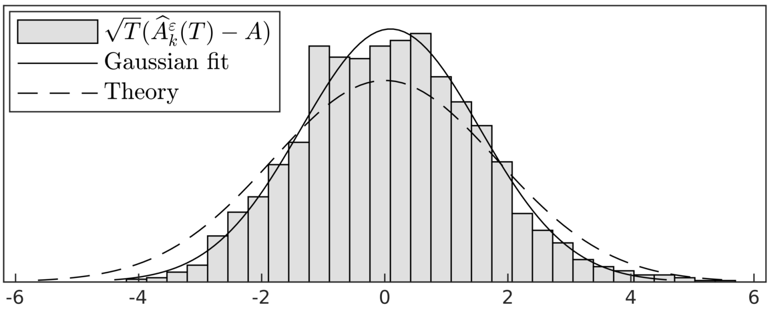

Figure 1: Central limit theorem result. The histogram represents numerical results, the solid curve a Gaussian fit to the latter and the dashed curve the theoretical estimate given in Theorem 0.1.

0.1 Central limit theorem

Text to be added with motivation, proof in Appendix 1.

Theorem 0.1. *Central limit theorem statement:*

$$\lim_{\varepsilon\to 0}\lim_{T\to\infty} \sqrt{T}\left(\widehat{A}_k^\varepsilon(T) - A\right) = \Lambda, \quad \textit{in law,}$$

where $\Lambda \sim \mathcal{N}(0,\Gamma)$ *and*

$$\Gamma = 2\sigma\, \mathbb{E}^{\rho^0}[V'(Z)\otimes V'(X)]^{-1}\, \mathbb{E}^{\rho^0}[V'(Z)\otimes V'(Z)]\, \mathbb{E}^{\rho^0}[V'(X)\otimes V'(Z)]^{-1}.$$

This covariance may not be correct (see proof in the Appendix) but it is just the limit of the "third term"

0.2 Central limit theorem

In this experiment we wish to confirm the validity of Theorem 0.1. We consider the same test equation as for Section **??**, i.e., the quadratic potential $V(x) = x^2/2$ with fluctuating potential $p(y) = \sin(y)$, multiscale parameter $\varepsilon = 0.05$ and diffusion coefficient $\sigma = 1$. The parameters of the filter are set to $\beta = 1$ and $\delta = 1$. We compute the estimator $\widehat{A}_k^\varepsilon(T)$ with final time $T = 10^3$ on 2000 realizations of the solution and estimate the quantity

$$\Delta_A^\varepsilon(T) \coloneqq \sqrt{T}\left(\widehat{A}_k^\varepsilon(T) - A\right),$$

where A is the drift coefficient of the homogenized equation. Results, depicted in Figure 1, show that the distribution of $\Delta_A^\varepsilon(T)$ indeed follows a zero-mean Gaussian law, whose covariance agrees with the theoretical results (?).

1 Proof of Theorem 0.1

Let us first introduce a technical Lemma.

Lemma 1.1. *Let* $\mathcal{L}_\varepsilon$ *be the generator of the couple* $(X^\varepsilon, Z^\varepsilon)^\top$*, i.e.,*

$$\mathcal{L}_\varepsilon = -\left(\alpha\cdot V'(x) + \frac{1}{\varepsilon}p'\left(\frac{x}{\varepsilon}\right)\right)\partial_x + \frac{1}{\delta}(x-z)\partial_z + \sigma\partial_{xx}^2.$$

Moreover, let ρ^ε *be the density of the invariant measure of* $(X^\varepsilon, Z^\varepsilon)^\top$ *and* $u^\varepsilon\colon \mathbb{R}^2 \to \mathbb{R}^N$ *be the solution of*

$$-\mathcal{L}_\varepsilon u^\varepsilon = \chi^\varepsilon - \mathbb{E}^{\rho^\varepsilon}[\chi^\varepsilon(X^\varepsilon, Z^\varepsilon)], \tag{1.1}$$

satisfying $\mathbb{E}^{\rho^\varepsilon}[u^\varepsilon(X^\varepsilon, Z^\varepsilon)] = 0$ *for* $\chi^\varepsilon \colon \mathbb{R}^2 \to \mathbb{R}^N$. *Then, it holds*

$$\frac{1}{T}\int_0^T \chi^\varepsilon(X_t^\varepsilon, Z_t^\varepsilon)\,\mathrm{d}t = \mathbb{E}^{\rho^\varepsilon}[\chi^\varepsilon(X^\varepsilon, Z^\varepsilon)] - \frac{R^\varepsilon(T)}{T} + \sqrt{2\sigma}\frac{S^\varepsilon(T)}{T}, \tag{1.2}$$

where

$$R^\varepsilon(T) \coloneqq u^\varepsilon(X_T^\varepsilon, Z_T^\varepsilon) - u^\varepsilon(X_0^\varepsilon, Z_0^\varepsilon), \qquad S^\varepsilon(T) \coloneqq \int_0^T \partial_x u^\varepsilon(X_t^\varepsilon, Z_t^\varepsilon)\,\mathrm{d}W_t. \tag{1.3}$$

Moreover, it holds

$$\begin{aligned} 2\sigma\,\mathbb{E}^{\rho^\varepsilon}[\partial_x u^\varepsilon(X^\varepsilon, Z^\varepsilon) \otimes \partial_x u^\varepsilon(X^\varepsilon, Z^\varepsilon)] = \mathbb{E}^{\rho^\varepsilon}\,[&\chi^\varepsilon(X^\varepsilon, Z^\varepsilon) \otimes u^\varepsilon(X^\varepsilon, Z^\varepsilon) \\ &+ u^\varepsilon(X^\varepsilon, Z^\varepsilon) \otimes \chi^\varepsilon(X^\varepsilon, Z^\varepsilon)]. \end{aligned} \tag{1.4}$$

Proof. The proof of (1.2) and (1.3) is an application of the Itô formula (see e.g. [**?**, Remark 6.17]). For (1.4), it is possible to show that since $\mathcal{L}_\varepsilon^*\rho^\varepsilon = 0$ it holds

$$\mathcal{L}_\varepsilon^*(u^\varepsilon\rho^\varepsilon) = 2\sigma\rho^\varepsilon\partial_{xx}^2 u^\varepsilon - \rho^\varepsilon\mathcal{L}_\varepsilon u^\varepsilon + 2\sigma\partial_x u^\varepsilon\partial_x\rho^\varepsilon.$$

Therefore, an integration by parts yields

$$\begin{aligned} \mathbb{E}^{\rho^\varepsilon}[\mathcal{L}_\varepsilon u^\varepsilon(X^\varepsilon, Z^\varepsilon) \otimes u^\varepsilon(X^\varepsilon, Z^\varepsilon)] &= \int_\mathbb{R}\int_\mathbb{R} u^\varepsilon \otimes \mathcal{L}_\varepsilon^*\,(u^\varepsilon\rho^\varepsilon)\,\mathrm{d}x\,\mathrm{d}z \\ &= -\int_\mathbb{R}\int_\mathbb{R} u^\varepsilon \otimes \mathcal{L}_\varepsilon u^\varepsilon\rho^\varepsilon\,\mathrm{d}x\,\mathrm{d}z - 2\sigma\int_\mathbb{R}\int_\mathbb{R}\partial_x u^\varepsilon \otimes \partial_x u^\varepsilon\rho^\varepsilon\,\mathrm{d}x\,\mathrm{d}z \\ &= -\,\mathbb{E}^{\rho^\varepsilon}[u^\varepsilon(X^\varepsilon, Z^\varepsilon) \otimes \mathcal{L}_\varepsilon u^\varepsilon(X^\varepsilon, Z^\varepsilon)] \\ &\quad - 2\sigma\,\mathbb{E}^{\rho^\varepsilon}[\partial_x u^\varepsilon(X^\varepsilon, Z^\varepsilon) \otimes \partial_x u^\varepsilon(X^\varepsilon, Z^\varepsilon)]. \end{aligned}$$

Finally, since $\mathbb{E}^{\rho^\varepsilon}[u(X^\varepsilon, Z^\varepsilon)] = 0$

$$\begin{aligned} 2\sigma\,\mathbb{E}^{\rho^\varepsilon}[\partial_x u^\varepsilon(X^\varepsilon, Z^\varepsilon) \otimes \partial_x u^\varepsilon(X^\varepsilon, Z^\varepsilon)] &= -\,\mathbb{E}^{\rho^\varepsilon}\,[\mathcal{L}_\varepsilon u^\varepsilon(X^\varepsilon, Z^\varepsilon) \otimes u^\varepsilon(X^\varepsilon, Z^\varepsilon) \\ &\qquad + u^\varepsilon(X^\varepsilon, Z^\varepsilon) \otimes \mathcal{L}_\varepsilon u^\varepsilon(X^\varepsilon, Z^\varepsilon)] \\ &= \mathbb{E}^{\rho^\varepsilon}\,[\chi^\varepsilon(X^\varepsilon, Z^\varepsilon) \otimes u^\varepsilon(X^\varepsilon, Z^\varepsilon) \\ &\qquad + u^\varepsilon(X^\varepsilon, Z^\varepsilon) \otimes \chi^\varepsilon(X^\varepsilon, Z^\varepsilon)], \end{aligned}$$

which is the desired result. □

Lemma 1.2. *Let* u^ε *be the solution of* (1.1) *with*

$$\chi^\varepsilon(x, z) = \frac{1}{\varepsilon}p'\left(\frac{x}{\varepsilon}\right)V'(z) - V'(z) \otimes V'(x)\mathcal{M}_\varepsilon^{-1}\mathfrak{p}_\varepsilon,$$

where

$$\mathcal{M}_\varepsilon \coloneqq \mathbb{E}^{\rho^\varepsilon}[V'(Z^\varepsilon) \otimes V'(X^\varepsilon)], \qquad \mathfrak{p}_\varepsilon \coloneqq \mathbb{E}^{\rho^\varepsilon}\left[\frac{1}{\varepsilon}p'\left(\frac{X^\varepsilon}{\varepsilon}\right)V'(Z^\varepsilon)\right].$$

Then, $u^\varepsilon \to 0$ *in which sense? for* $\varepsilon \to 0$.

Proof. We here present a formal proof based on asymptotic expansion with respect to ε. Let us first remark that by definition

$$\mathbb{E}^{\rho^\varepsilon}[\chi^\varepsilon(X^\varepsilon, Z^\varepsilon)] = 0,$$

and therefore problem (1.1) reads

$$-\,\mathcal{L}^\varepsilon u^\varepsilon = \chi^\varepsilon.$$

Let us now denote $y = x/\varepsilon$ and write

$$u^\varepsilon(x, z) = u_0(x, y, z) + \varepsilon u_1(x, y, z) + \varepsilon^2 u_2(x, y, z) + \dots,$$

which implies that

$$\partial_x u^\varepsilon = \partial_x(u_0(x,y,z) + \varepsilon u_1(x,y,z) + \varepsilon^2 u_2(x,y,z) + \dots) + \frac{1}{\varepsilon}\partial_y(u_0(x,y,z) + \varepsilon u_1(x,y,z) + \varepsilon^2 u_2(x,y,z) + \dots).$$

Let us first remark that from the proof of Theorem **??** we have that

$$\lim_{\varepsilon\to 0} \mathcal{M}_\varepsilon^{-1}\mathfrak{p}_\varepsilon = A - \alpha.$$

Replacing u^ε in (1.1) and grouping the terms of order ε^0, ε^{-1} and ε^{-2} we get the system

$$L_0 u_0 = 0, \tag{1.5}$$

$$L_1 u_0 + L_0 u_1 = -p'(y)V'(z), \tag{1.6}$$

$$L_0 u_2 + L_1 u_1 + L_2 u_0 = (V'(z) \otimes V'(x))\,(A - \alpha), \tag{1.7}$$

where

$$\begin{aligned} L_0 &= -p'(y)\partial_y + \sigma\partial^2_{yy}, \\ L_1 &= -p'(y)\partial_x - \alpha\cdot V'(x)\partial_y + 2\sigma\partial^2_{xy}, \\ L_2 &= -\alpha\cdot V'(x)\partial_x - \frac{1}{\delta}(z-x)\partial_z + \sigma\partial^2_{xx}. \end{aligned}$$

Let us first remark that equation (1.5) is satisfied for $u_0 = u_0(x,z)$ independent of y. In particular, the kernel of L_0 is made of constants and the kernel of L_0^* is one-dimensional and $\mathrm{Ker}(L_0^*) = \mathrm{Span}\{\rho\}$ where

$$\rho(y) = \frac{1}{Z}e^{-p(y)/\sigma}, \quad \text{where} \quad Z = \int_0^L e^{-p(y)/\sigma}\,\mathrm{d}y, \tag{1.8}$$

where L is the period of p. Since u_0 is independent of y equation (1.6) reduces to

$$L_0 u_1 = p'(y)\left(\partial_x u_0 - V'(z)\right).$$

Let us remark that the general solution u_1 can be written as

$$u_1(x,y,z) = \Phi(y)\left(\partial_x u_0(x,z) - V'(z)\right),$$

where $\Phi\colon \mathbb{R}\to\mathbb{R}$ satisfies

$$-L_0\Phi = -p'(y).$$

Let us remark that this cell problem is the same as (**??**). We consider now equation (1.7) and remark that we can rewrite it as

$$L_0 u_2 = (V'(z)\otimes V'(x))(A-\alpha) - \left(2\Phi'(y) - \frac{1}{\sigma}p'(y)\Phi(y) + 1\right)\sigma\partial^2_{xx}u_0 + (\Phi'(y)+1)\left(\alpha\cdot V'(x)\right)\partial_x u_0 - (V'(z)\otimes V'(x))\,\alpha\Phi'(y) + \frac{1}{\delta}(z-x)\partial_z u_0.$$

By the Fredholm alternative, in order for (1.7) to have a solution we need the right hand side to have zero-mean with respect to ρ in (1.8). Therefore, recalling that the homogenization coefficient K in (**??**) is given by

$$K = \int_0^L (1+\Phi'(y))^2\rho(y)\,\mathrm{d}y$$

and remarking that integrations by part allow to rewrite K as

$$K = \int_0^L (1+\Phi'(y))\rho(y)\,\mathrm{d}y, \quad K = \int_0^L \left(2\Phi'(y) - \frac{1}{\sigma}p'(y)\Phi(y) + 1\right)\rho(y)\,\mathrm{d}y,$$

we get

$$0 = (V'(z)\otimes V'(x))(A - \alpha - (K-1)\alpha) - K\sigma\partial^2_{xx}u_0 + \left(K\alpha\cdot V'(x)\right)\partial_x u_0 + \frac{1}{\delta}(z-x)\partial_z u_0.$$

Moreover, since $A = K\alpha$ and $\Sigma = K\sigma$, it implies

$$0 = -\Sigma \partial_{xx}^2 u_0 + (A \cdot V'(x))\, \partial_x u_0 + \frac{1}{\delta}(z-x)\partial_z u_0,$$

which can be written as

$$-\mathcal{L}_0 u_0 = 0, \tag{1.9}$$

where $\mathcal{L}_0$ is the generator of the couple $(X^0, Z^0)^\top$. Finally, the unique solution of (1.9) satisfying

$$\mathbb{E}^{\rho^0}[u_0(X,Z)] = 0,$$

is given by

$$u_0(x,z) = 0.$$

□

We can now proceed with the main proof.

Proof of Theorem 0.1. Let us introduce the notation

$$\mathcal{M}_\varepsilon := \mathbb{E}^{\rho^\varepsilon}[V'(Z^\varepsilon) \otimes V'(X^\varepsilon)], \quad \mathcal{M}_0 := \mathbb{E}^{\rho^0}[V'(Z) \otimes V'(X)],$$

and the notation

$$\mathfrak{p}_\varepsilon := \mathbb{E}^{\rho^\varepsilon}\left[\frac{1}{\varepsilon} p'\left(\frac{X^\varepsilon}{\varepsilon}\right) V'(Z^\varepsilon)\right]$$

and let us remark that from the proof of Theorem **??** one can deduce the equalities

$$A = \frac{1}{\delta}\mathcal{M}_0^{-1}\, \mathbb{E}^{\rho^0}[(X-Z)^2 V''(Z)], \tag{1.10}$$

and

$$\alpha = \mathcal{M}_\varepsilon^{-1}\left(\frac{1}{\delta}\,\mathbb{E}^{\rho^\varepsilon}[(X^\varepsilon - Z^\varepsilon)^2 V''(Z^\varepsilon)] - \mathfrak{p}_\varepsilon\right). \tag{1.11}$$

Let us furthermore remark that the decomposition (**??**) yields

$$\sqrt{T}\left(\widehat{A}_k^\varepsilon(T) - A\right) = \sqrt{T}\left(\alpha - A + I_1^\varepsilon - I_2^\varepsilon\right).$$

Replacing the expression for A and α given in (1.10) and (1.11), we get

$$\begin{aligned}\sqrt{T}\left(\widehat{A}_k^\varepsilon(T) - A\right) &= \sqrt{T}\left(I_1^\varepsilon(T) - \mathcal{M}_\varepsilon^{-1}\mathfrak{p}_\varepsilon\right) \\ &+ \frac{\sqrt{T}}{\delta}\left(\mathcal{M}_\varepsilon^{-1}\,\mathbb{E}^{\rho^\varepsilon}[(X^\varepsilon - Z^\varepsilon)^2 V''(Z^\varepsilon)] - \mathcal{M}_0^{-1}\,\mathbb{E}^{\rho^0}[(X-Z)^2 V''(Z)]\right) \\ &- \sqrt{T} I_2^\varepsilon(T).\end{aligned}$$

Third term is OK: We rewrite the term involving $I_2^\varepsilon(T)$ as

$$\sqrt{T} I_2^\varepsilon(T) = \frac{\sqrt{2\sigma}}{\sqrt{T}} \widetilde{M}^{-1} Q^\varepsilon(T),$$

where, since Z_t^ε is adapted with respect to the natural filtration $\mathcal{F}_t$ of the Wiener process $W := (W_t, t \geq 0)$, the quantity

$$Q^\varepsilon(T) := \int_0^T V'(Z_t^\varepsilon)\,\mathrm{d}W_t,$$

is a martingale whose quadratic variation is given by

$$\langle Q^\varepsilon \rangle_T = \int_0^T V'(Z_t^\varepsilon) \otimes V'(Z_t^\varepsilon)\,\mathrm{d}t.$$

Since the ergodic theorem guarantees that

$$\lim_{T\to\infty} \frac{\langle Q^\varepsilon \rangle_T}{T} = \mathbb{E}^{\rho^\varepsilon}[V'(Z^\varepsilon) \otimes V'(Z^\varepsilon)], \quad \text{in } L^1(\mu^\varepsilon),$$

the martingale central limit theorem gives

$$\lim_{T\to\infty} \frac{1}{\sqrt{T}} Q^\varepsilon(T) = \Xi^\varepsilon, \quad \text{in law},$$

where $\Xi^\varepsilon \sim \mathcal{N}(0, \mathbb{E}^{\rho^\varepsilon}[V'(Z^\varepsilon) \otimes V'(Z^\varepsilon)])$. Now, by the ergodic theorem

$$\lim_{T\to\infty} \widetilde{M}^{-1} = \mathcal{M}_\varepsilon^{-1}, \quad \text{a.s.}$$

Therefore, we have by Slutsky's theorem that

$$\lim_{T\to\infty} \sqrt{T} I_2^\varepsilon(T) = \Lambda^\varepsilon \sim \mathcal{N}\left(0, \Gamma^\varepsilon\right), \quad \text{in law.}$$

where the covariance matrix Γ^ε is given by

$$\Gamma^\varepsilon = 2\sigma \mathcal{M}_\varepsilon^{-1} \, \mathbb{E}^{\rho^\varepsilon}[V'(Z^\varepsilon) \otimes V'(Z^\varepsilon)] \mathcal{M}_\varepsilon^{-\top}.$$

Finally, we denote by Λ the limit for $\varepsilon \to 0$ of the sequence of Gaussian random variables Λ^ε.

Idea for the first term: Let us introduce the notation

$$J_1^\varepsilon(T) \coloneqq \sqrt{T}\left(I_1^\varepsilon(T) - \mathcal{M}_\varepsilon^{-1}\mathfrak{p}_\varepsilon\right),$$

and let us remark that we can rewrite

$$J_1^\varepsilon(T) = \sqrt{T}\left(\frac{1}{T}\int_0^T V'(Z_t^\varepsilon) \otimes V'(X_t^\varepsilon) \,\mathrm{d}t\right)^{-1} \\ \times \left(\frac{1}{T}\int_0^T \left(\frac{1}{\varepsilon}p'\left(\frac{X_t^\varepsilon}{\varepsilon}\right) V'(Z_t^\varepsilon) - V'(Z_t^\varepsilon) \otimes V'(X_t^\varepsilon)\mathcal{M}_\varepsilon^{-1}\mathfrak{p}_\varepsilon\right) \mathrm{d}t\right).$$

Let us denote by u^ε the solution of (1.1) with right hand side

$$\chi^\varepsilon(x,z) = \frac{1}{\varepsilon}p'\left(\frac{x}{\varepsilon}\right) V'(z) - V'(z) \otimes V'(x)\mathcal{M}_\varepsilon^{-1}\mathfrak{p}_\varepsilon,$$

and note that $\mathbb{E}^{\rho^\varepsilon}[\chi^\varepsilon(X^\varepsilon, Z^\varepsilon)] = 0$. Then, by Lemma 1.1, we have

$$J_1^\varepsilon(T) = \sqrt{T}\left(\frac{1}{T}\int_0^T V'(Z_t^\varepsilon) \otimes V'(X_t^\varepsilon) \,\mathrm{d}t\right)^{-1} \left(-\frac{R^\varepsilon(T)}{T} + \sqrt{2\sigma}\frac{S^\varepsilon(T)}{T}\right),$$

where $R^\varepsilon(T)$ and $S^\varepsilon(T)$ are defined in (1.3). Since $R^\varepsilon(T)$ is bounded independently of ε (is it clear?), we first get by the ergodic theorem

$$\lim_{T\to\infty} \left(\frac{1}{T}\int_0^T V'(Z_t^\varepsilon) \otimes V'(X_t^\varepsilon) \,\mathrm{d}t\right)^{-1} \frac{R^\varepsilon(T)}{\sqrt{T}} = 0, \quad \text{a.s.}$$

Repeating the same reasoning as for $Q^\varepsilon(T)$ and employing the ergodic theorem and the Slutsky's theorem, we get

$$\lim_{T\to\infty} J_1^\varepsilon(T) = \lim_{T\to\infty} \sqrt{2\sigma}\left(\frac{1}{T}\int_0^T V'(Z_t^\varepsilon) \otimes V'(X_t^\varepsilon) \,\mathrm{d}t\right)^{-1} \frac{S^\varepsilon(T)}{\sqrt{T}} = \Lambda_1^\varepsilon \sim \mathcal{N}(0, \Gamma_1^\varepsilon),$$

where the covariance is given by due to (1.4)

$$\begin{aligned}\Gamma_1^\varepsilon &= 2\sigma \mathcal{M}_\varepsilon^{-1}\, \mathbb{E}^{\rho^\varepsilon}[\partial_x u^\varepsilon(X^\varepsilon, Z^\varepsilon) \otimes \partial_x u^\varepsilon(X^\varepsilon, Z^\varepsilon)] \mathcal{M}_\varepsilon^{-\top} \\ &= \mathcal{M}_\varepsilon^{-1}\, \mathbb{E}^{\rho^\varepsilon}[u^\varepsilon(X^\varepsilon, Z^\varepsilon) \otimes \chi^\varepsilon(X^\varepsilon, Z^\varepsilon) + \chi^\varepsilon(X^\varepsilon, Z^\varepsilon) \otimes u^\varepsilon(X^\varepsilon, Z^\varepsilon)] \mathcal{M}_\varepsilon^{-\top}.\end{aligned}$$

For the covariance Γ_1^ε we don't know how it behaves in the limit $\varepsilon \to 0$. It would be nice if it was vanishing since it involves the solution u^ε, which goes to zero (in some sense) due to Lemma 1.2.

Idea for the second term: Let us now introduce the notation

$$J_2^\varepsilon(T) = \frac{\sqrt{T}}{\delta} \left(\mathcal{M}_\varepsilon^{-1}\, \mathbb{E}^{\rho^\varepsilon}[(X^\varepsilon - Z^\varepsilon)^2 V''(Z^\varepsilon)] - \mathcal{M}_0^{-1}\, \mathbb{E}^{\rho^0}[(X - Z)^2 V''(Z)] \right).$$

We have the decomposition

$$\begin{aligned}J_2^\varepsilon(T) &= \frac{\sqrt{T}}{\delta} \left(\mathcal{M}_\varepsilon^{-1} - \mathcal{M}_0^{-1} \right) \mathbb{E}^{\rho^\varepsilon}[(X^\varepsilon - Z^\varepsilon)^2 V''(Z^\varepsilon)] \\ &\quad + \frac{\sqrt{T}}{\delta} \mathcal{M}_0^{-1} \left(\mathbb{E}^{\rho^\varepsilon}[(X^\varepsilon - Z^\varepsilon)^2 V''(Z^\varepsilon)] - \mathbb{E}^{\rho^0}[(X - Z)^2 V''(Z)] \right) \\ &=: J_{2,1}^\varepsilon(T) + J_{2,2}^\varepsilon(T).\end{aligned}$$

In order to send the terms above to zero, we need the convergence rate of μ^ε to μ^0 w.r.t. ε (by fixing $T = \varepsilon^{-\gamma}$ with an appropriate γ). Some arguments in this sense seem to appear in [*Periodic Homogenization for Hypoelliptic Diffusions* – M. Hairer and G.A. Pavliotis 2004] but it is not clear to us whether it applies to this case too. □