Idea

Let us consider $f\colon \mathbb{R}^d \to \mathbb{R}^d$ and the differential equation

$$u' = f(u), \quad u(0) = u_0 \in \mathbb{R}^d. \tag{1}$$

Moreover, let us consider the Runge-Kutta approximation $U_k(h)$ at time $t_k = kh$ depending on the time step $h > 0$, i.e.

$$U_k(h) = \Psi_h(U_{k-1}), \quad U_0(h) = u_0, \tag{2}$$

where $\Psi_t(\cdot)$ is the numerical flow of the RK method applied to (1). Finally, we consider $U_k^\xi(h)$ the solution given by the probabilistic solver with time step h and r.v. ξ, i.e.

$$U_k^\xi(h) = \Psi_h(U_{k-1}) + \xi_k(h),$$

where we use the same RK method as deterministic part. Now consider (1) to be chaotic (e.g. Lorenz). In this case, since the system is chaotic, integrating with two different time steps h_1 and h_2 we get two different numerical solutions, hence the punctual solution with obtained with any time steps is not reliable (motivation of Probabilistic method).

Given a limit time step $h_{\max}$ (e.g., the maximum time step for which Ψ is stable applied to (1)), we could think of the numerical solution as depending continuously on the time step h, and then we could consider the long time behavior of the numerical solution w.r.t. h. For example, we could investigate if there exists a measure $\mu^{h_{\max}}$ such that for a function $\Phi\colon \mathbb{R}^d \to \mathbb{R}$ we have

$$\lim_{N\to\infty} \frac{1}{h_{\max}} \int_0^{h_{\max}} \Phi(U_N(s))\mathrm{d}s = \int_{\mathbb{R}^d} \Phi(y)\mathrm{d}\mu^{h_{\max}}(y), \tag{3}$$

where U_N is the **deterministic** numerical solution (2). It is clear that if the ODE is not chaotic (e.g., test equation with $\lambda < 0$), at long term all the numerical solutions with different time steps will converge to the same stable value (e.g., zero), therefore $\mu = \delta$ for some Dirac delta. Now considering the probabilistic solver with fixed time step h, we could have a measure $\nu^{h,M}$ such that

$$\lim_{N\to\infty} \frac{1}{M} \sum_{i=1}^{M} \Phi(U_N^{\xi^{(i)}}(h)) = \int_{\mathbb{R}^d} \Phi(y)\mathrm{d}\nu^{h,M}(y), \tag{4}$$

where $\xi^{(i)}$ are realizations of ξ for $i = 1, \ldots, M$. Now, are $\mu^{h_{\max}}$ and $\nu^{h,M}$ approximations of the same probability measure?

Numerical example

Consider the Lorenz system

$$\begin{aligned} x' &= \sigma(y - x), & x(0) &= -10, \\ y' &= x(\rho - z) - y, & y(0) &= -1, \\ z' &= xy - \beta z, & z(0) &= 40. \end{aligned}$$

with $\sigma = 10$, $\rho = 28$, $\beta = 8/3$, so that the system has chaotic behavior. We consider final time $T = 200$ and the explicit midpoint as RK. Moreover, we consider

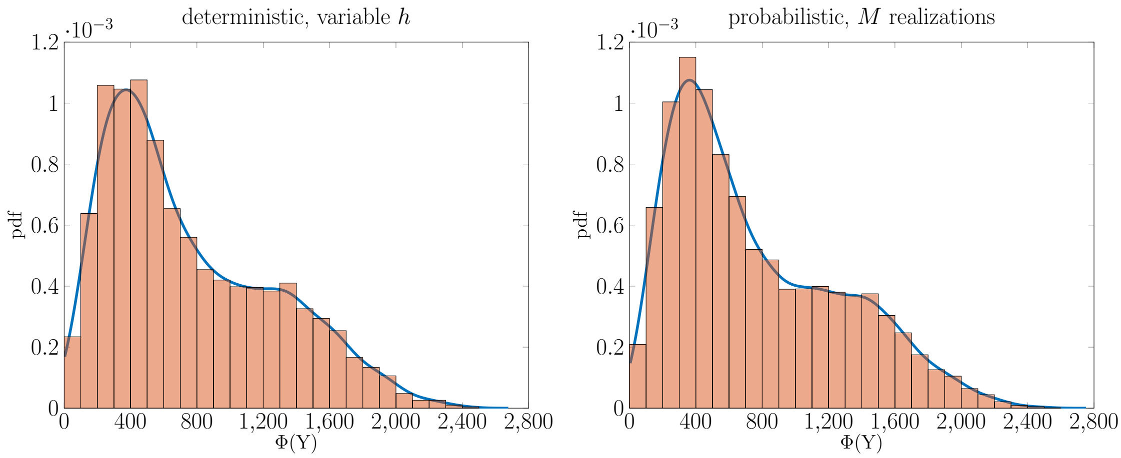

Figure 1: Density of $\Phi(Y)$.

$\frac{1}{h_{\max}-h_{\min}} \int_{h_{\min}}^{h_{\max}} \Phi(U_N(s))\mathrm{d}s$	$\int_{\mathbb{R}^3} \Phi(y)\mathrm{d}\mu^{h_{\max}}(y)$	$\frac{1}{M}\sum_{i=1}^{M} \Phi(U_N^{\xi^{(i)}}(h))$	$\int_{\mathbb{R}^3} \Phi(y)\mathrm{d}\nu^{h,M}(y)$
769.9	759.0	763.1	763.3

Table 1: Numerical results.

- deterministic solver with variable h (4000 different values of h in the range $[1.68 \cdot 10^{-4}, 2.5 \cdot 10^{-2}]$)
- probabilistic solver with fixed $h = 1 \cdot 10^{-3}$, and $M = 10^4$ trajectories
- the function $\Phi\colon \mathbb{R}^3 \to \mathbb{R}$ defined as

$$\Phi(x) = x^T x.$$

Then we consider the value of the functional Φ applied to the solution at final time obtained with the deterministic solver and probabilistic solver (Figure 1). Numerical results of the integrals in (3) and (4) displayed in Table 1 seem to confirm the idea that $\mu^{h_{\max}}$ and $\nu^{h,M}$ tend to the same measure. The integrals with respect to the measure are computed assuming the two measures admit a density function and approximating the density function from data (blue lines in plots).