1 Random time stepping

Let us consider $f\colon \mathbb{R}^d \to \mathbb{R}^d$ and the ODE

$$y' = f(y), \quad y(0) = y_0 \in \mathbb{R}^d.$$

Given a time step $h > 0$, the Explicit Euler method applied to this ODE reads

$$y_{n+1} = y_n + hf(y_n).$$

Let us consider the ODE to be chaotic, i.e., integrating the equation with different time steps leads to different numerical solutions. The idea is to choose at each time t_n the time step as the realization of a random variable H_n in order to quantify the uncertainty introduced by the numerical method. The following assumptions are needed in the following.

Assumption 1.1. *The i.i.d. random variables H_n satisfy*

1. *$H_n > 0$ a.s.,*
2. *there exists $h > 0$ such that $\mathbb{E}[H_n] = h$ for all $n = 0, 1, \ldots$,*
3. *there exists a function $A\colon \mathbb{R} \to \mathbb{R}$ with $A(h) \to 0$ when $h \to 0$ such that the first three integer moments of the random variables Z_n defined as*

$$Z_n \coloneqq \frac{1}{A(h)}(H_n - h)$$

satisfy

$$\mathbb{E}[Z_n] = 0, \quad \mathbb{E}[Z_n^2] = h, \quad \mathbb{E}[Z_n^3] = 0.$$

Integrating with variable time step H_n gives the discrete stochastic process $\{Y_n\}_{n\geq 0}$ defined as

$$Y_{n+1} = Y_n + H_n f(Y_n). \tag{1}$$

We can write equivalently Y_n as

$$\begin{aligned} Y_{n+1} &= Y_n + H_n f(Y_n) \\ &= Y_n + hf(Y_n) + (H_n - h)f(Y_n) \\ &= Y_n + hf(Y_n) + A(h)f(Y_n)Z_n, \end{aligned}$$

where Z_n are the r.v. of Assumption 1.1. Let us consider $\tilde{Y}$ the solution of the SDE

$$\mathrm{d}\tilde{Y} = f(\tilde{Y})\mathrm{d}t + A(h)f(\tilde{Y})\mathrm{d}W(t), \quad \tilde{Y}(0) = y_0, \tag{2}$$

where $W(t)$ for $t > 0$ is a standard one-dimensional Wiener process. Then, the following result of local weak convergence holds.

Proposition 1.1. *If the sequence $\{H_n\}_{n\geq 0}$ satisfies Assumption 1.1, then for any smooth function φ and $n = 1, 2, \ldots$, the numerical solution Y_n given by* (1) *and the solution $\tilde{Y}$ of* (2) *satisfy*

$$\left|\mathbb{E}[\varphi(Y_n) - \varphi(\tilde{Y}(nh))]\right| \leq Ch, \tag{3}$$

where C is a positive real constant.

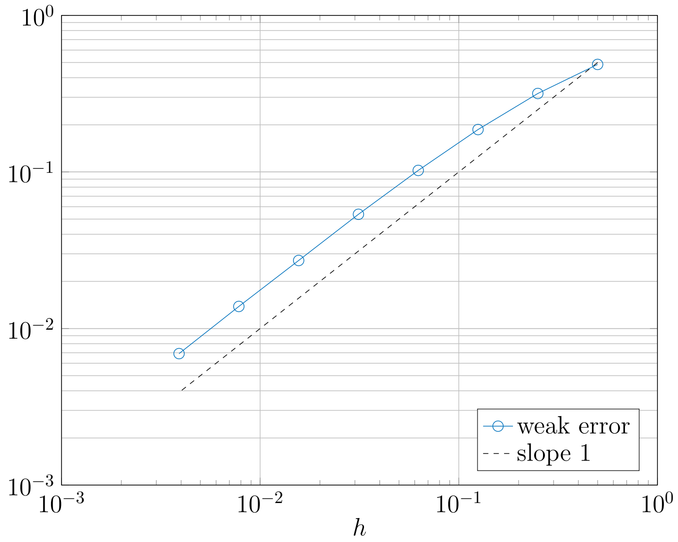

Figure 1: Weak order

Example 1.1. Given $h > 0$, let us consider the random variables H_n to be drawn from an uniform distribution

$$H_n \overset{\text{i.i.d.}}{\sim} \mathcal{U}(0, 2h).$$

Then, points 1. and 2. of Assumption 1.1 are trivially verified. Point 3. is verified considering

$$A(h) = \sqrt{\frac{h}{3}},$$

as in this case the random variables Z_n are distributed following

$$Z_n \overset{\text{i.i.d.}}{\sim} \mathcal{U}(-\sqrt{3h}, \sqrt{3h}),$$

and therefore the odd moments are trivially equal to zero and

$$\mathbb{E}[Z_n^2] = \frac{1}{12}(2\sqrt{3h})^2 = h.$$

1.1 Numerical examples

1.1.1 Weak order

In this first example we consider the one-dimensional ODE

$$f'(y) = \lambda y, \quad y(0) = 1.$$

with $\lambda = 1$, and final time $T = 1$. Let us remark that in practice we adapt the last time step so that the numerical solution is computed exactly up to time T. We consider the random variables H_n and the function $A(h)$ given in example 1.1. The SDE (2) then reads

$$\tilde{Y} = \lambda \tilde{Y} \mathrm{d}t + \sqrt{\frac{h}{3}} \lambda \tilde{Y} \mathrm{d}W(t),$$

and therefore the exact solution is given by

$$\tilde{Y}(t) = \exp\left(\left(\lambda - \frac{h}{6}\lambda^2\right) t + \sqrt{\frac{h}{3}}\lambda W(t)\right).$$

We consider φ in (3) to be the identity function and compute the weak order for $h = 0.5 \cdot 2^{-i}$, with $i = 0, \ldots, 7$ with $M = 10^4$ trajectories of the numerical solution and 10^6 realizations of the exact solution $\tilde{Y}(T)$. Results (Figure 1) show that the weak order of convergence one seems to be verified.

1.1.2 Distribution of the solution

We consider the Lorenz system

$$\begin{aligned} x' &= \sigma(y - x), & x(0) &= -10, \\ y' &= x(\rho - z) - y, & y(0) &= -1, \\ z' &= xy - \beta z, & z(0) &= 40. \end{aligned}$$

with $\sigma = 10$, $\rho = 28$, $\beta = 8/3$, so that the system has chaotic behavior. Moreover, we consider final time $T = 100$ and the function $\varphi(x) = x^T x$. We consider $h = 0.01$ and $5 \cdot 10^3$ trajectories of the following numerical solutions

- ODE solver with random solver, time steps H_n distributed as in example 1.1,
- SDE (2) with Euler-Maruyama and fixed time step h,
- probabilistic ODE solver [1] with fixed time step h.

Thus, we consider the value of φ applied to the numerical solution at final time T. Empirical results (Figure 2) show that the distribution of $\varphi(\tilde{Y})$ and of Y_N present the same behavior for the three methods above.

References

[1] P. R. Conrad, M. Girolami, S. Särkkä, A. Stuart, and K. Zygalakis, *Statistical analysis of differential equations: introducing probability measures on numerical solutions*, Stat. Comput., (2016).

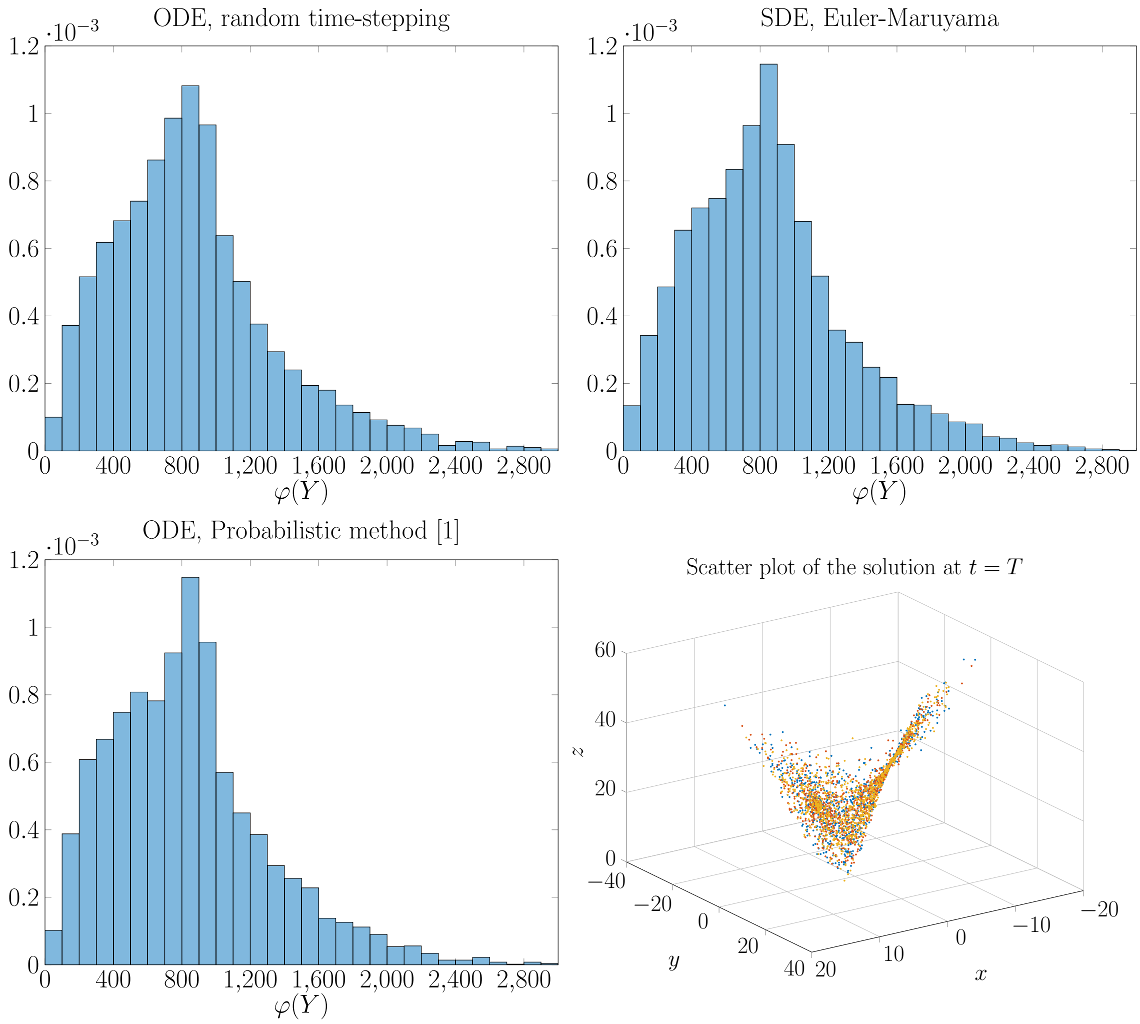

Figure 2: Empirical distribution of $\varphi(Y)$ for Y alternatively the solution of the ODE with stochastic time stepping, the SDE (2) and the probabilistic solver. Scatter plot of the solution at final time computed with the three methods (blue = ODE, random time stepping; red = SDE (2) with Euler-Maruyama; yellow = Probabilistic solver [1]).