

OPTIMIZATION-BASED COUPLING METHOD FOR MULTISCALE PROBLEMS

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GOAL

Assume that in some region ω of a domain Ω homogenization methods fail and a fine scale solver must be used instead. A coupling between the fine and coarse solvers should be done.

Question. What boundary data should be used on the interface?

Classical global-local. Use homogenized solution u^0 [3].

Drawbacks. Expensive to compute u^0 ; depends on the quality of u^0 .

Goal. Find better boundary data.

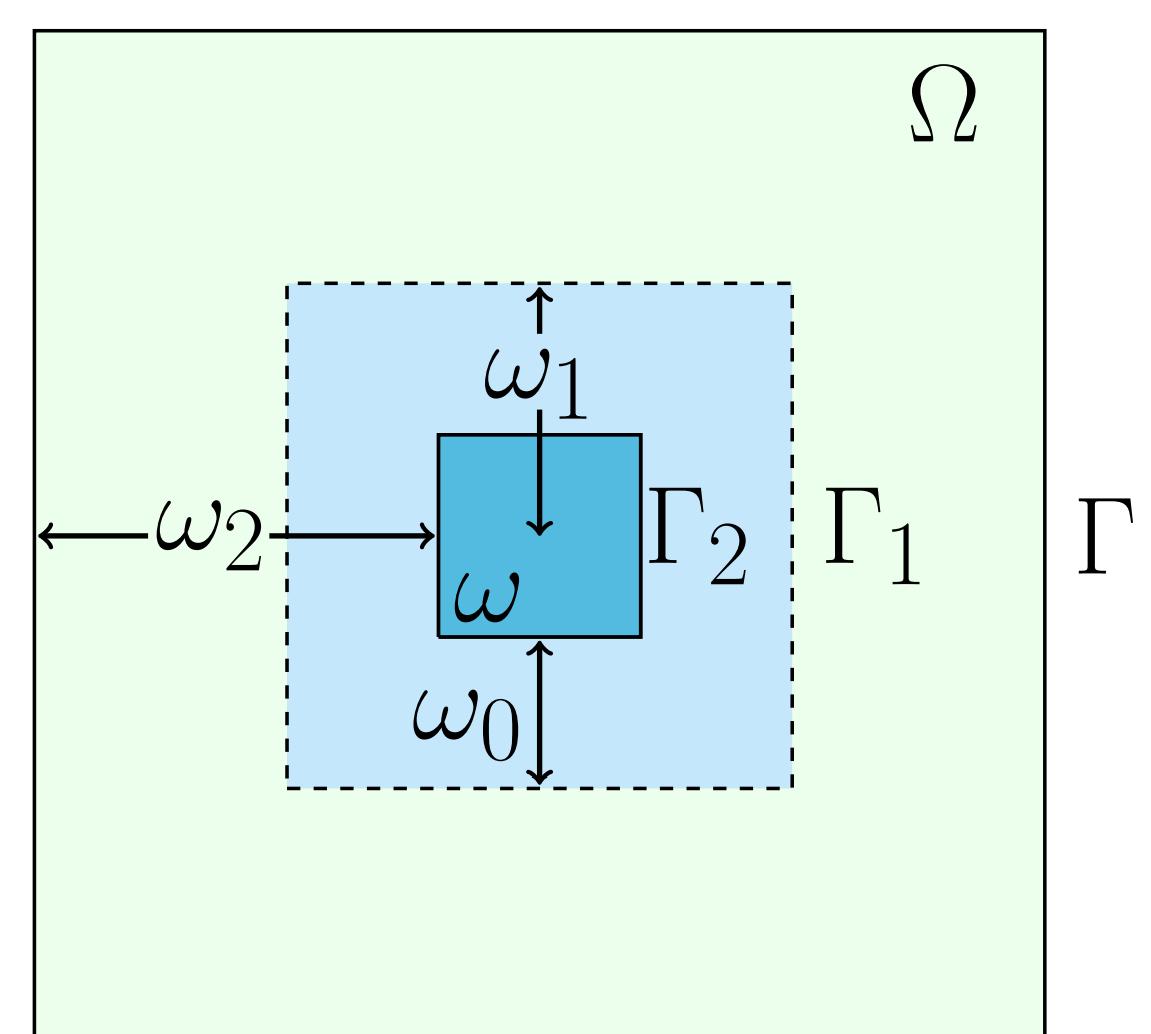
MULTISCALE ELLIPTIC PROBLEM

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and consider: find $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} -\operatorname{div}(a\nabla u) &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma, \end{aligned}$$

with highly heterogeneous uniformly bounded tensor a or source term f .

Idea. Introduce an overlap and solve a minimization problem with constraints [2].



OPTIMIZATION-BASED COUPLING METHOD

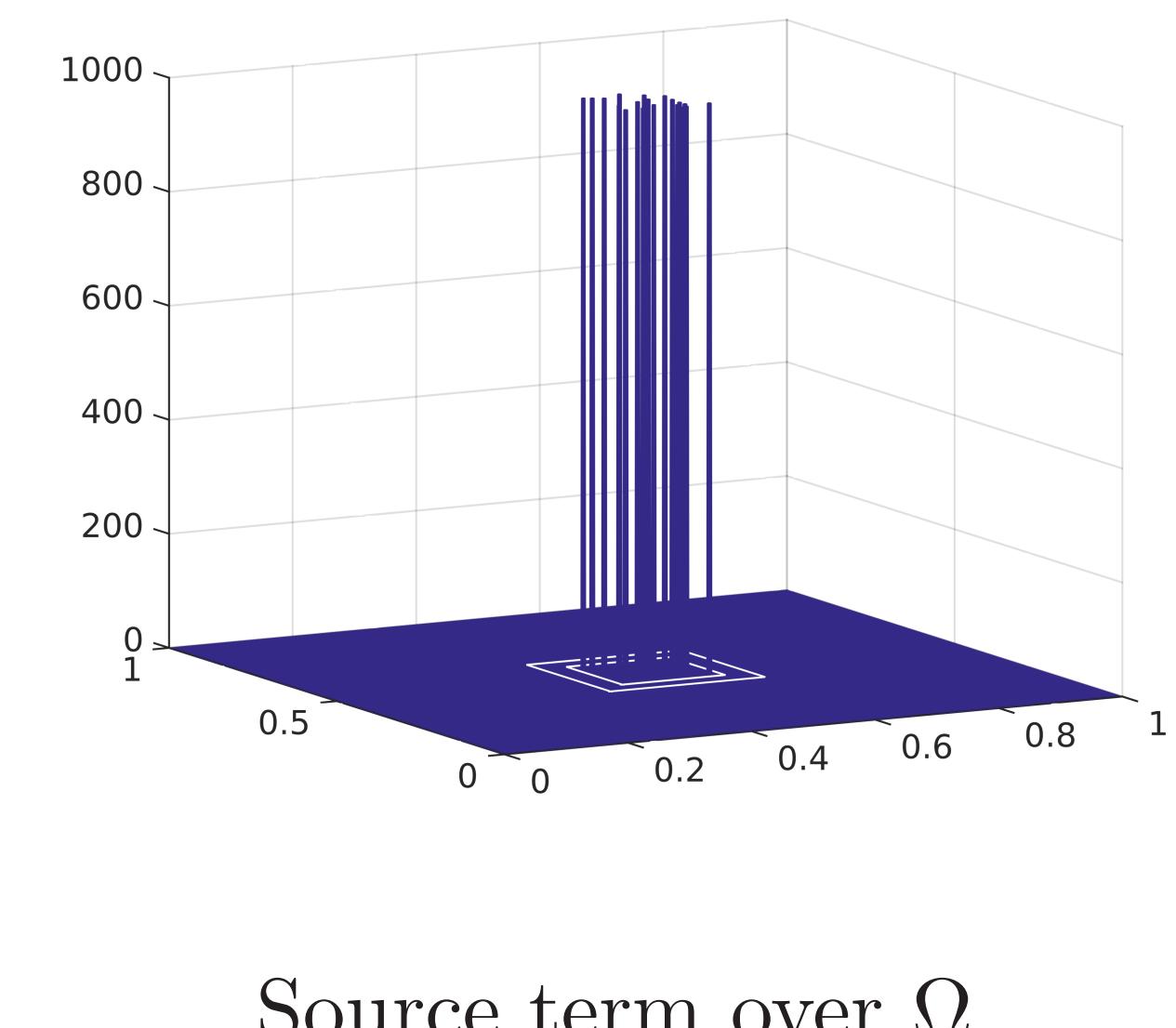
Optimization-based method [1]. Find $(u_1, u_2) \in H^1(\omega_1) \times H_\Gamma^1(\omega_2)$:

$$\min \frac{1}{2} \|u_1 - u_2\|_{L^2(\omega_0)}^2 \text{ such that} \begin{cases} -\operatorname{div}(a\nabla u_1) = f \text{ in } \omega_1, u_1 = \theta_1 \text{ on } \Gamma_1, \\ -\operatorname{div}(a^0 \nabla u_2) = f^0 \text{ in } \omega_2, u_2 = \theta_2 \text{ on } \Gamma_2. \end{cases}$$

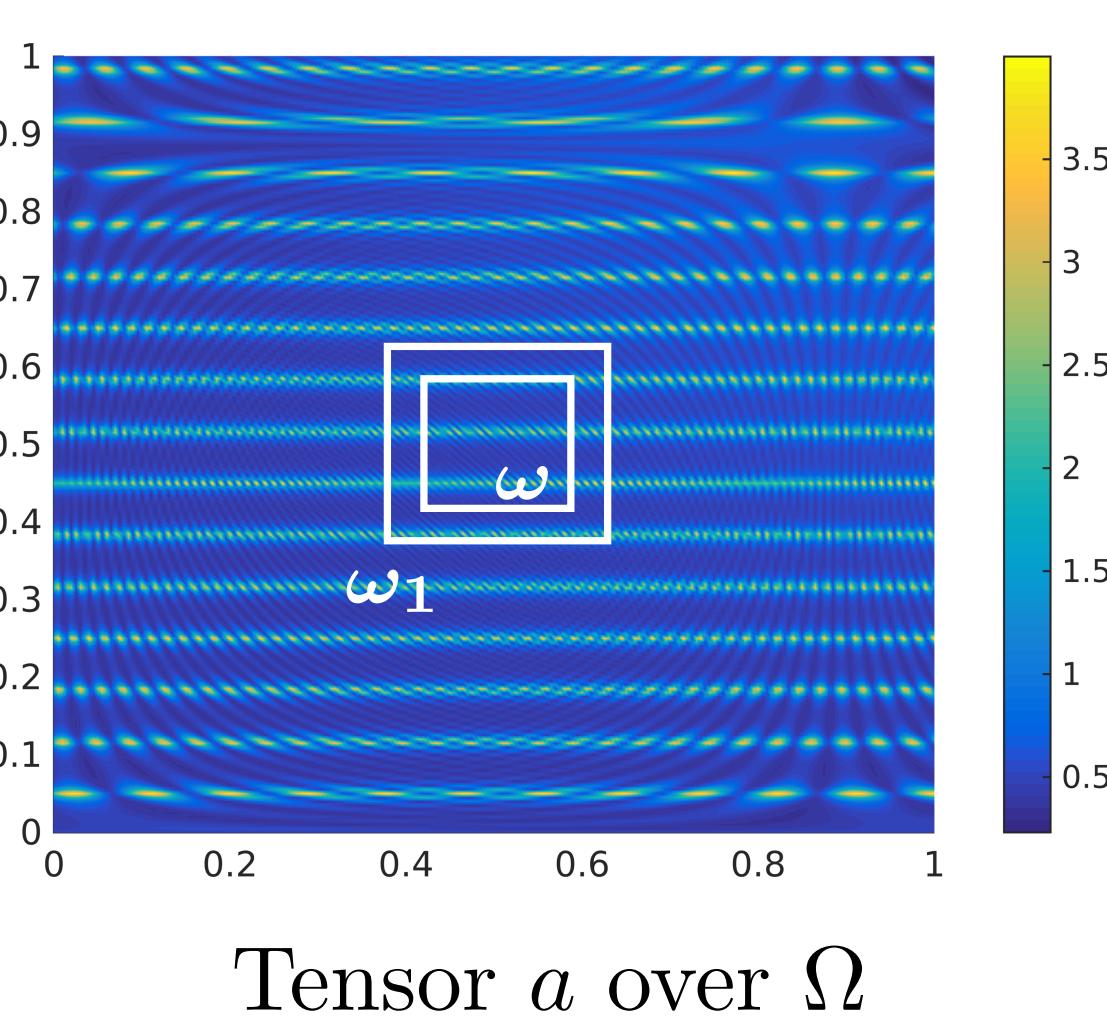
Introducing Lagrange multipliers for each constraint, (u_1, u_2) is given as a critical point of a saddle point problem.

NUMERICAL EXPERIMENTS

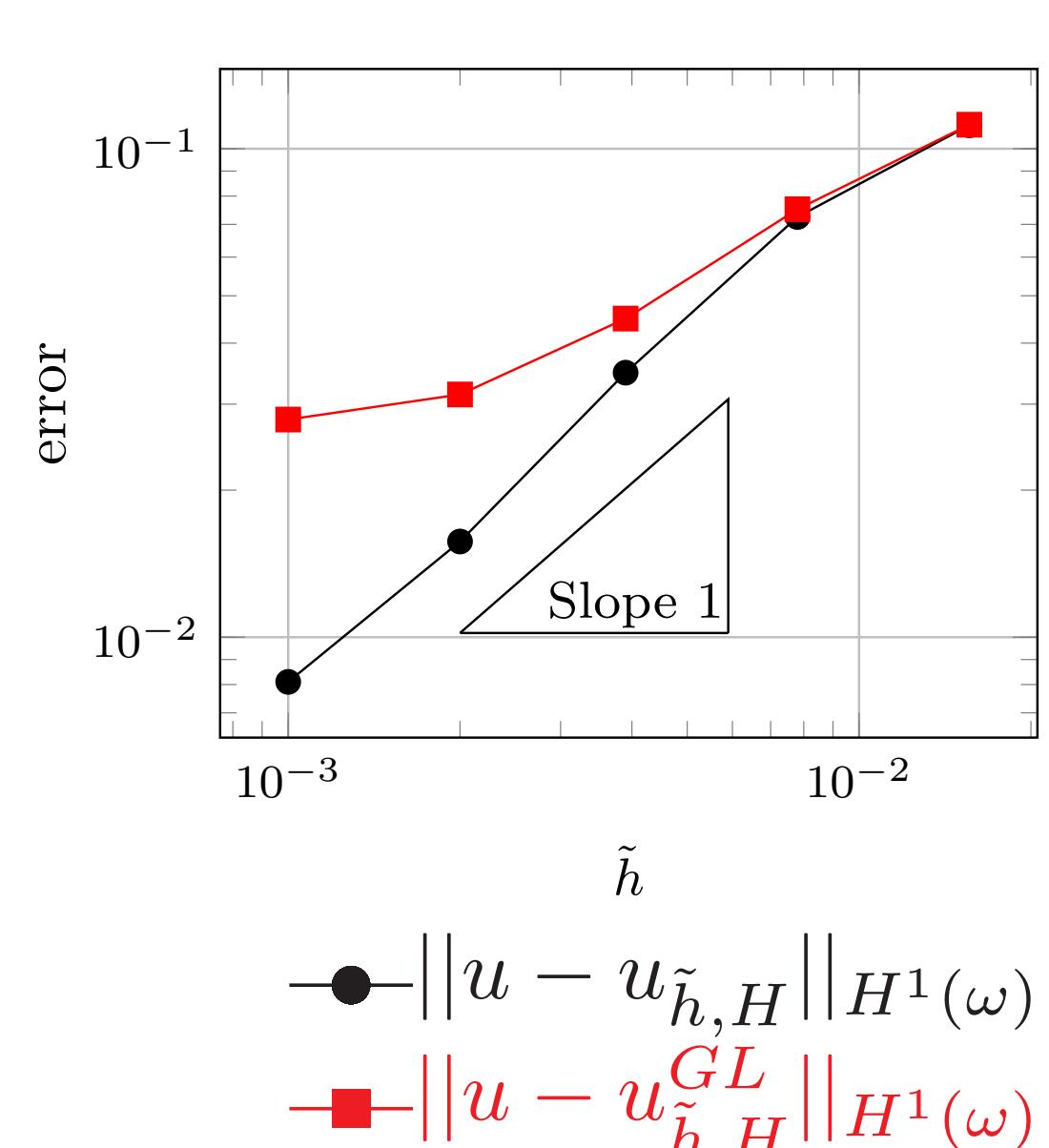
- f is generated by random peaks in ω ,
- consider discontinuities in the mesh \mathcal{T}_H on Γ_1 ,
- compute numerical HMM solution u_H^0 on Ω ,
- compute global-local solution $u_{h,H}^{GL}$ using u_H^0 as B.C on Γ_1 .



Source term over Ω

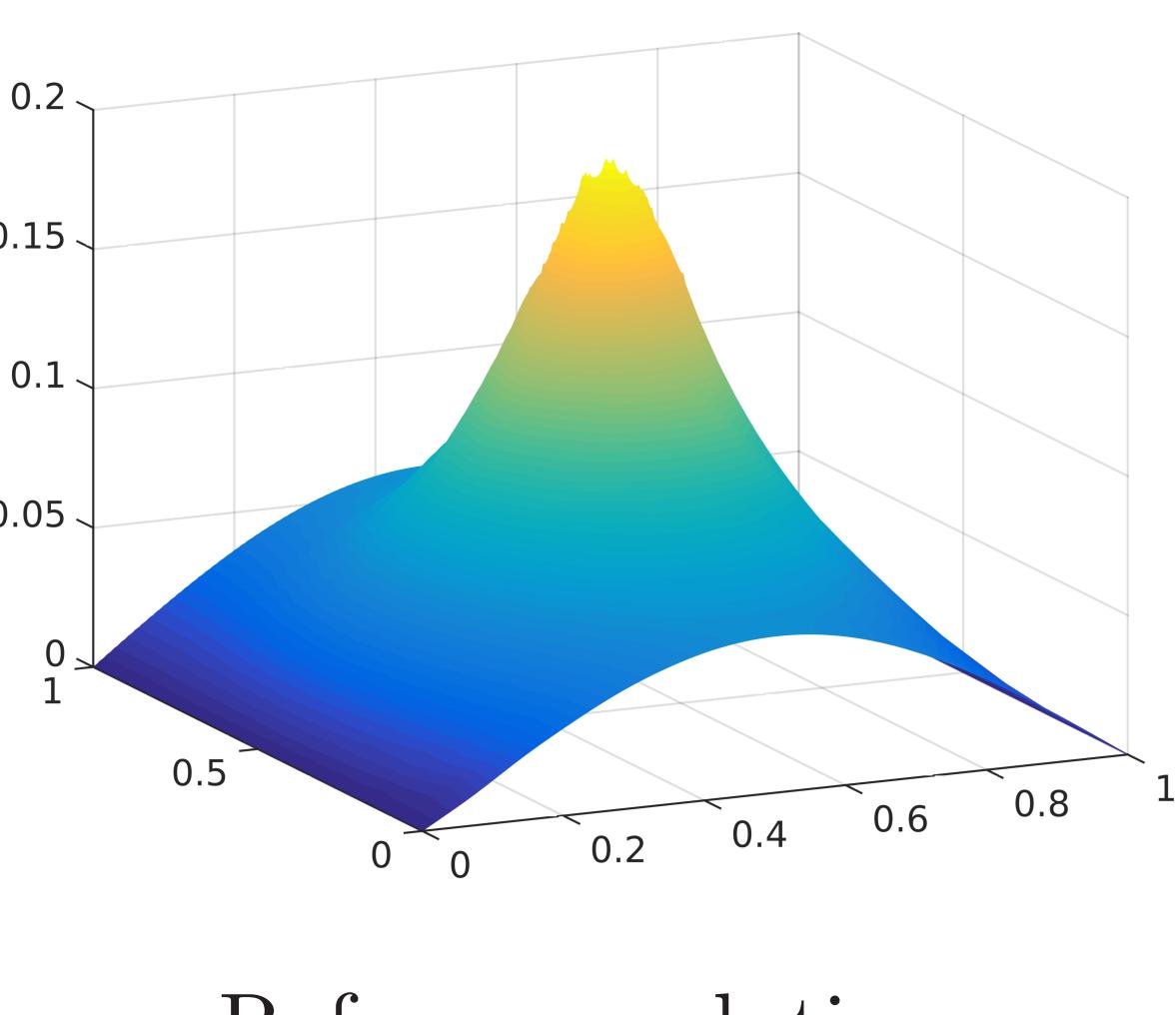


Tensor a over Ω

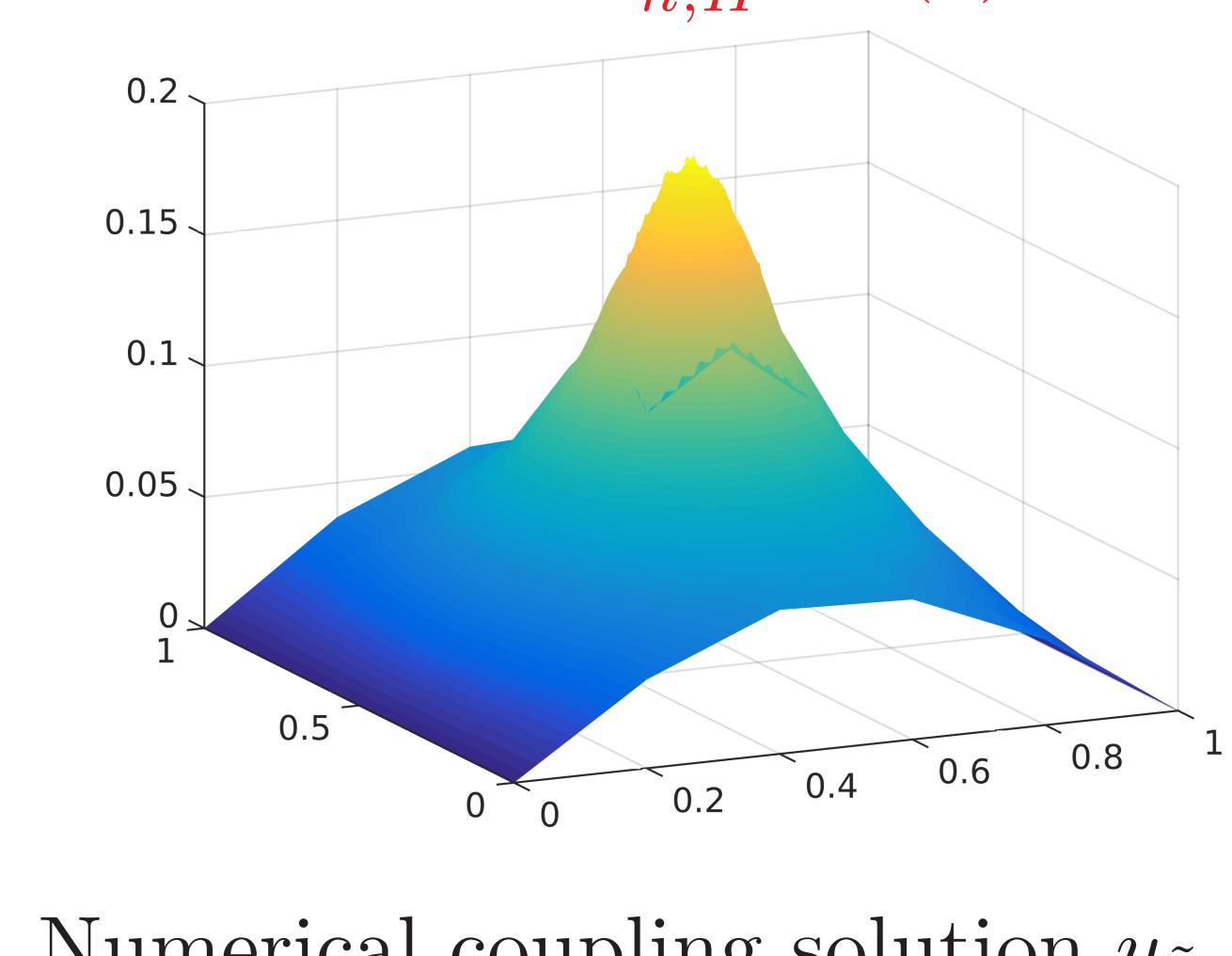


error

—●— $\|u - u_{h,H}\|_{H^1(\omega)}$
 —■— $\|u - u_{h,H}^{GL}\|_{H^1(\omega)}$



Reference solution

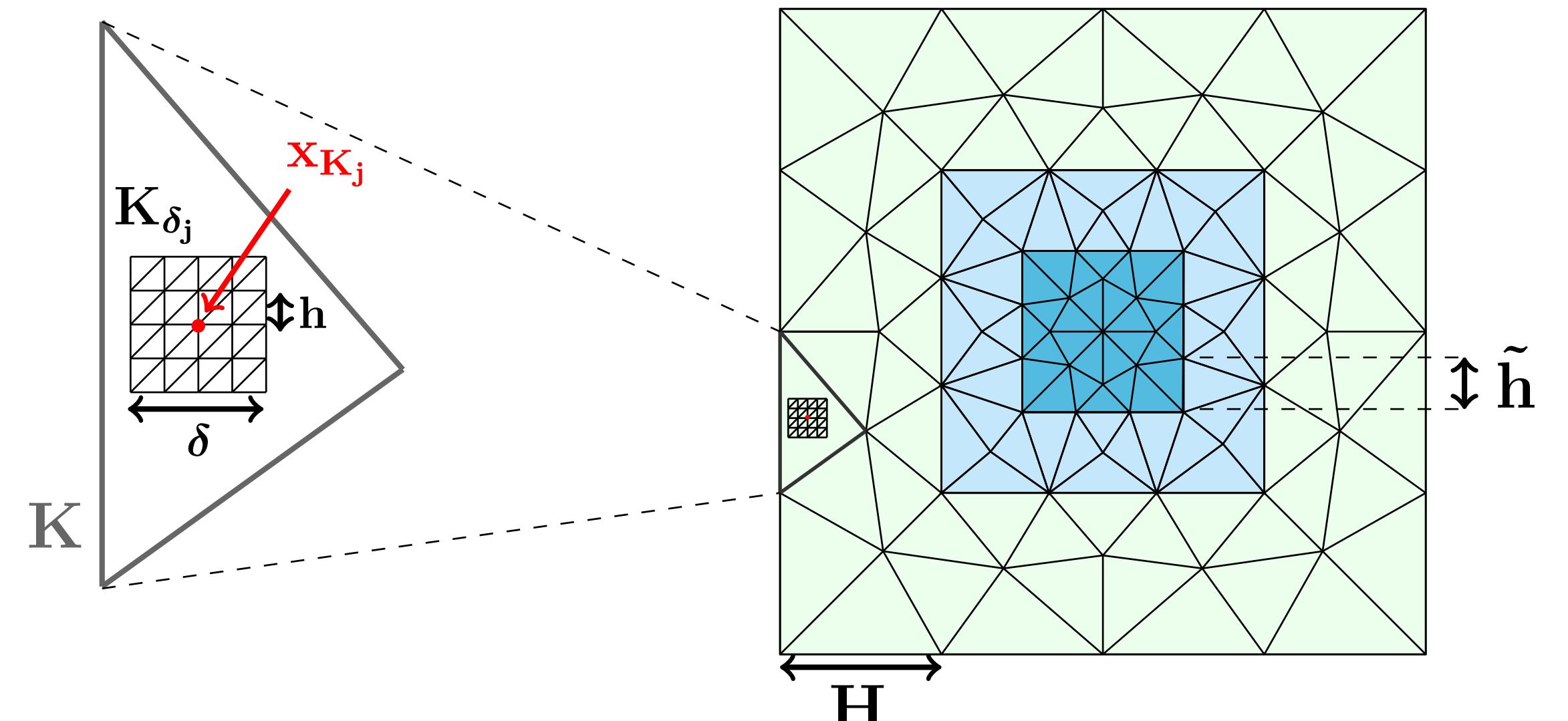


Numerical coupling solution $u_{h,H}$

FE OPTIMIZATION BASED COUPLING

Let $V^p(\omega_1, \mathcal{T}_{\tilde{h}})$, $V_\Gamma^p(\omega_2, \mathcal{T}_H)$ and $S^q(K_{\delta_j}, \mathcal{T}_h)$ be FE spaces with

$$h, \tilde{h} < \varepsilon, \text{ and } H \gg h.$$



Fine scale solver Find $u_{1,\tilde{h}} = u_{1,0,\tilde{h}} + v_{1,\tilde{h}}$, with $u_{1,0,\tilde{h}} \in V_0^p(\omega_1, \mathcal{T}_{\tilde{h}})$ such that

$$B_1(u_{1,0,\tilde{h}}, w_{\tilde{h}}) := \int_{\omega_1} a \nabla u_{1,0,\tilde{h}} \cdot \nabla w_{\tilde{h}} dx = \int_{\omega_1} f w_{\tilde{h}} dx, \quad \forall w_{\tilde{h}} \in V_0^p(\omega_1, \mathcal{T}_{\tilde{h}}).$$

FE-HMM solver Find $u_{2,H} = u_{2,0,H} + v_{2,H}$, with $u_{2,0,H} \in V_0^p(\omega_2, \mathcal{T}_H)$ such that, $\forall w_H \in V_0^p(\omega_2, \mathcal{T}_H)$

$$B_2(u_{2,0,H}, w_H) := \sum_{K \in \mathcal{T}_H} \sum_{j=1}^J \frac{w_{j,K}}{|K_{\delta_j}|} \int_{K_{\delta_j}} a \nabla u_j^h \cdot \nabla w_j^h dx = \int_{\omega_2} f^0 w_H dx,$$

where u_j^h, w_j^h are solutions of a micro problem involving the tensor a on K_{δ_j} . The solution $(v_{1,\tilde{h}}, v_{2,H})$ is the critical point of

$$\begin{aligned} \mathcal{L}(v_{1,\tilde{h}}, \lambda_{1,\tilde{h}}, v_{2,H}, \lambda_{2,H}) &= \frac{1}{2} \|(u_{1,0,\tilde{h}} + v_{1,\tilde{h}}) - (u_{2,0,H} + v_{2,H})\|_{L^2(\omega_0)}^2 \\ &\quad - B_1(v_{1,\tilde{h}}, \lambda_{1,\tilde{h}}) - B_2(v_{2,H}, \lambda_{2,H}). \end{aligned}$$

A PRIORI ERROR ESTIMATES

Let $u_{\tilde{h},H}$ be the numerical coupling solution. It holds

$$\|u - u_{\tilde{h},H}\|_{H^1(\omega)} \leq C \left(\tilde{h} |u|_{H^2(\omega_1)} + \frac{C_1}{\tau} (\varepsilon + \operatorname{err}_{HMM,L^2(\omega_2)}) \right),$$

where

$$\tau = \operatorname{dist}(\Gamma_1, \Gamma_2),$$

the ε term comes from the bound $\|u - u^0\|_{L^2(\omega_0)}$,

$$\operatorname{err}_{HMM,L^2(\omega_2)} \leq C \left(H^2 + \left(\frac{h}{\varepsilon} \right)^2 + \operatorname{err}_{mod} \right).$$

REFERENCES

- [1] Assyr Abdulle and Orane Jecker. An optimization-based heterogeneous to homogeneous coupling method. To appear in *Communications in Math Sciences*, 2015.
- [2] Paola Gervasio, Jacques-Louis Lions, and Alfio Quarteroni. Heterogeneous coupling by virtual control methods. *Numer. Math.*, 90:241–264, 2000.
- [3] J. Tinsley Oden and Kumar S. Vemaganti. Estimation of local modeling error and goal-oriented adaptive modeling of heterogeneous materials. I. Error estimates and adaptive algorithms. *J. Comput. Phys.*, 164(1):22–47, 2000.