The Continuous Stochastic Gain-Loss in Production Facilities as a Result of an Exogenous Financial Shock

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1 Introduction

This paper is a further expansion of the models presented in [1] and [2]. The first part remains basically the same; it introduces the model and draws initial conclusions in a deterministic environment. In the second part, the stochastic environment, described by a continuous probability distribution, is introduced.

The question to be answered is as follows:

How much do the firm's continuous expectations affect the output?

In line with the efficient market hypothesis, the expected value of the interest rate is assumed to be equal to the present time. Variance, on the other hand, is used as a proxy for investment riskiness and assumed as an independent variable. An attempt will then also be made to answer:

How much does the riskiness of the interest rate affect the expected output with respect to each production facility?

The use of the normal distribution and the complexity of the calculations made the use of the computer tool indispensable. Specifically, the computations are performed with Python language (via Scipy library). The full source code is available at the link given in the references.

2 Assumptions

Assume that the firm produces only one type of output, using only capital as input, and that the selling price is normalized to 1. In addition, the firm chooses the type of production from 3 available, which are mutually exclusive. Each production method is described by a production function. Thus, the profit maximization problem is:

$$\max_{f_i, K}(\pi) = f_i - p \cdot K \tag{1}$$

Such that:

$$f_i \in \{f_1, f_2, f_3\} \tag{2}$$

Where π denotes precisely the profit, K the capital and p the cost of capital, taken as constant. Each production function has the following form:

$$f_i(K) = \sum_{t=0}^{\infty}(\frac{1}{1+r})^t \cdot \phi_{i,t} \cdot K \tag{3}$$

In (3) $\phi_{i,t}$ represents the return (output) on capital of function i for each period, discounted by $\frac{1}{1+r}$, where r is the market interest rate. The firm thus faces two costs, the cost of acquiring capital, and the opportunity cost of immobilizing capital in a given facility.

Assume further that the firm at time $t = 0$ is indifferent with respect to each mode of production. Given then capital K^*:

$$\pi(f_i, K^*) = \pi(f_j, K^*) \tag{4}$$

Substituting:

$$\sum_{t=0}^{\infty}(\frac{1}{1+r})^t \cdot \phi_{i,t} = \sum_{t=0}^{\infty}(\frac{1}{1+r})^t \cdot \phi_{j,t} \tag{5}$$

Now, what distinguishes each production function is the series $\{\phi_t\}_1^3$, which can take infinite values. Assume that:

$$\phi_{1,t} < \phi_{1,t+1} \tag{6}$$

$$\phi_{2,t} > \phi_{2,t+1} \tag{7}$$

$$\phi_{3,t} = \phi_{3,t+1} \tag{8}$$

In other words, the capital has a decreasing, increasing and constant return, respectively.

Assume further:

$$t \in [0, 10] \tag{9}$$

And that the capital cannot be demobilized before time T.

To make the analysis clearer, approximate (3) to the continuous case:

$$f_i(K) = K \cdot \int_0^{10} e^{-rt} \cdot \phi_{i,t} \ dt \tag{10}$$

(6), (7) and (8) can now be explicitly described as follows:

$$\phi_{1,t} = -\alpha \cdot t + b \tag{11}$$

$$\phi_{2,t} = \gamma \cdot t \tag{12}$$

$$\phi_{3,t} = \delta \tag{13}$$

With $\alpha, \gamma, \delta > 0$. Given (9), and normalizing α to 1, (11) can be simplified as follows:

$$\phi_{1,t} = 10 - t \tag{14}$$

Thus, the production functions are:

$$f_1 = \int_0^{10} e^{-rt} \cdot (10 - t)) \ dt \tag{15}$$

$$f_2 = \int_0^{10} e^{-rt} \cdot \gamma \cdot t \ dt \tag{16}$$

$$f_3 = \int_0^{10} e^{-rt} \cdot \delta \ dt \tag{17}$$

K^* was omitted as equal for each f_i. Resolving the integrals:

$$f_1 = \frac{e^{-10r} + 10r - 1}{r^2} \tag{18}$$

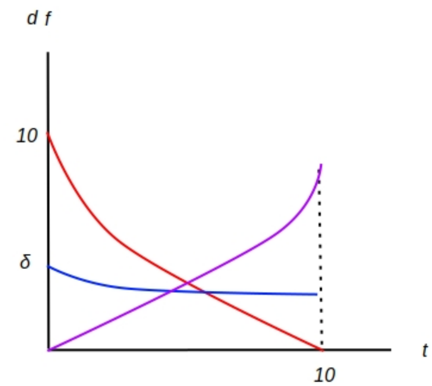

Figure 1: Productivity of capital in function of time

$$f_2 = \gamma \cdot \frac{1 - e^{-10r}(10r + 1)}{r^2} \tag{19}$$

$$f_3 = \delta \cdot \frac{1 - e^{-10r}}{r} \tag{20}$$

In Figure 1, it is possible to visualize the return on capital as a function of time; in red we have the argument of (15), in purple of (16) and in blue of (17). In contrast, with the same color scheme, (18), (19) and (20) are shown in Figure 2.

3 Expectations and Output after an Exogenous Financial Shock

Suppose that $r = 0.025$ and that the firm waits at most one changes of r, in a time sufficiently close to 0 [1]. The firm has inferred that r is distributed as a normal, with mean $\mu_r = 0.025$ and standard deviation σ. About the latter, the company has no specific prediction, and in the remainder of the discussion it will be taken as a parameter. What we describe is an application

[1] This way we can ignore the amount of output already produced

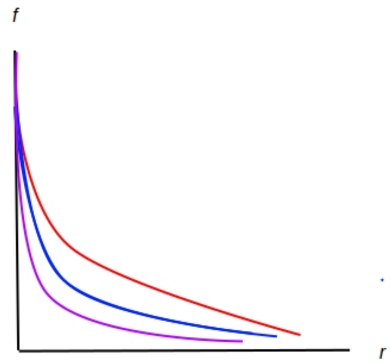

Figure 2: Productivity of capital in function of interest rate

of the efficient market hypothesis, which holds that, given an asset, there is no possibility of arbitrage, at least for significant periods [2]

Substituting $r = 0.025$ into (18), we find that the output at $t = 0$ is equal to 46; equalizing to (19) and (20) we find γ and δ equal to 1.1 and 5.2, respectively.

Given the Normal (or Gaussian) probability distribution, we use the LOTUS theorem to calculate the expected value of each output plant:

$$E(f_i, \sigma) = \int_{-\infty}^{+\infty} f_i(r) \cdot \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-0.5 \cdot (\frac{r-0.025}{\sigma})^2} \, dr \tag{21}$$

As (21) shows, the expected value is a function of variance.

We then substitute each production function into (21):

$$E(f_i, \sigma) = \int_{-\infty}^{+\infty} \frac{e^{-10r} + 10r - 1}{r^2} \cdot N(r, \sigma) \, dr \tag{22}$$

$$E(f_i, \sigma) = \int_{-\infty}^{+\infty} 1.1 \cdot \frac{1 - e^{-10r}(10r + 1)}{r^2} \cdot N(r, \sigma) \, dr \tag{23}$$

$$E(f_i, \sigma) = \int_{-\infty}^{+\infty} 5.2 \cdot \frac{e^{-10r} + 10r - 1}{r^2} \cdot N(r, \sigma) \, dr \tag{24}$$

Where $N(r, \sigma)$ denotes precisely the Gaussian.

[2]In economic theory, this principle is also called "no free lunch."

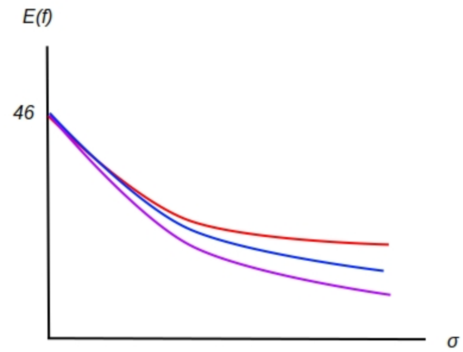

Figure 3: Expected outputs in function of variance of the interest rate

It is well known that the normal distribution is not integrable. So let us run a numerical simulation to solve (22), (23) and (24), assuming as the domain for the variance:

$$\sigma = \{0.01, 0.50\} \tag{25}$$

And:

$$\forall i, j \ (i \neq j) : |\sigma_i - \sigma j| = 0.01 \tag{26}$$

The expected value as a function of variance is shown in Figure 3, with the same color scheme as seen above. As can be seen, the greater the variance, the greater the difference in output among different production facilities. The latter is always in favor of f_1, penalizing f_2 more. This is quite reasonable, since f_1 takes advantage of the campital quickly, reducing the negative effects of an increasing change in the future interest rate.

However, as shown in Figure 4 for f_1 and f_2, the spread between the different expected values reaches its maximum when $\sigma = 0.37$, and then decreases slightly. This suggests that at high levels of riskiness, an effect favoring the increasing exploitation of capital comes into play.

From what we have seen, we can draw the following conclusion:

P4: *Given the normal probability distribution of the interest rate, with mean equal to present value, for nonexcessive levels of riskiness, an increase*

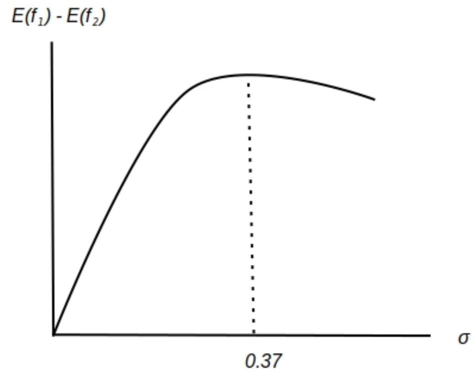

Figure 4: Expected output gap between in function of variance

in variability is positively correlated with the difference in expected autput between increasing and decreasing capital exploitation, in favor of the latter.

Thus, we have a different result than that seen for the discrete case in [2]. The reason is quite intuitable: in the discrete case, the probability distribution is skewed, placing different weights on the increasing and decreasing variation of r. In the continuous case, the distribution is symmetric, balancing the positive (negative) effects of decreasing (increasing) variation of r.

We can then generalize as follows:

P5: *Given the firm's expectations, described through a not skewed probability distribution and with expected value equal to present value, the decreasing capital exploitation strictly dominates over the others for each level of variance.*

The above does not hold in the case where the firm infers an average change in r other than present value. Such a case, however, invalidates the efficient market hypothesis: the firm might in fact profit by alienating capital (or buying more of it) instead of using it in production. Such a scenario is not analyzed in this article, but it is nevertheless modelable through the use of a skewed continuous distribution. Personally, I tried via the Beta distribution, which proved inadequate for this model. One suggestion might be to use the

Gaussian-Skewed function (O'Hagan, Leonard; 1976).

4 Conclusion

The analysis performed added important results to [1] and [2]. Essentially, given expectations about financial market performance that are continuous and centered on current market value, the decreasing exploitation of capital dominates over the other modes of production. This contrasts with the discrete case, where that mode of production has been seen to dominate over others only in a subset of possible expectations.

Riskiness does not invalidate the above result, but higher levels of riskiness, within non-excessive limits, further benefit decreasing exploitation at the expense of constant and increasing exploitation.

What we see is independent of the probability distribution used, as long as it is symmetric (or un-skewed) and centered on current market value. This result is important, both because it is consistent with an efficient financial market and because it is difficult to predict the distribution of a random variable [3]. In case this axiom is violated, however, it is possible to refer to the discrete case seen in [2], where the probability distribution is skewed in favor of increasing-decreasing variance.

In both the present work and [2], the results contrast with the deterministic case, where the company is shown to be totally indifferent to the plant used. However, without the axioms of [1], it is not possible to develop the model, as it lacks the underlying theory.

Reference

The cited articles on which this analysis is based are:

[1] *Gianmarco Corradini*, The Loss in Production Facilities as a Result of Exogenous Financial Shock; 2024

[2] ———, The Discrete Stochastic Gain-Loss in Production Facilities as a Result of an Exogenous Financial Shock; 2024

[3] Although the normal distribution is the best approximation

The model developed in this paper is based on microeconomic theory. An excellent reference text is:

Walter Nicholson, Christopher M. Snyder, Microeconomic Theory: Basic Principles and Extensions; 12th Edition, 2016, Cengage Learning.

The source code used for the computations is in the Python language, with the extension .ipynb (Jupyter Notebook file), and available at the following link:

https://giacorradini.github.io/archive