Caltech notes

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1 Introduction

Let $\varepsilon > 0$ and let us consider the one-dimensional multiscale stochastic differential equation (SDE)

$$\mathrm{d}X_t^\varepsilon = -\alpha V_0'(X_t^\varepsilon)\,\mathrm{d}t - \frac{1}{\varepsilon} V_1'\left(\frac{X_t^\varepsilon}{\varepsilon}\right) + \sqrt{2\sigma}\,\mathrm{d}W_t, \tag{1.1}$$

where the drift coefficient α and the diffusion coefficient σ are positive real parameters, possibly unknown, and W_t is a standard one-dimensional Brownian motion. The functions $V_0, V_1 \colon \mathbb{R} \to \mathbb{R}$ are slow and fast potentials driving the dynamics of the solution X_t^ε. In particular, we assume V_1 to be smooth and periodic of period L. Theory of homogenization [1] guarantees the existence of an SDE of the form

$$\mathrm{d}X_t^0 = -AV_0'(X)\,\mathrm{d}t + \sqrt{2\Sigma}\,\mathrm{d}W_t, \tag{1.2}$$

where the fast dynamics have been eliminated, such that $X_t^\varepsilon \to X_t^0$ in law as random variables with values in $\mathcal{C}^0((0,T))$. The drift and diffusion coefficients of the homogenized dynamics A and Σ are given by $A = K\alpha$ and $\Sigma = K\sigma$, where

$$K = \int_0^L (1 + \Phi'(y))^2\,\mu(\mathrm{d}y),$$

with

$$\mu(\mathrm{d}y) = \frac{1}{Z} \exp\left(-\frac{V_1'(y)}{\sigma}\right)\,\mathrm{d}y, \quad Z = \int_0^L \exp\left(-\frac{V_1'(y)}{\sigma}\right)\,\mathrm{d}y,$$

and Φ is the solution of the elliptic partial differential equation

$$-V'(y)\Phi'(y) + \sigma\Phi''(y) = V''(y), \quad 0 \leq y \leq L,$$

endowed with periodic boundary coefficients.

In order to estimate the drift coefficient, one considers the likelihood function

$$L_T(X_t) = \exp\left\{\int_0^T -AV_0'(X_t)\,\mathrm{d}X_t - \frac{1}{2}\int_0^T A^2 V_0'(X_t)^2\,\mathrm{d}t\right\},$$

whose logarithm $\ell_T(X_t) = \log L_T(X_t)$ can be maximised thus giving the estimator

$$\widehat{A} = -\frac{\int_0^T V_0'(X_t)\,\mathrm{d}X_t}{\int_0^T V_0'(X_t)^2\,\mathrm{d}t}. \tag{1.3}$$

The diffusion coefficient can be computed as the quadratic variation of the path, i.e., given a sequence of partitions $\mathcal{P}_h = \{t_k\}_{k=0}^{N_h}$, of the interval $[0,T]$, where $h \coloneqq \sup_k(t_k - t_{k-1})$, we have

$$\Sigma = \frac{1}{2T} \lim_{h\to 0} \sum_{k=1}^{N_h} (X_{t_k}^0 - X_{t_{k-1}}^0)^2, \tag{1.4}$$

in probability and for all $T > 0$.

In a Bayesian setting, we can fix a prior Λ with density λ and the posterior is then given by

$$\mu_T(B) = \frac{\int_B L_T(A)\lambda(A)\,\mathrm{d}A}{\int_{\mathcal{A}} L_T(A)\lambda(A)\,\mathrm{d}A}.$$

2 Point estimates from continuous data

In this section, we study the convergence with respect to the parameter ε of point estimates of the drift and the diffusion coefficients when the estimator is computed employing continuous data coming from the multiscale model.

2.1 Drift coefficient

Let $X^\varepsilon := (X_t^\varepsilon, 0 \le t \le T)$ be the solution of (1.1) and define $\mathcal{H}_\Delta(X^\varepsilon)$ as

$$\mathcal{H}_\Delta(X^\varepsilon)_t := \begin{cases} X_0, & t = 0, \\ \frac{1}{t}\int_0^t X_s \,\mathrm{d}s, & 0 < t < \Delta, \\ \frac{1}{\Delta}\int_{t-\Delta}^t X_s \,\mathrm{d}s, & \Delta \le t \le T, \end{cases} \tag{2.1}$$

with $\Delta > 0$. Let us denote for ease of notation, $Z_t^\varepsilon := \mathcal{H}_\Delta(X^\varepsilon)_t$. The maximum likelihood estimator of the drift coefficient is then

$$\widehat{A}_{T,\Delta}(Z_t^\varepsilon) = -\frac{\int_0^T V_0'(Z_t^\varepsilon)\,\mathrm{d}Z_t^\varepsilon}{\int_0^T V_0'(Z_t^\varepsilon)^2\,\mathrm{d}t}.$$

Let us remark that for $0 < t < \Delta$,

$$\mathrm{d}(tZ_t^\varepsilon) = X_t\,\mathrm{d}t,$$

which implies

$$\mathrm{d}Z_t^\varepsilon = \frac{1}{t}(X_t^\varepsilon - Z_t^\varepsilon)\,\mathrm{d}t.$$

For $\Delta \le t \le T$, instead

$$\mathrm{d}Z_t^\varepsilon = \frac{1}{\Delta}(X_t^\varepsilon - X_{t-\Delta}^\varepsilon)\,\mathrm{d}t.$$

We rewrite the estimator as

$$\widehat{A}_{T,\Delta}(Z_t^\varepsilon) = -\frac{\int_0^\Delta V_0'(Z_t^\varepsilon)\frac{1}{t}(X_t^\varepsilon - Z_t^\varepsilon)\,\mathrm{d}t}{\int_0^T V_0'(Z_t^\varepsilon)^2\,\mathrm{d}t} - \frac{\int_\Delta^T V_0'(Z_t^\varepsilon)(X_t^\varepsilon - X_{t-\Delta}^\varepsilon)\,\mathrm{d}t}{\Delta\int_0^T V_0'(Z_t^\varepsilon)^2\,\mathrm{d}t}.$$

The goal of this section is proving the following result.

Theorem 2.1. *Under assumption add assumptions, if there exists $\zeta \in (0,1)$ such that $\Delta = \varepsilon^\zeta$ and $\gamma > \zeta$ such that $T = \varepsilon^{-\gamma}$, it holds*

$$\lim_{\varepsilon \to 0} \widehat{A}_{T,\Delta}(Z_t^\varepsilon) = A, \quad \textit{in law.}$$

It is useful in the following to rewrite (1.1) as a system of two coupled SDEs. In particular, introducing the variable $Y_t^\varepsilon := X_t^\varepsilon/\varepsilon$, one has

$$\begin{aligned} \mathrm{d}X_t^\varepsilon &= -\alpha V_0'(X_t^\varepsilon)\,\mathrm{d}t - \frac{1}{\varepsilon}V_1'(Y_t^\varepsilon) + \sqrt{2\sigma}\,\mathrm{d}W_t, \\ \mathrm{d}Y_t^\varepsilon &= -\frac{\alpha}{\varepsilon}V_0'(X_t^\varepsilon)\,\mathrm{d}t - \frac{1}{\varepsilon^2}V_1'(Y_t^\varepsilon) + \sqrt{\frac{2\sigma}{\varepsilon^2}}\,\mathrm{d}W_t. \end{aligned}$$

The analysis necessary to prove Theorem 2.1 is based on the expansion

$$\begin{aligned} X_t^\varepsilon - X_{t-\Delta}^\varepsilon = &- \alpha \int_{t-\Delta}^t V_0'(X_s^\varepsilon)\big(1 + \Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}s \\ &+ \sqrt{2\sigma}\int_{t-\Delta}^t \big(1 + \Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}W_s \\ &- \varepsilon\big(\Phi(Y_t^\varepsilon) - \Phi(Y_{t-\Delta}^\varepsilon)\big), \end{aligned} \tag{2.2}$$

for $t \ge \Delta$ (see [2, Equation (5.8)]). The following lemma ensures that the process Z_t^ε has bounded moments.

Lemma 2.2. *The process Z_t^ε has bounded moments of all order, i.e., for all $p \geq 1$ and $t \geq 0$ it holds*

$$\mathbb{E}^{\mu^\varepsilon} |Z_t^\varepsilon|^p \leq C,$$

for $C > 0$ a constant uniform in $\varepsilon \to 0$.

Proof. The process X_t^ε has bounded moments (see [2, Corollary 5.4]), which implies the desired result with an application of the Hölder inequality. In fact, for $0 < t < \Delta$,

$$\begin{aligned}\mathbb{E}^{\mu^\varepsilon} |Z_t^\varepsilon|^p &\leq \frac{t^{p-1}}{t^p} \int_0^t \mathbb{E}^{\mu^\varepsilon} |X_s^\varepsilon|^p \,\mathrm{d}s \\ &\leq t^{-1} \int_0^t C \,\mathrm{d}s = C.\end{aligned}$$

For $\Delta \leq t \leq T$ the procedure is analogue. $\square$

In the following lemma the difference between the processes X_t^ε and Z_t^ε is bounded.

Lemma 2.3. *Under assumptions add assumptions*

$$\mathbb{E}^{\mu^\varepsilon} |X_t^\varepsilon - Z_t^\varepsilon|^p \leq C(\Delta^p + \Delta^{p/2} + \varepsilon^p),$$

where $C > 0$ is a constant independent of Δ and ε.

Proof. By definition of Z_t^ε for $\Delta \leq t \leq T$ and applying Hölder's inequality we have

$$\begin{aligned}\mathbb{E}^{\mu^\varepsilon} |X_t^\varepsilon - Z_t^\varepsilon|^p &= \Delta^{-p} \,\mathbb{E}^{\mu^\varepsilon} \left| \int_{t-\Delta}^t (X_t^\varepsilon - X_s^\varepsilon) \,\mathrm{d}s \right|^p \\ &\leq \Delta^{-1} \int_{t-\Delta}^t \mathbb{E}^{\mu^\varepsilon} |X_t^\varepsilon - X_s^\varepsilon|^p \,\mathrm{d}s\end{aligned}$$

We can now apply [2, Lemma 6.1] to the integrand to obtain

$$\mathbb{E}^{\mu^\varepsilon} |X_t^\varepsilon - Z_t^\varepsilon|^p \leq C\Delta^{-1} \int_{t-\Delta}^t (\Delta^p + \Delta^{p/2} + \varepsilon^p) \,\mathrm{d}s,$$

which implies the desired result. The case $0 < t \leq T$ can be proved analogously. $\square$

Lemma 2.4 (See [2, Proposition 5.8])**.** *Under assumptions add assumptions, it holds in law*

$$\alpha \int_{t-\Delta}^t V_0'(X_s^\varepsilon)\big(1 + \Phi'(Y_s^\varepsilon)\big) \,\mathrm{d}s = A\Delta V_0'(Z_t^\varepsilon) + R(\varepsilon, \Delta),$$

where for every $p > 0$ and if Δ and ε are sufficiently small, then

$$\left(\mathbb{E}^{\mu^\varepsilon} |R(\varepsilon, \Delta)|^p\right)^{1/p} \leq C(\varepsilon^2 + \Delta^{1/2}\varepsilon + \Delta^{3/2}),$$

where $C > 0$ is independent of ε and Δ.

Proof. Let us denote $\Psi(t) \coloneqq 1 + \Phi'(Y_t^\varepsilon)$. Then

$$\begin{aligned}\mathbb{E}^{\mu^\varepsilon} |R(\varepsilon, \Delta)|^p &= \mathbb{E}^{\mu^\varepsilon} \left| \int_{t-\Delta}^t \alpha V_0'(X_s^\varepsilon)\Psi(s) \,\mathrm{d}s - \Delta A V_0'(Z_t^\varepsilon) \right|^p \\ &\leq C\,\mathbb{E}^{\mu^\varepsilon} \left| V_0'(Z_t^\varepsilon) \int_{t-\Delta}^t (\alpha\Psi(s) - A) \,\mathrm{d}s \right|^p \\ &\quad + C\,\mathbb{E}^{\mu^\varepsilon} \left| \int_{t-\Delta}^t \alpha \left(V_0'(X_t^\varepsilon) - V_0'(Z_t^\varepsilon)\right) \Psi(s) \,\mathrm{d}s \right|^p.\end{aligned}$$

The result is then obtained following the proof of [2, Proposition 5.8] and replacing [2, Lemma 6.1] with Lemma 2.3, and [2, Corollary 4.1] with Lemma 2.2. $\square$

We can now prove Theorem 2.1.

Proof of Theorem 2.1. Consider the decomposition (2.2). Denoting

$$J_t := \sqrt{2\sigma} \int_{t-\Delta}^{t} \left(1 + \Phi'(Y_s^\varepsilon)\right) \mathrm{d}W_s,$$

we have due to Lemma 2.4 the equality in law

$$X_t^\varepsilon - X_{t-\Delta}^\varepsilon = -A\Delta V'(Z_t^\varepsilon) + J_t + \widehat{R}(\varepsilon, \Delta),$$

where, since $\zeta \in (0,1)$, we have

$$\left(\mathbb{E}^{\mu^\varepsilon} \left|\widehat{R}(\varepsilon, \Delta)\right|^p\right)^{1/p} \leq C(\varepsilon + \varepsilon^{3\zeta/2})$$

Therefore, we have that the estimator satisfies

$$\begin{aligned} \widehat{A}_{T,\Delta}(Z_t^\varepsilon) &= A - A\frac{\int_0^\Delta V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t}{\int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t} - \frac{\int_0^\Delta V_0'(Z_t^\varepsilon)\frac{1}{t}(X_t^\varepsilon - Z_t^\varepsilon) \,\mathrm{d}t}{\int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t} \\ &\quad - \frac{\int_\Delta^T V_0'(Z_t^\varepsilon) J_t \,\mathrm{d}t}{\Delta \int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t} - \frac{\widehat{R}(\varepsilon,\Delta) \int_\Delta^T V_0'(Z_t^\varepsilon) \,\mathrm{d}t}{\Delta \int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t} \\ &=: A - I_1 - I_2 - I_3 - I_4, \end{aligned} \tag{2.3}$$

in law. Let us analyse the terms I_i, $i = 1, \ldots, 4$ separately. Let us consider I_1 and multiply both the numerator and the denominator by $1/T$. Due to assumption add assumption and Lemma 2.2, we have

$$\frac{A}{T} \mathbb{E}^{\mu^\varepsilon} \left| \int_0^\Delta V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t \right| \leq C\varepsilon^{\gamma+\zeta},$$

for a constant $C > 0$ independent of Δ and ε. Hence the numerator vanishes in L^1 and thus in law for $\varepsilon \to 0$. We split the denominator as

$$\frac{1}{T}\int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t = \frac{1}{T}\int_0^T V_0'(X_t^\varepsilon)^2 \,\mathrm{d}t + \frac{1}{T}\int_0^T \left(V_0'(Z_t^\varepsilon)^2 - V_0'(X_t^\varepsilon)^2\right) \,\mathrm{d}t$$

For the first term, we have by the ergodic theorem

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T V_0'(X_t^\varepsilon)^2 \,\mathrm{d}t = \mathbb{E}^{\mu^\varepsilon} \left|V_0'\right|^2, \quad \text{a.s.}$$

For the second term, we have applying Cauchy–Schwarz's inequality and due to assumption add assumption and Lemma 2.3

$$\begin{aligned} \frac{1}{T}\mathbb{E}^{\mu^\varepsilon} \left| \int_0^T \left(V_0'(Z_t^\varepsilon)^2 - V_0'(X_t^\varepsilon)^2\right) \,\mathrm{d}t \right| &\leq \frac{C}{T}\int_0^T \left(\mathbb{E}^{\mu^\varepsilon} \left|V_0'(Z_t^\varepsilon) - V_0'(X_t^\varepsilon)\right|^2\right)^{1/2} \,\mathrm{d}t \\ &\leq C\left(\Delta + \Delta^{1/2} + \varepsilon\right), \end{aligned}$$

which implies that the denominator tends to a finite value in probability for $\varepsilon \to 0$. Therefore, by Slutsky's theorem,

$$\lim_{\varepsilon\to 0} I_1 = 0, \quad \text{in law.}$$

Let us now consider I_2 and multiply numerator and denominator by $1/T$. The denominator is the same as I_1, and therefore does not need to be treated further. The numerator can be bounded in L^1 as

$$\frac{1}{T}\mathbb{E}^{\mu^\varepsilon} \left| \int_0^\Delta V_0'(Z_t^\varepsilon)\frac{1}{t}(X_t^\varepsilon - Z_t^\varepsilon) \,\mathrm{d}t \right| \leq \frac{C}{\Delta T}\int_0^\Delta \frac{\Delta}{t} \mathbb{E}^{\mu^\varepsilon} \left|X_t^\varepsilon - Z_t^\varepsilon\right| \,\mathrm{d}t,$$

which, since $Z_0^\varepsilon = X_0^\varepsilon$, vanishes for $\varepsilon \to 0$. Hence, an application of Slutsky's theorem yields

$$\lim_{\varepsilon \to 0} I_2 = 0, \quad \text{in law}.$$

We consider now I_3, which can be rewritten as

$$\begin{aligned} I_3 &= \frac{1}{\sqrt{T\Delta}} \frac{\frac{1}{\sqrt{T\Delta}} \int_\Delta^T V_0'(Z_t^\varepsilon) J_t \,\mathrm{d}t}{\frac{1}{T} \int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t} \\ &= \varepsilon^{(\gamma-\zeta)/2} \frac{\frac{1}{\sqrt{T\Delta}} \int_\Delta^T V_0'(Z_t^\varepsilon) J_t \,\mathrm{d}t}{\frac{1}{T} \int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t} \end{aligned}$$

Let us remark that J_t is a martingale and that by Itô isometry

$$\mathbb{E}^{\mu^\varepsilon} \left|J_\Delta\right|^2 = 2\Sigma\Delta,$$

Therefore, we can apply the central limit theorem for martingales to the numerator and obtain the equality in law

$$\begin{aligned} \lim_{T \to \infty} \frac{1}{\sqrt{T\Delta}} \int_\Delta^T V_0'(Z_t^\varepsilon) J_t \,\mathrm{d}t &= \frac{1}{\sqrt{\Delta}} \mathcal{N}\left(0, \mathbb{E}^{\mu^\varepsilon} \left(\left|V_0'(X_0^\varepsilon)\right|^2 \left|J_\Delta\right|^2 \right)\right) \\ &= C\mathcal{N}(0,1). \end{aligned}$$

The denominator is the same as in I_2 and I_3 and tends in probability to a finite value. Hence, since by hypothesis $\gamma > \zeta$, we have

$$\lim_{\varepsilon \to 0} I_3 = 0, \quad \text{in law}.$$

For the last term I_4, we have

$$I_4 = \frac{\varepsilon^{\gamma-\zeta} \widehat{R}(\varepsilon, \Delta) \int_\Delta^T V_0'(Z_t^\varepsilon) \,\mathrm{d}t}{\frac{1}{T} \int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t}.$$

For the numerator, we have by the Cauchy–Schwarz inequality and due to Lemma 2.4

$$\begin{aligned} \varepsilon^{\gamma-\zeta} \, \mathbb{E}^{\mu^\varepsilon} \left| \widehat{R}(\varepsilon, \Delta) \int_\Delta^T V_0'(Z_t^\varepsilon) \,\mathrm{d}t \right| &\leq \varepsilon^{\gamma-\zeta} \left(\mathbb{E}^{\mu^\varepsilon} \left| \widehat{R}(\varepsilon, \Delta) \right|^2 \right)^{1/2} \left(\mathbb{E}^{\mu^\varepsilon} \left| \int_\Delta^T V_0'(Z_t^\varepsilon) \,\mathrm{d}t \right|^2 \right)^{1/2} \\ &\leq C\varepsilon^{\gamma-\zeta} (\varepsilon + \varepsilon^{3\zeta/2}) \varepsilon^{-\gamma} \\ &\leq C \left(\varepsilon^{1-\zeta} + \varepsilon^{\zeta/2} \right) \end{aligned}$$

which implies that, since the denominator is the same as before,

$$\lim_{\varepsilon \to 0} I_4 = 0, \quad \text{in law}.$$

The decomposition (2.3), together with the limits of I_i for $i = 1, \dots, 4$, prove the desired result. □

2.2 Diffusion coefficient

We now consider the same transformation of the data, i.e., we employ $Z_t^\varepsilon = \mathcal{H}_\Delta(X^\varepsilon)_t$ as defined in (2.1), to estimate the diffusion coefficient Σ of the homogenized model. In particular, we consider the estimator

$$\widehat{\Sigma}_{\Delta,T} = \frac{1}{2T} \lim_{h \to 0} \sum_{k=1}^{N_h} (Z_{t_k}^\varepsilon - Z_{t_{k-1}}^\varepsilon)^2, \tag{2.4}$$

where the limit has to be intended in probability and with respect to a series of refinements of partitions $\mathcal{P}_h = \{t_k\}$ of the interval $[0, T]$. Let us recall that if instead of Z_t^ε one employs a path from the homogenized model X_t^0, then formula (1.4) gives the exact value of Σ for any $T > 0$.

Let us introduce a theoretical result which will play the role of Lemma 2.4 in this framework.

Lemma 2.5 (See [2, Proposition 5.7])**.** *Under assumptions add assumptions, there exist a continuous standard Gaussian process $(\xi_t, \Delta \le t \le T)$ such that for $\Delta \le s \le t \le T$*

$$\mathbb{E}\left(\xi_t\, \xi_s\right) = \begin{cases} 0, & t-s \ge \Delta, \\ 1 - \frac{t-s}{\Delta}, & t-s < \Delta, \end{cases} \tag{2.5}$$

and such that for all $\Delta \le t \le T$ it holds in law

$$\sqrt{2\sigma} \int_{t-\Delta}^{t} \left(1 + \Phi'(Y_s^\varepsilon)\right) \mathrm{d}W_s = \sqrt{2\Sigma\Delta}\, \xi_t + S(\varepsilon),$$

where for every $p > 0$ and $\kappa \in (0, 1/2)$ it holds

$$\left(\mathbb{E}^{\mu^\varepsilon} \left|S(\varepsilon)\right|^p\right)^{1/p} \le C(\varepsilon^{2\kappa} + \varepsilon^{\kappa}).$$

Proof. The proof is identical to the proof of [2, Proposition 5.7] and is therefore omitted here. The process ξ_t is defined as

$$\xi_t = \frac{\widehat{W}_{2\Sigma t} - \widehat{W}_{2\Sigma(t-\Delta)}}{\sqrt{2\Sigma\Delta}},$$

where $\widehat{W}_t$ is a standard Brownian motion, and its covariance function (2.5) can be trivially derived from the basic properties of standard Brownian motion. $\square$

Let us now recall that the differential of the process Z_t^ε can be expressed as

$$\mathrm{d}Z_t^\varepsilon = \frac{X_t^\varepsilon - X_{t-\Delta}^\varepsilon}{\Delta}\, \mathrm{d}t,$$

for $\Delta \le t \le T$. Therefore, for any choice $\Delta \le s < t \le T$, we have

$$Z_t^\varepsilon - Z_s^\varepsilon = \frac{1}{\Delta} \int_s^t \left(X_r^\varepsilon - X_{r-\Delta}^\varepsilon\right) \mathrm{d}r. \tag{2.6}$$

We can now prove the main result.

Theorem 2.6. *Under the assumptions of Lemma 2.5, if $T = \mathcal{O}(1)$ with respect to ε and $\Delta = \varepsilon^\zeta$ for $\zeta \in (0, 1)$, then*

$$\lim_{\varepsilon \to 0} \widehat{\Sigma}_{\Delta,T} = \Sigma, \quad \textit{in law},$$

for $\widehat{\Sigma}_{\Delta,T}$ defined in (2.4).

Proof. Replacing (2.2) into (2.6) and considering Lemma 2.4 and Lemma 2.5, we have for $\Delta \le t_{k-1} < t_k \le T$ the equality in law

$$Z_{t_k}^\varepsilon - Z_{t_{k-1}}^\varepsilon = \frac{1}{\Delta} \int_{t_{k-1}}^{t_k} \left(\sqrt{2\Sigma\Delta}\xi_s + R(\varepsilon, \Delta)\right) \mathrm{d}s,$$

where the remainder $R(\varepsilon, \Delta)$ satisfies for any $\kappa \in (0, 1/2)$ and $p > 0$

$$\left(\mathbb{E}^{\mu^\varepsilon} \left|R(\varepsilon, \Delta)\right|^p\right)^{1/p} \le C\left(\Delta + \varepsilon^{\kappa}\right).$$

Let us consider the partition $\mathcal{P}_\Delta = \{t_k = k\Delta\}_{k=0}^{N_\Delta}$, with $T = \Delta N_\Delta$. Since we are interested in the limit $\varepsilon \to 0$ and $\Delta = \varepsilon^\zeta$, we can rewrite (2.4) as

$$\begin{aligned}
\widehat{\Sigma}_{\Delta,T} &= \lim_{\Delta\to 0} \frac{1}{2T\Delta^2} \sum_{k=0}^{N_\Delta-1} \left(\sqrt{2\Sigma\Delta} \int_{t_{k-1}}^{t_k} \xi_s \, \mathrm{d}s + \Delta R(\varepsilon,\Delta) \right)^2 \\
&= \lim_{\Delta\to 0} \left\{ \frac{\Sigma}{N_\Delta \Delta^2} \sum_{k=0}^{N_\Delta-1} \left(\int_{t_{k-1}}^{t_k} \xi_s \, \mathrm{d}s \right)^2 + \frac{1}{2T} \sum_{k=0}^{N_\Delta-1} R(\varepsilon,\Delta)^2 \right. \\
&\qquad \left. + \frac{\sqrt{2\Sigma}}{T\sqrt{\Delta}} \sum_{k=0}^{N_\Delta-1} R(\varepsilon,\Delta) \int_{t_{k-1}}^{t_k} \xi_s \, \mathrm{d}s \right\} \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Let us consider the first term. We have that

$$\int_{t_{k-1}}^{t_k} \xi_s \, \mathrm{d}s =: \Xi_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}\left(0, \Delta^2\right),$$

which, by the law of large numbers, implies that

$$\lim_{\varepsilon\to 0} I_1 = \Sigma, \quad \text{a.s.}$$

For the second term, we get

$$\mathbb{E}\,|I_2| \le C(\varepsilon^\zeta + \varepsilon^{2\kappa-\zeta}),$$

which implies that I_2 vanishes in L^1 for $\varepsilon \to 0$, and therefore in law, since we can choose κ as close as needed to $1/2$. Let us now consider the last term. The Cauchy–Schwarz inequality yields

$$\begin{aligned}
\mathbb{E}\,|I_2| &\le \frac{C}{2T\Delta} \sum_{k=0}^{N_\Delta-1} \left(\mathbb{E}\, R(\varepsilon,\Delta)^2\right)^{1/2} \left(\mathbb{E}\,\Xi_k^2\right)^{1/2} \\
&\le C\varepsilon^\zeta \left(\varepsilon^\zeta + \varepsilon^\kappa\right) \varepsilon^{-3\zeta/2} \\
&\le C(\varepsilon^{\zeta/2} + \varepsilon^{\kappa-\frac{\zeta}{2}}).
\end{aligned}$$

Hence, since κ can be chosen arbitrarily close to $1/2$, the conclusion follows. $\square$

3 Point estimates from discrete data

In practice, it is not possible to observe X_t^ε continuously, and data will therefore be given by a discrete sequence of time evaluations of the underlying continuous process. Let us consider data to be given by the discrete sequence $\mathbf{x}^\varepsilon = \{x_j^\varepsilon\}_{j=0}^N$ such that $x_j^\varepsilon = X_{t_j}^\varepsilon$, where X_t^ε is the solution of (1.1). We are interested in the case in which data is observed at high frequency, and therefore in the following we assume that $t_j = j\varepsilon^\beta$ for some exponent $\beta > 1$.

3.1 Drift coefficient

Let us first recall that for a general sequence $\mathbf{x} = \{x_j\}_{j=0}^N \in \mathbb{R}^{N+1}$ of evaluations on a time grid with spacing h, the estimator (1.3) is approximated by

$$\widehat{A}_N(\mathbf{x}) = -\frac{\sum_{n=1}^{N+1} V'(x_{n-1})(x_n - x_{n-1})}{\sum_{n=1}^{N+1} V'(x_{n-1})^2 h}, \tag{3.1}$$

see e.g. [2]. The discrete-time equivalent to the operator $\mathcal{H}_\Delta$ defined in (2.1) is the operator $H_\delta \colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$, defined as

$$H_\delta(\mathbf{x}^\varepsilon)_n = \begin{cases} x_0^\varepsilon, & n = 0, \\ \frac{1}{n+1}\sum_{j=0}^n x_j^\varepsilon, & 1 \leq n < \delta - 1, \\ \frac{1}{\delta}\sum_{j=0}^{\delta-1} x_{n-j}^\varepsilon, & \delta - 1 \leq n \leq N, \end{cases}$$

where, in this framework, $\delta \in \mathbb{N}_{>0}$ represents the size of the averaging window. In the following, we will employ the notation $\mathbf{z}^\varepsilon := H_\delta(\mathbf{x}^\varepsilon)$ and $\mathbf{z}^\varepsilon = \{z_n^\varepsilon\}_{n=0}^N$. Let us remark that

$$z_n^\varepsilon - z_{n-1}^\varepsilon = \frac{1}{\delta}(x_n^\varepsilon - x_{n-\delta}^\varepsilon), \tag{3.2}$$

for $\delta \leq n \leq N$ and for $1 \leq n < \delta - 1$

$$z_n^\varepsilon - z_{n-1}^\varepsilon = \frac{1}{n+1}x_n^\varepsilon - \frac{1}{n(n+1)}\sum_{j=0}^{n-1} x_j^\varepsilon.$$

Since the weight of the first $\delta - 1$ data points will be negligible in the theoretical results, we decide to modify the definition of the operator H_δ simply as

$$z_n^\varepsilon = H_\delta(\mathbf{x}^\varepsilon)_n = \frac{1}{\delta}\sum_{j=0}^{\delta-1} x_{n-j}^\varepsilon,$$

for $n = 0, 1, \dots, N$, through the introduction of δ fictitious points $x_j^\varepsilon = x_0^\varepsilon$, $j = -1, -2, \dots, -\delta + 1$. The choice of assigning to these "negative index" points the initial condition is arbitrary, but not influential. Therefore, with this choice, the difference $z_n^\varepsilon - z_{n-1}^\varepsilon$ is always given by (3.2).

Replacing $\mathbf{x}$ with the sequence $\mathbf{z}^\varepsilon$ in (3.1) and reminding that we consider the time grid to have spacing $h = \varepsilon^\beta$, we get the estimator

$$\widehat{A}_{N,\delta}(\mathbf{z}^\varepsilon) = -\frac{\sum_{n=1}^N V_0'(z_{n-1}^\varepsilon)(z_n^\varepsilon - z_{n-\delta}^\varepsilon)}{\sum_{n=1}^N \varepsilon^\beta V_0'(z_{n-1}^\varepsilon)^2}. \tag{3.3}$$

Employing the theoretical tools introduced in Section 2.1 it is possible to prove the following result.

Theorem 3.1. *Under assumption add assumptions, if there exists $\zeta \in (\beta - 1, \beta)$ such that $\delta = [\varepsilon^{-\zeta}]$ and $\gamma > 2\beta - \zeta$ such that $N = [\varepsilon^{-\gamma}]$, where $[\cdot]$ denotes the integer part of a real number, it holds*

$$\lim_{\varepsilon \to 0} \widehat{A}_{N,\delta}(\mathbf{z}^\varepsilon) = A, \quad \textit{in law.}$$

Let us first introduce a Lemma which replaces in the discrete case Lemma 2.3.

Lemma 3.2. *Under assumptions add assumptions*

$$\mathbb{E}^{\mu^\varepsilon}\left|X_s^\varepsilon - z_{n-1}^\varepsilon\right|^p \leq C(\varepsilon^{p\beta}\delta^p + \varepsilon^{p\beta/2}\delta^{p/2} + \varepsilon^p),$$

for $t \in [t_{n-\delta}, t_n]$, where $C > 0$ is independent of ε and δ.

Proof. Let us first consider $n \geq \delta$. We replace the definition of z_{n-1}^ε and apply the Hölder inequality to obtain

$$\begin{aligned}\mathbb{E}^{\mu^\varepsilon}\left|X_s^\varepsilon - z_{n-1}^\varepsilon\right|^p &= \delta^{-p}\,\mathbb{E}^{\mu^\varepsilon}\left|\sum_{j=0}^{\delta-1}\left(X_s^\varepsilon - x_{n-1-j}^\varepsilon\right)\right|^p \\ &\leq \delta^{-1}\sum_{j=0}^{\delta-1}\mathbb{E}^{\mu^\varepsilon}\left|X_s^\varepsilon - x_{n-1-j}^\varepsilon\right|^p.\end{aligned}$$

Applying on each element of the sum [2, Lemma 6.1], we obtain

$$\mathbb{E}^{\mu^\varepsilon} \left|X_s^\varepsilon - z_{n-1}^\varepsilon\right|^p \le C\delta^{-1} \sum_{j=0}^{\delta-1} \left(\varepsilon^{p\beta}\delta^p + \varepsilon^{p\beta/2}\delta^{p/2} + \varepsilon^p\right),$$

which implies the desired result. For $n < \delta$, we have equivalently

$$\mathbb{E}^{\mu^\varepsilon} \left|X_s^\varepsilon - z_{n-1}^\varepsilon\right|^p \le \delta^{-1} \sum_{j=0}^{n-1} \mathbb{E}^{\mu^\varepsilon} \left|X_s^\varepsilon - x_j^\varepsilon\right|^p + \delta^{-1}(\delta - n)\, \mathbb{E}^{\mu^\varepsilon} \left|X_s^\varepsilon - x_0^\varepsilon\right|^p,$$

which can be treated as above and therefore implies the desired result. $\square$

Lemma 3.3. *Under assumptions add assumptions, it holds in law*

$$\alpha \int_{t_{n-\delta}}^{t_n} V_0'(X_s^\varepsilon)\big(1 + \Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}s = \varepsilon^\beta \delta A V_0'(z_{n-1}^\varepsilon) + R(\varepsilon, \delta),$$

where for every $p > 0$ and if δ and ε are sufficiently small,

$$\left(\mathbb{E}^{\mu^\varepsilon} \left|R(\varepsilon, \delta)\right|^p\right)^{1/p} \le C\left(\varepsilon^2 + \varepsilon^{(2+\beta)/2}\delta^{1/2} + \varepsilon^{3\beta/2}\delta^{3/2}\right),$$

where $C > 0$ is a constant independent of ε and δ.

Proof. The proof follows from the proof of Lemma 2.4, with z_{n-1}^ε takes the role of Z_t^ε and replacing Δ by $\delta\varepsilon^q$. $\square$

Proof of Theorem 3.1. Let us recall the decomposition (2.2), which, combined with (3.2), reads

$$\begin{aligned} z_n^\varepsilon - z_{n-\delta}^\varepsilon = &- \frac{\alpha}{\delta} \int_{t_{n-\delta}}^{t_n} V_0'(X_s^\varepsilon)\big(1 + \Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}s \\ &+ \frac{\sqrt{2\sigma}}{\delta} \int_{t_{n-\delta}}^{t_n} \big(1 + \Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}W_s \\ &- \frac{\varepsilon}{\delta}\big(\Phi(Y_{t_n}^\varepsilon) - \Phi(Y_{t_{n-\delta}}^\varepsilon)\big). \end{aligned}$$

Hence, in light of 3.3 and denoting

$$J_n := \sqrt{2\sigma} \int_{t_{n-\delta}}^{t_n} \big(1 + \Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}W_s,$$

we have the equality in law

$$z_n^\varepsilon - z_{n-\delta}^\varepsilon = -\varepsilon^\beta A V_0'(z_{n-1}^\varepsilon) + \frac{\widehat{R}(\varepsilon, \delta)}{\delta} + \frac{J_n}{\delta},$$

where

$$\left(\mathbb{E}^{\mu^\varepsilon} \left|\widehat{R}(\varepsilon, \Delta)\right|^p\right)^{1/p} \le C(\varepsilon + \varepsilon^{(2+\beta)/2}\delta^{1/2} + \varepsilon^{3\beta/2}\delta^{3/2})$$

Replacing the equality above into (3.3), we get

$$\begin{aligned} \widehat{A}_{N,\delta}(\mathbf{z}^\varepsilon) &= A - \frac{1}{\delta\varepsilon^\beta} \frac{\sum_{n=1}^N V_0'(z_{n-1}^\varepsilon)\widehat{R}(\varepsilon, \delta)}{\sum_{n=1}^N V_0'(z_{n-1}^\varepsilon)^2} - \frac{1}{\delta\varepsilon^\beta} \frac{\sum_{n=1}^N V_0'(z_{n-1}^\varepsilon) J_n}{\sum_{n=1}^N V_0'(z_{n-1}^\varepsilon)^2} \\ &=: A - I_1 - I_2. \end{aligned}$$

Let us consider I_1 and multiply by $1/N$ both its numerator and denominator. For the denominator we apply the ergodic theorem and get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N V_0'(z_{n-1}^\varepsilon)^2 = \mathbb{E}^{\mu^\varepsilon}\left(V_0'(x)^2\right),$$

almost surely. Hence the denominator tends in probability to a finite value in the limit $\varepsilon \to 0$, which is equivalent to $N \to \infty$ since $N = [\varepsilon^{-\gamma}]$. We apply Hölder's inequality on the numerator and obtain for any $p > 1$

$$\begin{aligned}\mathbb{E}^{\mu^\varepsilon} \left| \varepsilon^{\zeta-\gamma-\beta} \sum_{n=1}^N V_0'(z_{n-1}^\varepsilon) \widehat{R}(\varepsilon, \delta) \right| &\leq C\varepsilon^{\zeta-\gamma-\beta} \sum_{n=1}^N \left(\mathbb{E}^{\mu^\varepsilon} \left| \widehat{R}(\varepsilon, \delta) \right|^p \right)^{1/p} \\ &\leq C\varepsilon^{\zeta-\beta} (\varepsilon + \varepsilon^{(2+\beta)/2} \delta^{1/2} + \varepsilon^{3\beta/2} \delta^{3/2}) \\ &\leq C(\varepsilon^{1+\zeta-\beta} + \varepsilon^{(2+\zeta-\beta)/2} + \varepsilon^{(\beta-\zeta)/2}),\end{aligned}$$

which, under the assumption $\zeta \in (\beta - 1, \beta)$ vanishes for $\varepsilon \to 0$. Hence I_1 tends to zero in L^1 and therefore by Slutsky's theorem

$$\lim_{\varepsilon \to 0} I_1 = 0, \quad \text{in law.}$$

Let us now consider I_2. Multiplying both the numerator and the denominator by $1/N$, we have that I_2 has the same denominator as I_1, and shall therefore not be treated further. For the numerator, we have (see the proof of Theorem 2.1 or of [2, Theorem 3.5])

$$\frac{1}{\sqrt{N\delta}} \sum_{n=1}^N V_0'(z_{n-1}^\varepsilon) J_n = c\mathcal{N}(0, 1), \quad \text{in law,}$$

where c is a constant independent of ε and δ. Therefore, we have the equality in law

$$\frac{1}{\sqrt{N\delta}\varepsilon^\beta} \frac{1}{\sqrt{N\delta}} \sum_{n=1}^N V_0'(z_{n-1}^\varepsilon) J_n = c\varepsilon^{(\gamma+\zeta-2\beta)/2} \mathcal{N}(0, 1).$$

This, together with the hypothesis $\gamma > 2\beta - \zeta$, yields

$$\lim_{\varepsilon \to 0} I_1 = 0, \quad \text{in law,}$$

which proves the desired result. □

4 Bayesian inference

Consider

$$L_T^0(A) = \exp \left\{ -\int_0^T AV_0'(X_t^0) \,\mathrm{d}X_t^0 - \frac{1}{2} \int_0^T A^2 V_0'(X_t^0)^2 \,\mathrm{d}t \right\},$$

and, denoting $Z_t^\varepsilon \coloneqq \mathcal{H}_\delta(X^\varepsilon)_t$, where $\mathcal{H}_\delta$ is defined in (2.1)

$$L_T^\varepsilon(A) = \exp \left\{ -\int_0^T AV_0'(Z_t^\varepsilon) \,\mathrm{d}Z_t^\varepsilon - \frac{1}{2} \int_0^T A^2 V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t \right\}.$$

Let the prior be denoted by Λ, with density λ and the corresponding posteriors μ_T^0 and μ_T^ε. Denote $\ell_t^0 = \log L_T^0$, respectively ℓ_t^ε the log-likelihoods.

Define

$$d_{\mathrm{TV}}(\mu, \nu) \coloneqq \sup_{B \in \mathcal{B}} |\mu(B) - \nu(B)|.$$

Compute for $B \in \mathcal{B}$

$$\left| \mu_T^0(B) - \mu_T^\varepsilon(B) \right| = \left| \frac{C^\varepsilon \int_B L_T^0(A)\lambda(A) \,\mathrm{d}A - Z^0 \int_B L_T^\varepsilon(A)\lambda(A) \,\mathrm{d}A}{C^0 C^\varepsilon} \right|,$$

where

$$C^0 = \int_{\mathcal{A}} L_T^0(A)\lambda(A) \,\mathrm{d}A,$$

and C^ε defined respectively. Then

$$\left|\mu_T^0(B) - \mu_T^\varepsilon(B)\right| \leq I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \frac{1}{C^0} \int_B \left|L_T^0(A) - L_T^\varepsilon(A)\right| \lambda(A) \,\mathrm{d}A, \\ I_2 &= \frac{\left|C^\varepsilon - C^0\right|}{C^0 C^\varepsilon} \mu_T^\varepsilon(B). \end{aligned}$$

Consider first I_1. Since $|\exp(a) - \exp(b)| \leq (\exp(a) + \exp(b)) \, |a - b|$, we have

$$I_1 \leq \frac{1}{C^0} \int_B \left(L_T^0(A) + L_T^\varepsilon(A)\right) \left|\ell_T^0(A) - \ell_T^\varepsilon(A)\right| \lambda(A) \,\mathrm{d}A.$$

Let us consider

$$\begin{aligned} \ell_T^0(A) - \ell_T^\varepsilon(A) = &- \int_0^T AV_0'(X_t^0) \,\mathrm{d}X_t^0 + \int_0^T AV_0'(Z_t^\varepsilon) \,\mathrm{d}Z_t^\varepsilon \\ &- \frac{1}{2} \int_0^T A^2 \left(V_0'(X_t^0)^2 - V_0'(Z_t^\varepsilon)^2\right) \mathrm{d}t. \end{aligned}$$

Lemma 4.1. *Under assumptions add assumptions, it holds*

$$\left|\ell_T^0(A) - \ell_T^\varepsilon(A)\right| \to 0,$$

for $\varepsilon \to 0$.

Proof. The triangle inequality

$$\begin{aligned} \left|\ell_T^0(A) - \ell_T^\varepsilon(A)\right| \leq &\left| \int_0^T AV_0'(X_t^0) \,\mathrm{d}X_t^0 - \int_0^T AV_0'(Z_t^\varepsilon) \,\mathrm{d}Z_t^\varepsilon \right| \\ &+ \left| \frac{1}{2} \int_0^T A^2 \left(V_0'(X_t^0)^2 - V_0'(Z_t^\varepsilon)^2\right) \mathrm{d}t \right| =: I_1 + I_2 \end{aligned}$$

Let us first consider I_1. From the definition of Z_t^ε, we divide

$$\begin{aligned} I_1 \leq &\left| \int_0^\delta AV_0'(X_t^0) \,\mathrm{d}X_t^0 - \int_0^\delta AV_0'(Z_t^\varepsilon) \frac{X_t^\varepsilon - Z_t^\varepsilon}{t} \,\mathrm{d}t \right| \\ &+ \left| \int_\delta^T AV_0'(X_t^0) \,\mathrm{d}X_t^0 - \int_\delta^T AV_0'(Z_t^\varepsilon) \frac{X_t^\varepsilon - X_{t-\delta}^\varepsilon}{\delta} \,\mathrm{d}t \right| =: I_1^1 + I_1^2. \end{aligned}$$

Let us first consider I_1^2. Replacing (2.2) we can write in law

$$I_1^2 = \left| \int_\delta^T AV_0'(X_t^0) \,\mathrm{d}X_t^0 - \int_\delta^T AV_0'(Z_t^\varepsilon) \frac{J_t - A\delta V_0'(Z_t^\varepsilon) + R(\varepsilon, \delta)}{\delta} \,\mathrm{d}t \right|,$$

where, due to Lemma 2.4, we have

$$\left(\mathbb{E}^{\mu^\varepsilon} \left|R(\varepsilon, \delta)\right|^p\right)^{1/p} \leq C(\varepsilon^2 + \delta^{1/2} + \delta^{3/2}).$$

Replacing $\mathrm{d}X_t^0$ with its definition given by (1.2), we can then split I_1^2 in three terms and apply the triangle inequality as

$$\begin{aligned} I_1^2 \leq A^2 &\left| \int_\delta^T \left(V_0'(X_t^0)^2 - V_0'(Z_t^\varepsilon)^2\right) \,\mathrm{d}t \right| + A \left| \int_\delta^T V_0'(X_t^0) \sqrt{2\Sigma} \,\mathrm{d}W_t - \frac{1}{\delta} \int_\delta^T V_0'(Z_t^\varepsilon) J_t \,\mathrm{d}t \right| \\ &+ A \left| \int_\delta^T V_0'(Z_t^\varepsilon) \frac{R(\varepsilon, \delta)}{\delta} \,\mathrm{d}t \right| =: R_1 + R_2 + R_3. \end{aligned}$$

$\square$

5 Numerical experiments

References

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