REPORT – A POSTERIORI ERROR ESTIMATOR FOR ODE

ASSYR ABDULLE* AND GIACOMO GAREGNANI†

1. Theory. Consider $f\colon \mathbb{R}^d \to \mathbb{R}^d$ and the ODE

$$\begin{cases} y'(t) = f(y(t)), & t \in (0,T], \\ y(0) = y_0, \end{cases} \tag{1}$$

whose flow is denoted by $y(t) = \varphi_t(y_0)$. Consider $N > 0$, $h = T/N$ and the additive noise numerical method

$$Y_{k+1} = \Psi_h(Y_k) + \xi_k(h), \quad k = 0, \dots, N, \tag{2}$$

where $\xi_k(h)$ satisfies

$$\mathbb{E}\,\xi_k(h) = 0, \tag{3}$$

$$\mathbb{E}\,\xi_k(h)\xi_k(h)^\top = Qh^{2p+1}, \quad \forall k = 1, \dots, N, \tag{4}$$

for some symmetric positive definite matrix Q and exponent $p \geq 1$.

THEOREM 1.1 (Mean square order). *Report the result with precise constants.*

Proof. From the proof of [**?**, Theorem 2.2], denoting the exact solution by $y_k = y(kh)$, the global error by $e_k = Y_k - y_k$ and the local error by $\varepsilon_k = \Psi_h(Y_k) - \varphi_h(Y_k)$, we have

$$\mathbb{E}|e_{k+1}|^2 \leq \mathbb{E}|\varphi_h(y_k) - \varphi_h(y_k - e_k) - \varepsilon_k|^2 + |Q|h^{2p+1}.$$

Hence, developing the square in the first term of the right hand side, considering that φ_h is Lipschitz with constant $1 + Lh$ and that $\mathbb{E}|\varepsilon_k|^2 \leq C_{\mathrm{loc}}^2 h^{2q+2}$, we have

$$\mathbb{E}|e_{k+1}|^2 \leq (1+Lh)^2\,\mathbb{E}|e_k|^2 + \mathbb{E}\big|\big(h^{1/2}(\varphi_h(y_k) - \varphi_h(y_k - e_k)), h^{-1/2}\varepsilon_k\big)\big| + C_{\mathrm{loc}}^2 h^{2q+2} + |Q|h^{2p+1}.$$

Applying Cauchy-Schwarz and Young inequalities on the inner product we get

$$\mathbb{E}|e_{k+1}|^2 \leq (1+h)(1+Lh)^2\,\mathbb{E}|e_k|^2 + (1+h)C_{\mathrm{loc}}^2 h^{2q+1} + |Q|h^{2p+1}.$$

Setting $p = q$, we rewrite this bound as

$$\mathbb{E}|e_{k+1}|^2 \leq \Big(1 + \big(1 + 2L + (L^2+2L)h + L^2h^2\big)h\Big)\,\mathbb{E}|e_k|^2 + \big((1+h)C_{\mathrm{loc}}^2 + |Q|\big)h^{2q+1}.$$

Denoting by $C_{\mathrm{err}} = C_{\mathrm{err}}(L,h) = 1 + 2L + (L^2+2L)h + L^2h^2$, we get by Gronwall's inequality

$$\mathbb{E}|e_k|^2 \leq \frac{(1+h)C_{\mathrm{loc}}^2 + |Q|}{C_{\mathrm{err}}}\Big(e^{C_{\mathrm{err}}T} - 1\Big)h^{2q},$$

which proves the desired result. ∎

THEOREM 1.2 (Variance). *The bound for the variance.*

Proof. Thanks to independence of Y_k and ξ_k we have

$$\operatorname{Var} Y_{k+1} = \operatorname{Var}\Psi_h(Y_k) + |Q|h^{2q+1}.$$

The Lipschitz constant of Ψ_h being $1 + L_\Psi h$, we have

$$\operatorname{Var} Y_{k+1} \leq (1 + L_\Psi h)^2 \operatorname{Var} Y_k + |Q|h^{2q+1}.$$

*Institute of Mathematics, École Polytechnique Fédérale de Lausanne (assyr.abdulle@epfl.ch)
†Institute of Mathematics, École Polytechnique Fédérale de Lausanne (giacomo.garegnani@epfl.ch)

Hence, denoting by $C_{\text{var}} = C_{\text{var}}(L, h) = 1 + (2L_\Psi + L_\Psi^2 h)h$, we obtain by Gronwall's inequality

$$\operatorname{Var} Y_k \leq \frac{|Q|}{C_{\text{var}}}\Big(e^{C_{\text{var}} T} - 1\Big) h^{2q}. \tag{5}$$

□

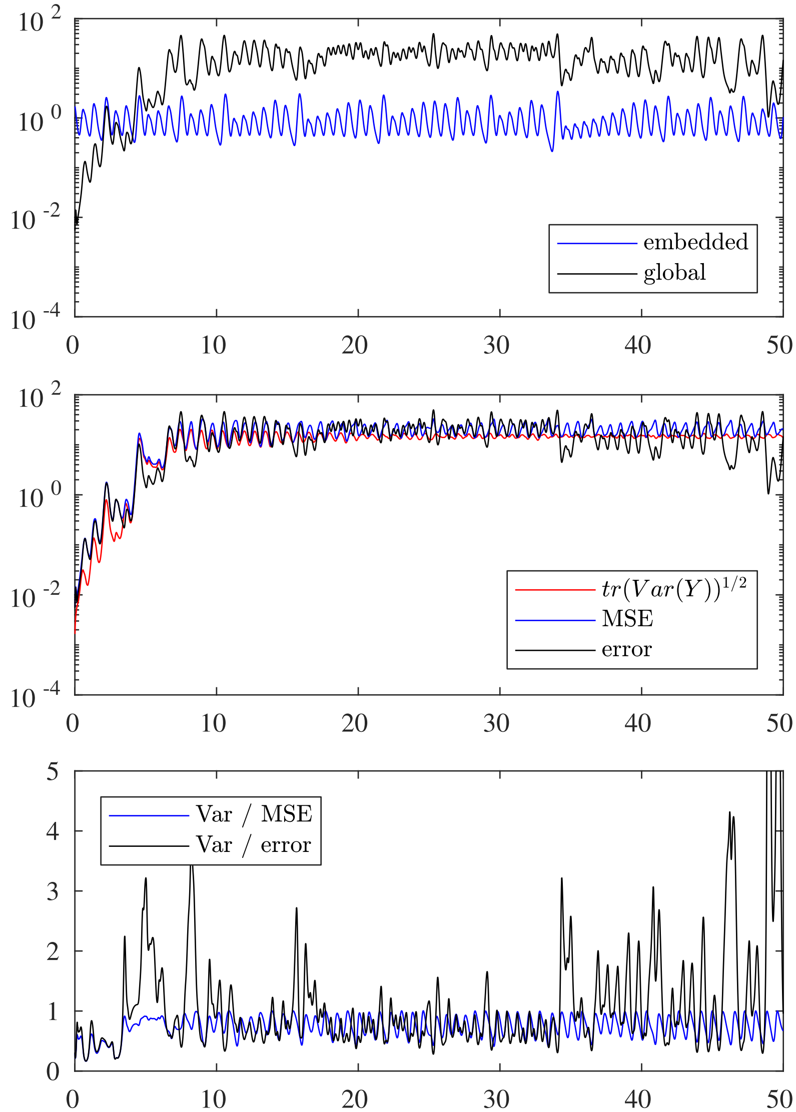

Fig. 1: Lorenz system with tuning of Q. Here $h = 0.01$, $T = 50$, Heun as deterministic integrator.

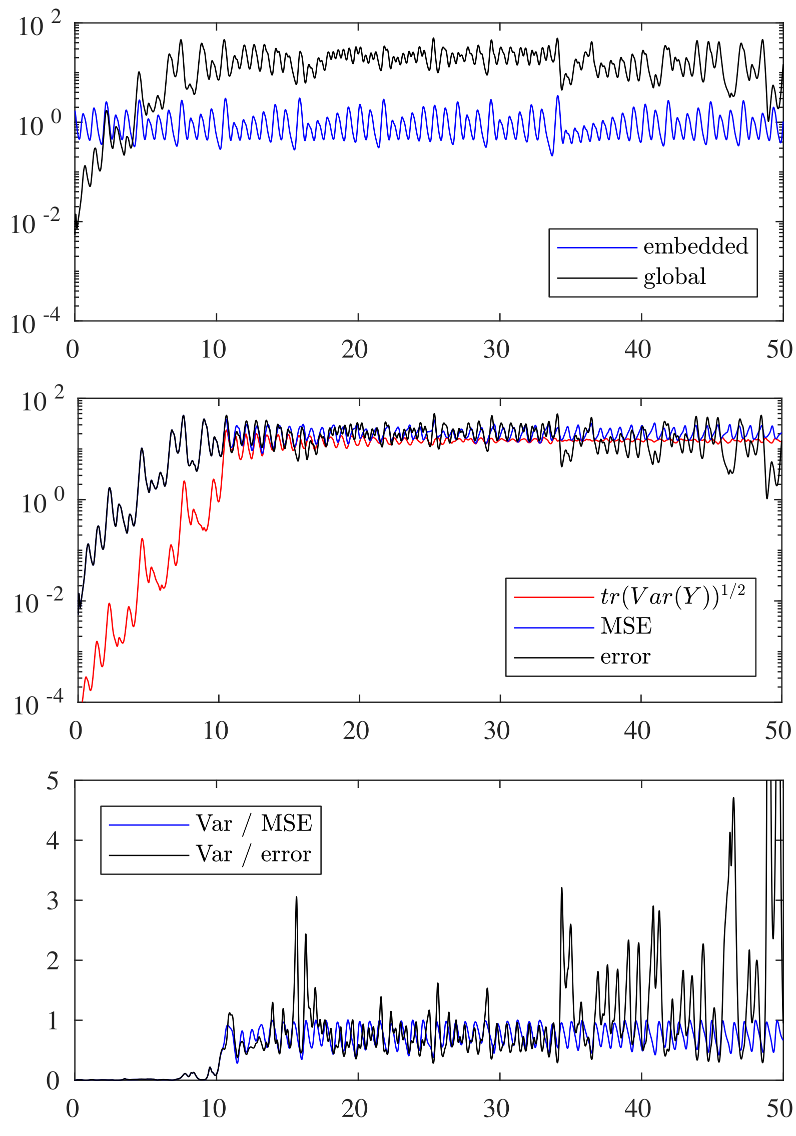

Fig. 2: Lorenz system without tuning of Q. Here $h = 0.01$, $T = 50$, Heun as deterministic integrator.

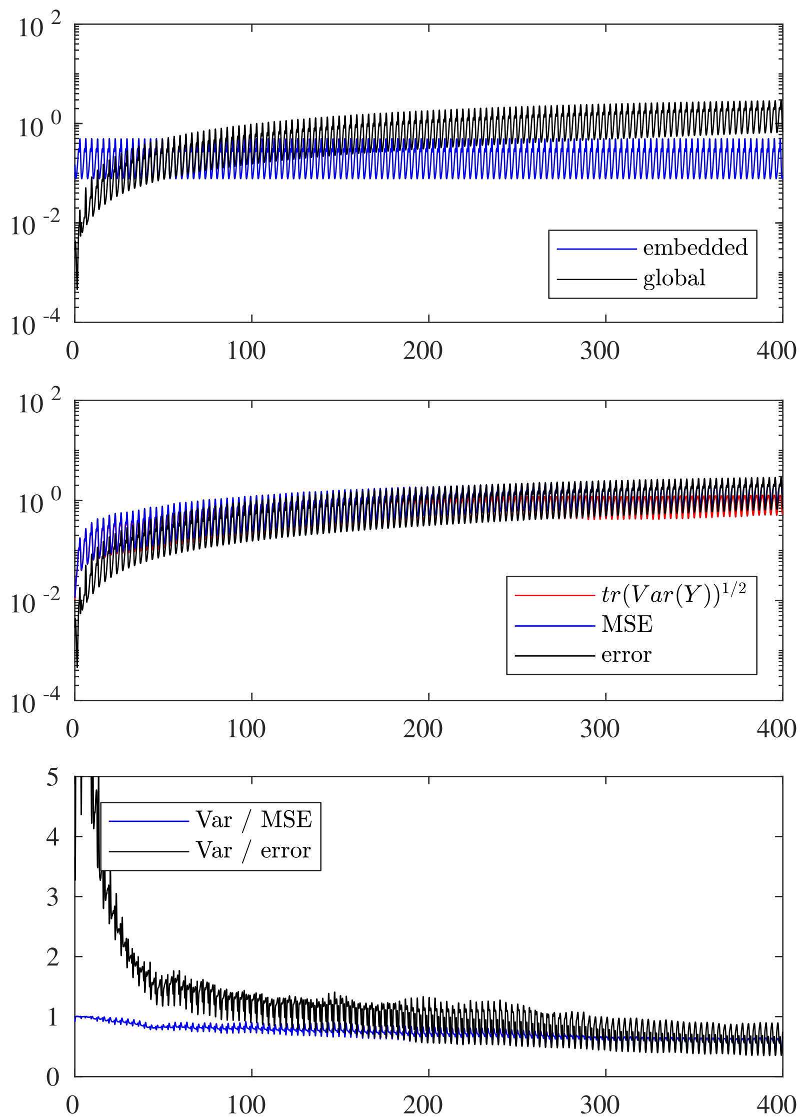

Fig. 3: Van der Pol system ($\varepsilon = 1$) with tuning of Q. Here $h = 0.1$, $T = 400$, Heun as deterministic integrator.

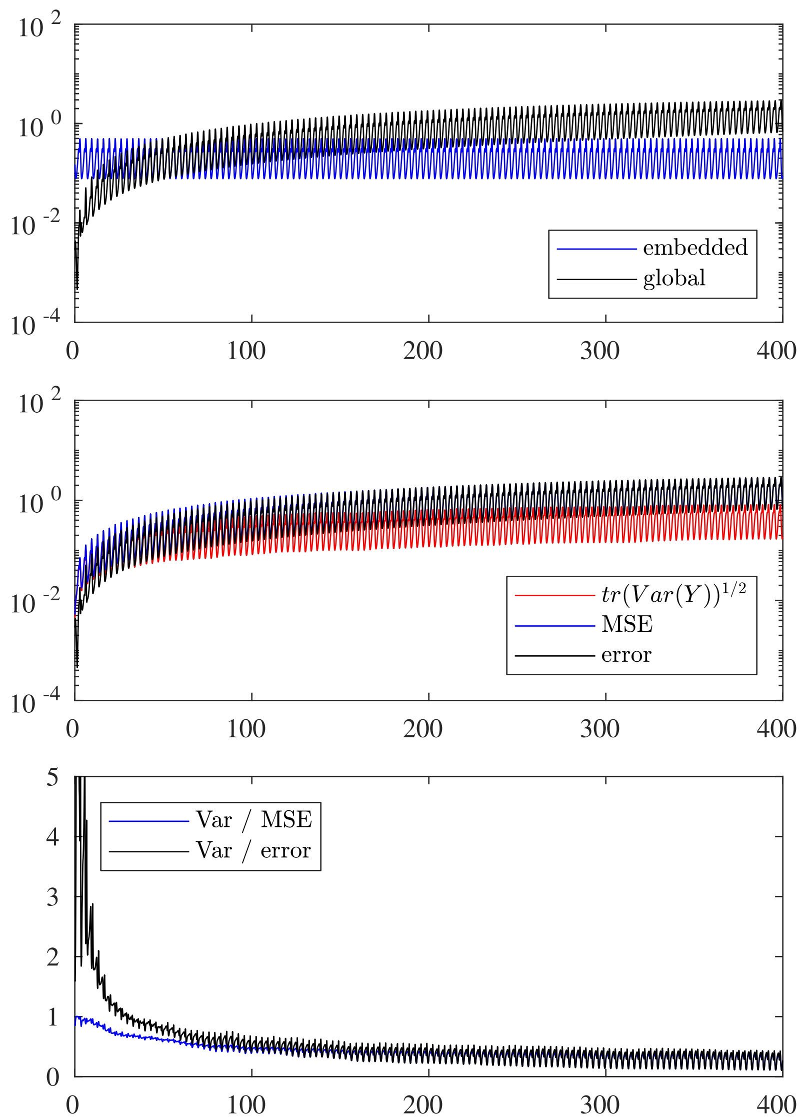

Fig. 4: Van der Pol system ($\varepsilon = 1$) without tuning of Q. Here $h = 0.1$, $T = 400$, Heun as deterministic integrator.