## Solutions to Homework 2

1

The rank of the resulting tensor is simply the number of open legs in the diagram.

(a) 
$$\sum_{i} A_{i}B_{i} = \underbrace{A}_{i} \underbrace{B}_{i}$$

(b)  $\sum_{j} A_{ij}B_{jk} = \underbrace{A}_{j} \underbrace{B}_{k}$ 

(c)  $\sum_{i,j,k,l,n} A_{ij}B_{ijkl}C_{km}D_{lnn} = \underbrace{A}_{j} \underbrace{B}_{l}$ 

$$\operatorname{tr}(ABCDE) = \underbrace{A}_{i} \underbrace{B}_{l}$$

In practice we will seldom write explicitly the indices of a tensor networks. This can lead to some confusion as to which contraction a diagram actually represents since it can be unclear which index is which. For example we may be incorrectly lead to believe that figure above also represents the contraction  $\operatorname{tr}(AEDCB)$ . By writing down the indices explicitly, one can check that it is indeed a different contraction.

2

Act I Notice that the maximum number of Schmidt values (i.e. the Schmidt rank) is necessarily bounded by  $\operatorname{rank}(c)$ . It follow that  $E(|\psi\rangle) \leq \log_2 D_{\min}$ , where  $D_{\min} = \min(D_1, D_2)$ . By applying the mapping, we obtain  $P|\psi\rangle = (P_1cP_2^\top)_{n',m'}|n'\rangle|m'\rangle$ . Using the property  $\operatorname{rank}(AB) \leq \min(\operatorname{rank}(A), \operatorname{rank}(B))$ , the Schmidt rank is then bounded by

$$\operatorname{rank}(P_1 c P_2^{\top}) \leq \min \left( \operatorname{rank}(c), \operatorname{rank}(P_1), \operatorname{rank}(P_2) \right) \leq D_{\min}$$

Hence,  $E(P|\psi\rangle) \leq \log_2 D_{\min}$ .

Act II

- Using the previous point,  $E(\bullet \bullet) \leq \log_2 D$ .
- Divide  $\mathcal{A}$  into the boundary and the bulk:  $\mathcal{A} = \partial \mathcal{A} + \mathcal{A}^{\circ}$ . The bulk is in a pure state  $(S_{\mathcal{A}^{\circ}} = 0)$  and will not contribute to the entanglement:  $S_{A} \leq S_{\partial \mathcal{A}} + S_{\mathcal{A}^{\circ}}$ , hence  $S_{\mathcal{A}} = S_{\partial \mathcal{A}}$ . A similar procedure applies to the pairs in  $\bar{\mathcal{A}}$ .
- We are cutting  $|\partial A|$  entangled pairs, hence the entanglement will be bounded by  $E_A \leq |\partial A| \log_2 D$ .

Act III As we proved in Act I, local operations do not increase the entanglement. We can split the total mapping into the terms acting inside  $\mathcal{A}$  and the ones acting outside:  $A_1 \otimes A_2 \otimes \cdots \otimes A_N = M_{\mathcal{A}} \otimes M_{\bar{\mathcal{A}}}$ . By splitting the Hilbert space accordingly into  $|\varphi\rangle = \sum c_{n,m} |n\rangle_{\mathcal{A}} |m\rangle_{\bar{\mathcal{A}}}$ , we see that the mapping acts locally — much as in Act I — and the entropy cannot increase. Therefore, we obtain the area law for PEPS

$$E_{\mathcal{A}} \leq |\partial \mathcal{A}| \log_2 D.$$

3

The mapping M projects on the spin-1 subspace:

$$M^{+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M^{0} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M^{-} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

The state  $|\Psi\rangle = \mathcal{M} |\Phi\rangle$  is then

$$|\Psi\rangle = \sum_{\boldsymbol{\sigma},\mathbf{a},\mathbf{b}} M_{a_1,b_1}^{\sigma_1} \Sigma_{b_1,a_2} M_{a_2,b_2}^{\sigma_2} \Sigma_{b_2,a_3} \dots \Sigma_{b_{L-1},a_L} A_{a_L,b_L}^{\sigma_L} \Sigma_{b_L,a_1} |\boldsymbol{\sigma}\rangle = \sum_{\boldsymbol{\sigma}} \operatorname{tr} \left( M^{\sigma_1} \Sigma M^{\sigma_2} \Sigma \dots M^{\sigma_L} \Sigma \right) |\boldsymbol{\sigma}\rangle$$

By introducing  $A^{\sigma} = M^{\sigma} \Sigma$  we obtain the desired form.

$$A^{+} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^{0} = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^{-} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

4

**Note** The choice of the tensors is not unique, you may well find a different combination that works.

- (a) A bond dimension D=1 is sufficient:  $A^0=0, A^1=1$ .
- (b) Here we need to introduce a minimum of D=2

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

.

(c) The most compact representation of the  $|W\rangle$  state is not translationally invariant. We then have to resort to different tensors and write  $|W\rangle = \sum \operatorname{tr}(A^{i_A}B^{i_B}C^{i_C})|i_A,i_B,i_C\rangle$ , with

$$A^{0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B^{0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C^{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$A^{1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B^{1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C^{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5

The proposed solution is written as a MATLAB script, see file multent.m provided. The easiest way to approach the problem is to define the tensor  $c_{ijkl}$ , reshape it into the desired matrix, and then perform an SVD decomposition. Care should be taken when constructing the state, since most routines use column-major versus row-major reshaping.

6

Applying the definition of  $\hat{b}_k$ ,

$$\epsilon_k = -\frac{2t}{R}\cos\left(\frac{2\pi k}{R}\right).$$

In the thermodynamic limit  $R \to \infty$ , the gap goes to zero. Hence this Hamiltonian is gapless. Doing the inverse Fourier transform, we obtain

$$|\Psi_0\rangle = \frac{1}{\sqrt{N!}} \left( \frac{1}{\sqrt{R}} \sum_{i=1}^R \hat{a}_i^{\dagger} \right)^N |0\rangle .$$

We split the sum and write

$$\begin{split} |\Psi_0\rangle &= \frac{R^{-N/2}}{\sqrt{N!}} \left(\sqrt{L}\hat{a}_A^\dagger + \sqrt{R-L}\hat{a}_{\bar{A}}^\dagger\right)^N |0\rangle \\ &= \frac{R^{-N/2}}{\sqrt{N!}} \sum_{n=0}^N \binom{N}{n} \left(\sqrt{L}\hat{a}_A^\dagger\right)^n \left(\sqrt{R-L}\hat{a}_{\bar{A}}^\dagger\right)^{N-n} |0\rangle \\ &= \sum_{n=0}^N \frac{\sqrt{N!}}{\sqrt{n!}\sqrt{(N-n)!}} \left(\frac{L}{R}\right)^{\frac{n}{2}} \left(\frac{R-L}{R}\right)^{\frac{N-n}{2}} \frac{(\hat{a}_A^\dagger)^n}{\sqrt{n!}} \frac{(\hat{a}_{\bar{A}}^\dagger)^{N-n}}{\sqrt{(N-n)!}} |0\rangle \\ &= \sum_{n=0}^N \sqrt{\lambda_n} \left|n\right\rangle_A |N-n\rangle_{\bar{A}}, \quad \lambda_n = \binom{N}{n} \left(\frac{L}{R}\right)^n \left(1-\frac{L}{R}\right)^{N-n} \end{split}$$

This is explicitly a Schmidt decomposition, i.e.  $\rho_n = \lambda_n$ . Notice also that  $\lambda_n$  follows a binomial distribution with parameter p = L/R, therefore as  $N \to \infty$  it approaches a normal distribution  $\mathcal{N}(Np,Np(1-p))$ . Computing the entropy is a Gaussian integral. Using  $\int_{-\infty}^{\infty} e^{-x^2/\alpha^2} = \sqrt{\pi/\alpha}$  and  $\int_{-\infty}^{\infty} x^2 e^{-x^2/\alpha^2} = \sqrt{\pi/\alpha^{3/2}}$ ,

$$E_A = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{n^2}{2\sigma^2}} \left( \frac{1}{2\sigma^2} n^2 + \log \sqrt{2\pi\sigma^2} \right) dn = \frac{1}{2} + \log \sqrt{2\pi\sigma^2} = \log \sigma + \frac{1 + \log(2\pi)}{2}.$$

Since  $\sigma^2 = \frac{N}{R}(1 - \frac{L}{R})L \to \frac{N}{R}L$  when  $N, R \to \infty$ 

$$E_A = \frac{1}{2}\log(L) + \frac{\log(N/R) + \log(2\pi) + 1}{2}.$$

In this case, the area law is violated. From the calculations shown above, the Hamiltonian is gapless, so we can't apply the known theorems to show an area law.

**Note** If you used the base-2 logarithm, you have to rescale this result by a factor  $\log_2 e$ .