MINIMAL SURFACES VIA EIGENVALUE OPTIMIZATION
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Abstract. These lectures will explore the relationship between minimal surfaces and eigenvalue optimization. We will mainly focus on free boundary minimal surfaces and their connection to the Steklov eigenvalue problem. We will prove important basic properties and then study some recent results about the behavior of Steklov optimizers for surfaces with fixed genus and number of boundary components converging to infinity. We will see how, in this asymptotic regime, Steklov optimization is related to Laplace optimization and closed minimal surfaces.

1. Correspondence between eigenvalue optimisation and minimal surfaces

Abstract. In this first lecture, we will introduce the Laplace and Steklov eigenvalue problems on a surface (without or with boundary, respectively). Then, we will explain how these problems are related to minimal surfaces in spheres and free boundary minimal surfaces in balls. In particular, we will focus on the relationship between Steklov eigenvalue optimisation and free boundary minimal surfaces in the unit ball, with proofs and basic properties.

Literature. Sections 1 and 2 of [FS13], in particular proof of Proposition 2.4 therein.

1.1. **Laplace eigenvalue problem on closed surfaces.** Let M^2 be a smooth compact surface and let g be a Riemmanian metric on M. Then, the Laplace operator Δ_g admits a discrete spectrum

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots \to \infty,$$

which satisfies the variational characterization

$$\lambda_k(M) = \min_{E\in\mathcal{E}_k} \max_{0\neq\tau\in E} \frac{\int_M |\nabla\tau|^2 \, dA_g}{\int_M \tau^2 \, dA_g},$$

where $\mathcal{E}_k$ is the set of all k-dimensional subspaces of $H^1(M)$ that are orthogonal to the constant functions on M.

It is a natural question to wonder what is the metric g maximizing the eigenvalue λ_k. However, observe that the eigenvalues are not scale invariant. Therefore, it is natural to consider the normalized Laplace eigenvalues

$$\overline{\lambda}_k(M, g) \coloneqq \lambda_k(M, g)\mathrm{area}(M, g).$$

Theorem 1.1 (cf. [FS13, Proposition 2.1]). *Let M be a smooth compact surface (without boundary). Assume that g_0 is a metric on M such that*

$$\overline{\lambda}_k(M, g_0) = \max_g \overline{\lambda}_k(M, g).$$

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Then there exist linearly independent eigenfunctions $u_1, \dots, u_n$ relative to the kth Laplace eigenvalue such that (after possibly rescaling the metric) the map $u = (u_1, \dots, u_n)\colon M \to \mathbb{R}^n$ gives an isometric minimal immersion into the unit sphere S^{n-1}.

1.2. **Steklov eigenvalue problem on surfaces with boundary.** Let M^2 be a smooth compact surface with boundary and let g be a Riemmanian metric on M. The Steklov eigenvalue problem on M is

$$\begin{cases} \Delta_g u = 0 & \text{on } M \\ \frac{\partial u}{\partial \eta} = \sigma u & \text{on } \partial M \end{cases}$$

where η is the outward unit normal to ∂M. The Steklov problem admits a discrete spectrum

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_k \leq \ldots \to \infty,$$

which satisfies the variational characterization

$$\sigma_k(M) = \min_{E \in \mathcal{E}_k} \max_{0 \neq \tau \in E} \frac{\int_M |\nabla \tau|^2 \, dA_g}{\int_{\partial M} \tau^2 \, ds_g},$$

where $\mathcal{E}_k$ is the set of all k-dimensional subspaces of $H^1(M)$ that are orthogonal to the constant functions on the boundary.

Again, these eigenvalues are not scaling invariants. Therefore, it is useful to define their scaled invariant version

$$\overline{\sigma}_k(M, g) \coloneqq \sigma_k(M, g)\text{length}(\partial M).$$

The quantity $\sigma_k(M, g)\sqrt{\text{area}(M)}$ is also scale invariant, but the motivation for using the length comes from the denominator in the Rayleigh quotient, and the conformal invariance of the Dirichlet energy in the numerator.

Theorem 1.2 (cf. [FS13, Proposition 2.4])**.** *Let M be a smooth compact surface with boundary. Assume that g_0 is a metric on M such that*

$$\overline{\sigma}_k(M, g_0) = \max_g \overline{\sigma}_k(M, g).$$

Then there exist linearly independent eigenfunctions $u_1, \dots, u_n$ relative to the kth Steklov eigenvalue such that (after possibly rescaling the metric) the map $u = (u_1, \dots, u_n)\colon M \to \mathbb{R}^n$ gives a conformal free boundary minimal immersion into the unit ball B^n, which is an isometry when restricted to ∂M.

Remark: We emphasise the hypothesis – *if there exists a maximising metric*. It is known from the works of [KKMS] that a maximiser exists for the first normalised Steklov eigenvalue for surfaces with a given genus equal to 0 or 1 and any number of boundary components. Some explicit known examples include

- The Euclidean unit disk $\mathbb{B}^2$ among the topological balls. FBMS in $\mathbb{B}^3$.
- The critical catenoid among annuli. FBMS in $\mathbb{B}^3$.
- The critical Möbius band among Möbius bands. FBMS in $\mathbb{B}^4$.

(see the website of Mario Schulz for more interesting visual examples, some of which are established rigorously). Likewise, the existence is known for any given closed surface, orientable or non-orientable, from the works of [KKMS, Pet, KPS], for the first normalised Laplace eigenvalue. However, the metric is only smooth up to finitely many conical singularities. Explicit examples include

- Genus 0: The round unit sphere among spheres. Minimal in $\mathbb{S}^2$ itself.
- Genus 1: The equilateral flat torus among torii. Minimal in $\mathbb{S}^5$.
- Genus 2: The Bolza surface, as a degree 2 branched cover of $\mathbb{S}^2$.

- $\mathbb{RP}^2$: The constant curvature metric. Minimal in $\mathbb{S}^4$.

More on the existence results later in Giada's talk.

Why only for surfaces? In the proof of the theorem, the stress-energy tensor being zero implies that the constructed map is conformal, which is only true in dimension 2. Further, we use a 2D fact that a harmonic+conformal map is minimal.

2. Steklov eigenvalues and the topology of a surface

Abstract. In this talk, we will prove upper bounds for the first (renormalized) Steklov eigenvalue, and its multiplicity, in terms of the topology of the surface. This shows an interesting connection between analytic properties of a surface (i.e. the Steklov eigenvalues) and topological ones.

Literature. Section 2 in [FS16], in particular Theorems 2.2 and 2.3.

2.1. Topological upper bound on the first Steklov eigenvalue.

Theorem 2.1 (cf. [FS16, Theorem 2.2]). *Let M be a smooth compact oriented surface with genus γ and β boundary components. Then*

$$\overline{\sigma}_1(M,g) \leq \min\left\{2\pi(\gamma+\beta), 8\pi\left\lfloor\frac{\gamma+3}{2}\right\rfloor\right\}.$$

The main idea in this proof is to find good functions to test against the variational characterization of the first eigenvalue.

To find the first bound, we use the existence of a proper conformal branched cover $\varphi\colon M \to D$ of degree at most $\gamma+\beta$, where D is the disk. This is a result due to Ahlfors and Gabard. For the second bound, we use a version of the Riemann-Roch theorem, which gives the existence of a holomorphic map $\varphi\colon M \to S^2$ of degree at most $\lfloor\frac{\gamma+3}{2}\rfloor$. In both of these, we will use the Hersch lemma, which is phrased as follows in [Kok14]:

Lemma 2.2 (Hersch lemma [Her70]). *Let x^i, $i=1,2,3$, be coordinate functions in $\mathbb{R}^3$ and $\varphi\colon M \to S^2 \subset \mathbb{R}^3$ be a conformal map to the unit sphere centred at the origin. Then for any continuous Radon measure μ on M there exists a conformal diffeomorphism $s\colon S^2 \to S^2$ such that*

$$\int_M (x^i \circ s \circ \varphi)d\mu = 0 \quad \text{for any } i=1,2,3.$$

Example 2.3. On the disk D, the previous bound is $\overline{\sigma}_1(D,g) \leq 2\pi$. This bound is sharp, as it is attained by the standard metric. For genus 0 surfaces, we get the bound $\overline{\sigma}_1(M,g) \leq 8\pi$, which turns out to be sharp in the asymptotic regime where the number of boundary components β goes to infinity, as we will see in Mateus' and Luca's talks.

2.2. Topological upper bound on the multiplicity of the eigenvalue.

Theorem 2.4 (cf. [FS16, Theorem 2.3]). *Let M be a smooth compact oriented surface with genus γ and β boundary components. Then the multiplicity of the eigenvalue σ_k is at most* $4\gamma+2k+1$.

The main ingredient in this proof is the Courant nodal domain theorem. We use that the nodal set of aany Steklov eigenfunction (namely the set where the eigenfunction vanishes) consists of C^2-immersed one dimensional submanifolds which meet only at a finite number of points. These nodal lines of an i-th eigenfunction separate M into at most $i+1$ connected components.

We will use this Courant nodal domain theorem to argue that the i-th eigenfunction cannot vanish to an order higher than $2\gamma+i$. The proof then concludes by contradiction: if the i-th eigenvalue has high multiplicity, we can construct an i-th eigenfunction with a high order of vanishing at some point of M.

3. Free boundary minimal surfaces in $\mathbb{B}^3$: area bounds, compactness, uniqueness

Abstract. In this third lecture, we will give an overview on properties of free boundary minimal surfaces in $\mathbb{B}^3$ whose proof relies on the study of the Steklov eigenvalue problem on the surfaces. In particular, we will discuss area bounds of a surface in terms of its topology, compactness of the space of free boundary minimal surfaces with a fixed topology, and uniqueness results for free boundary minimal annuli with reflection symmetries.

Literature. Proposition 3.4, Corollary 3.6, and Theorem 1.2 in [FL14]. Time permitting, proof of Theorem 3.1 (specialized to surfaces in $\mathbb{B}^3$). Theorem 6.6 in [FS16] and Corollary 1 in [McG18].

3.1. **Introduction.** In the first talk we established the connection between Steklov eigenvalue problems and free boundary minimal surfaces in $\mathbb{B}^n$. ([FS13], Proposition 2.4) It is reasonable to believe then, that properties of free boundary minimal surfaces in the Euclidean ball can be proved via the study of the Steklov eigenvalue problem. In this talk we are going to focus on this kind of properties of free boundary minimal surfaces in $\mathbb{B}^n$ and in particular the case of $\mathbb{B}^3$, that is particularly interesting since free boundary minimal surfaces in the 3-ball are 2-dimensional and we have explicit examples. The easiest ones are the equatorial disk and the critical catenoid, that is the unique portion of a suitably scaled catenoid that can be embedded in $\mathbb{B}^3$ as a free boundary minimal surface. There are a lot of other examples (see some nice pictures on Mario Schulz's webpage [Sch24]).

In this third lecture we are going to see two results:

- A compactness theorem: we are going to "prove" that free boundary minimal surfaces with fixed topology are a compact family in the C^∞ topology. This result can be seen as a first step toward a more ambitious conjecture (still open!): free boundary minimal surfaces of the same topological type are, up to congruences, a finite family. Intuitively, the result of Fraser and Li means that a sequence of such surfaces cannot "blow up", escape to infinity, or develop "wild" singularities. After possibly passing to a subsequence, it must always converge smoothly to another free boundary minimal surface of the same topological type.
- A uniqueness theorem: we are going to prove that the critical catenoid (mentioned before) is the only embedded minimal annulus ($\gamma = 0, \beta = 2$) in $\mathbb{B}^3$ under some additional hypothesis of symmetry.

3.2. **Uniquenes results for free boundary minimal annuli in the Euclidean unit ball.** A classical theorem of Nitsche [Nit85] states that the only free boundary minimal disk ($\gamma = 0, \beta = 1$) in the unit ball are flat equators.

It is interesting to see whether similar rigidity results hold for larger genus or larger number of boundary components. In general, as mentioned before, we expect to have a finite number of free boundary minimal surfaces with fixed topology, but we do not expect that we have uniqueness for larger genus or number of boundary components. We already know that the answer is negative for some topologies. There are explicit examples of families of minimal surfaces with the same topology and same symmetries that are not isometric. These examples have 3 boundary components ($\beta = 3$), different increasing genus, one converges to the union of the catenoid and the equatorial disk (cf. [KL21]), the other one converges to a disk and pieces of some other surface (cf. [CSW]).(again, you can find some nice pictures on Mario Schulze's webpage). For "low" topology we expect to have uniqueness (for example for annuli or for genus one surfaces with one boundary component).

The conjecture we are going to analyze is that up to congruences the critical catenoid is the unique properly embedded free boundary minimal annulus in $\mathbb{B}^3$. We remark that, if we relax the hypothesis "embedded" and ask for the free boundary minimal annulus to be only immersed, this conjecture is not true. There are examples, constructed by Fernandez, Hauswirth and Mira [FHM] and Kapouleas and McGrath [KM], of immersed (not embedded!) free boundary minimal annuli in the ball that are not critical catenoids.

The critical catenoid conjecture in the form mentioned before is still open, but there are partial results in this direction.

Theorem 3.1 (Fraser and Schoen [FS16, Theorem 1.2])**.** *Let $M \subset \mathbb{B}^n$ a free boundary minimal annulus such that the coordinate functions are first eigenfunctions. Then $n = 3$ and M is congruent to the critical catenoid.*

Theorem 3.2 (McGrath [McG18, Theorem 1])**.** *Let $M \subset \mathbb{B}^n$, $n \geq 3$ be an embedded free boundary minimal annulus. Suppose that M is invariant under reflection through 3 orthogonal hyperplanes $\Pi_i, i = 1, 2, 3$ and*

$$\partial M \cap \left(\mathbb{B}^n \setminus \cup_{i=1}^3 \Pi_i\right) \neq \emptyset$$

Then M is congruent to the critical catenoid.

Remark 3.3. There is a version of the McGrath's theorem that uses less planes of symmetries [seo, Theorem 1.2].

Note that McGrath's theorem follows from Fraser and Schoen's one if we manage to prove that the first Steklov eigenvalue is equal to one under the reflection symmetries hypothesis.

We need some preliminaries:

Definition 3.4. Suppose u is a Steklov eigenfunction, we define the *nodal set* of u as

$$\mathcal{N} = \{p \in M : u(p) = 0\}$$

and we call a *nodal domain* a connected component of $M \setminus \mathcal{N}$. Moreover we say *nodal line* a C^1 arc in $\mathcal{N}$.

It is a standard fact that when $n = 2$, then $\mathcal{N}$ is the union of finitely many C^1 arcs, that intersect at a finite set of points.

Theorem 3.5 (Courant)**.** *If u is a first eigenfunction, then u has exactly two nodal domains.*

The idea of the theorem is to prove that $\sigma_1(M) = 1$. We assume by contradiction that $\sigma_1(M) < 1$ and we let u be its first eigenfunction. The symmetry of M imply that a u has "too many" nodal domains.

Let $\Pi \subset \mathbb{R}^n$ be a hyperplane and let $R_\Pi : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal reflection through Π. We say a surface M is invariant under reflection through Π, or R_Π-invariant, if $R_\Pi(M) = M$. Suppose M is R_Π-invariant. We define operators $A_\Pi, S_\Pi : C^0(M) \to C^0(M)$ by

$$A_\Pi u = \frac{1}{2}\left(u - u \circ R_\Pi\right), \quad S_\Pi u = \frac{1}{2}\left(u + u \circ R_\Pi\right).$$

Clearly $A_\Pi u$ and $S_\Pi u$ are the antisymmetric and symmetric parts of u about Π, i.e.,

$$u = A_\Pi u + S_\Pi u$$

and

$$A_\Pi \circ R_\Pi u = -A_\Pi u, \quad S_\Pi \circ R_\Pi u = S_\Pi u.$$

If u is a first eigenfunction, $A_\Pi u$ and $S_\Pi u$ are eigenfunctions, in fact first eigenfunctions unless $A_\Pi u \equiv 0$ or $S_\Pi u \equiv 0$, respectively.

Lemma 3.6. *If $M^2 \subset \mathbb{R}^n$ is R_Π-invariant, $\sigma_1 < 1$, and u is a first eigenfunction, then $u = S_\Pi u$.*

Proof. Suppose for a contradiction that $A_\Pi u \not\equiv 0$. Then $A_\Pi u$ is a first eigenfunction. By Courant's theorem, $A_\Pi u$ has exactly two nodal domains. Let Ω be the nodal domain where $u > 0$. Since $A_\Pi u$ is antisymmetric, its nodal set N contains $M \cap \Pi$ and the second nodal domain is $R_\Pi(\Omega)$. Since Π disconnects M, Ω lies on one side of Π.
Define $\varphi := \langle X, w\rangle$, where $X \in \mathbb{R}^n$ is the position vector field and $w \in \mathbb{R}^n$ is a normal vector to Π such that $\Omega \subset \{\varphi > 0\}$. It follows that

$$\int_\Omega \varphi A_\Pi u > 0.$$

Since $A_\Pi \varphi = \varphi$, we then have

$$\int_{\partial M} \varphi A_\Pi u = \int_{\partial M \cap \Omega} \varphi A_\Pi u + \int_{\partial M \cap R_\Pi(\Omega)} A_\Pi \varphi A_\Pi u = 2 \int_{\partial M \cap \Omega} \varphi A_\Pi u > 0.$$

On the other hand, φ is an eigenfunction with eigenvalue 1. Since L is self-adjoint and $\sigma_1 < 1$, we have

$$\int_{\partial M} \varphi A_\Pi u = 0,$$

which gives a contradiction. □

Proof of Theorem 1. Let $M^2 \subset \mathbb{B}^n$ be an embedded, G-invariant free boundary annulus, where $G = \{R_{\Pi_1}, R_{\Pi_2}, R_{\Pi_3}\}$. Without loss of generality, we may assume Π_i, $i = 1, 2, 3$ are the first three coordinate planes. By contradiction, suppose $\sigma_1(M) < 1$ and let u be a first eigenfunction.

Since $\partial M \cap \left(\mathbb{B}^n \setminus \bigcup_{i=1}^3 \Pi_i\right) \neq \emptyset$, it follows from the symmetries that $\partial M \cap \mathrm{int}(O) \neq \emptyset$, where

$$O = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1, x_2, x_3 \geq 0\}.$$

Let $\Gamma \subset \partial \mathbb{B}^n$ be a boundary component of ∂M. Since M is embedded, any intersection of Γ with one of the planes Π_i is orthogonal. Since M is an annulus, the symmetries imply there is at least one such intersection.

Now consider the curve $\gamma := \Gamma \cap O$. Since Γ is an embedded G-invariant circle, γ is connected, embedded, and intersects exactly two (at first possibly indistinct) coordinate planes Π_i, $i = 1, 2, 3$. We parametrize γ on $[0, 1]$ and may suppose without loss of generality that $\gamma(0) \in \Pi_1$. Since M is an annulus, the symmetries imply that $\gamma(1) \notin \Pi_1$. Without loss of generality, we can assume $\gamma(1) \in \Pi_2$.

By the symmetries, the other boundary component of ∂M is $R_{\Pi_3}(\Gamma)$. Since M is an annulus, this implies $M \cap \Pi_3$ is a circle, and moreover that

$$D := M \cap O$$

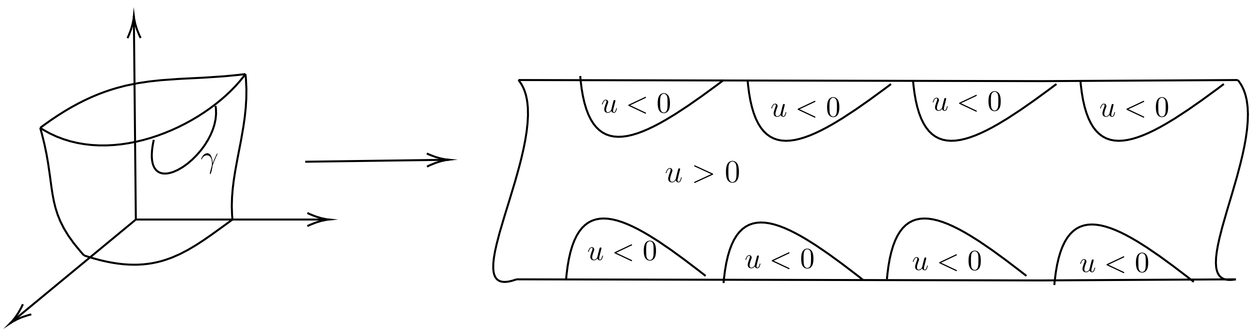
is bounded by four arcs γ, e_1, e_2, e_3, with $e_i \subset M \cap \Pi_i$, $i = 1, 2, 3$. In particular, D is a fundamental domain for M.

We first claim that $\mathcal{N}$ contains a nodal line ℓ which intersects γ at an interior point. If not, then u would not change sign on γ, and since u is G-invariant and D is a fundamental domain for M, we would have

$$\int_{\partial M} u = 0,$$

which is impossible. There are four cases, depending on whether the second intersection of ℓ with ∂D is on γ, e_1, e_2, or e_3.

By the lemma, $\mathcal{N}$ is G-invariant. If ℓ ends on γ, G-invariance of $\mathcal{N}$ implies u has at least 9 nodal domains. (see picture below)



If ℓ ends on e_1, G-invariance of N implies u has at least 5 nodal domains. In the same fashion, u has at least 5 nodal domains if ℓ ends on e_2. Finally, if ℓ ends on e_3, G-invariance of N implies u has at least 4 nodal domains. Each of these cases contradicts Courant's theorem, so it must be that $\sigma_1 = 1$. □

3.3. **Compactness of the space of compact properly embedded free boundary minimal surfaces of fixed topological type.** As we mentioned before, the topic of classification of free boundary minimal surfaces in $\mathbb{B}^3$ with fixed topology is a wide and complicated topic. We expect to have finite number of these surfaces once the genus and boundary component are fixed, but this conjecture is not yet proven, even for "small" topologies. A "weaker" version of this conjecture was proved by Fraser and Li, as stated in the following compactness theorem.

Theorem 3.7 (Fraser and Li [FL14, Theorem 1.2])**.** *The space of compact properly embedded minimal surfaces of fixed topological type in $\mathbb{B}^3$ with free boundary on $\partial\mathbb{B}^3$ is compact in the C^k topology for any $k \geq 2$.*

We first remark that this theorem holds in the more general setting of M free boundary minimal surface in N^3 with nonnegative Ricci curvature and ∂N strictly convex.
We consider a sequence M_i of free boundary minimal surfaces in the Euclidean ball $\mathbb{B}^3$. Our goal is to show that there exists a limiting surface M, which is itself a free boundary minimal surface in $\mathbb{B}^3$. To this end, we aim to control the second fundamental form. If a uniform bound were available, we could locally represent the surfaces as graphs and obtain smooth convergence directly. However, since in our setting such a uniform bound cannot be achieved a priori, we proceed differently.

We establish instead an L^2 bound on the norm of the second fundamental form, expressed in terms of the topology of the surfaces. This bound enables us to extract a subsequence M_{i_k} that converges to a free boundary minimal surface in $\mathbb{B}^3$ away from a finite set of points. Finally, by carefully analyzing the behavior near these points and performing the necessary adjustments, we can extend the limit across them, thereby completing the proof.

Proof. (just a sketch) *First step: L^2 bound of the norm of the second fundamental form.*

From the Gauss equation and the minimality of M, for any $x \in M$, we have

$$\frac{1}{2}\|h_M\|^2(x) = 1 - K_M(x),$$

where $K_M(x)$ is the sectional curvatures of the plane T_xM with respect to M. We can integrate the equality above over M and apply the Gauss-Bonnet theorem to obtain

$$\frac{1}{2}\int_M \|h_M\|^2 = A(M) - \int_{\partial M} k_g - 2\pi\chi(M),$$

where k_g is the geodesic curvature of ∂M with respect to M and $\chi(M)$ is the Euler characteristic of M. Since M meets $\partial\mathbb{B}^3$ orthogonally along ∂M, k_g is equal to $h^{\partial\mathbb{B}^3}(u,u)$, where u is the unit tangent vector for ∂M. Therefore, there exists a constant $C > 0$ depending only on the upper bound of the sectional curvature of M and the principal curvatures of ∂M so that

$$\frac{1}{2}\int_M \|h_M\|^2 \leq CA(M) + CL(\partial M) - 2\pi(2 - 2\gamma - \beta).$$

where C is a constant that depends on the principal curvatures of ∂M.

To conclude this first step we need a bound for the length of the boundary of M and an area bound. First, using the divergence theorem and the fact that M is a free boundary minimal surface in $\mathbb{B}^3$, we get

$$2A(M) = \int_M div_M x = \int_{\partial M} x \cdot \nu_{\partial M} = \int_M x^2 = L(\partial M)$$

Then, it is enough to bound the length of ∂M. We recall from the previous talk that

$$\sigma_1(M,g)L(\partial M, g) \leq \min\left\{2(\gamma+\beta)\pi, 8\pi\left\lfloor\frac{\gamma+3}{2}\right\rfloor\right\}.$$

and we have a lower bound for the first eigenvalue in the ball [FL14, Theorem 3.1], that is

$$\sigma_1(M) \geq \frac{1}{2}$$

(There is a conjecture that $\sigma_1(M) = 1$!)

This implies the length bound

$$L(\partial M) \leq \min\left\{4\pi(\beta+\gamma), 16\pi\left[\frac{\gamma+3}{2}\right]\right\}$$

Finally, we obtain this second fundamental form bound in terms of the topology of the surface.

$$\int_M \|h_M\|^2 \leq C(\beta+\gamma),$$

where C is a constant depending only on the geometry of $\mathbb{B}^3$.

Second step: construction of a converging subsequence. Let $\{M_i\}$ be a sequence of compact properly embedded minimal surfaces of fixed topological type. Using a covering argument and the L^2 bound, we can extract a subsequence of $\{M_i\}$ such that M_i converges in the C^∞ topology to some M_0, a properly embedded minimal surface in $\mathbb{B}^3 \setminus \{x_1, \ldots, x_\ell\}$ with free boundary on $\partial\mathbb{B}^3 \setminus \{x_1, \ldots, x_\ell\}$.

Third step: "some adjustments", namely removing the finitely many singularities.

We define $M = M_0 \cup \{x_1, ..., x_l\}$. By a removable singularity theorem, $M = M_0 \cup \{x_1, \ldots, x_\ell\}$ is a compact properly embedded minimal surface with free boundary. The only thing left to prove is that M has multiplicity 1 as the limit of M_i. Proceeding by contradiction, we assume that M has $\Gamma_1, ..., \Gamma_k$ sheets. Using the fact that M has more than one sheet, we can construct a function φ such that $\int_{\partial M_i} \varphi = 0$, $\int_{M_i} \|\nabla_{M_i}\varphi\|^2 \to 0 \quad$ as $\varepsilon \to 0$, $\int_{\partial M_i} \varphi^2 \to C \neq 0$ as $\varepsilon \to 0$. By the variational characterization of the first Steklov eigenvalue mentioned in the first lecture

$$\sigma_1(M_i) = \inf_{f \not\equiv 0, \int_{M_i} f \neq 0} \frac{\int_{M_i} \|\nabla_{M_i} f\|^2}{\int_{\partial M_i} f^2},$$

we see that $\sigma_1(M_i) \to 0$ as $\varepsilon \to 0$. This contradicts the estimate $\sigma_1 \geq \frac{1}{2}$. Finally, using the fact that M is of multiplicity one and a version of Allard's regularity theorem for minimal surfaces with free boundary, we see that M_i converges to M in the C^∞ topology even across the points $x_1, \ldots, x_\ell$. □

4. Relation between Steklov and Laplacian eigenvalue optimization

Abstract. In this talk, we will present a relation between Laplacian eigenvalue optimization on a closed surface and Steklov eigenvalue optimization on a subset with boundary of the initial surface. If time allows, we will see how this can be proven as a consequence of a min-max characterization of conformal eigenvalues by Karpukhin–Stern.

Literature. Theorem 1.6 in [KS24b], Corollary 1.6 in [GL21]. Time permitting, proof of Theorem 1.6 in [KS24b] in Section 5.3 of the same paper.

4.1. Setup.

Let (M, g) be a closed surfaces, let $\Omega \subset M$ be a compact domain with boundary $\partial\Omega$, sufficiently regular, assume $\partial\Omega$ is C^∞. Then (Ω, g) is a compact surface with boundary, then make sense to compute the Steklov eigenvalue of Ω.

Question: Is there a relation between maximal metrics for $\sigma_1(\Omega, g)$ and $\lambda_1(M, g)$?

4.2. Eigenvalues of a Radon measure.

This section is inspired to [Kok14]. Let (M, g) be a closed surface, $[g]$ a conformal classe and μ a Radon measure. We define the first eigenvalue of a Radon measure,

$$\lambda_1(M, [g], \mu) = \inf_{u \neq 0} \left\{ \frac{\int_M |\nabla^g u|^2 dv_g}{\int_M u^2 d\mu} \mid \int_M u d\mu = 0 \right\}.$$

- $\mu = dv_g$ we recover the firts Laplace eigenvalue.
- $\mu = \mathcal{H}^1|_{\partial\Omega}$ we recover the first Steklov eigenvalue.

We call μ an **admissible** if the trace operator

$$T : W^{1,2}(M, g) \to L^2(M, \mu)$$

is compact. Let us denote by

$$\bar{\lambda}_1(M, [g], \mu) = \lambda_1(M, [g], \mu)|M|_\mu,$$

the normalized eigenvalue of a Radon measure.

Lemma. *Let $\Omega \subset M$ be compact domain with boundary, suppose that $\partial\Omega$ is C^∞, then $\mu = \mathcal{H}^1|_{\partial\Omega}$ is regular and*

$$\sigma_1(\Omega, g) \leq \lambda_1(M, [g], \mu)$$

Proof. $\partial\Omega$ is smooth then the trace operator is compact. Let $u \in W^{1,2}(M,g)$ be such that $\int_{\partial\Omega} T(u) d\mathcal{H}^1_{\partial\Omega} = 0$, is the first eigenfution of a Radon measure, then we have

$$\begin{aligned}\lambda_1(M,[g],\mu) &= \frac{\int_M |\nabla^g u|^2 dv_g}{\int_{\partial\Omega} u^2 d\mathcal{H}^1_{\partial\Omega}} \geq \frac{\int_\Omega |\nabla^g u|^2 dv_g}{\int_{\partial\Omega} u^2 d\mathcal{H}^1_{\partial\Omega}} \\ &\geq \inf_{u\neq 0} \left\{ \frac{\int_\Omega |\nabla^g u|^2 dv_g}{\int_{\partial\Omega} u^2 d\mathcal{H}^1_{\partial\Omega}} \mid \int_{\partial\Omega} u d\mathcal{H}_{\partial\Omega} = 0 \right\} = \sigma_1(\Omega, g).\end{aligned}$$

□

Let

$$\Lambda_1(M,[g]) = \sup_{g\in[g]} \lambda_1(M,g)|M|_g$$

$$\Lambda_1(M) = \sup_{g\in Riem(M)} \lambda_1(M,g)|M|_g$$

Theorem 4.1 (Regularity of maximal metric, Karpukhin–Stern [KS24b]). *Let μ a admissible Radon measure, then*

$$\overline{\lambda}_1(M,[g],\mu) \leq \Lambda_1(M,[g]) \tag{4.1}$$

Moreover if we have equality on (4.1), then there exist a harmonic map $\Phi : (M,g) \to \mathbb{S}^n$, and

$$d\mu = \frac{1}{\lambda_1(M,[g],\mu)} |d\Phi|_g dv_g = \frac{1}{\lambda_1(M,[g],\mu)} dv_{\Phi^* can}, \tag{4.2}$$

where the components of Φ are first eigenfunction of Δ_g.

Before present a proof of the previous theorem, we present some consequence. Let

$$\Sigma_1(M) = \sup_{\Omega\subset M} \sup_{g\in Riem(\Omega)} \overline{\sigma}_1(\Omega, g).$$

We have

Theorem 4.2 (Karpukhin–Stern [KS24b]). *Let $\Omega \subset M$ a compact domain. Then one has*

$$\overline{\sigma}_1(\Omega, g) < \Lambda_1(M,[g]) \leq \Lambda_1(M).$$

Therefore

$$\Sigma_1(M) \leq \Lambda_1(M).$$

Proof. The first part is direct of the above results. Observe that the inequality is strict because $\mathcal{H}^1_{\partial\Omega}$ does not have full support. The second one is direct from the definition. □

On the other hand, the work of Girouard–Lagacé estabilish the other direction. They prove that

Theorem (Girouard–Lagacé [GL21]). *Given a surface (M,g), there exites a sequece of smooth domain $\Omega_n \subset M$ such that one has*

$$\lim_{n\to\infty} \overline{\sigma}_1(\Omega_n, g) \to \overline{\lambda}_1(M,g).$$

Therefore we have the following

Theorem (Girouard-Lagacé and Karpukhin-Stern).

$$\Lambda_1(M) = \Sigma_1(M).$$

4.3. **Proof of the regularity theorem.** To prove the theorem, we need introduce the *min-max energy* via a "Ginzburg–Landau" type relaxation.

Let

$$\Gamma_n(M) = \{F \in C^0(\overline{\mathbb{B}}^{n+1}, W^{1,2}(M, \mathbb{R}^{n+1})) \mid F_a \equiv a, \ \forall a \in \mathbb{S}^n\},$$

Given $\varepsilon > 0$ consider the modified harmonic energy

$$\mathcal{E}_{n,\varepsilon}(M, g) = \inf_{F \in \Gamma_n(M)} \sup_{a \in \mathbb{B}^{n+1}} E_\varepsilon(F_a),$$

where for $u \in W^{1,2}(M, \mathbb{R}^{n+1})$

$$E_\varepsilon(u) = \int_M \frac{1}{2}|\nabla^g u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2 \, dv_g.$$

Observe that E_ε decrease in ε, we define the min-max energy as

$$\mathcal{E}_n(M, g) = \sup_{\varepsilon > 0} \mathcal{E}_{n,\varepsilon}(M, g) = \lim_{\varepsilon \to 0} \mathcal{E}_{n,\varepsilon}(M, g).$$

Karpukhin-Stern proved that

- $\mathcal{E}_n(M, g)$ is conformal invariant, then we write $\mathcal{E}_n(M, g) = \mathcal{E}_n(M, [g])$
- for $n \geq 2$, $\mathcal{E}_n(M, g) \leq V_c(n, M, [g]) < \infty$ is finite
- $2\mathcal{E}_n(M, [g]) = \Lambda_1(M, [g])$.

Proof. (sketch)

Assume WLOG that $\mu(M) = 1$.

Take a sequence $\varepsilon_j \to 0$ a sequence of $F^j \in \Gamma_n(M)$ such that

$$\lim_{j \to \infty} \max_{y \in \mathbb{B}^{n+1}} E_{\varepsilon_j}(F^j_y) = \mathcal{E}_n(M, [g \ ])$$

. Use the Hersch trick for balance the maps. Use this balanced maps as test functions for $\lambda_1(M, [g], \mu)$. □

5. Behaviour of Steklov optimisers for large number of boundary components

Abstract. In this talk, we will show that free boundary minimal surfaces in the ball, obtained from maximisation of the first Steklov eigenvalue, converge to minimal surfaces in the sphere, as the number of boundary components converge to infinity and the genus is fixed. This gives an interesting relationship between free boundary minimal surfaces in balls and minimal surfaces in spheres. The most explicit example of this behavior are a sequence of free boundary minimal surfaces with genus zero in $\mathbb{B}^3$, which converge to the sphere $\mathbb{S}^2$.

Literature. Theorem 1.1 in [KS24a] (potentially with idea of the proof, Section 3 therein, or related results). Lemma 2.1 and Theorem 1.5 in [MZ24].

5.1. **Introduction.** We have just seen that $\sup_{\Omega \subset M} \overline{\sigma_1}(\Omega) \leq \Lambda_1(M)$; in other words, there is a sequence of domains $\Omega_n \subset M$ such that

$$\overline{\sigma_1}(\Omega_n) \to \overline{\lambda_1}(M, g_0),$$

where g_0 is the metric realising the maximal value of $\overline{\lambda_1}$.

Recall that, associated to each $\overline{\sigma_1}(\Omega_n)$ on the one hand, and associated to $\overline{\lambda_1}(M, g_0)$ on the other, there are maps $u_n : \Omega_n \to B^{N+1}$ and $u : M \to S^N$, associated respectively, such that the u_n are free boundary minimal surfaces in B^{N+1}, and u is a minimal surface in S^N. Naturally, then:

Question. What is the relation between u_n and u?

Our talk will be concerned with this question. Indeed, there is a very general result in this direction.

Let us quickly recall that M is a Riemann surface, possibly with a metric g (which may or may not be Laplace-maximising), and by Ω_n we denote some arbitrary subsets of M obtained by removing n disjoint discs.

(For purposes of exposition, we have slightly simplified the below theorem.)

Theorem 5.1 ([KS24a, Thm. 1.1]). *Suppose M is a closed surface and that there is a sequence of integers $k_n \to \infty$ and domains Ω_{k_n} of N such that $\overline{\sigma_1}$-maximising metrics on Ω_{k_n} exist. Then there is some dimension $N > 0$ such that the associated free boundary minimal immersions $u_n : \Omega_n \to B^{N+1}$ to these Steklov-maximising metrics varifold-converge (up to sequence) to the branched minimal immersion $u : M \to S^N$ associated to the Laplace-maximising metric.*

There is an assumption here about existence of Steklov-maximising metrics, so *a priori* it is possible that this phenomenon not occur.

Nevertheless, we have the following theorem.

Theorem 5.2. *There exists a sequence Ω_n of genus zero embedded free boundary minimal surfaces in B^3, obtained by Steklov eigenvalue optimisation, which varifold-converge to S^2, obtained by Laplace eigenvalue optimisation.*

Proof. Recall from the previous lecture[1] that there exist a sequence of domains $\Omega_n \subset S^2$ such that

$$\overline{\sigma_1}(\Omega_n, g) \to 8\pi;$$

here, we have used the fact that $\Lambda_1(S^2) = 8\pi$ (see for example [Her70]).

On the other hand, let h_n denote the Steklov-maximising metrics on Ω_n. Then certainly $\overline{\sigma_1}(\Omega_n, g) \leq \overline{\sigma_1}(\Omega_n, h_n)$, and moreover we have $\overline{\sigma_1}(\Omega_n, h_n) \leq \Lambda_1(S^2, [h_n]) \leq \Lambda_1(S^2) = 8\pi$ by [KS24a, Thm. 1.6] mentioned in the previous lecture. Finally, we have

$$\overline{\sigma_1}(\Omega_n, g) = \text{length}(\partial\Omega_n, \tilde{g}_n) = 2\text{area}(\Omega_n, \tilde{g}_n),$$

where the first equality follows by rescaling the metric to $\tilde{g}_n = \sigma_1(\Omega_n, g)^2 g$, and the second equality follows from a trick in Claudia's talk. This trick also gives $\overline{\sigma_1}(\Omega_n, h_n) = 2\text{area}(\Omega_n, h_n)$.

Putting this all together, we have

$$\text{area}(\Omega_n, \tilde{g}_n) \leq \text{area}(\Omega_n, h_n) \leq 4\pi.$$

Taking $n \to \infty$ gives $\text{area}(\Omega, h_n) \to 4\pi$, since the left hand side of the above inequality, being in fact equal to the Steklov eigenvalue of (Ω_n, g) obtained from Girouard-Lagacé, converges to 4π.

From the multiplicity bound, we have that the dimension of the ball into which Ω_k are immersed is at most three. We argue further that Ω_k is actually embedded in B^3. This lets us apply [MZ24, Thm. 1.3], so that

$$|\Omega_k| \leq |\text{Rad}\,\Omega_k| \leq 4\pi,$$

where Rad is the radial projection of Ω_k onto S^2. In other words,

$$\text{area}(\text{Rad}\,\Omega_k, g_{S^2}) - \text{area}(\Omega_k, g_{S^2}) \to 0.$$

Applying the following theorem gives the result. □

Again, we slightly simplify the theorem.

[1]The result is in [GL21], see also [KS24b, Remark 1.7]

Theorem 5.3 ([MZ24, Thm. 1.5]). *For each n, let $\Omega_n \subset B^3$ be embedded free boundary minimal surfaces of genus zero and at least two boundary components. If $|Rad(\Omega_n)| - |\Omega_n| \to 0$ then Ω_n varifold-converges to S^2.*

Sketch of proof. Associated to each Ω_n, let $\pi_n : \Omega_n \to S^2$ be the radial projection, and ν_n the outward pointing unit normal to Ω_n.

Let $\delta > 0$, and denote by $\mathcal{C}_n$ the cone over Ω_n. □

6. Recap, recent results and future directions

Abstract. In this final lecture, we will give a brief recap, and present recent results and future directions in the field. In particular, we will focus on recent papers about existence of maximising metrics for the first (renormalized) Laplace eigenvalue, and existence of many new examples of minimal surfaces in $\mathbb{S}^3$ and free boundary minimal surfaces in $\mathbb{B}^3$ by eigenvalue optimisation.

Literature. [KKMS], [Pet], [KPS].

6.1. **Existence of maximizing metrics.** Most of what we did so far, starting from Theorems 1.1 and 1.2, work under the assumption that we are able to find a metric maximizing the first (or, more in general, the kth) renormalized Laplace or Steklov eigenvalue. This turns out to be a delicate question, as pointed out in the first lecture. Thanks to some recent important results, we now know existence of a maximizing metric for the first Laplace eigenvalue in every closed surface.

Theorem 6.1 (cf. [KKMS, Pet, KPS]). *For every closed smooth surface M^2, there exists a metric g_0 (smooth away from finitely many conical singularities) maximizing the first renormalized Laplace eigenvalue $\bar{\lambda}_1$ among all smooth metrics on M, namely*

$$\bar{\lambda}_1(M, g_0) = \Lambda_1(M) = \sup_{g \in \mathrm{Met}(M)} \bar{\lambda}_1(M, g).$$

Proof (sketch). Here is a very synthetic outline of the proof.

Step 1: Proof the existence of a maximizer in every conformal class. There are different methods, one is to use the min-max characterization of the first conformal eigenvalue by Karpukhin–Stern [KS24b].

Step 2: Let g_n be a sequence of metrics such that $\Lambda_1(M, [g_n]) = \bar{\lambda}_1(M, g_n) \to \Lambda_1(M)$. The second step is to use some compactness results for the moduli space of conformal classes on M to prove that, up to subsequence and up to diffeomorphisms, (M, g_n) converges to (M', g'), where M' is a "topological degeneration" of M, namely it is obtained by M by neck surgeries. Moreover, we have that

$$\Lambda_1(M') \geq \bar{\lambda}_1(M', g') = \limsup_{n\to\infty} \bar{\lambda}_1(M, g_n) = \Lambda_1(M).$$

Step 3: Show that $\Lambda_1(M) > \Lambda_1(M')$ for every nontrivial topological degeneration M' of M. This implies that the M' obtain above has to be a trivial topological degeneration of M, namely $M' = M$. This conclude the proof by taking $g_0 \in [g']$ to be the maximizer of $\bar{\lambda}_1$ in the conformal class of $[g']$. Indeed, we have

$$\bar{\lambda}_1(M, g_0) = \Lambda_1(M, [g']) = \limsup_{n\to\infty} \bar{\lambda}_1(M, g_n) = \Lambda_1(M).$$

The first two steps were already carried out in [Pet14], while the last step turns out to be the most delicate, proved only recently by [KKMS, Pet, KPS]. □

The problem of finding a maximizer for the first renormalized Steklov eigenvalue on general surfaces with boundary is still open. However, it is expected that the methods developed recently are strong enough to be soon able to solve the Steklov setting too. In fact, [KKMS] were already able to prove partial results in this direction.

Theorem 6.2 (cf. [KKMS, Corollary 1.11])**.** *For every compact smooth surface M^2 with boundary and genus zero or one, there exists a metric g_0 (smooth away from finitely many conical singularities) maximizing $\bar{\sigma}_1$ among all smooth metrics on M, namely*

$$\bar{\sigma}_1(M, g_0) = \Sigma_1(M) = \sup_{g \in \mathrm{Met}(M)} \bar{\sigma}_1(M, g).$$

6.2. **Equivariant eigenvalue optimization.** Thanks to the previous existence theorems, together with Theorems 1.1 and 1.2, we get existence of (free boundary) minimal (branched) immersions in $\mathbb{S}^n$ (resp. $\mathbb{B}^n$). However, in general we have little control on the embeddedness of these surfaces and on the dimension of the target space (apart from some upper bounds, such as the one in Theorem 2.4).

Assume now instead that we want to study minimal surfaces in $\mathbb{S}^3$ and free boundary minimal surfaces in $\mathbb{B}^3$. Recall that it has been conjectured by Yau that the first Laplace eigenvalue of any *embedded* minimal surface in $\mathbb{S}^3$ is 2, and the first Steklov eigenvalue of any *embedded* free boundary minimal surface in $\mathbb{B}^3$ is 1. Therefore, a priori, it could be possible to find all embedded minimal surfaces in $\mathbb{S}^3$ as *critical points* of the first renormalized Laplace eigenvalue on (orientable) surfaces. Same for embedded free boundary minimal surfaces in $\mathbb{B}^3$ and the Steklov eigenvalue.

There are still many technical ingredients missing in order to study critical points that are not maximizers. One approach that has been fruitfully taken so far is to study the eigenvalue problems under some suitable symmetry assumption. Indeed, maximizers of the first renormalized eigenvalue (either Laplace or Steklov) among metrics satisfying some symmetry are also unconstrained critical points (by the principle of symmetric criticality by Palais [Pal79]). In particular, they give rise to (free boundary) minimal surfaces by Theorems 1.1 and 1.2.

Karpukhin–Kusner–McGrath–Stern [KKMS] used equivariant eigenvalue optimization with suitable symmetry groups to get many new examples of minimal surfaces in $\mathbb{S}^3$ and free boundary minimal surfaces in $\mathbb{B}^3$, as stated in the following two theorems.

Theorem 6.3 ([KKMS, Theorem 1.1])**.** *For every $\gamma \in \mathbb{N}$, there exist at least $\lfloor \frac{\gamma-1}{4} \rfloor + 1$ nonisometric, embedded, orientable, minimal surfaces in $\mathbb{S}^3$ with genus γ and area less than 8π. Moreover, these surfaces are doublings of an equatorial sphere $\mathbb{S}^2 \subset \mathbb{S}^3$.*

Theorem 6.4 ([KKMS, Theorem 1.2])**.** *For every $\gamma, \beta \in \mathbb{N}$, $\beta \geq 1$, there exists at least $\lfloor \frac{\gamma-2}{4} \rfloor + 1$ nonisometric, embedded, orientable, free boundary minimal surfaces in $\mathbb{B}^3$ with genus γ and β boundary components. Moreover, such embeddings have area less than 2π and are doublings of an equatorial disk $\mathbb{B}^2 \subset \mathbb{B}^3$.*

Remark 6.5. See [Pet23a, Pet23b] for other examples of equivariant eigenvalue optimization, in this case to construct minimal surfaces and free boundary minimal surfaces in ellipsoids.

Idea of the proof for closed surfaces. One of the main ideas in the proof is to consider a (closed, smooth, orientable) surface M symmetric with respect to a reflection τ such that $M/\langle\tau\rangle$ has genus zero. With ideas similar to Theorem 2.4, this is sufficient to prove that the first eigenvalue of the Laplace on M with respect to a τ-equivariant metric has multiplicity 3. Moreover, with ideas similar to Theorem 2.1, one can prove that the first renormalized Laplace eigenvalue is at most 16π.

Proving existence of a maximizing metric among τ-equivariant metrics is a delicate part of the paper [KKMS], and spurred the ideas to prove Theorems 6.1 and 6.2. $\square$

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