Solutions to Homework 1

1

Remember that $\mathrm{tr}_B(\bullet) = \sum_n \langle\varphi_n|_B \bullet |\varphi_n\rangle_B$. Then

$$\mathrm{tr}_B\left(|\Psi\rangle_{AB}\langle\Psi|\right) = \sum_{n,n',m} \sqrt{\lambda_n}\sqrt{\lambda_{n'}}\, |\varphi_n\rangle_A \underbrace{\langle\varphi_m|\varphi_n\rangle_B}_{\delta_{m,n}} \underbrace{\langle\varphi_{n'}|\varphi_m\rangle_B}_{\delta_{n',m}} \langle\varphi_{n'}|_A = \sum_m \lambda_m |\varphi_m\rangle_A \langle\varphi_m|_A = \rho.$$

We notice that the dimension of the ancilla corresponds to the rank of ρ.

2

The inequality (a) is just a restatement of the strong subadditivity

$$\begin{gathered} I(A,B:C) \geq I(B:C) \\ S(A,B) + S(C) - S(A,B,C) \geq S(B) + S(C) - S(B,C) \\ S(A,B) + S(B,C) \geq S(A,B,C) + S(B) \end{gathered}$$

Hence adding a system never decreases the mutual information. For (b), we use the properties $S(A,B) \leq S(A) + S(B)$ and $S(A,B) \geq |S(A) - S(B)| \geq S(A) - S(B)$.

$$I(A,B:C) = \underbrace{S(A,B)}_{\leq S(A)+S(B)} + S(C) - \underbrace{S(A,B,C)}_{\geq |S(B,C)-S(A)|} \leq 2S(A) + S(B) + S(C) - S(B,C) = I(B:C) + 2S(A).$$

Combined, these two inequalities yield $I(B:C) \leq I(A,B:C) \leq I(B:C) + 2S(A)$. Intuitively, it means the following: by adding into consideration system A, we will increase the mutual information, or correlations. However, this increase is bounded by twice the entropy of A.

3

In the computational basis we write the state as $|\Psi\rangle = \sum_{ij} c_{ij} |i,j\rangle$. We then compute $c^\dagger c$ or $cc^\dagger$

$$c = \frac{1}{\sqrt{30}} \begin{pmatrix} 2\sqrt{2}-1 & 2\sqrt{2}+1 \\ -2-\sqrt{2} & 2-\sqrt{2} \end{pmatrix}, \quad c^\dagger c = \frac{1}{6}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

The eigenvalues of $c^\dagger c$ are given by the roots of the characteristic polynomial $p(\lambda) = (1/2-\lambda)^2 - (1/6)^2$, from which we obtain $\lambda_1 = 2/3, \lambda_2 = 1/3$. The entanglement is then

$$E = -\sum_n \lambda_n \log_2 \lambda_n = \frac{2}{3}\log_2\frac{3}{2} + \frac{1}{3}\log_2 3 = \log_2 3 - \frac{2}{3} \approx 0.9183.$$

4

Computing each c_{ij} yields

$$c = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

We notice that $cc^\dagger$ is diagonal, so the eigenvalues are $1/3$. Hence the entanglement is

$$E = -3(1/3)\log_2(1/3) = \log_2 3 \approx 1.585.$$