Caltech notes

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Abstract

AMS subject classifications.

Keywords.

1 Introduction

Let $\varepsilon > 0$ and let us consider the one-dimensional multiscale stochastic differential equation (SDE)

$$\mathrm{d}X_t^\varepsilon = -\alpha V_0'(X_t^\varepsilon)\,\mathrm{d}t - \frac{1}{\varepsilon} V_1'\left(\frac{X_t^\varepsilon}{\varepsilon}\right) + \sqrt{2\sigma}\,\mathrm{d}W_t, \tag{1.1}$$

where the drift coefficient α and the diffusion coefficient σ are positive real parameters, possibly unknown, and W_t is a standard one-dimensional Brownian motion. The functions $V_0, V_1 \colon \mathbb{R} \to \mathbb{R}$ are slow and fast potentials driving the dynamics of the solution X_t^ε. In particular, we assume V_1 to be smooth and periodic of period L. Theory of homogenization [**?**] guarantees the existence of an SDE of the form

$$\mathrm{d}X_t^0 = -AV_0'(X)\,\mathrm{d}t + \sqrt{2\Sigma}\,\mathrm{d}W_t, \tag{1.2}$$

where W_t is the same Brownian motion and where the fast dynamics have been eliminated, such that $X_t^\varepsilon \to X_t^0$ for $\varepsilon \to 0$ in law as random variables in $\mathcal{C}^0((0,T),\mathbb{R})$. The drift and diffusion coefficients of the homogenized dynamics A and Σ are given by $A = K\alpha$ and $\Sigma = K\sigma$, where

$$K = \int_0^L (1 + \Phi'(y))^2\,\mu(\mathrm{d}y), \tag{1.3}$$

with

$$\mu(\mathrm{d}y) = \frac{1}{Z}\exp\left(-\frac{V_1'(y)}{\sigma}\right)\mathrm{d}y, \quad Z = \int_0^L \exp\left(-\frac{V_1'(y)}{\sigma}\right)\mathrm{d}y, \tag{1.4}$$

and Φ is the solution of the elliptic partial differential equation

$$-V'(y)\Phi'(y) + \sigma\Phi''(y) = V''(y), \quad 0 \le y \le L, \tag{1.5}$$

endowed with periodic boundary coefficients.

In order to estimate the drift coefficient, one considers the likelihood function

$$L_T(X_t) = \exp\left\{\int_0^T -AV_0'(X_t)\,\mathrm{d}X_t - \frac{1}{2}\int_0^T A^2 V_0'(X_t)^2\,\mathrm{d}t\right\}, \tag{1.6}$$

whose logarithm $\ell_T(X_t) = \log L_T(X_t)$ can be maximised thus giving the estimator

$$\widehat{A} = -\frac{\int_0^T V_0'(X_t)\,\mathrm{d}X_t}{\int_0^T V_0'(X_t)^2\,\mathrm{d}t}. \tag{1.7}$$

The diffusion coefficient can be computed as the quadratic variation of the path, i.e., given a sequence of partitions $\mathcal{P}_h = \{t_k\}_{k=0}^{N_h}$, of the interval $[0,T]$, where $h \coloneqq \sup_k(t_k - t_{k-1})$, we have

$$\Sigma = \frac{1}{2T} \lim_{h\to 0} \sum_{k=1}^{N_h} (X_{t_k}^0 - X_{t_{k-1}}^0)^2, \tag{1.8}$$

in probability and for all $T > 0$.

In a Bayesian setting, we can fix a prior Λ with density λ and the posterior is then given by

$$\mu_T(B) = \frac{\int_B L_T(A)\lambda(A)\,\mathrm{d}A}{\int_{\mathcal{A}} L_T(A)\lambda(A)\,\mathrm{d}A}. \tag{1.9}$$

2 Convergence analysis

Let us consider a Gaussian prior $\mu_0 = \mathcal{N}(A_0, \sigma_0^2)$. The posterior then reads

$$\begin{aligned}\mathbb{P}(A \mid X_t) &\propto \mathbb{P}(X_t \mid A)\ \mathbb{P}(A) \\ &\propto \exp\left\{ -A \int_0^T V_0'(X_t)\,\mathrm{d}X_t - \frac{A^2}{2}\int_0^T V_0'(X_t)^2\,\mathrm{d}t - \frac{(A-A_0)^2}{2\sigma_0^2} \right\}.\end{aligned} \tag{2.1}$$

The posterior is therefore clearly a Gaussian distribution $\mathbb{P}(A \mid X_t^\varepsilon) = \mathcal{N}(m, \sigma^2)$, whose mean and covariance are given by

$$\begin{aligned} m &= \frac{A_0}{1 + \sigma_0^2 \int_0^T V_0'(X_t)^2\,\mathrm{d}t} - \frac{\sigma_0^2 \int_0^T V_0'(X_t)\,\mathrm{d}X_t}{1 + \sigma_0^2 \int_0^T V_0'(X_t)^2\,\mathrm{d}t}, \\ \sigma^{-2} &= \sigma_0^{-2} + \int_0^T V_0'(X_t)^2\,\mathrm{d}t. \end{aligned} \tag{2.2}$$

In the asymptotic limit $\varepsilon \to 0$, and therefore $T \to \infty$, we clearly have that the posterior variance vanishes and that the posterior mean satisfies

$$\lim_{T\to\infty} m_T = \lim_{T\to\infty} \widehat{A}_T, \tag{2.3}$$

where $\widehat{A}_T$ is given in (1.7).

Let $Z_t^{\varepsilon,k}$ be defined as

$$Z_t^{\varepsilon,k} \coloneqq \int_0^t k^\varepsilon(t,s) X_s^\varepsilon \,\mathrm{d}s, \tag{2.4}$$

where $k^\varepsilon(t,s)$ is a kernel satisfying the following assumption.

Assumption 1. The kernel $k\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ satisfies either for all $T > 0$, $t, s \in (0,T)$ and $\alpha \geq 1/2$

(i) (regularity) $k^\varepsilon(t,s), \partial_t k^\varepsilon(t,s) \in \mathcal{C}^0(\mathbb{R}^+ \times \mathbb{R}^+; \mathbb{R})$

(ii) (divergence–free) $\partial_t k^\varepsilon(t,s) + \partial_s k^\varepsilon(t,s) = 0$,

(iii) (normalization) there exists a function $\psi(t,\varepsilon)$ satisfying for all $p \geq 0$ and for a constant $C > 0$ independent of ε

$$\lim_{t\to\infty,\varepsilon\to 0} \psi(t,\varepsilon) = 0, \qquad \int_0^T |\psi(t,\varepsilon)|^p \,\mathrm{d}t \leq C, \tag{2.5}$$

such that k^ε satisfies

$$\int_0^t (t-s)\,\partial_t k^\varepsilon(t,s)\,\mathrm{d}s = -1 + \psi(t,\varepsilon), \tag{2.6}$$

(iv) (vanishing condition) $\lim_{t\to\infty} k^\varepsilon(t,0) = 0$.

or it can be written as

$$k^\varepsilon(t,s) = \chi_{[t-\delta,t]}(s), \tag{2.7}$$

where $\chi_{[t',t]}$ is the indicator function of the interval $[t',t]$ for $t' < t$ and $\delta = \varepsilon^\zeta$ for $\zeta \in (0,1)$.

Example 1. Let $\delta = \varepsilon^\zeta$ for $\zeta \in (0,1)$, $\beta > 0$ and

$$k^\varepsilon(t,s) = C_\beta \delta^{-1/\beta} e^{-(t-s)^\beta/\delta}, \tag{2.8}$$

where C_β is a normalizing constant. It is possible to verify that for each β there exists C_β such that (i)–(iv) are all verified. For example, for $\beta = 1$ we have $C_1 = 1$, whereas for $\beta = 2$ we have $C_2 = 2/\sqrt{\pi}$.

Due to assumption ((i)) we can apply Leibniz's integral rule and obtain the following representation

$$\frac{\mathrm{d}Z_t^{\varepsilon,k}}{\mathrm{d}t} = k^\varepsilon(t,t)X_t^\varepsilon + \int_0^t \partial_t k^\varepsilon(t,s) X_s^\varepsilon \,\mathrm{d}s. \tag{2.9}$$

Adding and subtracting X_t^ε inside the integral yields

$$\frac{\mathrm{d}Z_t^{\varepsilon,k}}{\mathrm{d}t} = \int_0^t \partial_t k^\varepsilon(t,s)(X_s^\varepsilon - X_t^\varepsilon)\,\mathrm{d}s + \left(k^\varepsilon(t,t) + \int_0^t \partial_t k^\varepsilon(t,s)\,\mathrm{d}s\right) X_t^\varepsilon. \tag{2.10}$$

Due to ((ii)), we have

$$\begin{aligned} k^\varepsilon(t,t) + \int_0^t \partial_t k^\varepsilon(t,s)\,\mathrm{d}s &= k^\varepsilon(t,0) + \int_0^t \left(\partial_s k^\varepsilon(t,s) + \partial_t k^\varepsilon(t,s)\right)\,\mathrm{d}s \\ &= k^\varepsilon(t,0). \end{aligned} \tag{2.11}$$

Therefore,

$$\frac{\mathrm{d}Z_t^{\varepsilon,k}}{\mathrm{d}t} = \int_0^t \partial_t k^\varepsilon(t,s)(X_s^\varepsilon - X_t^\varepsilon)\,\mathrm{d}s + k^\varepsilon(t,0)X_t^\varepsilon. \tag{2.12}$$

Then

$$\begin{aligned} \frac{\mathrm{d}Z_t^{\varepsilon,k}}{\mathrm{d}t} =& \alpha \int_0^t \int_s^t \partial_t k^\varepsilon(t,s)\, V_0'(X_r^\varepsilon)\left(1 + \Phi'(Y_r^\varepsilon)\right)\mathrm{d}r\,\mathrm{d}s \\ &- \sqrt{2\sigma} \int_0^t \int_s^t \partial_t k^\varepsilon(t,s)\big(1 + \Phi'(Y_r^\varepsilon)\big)\,\mathrm{d}W_r\,\mathrm{d}s \\ &+ \varepsilon \int_0^t \partial_t k^\varepsilon(t,s)\big(\Phi(Y_t^\varepsilon) - \Phi(Y_s^\varepsilon)\big)\,\mathrm{d}s + k^\varepsilon(t,0)X_t^\varepsilon \end{aligned} \tag{2.13}$$

Lemma 2.1. *Lemma text*

$$\alpha \int_s^t V_0'(X_r^\varepsilon)\left(1 + \Phi'(Y_r^\varepsilon)\right)\mathrm{d}r = A(t-s)V_0'(Z_t^{\varepsilon,k}) + R(\varepsilon, t-s), \tag{2.14}$$

where for all $p \geq 1$ it holds

$$\left(\mathbb{E}^{\mu^\varepsilon} |R(\varepsilon,t)|^p\right)^{1/p} \leq C(\varepsilon^2 + \varepsilon t^{1/2} + \varepsilon t + t^{3/2}), \tag{2.15}$$

3 Point estimates from continuous data

In this section, we study the convergence with respect to the parameter ε of point estimates of the drift and the diffusion coefficients when the estimator is computed employing continuous data coming from the multiscale model.

3.1 Drift coefficient

Let $X^\varepsilon := (X_t^\varepsilon, 0 \le t \le T)$ be the solution of (1.1) and define $\mathcal{H}_\Delta(X^\varepsilon)$ as

$$\mathcal{H}_\Delta(X^\varepsilon)_t := \begin{cases} X_0, & t = 0, \\ \frac{1}{t}\int_0^t X_s \,\mathrm{d}s, & 0 < t < \Delta, \\ \frac{1}{\Delta}\int_{t-\Delta}^t X_s \,\mathrm{d}s, & \Delta \le t \le T, \end{cases} \tag{3.1}$$

with $\Delta > 0$. Let us denote for ease of notation, $Z_t^\varepsilon := \mathcal{H}_\Delta(X^\varepsilon)_t$. The maximum likelihood estimator of the drift coefficient is then

$$\widehat{A}_{T,\Delta}(Z_t^\varepsilon) = -\frac{\int_0^T V_0'(Z_t^\varepsilon)\,\mathrm{d}Z_t^\varepsilon}{\int_0^T V_0'(Z_t^\varepsilon)^2\,\mathrm{d}t}. \tag{3.2}$$

Let us remark that for $0 < t < \Delta$,

$$\mathrm{d}(tZ_t^\varepsilon) = X_t\,\mathrm{d}t, \tag{3.3}$$

which implies

$$\mathrm{d}Z_t^\varepsilon = \frac{1}{t}(X_t^\varepsilon - Z_t^\varepsilon)\,\mathrm{d}t. \tag{3.4}$$

For $\Delta \le t \le T$, instead

$$\mathrm{d}Z_t^\varepsilon = \frac{1}{\Delta}(X_t^\varepsilon - X_{t-\Delta}^\varepsilon)\,\mathrm{d}t. \tag{3.5}$$

We rewrite the estimator as

$$\widehat{A}_{T,\Delta}(Z_t^\varepsilon) = -\frac{\int_0^\Delta V_0'(Z_t^\varepsilon)\frac{1}{t}(X_t^\varepsilon - Z_t^\varepsilon)\,\mathrm{d}t}{\int_0^T V_0'(Z_t^\varepsilon)^2\,\mathrm{d}t} - \frac{\int_\Delta^T V_0'(Z_t^\varepsilon)(X_t^\varepsilon - X_{t-\Delta}^\varepsilon)\,\mathrm{d}t}{\Delta\int_0^T V_0'(Z_t^\varepsilon)^2\,\mathrm{d}t}. \tag{3.6}$$

The goal of this section is proving the following result.

Theorem 3.1. *Under assumption add assumptions, if there exists $\zeta \in (0,1)$ such that $\Delta = \varepsilon^\zeta$ and $\gamma > \zeta$ such that $T = \varepsilon^{-\gamma}$, it holds*

$$\lim_{\varepsilon\to 0} \widehat{A}_{T,\Delta}(Z_t^\varepsilon) = A, \quad \textit{in law.} \tag{3.7}$$

It is useful in the following to rewrite (1.1) as a system of two coupled SDEs. In particular, introducing the variable $Y_t^\varepsilon := X_t^\varepsilon/\varepsilon$, one has

$$\begin{aligned} \mathrm{d}X_t^\varepsilon &= -\alpha V_0'(X_t^\varepsilon)\,\mathrm{d}t - \frac{1}{\varepsilon}V_1'(Y_t^\varepsilon) + \sqrt{2\sigma}\,\mathrm{d}W_t, \\ \mathrm{d}Y_t^\varepsilon &= -\frac{\alpha}{\varepsilon}V_0'(X_t^\varepsilon)\,\mathrm{d}t - \frac{1}{\varepsilon^2}V_1'(Y_t^\varepsilon) + \sqrt{\frac{2\sigma}{\varepsilon^2}}\,\mathrm{d}W_t. \end{aligned} \tag{3.8}$$

The analysis necessary to prove Theorem 3.1 is based on the expansion

$$\begin{aligned} X_t^\varepsilon - X_{t-\Delta}^\varepsilon = &-\alpha\int_{t-\Delta}^t V_0'(X_s^\varepsilon)\big(1 + \Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}s \\ &+ \sqrt{2\sigma}\int_{t-\Delta}^t \big(1 + \Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}W_s \\ &- \varepsilon\big(\Phi(Y_t^\varepsilon) - \Phi(Y_{t-\Delta}^\varepsilon)\big), \end{aligned} \tag{3.9}$$

for $t \ge \Delta$ (see [**?**, Equation (5.8)]). The following lemma ensures that the process Z_t^ε has bounded moments.

Lemma 3.2. *The process Z_t^ε has bounded moments of all order, i.e., for all $p \ge 1$ and $t \ge 0$ it holds*

$$\mathbb{E}^{\mu^\varepsilon}|Z_t^\varepsilon|^p \le C, \tag{3.10}$$

for $C > 0$ a constant uniform in $\varepsilon \to 0$.

Proof. The process X_t^ε has bounded moments (see [?, Corollary 5.4]), which implies the desired result with an application of the Hölder inequality. In fact, for $0 < t < \Delta$,

$$\begin{aligned} \mathbb{E}^{\mu^\varepsilon} \left|Z_t^\varepsilon\right|^p &\leq \frac{t^{p-1}}{t^p} \int_0^t \mathbb{E}^{\mu^\varepsilon} \left|X_s^\varepsilon\right|^p \,\mathrm{d}s \\ &\leq t^{-1} \int_0^t C \,\mathrm{d}s = C. \end{aligned} \tag{3.11}$$

For $\Delta \leq t \leq T$ the procedure is analogue. □

In the following lemma the difference between the processes X_t^ε and Z_t^ε is bounded.

Lemma 3.3. *Under assumptions add assumptions*

$$\mathbb{E}^{\mu^\varepsilon} \left|X_t^\varepsilon - Z_t^\varepsilon\right|^p \leq C(\Delta^p + \Delta^{p/2} + \varepsilon^p), \tag{3.12}$$

where $C > 0$ is a constant independent of Δ and ε.

Proof. By definition of Z_t^ε for $\Delta \leq t \leq T$ and applying Hölder's inequality we have

$$\begin{aligned} \mathbb{E}^{\mu^\varepsilon} \left|X_t^\varepsilon - Z_t^\varepsilon\right|^p &= \Delta^{-p}\, \mathbb{E}^{\mu^\varepsilon} \left|\int_{t-\Delta}^t (X_t^\varepsilon - X_s^\varepsilon) \,\mathrm{d}s\right|^p \\ &\leq \Delta^{-1} \int_{t-\Delta}^t \mathbb{E}^{\mu^\varepsilon} \left|X_t^\varepsilon - X_s^\varepsilon\right|^p \,\mathrm{d}s \end{aligned} \tag{3.13}$$

We can now apply [?, Lemma 6.1] to the integrand to obtain

$$\mathbb{E}^{\mu^\varepsilon} \left|X_t^\varepsilon - Z_t^\varepsilon\right|^p \leq C\Delta^{-1} \int_{t-\Delta}^t (\Delta^p + \Delta^{p/2} + \varepsilon^p) \,\mathrm{d}s, \tag{3.14}$$

which implies the desired result. The case $0 < t \leq \Delta$ can be proved analogously. □

Lemma 3.4 (See [?, Proposition 5.8])**.** *Under assumptions add assumptions, it holds in law*

$$\alpha \int_{t-\Delta}^t V_0'(X_s^\varepsilon)\big(1 + \Phi'(Y_s^\varepsilon)\big) \,\mathrm{d}s = A\Delta V_0'(Z_t^\varepsilon) + R(\varepsilon, \Delta), \tag{3.15}$$

where for every $p > 0$ and if Δ and ε are sufficiently small, then

$$\left(\mathbb{E}^{\mu^\varepsilon} \left|R(\varepsilon, \Delta)\right|^p\right)^{1/p} \leq C(\varepsilon^2 + \Delta^{1/2}\varepsilon + \Delta^{3/2}), \tag{3.16}$$

where $C > 0$ is independent of ε and Δ.

Proof. Let us denote $\Psi(t) \coloneqq 1 + \Phi'(Y_t^\varepsilon)$. Then

$$\begin{aligned} \mathbb{E}^{\mu^\varepsilon} \left|R(\varepsilon, \Delta)\right|^p &= \mathbb{E}^{\mu^\varepsilon} \left|\int_{t-\Delta}^t \alpha V_0'(X_s^\varepsilon)\Psi(s) \,\mathrm{d}s - \Delta A V_0'(Z_t^\varepsilon)\right|^p \\ &\leq C\, \mathbb{E}^{\mu^\varepsilon} \left|V_0'(Z_t^\varepsilon) \int_{t-\Delta}^t (\alpha\Psi(s) - A) \,\mathrm{d}s\right|^p \\ &\quad + C\, \mathbb{E}^{\mu^\varepsilon} \left|\int_{t-\Delta}^t \alpha \left(V_0'(X_t^\varepsilon) - V_0'(Z_t^\varepsilon)\right) \Psi(s) \,\mathrm{d}s\right|^p. \end{aligned} \tag{3.17}$$

The result is then obtained following the proof of [?, Proposition 5.8] and replacing [?, Lemma 6.1] with Lemma 3.3, and [?, Corollary 4.1] with Lemma 3.2. □

We can now prove Theorem 3.1.

Proof of Theorem 3.1. Consider the decomposition (3.9). Denoting

$$J_t := \sqrt{2\sigma}\int_{t-\Delta}^{t} \left(1+\Phi'(Y_s^\varepsilon)\right) \mathrm{d}W_s, \tag{3.18}$$

we have due to Lemma 3.4 the equality in law

$$X_t^\varepsilon - X_{t-\Delta}^\varepsilon = -A\Delta V'(Z_t^\varepsilon) + J_t + \widehat{R}(\varepsilon,\Delta), \tag{3.19}$$

where, since $\zeta \in (0,1)$, we have

$$\left(\mathbb{E}^{\mu^\varepsilon}\left|\widehat{R}(\varepsilon,\Delta)\right|^p\right)^{1/p} \leq C(\varepsilon + \varepsilon^{3\zeta/2}) \tag{3.20}$$

Therefore, we have that the estimator satisfies

$$\begin{aligned}\widehat{A}_{T,\Delta}(Z_t^\varepsilon) &= A - A\frac{\int_0^\Delta V_0'(Z_t^\varepsilon)^2\,\mathrm{d}t}{\int_0^T V_0'(Z_t^\varepsilon)^2\,\mathrm{d}t} - \frac{\int_0^\Delta V_0'(Z_t^\varepsilon)\frac{1}{t}(X_t^\varepsilon - Z_t^\varepsilon)\,\mathrm{d}t}{\int_0^T V_0'(Z_t^\varepsilon)^2\,\mathrm{d}t} \\ &\quad - \frac{\int_\Delta^T V_0'(Z_t^\varepsilon)J_t\,\mathrm{d}t}{\Delta\int_0^T V_0'(Z_t^\varepsilon)^2\,\mathrm{d}t} - \frac{\widehat{R}(\varepsilon,\Delta)\int_\Delta^T V_0'(Z_t^\varepsilon)\,\mathrm{d}t}{\Delta\int_0^T V_0'(Z_t^\varepsilon)^2\,\mathrm{d}t} \\ &=: A - I_1 - I_2 - I_3 - I_4,\end{aligned} \tag{3.21}$$

in law. Let us analyse the terms I_i, $i = 1,\dots,4$ separately. Let us consider I_1 and multiply both the numerator and the denominator by $1/T$. Due to assumption add assumption and Lemma 3.2, we have

$$\frac{A}{T}\mathbb{E}^{\mu^\varepsilon}\left|\int_0^\Delta V_0'(Z_t^\varepsilon)^2\,\mathrm{d}t\right| \leq C\varepsilon^{\gamma+\zeta}, \tag{3.22}$$

for a constant $C > 0$ independent of Δ and ε. Hence the numerator vanishes in L^1 and thus in law for $\varepsilon \to 0$. We split the denominator as

$$\frac{1}{T}\int_0^T V_0'(Z_t^\varepsilon)^2\,\mathrm{d}t = \frac{1}{T}\int_0^T V_0'(X_t^\varepsilon)^2\,\mathrm{d}t + \frac{1}{T}\int_0^T \left(V_0'(Z_t^\varepsilon)^2 - V_0'(X_t^\varepsilon)^2\right)\,\mathrm{d}t \tag{3.23}$$

For the first term, we have by the ergodic theorem

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T V_0'(X_t^\varepsilon)^2\,\mathrm{d}t = \mathbb{E}^{\mu^\varepsilon}\left|V_0'\right|^2, \quad \text{a.s.} \tag{3.24}$$

For the second term, we have applying Cauchy–Schwarz's inequality and due to assumption add assumption and Lemma 3.3

$$\begin{aligned}\frac{1}{T}\mathbb{E}^{\mu^\varepsilon}\left|\int_0^T \left(V_0'(Z_t^\varepsilon)^2 - V_0'(X_t^\varepsilon)^2\right)\,\mathrm{d}t\right| &\leq \frac{C}{T}\int_0^T \left(\mathbb{E}^{\mu^\varepsilon}\left|V_0'(Z_t^\varepsilon) - V_0'(X_t^\varepsilon)\right|^2\right)^{1/2}\mathrm{d}t \\ &\leq C\left(\Delta + \Delta^{1/2} + \varepsilon\right),\end{aligned} \tag{3.25}$$

which implies that the denominator tends to a finite value in probability for $\varepsilon \to 0$. Therefore, by Slutsky's theorem,

$$\lim_{\varepsilon\to 0} I_1 = 0, \quad \text{in law.} \tag{3.26}$$

Let us now consider I_2 and multiply numerator and denominator by $1/T$. The denominator is the same as I_1, and therefore does not need to be treated further. The numerator can be bounded in L^1 as

$$\frac{1}{T}\mathbb{E}^{\mu^\varepsilon}\left|\int_0^\Delta V_0'(Z_t^\varepsilon)\frac{1}{t}(X_t^\varepsilon - Z_t^\varepsilon)\,\mathrm{d}t\right| \leq \frac{C}{\Delta T}\int_0^\Delta \frac{\Delta}{t}\mathbb{E}^{\mu^\varepsilon}\left|X_t^\varepsilon - Z_t^\varepsilon\right|\,\mathrm{d}t, \tag{3.27}$$

which, since $Z_0^\varepsilon = X_0^\varepsilon$, vanishes for $\varepsilon \to 0$. Hence, an application of Slutsky's theorem yields

$$\lim_{\varepsilon \to 0} I_2 = 0, \quad \text{in law.} \tag{3.28}$$

We consider now I_3, which can be rewritten as

$$\begin{aligned} I_3 &= \frac{1}{\sqrt{T\Delta}} \frac{\frac{1}{\sqrt{T\Delta}} \int_\Delta^T V_0'(Z_t^\varepsilon) J_t \,\mathrm{d}t}{\frac{1}{T}\int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t} \\ &= \varepsilon^{(\gamma-\zeta)/2} \frac{\frac{1}{\sqrt{T\Delta}} \int_\Delta^T V_0'(Z_t^\varepsilon) J_t \,\mathrm{d}t}{\frac{1}{T}\int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t} \end{aligned} \tag{3.29}$$

Let us remark that J_t is a martingale and that by Itô isometry

$$\mathbb{E}^{\mu^\varepsilon} |J_\Delta|^2 = 2\Sigma\Delta, \tag{3.30}$$

Therefore, we can apply the central limit theorem for martingales to the numerator and obtain the equality in law

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{\sqrt{T\Delta}} \int_\Delta^T V_0'(Z_t^\varepsilon) J_t \,\mathrm{d}t &= \frac{1}{\sqrt{\Delta}} \mathcal{N}\left(0, \mathbb{E}^{\mu^\varepsilon}\left(\left|V_0'(X_0^\varepsilon)\right|^2 |J_\Delta|^2\right)\right) \\ &= C\mathcal{N}(0,1). \end{aligned} \tag{3.31}$$

The denominator is the same as in I_2 and I_3 and tends in probability to a finite value. Hence, since by hypothesis $\gamma > \zeta$, we have

$$\lim_{\varepsilon \to 0} I_3 = 0, \quad \text{in law.} \tag{3.32}$$

For the last term I_4, we have

$$I_4 = \frac{\varepsilon^{\gamma-\zeta} \widehat{R}(\varepsilon,\Delta) \int_\Delta^T V_0'(Z_t^\varepsilon) \,\mathrm{d}t}{\frac{1}{T}\int_0^T V_0'(Z_t^\varepsilon)^2 \,\mathrm{d}t}. \tag{3.33}$$

For the numerator, we have by the Cauchy–Schwarz inequality and due to Lemma 3.4

$$\begin{aligned} \varepsilon^{\gamma-\zeta} \mathbb{E}^{\mu^\varepsilon} \left| \widehat{R}(\varepsilon,\Delta) \int_\Delta^T V_0'(Z_t^\varepsilon) \,\mathrm{d}t \right| &\leq \varepsilon^{\gamma-\zeta} \left(\mathbb{E}^{\mu^\varepsilon} \left| \widehat{R}(\varepsilon,\Delta) \right|^2 \right)^{1/2} \left(\mathbb{E}^{\mu^\varepsilon} \left| \int_\Delta^T V_0'(Z_t^\varepsilon) \,\mathrm{d}t \right|^2 \right)^{1/2} \\ &\leq C\varepsilon^{\gamma-\zeta}(\varepsilon + \varepsilon^{3\zeta/2})\varepsilon^{-\gamma} \\ &\leq C\left(\varepsilon^{1-\zeta} + \varepsilon^{\zeta/2}\right) \end{aligned} \tag{3.34}$$

which implies that, since the denominator is the same as before,

$$\lim_{\varepsilon \to 0} I_4 = 0, \quad \text{in law.} \tag{3.35}$$

The decomposition (3.21), together with the limits of I_i for $i = 1, \dots, 4$, prove the desired result. □

3.2 Diffusion coefficient

We now consider the same transformation of the data, i.e., we employ $Z_t^\varepsilon = \mathcal{H}_\Delta(X^\varepsilon)_t$ as defined in (3.1), to estimate the diffusion coefficient Σ of the homogenized model. In particular, we consider the estimator

$$\widehat{\Sigma}_{\Delta,T} = \frac{1}{2T} \lim_{h\to 0} \sum_{k=1}^{N_h} (Z_{t_k}^\varepsilon - Z_{t_{k-1}}^\varepsilon)^2, \tag{3.36}$$

where the limit has to be intended in probability and with respect to a series of refinements of partitions $\mathcal{P}_h = \{t_k\}$ of the interval $[0,T]$. Let us recall that if instead of Z_t^ε one employs a path from the homogenized model X_t^0, then formula (1.8) gives the exact value of Σ for any $T > 0$.

Let us introduce a theoretical result which will play the role of Lemma 3.4 in this framework.

Lemma 3.5 (See [?, Proposition 5.7])**.** *Under assumptions add assumptions, there exist a continuous standard Gaussian process $(\xi_t, \Delta \le t \le T)$ such that for $\Delta \le s \le t \le T$*

$$\mathbb{E}\left(\xi_t\, \xi_s\right) = \begin{cases} 0, & t - s \ge \Delta, \\ 1 - \frac{t-s}{\Delta}, & t - s < \Delta, \end{cases} \tag{3.37}$$

and such that for all $\Delta \le t \le T$ it holds in law

$$\sqrt{2\sigma} \int_{t-\Delta}^{t} \left(1 + \Phi'(Y_s^\varepsilon)\right) \mathrm{d}W_s = \sqrt{2\Sigma\Delta}\, \xi_t + S(\varepsilon), \tag{3.38}$$

where for every $p > 0$ and $\kappa \in (0, 1/2)$ it holds

$$\left(\mathbb{E}^{\mu^\varepsilon} \left|S(\varepsilon)\right|^p\right)^{1/p} \le C(\varepsilon^{2\kappa} + \varepsilon^\kappa). \tag{3.39}$$

Proof. The proof is identical to the proof of [?, Proposition 5.7] and is therefore omitted here. The process ξ_t is defined as

$$\xi_t = \frac{\widehat{W}_{2\Sigma t} - \widehat{W}_{2\Sigma(t-\Delta)}}{\sqrt{2\Sigma\Delta}}, \tag{3.40}$$

where $\widehat{W}_t$ is a standard Brownian motion, and its covariance function (3.37) can be trivially derived from the basic properties of standard Brownian motion. □

Let us now recall that the differential of the process Z_t^ε can be expressed as

$$\mathrm{d}Z_t^\varepsilon = \frac{X_t^\varepsilon - X_{t-\Delta}^\varepsilon}{\Delta}\,\mathrm{d}t, \tag{3.41}$$

for $\Delta \le t \le T$. Therefore, for any choice $\Delta \le s < t \le T$, we have

$$Z_t^\varepsilon - Z_s^\varepsilon = \frac{1}{\Delta} \int_s^t \left(X_r^\varepsilon - X_{r-\Delta}^\varepsilon\right) \mathrm{d}r. \tag{3.42}$$

We can now prove the main result.

Theorem 3.6. *Under the assumptions of Lemma 3.5, if $T = \mathcal{O}(1)$ with respect to ε and $\Delta = \varepsilon^\zeta$ for $\zeta \in (0, 1)$, then*

$$\lim_{\varepsilon \to 0} \widehat{\Sigma}_{\Delta,T} = \Sigma, \quad \textit{in law}, \tag{3.43}$$

for $\widehat{\Sigma}_{\Delta,T}$ defined in (3.36)*.*

Proof. Replacing (3.9) into (3.42) and considering Lemma 3.4 and Lemma 3.5, we have for $\Delta \le t_{k-1} < t_k \le T$ the equality in law

$$Z_{t_k}^\varepsilon - Z_{t_{k-1}}^\varepsilon = \frac{1}{\Delta} \int_{t_{k-1}}^{t_k} \left(\sqrt{2\Sigma\Delta}\xi_s + R(\varepsilon, \Delta)\right) \mathrm{d}s, \tag{3.44}$$

where the remainder $R(\varepsilon, \Delta)$ satisfies for any $\kappa \in (0, 1/2)$ and $p > 0$

$$\left(\mathbb{E}^{\mu^\varepsilon} \left|R(\varepsilon, \Delta)\right|^p\right)^{1/p} \le C\left(\Delta + \varepsilon^\kappa\right). \tag{3.45}$$

Let us consider the partition $\mathcal{P}_\Delta = \{t_k = k\Delta\}_{k=0}^{N_\Delta}$, with $T = \Delta N_\Delta$. Since we are interested in the limit $\varepsilon \to 0$ and $\Delta = \varepsilon^\zeta$, we can rewrite (3.36) as

$$\begin{aligned}
\widehat{\Sigma}_{\Delta,T} &= \lim_{\Delta\to 0} \frac{1}{2T\Delta^2} \sum_{k=0}^{N_\Delta-1} \left(\sqrt{2\Sigma\Delta} \int_{t_{k-1}}^{t_k} \xi_s \,\mathrm{d}s + \Delta R(\varepsilon,\Delta) \right)^2 \\
&= \lim_{\Delta\to 0} \left\{ \frac{\Sigma}{N_\Delta \Delta^2} \sum_{k=0}^{N_\Delta-1} \left(\int_{t_{k-1}}^{t_k} \xi_s \,\mathrm{d}s \right)^2 + \frac{1}{2T} \sum_{k=0}^{N_\Delta-1} R(\varepsilon,\Delta)^2 \right. \\
&\qquad \left. + \frac{\sqrt{2\Sigma}}{T\sqrt{\Delta}} \sum_{k=0}^{N_\Delta-1} R(\varepsilon,\Delta) \int_{t_{k-1}}^{t_k} \xi_s \,\mathrm{d}s \right\} \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{3.46}$$

Let us consider the first term. We have that

$$\int_{t_{k-1}}^{t_k} \xi_s \,\mathrm{d}s =: \Xi_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}\left(0, \Delta^2\right), \tag{3.47}$$

which, by the law of large numbers, implies that

$$\lim_{\varepsilon\to 0} I_1 = \Sigma, \quad \text{a.s.} \tag{3.48}$$

For the second term, we get

$$\mathbb{E}\,|I_2| \le C(\varepsilon^\zeta + \varepsilon^{2\kappa-\zeta}), \tag{3.49}$$

which implies that I_2 vanishes in L^1 for $\varepsilon \to 0$, and therefore in law, since we can choose κ as close as needed to $1/2$. Let us now consider the last term. The Cauchy–Schwarz inequality yields

$$\begin{aligned}
\mathbb{E}\,|I_2| &\le \frac{C}{2T\Delta} \sum_{k=0}^{N_\Delta-1} \left(\mathbb{E}\, R(\varepsilon,\Delta)^2\right)^{1/2} \left(\mathbb{E}\,\Xi_k^2\right)^{1/2} \\
&\le C\varepsilon^\zeta \left(\varepsilon^\zeta + \varepsilon^\kappa\right) \varepsilon^{-3\zeta/2} \\
&\le C(\varepsilon^{\zeta/2} + \varepsilon^{\kappa - \frac{\zeta}{2}}).
\end{aligned} \tag{3.50}$$

Hence, since κ can be chosen arbitrarily close to $1/2$, the conclusion follows. □

4 Point estimates from discrete data

In practice, it is not possible to observe X_t^ε continuously, and data will therefore be given by a discrete sequence of time evaluations of the underlying continuous process. Let us consider data to be given by the discrete sequence $\mathbf{x}^\varepsilon = \{x_j^\varepsilon\}_{j=0}^N$ such that $x_j^\varepsilon = X_{t_j}^\varepsilon$, where X_t^ε is the solution of (1.1). We are interested in the case in which data is observed at high frequency, and therefore in the following we assume that $t_j = j\varepsilon^\beta$ for some exponent $\beta > 1$.

4.1 Drift coefficient

Let us first recall that for a general sequence $\mathbf{x} = \{x_j\}_{j=0}^N \in \mathbb{R}^{N+1}$ of evaluations on a time grid with constant spacing h, the estimator (1.7) is approximated by

$$\widehat{A}_N(\mathbf{x}) = -\frac{\sum_{n=1}^{N+1} V'(x_{n-1})(x_n - x_{n-1})}{\sum_{n=1}^{N+1} V'(x_{n-1})^2 h}, \tag{4.1}$$

see e.g. [?]. The discrete-time equivalent to the operator $\mathcal{H}_\Delta$ defined in (3.1) is the operator $H_\delta \colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$, defined as

$$H_\delta(\mathbf{x}^\varepsilon)_n := \begin{cases} x_0^\varepsilon, & n = 0, \\ \frac{1}{n+1} \sum_{j=0}^n x_j^\varepsilon, & 1 \le n < \delta - 1, \\ \frac{1}{\delta} \sum_{j=0}^{\delta-1} x_{n-j}^\varepsilon, & \delta - 1 \le n \le N, \end{cases} \tag{4.2}$$

where, in this framework, $\delta \in \mathbb{N}_{>0}$ represents the size of the averaging window. Let us remark that, trivially, $H_1 = \mathrm{Id}$. In the following, we will employ the notation $\mathbf{z}^\varepsilon := H_\delta(\mathbf{x}^\varepsilon)$ and $\mathbf{z}^\varepsilon = \{z_n^\varepsilon\}_{n=0}^N$. Let us remark that, by definition

$$z_n^\varepsilon - z_{n-1}^\varepsilon = \frac{1}{\delta}(x_n^\varepsilon - x_{n-\delta}^\varepsilon), \tag{4.3}$$

for $\delta \le n \le N$ and for $1 \le n < \delta$

$$z_n^\varepsilon - z_{n-1}^\varepsilon = \frac{1}{n+1}\left(x_n^\varepsilon - z_{n-1}^\varepsilon\right). \tag{4.4}$$

Since the weight of the first $\delta - 1$ data points will be negligible in the theoretical results, we decide to modify the definition of the operator H_δ simply as

$$z_n^\varepsilon = H_\delta(\mathbf{x}^\varepsilon)_n = \frac{1}{\delta}\sum_{j=0}^{\delta-1} x_{n-j}^\varepsilon, \tag{4.5}$$

for $n = 0, 1, \dots, N$, through the introduction of δ fictitious points $x_j^\varepsilon = x_0^\varepsilon$, $j = -1, -2, \dots, -\delta + 1$. The choice of assigning to these "negative index" points the initial condition is natural and necessary to prove Lemma 4.2. Therefore, with this choice, the difference $z_n^\varepsilon - z_{n-1}^\varepsilon$ is always given by (4.3).

Remark 1. Let us remark that applying the operator H_δ on a sequence $\mathbf{x} \in \mathbb{R}^{N+1}$ has a computational complexity of $\mathcal{O}(N)$ simple operations.

Replacing $\mathbf{x}$ with the sequence $\mathbf{z}^\varepsilon$ in (4.1) and reminding that we consider the time grid to have spacing $h = \varepsilon^\beta$, we get the estimator

$$\widehat{A}_{N,\delta}(\mathbf{z}^\varepsilon) = -\frac{\sum_{n=1}^N V_0'(z_{n-1}^\varepsilon)(z_n^\varepsilon - z_{n-1}^\varepsilon)}{\sum_{n=1}^N \varepsilon^\beta V_0'(z_{n-1}^\varepsilon)^2}. \tag{4.6}$$

Employing the theoretical tools introduced in Section 3.1 it is possible to prove the following result.

Theorem 4.1. *Under assumption add assumptions, if there exists $\zeta \in (\beta - 1, \beta)$ such that $\delta = [\varepsilon^{-\zeta}]$ and $\gamma > \beta - \zeta$ such that $N = [\varepsilon^{-\gamma}]$, where $[\cdot]$ denotes the integer part of a real number, it holds*

$$\lim_{\varepsilon \to 0} \widehat{A}_{N,\delta}(\mathbf{z}^\varepsilon) = A, \quad \textit{in law.} \tag{4.7}$$

Let us first introduce a Lemma which replaces in the discrete case Lemma 3.3.

Lemma 4.2. *Under assumptions add assumptions, it holds for all $p \ge 1$*

$$\left(\mathbb{E}^{\mu^\varepsilon}\left|X_s^\varepsilon - z_{n-1}^\varepsilon\right|^p\right)^{1/p} \le C(\varepsilon^\beta\delta + \varepsilon^{\beta/2}\delta^{1/2} + \varepsilon), \tag{4.8}$$

for $s \in [(n-\delta)\varepsilon^\beta, n\varepsilon^\beta]$, where $C > 0$ is independent of ε and δ.

Proof. Let us first consider $n \ge \delta$. We replace the definition of z_{n-1}^ε and apply the Hölder inequality to obtain

$$\begin{aligned} \mathbb{E}^{\mu^\varepsilon}\left|X_s^\varepsilon - z_{n-1}^\varepsilon\right|^p &= \delta^{-p}\,\mathbb{E}^{\mu^\varepsilon}\left|\sum_{j=0}^{\delta-1}\left(X_s^\varepsilon - x_{n-1-j}^\varepsilon\right)\right|^p \\ &\le \delta^{-1}\sum_{j=0}^{\delta-1}\mathbb{E}^{\mu^\varepsilon}\left|X_s^\varepsilon - x_{n-1-j}^\varepsilon\right|^p. \end{aligned} \tag{4.9}$$

Applying on each element of the sum [**?**, Lemma 6.1], we obtain

$$\mathbb{E}^{\mu^\varepsilon} \left|X_s^\varepsilon - z_{n-1}^\varepsilon\right|^p \leq C\delta^{-1} \sum_{j=0}^{\delta-1} \left(\varepsilon^{p\beta}\delta^p + \varepsilon^{p\beta/2}\delta^{p/2} + \varepsilon^p\right), \tag{4.10}$$

which implies the desired result. For $n < \delta$, we have equivalently

$$\mathbb{E}^{\mu^\varepsilon} \left|X_s^\varepsilon - z_{n-1}^\varepsilon\right|^p \leq \delta^{-1} \sum_{j=0}^{n-1} \mathbb{E}^{\mu^\varepsilon} \left|X_s^\varepsilon - x_j^\varepsilon\right|^p + \delta^{-1}(\delta - n)\, \mathbb{E}^{\mu^\varepsilon} \left|X_s^\varepsilon - x_0^\varepsilon\right|^p, \tag{4.11}$$

which can be treated as above and therefore implies the desired result. ☐

Lemma 4.3. *Under assumptions add assumptions, let $t_k = k\varepsilon^\beta$. Then, it holds in law*

$$\alpha \int_{t_{n-\delta}}^{t_n} V_0'(X_s^\varepsilon)\big(1 + \Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}s = \varepsilon^\beta \delta A V_0'(z_{n-1}^\varepsilon) + R(\varepsilon, \delta), \tag{4.12}$$

where for every $p > 0$ and if δ and ε are sufficiently small,

$$\left(\mathbb{E}^{\mu^\varepsilon} \left|R(\varepsilon, \delta)\right|^p\right)^{1/p} \leq C\left(\varepsilon^2 + \varepsilon^{(2+\beta)/2}\delta^{1/2} + \varepsilon^{3\beta/2}\delta^{3/2}\right), \tag{4.13}$$

where $C > 0$ is a constant independent of ε and δ.

Proof. The proof follows from the proof of Lemma 3.4, with z_{n-1}^ε taking the role of Z_t^ε and replacing Δ by $\varepsilon^\beta\delta$. ☐

We can now prove the main result.

Proof of Theorem 4.1. Let us recall the decomposition (3.9), which, combined with (4.3), reads

$$\begin{aligned} z_n^\varepsilon - z_{n-1}^\varepsilon = &-\frac{\alpha}{\delta} \int_{t_{n-\delta}}^{t_n} V_0'(X_s^\varepsilon)\big(1 + \Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}s \\ &+ \frac{\sqrt{2\sigma}}{\delta} \int_{t_{n-\delta}}^{t_n} \big(1 + \Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}W_s - \frac{\varepsilon}{\delta}\big(\Phi(Y_{t_n}^\varepsilon) - \Phi(Y_{t_{n-\delta}}^\varepsilon)\big). \end{aligned} \tag{4.14}$$

Hence, in light of Lemma 4.3 and denoting

$$J_n \coloneqq \sqrt{2\sigma} \int_{t_{n-\delta}}^{t_n} \big(1 + \Phi'(Y_s^\varepsilon)\big)\,\mathrm{d}W_s, \tag{4.15}$$

we have the equality in law

$$z_n^\varepsilon - z_{n-1}^\varepsilon = -\varepsilon^\beta A V_0'(z_{n-1}^\varepsilon) + \frac{\widehat{R}(\varepsilon, \delta)}{\delta} + \frac{J_n}{\delta}, \tag{4.16}$$

where

$$\left(\mathbb{E}^{\mu^\varepsilon} \left|\widehat{R}(\varepsilon, \Delta)\right|^p\right)^{1/p} \leq C(\varepsilon + \varepsilon^{(2+\beta)/2}\delta^{1/2} + \varepsilon^{3\beta/2}\delta^{3/2}) \tag{4.17}$$

Replacing the equality above into (4.6), we get

$$\begin{aligned} \widehat{A}_{N,\delta}(\mathbf{z}^\varepsilon) &= A - \frac{1}{\delta\varepsilon^\beta} \frac{\sum_{n=1}^N V_0'(z_{n-1}^\varepsilon)\widehat{R}(\varepsilon, \delta)}{\sum_{n=1}^N V_0'(z_{n-1}^\varepsilon)^2} - \frac{1}{\delta\varepsilon^\beta} \frac{\sum_{n=1}^N V_0'(z_{n-1}^\varepsilon) J_n}{\sum_{n=1}^N V_0'(z_{n-1}^\varepsilon)^2} \\ &=: A - I_1 - I_2. \end{aligned} \tag{4.18}$$

Let us consider I_1 and multiply by $1/N$ both its numerator and denominator. For the denominator, we have

$$\frac{1}{N}\sum_{n=1}^{N} V_0'(z_{n-1}^{\varepsilon})^2 = \frac{1}{N}\sum_{n=1}^{N} V_0'(x_{n-1}^{\varepsilon})^2 + \frac{1}{N}\sum_{n=1}^{N}\left(V_0'(z_{n-1}^{\varepsilon})^2 - V_0'(x_{n-1}^{\varepsilon})^2\right). \tag{4.19}$$

Applying the ergodic theorem on the first term we get

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} V_0'(x_{n-1}^{\varepsilon})^2 = \mathbb{E}^{\mu^{\varepsilon}}\left(V_0'(x)^2\right), \tag{4.20}$$

almost surely. For the second term, we can employ assumption add assumption and Lemma 4.2 to get

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left(V_0'(z_{n-1}^{\varepsilon})^2 - V_0'(x_{n-1}^{\varepsilon})^2\right) = 0, \tag{4.21}$$

in L^p for all $p \geq 1$. Hence, the denominator tends in probability to a finite value in the limit $\varepsilon \to 0$, which is equivalent to $N \to \infty$ since $N = [\varepsilon^{-\gamma}]$. We apply Hölder's inequality on the numerator and obtain for any $p > 1$

$$\begin{aligned}
\mathbb{E}^{\mu^{\varepsilon}}\left|\varepsilon^{\zeta-\gamma-\beta}\sum_{n=1}^{N} V_0'(z_{n-1}^{\varepsilon})\widehat{R}(\varepsilon,\delta)\right| &\leq C\varepsilon^{\zeta-\gamma-\beta}\sum_{n=1}^{N}\left(\mathbb{E}^{\mu^{\varepsilon}}\left|\widehat{R}(\varepsilon,\delta)\right|^p\right)^{1/p} \\
&\leq C\varepsilon^{\zeta-\beta}(\varepsilon + \varepsilon^{(2+\beta)/2}\delta^{1/2} + \varepsilon^{3\beta/2}\delta^{3/2}) \\
&\leq C(\varepsilon^{1+\zeta-\beta} + \varepsilon^{(2+\zeta-\beta)/2} + \varepsilon^{(\beta-\zeta)/2}),
\end{aligned} \tag{4.22}$$

which, under the assumption $\zeta \in (\beta - 1, \beta)$, vanishes for $\varepsilon \to 0$. Hence I_1 tends to zero in L^1 and therefore by Slutsky's theorem

$$\lim_{\varepsilon\to 0} I_1 = 0, \quad \text{in law}. \tag{4.23}$$

Let us now consider I_2. Multiplying both the numerator and the denominator by $1/N$, we have that I_2 has the same denominator as I_1, and shall therefore not be treated further. For the numerator, we have (see the proof of Theorem 3.1 or of [**?**, Theorem 3.5])

$$\frac{1}{\sqrt{N\delta\varepsilon^{\beta}}}\sum_{n=1}^{N} V_0'(z_{n-1}^{\varepsilon})J_n = C\mathcal{N}(0,1), \quad \text{in law}, \tag{4.24}$$

where C is a constant independent of ε and δ. Therefore, we have the equality in law

$$\frac{1}{\sqrt{N\delta\varepsilon^{\beta}}}\frac{1}{\sqrt{N\delta\varepsilon^{\beta}}}\sum_{n=1}^{N} V_0'(z_{n-1}^{\varepsilon})J_n = C\varepsilon^{(\gamma+\zeta-\beta)/2}\mathcal{N}(0,1). \tag{4.25}$$

This, together with the hypothesis $\gamma > \beta - \zeta$, yields

$$\lim_{\varepsilon\to 0} I_2 = 0, \quad \text{in law}, \tag{4.26}$$

which proves the desired result. □

4.2 Diffusion coefficient

We now consider the estimator of the diffusion coefficient based on a discrete-time approximation of the quadratic variation. For a generic sequence $\mathbf{x} \in \mathbb{R}^{N+1}$ based on a time sequence with spacing h, so that $T = Nh$, a discrete approximation to (1.8) is given by

$$\widehat{\Sigma}_N(\mathbf{x}) = \frac{1}{2Nh}\sum_{n=0}^{N}(x_{n+1} - x_n)^2. \tag{4.27}$$

As in the previous section, let us consider $h = \varepsilon^\beta$ for $\beta \geq 1$ and the sequence $\mathbf{z}^\varepsilon = H_\delta(\mathbf{x}^\varepsilon)$. We then consider the estimator

$$\widehat{\Sigma}_{N,\delta}(\mathbf{z}^\varepsilon) = \frac{\delta}{2N\varepsilon^\beta} \sum_{n=0}^{N} (z_{n+1}^\varepsilon - z_n^\varepsilon)^2. \tag{4.28}$$

Let us remark that an additional δ appears at the numerator of the estimator above. This additional term, which may appear rather unnatural, has to be introduced in order to guarantee asymptotic unbiasedness of the estimator.

Theorem 4.4. *Under assumptions add assumptions, if $\delta = \varepsilon^{-\zeta}$ for $\zeta \in (0, 1+\beta)$ and if $N = [\varepsilon^{-\gamma}]$ for $\gamma > 0$, then*

$$\lim_{\varepsilon \to 0} \widehat{\Sigma}_{N,\delta} = \Sigma, \quad \text{in law}, \tag{4.29}$$

for $\widehat{\Sigma}_{N,\delta}$ defined in (4.28).

Proof. Due to ..., we have

$$z_{n+1} - z_n = \sqrt{\frac{2\Sigma\varepsilon^\beta}{\delta}}\xi_n + \frac{\widehat{R}(\varepsilon, \delta)}{\delta}, \tag{4.30}$$

where for all $\kappa \in (0, 1/2)$ it holds for ε and δ sufficiently small

$$\left(\mathbb{E}^{\mu^\varepsilon} \left|\widehat{R}(\varepsilon, \Delta)\right|^p\right)^{1/p} \leq C\left(\varepsilon^\kappa + \varepsilon^{(2+\beta)/2}\delta^{1/2}\right). \tag{4.31}$$

Then, replacing (4.30) into (4.28) yields

$$\begin{aligned} \widehat{\Sigma}_{N,\delta}(\mathbf{z}^\varepsilon) &= \Sigma\frac{1}{N}\sum_{n=0}^{N}\xi_n^2 + \sqrt{\frac{2\Sigma}{\delta\varepsilon^\beta}}\frac{1}{N}\sum_{n=0}^{N}\hat{R}(\varepsilon,\delta)\xi_n + \frac{1}{2N\varepsilon^\beta\delta}\sum_{n=0}^{N}\hat{R}(\varepsilon,\delta)^2 \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{4.32}$$

The law of large numbers implies that, since $N \to \infty$ when $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} I_1 = \Sigma, \quad \text{a.s.} \tag{4.33}$$

Let us now consider I_2. We apply the Cauchy–Schwarz inequality and obtain for a constant $C > 0$ independent of δ and ε

$$\begin{aligned} \mathbb{E}^{\mu^\varepsilon}|I_2| &\leq C\frac{\varepsilon^{(\beta-\zeta)/2}}{N}\sum_{n=0}^{N}\left(\mathbb{E}^{\mu^\varepsilon}\left|\widehat{R}(\varepsilon,\delta)\right|^2\right)^{1/2} \\ &\leq C\varepsilon^{(\beta-\zeta)/2}\left(\varepsilon^\kappa + \varepsilon^{(2+\beta+\zeta)/2}\right) \\ &\leq C\left(\varepsilon^{(2\kappa+\beta-\zeta)/2} + \varepsilon^{1+\beta}\right). \end{aligned} \tag{4.34}$$

Let us remark that the first term tends to zero when

$$\zeta - \beta < 2\kappa, \tag{4.35}$$

which is verified since $0 < \zeta < 1 + \beta$ and κ can be chosen arbitrarily close to $1/2$. Hence, I_2 vanishes in L^1 and therefore

$$\lim_{\varepsilon \to 0} I_2 = 0, \quad \text{in law}. \tag{4.36}$$

We now consider I_3, for which similarly

$$\begin{aligned} \mathbb{E}^{\mu^\varepsilon}|I_3| &\leq C\varepsilon^{\beta-\zeta}\left(\varepsilon^{2\kappa} + \varepsilon^{2+\beta+\zeta}\right) \\ &\leq C\left(\varepsilon^{2\kappa+\beta-\zeta} + \varepsilon^{2+2\beta}\right), \end{aligned} \tag{4.37}$$

which, for the same reason as above, yields

$$\lim_{\varepsilon \to 0} I_3 = 0, \quad \text{in law}, \tag{4.38}$$

and which concludes the proof. □

Figure 1: Estimation of the drift coefficient A of (1.2) with (4.1). On both figures, the solid horizontal line is the true value of the homogenized drift coefficient A, while the dashed line is the value of the drift coefficient α of the multiscale equation. On the left, the estimation is obtained varying ε for fixed values of the coefficients (β, γ, ζ). On the right, we fix $\varepsilon = 5 \cdot 10^{-3}$, the exponent $\beta = 2$ and vary ζ and $\gamma = 2\beta - \zeta$.

5 Numerical experiments

In this section, we display a series of numerical experiment confirming in practice the validity of our theoretical results. For feasibility reasons, we will focus on the results for discrete sequences shown in Section 4.

5.1 Ornstein–Uhlenbeck process

Let us consider the case $V_0(x) = x$ and $V_1(x) = \cos(x)$, so that the homogenized model (1.2) is the Ornstein–Uhlenbeck equation. We fix $\alpha = 1$ and $\sigma = 0.5$ and wish to retrieve the drift coefficient A of (1.2). In this case, the value of K in (1.3) is given by

$$K = \frac{L^2}{Z\widehat{Z}}, \tag{5.1}$$

with $L = 2\pi$ and

$$Z = \int_0^L \exp\left(-\frac{V_1(y)}{\sigma}\right) \mathrm{d}y, \quad \widehat{Z} = \int_0^L \exp\left(\frac{V_1(y)}{\sigma}\right) \mathrm{d}y. \tag{5.2}$$

It is therefore possible to compute cheaply the coefficients of the homogenized model and obtain a comparison for numerical results. Following Theorem 4.1, we consider a set of values of ε, in particular we choose $\varepsilon_i = 1.3^i \cdot 10^{-3}$ for $i = 0, 1, \ldots, 20$ and fix $h_i = \varepsilon_i^\beta$ with $\beta = 2$, the averaging window $\delta_i = \varepsilon^\zeta$ for $\zeta = \beta - 0.7$, and $N = [\varepsilon^{-\gamma}]$ with $\gamma = 2\beta - \zeta$. Results, displayed in Figure 1, show how the estimated drift coefficient tends towards the value of the homogenized coefficient. Nonetheless, given values of ε and β, the choice of $\zeta \in (\beta - 1, \beta)$ is arbitrary. Theoretically, all choices in this interval should lead asymptotically with respect to ε to the value A of the homogenized equation, in law. In order to test numerically this property, we fix $\varepsilon = 5 \cdot 10^{-3}$, the exponent $\beta = 2$ and vary ζ linearly in the range $[\beta - 1, \beta]$, thus fixing $\gamma = 2\beta - \zeta$. Results, displayed in Figure 1, show that the best estimations are obtained for values close to the bounds of the interval $(\beta - 1, \beta)$.

We now consider inference of the diffusion coefficient Σ of (1.2). Adopting the notation of Theorem 4.4, we first vary $\varepsilon \in [0.01, 0.2]$ and choose $N = [\varepsilon^{-\gamma}]$ with $\gamma = 4$. We then fix $\beta = 3$ and $\zeta = 2$. Let us remark that Theorem 4.4 does only require γ to be positive. Nonetheless, the term I_1 in its proof tends to Σ for $N \to \infty$, and taking a relatively high value for γ allows to observe convergence. Results, shown in Figure 2, demonstrate the validity of the theoretical result. As for the drift coefficient, we consider the effect of varying the averaging window width, i.e., the exponent ζ such that $\delta = [\varepsilon^\zeta]$. We fix $\varepsilon = 0.02$, $\gamma = 4$ and $\beta = 3$, and vary $\zeta \in [0, \beta + 1]$. For $\zeta = 0$, we retrieve the parameter σ of the multiscale model, as it should be expected. For $\zeta > \beta$, the estimator rapidly tends to zero. The estimation appears to be correct for values of ζ in the interval $(\beta - 1, \beta)$, which is the same range of validity as for the estimation of the coefficient A.

Figure 2: Estimation of the diffusion coefficient Σ of (1.2) with (4.28). The dashed horizontal line corresponds to the diffusion coefficient α of the multiscale model, while the solid line corresponds to the true homogenized value A. On the left, the estimation is obtained varying ε for fixed values of the coefficients (β, γ, ζ). On the right, we fix $\varepsilon = 2 \cdot 10^{-2}$, the exponents $\beta = 3$ and $\gamma = 4$ and vary ζ.