

Act III As we proved in Act I, local operations do not increase the entanglement. We can split the total mapping into the terms acting inside \mathcal{A} and the ones acting outside: $A_1 \otimes A_2 \otimes \dots \otimes A_N = M_{\mathcal{A}} \otimes M_{\bar{\mathcal{A}}}$. By splitting the Hilbert space accordingly into $|\varphi\rangle = \sum c_{n,m} |n\rangle_{\mathcal{A}} |m\rangle_{\bar{\mathcal{A}}}$, we see that the mapping acts locally — much as in Act I — and the entropy cannot increase. Therefore, we obtain the area law for PEPS

$$E_{\mathcal{A}} \leq |\partial\mathcal{A}| \log_2 D.$$

3

The mapping M projects on the spin-1 subspace:

$$M^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

The state $|\Psi\rangle = \mathcal{M}|\Phi\rangle$ is then

$$|\Psi\rangle = \sum_{\sigma, \mathbf{a}, \mathbf{b}} M_{a_1, b_1}^{\sigma_1} \Sigma_{b_1, a_2} M_{a_2, b_2}^{\sigma_2} \Sigma_{b_2, a_3} \dots \Sigma_{b_{L-1}, a_L} A_{a_L, b_L}^{\sigma_L} \Sigma_{b_L, a_1} |\sigma\rangle = \sum_{\sigma} \text{tr}(M^{\sigma_1} \Sigma M^{\sigma_2} \Sigma \dots M^{\sigma_L} \Sigma) |\sigma\rangle$$

By introducing $A^{\sigma} = M^{\sigma} \Sigma$ we obtain the desired form.

$$A^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^0 = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

4

Note The choice of the tensors is not unique, you may well find a different combination that works.

(a) A bond dimension $D = 1$ is sufficient: $A^0 = 0$, $A^1 = 1$.

(b) Here we need to introduce a minimum of $D = 2$

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

.

(c) The most compact representation of the $|\mathbf{W}\rangle$ state is not translationally invariant. We then have to resort to different tensors and write $|\mathbf{W}\rangle = \sum \text{tr}(A^{i_A} B^{i_B} C^{i_C}) |i_A, i_B, i_C\rangle$, with

$$A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5

The proposed solution is written as a MATLAB script, see file `multent.m` provided. The easiest way to approach the problem is to define the tensor c_{ijkl} , reshape it into the desired matrix, and then perform an SVD decomposition. Care should be taken when constructing the state, since most routines use column-major versus row-major reshaping.

6

Applying the definition of \hat{b}_k ,

$$\epsilon_k = -\frac{2t}{R} \cos\left(\frac{2\pi k}{R}\right).$$

In the thermodynamic limit $R \rightarrow \infty$, the gap goes to zero. Hence this Hamiltonian is *gapless*. Doing the inverse Fourier transform, we obtain

$$|\Psi_0\rangle = \frac{1}{\sqrt{N!}} \left(\frac{1}{\sqrt{R}} \sum_{i=1}^R \hat{a}_i^\dagger \right)^N |0\rangle.$$

We split the sum and write

$$\begin{aligned} |\Psi_0\rangle &= \frac{R^{-N/2}}{\sqrt{N!}} \left(\sqrt{L} \hat{a}_A^\dagger + \sqrt{R-L} \hat{a}_{\bar{A}}^\dagger \right)^N |0\rangle \\ &= \frac{R^{-N/2}}{\sqrt{N!}} \sum_{n=0}^N \binom{N}{n} \left(\sqrt{L} \hat{a}_A^\dagger \right)^n \left(\sqrt{R-L} \hat{a}_{\bar{A}}^\dagger \right)^{N-n} |0\rangle \\ &= \sum_{n=0}^N \frac{\sqrt{N!}}{\sqrt{n!} \sqrt{(N-n)!}} \left(\frac{L}{R} \right)^{\frac{n}{2}} \left(\frac{R-L}{R} \right)^{\frac{N-n}{2}} \frac{(\hat{a}_A^\dagger)^n}{\sqrt{n!}} \frac{(\hat{a}_{\bar{A}}^\dagger)^{N-n}}{\sqrt{(N-n)!}} |0\rangle \\ &= \sum_{n=0}^N \sqrt{\lambda_n} |n\rangle_A |N-n\rangle_{\bar{A}}, \quad \lambda_n = \binom{N}{n} \left(\frac{L}{R} \right)^n \left(1 - \frac{L}{R} \right)^{N-n} \end{aligned}$$

This is explicitly a Schmidt decomposition, i.e. $\rho_n = \lambda_n$. Notice also that λ_n follows a binomial distribution with parameter $p = L/R$, therefore as $N \rightarrow \infty$ it approaches a normal distribution $\mathcal{N}(Np, Np(1-p))$. Computing the entropy is a Gaussian integral. Using $\int_{-\infty}^{\infty} e^{-x^2/\alpha^2} = \sqrt{\pi/\alpha}$ and $\int_{-\infty}^{\infty} x^2 e^{-x^2/\alpha^2} = \sqrt{\pi}/\alpha^{3/2}$,

$$E_A = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{n^2}{2\sigma^2}} \left(\frac{1}{2\sigma^2} n^2 + \log \sqrt{2\pi\sigma^2} \right) dn = \frac{1}{2} + \log \sqrt{2\pi\sigma^2} = \log \sigma + \frac{1 + \log(2\pi)}{2}.$$

Since $\sigma^2 = \frac{N}{R}(1 - \frac{L}{R})L \rightarrow \frac{N}{R}L$ when $N, R \rightarrow \infty$

$$E_A = \frac{1}{2} \log(L) + \frac{\log(N/R) + \log(2\pi) + 1}{2}.$$

In this case, *the area law is violated*. From the calculations shown above, the Hamiltonian is gapless, so we can't apply the known theorems to show an area law.

Note If you used the base-2 logarithm, you have to rescale this result by a factor $\log_2 e$.