Solutions to Homework 4

1

(a) By defining $v_x = X$, $v_z = Z$, $v_{xz} = Y$, we obtain the following *factor system* $e^{i\omega(g,h)}$, represented as a matrix,

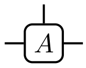
$$\left\{e^{i\omega(g,h)}\right\}_{g,h} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -i & i \\ 1 & i & 1 & -i \\ 1 & i & -i & 1 \end{pmatrix}, \quad g,h = \mathbf{1}, x, z, xz.$$

(b) See above.

(c) By changing the phase $v_g \to e^{i\phi_g} v_g$ the commutator becomes $e^{i(\phi_x+\phi_z)}(v_x v_z - v_z v_x)$ which is non-zero for every value of $\phi_x + \phi_z$. Since no rephasing can bring the commutator to zero, we deduce that the Pauli matrices cannot form a linear representation of this group.

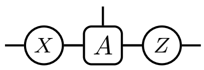
The symmetry of the cluster state is $G = \{\mathbb{1}, X_{\text{odd}}, X_{\text{even}}, X_{\text{all}}\}$, where $X_{\text{all}} = X_{\text{even}} \circ X_{\text{odd}}$ (the group is abelian and these two terms commute). Notice that this group is $\mathrm{Z}_2 \times \mathrm{Z}_2$, since $X_{\text{even}}^2 = X_{\text{odd}}^2 = \mathbb{1}$, $[X_{\text{even}}, X_{\text{odd}}] = 0$.

The properties of the cluster state are easily checked by using the properties of the Pauli matrices. By representing each entry A^i in a vector,

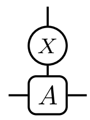


$$= \begin{pmatrix} A^0 \\ A^1 \end{pmatrix}$$

we obtain



$$= \begin{pmatrix} X\,|0\rangle\langle +|\,Z \\ X\,|1\rangle\langle -|\,Z \end{pmatrix} = \begin{pmatrix} |1\rangle\langle -| \\ |0\rangle\langle +| \end{pmatrix} =$$



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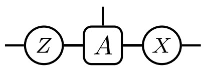
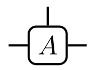


$$= \begin{pmatrix} Z\,|0\rangle\langle +|\,X \\ Z\,|1\rangle\langle -|\,X \end{pmatrix} = \begin{pmatrix} |0\rangle\langle +| \\ |1\rangle\langle -| \end{pmatrix} =$$



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We can use these properties to define the action of the symmetries on the pair of sites:

$$\begin{aligned}
(X_{i,i'} \otimes \mathbb{1}_{j,j'})(A^{i'}A^{j'}) &= XA^iZA^j(XX) = X(A^iA^j)X, \\
(\mathbb{1}_{i,i'} \otimes X_{j,j'})(A^{i'}A^{j'}) &= (ZZ)A^iXA^jZ = Z(A^iA^j)Z, \\
(X_{i,i'} \otimes X_{j,j'})(A^{i'}A^{j'}) &= XA^iZXA^jZ = XZA^i(XZ)^2A^jXZ = -iY(A^iA^j)iY.
\end{aligned}$$

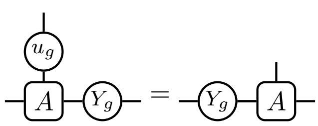
Notice that the last one can be deduced immediately, since it the composition of the two previous symmetries, therefore $ZX = -iY$. The different local representations u_g and their actions on the virtual level can be summarized in the following table:

u_g	$\mathbb{1} \otimes \mathbb{1}$	$X \otimes \mathbb{1}$	$\mathbb{1} \otimes X$	$X \otimes X$
V_g	$\mathbb{1}$	X	Z	$-iY$

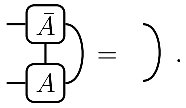
These local operations are the building blocks for the global symmetries in G, as one can easily verify graphically.

2

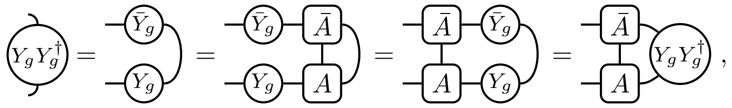
We can think of the action of u_g as "passing Y_g through A"



In the right canonical form, the *only* fixed point is



By passing Y_g through A we obtain



where the two u_g cancel out since they are unitary. Now, this equation means that $Y_g Y_G^\dagger$ is a fixed point of the map. By the unicity of the fixed point, $Y_g Y_G^\dagger = \alpha \mathbb{1}$, with α a positive, real constant (since $Y_g Y_G^\dagger \succeq 0$) which can be absorbed in Y_g.

3

Since $(\vec{S}_1 + \vec{S}_2)^2$ is the total spin has the eigenvalue equation

$$(\vec{S}_1 + \vec{S}_2)^2 \left|s, m_s\right\rangle = \left(|S_1|^2 + 2\vec{S}_1 \cdot \vec{S}_2 + |S_2|^2\right) \left|s, m_s\right\rangle = s(s+1) \left|s, m_s\right\rangle, \quad s = 0, 1, 2.$$

Since $|S_1|^2 = |S_2|^2 = 2$, the scalar product can have the eigenvalues

$$\vec{S}_1 \cdot \vec{S}_2 = \begin{cases} -2 & s = 0 \\ -1 & s = 1 \\ 2 & s = 2 \end{cases}$$

Method 1 Let $X = \vec{S}_1 \cdot \vec{S}_2$, we construct a second-degree polynomial $p(X)$ with the following properties

$$p(X) = \begin{cases} 0 & X = -2 \\ 0 & X = -1 \\ 2 & X = 2 \end{cases}$$

This can be constructed as

$$p(X) = \frac{1}{6}(X+2)(X+1) = \frac{X^2}{6} + \frac{X}{2} + \frac{1}{3}$$

which is the desired expression.

Method 2 Alternatively, we construct a projector in terms of the total angular momentum that gives $\Pi_{s=2} = 0$ for $s = 0, 1$ and $\Pi_{s=2} = 1$ for $s = 2$.

$$\Pi_{s=2} = \frac{1}{24} \underbrace{\left((\vec{S}_1 + \vec{S}_2)^2 - 2\right)}_{=0 \text{ for } |s=1\rangle} \underbrace{\left(\vec{S}_1 + \vec{S}_2)^2\right)}_{=0 \text{ for } |s=0\rangle} = \frac{1}{2}\vec{S}_1 \cdot \vec{S}_2 + \frac{1}{6}(\vec{S}_1 \cdot \vec{S}_2)^2 + \frac{1}{3} = h_{1,2}.$$