RANDOM MESH FEM

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1. Idea. Consider Ω a convex polygon in $\mathbb{R}^d$, with $d = 1, 2, 3$ and the elliptic PDE with Dirichlet boundary conditions

$$\begin{aligned} -\mathcal{L}u &= f, \quad \text{in } \Omega, \\ u &= g, \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Given a Hilbert space V weak formulation (assume $a(u,u) = \|u\|_a^2$)

$$\text{Find } u \in V \text{ such that } a(u,v) = F(v) \text{ for all } v \in V. \tag{2}$$

Galerkin formulation. Consider discretization parameter $h > 0$ and a mesh T_h (usual hypotheses). Consider the space $V_h \subset V$ defined as

$$V_h = \{v \in \mathcal{C}^0(\Omega) \colon v|_K \in \mathcal{P}_1, \ \forall K \in T_h\} \cap V. \tag{3}$$

Given internal vertices $\{x_i\}_{i=1}^N$, then

$$V_h = \operatorname{span}\{\varphi_i\}_{i=1}^N, \tag{4}$$

where $\varphi_i \in V_h$ and $\varphi_i(x_k) = \delta_{ik}$ for $i, k = 1, \dots, N$. Galerkin formulation then reads

$$\text{Find } u_h \in V_h \text{ such that } a(u_h, v_h) = F(v_h) \text{ for all } v_h \in V_h. \tag{5}$$

Consider now a new set of random internal vertices $\{X_i\}_{i=1}^N$ such that

(i) $\mathbb{E}\, X_i = x_i$,

(ii) $\operatorname{Var} X_i = Ch^{2p}$, for a constant $C > 0$.

for all $i = 1, \dots, N$. Random mesh $\mathcal{T}_h$ is built using the nodes $\{X_i\}_{i=1}^N$ from T_h maintaining connections between vertices with same indices (in 1D it is easy, in 2D/3D is it possible to maintain hypotheses of mesh quality?). Then consider

$$\mathcal{V}_h = \{v \in \mathcal{C}^0(\Omega) \colon v|_K \in \mathcal{P}_1, \ \forall K \in \mathcal{T}_h\} \cap V. \tag{6}$$

i.e., $\mathcal{V}_h = \operatorname{span}\{\Phi_i\}_{i=1}^N$, where $\Phi_i \in \mathcal{V}_h$ and $\Phi_i(X_k) = \delta_{ik}$. We then have the random-mesh Galerkin formulation

$$\text{Find } U_h \in \mathcal{V}_h \text{ such that } a(U_h, V_h) = F(V_h) \text{ for all } V_h \in \mathcal{V}_h. \tag{7}$$

<u>Goal.</u> What is

$$\mathbb{E}\|U_h - u\|_V, \tag{8}$$

$$|\mathbb{E}\, G(U_h) - G(u)|. \tag{9}$$

2. One-dimensional case. Consider deterministic uniform mesh (spacing h) and perturbation r.v.s such that

$$X_i = x_i + hP_i, \quad P_i \sim \mathcal{U}(-h^{p-1}/2, h^{p-1}/2). \tag{10}$$

(1/2 so that the ordering does not change). Consider basis functions deterministic case

$$\varphi_i(x) = \underbrace{\frac{x - x_{i-1}}{h} \mathbb{1}_{(x_{i-1}, x_i)}(x)}_{\varphi_{i,1}(x)} + \underbrace{\frac{x_{i+1} - x}{h} \mathbb{1}_{(x_i, x_{i+1})}(x)}_{\varphi_{i,2}(x)}. \tag{11}$$

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The random basis functions are given analogously by

$$\Phi_i(x) = \frac{x - X_{i-1}}{X_i - X_{i-1}} \mathbb{1}_{(X_{i-1},X_i)}(x) + \frac{X_{i+1} - x}{X_{i+1} - X_i} \mathbb{1}_{(X_i,X_{i+1})}(x). \tag{12}$$

Let us denote by $\Phi_{i,1}(x)$ and $\Phi_{i,1}(x)$ the two components of the sum above so that $\Phi_i(x) = \Phi_{i,1}(x) + \Phi_{i,2}(x)$. Via the definition of the random variables we rewrite $\Phi_{i,1}(x)$ with elementary operations as

$$\begin{aligned}\Phi_{i,1}(x) &= \frac{x - x_{i-1} - hP_{i-1}}{x_i - x_{i-1} + h(P_i - P_{i-1})} \mathbb{1}_{(X_{i-1},X_i)}(x)\\ &= \frac{x - x_{i-1} - hP_{i-1}}{h(1 + P_i - P_{i-1})} \mathbb{1}_{(X_{i-1},X_i)}(x)\\ &= \underbrace{\frac{1}{1 + P_i - P_{i-1}}}_{C_{i,i-1}} \Big(\frac{x - x_{i-1}}{h} - P_{i-1}\Big) \mathbb{1}_{(X_{i-1},X_i)}(x).\end{aligned} \tag{13}$$

Analogously,

$$\Phi_{i,2}(x) = \underbrace{\frac{1}{1 + P_{i+1} - P_i}}_{C_{i+1,i}} \Big(\frac{x_{i+1} - x}{h} + P_{i+1}\Big) \mathbb{1}_{(X_i,X_{i+1})}(x). \tag{14}$$

Consider the indicator function in $\Phi_{i,1}$. Let us denote by $A = (x_{i-1}, x_i)$. Then (dropping the dependence on x of the indicator functions)

$$\begin{aligned}\mathbb{1}_{(X_{i-1},X_i)} =&\mathbb{1}_{A\cup(X_{i-1},x_{i-1})\cup(x_i,X_i)} \mathbb{1}_{\{P_{i-1}<0\}} \mathbb{1}_{\{P_i>0\}}\\ &+\mathbb{1}_{(A\cup(X_{i-1},x_{i-1}))\cap(X_i,x_i)^C} \mathbb{1}_{\{P_{i-1}<0\}} \mathbb{1}_{\{P_i<0\}}\\ &+\mathbb{1}_{(A\cup(x_i,X_i))\cap(x_{i-1},X_{i-1})^C} \mathbb{1}_{\{P_{i-1}>0\}} \mathbb{1}_{\{P_i>0\}}\\ &+\mathbb{1}_{A\cap(x_{i-1},X_{i-1})^C\cap(X_i,x_i)^C} \mathbb{1}_{\{P_{i-1}>0\}} \mathbb{1}_{\{P_i<0\}},\end{aligned} \tag{15}$$

applying the properties of the indicator function we thus have

$$\begin{aligned}\mathbb{1}_{(X_{i-1},X_i)} =&\Big(\mathbb{1}_A + \mathbb{1}_{(X_{i-1},x_{i-1})} + \mathbb{1}_{(x_i,X_i)}\Big) \mathbb{1}_{\{P_{i-1}<0\}} \mathbb{1}_{\{P_i>0\}}\\ &+\Big(\big(\mathbb{1}_A + \mathbb{1}_{(X_{i-1},x_{i-1})}\big)\big(1 - \mathbb{1}_{(X_i,x_i)}\big)\Big) \mathbb{1}_{\{P_{i-1}<0\}} \mathbb{1}_{\{P_i<0\}}\\ &+\Big(\big(\mathbb{1}_A + \mathbb{1}_{(x_i,X_i)}\big)\big(1 - \mathbb{1}_{(x_{i-1},X_{i-1})}\big)\Big) \mathbb{1}_{\{P_{i-1}>0\}} \mathbb{1}_{\{P_i>0\}}\\ &+\Big(\mathbb{1}_A\big(1 - \mathbb{1}_{(x_{i-1},X_{i-1})}\big)\big(1 - \mathbb{1}_{(X_i,x_i)}\big)\Big) \mathbb{1}_{\{P_{i-1}>0\}} \mathbb{1}_{\{P_i<0\}}.\end{aligned} \tag{16}$$

Hence

$$\begin{aligned}\mathbb{1}_{(X_{i-1},X_i)} = \mathbb{1}_A&+\big(\mathbb{1}_{(X_{i-1},x_{i-1})} + \mathbb{1}_{(x_i,X_i)}\big) \mathbb{1}_{\{P_{i-1}<0\}} \mathbb{1}_{\{P_i>0\}}\\ &+\big(\mathbb{1}_{(X_{i-1},x_{i-1})} - \mathbb{1}_{(X_{i-1},x_{i-1})}\mathbb{1}_{(X_i,x_i)} - \mathbb{1}_A \mathbb{1}_{(X_i,x_i)}\big) \mathbb{1}_{\{P_{i-1}<0\}} \mathbb{1}_{\{P_i<0\}}\\ &+\big(\mathbb{1}_{(x_i,X_i)} - \mathbb{1}_{(x_i,X_i)}\mathbb{1}_{(x_{i-1},X_{i-1})} - \mathbb{1}_A \mathbb{1}_{(x_{i-1},X_{i-1})}\big) \mathbb{1}_{\{P_{i-1}>0\}} \mathbb{1}_{\{P_i>0\}}\\ &+\big(\mathbb{1}_A \mathbb{1}_{(x_{i-1},X_{i-1})}\mathbb{1}_{(X_i,x_i)} - \mathbb{1}_A \mathbb{1}_{(X_i,x_i)} - \mathbb{1}_A \mathbb{1}_{(x_{i-1},X_{i-1})}\big) \mathbb{1}_{\{P_{i-1}>0\}} \mathbb{1}_{\{P_i<0\}}.\end{aligned} \tag{17}$$

This expression can be simplified as

$$\begin{aligned}\mathbb{1}_{(X_{i-1},X_i)} = \mathbb{1}_A&+\big(\mathbb{1}_{(X_{i-1},x_{i-1})} + \mathbb{1}_{(x_i,X_i)}\big) \mathbb{1}_{\{P_{i-1}<0\}} \mathbb{1}_{\{P_i>0\}}\\ &+\big(\mathbb{1}_{(X_{i-1},x_{i-1})} - \mathbb{1}_{(X_i,x_i)}\big) \mathbb{1}_{\{P_{i-1}<0\}} \mathbb{1}_{\{P_i<0\}}\\ &+\big(\mathbb{1}_{(x_i,X_i)} - \mathbb{1}_{(x_{i-1},X_{i-1})}\big) \mathbb{1}_{\{P_{i-1}>0\}} \mathbb{1}_{\{P_i>0\}}\\ &-\big(\mathbb{1}_{(X_i,x_i)} + \mathbb{1}_{(x_{i-1},X_{i-1})}\big) \mathbb{1}_{\{P_{i-1}>0\}} \mathbb{1}_{\{P_i<0\}}.\end{aligned} \tag{18}$$

Regrouping the terms

$$
\begin{aligned}
\mathbb{1}_{(X_{i-1},X_i)} = \mathbb{1}_A + &\mathbb{1}_{(X_{i-1},x_{i-1})}\mathbb{1}_{\{P_{i-1}<0\}}\big(\mathbb{1}_{\{P_i>0\}} + \mathbb{1}_{\{P_i<0\}}\big)\\
-&\mathbb{1}_{(x_{i-1},X_{i-1})}\mathbb{1}_{\{P_{i-1}>0\}}\big(\mathbb{1}_{\{P_i>0\}} + \mathbb{1}_{\{P_i<0\}}\big)\\
+&\mathbb{1}_{(x_i,X_i)}\mathbb{1}_{\{P_i>0\}}\big(\mathbb{1}_{\{P_{i-1}>0\}} + \mathbb{1}_{\{P_{i-1}<0\}}\big)\\
-&\mathbb{1}_{(X_i,x_i)}\mathbb{1}_{\{P_i<0\}}\big(\mathbb{1}_{\{P_{i-1}>0\}} + \mathbb{1}_{\{P_{i-1}<0\}}\big).
\end{aligned} \tag{19}
$$

Hence, we get the final expression

$$
\begin{aligned}
\mathbb{1}_{(X_{i-1},X_i)} = \mathbb{1}_A &+ \mathbb{1}_{(X_{i-1},x_{i-1})}\mathbb{1}_{\{P_{i-1}<0\}} - \mathbb{1}_{(x_{i-1},X_{i-1})}\mathbb{1}_{\{P_{i-1}>0\}}\\
&+ \mathbb{1}_{(x_i,X_i)}\mathbb{1}_{\{P_i>0\}} - \mathbb{1}_{(X_i,x_i)}\mathbb{1}_{\{P_i<0\}}.
\end{aligned} \tag{20}
$$

Plugging the expression of $\mathbb{1}_{(X_{i-1},X_i)}$ into the randomized basis functions (13) and recalling that $A = (x_{i-1}, x_i)$ we get

$$
\begin{aligned}
\Phi_{i,1}(x) = C_{i,i-1}&\Big(\frac{x-x_{i-1}}{h} - P_{i-1}\Big)\mathbb{1}_{(x_{i-1},x_i)}(x)\\
+ C_{i,i-1}&\Big(\frac{x-x_{i-1}}{h} - P_{i-1}\Big)\big(\mathbb{1}_{(X_{i-1},x_{i-1})}(x)\mathbb{1}_{\{P_{i-1}<0\}} - \mathbb{1}_{(x_{i-1},X_{i-1})}(x)\mathbb{1}_{\{P_{i-1}>0\}}\big)\\
+ C_{i,i-1}&\Big(\frac{x-x_{i-1}}{h} - P_{i-1}\Big)\big(\mathbb{1}_{(x_i,X_i)}(x)\mathbb{1}_{\{P_i>0\}} - \mathbb{1}_{(X_i,x_i)}(x)\mathbb{1}_{\{P_i<0\}}\big).
\end{aligned} \tag{21}
$$

Replacing the definition of $\varphi_{i,1}$, we get

$$
\begin{aligned}
\Phi_{i,1}(x) = C_{i,i-1}&\varphi_{i,1}(x) - C_{i,i-1}P_{i-1}\mathbb{1}_{(x_{i-1},x_i)}(x)\\
+ C_{i,i-1}&\Big(\frac{x-x_{i-1}}{h} - P_{i-1}\Big)\big(\mathbb{1}_{(X_{i-1},x_{i-1})}(x)\mathbb{1}_{\{P_{i-1}<0\}} - \mathbb{1}_{(x_{i-1},X_{i-1})}(x)\mathbb{1}_{\{P_{i-1}>0\}}\big)\\
+ C_{i,i-1}&\Big(\frac{x-x_{i-1}}{h} - P_{i-1}\Big)\big(\mathbb{1}_{(x_i,X_i)}(x)\mathbb{1}_{\{P_i>0\}} - \mathbb{1}_{(X_i,x_i)}(x)\mathbb{1}_{\{P_i<0\}}\big).
\end{aligned} \tag{22}
$$

We can apply the same reasoning to $\Phi_{i,2}(x)$. In particular ($A = (x_i, x_{i+1})$ in this case)

$$
\begin{aligned}
\mathbb{1}_{(X_i,X_{i+1})} = \mathbb{1}_A &+ \mathbb{1}_{(X_i,x_i)}\mathbb{1}_{\{P_i<0\}} - \mathbb{1}_{(x_i,X_i)}\mathbb{1}_{\{P_i>0\}}\\
&+ \mathbb{1}_{(x_{i+1},X_{i+1})}\mathbb{1}_{\{P_{i+1}>0\}} - \mathbb{1}_{(X_{i+1},x_{i+1})}\mathbb{1}_{\{P_{i+1}<0\}}.
\end{aligned} \tag{23}
$$

Then

$$
\begin{aligned}
\Phi_{i,2}(x) = C_{i+1,i}&\varphi_{i,1}(x) + C_{i+1,i}P_{i+1}\mathbb{1}_{(x_i,x_{i+1})}(x)\\
+ C_{i+1,i}&\Big(\frac{x_{i+1}-x}{h} + P_{i+1}\Big)\big(\mathbb{1}_{(X_i,x_i)}(x)\mathbb{1}_{\{P_i<0\}} - \mathbb{1}_{(x_i,X_i)}(x)\mathbb{1}_{\{P_i>0\}}\big)\\
+ C_{i+1,i}&\Big(\frac{x_{i+1}-x}{h} + P_{i+1}\Big)\big(\mathbb{1}_{(x_{i+1},X_{i+1})}(x)\mathbb{1}_{\{P_{i+1}>0\}} - \mathbb{1}_{(X_{i+1},x_{i+1})}(x)\mathbb{1}_{\{P_{i+1}<0\}}\big).
\end{aligned} \tag{24}
$$

Let us remark that the random coefficient $C_{i,i-1}$ can be expanded (for any i) as

$$
C_{i,i-1} = \frac{1}{1-(P_{i-1}-P_i)} = 1 + \underbrace{\sum_{n=1}^{\infty}(P_{i-1}-P_i)^n}_{\tilde{C}_{i,i-1}}. \tag{25}
$$

In expectation (for $n = 1$ we have a zero)

$$
\mathbb{E}\,\tilde{C}_{i,i-1} = \mathcal{O}(h^{2p-2}). \tag{26}
$$

Hence,

$$
\begin{aligned}
\Phi_i(x) &= \varphi_i(x) + \tilde{C}_{i,i-1}\varphi_{i,1}(x) + \tilde{C}_{i+1,i}\varphi_{i,2}(x) \\
&\quad - C_{i,i-1}P_{i-1}\mathbb{1}_{(x_{i-1},x_i)}(x) + C_{i+1,i}P_{i+1}\mathbb{1}_{(x_i,x_{i+1})}(x) \\
&\quad + C_{i,i-1}\Big(\frac{x-x_{i-1}}{h} - P_{i-1}\Big)\big(\mathbb{1}_{(X_{i-1},x_{i-1})}(x)\mathbb{1}_{\{P_{i-1}<0\}} - \mathbb{1}_{(x_{i-1},X_{i-1})}(x)\mathbb{1}_{\{P_{i-1}>0\}}\big) \\
&\quad + C_{i,i-1}\Big(\frac{x-x_{i-1}}{h} - P_{i-1}\Big)\big(\mathbb{1}_{(x_i,X_i)}(x)\mathbb{1}_{\{P_i>0\}} - \mathbb{1}_{(X_i,x_i)}(x)\mathbb{1}_{\{P_i<0\}}\big) \\
&\quad + C_{i+1,i}\Big(\frac{x_{i+1}-x}{h} + P_{i+1}\Big)\big(\mathbb{1}_{(X_i,x_i)}(x)\mathbb{1}_{\{P_i<0\}} - \mathbb{1}_{(x_i,X_i)}(x)\mathbb{1}_{\{P_i>0\}}\big) \\
&\quad + C_{i+1,i}\Big(\frac{x_{i+1}-x}{h} + P_{i+1}\Big)\big(\mathbb{1}_{(x_{i+1},X_{i+1})}(x)\mathbb{1}_{\{P_{i+1}>0\}} - \mathbb{1}_{(X_{i+1},x_{i+1})}(x)\mathbb{1}_{\{P_{i+1}<0\}}\big).
\end{aligned}
\tag{27}
$$

In expectation

$$
\mathbb{E}\,\Phi_i(x) = \varphi_i(x) + (1 + \mathcal{O}(h^{2p-2}))(\varphi_{i,1}(x) + \varphi_{i,2}(x))
\tag{28}
$$