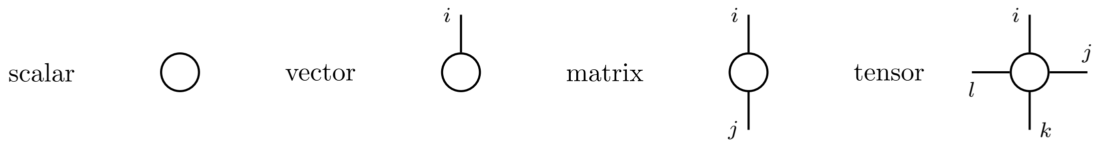
Homework 2

1 Tensor Network Diagrams

Let us review the diagrammatic notation for tensors. As you may remember tensors can be represented by a circle with a number of legs, each leg corresponding to one of its indices. The number of indices of tensor is called its *rank*. A *contraction* between a set of tensors is the sum over all possible values of the repeated indices.



Draw the diagrams associated to the contractions

$$\text{(a)}\ \sum_i A_i B_i\,, \qquad \text{(b)}\ \sum_j A_{ij} B_{jk}\,, \qquad \text{(c)}\ \sum_{i,j,k,l,n} A_{ij} B_{ijkl} C_{km} D_{lnn}$$

and make sure to write down the indices explicitly. What is the rank of each resulting tensor?

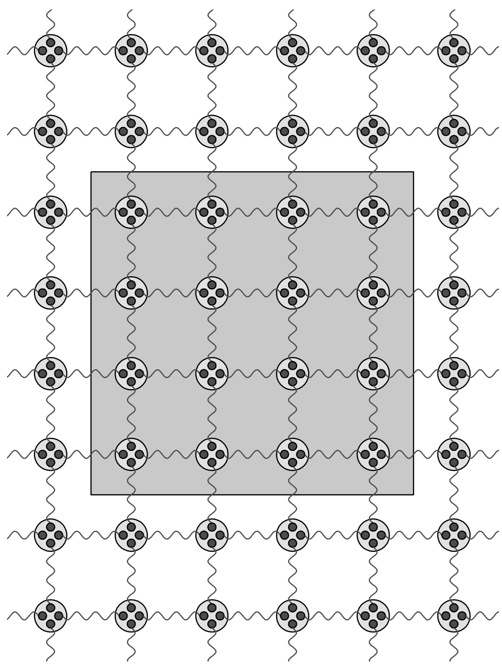
Let A, B, C, D, E be five square matrices. Draw the diagram of $\mathrm{tr}(ABCDE)$. Can you illustrate the cyclic property of the trace in this representation?

2 PEPS Satisfy the Area Law — In Three Acts

Act I Consider a general pair of entangled systems: $|\psi\rangle = \sum_{n,m} c_{n,m} |n\rangle |m\rangle$, with $n = 1, \ldots, D_1$ and $m = 1, \ldots, D_2$. Show that the entanglement is upper-bounded by $\log_2 \min(D_1, D_2)$. Show that this bound is still valid after acting with a local map $P = P_1 \otimes P_2$, where $P_{1,2} : \mathbb{C}^{D_{1,2}} \to \mathbb{C}^{D'_{1,2}}$.

Act II Now consider a lattice, composed of N nodes of coordination number z. On each node we assign z D-level systems, each entangled with a system on a neighboring node. The total wavefunction is then $|\varphi\rangle = |\bullet\!\!-\!\!\bullet\rangle^{\otimes zN/2}$. Let $\mathcal{A}$ be an region of this lattice. Let us now show that this construction satisfies the area law.

- What is the maximum entanglement of a single pair $|\bullet\!\!-\!\!\bullet\rangle$?
- Show that the entanglement of $\mathcal{A}$ with respect to its surrounding $\bar{\mathcal{A}}$ depends only on the pairs cut by $\partial\mathcal{A}$.
- Derive an upper-bound for the entanglement entropy of $\mathcal{A}$ that is an area law.



Act III We now construct a *Projected Entangled Pair State* (PEPS). We apply a map A: $(\mathbb{C}^D)^z \to \mathbb{C}^d$, on every node: $|\mathrm{PEPS}\rangle = A_1 A_2 \ldots A_N |\varphi\rangle$. Show that the entanglement of $\mathcal{A}$ still obeys an area law.

Tip If you are stuck, start by considering a $z = 2$ lattice and then going to higher dimensions.

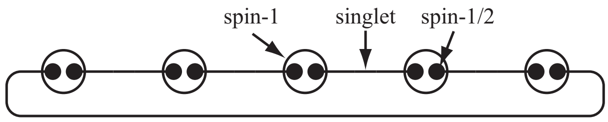
3 The AKLT State

Remember the addition of angular momentum from your quantum mechanics course? Or did you think you could forget Clebsch–Gordon coefficients? Convince yourself that by combining two spin-1/2 you obtain the triplet

$$|+\rangle = |\uparrow\uparrow\rangle\,, \qquad |0\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}, \qquad |-\rangle = |\downarrow\downarrow\rangle\,,$$

associated to spin-1, and the spin-0 singlet

$$|\bullet\!\!-\!\!\bullet\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}.$$



Let us now construct a state by using $2L$ auxiliary spin-1/2 states for a chain of L sites. In this construction

$$|\Phi\rangle = \sum_{\mathbf{a},\mathbf{b}} c_{\mathbf{ab}}\,|\mathbf{a},\mathbf{b}\rangle\,,$$

where $|\mathbf{a}\rangle = |a_1,\ldots,a_L\rangle$ and $|\mathbf{b}\rangle = |b_1,\ldots,b_L\rangle$ represent the first and second spin-1/2 on each site. To encode the singlet bonds between b_i and a_{i+1} we introduce a matrix

$$\Sigma = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and write down the state with singlets as

$$|\Phi\rangle = \sum_{\mathbf{a},\mathbf{b}} \Sigma_{b_1a_2}\Sigma_{b_2a_3}\ldots\Sigma_{b_{L-1}a_L}\Sigma_{b_La_1}\,|\mathbf{a},\mathbf{b}\rangle\,,$$

To perform the spin addition, we introduce a mapping M from the state of the two auxiliary spins-1/2 $|a_i\rangle\,|b_i\rangle \in \{|\uparrow\rangle\,,|\downarrow\rangle\}^{\otimes 2}$ to the states of the physical spin-1 $|\sigma_i\rangle \in \{|+\rangle\,,|0\rangle\,,|-\rangle\}$. Write the local mapping $M^{\sigma}_{ab}\,|\sigma\rangle\,\langle a,b|$ as three 2×2 matrices, one for each σ.

The total mapping is then

$$\mathcal{M} = \sum_{\boldsymbol{\sigma}}\sum_{\mathbf{a},\mathbf{b}} M^{\sigma_1}_{a_1b_1}M^{\sigma_2}_{a_2b_2}\ldots M^{\sigma_L}_{a_Lb_L}\,|\boldsymbol{\sigma}\rangle\,\langle\mathbf{a},\mathbf{b}|$$

Deduce the form of the state after applying the mapping and write it as

$$|\Psi\rangle = \sum_{\boldsymbol{\sigma}} \mathrm{tr}\left(A^{\sigma_1}A^{\sigma_2}\ldots A^{\sigma_L}\right)|\boldsymbol{\sigma}\rangle$$

where each A is a rank-3 tensor. Give the explicit form of A^{σ} for each σ. Finally, draw the associated diagram.

Note Notice how the trace is used to express the periodic boundary conditions.

4 Some Matrix Product States

A *Matrix Product State* (MPS) is a state describing a one-dimensional system expressed in the form

$$|\mathrm{MPS}\rangle = \sum_{n_1,\ldots n_N} \mathrm{tr}\left(A^{n_1}A^{n_2}\ldots A^{n_N}\right)|n_1,\ldots n_N\rangle$$

where each A^{n_i} can be seen as a matrix for each entry n_i. More precisely it is the decomposition of a high-rank tensor into a series of rank-3 tensors (not necessarily the same) contracted sequentially, hence the name *Tensor Train decomposition*, used in mathematics and computer science. Construct the MPS — i.e. write down the tensors A^{n_i} — associated to the following N-qubit states:

(a) $|\mathrm{Prod}\rangle = |11111\rangle\,,$ (b) $|\mathrm{GHZ}\rangle = |0000\rangle + |1111\rangle\,,$ (c) $|\mathrm{W}\rangle = |100\rangle + |010\rangle + |001\rangle\,.$

Do not worry about normalizing the states.

Bonus Construct the first two states for a two-dimensional square lattice in the thermodynamic limit.

5 Multipartite Entanglement

Let $|\psi\rangle$ be a 4-qubit state defined in terms of the Bell states

$$|\psi\rangle \propto |\Phi^+\rangle_{AB} \otimes |\Phi^+\rangle_{CD} + |\Psi^+\rangle_{AB} \otimes |\Psi^+\rangle_{CD}\,.$$

Normalize the state and numerically compute the entanglement of the partitions of $(A)(BCD)$, $(AB)(CD)$ and $(ABC)(D)$ by Schmidt decomposition.

Tip Test your code with states like $|\Phi^+\rangle_{AB}\,|00\rangle_{CD}$. Which are the entangled qubits in this case?

6 Free Bosons on a Lattice

Consider $N \in \mathbb{N}$ bosons on a one dimensional lattice of R sites described by the Hamiltonian

$$\hat{H} = -t\sum_{j=1}^{R}\left(\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_j \hat{a}_{j+1}^\dagger\right)$$

where $\hat{a}_j$ are the bosonic operators satisfying $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{i,j}$, and $t > 0$. Assume periodic boundary conditions, i.e. $\hat{a}_{N+1} = \hat{a}_1$. In momentum-space this Hamiltonian is diagonal

$$\hat{H} = \sum_k \epsilon_k \hat{b}_k^\dagger \hat{b}_k$$

where $\hat{b}_k = \frac{1}{\sqrt{R}}\sum_j e^{-2\pi i jk/R}\hat{a}_j$ are the Fourier-transformed annihilation operators. Compute the dispersion relation ϵ_k — is the system gapped? The ground state is given by all particles at $k = 0$,

$$|\Psi_0\rangle = \frac{1}{\sqrt{N!}}(\hat{b}_0^\dagger)^N|0\rangle$$

What is $|\Psi_0\rangle$ in real space?

We now separate the system into two parts: A, composed of the first L sites, and $\bar{A}$, composed of $R - L$ sites. The quantity of interest is the entanglement between A sites and the rest, where $L \ll R$. This bipartition of the Hilbert space is achieved by defining the operators

$$\hat{a}_A^\dagger = \frac{1}{\sqrt{L}}\sum_{j\in A}\hat{a}_j^\dagger, \qquad \hat{a}_{\bar{A}}^\dagger = \frac{1}{\sqrt{R-L}}\sum_{j\in\bar{A}}\hat{a}_j^\dagger.$$

that divide the Fock space into the tensor product of the first L modes with the rest. Show that, using the binomial formula,

$$|\Psi_0\rangle = \sum_{n=0}^{N}\sqrt{\binom{N}{n}}\left(\frac{L}{R}\right)^{\frac{n}{2}}\left(\frac{R-L}{R}\right)^{\frac{N-n}{2}}|n\rangle_A\,|N-n\rangle_{\bar{A}}\,,$$

where $|n\rangle_A = (1/\sqrt{n!})(\hat{a}_A^\dagger)^n\,|0\rangle$ and similarly for $\bar{A}$. In this representation, the density operator for the first subsystem is immediate: $\hat{\rho}_A = \sum_n \rho_n\,|n\rangle\langle n|$. When N is large, we can replace the sum by an integral, and show that

$$\rho(n) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(n-n_0)^2}{2\sigma^2}\right).$$

with $n_0 = NL/R$ and $\sigma = NL(R-L)/R^2$. The entanglement entropy will then be

$$E_A = -\int_{-\infty}^{\infty}\mathrm{d}n\,\rho(n)\log\rho(n),$$

Perform the integral, the leading order should be $E_A \sim \log\sigma$. Finally, take the limit $N, R \to \infty$ with the ratio N/R fixed, and obtain

$$E_A = \frac{1}{2}\log L + \mathcal{O}(1)$$

Briefly comment on this result.