Solutions to Homework 3

1

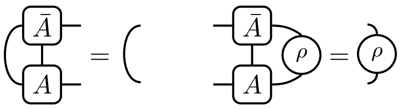
Using the definition of an MPS,

$$\langle\Psi|\Psi\rangle = \sum_{\{i_n\}} \overline{\operatorname{tr}(A^{i_1}\dots A^{i_N})}\operatorname{tr}(A^{i_1}\dots A^{i_N}) = \operatorname{tr}\left(\sum_{i_1}\bar{A}^{i_1}\otimes A^{i_1}\right)\dots\left(\sum_{i_N}\bar{A}^{i_N}\otimes A^{i_N}\right) = \operatorname{tr}\mathbb{E}^N.$$

Using the fact that r and ℓ are the fixed points of ε and ε^* respectively,

$$\sum_i \tilde{A}^{i\dagger}\tilde{A}^i = \sum_i \ell^{-1/2}\underbrace{{A^i}^\dagger\ell^{1/2}\ell^{1/2}A^i}_{=\ell}\ell^{-1/2} = \mathbb{1}, \quad \sum_i \tilde{A}^i\tilde{A}^{i\dagger} = \sum_i r^{-1/2}\underbrace{A^i r^{1/2}r^{1/2}{A^i}^\dagger}_{=r}r^{-1/2} = \mathbb{1}.$$

Graphically, the left-canonical form corresponds to



Theorem 1.1 (Fundamental theorem of MPS)

Proof. In the large N limit, the overlap is $|\langle\Psi_B|\Psi_A\rangle| \to |\lambda|^N$. Clearly, $|\lambda| \le 1$, and there are two possible scenarios in the termodynamic limit:

- $|\langle\Psi_B|\Psi_A\rangle| \to 0$, if the two states are different ($\lambda < 1$),
- $|\langle\Psi_B|\Psi_A\rangle| = 1$, if the two states are the same ($\lambda = 1$).

Hence, we are interested in the case where $\lambda = 1$. Using the fact that the fixed point ρ is strictly positive,

$$\begin{aligned}\left|\lambda\operatorname{tr}(X\rho X^\dagger)\right|^2 &= \left|\operatorname{tr}(XA^i\sqrt{\rho}\sqrt{\rho}X^\dagger {B^i}^\dagger)\right|^2 \\ &\le \left(\sum_i \operatorname{tr}(XA^i\rho {A^i}^\dagger X^\dagger)\right)\left(\sum_i \operatorname{tr}(B^iX\rho X^\dagger {B^i}^\dagger)\right) = \left|\operatorname{tr}(X\rho X^\dagger)\right|^2\end{aligned}$$

Since $\lambda = 1$, the Cauchy–Schwarz inequality is satisfied with an equal sign, that means that $u \propto v$, i.e. $XA^i\sqrt{\rho} = \alpha B^iX\sqrt{\rho}$. and show that α must be unity since

$$X = \sum_i {B^i}^\dagger B^iX = \alpha\sum {B^i}^\dagger XA^i = \alpha\lambda X.$$

We can also absorb it into X. Since ρ is full rank, it can be inverted,

$$XA^i = B^iX.$$

□

Note A diagrammatic proof has been shown during the exercise session.

2

Calculating the sum explicitly yields $\sum_\sigma {A^\sigma}^\dagger A^\sigma = \frac{3}{4}\mathbb{1}$. Therefore we rescale all A^σ by $2/\sqrt{3}$,

$$A^+ = \frac{2}{\sqrt{3}}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad A^0 = \frac{1}{\sqrt{3}}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \quad A^- = \frac{2}{\sqrt{3}}\begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix}.$$

Calculating the transfer matrix yields

$$\mathbb{E} = \frac{1}{4}\begin{pmatrix}1 & 0 & 0 & 2\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 2 & 0 & 0 & 1\end{pmatrix}.$$

The eigenvalues are 1, $-1/3$, $-1/3$, $-1/3$. Therefore, $\langle\Psi|\Psi\rangle = 1 + 3(1/3)^L \to 1$ as $L \to \infty$.

3

After blocking L sites, we still have only two linear independent vectors, namely $|0,\ldots,0\rangle = A^0 \ldots A^0$ and $|1,\ldots,1\rangle = A^1 \ldots A^1$. Therefore, we will never be able to span a space of 2×2 with two vectors, hence this state will never be injective by blocking, we say its *injectivity length* is infinite. We also notice that GHZ state is scale-invariant. The transfer matrix is

$$\mathbb{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad |\rho_\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\rho_\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

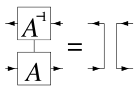
it has two degenerate dominant eigenvalues 1, with associated eigenvectors $|\rho_\uparrow\rangle$ and $|\rho_\downarrow\rangle$. Notice that $\mathbb{E}^N = \mathbb{E} = |\rho_\uparrow\rangle\langle\rho_\uparrow| + |\rho_\downarrow\rangle\langle\rho_\downarrow|$. Therefore,

$$\begin{aligned}
\langle \mathrm{GHZ}|\mathrm{GHZ}\rangle &= \operatorname{tr}\mathbb{E}^N = \sum_{\mu=\uparrow,\downarrow} \langle\rho_\mu|\rho_\mu\rangle = 2 \\
\langle O_n\rangle &= \frac{1}{2}\operatorname{tr}(\mathbb{E}_O\mathbb{E}^N) = \frac{1}{2}\sum_{\mu=\uparrow,\downarrow}\langle\rho_\mu|\mathbb{E}_O|\rho_\mu\rangle\,, \\
\langle O_n O_{L+n}\rangle &= \frac{1}{2}\operatorname{tr}(\mathbb{E}_O\mathbb{E}^{L-1}\mathbb{E}_O\mathbb{E}^N) = \frac{1}{2}\sum_{\mu,\nu=\uparrow,\downarrow}\langle\rho_\mu|\mathbb{E}_O|\rho_\nu\rangle\langle\rho_\nu|\mathbb{E}_O|\rho_\mu\rangle \\
C_L &= \frac{1}{4}\sum_{\mu,\nu=\uparrow,\downarrow} 2\langle\rho_\mu|\mathbb{E}_O|\rho_\nu\rangle\langle\rho_\nu|\mathbb{E}_O|\rho_\mu\rangle - \langle\rho_\mu|\mathbb{E}_O|\rho_\mu\rangle\langle\rho_\nu|\mathbb{E}_O|\rho_\nu\rangle \\
&= \frac{1}{4}\Big(|(\mathbb{E}_O)_{\uparrow\uparrow}|^2 + 4(\mathbb{E}_O)_{\uparrow\downarrow}(\mathbb{E}_O)_{\downarrow\uparrow} - 2(\mathbb{E}_O)_{\uparrow\uparrow}(\mathbb{E}_O)_{\downarrow\downarrow} + |(\mathbb{E}_O)_{\downarrow\downarrow}|^2\Big), \quad (\mathbb{E}_O)_{\mu\nu} = \langle\rho_\mu|\mathbb{E}_O|\rho_\nu\rangle\,.
\end{aligned}$$

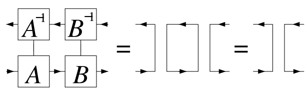
Notice how the correlation is independent of L. The GHZ has *infinite-range correlation*, and, somewhat surprisingly, we can describe that with a non-normal MPS.

4

We simply repeat the results in the original paper[1]. If some steps are not clear, you should take a look at it. Graphically, the inverse acts as



The injectivity is preserved by concatenating tensors, since



where we omit the scalar, corresponding to the contraction of the loop $\operatorname{tr}\mathbb{1} = D$.

The k-site state is

$$\left|\Psi^{[k]}_{i_{k+1},\ldots,i_L}\right\rangle = \sum_{i_1,\ldots,i_k}\operatorname{tr}(A^{i_1}\ldots A^{i_k}X^{i_{k+1},\ldots,i_L})\,|i_1,\ldots,i_k\rangle\,, \qquad X^{i_{k+1},\ldots,i_L} = A^{i_{k+1}}\ldots A^{i_L}.$$

The k-site density matrix ρ_k is then

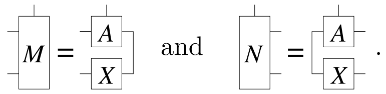
$$\rho_k = \sum_{i_{k+1},\ldots,i_L}\left|\Psi^{[k]}_{i_{k+1},\ldots,i_L}\right\rangle\left\langle\Psi^{[k]}_{i_{k+1},\ldots,i_L}\right|$$

We can think of this operator as a linear combination of outer products of elements in $\mathcal{S}_k$, hence a map $\rho_k : \mathcal{S}_k \mapsto \mathcal{S}_k$. Moreover, it can be shown that the injectivity of A implies that $X^{i_{k+1},\ldots,i_L}$ spans the whole space $\mathbf{L}(\mathbb{C}^{D\times D})$.

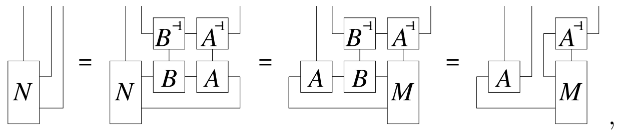
[1] N. Schuch, J.I. Cirac, D. Peréz-García, Annals of Physics (2010), arXiv:1001.3807.
All diagrams of this exercise come from this paper.

Theorem 4.1 (Intersection property)

Proof. To prove rhs $\subseteq$ lhs we choose



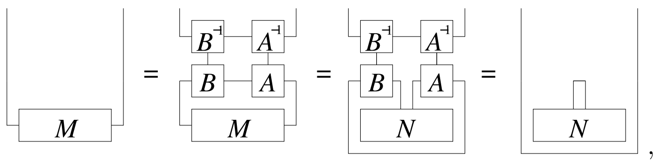
In the opposite direction, we obtain a relationship between M and N



which we plug back into $|\psi\rangle$ to obtain $|\psi\rangle \in$ rhs. Hence lhs $\subseteq$ rhs $\subseteq$ lhs $\Leftrightarrow$ lhs $=$ rhs. $\square$

Theorem 4.2 (Closure property)

Proof. The fact that rhs $\subseteq$ lhs is verified immediately by taking $M = N = \mathbb{1}$. Conversely,



shows that lhs $\subseteq$ rhs. Putting these inclusions together, we obtain lhs $=$ rhs. $\square$

Theorem 4.3 (Parent Hamiltonian)

Proof. Take $B^\alpha = A^{i+1} \dots A^{L-1}$, with the index $\alpha = (i+1, \dots, L-1)$,

$$\mathcal{S}_{L-1} \otimes \mathbb{C}^d = \left\{ \sum_{i_1, \alpha, i_L} \operatorname{tr}(A^{i_1} B^\alpha M^{i_L}) \, |i_1, \alpha, i_L\rangle \middle| M \in \mathbb{C}^d \otimes \mathbf{L}(\mathbb{C}^{D \times D}) \right\}$$

and similarly for $\mathbb{C}^d \otimes \mathcal{S}_{L-1}$. Then, applying the intersection property, $\mathcal{S}_L = \left(\mathcal{S}_{L-1} \otimes \mathbb{C}^d\right) \cap \left(\mathbb{C}^d \otimes \mathcal{S}_{L-1}\right)$. By repeatedly we span the whole space $\mathcal{S}_L$.

$$\mathcal{S}_L = \Big(\mathcal{S}_2 \otimes (\mathbb{C}^d)^{\otimes(L-2)}\Big) \cap \dots \cap \Big((\mathbb{C}^d)^{\otimes(L-2)} \otimes \mathcal{S}_2\Big) = \bigcap_{k=1}^{L-1} (\mathbb{C}^d)^{\otimes(k-1)} \otimes \mathcal{S}_2 \otimes (\mathbb{C}^d)^{\otimes(L-k-1)}$$

Since $\mathcal{S}_2$ corresponds to the ground space of every $h_{i,i+1}$, $\mathcal{S}_L$ corresponds to the ground space of $H' = \sum_{i=1}^{N-1} h_{i,i+1}$ (notice the open boundary conditions). We now use. Now consider the ground spaces of the two operators $H_{\text{left}} = h_{1,2} + \frac{1}{2} h_{2,3} + \dots \frac{1}{2} h_{L-1,L}$ and $H_{\text{right}} = \frac{1}{2} h_{2,3} + \dots \frac{1}{2} h_{L-1,L} + h_{L,1}$. The ground space of each of these operators is the lhs of Theorem 4.2, from which we conclude that the ground space of $H_{\text{left}} + H_{\text{right}} = H$ is $\operatorname{span}\{|\Psi\rangle\}$. $\square$