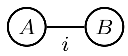
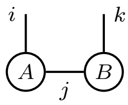
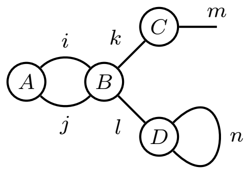
Solutions to Homework 2

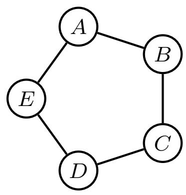
1

The rank of the resulting tensor is simply the number of open legs in the diagram.

(a) $\sum_i A_i B_i =$ 

(b) $\sum_j A_{ij} B_{jk} =$ 

(c) $\sum_{i,j,k,l,n} A_{ij} B_{ijkl} C_{km} D_{lnn} =$ 

$\mathrm{tr}(ABCDE) =$ 

In practice we will seldom write explicitly the indices of a tensor networks. This can lead to some confusion as to which contraction a diagram actually represents since it can be unclear which index is which. For example we may be incorrectly lead to believe that figure above also represents the contraction $\mathrm{tr}(AEDCB)$. By writing down the indices explicitly, one can check that it is indeed a different contraction.

2

Act I Notice that the maximum number of Schmidt values (i.e. the Schmidt rank) is necessarily bounded by $\mathrm{rank}(c)$. It follow that $E(|\psi\rangle) \leq \log_2 D_{\min}$, where $D_{\min} = \min(D_1, D_2)$. By applying the mapping, we obtain $P|\psi\rangle = (P_1 c P_2^\top)_{n',m'} |n'\rangle |m'\rangle$. Using the property $\mathrm{rank}(AB) \leq \min\big(\mathrm{rank}(A), \mathrm{rank}(B)\big)$, the Schmidt rank is then bounded by

$$\mathrm{rank}(P_1 c P_2^\top) \leq \min\big(\mathrm{rank}(c), \mathrm{rank}(P_1), \mathrm{rank}(P_2)\big) \leq D_{\min}$$

Hence, $E(P|\psi\rangle) \leq \log_2 D_{\min}$.

Act II

- Using the previous point, $E(\bullet\!\!-\!\!\bullet) \leq \log_2 D$.
- Divide $\mathcal{A}$ into the boundary and the bulk: $\mathcal{A} = \partial\mathcal{A} + \mathcal{A}^\circ$. The bulk is in a pure state ($S_{\mathcal{A}^\circ} = 0$) and will not contribute to the entanglement: $S_A \leq S_{\partial\mathcal{A}} + S_{\mathcal{A}^\circ}$, hence $S_{\mathcal{A}} = S_{\partial\mathcal{A}}$. A similar procedure applies to the pairs in $\bar{\mathcal{A}}$.
- We are cutting $|\partial A|$ entangled pairs, hence the entanglement will be bounded by $E_{\mathcal{A}} \leq |\partial\mathcal{A}| \log_2 D$.

Act III As we proved in Act I, local operations do not increase the entanglement. We can split the total mapping into the terms acting inside $\mathcal{A}$ and the ones acting outside: $A_1 \otimes A_2 \otimes \cdots \otimes A_N = M_{\mathcal{A}} \otimes M_{\bar{\mathcal{A}}}$. By splitting the Hilbert space accordingly into $|\varphi\rangle = \sum c_{n,m} |n\rangle_{\mathcal{A}} |m\rangle_{\bar{\mathcal{A}}}$, we see that the mapping acts locally — much as in Act I — and the entropy cannot increase. Therefore, we obtain the area law for PEPS

$$E_{\mathcal{A}} \leq |\partial \mathcal{A}| \log_2 D.$$

3

The mapping M projects on the spin-1 subspace:

$$M^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

The state $|\Psi\rangle = \mathcal{M} |\Phi\rangle$ is then

$$|\Psi\rangle = \sum_{\boldsymbol{\sigma},\mathbf{a},\mathbf{b}} M^{\sigma_1}_{a_1,b_1} \Sigma_{b_1,a_2} M^{\sigma_2}_{a_2,b_2} \Sigma_{b_2,a_3} \dots \Sigma_{b_{L-1},a_L} A^{\sigma_L}_{a_L,b_L} \Sigma_{b_L,a_1} |\boldsymbol{\sigma}\rangle = \sum_{\boldsymbol{\sigma}} \mathrm{tr}\left(M^{\sigma_1} \Sigma M^{\sigma_2} \Sigma \dots M^{\sigma_L} \Sigma\right) |\boldsymbol{\sigma}\rangle$$

By introducing $A^\sigma = M^\sigma \Sigma$ we obtain the desired form.

$$A^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^0 = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

4

(a) A bond dimension $D = 1$ is sufficient: $A^0 = 0$, $A^1 = 1$.

(b) Here we need to introduce a mimimum of $D = 2$

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

.

(c) The most compact representation of the $|\mathrm{W}\rangle$ state is not translationally invariant. We then have to resort to different tensors and write $|\mathrm{W}\rangle = \sum \mathrm{tr}(A^{i_A} B^{i_B} C^{i_C}) |i_A, i_B, i_C\rangle$, with

$$A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$A^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note The choice of the tensors is not unique, you may well find a different combination that works.

5

The proposed solution the MATLAB script `multent.m` provided. The easiest way to approach the problem is to define the tensor c_{ijkl}, reshape it into the desired matrix, and then perform an SVD decomposition. Care should be taken when constructing the state, since most routines use column-major versus row-major reshaping.

6

Applying the definition of $\hat{b}_k$,

$$\epsilon_k = -\frac{2t}{R}\cos\left(\frac{2\pi k}{R}\right).$$

In the thermodynamic limit $R \to \infty$, the gap goes to zero. Hence this Hamiltonian is *gapless*. Doing the inverse Fourier transform, we obtain

$$|\Psi_0\rangle = \frac{1}{\sqrt{N!}}\left(\frac{1}{\sqrt{R}}\sum_{i=1}^{R}\hat{a}_i^\dagger\right)^N |0\rangle\,.$$

We split the sum and write

$$\begin{aligned}
|\Psi_0\rangle &= \frac{R^{-N/2}}{\sqrt{N!}}\left(\sqrt{L}\hat{a}_A^\dagger + \sqrt{R-L}\hat{a}_{\bar{A}}^\dagger\right)^N |0\rangle \\
&= \frac{R^{-N/2}}{\sqrt{N!}}\sum_{n=0}^{N}\binom{N}{n}\left(\sqrt{L}\hat{a}_A^\dagger\right)^n\left(\sqrt{R-L}\hat{a}_{\bar{A}}^\dagger\right)^{N-n}|0\rangle \\
&= \sum_{n=0}^{N}\frac{\sqrt{N!}}{\sqrt{n!}\sqrt{(N-n)!}}\left(\frac{L}{R}\right)^{\frac{n}{2}}\left(\frac{R-L}{R}\right)^{\frac{N-n}{2}}\frac{(\hat{a}_A^\dagger)^n}{\sqrt{n!}}\frac{(\hat{a}_{\bar{A}}^\dagger)^{N-n}}{\sqrt{(N-n)!}}|0\rangle \\
&= \sum_{n=0}^{N}\sqrt{\lambda_n}\,|n\rangle_A\,|N-n\rangle_{\bar{A}}\,,\quad \lambda_n = \binom{N}{n}\left(\frac{L}{R}\right)^n\left(1-\frac{L}{R}\right)^{N-n}
\end{aligned}$$

This is explicitly a Schmidt decomposition, i.e. $\rho_n = \lambda_n$. Notice also that λ_n follows a binomial distribution with parameter $p = L/R$, therefore as $N \to \infty$ it approaches a normal distribution $\mathcal{N}\big(Np, Np(1-p)\big)$. Computing the entropy is a Gaussian integral. Using $\int_{-\infty}^{\infty} e^{-x^2/\alpha^2} = \sqrt{\pi/\alpha}$ and $\int_{-\infty}^{\infty} x^2 e^{-x^2/\alpha^2} = \sqrt{\pi}/\alpha^{3/2}$,

$$E_A = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{n^2}{2\sigma^2}}\left(\frac{1}{2\sigma^2}n^2 + \log\sqrt{2\pi\sigma^2}\right)\mathrm{d}n = \frac{1}{2} + \log\sqrt{2\pi\sigma^2} = \log\sigma + \frac{1+\log(2\pi)}{2}.$$

Since $\sigma^2 = \frac{N}{R}(1-\frac{L}{R})L \to \frac{N}{R}L$ when $N, R \to \infty$

$$E_A = \frac{1}{2}\log(L) + \frac{\log(N/R) + \log(2\pi) + 1}{2}.$$

In this case, *the area law is violated.* From the calculations shown above, the Hamiltonian is gapless, so we can't apply the known theorems to show an area law.

Note If you used the base-2 logarithm, you have to rescale this result by a factor $\log_2 e$.