

Uncertain Darcy's problem and the stochastic particle transport

Semester Project - Master in CSE

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Problem statement

Underground flow \rightarrow Uncertain Darcy's problem

$$\left\{ \begin{array}{ll} u = -A\nabla p, & \text{in } D, \\ \nabla \cdot u = f, & \text{in } D, \\ p = p_0, & \text{on } \Gamma_{in}, \\ p = 0, & \text{on } \Gamma_{out}, \\ \nabla p \cdot n = 0, & \text{on } \Gamma_N, \end{array} \right.$$

Stochastic particle transport \rightarrow Ornstein–Uhlenbeck process

$$\left\{ \begin{array}{l} dX(t) = u(X(t))dt + \sigma dW(t), \quad 0 \leq t \leq T, \\ X(0) = X_0 \in D, \end{array} \right.$$

Outline

- ▶ Expected exit time from a domain
- ▶ Theoretical investigation: perturbed SDE's
- ▶ The uncertain Darcy's problem

Mean exit time. Setting

Given a domain $D \subset \mathbb{R}^d$, $f: D \rightarrow \mathbb{R}^d$, $g: D \rightarrow \mathbb{R}^{d \times m}$ and an m -dimensional standard Wiener process $W(t)$, consider

$$\begin{cases} dX(t) = f(X(t))dt + g(X(t))dW(t), & 0 < t \leq T, \\ X(0) = X_0, & X_0 \in D. \end{cases}$$

The equation is defined in a domain $D \rightarrow$ boundary conditions

- ▶ *killing boundaries*: $X(t)$ is stopped,
- ▶ *reflecting boundaries*: $X(t)$ is reflected normally inside D .

Mean exit time. Problem statement

Problem. Estimate the exit time of the trajectories

$$\tau = \min\{\tau_e, T\}, \text{ where } \tau_e = \min\{t: X(t) \notin D\}.$$

Another quantity of interest, given $F: D \rightarrow \mathbb{R}$

$$\varphi = \varphi(T, X_0, F) = \mathbb{1}_{\{T < \tau_e\}} F(X(T)).$$

If $F \equiv 1$, exit probability

$$\Phi(T, X_0) := \Pr(\tau < T | X(0) = X_0) = 1 - \mathbb{E}(\varphi(T, X_0, 1)).$$

Goal. Estimate numerically τ and φ .

Discrete Euler-Maruyama (DEM)

Method:

$$\begin{cases} X_h^d(t_{i+1}) = f(X(t_i))h + g(X(t_i))(W(t_{i+1}) - W(t_i)), \\ X_h^d(0) = X_0. \end{cases}$$

Parameters of interest computed naively

$$\begin{aligned} \tau_h^d &= \min\{\tau_{h,e}^d, T\}, \text{ where } \tau_{h,e}^d = \min\{t_i: X_h^d(t_i) \notin D\}, \\ \varphi_h^d &= \mathbb{1}_{\{T < \tau_{h,e}^d\}} F(X_h^d(T)). \end{aligned}$$

Missed exits \Rightarrow 1/2 loss in weak order:

$$\begin{aligned} |\mathbb{E}(\tau_h^d) - \mathbb{E}(\tau)| &= O(\sqrt{h}), \\ |\mathbb{E}(\varphi_h^d) - \mathbb{E}(\varphi)| &= O(\sqrt{h}). \end{aligned}$$

Continuous Euler-Maruyama (CEM)

Goal. Restore the weak order of convergence 1 of Euler-Maruyama
 \Rightarrow Brownian bridge approach.

Method:

$$\begin{cases} X_h^c(t) = f(X(t_i))(t - t_i) + g(X(t_i))(W(t) - W(t_i)), & t_i < t \leq t_{i+1}, \\ X_h^c(0) = X_0. \end{cases}$$

Estimate at each time step the probability of exit. If D is an half-space

$$\begin{aligned} & \Pr(\exists t \in [t_i, t_{i+1}] \quad X_h^d(t) \notin D | X_h^d(t_i) = x_i, X_h^d(t_{i+1}) = x_{i+1}) \\ &= p(x_i, x_{i+1}, h) \\ &= \exp\left(-2 \frac{[n \cdot (x_i - z_i)][n \cdot (x_{i+1} - z_i)]}{hn \cdot (gg^T(x_i)n)}\right). \end{aligned}$$

Continuous Euler-Maruyama (CEM)

Parameters of interest. Given u a realization of U uniform r.v. in $(0, 1)$

$$\tau_h^c = \min\{T, \tau_{h,e}^c\},$$

$$\text{where } \tau_{h,e}^c = \min\{\tau_{h,e1}^c, \tau_{h,e2}^c\},$$

$$\tau_{h,e1}^c = \min\{t_i = hi : X_h(t_i) \notin D\},$$

$$\tau_{h,e2}^c = \min\{t_i = hi : u < p(x_{i-1}, x_i, h)\},$$

$$\varphi_h^c = \mathbb{1}_{\{T < \tau_{h,e}^c\}} F(X_h^c(T)).$$

Weak order 1 is restored:

$$|\mathbb{E}(\tau_h^c) - \mathbb{E}(\tau)| = O(h),$$

$$|\mathbb{E}(\varphi_h^c) - \mathbb{E}(\varphi)| = O(h).$$

An adaptive procedure

Goal. Restore the weak order of convergence 1 of Euler-Maruyama
 \Rightarrow Adaptivity in space.

Setting. Consider $\sigma \in \mathbb{R}$, I the identity in $\mathbb{R}^{d \times d}$ and

$$\begin{cases} dX(t) = f(X(t))dt + \sigma IdW(t), & 0 < t \leq T, \\ X(0) = X_0, & X_0 \in D. \end{cases}$$

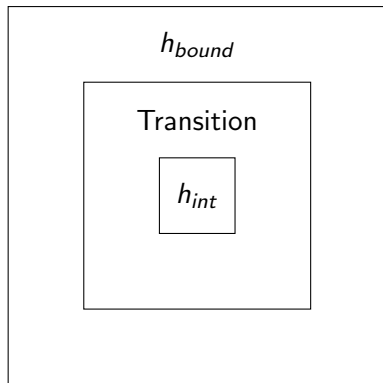
Idea. Given $I \in \mathbb{N}$, $h_0 > 0$, adapt step size h in DEM as follows

$$h = \max \left\{ h_{bound}, \min \left\{ h_{int}, \left(\frac{d}{(I+3)\sigma} \right)^2 \right\} \right\}.$$

where

$$\begin{aligned} h_{bound} &= 2^{-2I} h_0, \\ h_{int} &= 2^{-I} h_0. \end{aligned}$$

An adaptive procedure



Quel porco di mazzinga la vede.