RANDOM MESH FEM

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1. Idea. Consider Ω a convex polygon in $\mathbb{R}^d$, with $d = 1, 2, 3$ and the elliptic PDE with Dirichlet boundary conditions

$$\begin{aligned} -\mathcal{L}u &= f, \quad \text{in } \Omega, \\ u &= g, \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Given a Hilbert space V weak formulation (assume $a(u, u) = \|u\|_a^2$)

$$\text{Find } u \in V \text{ such that } a(u, v) = F(v) \text{ for all } v \in V. \tag{2}$$

Galerkin formulation. Consider discretization parameter $h > 0$ and a mesh T_h (usual hypotheses). Consider the space $V_h \subset V$ defined as

$$V_h = \{v \in \mathcal{C}^0(\Omega) \colon v|_K \in \mathcal{P}_1, \ \forall K \in T_h\} \cap V. \tag{3}$$

Given internal vertices $\{x_i\}_{i=1}^N$, then

$$V_h = \operatorname{span}\{\varphi_i\}_{i=1}^N, \tag{4}$$

where $\varphi_i \in V_h$ and $\varphi_i(x_k) = \delta_{ik}$ for $i, k = 1, \dots, N$. Galerkin formulation then reads

$$\text{Find } u_h \in V_h \text{ such that } a(u_h, v_h) = F(v_h) \text{ for all } v_h \in V_h. \tag{5}$$

Consider now a new set of random internal vertices $\{X_i\}_{i=1}^N$ such that

(i) $\mathbb{E}\, X_i = x_i$,
(ii) $\operatorname{Var} X_i = Ch^{2p}$, for a constant $C > 0$.

for all $i = 1, \dots, N$. Random mesh $\mathcal{T}_h$ is built using the nodes $\{X_i\}_{i=1}^N$ from T_h maintaining connections between vertices with same indices (in 1D it is easy, in 2D/3D is it possible to maintain hypotheses of mesh quality?). Then consider

$$\mathcal{V}_h = \{v \in \mathcal{C}^0(\Omega) \colon v|_K \in \mathcal{P}_1, \ \forall K \in \mathcal{T}_h\} \cap V. \tag{6}$$

i.e., $\mathcal{V}_h = \operatorname{span}\{\Phi_i\}_{i=1}^N$, where $\Phi_i \in \mathcal{V}_h$ and $\Phi_i(X_k) = \delta_{ik}$. We then have the random-mesh Galerkin formulation

$$\text{Find } U_h \in \mathcal{V}_h \text{ such that } a(U_h, V_h) = F(V_h) \text{ for all } V_h \in \mathcal{V}_h. \tag{7}$$

<u>Goal.</u> What is

$$\mathbb{E}\|U_h - u\|_V, \tag{8}$$
$$|\mathbb{E}\, G(U_h) - G(u)|. \tag{9}$$

2. One-dimensional case. Consider deterministic uniform mesh (spacing h) and perturbation r.v.s such that

$$X_i = x_i + hP_i, \quad P_i \sim \mathcal{U}(-h^{p-1}/2, h^{p-1}/2). \tag{10}$$

(1/2 so that the ordering does not change). Consider basis functions deterministic case

$$\varphi_i(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}} \mathbb{1}_{(x_{i-1}, x_i)}(x) + \frac{x_{i+1} - x}{x_{i+1} - x_i} \mathbb{1}_{(x_i, x_{i+1})}(x). \tag{11}$$

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The random basis functions are given analogously by

$$\Phi_i(x) = \frac{x - X_{i-1}}{X_i - X_{i-1}} \mathbb{1}_{(X_{i-1}, X_i)}(x) + \frac{X_{i+1} - x}{X_{i+1} - X_i} \mathbb{1}_{(X_i, X_{i+1})}(x). \tag{12}$$

Let us denote by $\Phi_{i,1}(x)$ and $\Phi_{i,1}(x)$ the two components of the sum above so that $\Phi_i(x) = \Phi_{i,1}(x) + \Phi_{i,2}(x)$. Via the definition of the random variables we rewrite $\Phi_{i,1}(x)$ with elementary operations as

$$\begin{aligned} \Phi_{i,1}(x) &= \frac{x - x_{i-1} - hP_{i-1}}{x_i - x_{i-1} + h(P_i - P_{i-1})} \mathbb{1}_{(X_{i-1}, X_i)}(x) \\ &= \frac{x - x_{i-1} - hP_{i-1}}{h(1 + P_i - P_{i-1})} \mathbb{1}_{(X_{i-1}, X_i)}(x) \\ &= \frac{1}{1 + P_i - P_{i-1}} \Big(\frac{x - x_{i-1}}{h} - P_{i-1} \Big) \mathbb{1}_{(X_{i-1}, X_i)}(x) \end{aligned} \tag{13}$$

Analogously

$$\Phi_{i,2}(x) = \frac{1}{1 + P_{i+1} - P_i} \Big(\frac{x_{i+1} - x}{h} + P_{i+1} \Big) \mathbb{1}_{(X_i, X_{i+1})}(x) \tag{14}$$