Homework 1

1 Every Mixed State is a Reduced State

A generic density operator ρ can be written as $\rho = \sum_n \lambda_n |\varphi_n\rangle \langle\varphi_n|$, with $\{\varphi_n\}$ a complete set of basis states satisfying $\langle\varphi_n|\varphi_m\rangle = \delta_{n,m}$. Explain why this is possible and why all the eigenvalues λ_n are real and $\lambda_n \geq 0$.

We will now show that it is always possible to construct a purification of some system A by introducing an additional copy of the system called B. Starting from the pure state

$$|\Psi\rangle_{AB} = \sum_n \sqrt{\lambda_n}\, |\varphi_n\rangle_A |\varphi_n\rangle_B$$

show that we obtain the reduced state ρ for A by tracing out B.

Bonus This was obtained with an ancilla of the same dimension as the system. Can we obtain a purification with an ancilla of a smaller dimension? What constraints would that introduce on ρ?

2 Mutual Information Inequalities

Let A_1, B and C be three quantum systems. Prove the following inequalities:

(a) $I(A, B : C) \geq I(B : C)$,

(b) $I(A, B : C) \leq I(B : C) + 2S(A)$.

3 A Two-Qubit State

Compute the entanglement of the two-qubit state

$$|\Psi\rangle = \frac{-(1-2\sqrt{2})\,|00\rangle - (2+\sqrt{2})\,|10\rangle + (1+2\sqrt{2})\,|01\rangle + (2-\sqrt{2})\,|11\rangle}{\sqrt{30}}.$$

4 Something That Will Come Back Later On

For spin-1/2 systems, we denote the Pauli matrices as

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

There, I wrote them down once, and I won't bother doing it in the next assignments. Additionally, we define the raising and lowering operators as $\sigma^\pm = (\sigma^x \pm i\sigma^y)/2$.

Let $\{|+\rangle, |0\rangle, |-\rangle\}$ be the standard basis for a spin-1 system. We will now construct a two-particle state in a slightly unusual way as

$$|\Psi\rangle = \sum_{i,j=+,0,-} c_{ij}\, |ij\rangle, \quad \text{where } c_{ij} = \frac{1}{\sqrt{3}} \operatorname{tr}\left(A^i A^j\right)$$

and the matrices $\{A^i\}$ are defined as

$$A^+ = \sigma^+, \quad A^0 = -\frac{1}{\sqrt{2}}\sigma^z, \quad A^- = -\sigma^-.$$

Compute the entanglement between the two particles.