The Real Effects of the Business Cycle on the Probability of Corporate Bankruptcy

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1 Introduction

The purpose of this article is to measure how the business cycle affects the business riskiness. Specifically, an attempt will be made to answer the following question:

> *What are the effects of the business cycle on the probability of bankruptcy*?

The model is based on microeconomic and corporate finance theory and the value of assets is estimated not through accounting procedures, but stochastically. By contrast, accounts payable are fixed and expressed in nominal value.

As will be seen below, the complexity of the corporate balance sheet made it necessary to place some simplifications, such as a small number of assets and assigning numerical values to the model parameters. However, the model is not so trivialized, as the central point is probability distributions, to which few simplifications are allowed. Moreover, it became necessary to use computational techniques, especially because some functions cannot be integrated except by numerical methods.

The source code used for the computations is available at the link in the references.

2 Assumptions

Assume a firm with a balance sheet named at his real value. It is then composed of the sum of n assets divided not by accounting category, but by risk and time to claim settlement. In contrast, the company's debts to third parties are fixed. Equity is thus equal to:

$$K(\mathbf{r}, \mathbf{t}) = A^*(\mathbf{r}, \mathbf{t}) - L^* \tag{1}$$

With:

$$\begin{aligned} \mathbf{r} &= < r_1, ..., r_n > \\ \mathbf{t} &= < t_1, ..., t_n > \end{aligned} \tag{2}$$

Where $K(.)$ denotes precisely equity, or capital stock, $A(.)$ the sum of the real value of the assets, and L^* the debts owed to third parties; $\mathbf{r}$ represents the vector of the returns of all the debtor firms, and $\mathbf{t}$ the different bond maturity times. Assuming that for each company the bonds have the same maturity time, each receivable from company i is described by the tuple (r_i, t_i).

The value of each asset is given by the face value of the debt, appropriately discounted by risk and time to maturity. In formulas:

$$A_i^* = \left(\frac{1}{1 + r_i}\right)^{t_i} \cdot A_i \tag{3}$$

In (3), the element in parentheses is the discount factor and A_i the face value of the loan.

Substituting (3) into (1) we get:

$$K(\mathbf{r}, \mathbf{t}) = \left[\sum_{i=1}^{n} \left(\frac{1}{1 + r_i}\right)^{t_i} \cdot A_i\right] - L^* \tag{4}$$

Assume further that for each firm i:

$$r_i = r_{i,u} + r_{i,s} \tag{5}$$

(5) indicates that each firm's return is given by the sum of unsystemic (firm specific) and systemic (market specific) return.

Suppose that:

$$r_{i,s} = s_i \cdot r_m \tag{6}$$

In (5) and (6) $r_{i,u}$ and s_i are random variables, while r_m is a deterministic variable. The latter describes the financial market trend, while s_i is the sensitivity of asset i to the market.

Assume that:

$$r_{i,u} \sim N(\mu_{i,u}, \sigma^2_{i,u}) \tag{7}$$

$$s_i \sim N(\mu_{i,s}, \sigma^2_{i,s}) \tag{8}$$

In other words, $r_{i,u}$ and s_i have a normal probability distribution. Consequently, r_m in (6) acts as a scaling factor and $r_{i,s}$ still distributes normally, with the following parameters:

$$r_{i,s} \sim N(r_m \cdot \mu_{i,s}, r_m^2 \cdot \sigma^2_{i,s}) \tag{9}$$

Moreover, it is reasonable to assume that $r_{i,u}$ and s_i are independent, at least in the short run.

In (5), r_i is the sum of two normally distributed random variables, so it too is normal, with the following parameters:

$$r_i \sim N(\mu_{i,u} + r_m \cdot \mu_{i,s}, \sigma^2_{i,u} + r_m^2 \cdot \sigma^2_{i,s}) \tag{10}$$

To make the discussion clearer, let's assume the following parameters:

$$n = 2$$

$$t_1 = t_2 = 1$$

In other words, we have two assets in the balance sheet with the same maturity. This, as will be seen, simplifies the discussion and does not trivialize the model, since it is possible to imagine a higher maturity embedded directly in the interest rate [1].

We also assume that the book value of the capital stock is a proportion of the book value of the assets:

$$K_{book} = (1 - \alpha) \cdot (A_1 + A_2) \tag{11}$$

[1] In other words, given maturity n, interest is expressible as: $r_i^* = (1 + r_i)^n - 1$.

As a result, the value of accounts payable is:

$$L = \alpha \ (A_1 + A_2) \tag{12}$$

It is important to note that (11) is different from (4), in that the former expresses a book value and thus partial, while the latter expresses a real value [2]. This is quite in line with accounting practice, where it is required by law that the shareholders' investment not be nominally less than a proportion of the capital assets.

In light of (12), we define:

$$K_i^* = A_i^* - \alpha A_i \tag{13}$$

Equation (13) expresses the net share of assets owned by shareholders. It is a random variable with probability distribution:

$$f_i = \frac{A_i}{(K_i^* + \alpha)^2} \cdot \frac{1}{\sqrt{2\pi} \cdot \sigma_i} \cdot exp \left(-0.5 \left[\frac{A_i - (K_i^* + \alpha)(1 + \mu_i)}{(K_i^* + \alpha) \cdot \sigma_i} \right]^2 \right) \tag{14}$$

With:

$$\sigma_i^2 = \sigma_{i,u}^2 + r_m^2 \cdot \sigma_{i,s}^2 \tag{15}$$

And:

$$\mu_i = \mu_{i,u} + r_m \cdot \mu_{i,s} \tag{16}$$

Equation (14) was obtained by the change of variables[3].

K_i^* and A_i^* have distributions that belong to the same family, where α acts as a shift parameter, not modifying the shape of the distribution.

Equation (4) then becomes:

$$K_i^*(\alpha) = K_1^* + K_2^* \tag{17}$$

Assume that r_1 and r_2, are also independent of each other and use the convolution [4] to find its probability distribution:

[2] In our model, only third-party payables (L) are expressed in book value. However, they should also be discounted by market interest

[3] if $x \sim f$ and $y = g(x)$, then $y \sim \frac{d}{dy} \cdot |g^{-1}(y)| \cdot f(g^{-1}(y))$.

[4] If $z = x + y$ and $x \sim f_x$ and $y \sim f_y$, then $z \sim \int_{-\infty}^{\infty} f_x(x) f_y(z - x) dx$.

$$f_K = \int_{-\infty}^{\infty} \frac{A_1 \cdot A_2}{((K_1^* + \alpha)(K - (K_1^* + \alpha))^2 \cdot 2\pi \cdot \sigma_1 \sigma_2} exp(-0.5\beta(K)) dA_1^* \quad (18)$$

With:

$$\beta(K) = \left[\frac{A_1 - (K_1^* + \alpha)\ (\mu_1 + 1)}{(K_1^* + \alpha)\ \sigma_1} \right]^2 \quad (19)$$

$$+ \left[\frac{A_2 - (K - (K_1^* + \alpha))\ (\mu_1 + 1)}{(K - (K_1^* + \alpha))\ \sigma_2} \right]^2$$

Equation (18) is not normally distributed, yet it belongs to the same family as (14).

3 The Corporate Financial Situation before the Bisiness Cycle

Given the assumptions of the previous section, we know that the company is bankrupt as the capital stock is less or equal than zero. In other words, the reduction in assets is greater than the capital stock. Since the latter is a random variable, we can calculate the probability of the firm's insolvency as follows:

$$P_{br} = \int_{-\infty}^{0} f_K(K)\ dK \quad (20)$$

Equation (20) assumes a static perspective of the firm, where profit and loss, which respectively increase and decrease the capital stock, are not taken into account. However, the analysis aims to analyze an exogenous shock, thus immediate and unrelated to normal firm operations.

The f_K distribution has too many parameters to give us any useful information. Let us assume an initial situation with the numerical values expressed in Table 1, where, as can be seen, the book value of assets A_1 and A_2 has been normalized to 1.

Let us then proceed by calculating the mean and variance expressed by equations (13) and (14); the result is the values shown in Table 2.

V	μ	σ
r_m	0.05	0
$r_{1,u}$	0.03	0.3
$r_{2,u}$	0.02	0.2
s_1	0.8	8
s_2	0.5	5
A_1	1	0
A_2	1	0
α	0.75	0

Table 1: Numerical values for the parameters

V	μ	σ
r_1	0.07	0.5
r_2	0.045	0.32016

Table 2: Calculated values for the assets returns and riskyness

Substituting the numerical values in Table 1 into (11), we find the probability distribution of the real value of each asset. Graphically, both are represented in Figure 1. As can be seen, the distributions are not normal and exhibit skewness toward the right. This indicates that the expected value shifts to the left and, with decreasing probabilities, the real value takes on extremely large values.

Equation (18) is not integrable. We then run a numerical simulation to represent the distribution as in Figure 2. As we observe, the function vaguely resembles a normal distribution and exhibits skewness toward the right. We then calculate the bankruptcy probability as in (20):

$$P_{br} = 0.13158 \tag{21}$$

In other words, given the structure of the balance sheet and market conditions, the firm has 13% probability of bankruptcy.

Finally, we calculate the mean and standard deviation of K^*:

$$\mu_K = 0.96 \quad \sigma_K = 0.201 \tag{22}$$

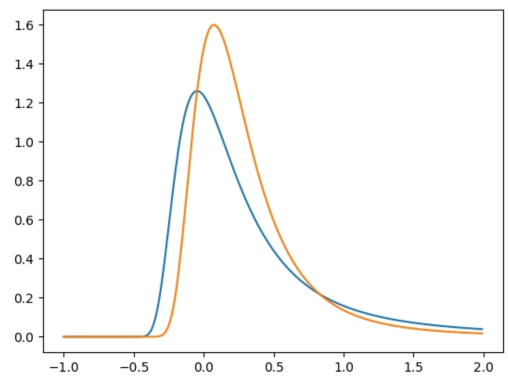

Figure 1: Probability distribution of the real value of the asset K_1^* and $K{*}_2$

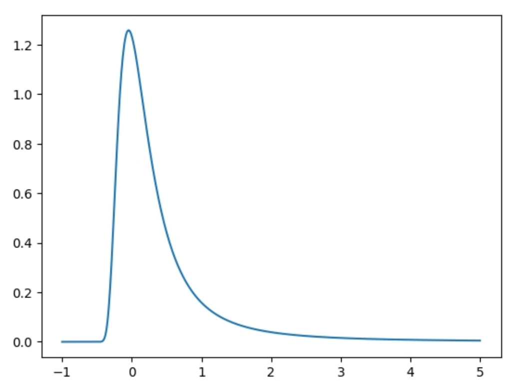

Figure 2: Probability distribution of the capital

As (22) shows, before the exogenous shock, the company's capital has a real value almost double the book value [5].

4 The Expansionary Phase of the Business Cycle

Suppose now we enter the expansionary phase of the business cycle. This in our model corresponds to an exogenous shock that increases the parameter r_m. Let us imagine the following three scenarios:

$$r_{m,1} = 0.0625$$
$$r_{m,2} = 0.075$$
$$r_{m,3} = 1.000$$

These correspond to an increase in r_m of 25, 50 and 100 percent, respectively.

We begin by graphically representing the different probability distributions, as in Figure 3. As can be seen, (18) shifts to the left with each increase in r_m. This suggests a gradual increase in the probability of default. However, the shape of the curve also changes, and to get a definitive answer we proceed by calculating the probability of bankruptcy for each of the assumed scenarios. The results are as follow:

$$P_{br}(r_m = 0.0625) = 0.1634 \tag{23}$$

$$P_{br}(r_m = 0.0625) = 0.1908 \tag{24}$$

$$P_{br}(r_m = 0.0625) = 0.2331 \tag{25}$$

Given the above results, we can draw the following conclusion:

> **T1**: *Given a company with assets valued in real terms and liabilities in book value, the expansive phase of the business cycle worsens the balance sheet structure, increasing the probability of bankruptcy.*

[5] The book value is, as shown in (11), equal to: $(1 - 0.75) \cdot (1 + 1) = 0.5$.

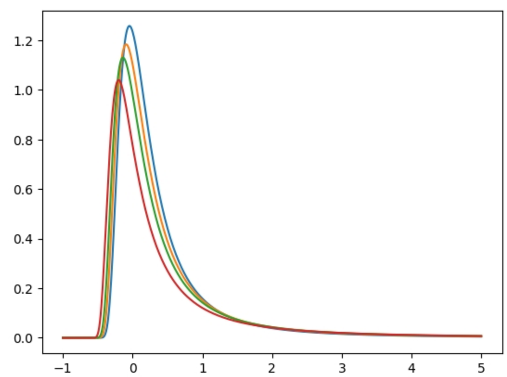

Figure 3: Probability distributions before and after the exogenous shocks

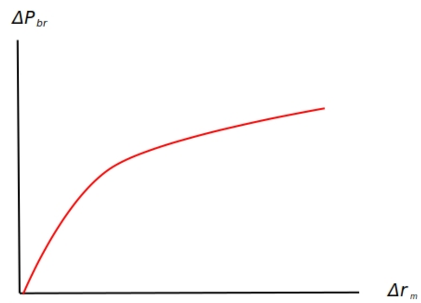

Figure 4: Relationship between probability of bankruptcy and market portfolio return

Note, however, that the increase in the probability of bankruptcy is not linear with respect to the increase in r_m. It is therefore possible to assume a relationship between ΔP_{br} and Δr_m decreasing as shown in Figure 4.

Finally, we calculate the expected value and standard deviation of the capital stock:

$$\mu_{K,1} = 0.93 \quad \sigma_{K,1} = 0.2583$$

$$\mu_{K,2} = 0.89 \quad \sigma_{K,2} = 0.3061$$

$$\mu_{K,3} = 0.82 \quad \sigma_{K,3} = 0.3823$$

As the probability of bankruptcy increases, it is not surprising that the expected value of capital decreases. However, it is noteworthy that the expected value decreases faster than the increase of r_m. We can therefore draw the following conclusion:

> **T2**: *Given the previous assumptions, an increase in the market portfolio return decreases the expected value of the firm's capital exponentially.*

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Same for riskiness, or standard deviation, which grows exponentially with r_m.

Conclusion

To the question we posed in the introduction, namely how the business cycle affects the probability of bankruptcy, we can give a partially definitive answer. On the one hand, the expansive phase of the business cycle reduces the real value of assets, increasing the probability of insolvency. On the other hand, this is true for a company where only assets are expressed in real value.

What we've seen runs counter to expectations, where economic growth is viewed favorably with regard to corporate strength. This is because the model is based on a short-term perspective, where revenues are not taken into account. This assumption is in line with economic theory, where profits are dependent on the technical conditions of production and not on monetary variables.

Reference

The model developed in this paper is based on microeconomic and corporate financial theory. Eexcellent reference texts are:

Walter Nicholson, Christopher M. Snyder, Microeconomic Theory: Basic Principles and Extensions; 12th Edition, 2016, Cengage Learning.

Jonathan Berk, Peter DeMarzo, Corporate Finance, 5th Edition, 2019, Pearson.

The source code used for the computations is in the Python language, with the extension .ipynb (Jupyter Notebook file), and available at the following link:

https://giacorradini.github.io/archive