A random title for a random problem

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1 Formulation

2 A priori error analysis

Galerkin

Define $\Pi_h \colon \widetilde{V}_h \to V_h$ such that if $\tilde v_h(x) = \sum_i \tilde v_i \tilde\varphi_i(x)$ then $\Pi_h \tilde v_h(x) = \sum_i \tilde v_i \varphi_i(x)$. Analogously, we denote by $\Pi_h^{-1} \colon V_h \to \widetilde{V}_h$ the inverse operator such that if $v_h = \sum_i v_i \varphi_i(x)$, then $\Pi_h^{-1} v_h(x) = \sum_i v_i \tilde\varphi_i(x)$. It is clear that $\Pi_h \circ \Pi_h^{-1} = \mathrm{Id}$. Consider

$$
\begin{aligned}
a(u_h, v_h) &= f(v_h), \quad \forall v_h \in V_h,\\
a(\tilde u_h, \tilde v_h) &= f(\tilde v_h), \quad \forall \tilde v_h \in \tilde V_h.
\end{aligned} \tag{1}
$$

Galerkin "orthogonality", $\forall v_h \in V_h$

$$
\begin{aligned}
a(u_h - \tilde u_h, v_h) &= f(v_h - \tilde v_h) + a(\tilde u_h, \tilde v_h - v_h)\\
&\leq \big(C_f + C_a \|\tilde u_h\|_V\big) \|v_h - \tilde v_h\|_V.
\end{aligned} \tag{2}
$$

for all $\tilde v_h \in \widetilde{V}_h$. Choose $\tilde v_h = \Pi_h^{-1} v_h$,

$$
a(u_h - \tilde u_h, v_h) \leq \big(C_f + C_a \|\tilde u_h\|_V\big) \|v_h - \Pi_h^{-1} v_h\|_V. \tag{3}
$$

Interpolation estimates

Goal: Estimate $\|\nabla(v_h - \Pi_h v_h)\|_{\mathrm{L}^2}$.

$$
\begin{aligned}
\|\nabla(v_h - \Pi_h v_h)\|_{\mathrm{L}^2} &= \|\sum_i v_i \nabla(\varphi_i - \tilde\varphi_i)\|_{\mathrm{L}^2}\\
&\leq \sum_i |v_i| \|\nabla(\varphi_i - \tilde\varphi_i)\|_{\mathrm{L}^2}.
\end{aligned} \tag{4}
$$

If S_i support of φ_i and $\widetilde{S}_i$ support of $\tilde\varphi_i$

$$
\begin{aligned}
\|\nabla(\varphi_i - \tilde\varphi_i)\|_{\mathrm{L}^2}^2 &= \int_D |\nabla(\varphi_i - \tilde\varphi_i)|^2\\
&= \int_{S_i} |\nabla(\varphi_i - \tilde\varphi_i)|^2 + \int_{\tilde S_i \setminus S_i} |\nabla \tilde\varphi_i|^2.
\end{aligned} \tag{5}
$$

One-dimensional case. Define

$$
\begin{aligned}
&\bar h_i = \frac{h_{i+1} - h_i}{2}, \quad i = 1, \dots, N-1\\
&\bar h_0 = 0, \quad \bar h_N = 0.
\end{aligned} \tag{6}
$$

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And the points $\tilde{x}_i = x_i + \alpha_i \bar{h}_i^{p+1}$ for some $p > 1$, α_i such that $|\alpha_i| \leq M$ and $i = 0, \ldots, N$. Estimate the two terms separately. Linear basis functions

$$\nabla \tilde{\varphi}_i = \begin{cases} \dfrac{1}{h_i + (\alpha_i \bar{h}_i^{p+1} - C_{i-1} \bar{h}_{i-1}^{p+1})}, & \text{in } (\tilde{x}_{i-1}, \tilde{x}_i), \\ \dfrac{-1}{h_{i+1} + (\alpha_{i+1} \bar{h}_{i+1}^{p+1} - \alpha_i \bar{h}_i^{p+1})}, & \text{in } (\tilde{x}_i, \tilde{x}_{i+1}) \end{cases} \tag{7}$$

Then

$$\begin{aligned} \int_{\tilde{S}_i \setminus S_i} |\nabla \tilde{\varphi}_i|^2 &= \int_{\tilde{x}_{i-1}}^{x_{i-1}} |\nabla \tilde{\varphi}_i|^2 + \int_{\tilde{x}_{i+1}}^{x_{i+1}} |\nabla \tilde{\varphi}_i|^2 \\ &= \frac{1}{(h_i + (\alpha_i \bar{h}_i^{p+1} - \alpha_{i-1} \bar{h}_{i-1}^{p+1}))^2} \alpha_{i-1} \bar{h}_{i-1}^{p+1} \\ &\quad + \frac{1}{(h_{i+1} + (\alpha_{i+1} \bar{h}_{i+1}^{p+1} - \alpha_i \bar{h}_i^{p+1}))^2} \alpha_{i+1} \bar{h}_{i+1}^{p+1} \\ &\leq \Big(\frac{1}{(h_i - M(\bar{h}_i^{p+1} + \bar{h}_{i-1}^{p+1}))^2} + \frac{1}{(h_{i+1} - M(\bar{h}_{i+1}^{p+1} + \bar{h}_i^{p+1}))^2} \Big) M h^{p+1} \\ &\leq C \Big(\frac{1}{h_i^2} + \frac{1}{h_{i+1}^2} \Big) h^{p+1} \leq C h^{p-1}, \end{aligned} \tag{8}$$

where in the last step we used the fact that $h \leq C h_i$ (quasi-uniform).

Almost sure convergence

Finally

$$\begin{aligned} \alpha \|u_h - U_h\|_V^2 &\leq a(u_h - \tilde{u}_h, u_h - U_h) + a(\tilde{u}_h - U_h, u_h - U_h) \\ &\leq \big(C_f + C_a \|\tilde{u}_h\|_V\big) \|u_h - U_h - \Pi_h^{-1}(u_h - U_h)\|_V + C_a \|\tilde{u}_h - U_h\|_V \|u_h - U_h\|_V. \end{aligned} \tag{9}$$

Hence

$$\|u_h - U_h\|_V \leq \frac{1}{\alpha} \big(C_f + C_a \|\tilde{u}_h\|_V\big) \text{int.estimate} + C_a \|\tilde{u}_h - U_h\|_V. \tag{10}$$

$\mathrm{L}^2(\Omega)$ convergence

3 A posteriori error analysis

4 Inverse problems

References