-\zeta} + \varepsilon^{\zeta/2}\right) \end{aligned}$$

which implies that, since the denominator is the same as before,

$$\lim_{\varepsilon\to 0} I_4 = 0, \quad \text{in law.}$$

The decomposition (2.4), together with the limits of I_i for $i = 1,\dots,4$, prove the desired result. $\square$

2.2 Diffusion coefficient

We now consider the same transformation of the data, i.e., we employ $Z_t^\varepsilon = \mathcal{H}_\Delta(X)_t$ as defined in (2.1), to estimate the diffusion coefficient Σ of the homogenized model. In particular, we consider the estimator

$$\widehat{\Sigma}_{\Delta,T} = \frac{1}{2T} \lim_{h\to 0} \sum_{k=1}^{N_h} (Z_{t_k}^\varepsilon - Z_{t_{k-1}}^\varepsilon)^2,$$

where the limit has to be intended in probability and with respect to a series of refinements of partitions $\mathcal{P}_h = \{t_k\}$ of the interval $[0,T]$. Let us recall that if instead of Z_t^ε one employs a path from the homogenized model X_t^0, then formula (1.3) gives the exact value of Σ for any $T > 0$.

Let us introduce a theoretical result which will play the role of Lemma 2.3 in this framework.

Lemma 2.4 (See [2, Proposition 5.7])**.** *Under assumptions add assumptions, there exist random variables $\xi_t \sim \mathcal{N}(0,1)$ such that for all $0 \le t' < t \le T$ it holds in law*

$$\sqrt{2\sigma}\int_{t'}^{t} \left(1 + \Phi'(Y_s^\varepsilon)\right) \mathrm{d}W_s = \sqrt{2\Sigma(t-t')}\,\xi_t + S(\varepsilon),$$

where for every $p > 0$ and $\kappa \in (0, \frac{1}{2})$ it holds

$$\left(\mathbb{E}^{\mu^\varepsilon} |S(\varepsilon)|^p\right)^{1/p} \le C(\varepsilon^{2\kappa} + \varepsilon^{\kappa}).$$

Proof. The proof is identical to the proof of [2, Proposition 5.7] and is therefore omitted here. □

We now need a decomposition similar to (2.3) for the process Z_t^ε. A first step is given by the following lemma.

Lemma 2.5. *The process $Z_t^\varepsilon := \mathcal{H}_\Delta(X_t^\varepsilon)$, where $\mathcal{H}_\Delta$ is defined in (2.1), admits for $\Delta \le t \le T$ the representation*

$$Z_t^\varepsilon = X_{t-\Delta}^\varepsilon - \frac{1}{\Delta}\int_{t-\Delta}^{t} (t-s)\left(\alpha V_0'(X_s^\varepsilon) + \frac{1}{\varepsilon} V_1'\left(Y_s^\varepsilon\right)\right) \mathrm{d}s + \frac{1}{\Delta}\int_{t-\Delta}^{t} \sqrt{2\sigma}(t-s)\,\mathrm{d}W_s,$$

where $(X_t^\varepsilon, Y_t^\varepsilon)$ is the solution of (2.2).

Proof. Let us for ease of notation denote $Z_t := Z_t^\varepsilon$, $X_t := X_t^\varepsilon$ and

$$f(X_t) := -\alpha V_0(X_t) - \frac{1}{\varepsilon} V_1\left(Y_t^\varepsilon\right).$$

Due to the definition of Z_t, we have

$$\Delta Z_t = \Delta X_{t-\Delta} + \int_{t-\Delta}^{t}\int_{t-\Delta}^{s} f(X_r)\,\mathrm{d}r\,\mathrm{d}s + \int_{t-\Delta}^{t}\int_{t-\Delta}^{s} \sqrt{2\sigma}\,\mathrm{d}W_r\,\mathrm{d}s, \tag{2.5}$$

where, exchanging the order of integration, we obtain for the deterministic integral

$$\begin{aligned}\int_{t-\Delta}^{t}\int_{t-\Delta}^{s} f(X_r)\,\mathrm{d}r\,\mathrm{d}s &= \int_{t-\Delta}^{t}\int_{r}^{t} f(X_r)\,\mathrm{d}s\,\mathrm{d}r \\ &= \int_{t-\Delta}^{t} (t-s) f(X_s)\,\mathrm{d}s.\end{aligned} \tag{2.6}$$

For the stochastic integral, we can write

$$\begin{aligned}\int_{t-\Delta}^{t}\int_{t-\Delta}^{s} \mathrm{d}W_r\,\mathrm{d}s &= \int_{t-\Delta}^{t} (W_s - W_{t-\Delta})\,\mathrm{d}s \\ &= \int_{t-\Delta}^{t} W_s\,\mathrm{d}s - \Delta W_{t-\Delta}.\end{aligned}$$

The formula $\mathrm{d}(tW_t) = t\,\mathrm{d}W_t + W_t\,\mathrm{d}t$ yields

$$\begin{aligned}\int_{t-\Delta}^{t} W_s\,\mathrm{d}s &= \left(tW_t - (t-\Delta)W_{t-\Delta}\right) - \int_{t-\Delta}^{t} s\,\mathrm{d}W_s \\ &= t(W_t - W_{t-\Delta}) - \int_{t-\Delta}^{t} s\,\mathrm{d}W_s + \Delta W_{t-\Delta} \\ &= \int_{t-\Delta}^{t} (t-s)\,\mathrm{d}W_s + \Delta W_{t-\Delta},\end{aligned}$$

which implies

$$\int_{t-\Delta}^{t}\int_{t-\Delta}^{s} \mathrm{d}W_r\,\mathrm{d}s = \int_{t-\Delta}^{t} (t-s)\,\mathrm{d}W_s. \tag{2.7}$$

Replacing (2.6) and (2.7) into (2.5) then gives the desired result. □

3 Point estimates from discrete data

Let us consider high-frequency data to be given by the discrete sequence $\mathbf{x}^\varepsilon = \{x_j^\varepsilon\}_{j=0}^N$ such that $x_j^\varepsilon = X_{j\varepsilon^q}^\varepsilon$, where X_t^ε is a realization of the solution of (1.1) and $q \geq 2$. Hence, we set $T = N\varepsilon^q$. Moreover, let us consider $\Delta \in \mathbb{N}$, $\Delta \geq 1$ and the discrete operator $H_\Delta \colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$, $H_\Delta \colon \mathbf{x}^\varepsilon \mapsto \langle \mathbf{x}^\varepsilon \rangle^\Delta$, defined by

$$\langle \mathbf{x}^\varepsilon \rangle_n^\Delta = \begin{cases} x_0^\varepsilon, & n = 0, \\ \frac{1}{n+1}\sum_{j=0}^n x_{n-j}^\varepsilon, & 1 \leq n < \Delta - 1, \\ \frac{1}{\Delta}\sum_{j=0}^{\Delta-1} x_{n-j}^\varepsilon, & \Delta - 1 \leq n \leq N. \end{cases}$$

In the following, we will always consider $n \geq \Delta - 1$. In this case, the maximum likelihood estimator $\widehat{A}_{N,\Delta}$ of the coefficient A of (1.2) based on the sequence $\langle \mathbf{x}^\varepsilon \rangle^\Delta$ is given by

$$\widehat{A}_{N,\Delta} = -\frac{\sum_{i=0}^{N-1} V_0'(\langle \mathbf{x}^\varepsilon \rangle_n^\Delta)\big(\langle \mathbf{x}^\varepsilon \rangle_{n+1}^\Delta - \langle \mathbf{x}^\varepsilon \rangle_n^\Delta\big)}{\sum_{i=0}^{N-1} \varepsilon^q V_0'(\langle \mathbf{x}^\varepsilon \rangle_n^\Delta)^2}.$$

We have

$$\langle \mathbf{x} \rangle_{n+1}^\Delta - \langle \mathbf{x} \rangle_n^\Delta = \frac{1}{\Delta}(x_{n+1}^\varepsilon - x_{n-\Delta+1}^\varepsilon),$$

and therefore (equivalent to equation (5.8) in [2]) by Itô formula on $\Phi(Y_s^\varepsilon)$

$$\begin{aligned} \langle \mathbf{x}^\varepsilon \rangle_{n+1}^\Delta - \langle \mathbf{x}^\varepsilon \rangle_n^\Delta = &-\frac{\alpha}{\Delta}\int_{(n-\Delta+1)\varepsilon^q}^{(n+1)\varepsilon^q} V_0'(X_s^\varepsilon)\big(1 + \Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}s \\ &+ \frac{\sqrt{2\sigma}}{\Delta}\int_{(n-\Delta+1)\varepsilon^q}^{(n+1)\varepsilon^q} \big(1 + \Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}W_s \\ &- \frac{\varepsilon}{\Delta}\big(\Phi'(Y_{(n+1)\varepsilon^q}^\varepsilon) - \Phi'(Y_{(n-\Delta+1)\varepsilon^q}^\varepsilon)\big). \end{aligned}$$

The properties of the maximum likelihood estimator obtained replacing $\mathbf{x}^\varepsilon$ with $\langle \mathbf{x}^\varepsilon \rangle^\Delta$ can be determined analysing the terms in the decomposition above.

Lemma 3.1 (Equivalent to [2, Lemma 6.1])**.** *Lemma text*

$$\mathbb{E}^{\mu^\varepsilon}\left|X_s^\varepsilon - \langle \mathbf{x}^\varepsilon \rangle_n^\Delta\right|^p \leq C(\varepsilon^{pq}\Delta^p + \varepsilon^{pq/2}\Delta^{p/2} + \varepsilon^p\Delta^{-p}),$$

for $s \in [(n - \Delta + 1)\varepsilon^q, (n+1)\varepsilon^q]$.

Proof. We replace the definition of $\langle \cdot \rangle^\Delta$ and apply the Hölder inequality to obtain

$$\begin{aligned} \mathbb{E}^{\mu^\varepsilon}\left|X_s^\varepsilon - \langle \mathbf{x} \rangle_n^\Delta\right|^p &= \Delta^{-p}\,\mathbb{E}^{\mu^\varepsilon}\left|\sum_{j=0}^{\Delta-1}\big(X_s^\varepsilon - x_{n-j}^\varepsilon\big)\right|^p \\ &\leq \Delta^{-1}\sum_{j=0}^{\Delta-1}\left|X_s^\varepsilon - x_{n-j}^\varepsilon\right|^p. \end{aligned}$$

Applying on each element of the sum [2, Lemma 6.1], we obtain

$$\mathbb{E}^{\mu^\varepsilon}\left|X_s^\varepsilon - \langle \mathbf{x} \rangle_n^\Delta\right|^p \leq C\Delta^{-1}\sum_{j=0}^{\Delta-1}\Big(\varepsilon^{pq}\Delta^p + \varepsilon^{pq/2}\Delta^{p/2} + \varepsilon^p\Delta^{-p}\Big),$$

which implies the desired result. □

Lemma 3.2 (Equivalent to [2, Proposition 5.8])**.** *Lemma text*

$$\frac{\alpha}{\Delta}\int_{(n-\Delta+1)\varepsilon^q}^{(n+1)\varepsilon^q} V_0'(X_s^\varepsilon)\big(1+\Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}s = \varepsilon^q A V_0'(\langle \mathbf{x}\rangle_n^\Delta) + R_2(\varepsilon,\Delta),$$

in law, where for a constant $C>0$ it holds

$$\left(\mathbb{E}^{\mu^\varepsilon}|R_2(\varepsilon,\Delta)|^p\right)^{1/p} \le C\left(\varepsilon^2\Delta^{-1}+\varepsilon^{q+1/2}\Delta^{-1/2}+\varepsilon^{q+1}+\varepsilon^{3q/2}\Delta^{1/2}+\varepsilon^{2q}\Delta\right).$$

Proof. We compute

$$\begin{aligned}
\mathbb{E}^{\mu^\varepsilon}|R_2(\varepsilon,\Delta)|^p &= \mathbb{E}^{\mu^\varepsilon}\left|\frac{\alpha}{\Delta}\int_{(n-\Delta+1)\varepsilon^q}^{(n+1)\varepsilon^q} V_0'(X_s^\varepsilon)\big(1+\Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}s - A\varepsilon^q V_0'(\langle \mathbf{x}^\varepsilon\rangle_n^\Delta)\right|^p\\
&= \mathbb{E}^{\mu^\varepsilon}\left|\frac{\alpha}{\Delta}\int_{(n-\Delta+1)\varepsilon^q}^{(n+1)\varepsilon^q} V_0'(\langle \mathbf{x}^\varepsilon\rangle_n^\Delta)\big(1+\Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}s - \frac{A}{\Delta}\int_{(n-\Delta+1)\varepsilon^q}^{(n+1)\varepsilon^q} V_0'(\langle \mathbf{x}^\varepsilon\rangle_n^\Delta))\,\mathrm{d}s\right.\\
&\quad\left. +\frac{\alpha}{\Delta}\int_{(n-\Delta+1)\varepsilon^q}^{(n+1)\varepsilon^q}\left(V_0'(X_s^\varepsilon)-V_0'(\langle \mathbf{x}^\varepsilon\rangle_n^\Delta)\right)\left(1+\Phi'(Y_s^\varepsilon)\right)\mathrm{d}s\right|^p\\
&\le C\Delta^{-p}\,\mathbb{E}^{\mu^\varepsilon}\left|V_0'(\langle \mathbf{x}^\varepsilon\rangle_n^\Delta)\int_{(n-\Delta+1)\varepsilon^q}^{(n+1)\varepsilon^q}\left(\alpha\big(1+\Phi'(Y_s^\varepsilon)\big)-A\right)\right|^p\\
&\quad + C\alpha^p\Delta^{-p}\,\mathbb{E}^{\mu^\varepsilon}\left|\int_{(n-\Delta+1)\varepsilon^q}^{(n+1)\varepsilon^q}\left(V_0'(X_s^\varepsilon)-V_0'(\langle \mathbf{x}^\varepsilon\rangle_n^\Delta)\right)\left(1+\Phi'(Y_s^\varepsilon)\right)\mathrm{d}s\right|^p\\
&=: I^1_{\varepsilon,\Delta}+I^2_{\varepsilon,\Delta},
\end{aligned}$$

where C only depends on p. Then, Hölder's inequality and Lemma 3.1 give

$$\begin{aligned}
I^2_{\varepsilon,\Delta} &\le C\Delta^{-1}\varepsilon^{q(p-1)}\int_{(n-\Delta+1)\varepsilon^q}^{(n+1)\varepsilon^q}\mathbb{E}^{\mu^\varepsilon}\left|X_s^\varepsilon-\langle \mathbf{x}^\varepsilon\rangle_n^\Delta\right|^p\,\mathrm{d}s\\
&\le C\Delta^{-1}\varepsilon^{q(p-1)}\int_{(n-\Delta+1)\varepsilon^q}^{(n+1)\varepsilon^q}\left(\varepsilon^{pq}\Delta^p+\varepsilon^{pq/2}\Delta^{p/2}+\varepsilon^p\Delta^{-p}\right)\mathrm{d}s\\
&\le C\left(\varepsilon^{2pq}\Delta^p+\varepsilon^{3pq/2}\Delta^{p/2}+\varepsilon^{p(q+1)}\Delta^{-p}\right),
\end{aligned}$$

which implies

$$\left(I^2_{\varepsilon,\Delta}\right)^{1/p} \le C\left(\varepsilon^{2q}\Delta+\varepsilon^{3q/2}\Delta^{1/2}+\varepsilon^{q+1}\Delta^{-1}\right).$$

Let us now consider $I^1_{\varepsilon,\Delta}$. Due to [2, Lemma 5.6], we have

$$\mathbb{E}^{\mu^\varepsilon}\left|\int_{(n-\Delta+1)\varepsilon^q}^{(n+1)\varepsilon^q}\left(\alpha\big(1+\Phi'(Y_s^\varepsilon)\big)-A\right)\right|^p \le C(\varepsilon^{2p}+\varepsilon^{p(q+1)}\Delta^p+\varepsilon^{p(1+q/2)}\Delta^{p/2}),$$

which, in light of equation (3.1) and Corollary 5.4 in [2], yields

$$\left(I^1_{\varepsilon,\Delta}\right)^{1/p} \le C(\Delta^{-1}\varepsilon^2+\varepsilon^{q+1}+\varepsilon^{q+1/2}\Delta^{-1/2}).$$

Hence, since $q\ge 2$ and $\varepsilon<1$,

$$\begin{aligned}
\mathbb{E}^{\mu^\varepsilon}|R_2(\varepsilon,\Delta)|^p &\le C\left((\varepsilon^2+\varepsilon^{q+1})\Delta^{-1}+\varepsilon^{q+1/2}\Delta^{-1/2}+\varepsilon^{q+1}+\varepsilon^{3q/2}\Delta^{1/2}+\varepsilon^{2q}\Delta\right)\\
&\le C\left(\varepsilon^2\Delta^{-1}+\varepsilon^{q+1/2}\Delta^{-1/2}+\varepsilon^{q+1}+\varepsilon^{3q/2}\Delta^{1/2}+\varepsilon^{2q}\Delta\right),
\end{aligned}$$

which concludes the proof. ∎

Theorem 3.1. *Equivalent to Theorem 2.1 in discrete case.*

Proof. Since

$$\langle \mathbf{x}^\varepsilon \rangle_{n+1}^{\Delta} - \langle \mathbf{x}^\varepsilon \rangle_n^{\Delta} = -\varepsilon^q A V'(\langle \mathbf{x}^\varepsilon \rangle_n^{\Delta}) + R_1 + R_2,$$

(determine R_1 and R_2 well), we have

$$\widehat{A}_{N,\Delta} = A + \dots$$

□

4 Bayesian inference

Consider

$$L_T^0(A) = \exp\left\{ -\int_0^T A V_0'(X_t^0) \,\mathrm{d}X_t^0 - \frac{1}{2}\int_0^T A^2 V_0'(X_t^0)^2 \,\mathrm{d}t \right\},$$

and, denoting $Z_t^\varepsilon \coloneqq \mathcal{H}_\Delta(X^\varepsilon)_t$, where $\mathcal{H}_\Delta$ is defined in (2.1)

$$L_T^\varepsilon(A) = \exp\left\{ -\int_0^T A V_0'(Z_t^\varepsilon) \,\mathrm{d}Z_t^\varepsilon - \frac{1}{2}\int_0^T A^2 V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t \right\}.$$

Let the prior be denoted by Λ, with density λ and the corresponding posteriors μ_T^0 and μ_T^ε. Denote $\ell_t^0 = \log L_T^0$, respectively ℓ_t^ε the log-likelihoods.

Define

$$d_{\mathrm{TV}}(\mu,\nu) \coloneqq \sup_{B\in\mathcal{B}} |\mu(B) - \nu(B)|\,.$$

Compute for $B \in \mathcal{B}$

$$\left|\mu_T^0(B) - \mu_T^\varepsilon(B)\right| = \left| \frac{C^\varepsilon \int_B L_T^0(A)\lambda(A)\,\mathrm{d}A - Z^0 \int_B L_T^\varepsilon(A)\lambda(A)\,\mathrm{d}A}{C^0 C^\varepsilon} \right|,$$

where

$$C^0 = \int_{\mathcal{A}} L_T^0(A)\lambda(A)\,\mathrm{d}A,$$

and C^ε defined respectively. Then

$$\left|\mu_T^0(B) - \mu_T^\varepsilon(B)\right| \le I_1 + I_2,$$

where

$$I_1 = \frac{1}{C^0}\int_B \left|L_T^0(A) - L_T^\varepsilon(A)\right| \lambda(A)\,\mathrm{d}A,$$
$$I_2 = \frac{\left|C^\varepsilon - C^0\right|}{C^0 C^\varepsilon}\mu_T^\varepsilon(B).$$

Consider first I_1. Since $|\exp(a) - \exp(b)| \le (\exp(a) + \exp(b))\,|a-b|$, we have

$$I_1 \le \frac{1}{C^0}\int_B \left(L_T^0(A) + L_T^\varepsilon(A)\right)\left|\ell_T^0(A) - \ell_T^\varepsilon(A)\right| \lambda(A)\,\mathrm{d}A.$$

Let us consider

$$\begin{aligned}\ell_T^0(A) - \ell_T^\varepsilon(A) = &-\int_0^T A V_0'(X_t^0)\,\mathrm{d}X_t^0 + \int_0^T A V_0'(Z_t^\varepsilon)\,\mathrm{d}Z_t^\varepsilon \\ &- \frac{1}{2}\int_0^T A^2\left(V_0'(X_t^0)^2 - V_0'(Z_t^\varepsilon)^2\right)\mathrm{d}t.\end{aligned}$$

Lemma 4.1. *Under assumptions add assumptions, it holds*

$$\left|\ell_T^0(A) - \ell_T^\varepsilon(A)\right| \to 0,$$

for $\varepsilon \to 0$.

Proof. The triangle inequality

$$\begin{aligned}\left|\ell_T^0(A) - \ell_T^\varepsilon(A)\right| &\le \left|\int_0^T AV_0'(X_t^0)\,\mathrm{d}X_t^0 - \int_0^T AV_0'(Z_t^\varepsilon)\,\mathrm{d}Z_t^\varepsilon\right| \\ &\quad + \left|\frac{1}{2}\int_0^T A^2\left(V_0'(X_t^0)^2 - V_0'(Z_t^\varepsilon)^2\right)\mathrm{d}t\right| =: I_1 + I_2\end{aligned}$$

Let us first consider I_1. From the definition of Z_t^ε, we divide

$$\begin{aligned}I_1 &\le \left|\int_0^\Delta AV_0'(X_t^0)\,\mathrm{d}X_t^0 - \int_0^\Delta AV_0'(Z_t^\varepsilon)\frac{X_t^\varepsilon - Z_t^\varepsilon}{t}\,\mathrm{d}t\right| \\ &\quad + \left|\int_\Delta^T AV_0'(X_t^0)\,\mathrm{d}X_t^0 - \int_\Delta^T AV_0'(Z_t^\varepsilon)\frac{X_t^\varepsilon - X_{t-\Delta}^\varepsilon}{\Delta}\,\mathrm{d}t\right| =: I_1^1 + I_1^2.\end{aligned}$$

Let us first consider I_1^2. Replacing (2.3) we can write in law

$$I_1^2 = \left|\int_\Delta^T AV_0'(X_t^0)\,\mathrm{d}X_t^0 - \int_\Delta^T AV_0'(Z_t^\varepsilon)\frac{J_t - A\Delta V_0'(Z_t^\varepsilon) + R(\varepsilon,\Delta)}{\Delta}\,\mathrm{d}t\right|,$$

where, due to Lemma 2.3, we have

$$\left(\mathbb{E}^{\mu^\varepsilon}\left|R(\varepsilon,\Delta)\right|^p\right)^{1/p} \le C(\varepsilon^2 + \Delta^{1/2} + \Delta^{3/2}).$$

Replacing $\mathrm{d}X_t^0$ with its definition given by (1.2), we can then split I_1^2 in three terms and apply the triangle inequality as

$$\begin{aligned}I_1^2 &\le A^2\left|\int_\Delta^T \left(V_0'(X_t^0)^2 - V_0'(Z_t^\varepsilon)^2\right)\mathrm{d}t\right| + A\left|\int_\Delta^T V_0'(X_t^0)\sqrt{2\Sigma}\,\mathrm{d}W_t - \frac{1}{\Delta}\int_\Delta^T V_0'(Z_t^\varepsilon)J_t\,\mathrm{d}t\right| \\ &\quad + A\left|\int_\Delta^T V_0'(Z_t^\varepsilon)\frac{R(\varepsilon,\Delta)}{\Delta}\,\mathrm{d}t\right| =: R_1 + R_2 + R_3.\end{aligned}$$

□

5 Numerical experiments

References

[1] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, North-Holland Publishing Co., Amsterdam, 1978.

[2] G. A. Pavliotis and A. M. Stuart, *Parameter estimation for multiscale diffusions*, J. Stat. Phys., 127 (2007), pp. 741–781.