Probabilistic methods for elliptic partial differential equations

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Abstract

AMS subject classification.

Keywords.

1 Introduction

Elliptic equations

$$
\begin{aligned}
-\nabla \cdot (\kappa \nabla u) &= f, \quad \text{in } D,\\
u &= g, \quad \text{on } \partial D.
\end{aligned} \tag{1}
$$

Prob methods [1–5] motivation

Main results

Outline

2 Method definition

Weak formulation: bilinear form $a\colon V \times V \to \mathbb{R}$ and a linear functional $F\colon V \to \mathbb{R}$ satisfying the usual continuity and coercivity constraints, look for $u \in V$ satisfying

$$
a(u, v) = F(v), \tag{2}
$$

for all functions $v \in V$. Galerkin formulation: for $V_h \subset V$ such that $\dim V_h < \infty$, find $u_h \in V_h$ such that

$$
a(u_h, v_h) = f(v_h), \quad \forall v_h \in V_h, \tag{3}
$$

for all $v_h \in V_h$. Given a triangulation $\mathcal{T}_h$ of the domain D, we choose V_h to be the space of linear finite elements, i.e., $V_h = X_h^1 \cap V$, where

$$
X_h^1 = \{v_h \in C^0(\overline{D})\colon v_h|_K \in \mathcal{P}_1, \text{ for all } K \in \mathcal{T}_h\}, \tag{4}
$$

and where $\mathcal{P}_1$ is the space of polynomials of degree at most one. The finite element space can be written then as $V_h = \operatorname{span}\{\varphi_i\}_{i=1}^N$, where the basis $\{\varphi_i\}_{i=1}^N$ are the Lagrange basis functions. Hence, each $v_h \in V_h$ can be written as $v_h = \sum_{i=1}^N v_i \varphi_i$, where v_i are the coefficients of v_h on the basis $\{\varphi_i\}_{i=1}^N$. Our probabilistic method is based on a randomly perturbed mesh $\widetilde{\mathcal{T}}_h$ which is defined as follows.

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Definition 1. Given $D \in \mathbb{R}^d$ and a mesh $\mathcal{T}_h$, the randomly perturbed mesh $\widetilde{\mathcal{T}}_h$ is defined by a sequence of random variables $\{\alpha_i\}_{i=1}^{N_{\text{int}}}$ with values in $\mathbb{R}^d$ and by its internal vertices $\{\tilde{x}_i\}_{i=1}^{N_{\text{int}}}$ as

$$\tilde{x}_i = x_i + \bar{h}_i^{p+1}\alpha_i, \tag{5}$$

where $p \geq 1$ and $\bar{h}_i$ is defined as the minimum diameter of the elements K having x_i as a vertex, i.e.

$$\bar{h}_i = \min_{K \in \mathcal{T}_{h,i}} h_K, \tag{6}$$

where $\mathcal{T}_{h,i}$ is such set of elements. The vertices laying on ∂D in $\mathcal{T}_h$ are the same in $\widetilde{\mathcal{T}}_h$.

Once the perturbed mesh $\widetilde{\mathcal{T}}_h$ is obtained, let us denote by $\widetilde{V}_h$ the finite element space defined on $\widetilde{\mathcal{T}}_h$, and by $\{\tilde{\varphi}_i\}_{i=1}^N$ its Lagrange basis. Moreover we define a linear operator $P_h \colon \widetilde{V}_h \to V_h$ as

$$P_h \tilde{v}_h = \sum_{i=1}^N \tilde{v}_i \varphi_i, \tag{7}$$

where $\{\tilde{v}_i\}_{i=1}^N$ are the coefficients of $\tilde{v}_h \in \widetilde{V}_h$ on the basis $\{\tilde{\varphi}_i\}_{i=1}^N$. The operator P_h is hence a mapping between the perturbed and the non-perturbed finite element spaces. We can now define the probabilistic finite element solution.

Definition 2. With the notation above, let $\tilde{u}_h \in \widetilde{V}_h$ be the random solution of

$$a(\tilde{u}_h, \tilde{v}_h) = F(\tilde{v}_h), \tag{8}$$

for all $\tilde{v}_h \in \widetilde{V}_h$. The probabilistic solution $U_h \in V_h$ is then defined as $U_h = P_h \tilde{u}_h$.

Let us finally introduce the following assumption on the random variables defining the mesh perturbation.

Assumption 1. The random variables α_i are chosen such that the perturbed mesh $\widetilde{\mathcal{T}}_h$ has the same topology of the mesh $\mathcal{T}_h$ (e.g., no exchange of vertices in one-dimension and no crossing edges in two-dimensions) almost surely.

3 A priori error analysis

Lemma 1. *Under Assumption 1, let us denote by $\delta a \colon V_h \times V_h \to \mathbb{R}$ the bilinear form defined as*

$$\delta a(w_h, v_h) = a(P_h^{-1} w_h, P_h^{-1} v_h) - a(w_h, v_h). \tag{9}$$

Then, it holds

$$\delta a(\varphi_i, \varphi_j) = C_{ij} h^p a(\varphi_i, \varphi_j), \tag{10}$$

where $|C_{ij}| \leq C$ with C independent of h for all i, j almost surely. Moreover, there exists a constant $C > 0$ independent of h such that

$$\delta a(w_h, v_h) \leq C h^p \|w_h\|_V \|v_h\|_V, \tag{11}$$

for all $v_h, w_h \in V_h$.

Proof. In the following, we prove (10) for different mesh constructions.

1d mesh with uniform spacing h and $\kappa(x) = 1$

In this simple case, it is known that for the original mesh we have

$$a(\varphi_i, \varphi_j) = \begin{cases} \dfrac{2}{h} & j = i, \\ -\dfrac{1}{h}, & j = i+1,\ j = i-1 \\ 0, & \text{otherwise}. \end{cases} \tag{12}$$

The modified basis functions $\tilde{\varphi}_i = P_h^{-1}\varphi_i$ have gradients given by

$$\nabla\tilde{\varphi}_i(x) = \begin{cases} \dfrac{1}{\tilde{x}_i - \tilde{x}_{i-1}}, & \tilde{x}_{i-1} < x \le \tilde{x}_i, \\ -\dfrac{1}{\tilde{x}_{i+1} - \tilde{x}_i}, & \tilde{x}_i < x \le \tilde{x}_{i+1}. \end{cases} \tag{13}$$

Replacing $\tilde{x}_j = x_j + \alpha_j h^{p+1}$ for $j = i-1, i, i+1$ we get

$$\nabla\tilde{\varphi}_i(x) = \frac{1}{h}\begin{cases} \dfrac{1}{1 + (\alpha_i - \alpha_{i-1})h^p}, & \tilde{x}_{i-1} < x \le \tilde{x}_i, \\ -\dfrac{1}{1 + (\alpha_{i+1} - \alpha_i)h^p}, & \tilde{x}_i < x \le \tilde{x}_{i+1}. \end{cases} \tag{14}$$

Let us now fix and index i and consider $j = i-1, i, i+1$. For $j = i$ it is possible to find

$$a(\tilde{\varphi}_i, \tilde{\varphi}_i) = \frac{2}{h}\Big(\frac{1 + (\alpha_{i+1} - \alpha_{i-1})h^p/2}{\big(1 + (\alpha_i - \alpha_{i-1})h^p\big)\big(1 + (\alpha_{i+1} - \alpha_i)h^p\big)}\Big) \tag{15}$$

Hence

$$\begin{aligned} \delta a(\varphi_i, \varphi_i) &= a(\varphi_i, \varphi_i)h^p\Big(\frac{(\alpha_{i+1} - \alpha_i)(\alpha_i - \alpha_{i-1})h^p - (\alpha_{i+1} - \alpha_{i-1})/2}{\big(1 + (\alpha_i - \alpha_{i-1})h^p\big)\big(1 + (\alpha_{i+1} - \alpha_i)h^p\big)}\Big) \\ &= C_{i,i}a(\varphi_i, \varphi_i)h^p. \end{aligned} \tag{16}$$

Let us now choose $j = i-1$. In this case

$$a(\tilde{\varphi}_i, \tilde{\varphi}_i) = -\frac{1}{h}\Big(\frac{1}{1 + (\alpha_i - \alpha_{i-1})h^p}\Big), \tag{17}$$

and hence

$$\begin{aligned} \delta a(\varphi_i, \varphi_{i-1}) &= a(\varphi_i, \varphi_{i-1})h^p\Big(-\frac{(\alpha_i - \alpha_{i-1})}{1 + (\alpha_i - \alpha_{i-1})h^p}\Big) \\ &= C_{i,i-1}a(\varphi_i, \varphi_{i-1})h^p. \end{aligned} \tag{18}$$

An analogous result can be found equivalently for $j = i+1$. In this case (10) is hence proved since $|C_{i,i}|$, $|C_{i,i-1}|$ and $|C_{i,i+1}|$ are bounded independently of h.

<u>1d mesh with non-uniform spacing and $\kappa(x) = 1$</u>

In this case

$$a(\varphi_i, \varphi_j) = \begin{cases} \dfrac{1}{h_i} + \dfrac{1}{h_{i+1}} & j = i, \\ -\dfrac{1}{h_{i+1}}, & j = i+1, \\ -\dfrac{1}{h_{i-1}}, & j = i-1 \\ 0, & \text{otherwise}. \end{cases} \tag{19}$$

Modified basis functions, rewrite

$$\nabla\tilde{\varphi}_i(x) = \begin{cases} \dfrac{1}{h_i(1 + R_i)}, & \tilde{x}_{i-1} < x \le \tilde{x}_i, \\ \dfrac{1}{h_{i+1}(1 + R_{i+1})}, & \tilde{x}_i < x \le \tilde{x}_{i+1}, \end{cases} \tag{20}$$

Where (Assumption 1: $\beta_{i,j} = h_i/h_j$)

$$R_i = \alpha_i \min\{1, \beta_{i+1,i}\}\bar{h}_i^p - \alpha_{i-1}\min\{1, \beta_{i-1,i}\}\bar{h}_{i-1}^p. \tag{21}$$

Then

$$\begin{aligned} a(\tilde{\varphi}_i, \tilde{\varphi}_i) =& \frac{1}{h_i(1 + R_i)} + \frac{1}{h_{i+1}(1 + R_{i+1})} \\ &= a(\varphi_i, \varphi_i)S_{i,i}. \end{aligned} \tag{22}$$

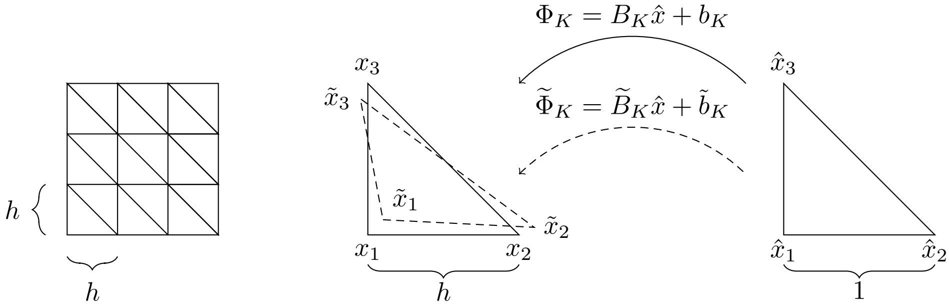

Figure 1: Structured mesh in the two-dimensional case.

where

$$S_{i,i} = \Big(\frac{h_{i+1}(1+R_{i+1}) + h_i(1+R_i)}{(1+R_i)(1+R_{i+1})(h_{i+1}+h_i)}\Big). \tag{23}$$

Hence $\delta a(\varphi_i, \varphi_i) = a(\varphi_i, \varphi_i)(S_{i,i} - 1)$. Simplifying

$$|S_{i,i} - 1| = \Big|\frac{R_i h_{i+1} + R_{i+1} h_i + R_i R_{i+1}(h_{i+1}+h_i)}{(1+R_i)(1+R_{i+1})(h_{i+1}+h_i)}\Big|. \tag{24}$$

The denominator is bounded from below by $C_D h$ for some constant C_D and the numerator can be bounded from above thanks to the definition of R_i by $C_N h^{p+1}$ for some constant C_N, hence

$$|S_{i,i} - 1| \leq C h^p, \tag{25}$$

which is the desired result for $C = C_N/C_D$. Now

$$a(\tilde{\varphi}_i, \tilde{\varphi}_{i-1}) = -\frac{1}{h_i}\Big(\frac{1}{1+R_i}\Big). \tag{26}$$

Hence

$$\delta a(\varphi_i, \varphi_{i-1}) = a(\varphi_i, \varphi_i)\Big(-\frac{R_i}{1+R_i}\Big), \tag{27}$$

which proves the result as there exists a constant C such that

$$\Big|\frac{R_i}{1+R_i}\Big| \leq C h^p. \tag{28}$$

Proceeding analogously for $a(\varphi_i, \varphi_{i-1})$ yields the desired result.

2d structured mesh with constant mesh size h and $\kappa(x) = 1$

Let us consider a two-dimensional square domain and a structured mesh with constant mesh size h, as the one of Fig. 1. The reference triangle $\widehat{K}$, with vertices $\hat{x}_1 = (0,0)^\top$, $\hat{x}_2 = (1,0)^\top$ and $\hat{x}_3 = (0,1)^\top$ is transformed into any element K of the mesh via an affine map $\Phi_K(\hat{x}) = B_K\hat{x} + b_K$. Likewise, the modified triangle $\widetilde{K}$ is obtained from $\widehat{K}$ via a modified map $\widetilde{\Phi}_K$, which is affine too and given by $\widetilde{\Phi}_K(\hat{x}) = \widetilde{B}_K\hat{x} + \tilde{b}_K$. For the structured mesh, the matrices B_K are all equal alternatively to $B_K = hI$ or $B_K = -hI$, where I is the identity in $\mathbb{R}^{2\times 2}$, while the translation vector b_K depends on the position inside the mesh. In the following, we will consider a single element K (respectively $\widetilde{K}$ in the perturbed mesh) such that $B_K = hI$ and denote $B = B_K$ (respectively $\widetilde{B} = \widetilde{B}_K$). Moreover, we will call its vertices x_1, x_2 and x_3 (respectively $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_3$). The matrix $\widetilde{B}$ is given by

$$\begin{aligned} \widetilde{B} &= \big(\tilde{x}_2 - \tilde{x}_1 \mid \tilde{x}_3 - \tilde{x}_1\big) \\ &= B + \Lambda h^{p+1}, \end{aligned} \tag{29}$$

where $\Lambda \in \mathbb{R}^{2\times 2}$ is defined as

$$\Lambda = \big(\alpha_2 - \alpha_1 \mid \alpha_3 - \alpha_1\big). \tag{30}$$

We can then compute the local contributions to $a(\tilde{\varphi}_i, \tilde{\varphi}_j)$ via integrals on the reference triangle as

$$
\begin{aligned}
\int_{\widetilde{K}} \nabla\tilde{\varphi}_i \cdot \nabla\tilde{\varphi}_j &= \int_{\widehat{K}} \widetilde{B}\nabla\hat{\varphi}_i \cdot \widetilde{B}\nabla\hat{\varphi}_j \frac{1}{|\det \widetilde{B}|} \\
&= \frac{|\widehat{K}|}{|\det \widetilde{B}|}(B + \Lambda h^{p+1})\nabla\hat{\varphi}_i \cdot (B + \Lambda h^{p+1})\nabla\hat{\varphi}_j \\
&= R\frac{|\widehat{K}|}{|\det B|} B\nabla\hat{\varphi}_i \cdot B\nabla\hat{\varphi}_j \\
&= R\int_K \nabla\varphi_i \cdot \nabla\varphi_j,
\end{aligned}
\tag{31}
$$

where $\hat{\varphi}_i$ and $\hat{\varphi}_j$ are the basis functions on the reference triangle corresponding to φ_i and φ_j and where

$$
R = \frac{|\det B|}{|\det \widetilde{B}|}\Big(1 + \frac{h^{p+1}(B\nabla\hat{\varphi}_i \cdot \Lambda\nabla\hat{\varphi}_j + B\nabla\hat{\varphi}_j \cdot \Lambda\nabla\hat{\varphi}_i) + h^{2p+2}\Lambda\nabla\hat{\varphi}_i \cdot \Lambda\nabla\hat{\varphi}_j}{B\nabla\hat{\varphi}_i \cdot B\nabla\hat{\varphi}_j}\Big), \tag{32}
$$

where $\hat{\nabla}\varphi_i$ and $\hat{\nabla}\varphi_j$ are not orthogonal. Being B and Λ in $\mathbb{R}^{2\times 2}$ and since $\det B = h^2$ and $B^{-1} = h^{-1}I$, it holds

$$
\begin{aligned}
\det \widetilde{B} &= \det B + h^{2p+2}\det\Lambda + \det B\,\mathrm{tr}(B^{-1}\Lambda)h^{p+1} \\
&= h^2 + h^{2p+2}\det\Lambda + \mathrm{tr}(\Lambda)h^{p+2}.
\end{aligned}
\tag{33}
$$

Hence, the ratio between the determinants is given by

$$
\begin{aligned}
\frac{|\det B|}{|\det \widetilde{B}|} &= \frac{1}{|1 + h^{2p}\det\Lambda + \mathrm{tr}(\Lambda)h^p|} \\
&\le 1 + |(h^{2p}\det\Lambda + \mathrm{tr}(\Lambda)h^p)\big(1 + \textstyle\sum_{l=1}^{\infty}(-1)^l(h^{2p}\det\Lambda + \mathrm{tr}(\Lambda)h^p)^l\big)| \\
&= 1 + C_1 h^p,
\end{aligned}
\tag{34}
$$

for the constant C_1 defined as

$$
C_1 = |(h^p\det\Lambda + \mathrm{tr}(\Lambda))\big(1 + \textstyle\sum_{l=1}^{\infty}(-1)^l(h^{2p}\det\Lambda + \mathrm{tr}(\Lambda)h^p)^l\big)|, \tag{35}
$$

which is bounded independently of h. Moreover,

$$
\begin{aligned}
\frac{(B\nabla\hat{\varphi}_i \cdot \Lambda\nabla\hat{\varphi}_j + B\nabla\hat{\varphi}_j \cdot \Lambda\nabla\hat{\varphi}_i)}{B\nabla\hat{\varphi}_i \cdot B\nabla\hat{\varphi}_j}h^{p+1} &= \frac{(\nabla\hat{\varphi}_i \cdot \Lambda\nabla\hat{\varphi}_j + \nabla\hat{\varphi}_j \cdot \Lambda\nabla\hat{\varphi}_i)}{\nabla\hat{\varphi}_i \cdot \nabla\hat{\varphi}_j}h^p \\
&= C_2 h^p,
\end{aligned}
\tag{36}
$$

where C_2 is bounded independently of h. Finally

$$
\begin{aligned}
\frac{\Lambda\nabla\hat{\varphi}_i \cdot \Lambda\nabla\hat{\varphi}_j}{B\nabla\hat{\varphi}_i \cdot B\nabla\hat{\varphi}_j}h^{2p+2} &= \frac{\Lambda\nabla\hat{\varphi}_i \cdot \Lambda\nabla\hat{\varphi}_j}{\nabla\hat{\varphi}_i \cdot \nabla\hat{\varphi}_j}h^{2p} \\
&= C_3 h^{2p},
\end{aligned}
\tag{37}
$$

where C_3 is bounded independently of h. Hence

$$
\int_{\widetilde{K}} \nabla\tilde{\varphi}_i \cdot \nabla\tilde{\varphi}_j - \int_K \nabla\varphi_i \cdot \nabla\varphi_j = \Big(\int_K \nabla\varphi_i \cdot \nabla\varphi_j\Big)Ch^p, \tag{38}
$$

where

$$
C = C_1 + C_2 + C_1C_2h^p + C_3h^p + C_1C_3h^{2p}, \tag{39}
$$

which proves the desired result.

2d mesh with $\kappa(x) = 1$

Let us now prove (11). Denoting by $\mathbf{w_h}$ and $\mathbf{v_h}$ the vector of the nodal values of w_h and v_h respectively, we have

$$\begin{aligned}\delta a(w_h, v_h) &= \sum_{i,j} (\mathbf{w_h})_i (\mathbf{v_h})_j \delta a(\varphi_i, \varphi_j) \\ &\leq Ch^p \sum_{i,j} (\mathbf{w_h})_i (\mathbf{v_h})_j a(\varphi_i, \varphi_j) \\ &\leq CMh^p \|w_h\|_V \|v_h\|_V,\end{aligned} \tag{40}$$

where we applied (10) and the continuity of a to get the desired result. □

Remark 1. Perturbation on the mesh is equivalent in algebraic formulation to a perturbation on the stiffness matrix A, as (10) yields

$$\widetilde{A}_{i,j} = A_{i,j} + C_{i,j} h^p. \tag{41}$$

If the $C_{i,j}$ can be written explicitly given the α_i, the stiffness matrix $\widetilde{A}_{i,j}$ does not have to be assembled for each perturbation of the mesh.

Lemma 2. *Under Assumption 1, there exists a constant $C_F > 0$ independent of h such that*

$$|F(P_h^{-1} v_h - v_h)| \leq C_F h^{p+1} \|v_h\|_V, \tag{42}$$

for all $v_h \in V_h$.

Proof. TO DO □

Lemma 3. *Under Assumption 1, let $\hat{u}_h \in \widetilde{V}_h$ be the unique solution of*

$$a(\hat{u}_h, P_h^{-1} v_h) = F(v_h), \tag{43}$$

for all $v_h \in V_h$. Then, if $\widehat{U}_h = P_h \hat{u}_h$,

$$\|\widehat{U}_h - u_h\|_V \leq C \|\widehat{U}_h\|_V h^p \quad a.s. \text{ in } \Omega, \tag{44}$$

where $C > 0$ is a constant independent of h.

Remark 2. Existence and uniqueness of $\hat{u}_h$ guaranteed as ...

Proof. For any $w_h, v_h \in V_h$

$$a(P_h^{-1} w_h, P_h^{-1} v_h) = a(w_h, v_h) + \delta a(w_h, v_h), \tag{45}$$

where $\delta a(w_h, v_h) = a(P_h^{-1} w_h, P_h^{-1} v_h) - a(w_h, v_h)$. From the definition of $\hat{u}_h$ and $\widehat{U}_h$, we have for all $v_h \in V_h$

$$a(P_h^{-1} \widehat{U}_h, P_h^{-1} v_h) = F(v_h), \tag{46}$$

which can be rewritten thanks to (45) as

$$a(\widehat{U}_h, v_h) + \delta a(\widehat{U}_h, v_h) = F(v_h). \tag{47}$$

Subtracting $a(u_h, v_h)$ on both sides gives

$$a(u_h - \widehat{U}_h, v_h) = \delta a(\widehat{U}_h, v_h). \tag{48}$$

By the coercivity of a and thanks to Lemma 1 we then have, defining $\hat{\varepsilon}_h = u_h - \widehat{U}_h$ and choosing $v_h = \hat{\varepsilon}_h$,

$$\alpha \|\hat{\varepsilon}_h\|_V^2 \leq Ch^p \|\hat{U}_h\|_V \|\hat{\varepsilon}_h\|_V, \tag{49}$$

which gives the desired result dividing both sides by $\alpha \|\hat{\varepsilon}_h\|_V$. □

Lemma 4. *Under Assumption 1, let $\tilde{u}_h \in \widetilde{V}_h$ be the unique solution of*

$$a(\tilde{u}_h, \tilde{v}_h) = F(\tilde{v}_h), \tag{50}$$

for all $\tilde{v}_h \in \widetilde{V}_h$. Then, if $U_h = P_h \tilde{u}_h$ and h is sufficiently small

$$\|U_h - \hat{U}_h\|_V \leq Ch^{p+1} \quad a.s. \text{ in } \Omega, \tag{51}$$

where $C > 0$ is a constant independent of h.

Proof. Thanks to the definition of $\tilde{u}_h$ and $\hat{u}_h$, we have for all $v_h \in V_h$

$$a(P_h^{-1} U_h - P_h^{-1} \widehat{U}_h, P_h^{-1} v_h) = F(P_h^{-1} v_h - v_h). \tag{52}$$

Thanks to (45) and the linearity of P_h^{-1}, we then obtain

$$a(U_h - \widehat{U}_h, v_h) = F(P_h^{-1} v_h - v_h) - \delta a(U_h - \widehat{U}_h, v_h). \tag{53}$$

Exploiting the coercivity of a, introducing the notation $\varepsilon_h = U_h - \widehat{U}_h$ and choosing $v_h = \varepsilon_h$, we get

$$\alpha \|\varepsilon_h\|_V^2 \leq F(P_h^{-1} \varepsilon_h - \varepsilon_h) - \delta a(\varepsilon_h, \varepsilon_h). \tag{54}$$

Let us consider the two terms on the right hand side separately. For the first term we apply directly Lemma 2 and for the second term we apply Lemma 1, thus obtaining

$$\alpha \|\varepsilon_h\|_V^2 \leq C_F h^{p+1} \|\varepsilon_h\|_V + C\|\varepsilon_h\|_V^2, \tag{55}$$

where C is given in Lemma 1. Dividing both sides by $\|\varepsilon_h\|_V$ we finally get

$$(\alpha - C)\|\varepsilon_h\|_V \leq C_F h^{p+1}. \tag{56}$$

Hence, if h satisfies

$$h < \left(\frac{\alpha}{C}\right)^{1/p}, \tag{57}$$

we obtain the desired result. □

Theorem 1. *Under Assumption 1,*

$$\|U_h - u_h\|_V \leq Ch^p \quad a.s. \text{ in } \Omega, \tag{58}$$

where $C > 0$ is a constant independent of h.

Proof. The triangular inequality yields

$$\begin{aligned} \|U_h - u_h\|_V &\leq \|U_h - \hat{U}_h\|_V + \|\hat{U}_h - u_h\|_V \\ &\leq C_1 h^p + C_2 h^{p+1}, \end{aligned} \tag{59}$$

where we applied Lemma 4 and Lemma 3. □

Theorem 2. *Under Assumption 1,*

$$\|U_h - u\|_V \leq Ch \quad a.s. \text{ in } \Omega, \tag{60}$$

where $C > 0$ is a constant independent of h.

Proof. The triangular inequality and the standard convergence analysis of the finite elements method yields

$$\begin{aligned} \|U_h - u\|_V &\leq \|U_h - u_h\|_V + \|u - u_h\|_V \\ &\leq C_1 h^p + C_2 h, \end{aligned} \tag{61}$$

which gives the desired result since $p \geq 1$. □

4 A posteriori error analysis

5 Mesh adaptivity

6 Inverse problems

7 Numerical experiments

7.1 Convergence

7.2 Error estimators

7.3 Mesh adaptivity

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