Solutions to Homework 6

1

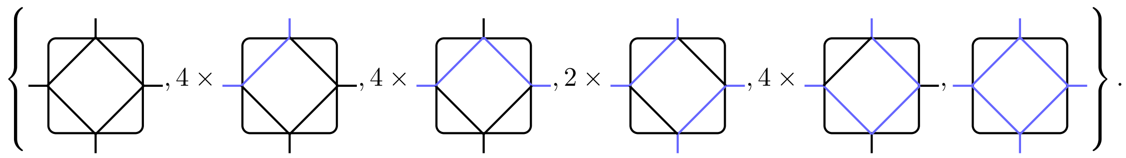
Consider a generic area of a square lattice with all links in $|0\rangle$. For this area, we can capture all loop configurations by applying B_v on each plaquette, there are $2^{|A|}$ possibilities.

Now let us consider the toric code. Since there are no endpoints, all boundary patterns have an even number of links. We can then join these links two-by-two, for example going around the boundary. We can then apply B_v repeatedly to describe all configurations in A. Since this number is constant, the number of loop configurations in the interior of A is independent of the boundary pattern.

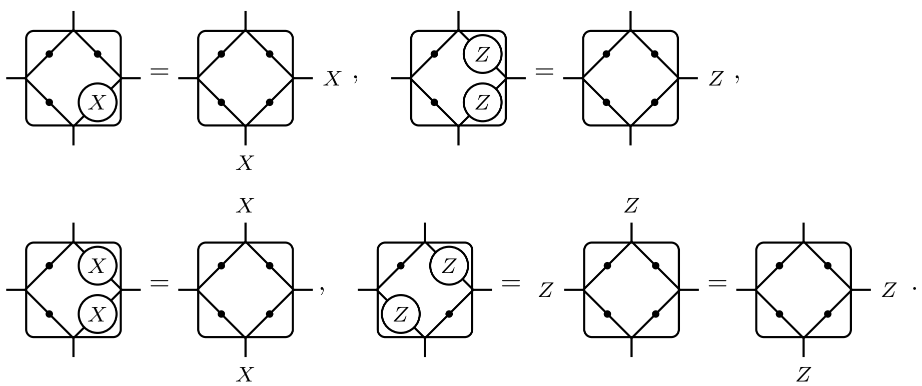
The Schmidt decomposition then reads $|\Psi\rangle = \mathcal{N}_B^{-1/2} \sum_i |\alpha_i\rangle |\beta_i\rangle$. The number of boundary patterns is $\mathcal{N}_B = 2^{|\partial A|}/2$, with the 1/2 factor coming from the fact that we require an even number of boundary links. We can easily deduce that $S_A = \log_2 \mathcal{N}_B = |\partial A| - 1$, from which we conclude that the toric code has a topological correction to the area law of $\gamma = 1$.

2

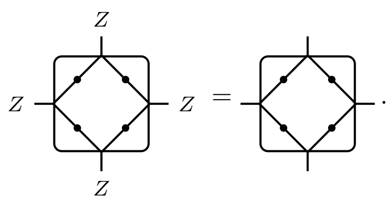
(a) Up to possible rotations, we have the following non-zero configurations



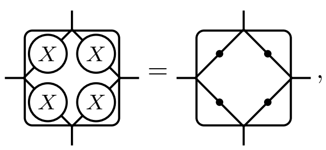
(b) Some diagrams are



(c) The Z_2 invariance means that on the index level



(d) Using the properties above one easily checks that $A_v |\Psi\rangle = |\Psi\rangle$. Then, we can check $B_p |\Psi\rangle = |\Psi\rangle$, since



One should also check the plaquette between 4 tensors, and using the properties in (b) to show invariance.

(e) The string $W_l^{(e)}$ generates a series of X matrices in between the tensors, along the path l. They all cancel out, except at the endpoint, where we have a new tensor

$$B^{i_1,i_2,i_3,i_4}_{\alpha,\beta,\gamma,\delta} = \sum X_{\delta,\delta'} A^{i_1,i_2,i_3,i_4}_{\alpha,\beta,\gamma,\delta'}.$$

Notice that $Z^{\otimes 4} B^i = -B^i$. The $W_{l^*}^{(m)}$ acts with Z on the physical level, generating a string of Zs on the virtual level. Notice that this string can be moved at will, except for the endpoints, where we have a new tensor

$$C^{i_1,i_2,i_3,i_4}_{\alpha,\beta,\gamma,\delta} = \sum Z_{i_4,i_4'} A^{i_1,i_2,i_3,i_4'}_{\alpha,\beta,\gamma,\delta}.$$

This tensor remains invariant under the symmetry: $Z^{\otimes 4} C^i = C^i$.

(f) From the structure of the tensors, we see that a non-zero term in the contraction of the tensor network corresponds to a closed loop configuration. Since there is a one-to-one correspondence between the physical indices $i_1, \ldots, i_4$ of the tensors and the lattice sites, we conclude that there is a bijection between a non-zero tensor contraction term and a loop configuration, meaning the tensor network corresponds to the ground state of the toric code.

3

No solution is presented for this exercise.