# Homework 3

## 1 The Cluster State

The group $\mathrm{Z}_2 \times \mathrm{Z}_2$ has the presentation $\mathrm{Z}_2 \times \mathrm{Z}_2 = \langle x, z \,|\, x^2 = z^2 = \mathbf{1}, xz = zx \rangle$, where $\mathbf{1}$ is the identity element.

(a) Show that the Pauli matrices $\{X, Y, Z\}$ form a *projective representation* of this group, i.e. $v_g v_h = e^{i\omega(g,h)} v_{gh}$.

(b) What are the possible values of $e^{i\omega(g,h)}$?

(c) Can the Pauli representation form a *linear representation*? That is, can we rephase $v_g \to e^{i\phi_g} v_g$ to make the commutator zero?

**Hint** Define $v_x = X$, $v_z = Z$, $v_{xz} = Y$.

The cluster state Hamiltonian is

$$H_c = -\sum_{i=1}^{N} Z_{i-1} X_i Z_{i+1}.$$

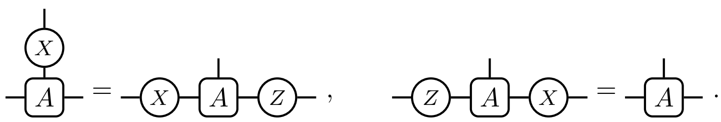
The ground state $|\psi_c\rangle$ is an MPS of bond dimension $D = 2$. In the computational basis,

$$A^0 = |0\rangle \langle +| , \quad A^1 = |1\rangle \langle -| , \quad \text{where } |\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}.$$

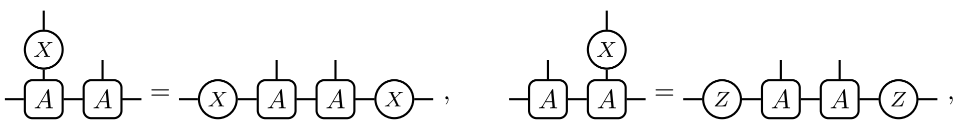
There are two operations that leave the state invariant, namely

$$X_{\text{odd}} = \bigotimes_{i \text{ odd}} X_i, \quad X_{\text{even}} = \bigotimes_{i \text{ even}} X_i.$$

Check if these operators commute with the Hamiltonian. What is the *symmetry group* of the cluster state? Show that it is $\mathrm{Z}_2 \times \mathrm{Z}_2$. Now prove the following properties:



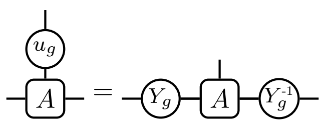
By blocking two sites, show that



and explicitly write the different local representations $u_g$ and their associated matrices $V_g$. Graphically argue that cluster state is invariant under the operations in the symmetry group.

## 2 Gauges and Symmetries

Remember that the gauge freedom for MPS is $A \mapsto GAG^{-1}$. Hence, we could imagine a unitary group acting as



However, by choosing a canonical form, this imposes the constraint that $Y_g$ be unitary. Show this, by taking the equation defining a canonical form, and assuming injective tensors.

**Hint** For injective tensor the left and right fixed points are unique.

## 3 The AKLT State, Because We Never Saw It Before

Construct the parent Hamiltonian of the AKLT state as the projector on the spin-2 subspace of two spin-1 particles (labeled 1 & 2). You should obtain

$$h_{1,2} = \frac{1}{2}\vec{S}_1 \cdot \vec{S}_2 + \frac{1}{6}(\vec{S}_1 \cdot \vec{S}_2)^2 + \frac{1}{3}$$

**Hint** You may do this by brute force, or by being a little bit smart. Start by expanding $(\vec{S}_1 \cdot \vec{S}_2)^2$. What are its possible eigenvalues? Now, either construct a projector onto $S = 2$ as a product of terms involving $(\vec{S}_1 + \vec{S}_2)^2$, or build a polynomial in terms of $X = \vec{S}_1 \cdot \vec{S}_2$ that satisfies the constraints above.