REPORT – ANALYTICAL POSTERIORS IN A SIMPLE CASE

Let us consider as a toy problem the one dimensional ODE

$$y' = -y, \quad y(0) = y_0.$$

We consider the inferential problem of determining the true initial condition $y_0^* = 1$ from the observations. We consider a single observation $d = y^{y_0^*}(1/2) + \eta$, where $y^{y_0^*}(1/2) = e^{-1/2}$ is the true solution at time $t = 1/2$ and $\eta \sim \mathcal{N}(0, \sigma^2)$ is a source of noise. If a Gaussian prior $\pi_0 = \mathcal{N}(0, 1)$ is given for y_0, the posterior distribution is computable analytically and is given by

$$\pi(y_0 \mid d) = \mathcal{N}\Big(y_0; \frac{de^{-1/2}}{\sigma^2 + e^{-1}}, \frac{\sigma^2}{\sigma^2 + e^{-1}}\Big),$$

where $\mathcal{N}(x; \mu, \alpha^2)$ is the density of a Gaussian random variable of mean μ and variance α^2 evaluated in x. Consistently, if $\sigma^2 \to 0$, we have that $d \to e^{-1/2}$ and therefore $\pi(y_0 \mid d) \to \delta_1 = \delta_{y_0^*}$.

If we approximate $y^{y_0}(1/2)$ for a given initial condition y_0 with a single step of the explicit Euler method (i.e., $h = 1/2$), we get $y^{y_0}(1/2) \approx y_0/2$. Computing the posterior distribution obtained with this approximation leads to

$$\pi_{\mathrm{EE}}(y_0 \mid d) = \mathcal{N}\Big(y_0; \frac{d}{2(\sigma^2 + 1/4)}, \frac{\sigma^2}{\sigma^2 + 1/4}\Big).$$

In the limit of $\sigma^2 \to 0$, we get in this case that the posterior distribution tends to $\pi_{\mathrm{EE}}(y_0 \mid d) \to \delta_{2e^{-1/2}} \approx \delta_{1.213}$. The posterior distribution is hence tending to a biased Dirac delta with respect to the true value.

Let us consider the additive noise explicit Euler (AN-EE), i.e., the approximation $y^{y_0}(1/2) \approx Y_1$, where $Y_1 = y_0/2 + \xi$ and ξ is a random variable $\mathcal{N}(0, h^3) = \mathcal{N}(0, 1/8)$, so that the method converges consistently with the deterministic method. In this case, the posterior distribution is given by

$$\pi_{\mathrm{AN-EE}}(y_0 \mid d) = \mathcal{N}\Big(y_0; \frac{d}{2(\tilde\sigma^2 + 1/4)}, \frac{\tilde\sigma^2}{\tilde\sigma^2 + 1/4}\Big),$$

where $\tilde\sigma^2 = \sigma^2 + h^3 = \sigma^2 + 1/8$. In this case, taking the limit $\sigma^2 \to 0$ leads to

$$\pi_{\mathrm{AN-EE}}(y_0 \mid d) \to \mathcal{N}\Big(y_0; \frac{e^{-1/2}}{2(h^3 + 1/4)}, \frac{h^3}{h^3 + 1/4}\Big),$$

which shows that while the asymptotic mean is still biased with respect to the true value (≈ 0.8087), the uncertainty in the forward model is reflected by a positive variance ($= 1/3$).

Let us now consider the random time step explicit Euler (RTS-EE) with step size distribution $H \sim \mathcal{U}(h - h^p, h + h^p)$. In this case, the forward model acts as

$$Y_1 = y_0 - Hy_0 = y_0 - hy_0 + (h - H)y_0 = \frac{y_0}{2} + Uy_0, \quad U \sim \mathcal{U}(-h^p, h^p).$$

The posterior distribution over y_0 can be computed as

$$\begin{aligned}\pi_{\mathrm{RTS-EE}}(y_0 \mid d) &\propto \pi_0(y_0)\, \mathbb{E}^U\, \pi(d \mid y_0)\\ &\propto \exp\Big(-\frac{y_0^2}{2}\Big)\, \mathbb{E}^U \exp\Big(-\frac{(d - y_0/2 - Uy_0)^2}{2\sigma^2}\Big).\end{aligned}$$

Let us compute the likelihood term. With a change of variable $z = Uy_0$ we obtain

$$\mathbb{E}^U\, \pi(d \mid y_0) = \frac{1}{2h^p y_0} \int_{y_0 h^p}^{y_0 h^p} \exp\Big(-\frac{(d - y_0/2 - z)^2}{2\sigma^2}\Big) \mathrm{d}z.$$

Now a change of variable $w = (z - (d - y_0/2))/\sigma$ gives

$$\mathbb{E}^U\,\pi(d \mid y_0) = \frac{\sigma}{2h^p y_0} \int_{(-y_0 h^p - (d - y_0/2))/\sigma}^{(y_0 h^p - (d - y_0/2))/\sigma} \exp\Big(-\frac{w^2}{2}\Big)\mathrm{d}z,$$

Hence the likelihood can be expressed in terms of the cumulative distribution function Φ of a standard Gaussian random variable, i.e.,

$$\mathbb{E}^U\,\pi(d \mid y_0) = \frac{\sigma\sqrt{2\pi}}{2h^p y_0}\Big(\Phi\Big(\frac{y_0 h^p - (d - y_0/2)}{\sigma}\Big) - \Phi\Big(\frac{-y_0 h^p - (d - y_0/2)}{\sigma}\Big)\Big).$$

Disregarding all multiplicative constant that are independent of y_0, we get the posterior

$$\pi_{\mathrm{RTS-EE}}(y_0 \mid d) \propto \exp\Big(-\frac{y_0^2}{2}\Big)\frac{1}{y_0}\Big(\Phi\Big(\frac{y_0 h^p - (d - y_0/2)}{\sigma}\Big) - \Phi\Big(\frac{-y_0 h^p - (d - y_0/2)}{\sigma}\Big)\Big).$$

The natural choice of p is $p = q + 1/2 = 3/2$, and being $h = 1/2$ we get

$$\pi_{\mathrm{RTS-EE}}(y_0 \mid d) \propto \exp\Big(-\frac{y_0^2}{2}\Big)\frac{1}{y_0}\Big(\Phi\Big(\frac{\frac{y_0}{2}(1 + \frac{1}{\sqrt{2}}) - d}{\sigma}\Big) - \Phi\Big(\frac{\frac{y_0}{2}(1 - \frac{1}{\sqrt{2}}) - d}{\sigma}\Big)\Big).$$

In the limit for $\sigma \to 0$, the difference between the cumulative distribution functions of tends to 1 or to 0 depending on the sign of the argument. In practice, the limiting distribution is

$$\pi_{\mathrm{RTS-EE}}(y_0 \mid d) \propto \exp\Big(-\frac{y_0^2}{2}\Big)\frac{1}{y_0}\chi_{\{y_0 > y_{\mathrm{lim}}\}},$$

where y_{lim} is given by

$$y_{\mathrm{lim}} = \frac{2\sqrt{2}d}{\sqrt{2}+1}.$$

The analytical posterior distributions for four values of $\sigma = \{0.1, 0.05, 0.025, 0.0125\}$ are plotted in Figure 1 and 2.

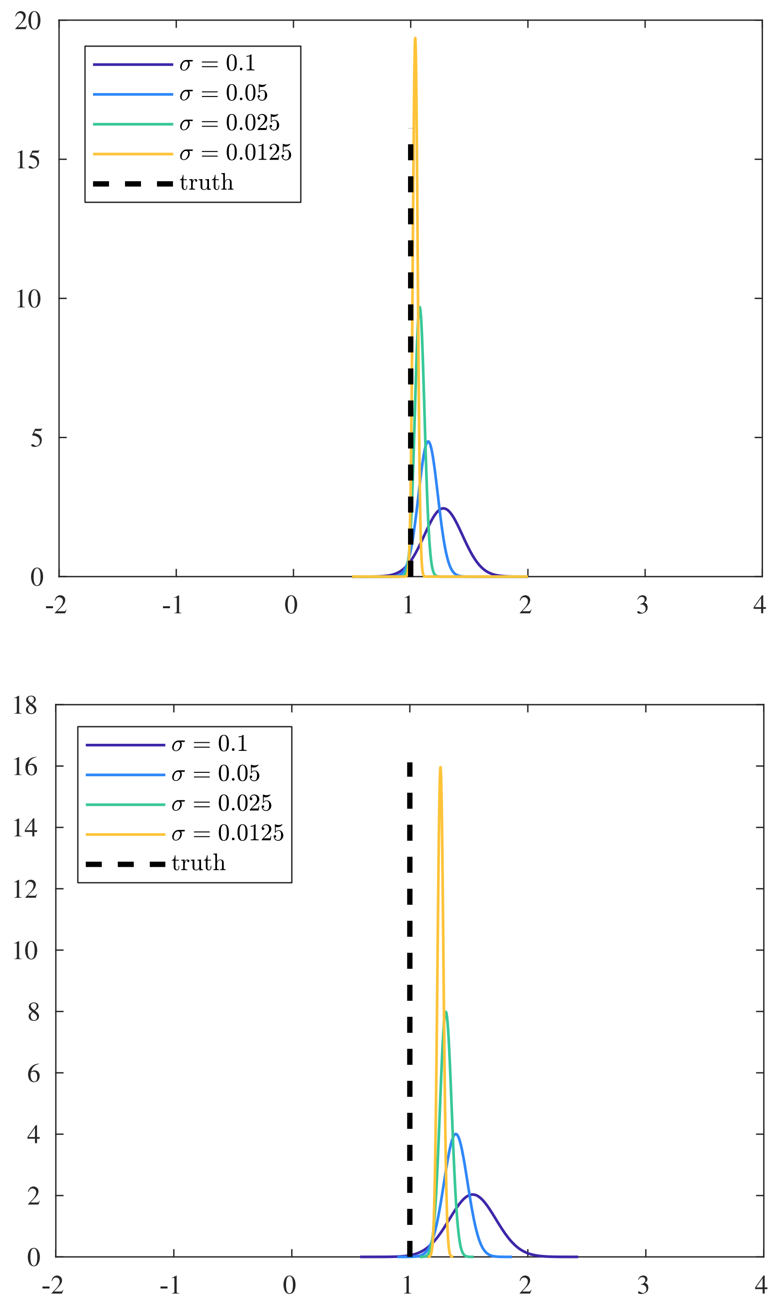

Fig. 1: Exact posterior distribution and explicit Euler posterior.

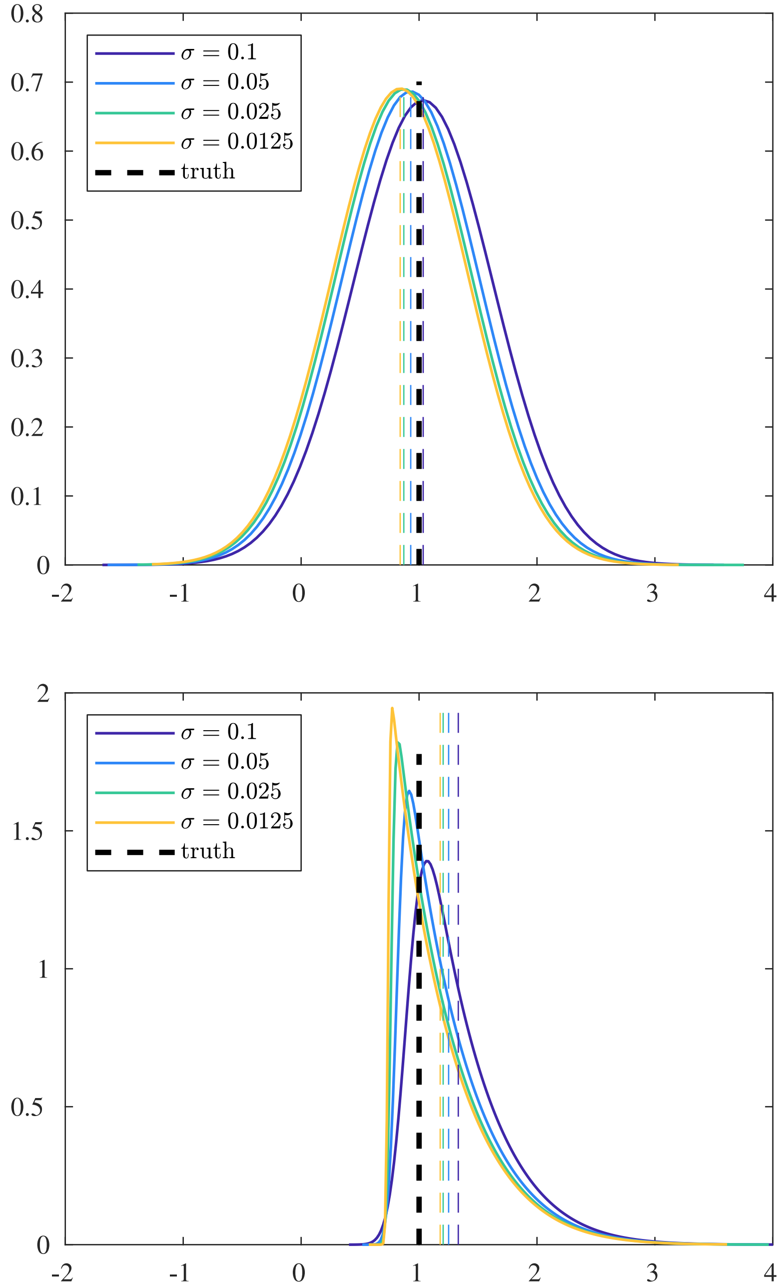

Fig. 2: Posterior distributions for the AN-EE (top) and the RTS-EE (bottom) methods. The mean of the posterior distribution is represented by vertical dashed lines for the different values of σ.