Uncertain Darcy's problem and the stochastic particle transport

Semester Project - Master in CSE

Giacomo Garegnani
Supervisors: Dr. Sebastian Krumscheid and Prof. Fabio Nobile

EPFL

16/06/2016

Problem statement

Underground flow $\rightarrow$ Uncertain Darcy's problem

$$\begin{cases} u = -A\nabla p, & \text{in } D, \\ \nabla \cdot u = f, & \text{in } D, \\ p = p_0, & \text{on } \Gamma_{in}, \\ p = 0, & \text{on } \Gamma_{out}, \\ \nabla p \cdot n = 0, & \text{on } \Gamma_N, \end{cases}$$

Stochastic particle transport $\rightarrow$ Ornstein–Uhlenbeck process

$$\begin{cases} dX(t) = u(X(t))dt + \sigma dW(t), \quad 0 \leq t \leq T, \\ X(0) = X_0 \in D, \end{cases}$$

Outline

- Expected exit time from a domain
- Theoretical investigation: perturbed SDE's
- The uncertain Darcy's problem

Mean exit time. Setting

Given a domain $D \subset \mathbb{R}^d$, $f\colon D \to \mathbb{R}^d, g\colon D \to \mathbb{R}^{d\times m}$ and an m-dimensional standard Wiener process $W(t)$, consider

$$\begin{cases} dX(t) = f(X(t))dt + g(X(t))dW(t), & 0 < t \leq T, \\ X(0) = X_0, & X_0 \in D. \end{cases}$$

The equation is defined in a domain $D \to$ boundary conditions

- *killing boundaries*: $X(t)$ is stopped,
- *reflecting boundaries*: $X(t)$ is reflected normally inside D.

Mean exit time. Problem statement

Problem. Estimate the exit time of the trajectories

$$\tau = \min\{\tau_e, T\}, \text{ where } \tau_e = \min\{t\colon X(t) \notin D\}.$$

Another quantity of interest, given $F\colon D \to \mathbb{R}$

$$\varphi = \varphi(T, X_0, F) = \mathbb{1}_{\{T<\tau_e\}} F(X(T)).$$

If $F \equiv 1$, exit probability

$$\Phi(T, X_0) := \Pr(\tau < T | X(0) = X_0) = 1 - \mathbb{E}(\varphi(T, X_0, 1)).$$

Goal. Estimate numerically τ and φ.

Discrete Euler-Maruyama (DEM)

Method:

$$\begin{cases} X_h^d(t_{i+1}) = f(X(t_i))h + g(X(t_i))(W(t_{i+1}) - W(t_i)), \\ \quad X_h^d(0) = X_0. \end{cases}$$

Parameters of interest computed naively

$$\tau_h^d = \min\{\tau_{h,e}^d, T\}, \text{ where } \tau_{h,e}^d = \min\{t_i \colon X_h^d(t_i) \notin D\},$$
$$\varphi_h^d = \mathbb{1}_{\{T < \tau_{h,e}^d\}} F(X_h^d(T)).$$

Missed exits $\Rightarrow 1/2$ loss in weak order:

$$|\mathbb{E}(\tau_h^d) - \mathbb{E}(\tau)| = O(\sqrt{h}),$$
$$|\mathbb{E}(\varphi_h^d) - \mathbb{E}(\varphi)| = O(\sqrt{h}).$$

Continuous Euler-Maruyama (CEM)

Goal. Restore the weak order of convergence 1 of Euler-Maruyama
$\Rightarrow$ Brownian bridge approach.
Method:

$$\begin{cases} X_h^c(t) = f(X(t_i))(t - t_i) + g(X(t_i))(W(t) - W(t_i)), \quad t_i < t \leq t_{i+1}, \\ X_h^c(0) = X_0. \end{cases}$$

Estimate at each time step the probability of exit. If D is an half-space

$$\begin{aligned} &\Pr(\exists t \in [t_i, t_{i+1}] \quad X_h^d(t) \notin D | X_h^d(t_i) = x_i, X_h^d(t_{i+1}) = x_{i+1}) \\ &\quad = p(x_i, x_{i+1}, h) \\ &\quad = \exp\Big(-2 \frac{[n \cdot (x_i - z_i)][n \cdot (x_{i+1} - z_i)]}{hn \cdot (gg^T(x_i)n)} \Big). \end{aligned}$$

Continuous Euler-Maruyama (CEM)

Parameters of interest. Given u a realization of U uniform r.v. in $(0, 1)$

$$\begin{aligned}
\tau_h^c &= \min\{T, \tau_{h,e}^c\},\\
\text{where } \tau_{h,e}^c &= \min\{\tau_{h,e1}^c, \tau_{h,e2}^c\},\\
\tau_{h,e1}^c &= \min\{t_i = hi \colon X_h(t_i) \notin D\},\\
\tau_{h,e2}^c &= \min\{t_i = hi \colon u < p(x_{i-1}, x_i, h)\},\\
\varphi_h^c &= \mathbb{1}_{\{T<\tau_{h,e}^c\}} F(X_h^c(T)).
\end{aligned}$$

Weak order 1 is restored:

$$\begin{aligned}
|\mathbb{E}(\tau_h^c) - \mathbb{E}(\tau)| &= O(h),\\
|\mathbb{E}(\varphi_h^c) - \mathbb{E}(\varphi)| &= O(h).
\end{aligned}$$

An adaptive procedure

Goal. Restore the weak order of convergence 1 of Euler-Maruyama
$\Rightarrow$ Adaptivity in space.
Setting. Consider $\sigma \in \mathbb{R}$, I the identity in $\mathbb{R}^{d\times d}$ and

$$\begin{cases} dX(t) = f(X(t))dt + \sigma I dW(t), & 0 < t \leq T, \\ X(0) = X_0, & X_0 \in D. \end{cases}$$

Idea. Given $l \in \mathbb{N}, h_0 > 0$, adapt step size h in DEM as follows

$$h = \max\left\{h_{bound}, \min\left\{h_{int}, \left(\frac{d}{(l+3)\sigma}\right)^2\right\}\right\}.$$

where

$$h_{bound} = 2^{-2l} h_0,$$
$$h_{int} = 2^{-l} h_0.$$

An adaptive procedure