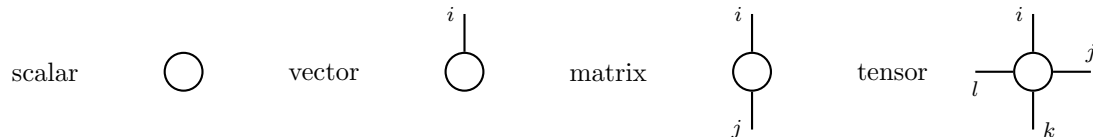


Homework 2

1 Tensor Network Diagrams

Let us review the diagrammatic notation for tensors. As you may remember tensors can be represented by a circle with a number of legs, each leg corresponding to one of its indices. The number of indices of tensor is called its *rank*. A *contraction* between a set of tensors is the sum over all possible values of the repeated indices.



Draw the diagrams associated to the contractions

$$(a) \sum_i A_i B_i, \quad (b) \sum_j A_{ij} B_{jk}, \quad (c) \sum_{i,j,k,l,n} A_{ij} B_{ijkl} C_{km} D_{lnn}$$

and make sure to write down the indices explicitly. What is the rank of each resulting tensor?

Let A, B, C, D, E be five square matrices. Draw the diagram of $\text{tr}(ABCDE)$. Can you illustrate the cyclic property of the trace in this representation?

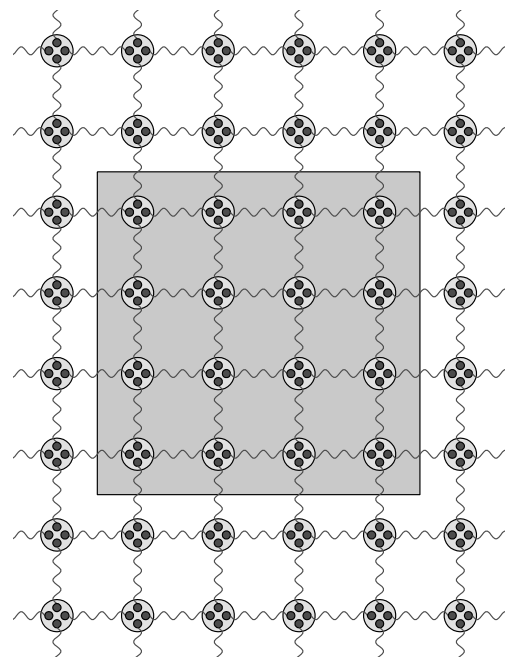
2 PEPS Satisfy the Area Law — In Three Acts

Act I Consider a general pair of entangled systems: $|\psi\rangle = \sum_{n,m} c_{n,m} |n\rangle |m\rangle$, with $n = 1, \dots, D_1$ and $m = 1, \dots, D_2$. Show that the entanglement is upper-bounded by $\log_2 \min(D_1, D_2)$. Show that this bound is still valid after acting with a local map $P = P_1 \otimes P_2$, where $P_{1,2} : \mathbb{C}^{D_{1,2}} \rightarrow \mathbb{C}^{D'_{1,2}}$.

Act II Now consider a lattice, composed of N nodes of coordination number z . On each node we assign z D -level systems, each entangled with a system on a neighboring node. The total wavefunction is then $|\varphi\rangle = |\bullet\bullet\rangle^{\otimes zN/2}$. Let \mathcal{A} be an region of this lattice. Let us now show that this construction satisfies the area law.

- What is the maximum entanglement of a single pair $|\bullet\bullet\rangle$?
- Show that the entanglement of \mathcal{A} with respect to its surrounding $\bar{\mathcal{A}}$ depends only on the pairs cut by $\partial\mathcal{A}$.
- Derive an upper-bound for the entanglement entropy of \mathcal{A} that is an area law.

Act III We now construct a *Projected Entangled Pair State* (PEPS). We apply a map $A: (\mathbb{C}^D)^z \rightarrow \mathbb{C}^d$, on every node: $|\text{PEPS}\rangle = A_1 A_2 \dots A_N |\varphi\rangle$. Show that the entanglement of \mathcal{A} still obeys an area law.



Tip If you are stuck, start by considering a $z = 2$ lattice and then going to higher dimensions.

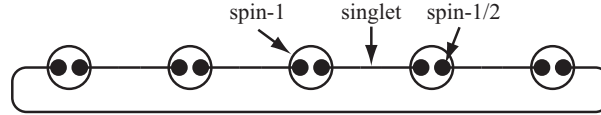
3 The AKLT State

Remember the addition of angular momentum from your quantum mechanics course? Or did you think you could forget Clebsch–Gordon coefficients? Convince yourself that by combining two spin-1/2 you obtain the triplet

$$|+\rangle = |\uparrow\uparrow\rangle, \quad |0\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}, \quad |-\rangle = |\downarrow\downarrow\rangle,$$

associated to spin-1, and the spin-0 singlet

$$|\bullet\bullet\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}.$$



Let us now construct a state by using $2L$ auxiliary spin-1/2 states for a chain of L sites. In this construction

$$|\Phi\rangle = \sum_{\mathbf{a}, \mathbf{b}} c_{\mathbf{a}\mathbf{b}} |\mathbf{a}, \mathbf{b}\rangle,$$

where $|\mathbf{a}\rangle = |a_1, \dots, a_L\rangle$ and $|\mathbf{b}\rangle = |b_1, \dots, b_L\rangle$ represent the first and second spin-1/2 on each site. To encode the singlet bonds between b_i and a_{i+1} we introduce a matrix

$$\Sigma = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and write down the state with singlets as

$$|\Phi\rangle = \sum_{\mathbf{a}, \mathbf{b}} \Sigma_{b_1 a_2} \Sigma_{b_2 a_3} \dots \Sigma_{b_{L-1} a_L} \Sigma_{b_L a_1} |\mathbf{a}, \mathbf{b}\rangle,$$

To perform the spin addition, we introduce a mapping M from the state of the two auxiliary spins-1/2 $|a_i\rangle |b_i\rangle \in \{|\uparrow\rangle, |\downarrow\rangle\}^{\otimes 2}$ to the states of the physical spin-1 $|\sigma_i\rangle \in \{|+\rangle, |0\rangle, |-\rangle\}$. Write the local mapping $M_{ab}^\sigma |\sigma\rangle \langle a, b|$ as three 2×2 matrices, one for each σ .

The total mapping is then

$$\mathcal{M} = \sum_{\sigma} \sum_{\mathbf{a}, \mathbf{b}} M_{a_1 b_1}^{\sigma_1} M_{a_2 b_2}^{\sigma_2} \dots M_{a_L b_L}^{\sigma_L} |\sigma\rangle \langle \mathbf{a}, \mathbf{b}|$$

Deduce the form of the state after applying the mapping and write it as

$$|\Psi\rangle = \sum_{\sigma} \text{tr}(A^{\sigma_1} A^{\sigma_2} \dots A^{\sigma_L}) |\sigma\rangle$$

where each A is a rank-3 tensor. Give the explicit form of A^σ for each σ . Finally, draw the associated diagram.

Note Notice how the trace is used to express the periodic boundary conditions.

4 Some Matrix Product States

A *Matrix Product State* (MPS) is a state describing a one-dimensional system expressed in the form

$$|\text{MPS}\rangle = \sum_{n_1, \dots, n_N} \text{tr}(A^{n_1} A^{n_2} \dots A^{n_N}) |n_1, \dots, n_N\rangle$$

where each A^{n_i} can be seen as a matrix for each entry n_i . More precisely it is the decomposition of a high-rank tensor into a series of rank-3 tensors (not necessarily the same) contracted sequentially, hence the name *Tensor Train decomposition*, used in mathematics and computer science. Construct the MPS — i.e. write down the tensors A^{n_i} — associated to the following N -qubit states:

$$(a) |\text{Prod}\rangle = |1111\rangle, \quad (b) |\text{GHZ}\rangle = |0000\rangle + |1111\rangle, \quad (c) |\text{W}\rangle = |100\rangle + |010\rangle + |001\rangle.$$

Do not worry about normalizing the states.

Bonus Construct the first two states for a two-dimensional square lattice in the thermodynamic limit.

5 Multipartite Entanglement

Let $|\psi\rangle$ be a 4-qubit state defined in terms of the Bell states

$$|\psi\rangle \propto |\Phi^+\rangle_{AB} \otimes |\Phi^+\rangle_{CD} + |\Psi^+\rangle_{AB} \otimes |\Psi^+\rangle_{CD}.$$

Normalize the state and numerically compute the entanglement of the partitions of $(A)(BCD)$, $(AB)(CD)$ and $(ABC)(D)$ by Schmidt decomposition.

Tip Test your code with states like $|\Phi^+\rangle_{AB} |00\rangle_{CD}$. Which are the entangled qubits in this case?

6 Free Bosons on a Lattice

Consider $N \in \mathbb{N}$ bosons on a one dimensional lattice of R sites described by the Hamiltonian

$$\hat{H} = -t \sum_{j=1}^R \left(\hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_j \hat{a}_{j+1}^\dagger \right)$$

where \hat{a}_j are the bosonic operators satisfying $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{i,j}$, and $t > 0$. Assume periodic boundary conditions, i.e. $\hat{a}_{N+1} = \hat{a}_1$. In momentum-space this Hamiltonian is diagonal

$$\hat{H} = \sum_k \epsilon_k \hat{b}_k^\dagger \hat{b}_k$$

where $\hat{b}_k = \frac{1}{\sqrt{R}} \sum_j e^{-2\pi i j k / R} \hat{a}_j$ are the Fourier-transformed annihilation operators. Compute the dispersion relation ϵ_k — is the system gapped? The ground state is given by all particles at $k = 0$,

$$|\Psi_0\rangle = \frac{1}{\sqrt{N!}} (\hat{b}_0^\dagger)^N |0\rangle$$

What is $|\Psi_0\rangle$ in real space?

We now separate the system into two parts: A , composed of the first L sites, and \bar{A} , composed of $R - L$ sites. The quantity of interest is the entanglement between A sites and the rest, where $L \ll R$. This bipartition of the Hilbert space is achieved by defining the operators

$$\hat{a}_A^\dagger = \frac{1}{\sqrt{L}} \sum_{j \in A} \hat{a}_j^\dagger, \quad \hat{a}_{\bar{A}}^\dagger = \frac{1}{\sqrt{R-L}} \sum_{j \in \bar{A}} \hat{a}_j^\dagger.$$

that divide the Fock space into the tensor product of the first L modes with the rest. Show that, using the binomial formula,

$$|\Psi_0\rangle = \sum_{n=0}^N \sqrt{\binom{N}{n}} \left(\frac{L}{R}\right)^{\frac{n}{2}} \left(\frac{R-L}{R}\right)^{\frac{N-n}{2}} |n\rangle_A |N-n\rangle_{\bar{A}},$$

where $|n\rangle_A = (1/\sqrt{n!})(\hat{a}_A^\dagger)^n |0\rangle$ and similarly for \bar{A} . In this representation, the density operator for the first subsystem is immediate: $\hat{\rho}_A = \sum_n \rho_n |n\rangle \langle n|$. When N is large, we can replace the sum by an integral, and show that

$$\rho(n) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(n-n_0)^2}{2\sigma^2}\right).$$

with $n_0 = NL/R$ and $\sigma = NL(R-L)/R^2$. The entanglement entropy will then be

$$E_A = - \int_{-\infty}^{\infty} dn \rho(n) \log \rho(n),$$

Perform the integral, the leading order should be $E_A \sim \log \sigma$. Finally, take the limit $N, R \rightarrow \infty$ with the ratio N/R fixed, and obtain

$$E_A = \frac{1}{2} \log L + \mathcal{O}(1)$$

Briefly comment on this result.