Optimization-based coupling method for multiscale problems

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Goal

Assume that in some region ω of a domain Ω homogenization methods fail and fine scale solver must be used instead. A coupling between the fine and coarse solvers should be done.

Question. What boundary data should be used on the interface?

Classical global-local. Use homogenized solution u^0 [2].

Drawbacks. expensive to compute u^0; depends on the quality of u^0.

Goal. Find better boundary data.

Multiscale elliptic problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and consider: find $u \in H_0^1(\Omega)$ such that

$$-\text{div}\,(a\nabla u) = f, \text{ in } \Omega,$$
$$+B.C., \text{ on } \Gamma,$$

with highly heterogeneous uniformly bounded tensor a or source term f.

Idea. Introduce an overlap and solve a minimization problem with constraints.

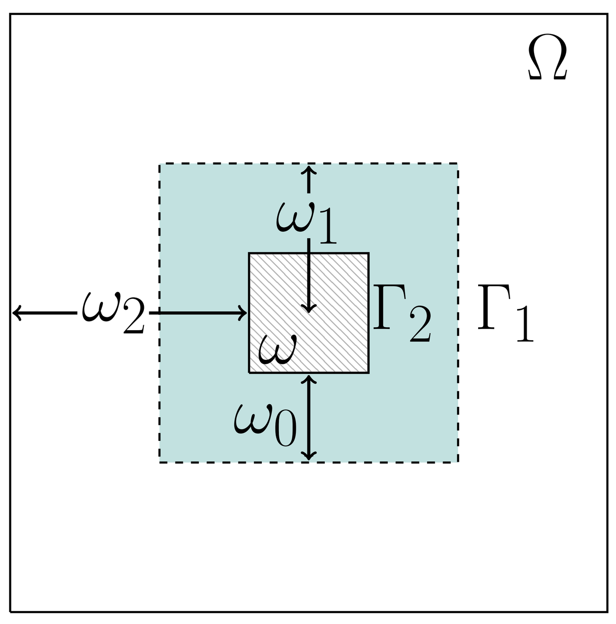


Figure 1: Domain decomposition

Optimization-Based coupling method

Optimization-based method [1]. Find $(u_1, u_2) \in H^1(\omega_1) \times H^1_\Gamma(\omega_2)$

$$\min \frac{1}{2}\|u_1 - u_2\|^2_{L^2(\omega_0)} \text{ such that } \begin{cases} -\text{div}\,(a\nabla u_1) = f \quad \text{in } \omega_1,\ u_1 = \theta_1 \text{ on } \Gamma_1, \\ -\text{div}\,(a^0\nabla u_2) = f^0 \text{ in } \omega_2,\ u_2 = \theta_2 \text{ on } \Gamma_2. \end{cases}$$

Introducing Lagrange multipliers for each constraints, (u_1, u_2) is given as a critical point of a saddle point problem.

FE Optimization Based Coupling

Let $V^p(\omega_1, \mathcal{T}_{\tilde{h}})$, $V^p_\Gamma(\omega_2, \mathcal{T}_H)$, and $S^q(K_{\delta_j}, \mathcal{T}_h)$ be FE spaces with $h, \tilde{h} < \varepsilon$, and $H >> h$.

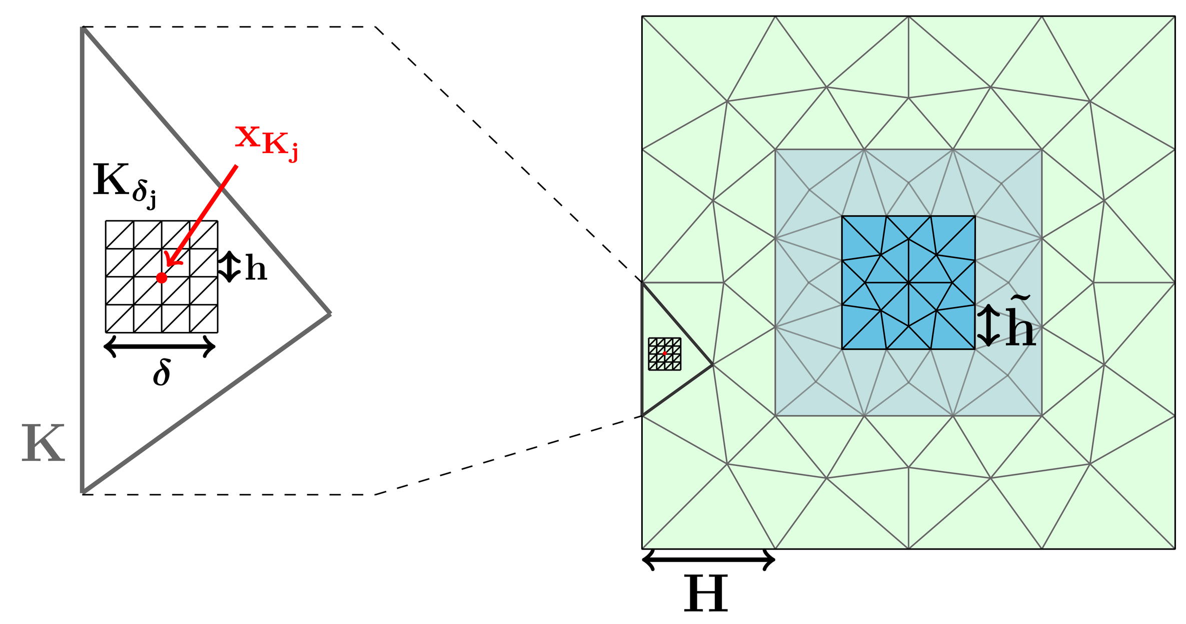


Figure 2: Mesh over Ω

Fine scale solver Find $u_{1,\tilde{h}} = u_{1,0,\tilde{h}} + v_{1,\tilde{h}}$, with $u_{1,0,\tilde{h}} \in V^p_0(\omega_1, \mathcal{T}_{\tilde{h}})$, such that

$$B_1(u_{1,0,\tilde{h}}, w_{\tilde{h}}) := \int_{\omega_1} a\nabla u_{1,0,\tilde{h}} \nabla w_{\tilde{h}} \mathrm{d}x = \int_{\omega_1} f w_{\tilde{h}} \mathrm{d}x, \forall w_{\tilde{h}} \in V^p_0(\omega_1, \mathcal{T}_{\tilde{h}}).$$

FE-HMM solver Find $u_{2,H} = u_{2,0,H} + v_{2,H}$, with $u_{2,0,H} \in V^p_0(\omega_2, \mathcal{T}_H)$ such that $\forall w_H \in V^p_0(\omega_2, \mathcal{T}_H)$

$$B_2(u_{2,0,H}, w_H) := \sum_{K\in\mathcal{T}_H} \sum_{j=1}^{J} \frac{w_{j,K}}{|K_{\delta_j}|} \int_{K_{\delta_j}} a\nabla u_j^h \nabla w_j^h \mathrm{d}x = \int_{\omega_2} f^0 w_H \mathrm{d}x,$$

where u_j^h, w_j^h are solutions of a micro problem involving a on K_{δ_j}. The solution $v_{1,\tilde{h}}$ and $v_{2,H}$ are the critical point of

$$\mathcal{L}(v_{1,\tilde{h}}, \lambda_{1,\tilde{h}}, v_{2,H}, \lambda_{2,H}) = \frac{1}{2}\|(u_{1,0,\tilde{h}} + v_{1,\tilde{h}}) - (u_{2,0,H} + v_{2,H})\|^2_{L^2(\omega_0)} - B_1(v_{1,\tilde{h}}, \lambda_{1,\tilde{h}}) - B_2(v_{2,H}, \lambda_{2,H}).$$

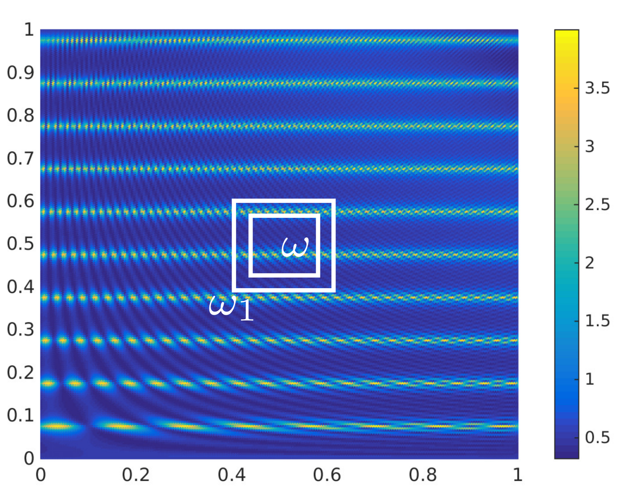
A Priori Error Estimates

Let $u_{\tilde{h},H}$ be the numerical coupling solution. It holds

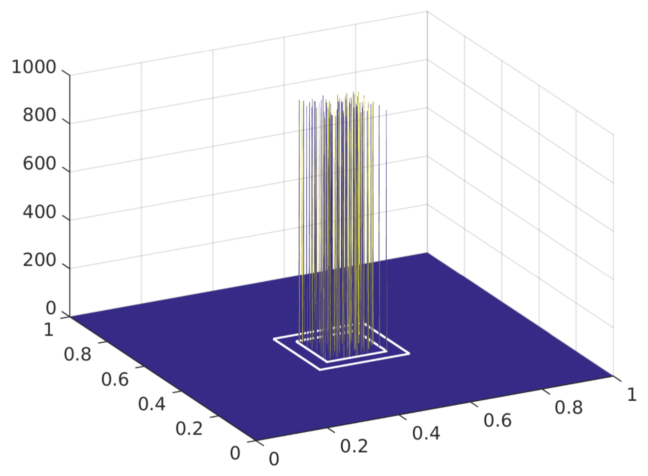
$$\|u - u_{\tilde{h},H}\|_{H^1(\omega)} \leqslant C\left(\tilde{h}|u|_{2,\omega_1} + \frac{C}{\tau}(\varepsilon + \text{err}_{HMM,L^2})\right),$$

where $\tau = \text{dist}(\Gamma_1, \Gamma_2)$. The ε term comes from the bound $\|u - u^0\|_{L^2(\omega_0)}$ and $\text{err}_{HMM,L^2} \leqslant C(H^2 + (\frac{h}{\varepsilon})^2 + \text{err}_{mod})$.

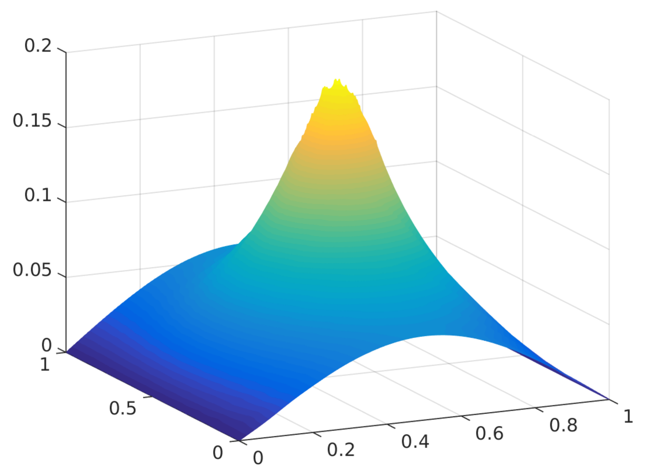
Numerical Experiments



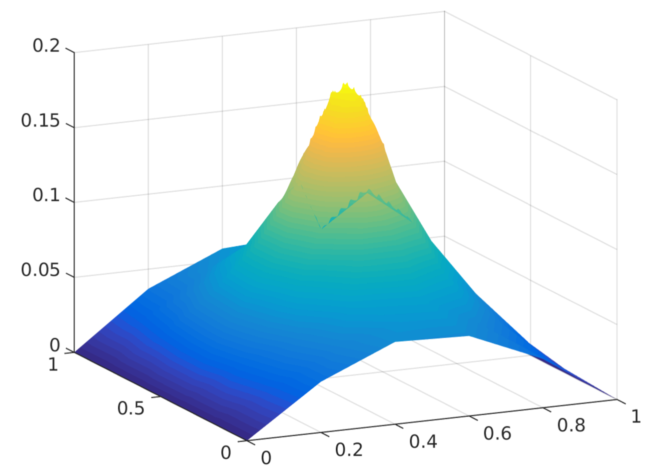
Tensor a over Ω



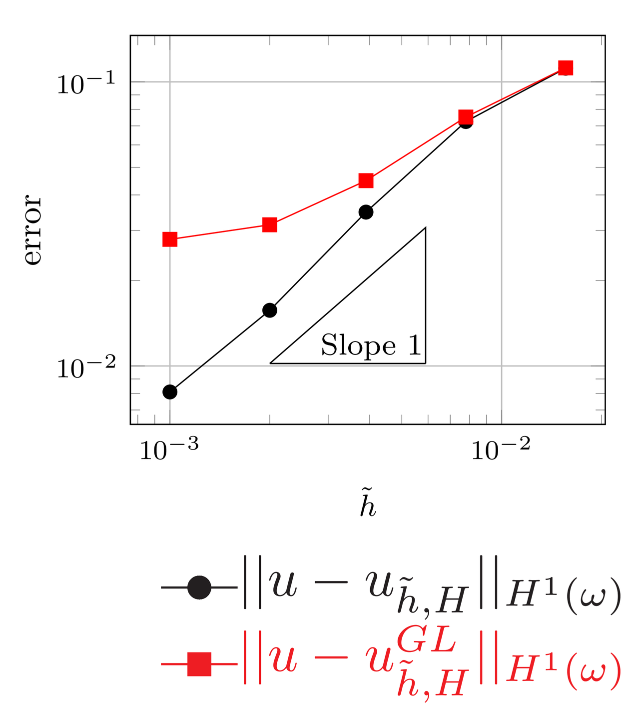
Source f over Ω



Reference solution



Numerical coupling solution



References

References

[1] Assyr Abdulle and Orane Jecker. An optimization-based heterogeneous to homogeneous coupling method. to appear in Communications in Math Sciences, 2015.

[2] J. Tinsley Oden and Kumar S. Vemaganti. Estimation of local modeling error and goal-oriented adaptive modeling of heterogeneous materials. I. Error estimates and adaptive algorithms. *J. Comput. Phys.*, 164(1):22–47, 2000.