Introduction to Particle MCMC methods

Giacomo Garegnani

Hidden Markov Models

Let us consider a parameter $\vartheta \in \mathbb{R}^p$ and the discrete dynamical system over $\mathbb{R}^d$ given by the transition density

$$X_{n+1} \sim f(\cdot \mid X_n, \vartheta), \tag{1}$$

which we assume defines a Markov Chain over $\mathbb{R}^d$, i.e.,

$$p(X_{n+1} \mid X_n, X_{n-1}, \dots, X_0, \vartheta) = p(X_{n+1} \mid X_n, \vartheta) = f(X_{n+1} \mid X_n, \vartheta).$$

For example, given a stochastic differential equations (SDE), the recursion given by a numerical integrator can be cast in the form (1). For example, if we have the SDE

$$\mathrm{d}X_t = F(X_t)\mathrm{d}t + G(X)\mathrm{d}W_t,$$

where $F\colon \mathbb{R}^d \to \mathbb{R}^d$ and $G\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are the drift and diffusion and W_t is a m-dimensional standard Brownian motion, the Euler–Maruyama method (EM) can be written as

$$X_{n+1} = X_n + F(X_n)h + G(X_n)\Delta_n W,$$

where $h > 0$ is the discretization step size and $\Delta_n W = W(t_{n+1}) - W(t_n)$. In this case, we have (1) with

$$f(\cdot \mid X_n, \vartheta) = \mathcal{N}\left(X_n + F(X_n)h, GG^\top(X_n)h\right).$$

Finally, we assume that we are not able to observe the dynamics (1) directly, but only a noisy observation model of the form

$$Y_n \sim g(X_n \mid \vartheta)$$

where $\{\xi_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables, that we assume to be Gaussian distributed $\xi_n \sim \mathcal{N}(0, \Sigma)$. The method can be extended to the case $Y_n = \gamma(X_n) + \xi_n$ for a function $\gamma\colon \mathbb{R}^d \to \mathbb{R}^{d'}$, with $d' \leq d$, in case only a functional of the state X_n is observable. For simplicity, we consider here the case $\gamma = \mathrm{Id}$.

Bootstrap particle filters

Advantages with respect to Monte Carlo

Importance sampling

Particle filters