MINIMAL SURFACES VIA EIGENVALUE OPTIMIZATION
BRITISH ISLES GRADUATE WORKSHOP VI – MINIMAL SURFACES

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Abstract. These lectures will explore the relationship between minimal surfaces and eigenvalue optimization. We will mainly focus on free boundary minimal surfaces and their connection to the Steklov eigenvalue problem. We will prove important basic properties and then study some recent results about the behavior of Steklov optimizers for surfaces with fixed genus and number of boundary components converging to infinity. We will see how, in this asymptotic regime, Steklov optimization is related to Laplace optimization and closed minimal surfaces.

1. Correspondence between eigenvalue optimisation and minimal surfaces

Abstract. In this first lecture, we will introduce the Laplace and Steklov eigenvalue problems on a surface (without or with boundary, respectively). Then, we will explain how these problems are related to minimal surfaces in spheres and free boundary minimal surfaces in balls. In particular, we will focus on the relationship between Steklov eigenvalue optimisation and free boundary minimal surfaces in the unit ball, with proofs and basic properties.

Literature. Sections 1 and 2 of [FS13], in particular proof of Proposition 2.4 therein.

1.1. **Laplace eigenvalue problem on closed surfaces.** Let M^2 be a smooth compact surface and let g be a Riemmanian metric on M. Then, the Laplace operator Δ_g admits a discrete spectrum

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots \to \infty,$$

which satisfies the variational characterization

$$\lambda_k(M) = \min_{E\in\mathcal{E}_k} \max_{0\neq\tau\in E} \frac{\int_M |\nabla\tau|^2 \, dA_g}{\int_M \tau^2 \, dA_g},$$

where $\mathcal{E}_k$ is the set of all k-dimensional subspaces of $H^1(M)$ that are orthogonal to the constant functions on M.

It is a natural question to wonder what is the metric g maximizing the eigenvalue λ_k. However, observe that the eigenvalues are not scale invariant. Therefore, it is natural to consider the normalized Laplace eigenvalues

$$\overline{\lambda}_k(M, g) \coloneqq \lambda_k(M, g)\mathrm{area}(M, g).$$

Theorem 1.1 (cf. [FS13, Proposition 2.1]). *Let M be a smooth compact surface (without boundary). Assume that g_0 is a metric on M such that*

$$\overline{\lambda}_k(M, g_0) = \max_g \overline{\lambda}_k(M, g).$$

Date: 1st – 5th September 2025.

Then there exist linearly independent eigenfunctions $u_1, \dots, u_n$ relative to the kth Laplace eigenvalue such that (after possibly rescaling the metric) the map $u = (u_1, \dots, u_n)\colon M \to \mathbb{R}^n$ gives an isometric minimal immersion into the unit sphere S^{n-1}.

1.2. **Steklov eigenvalue problem on surfaces with boundary.** Let M^2 be a smooth compact surface with boundary and let g be a Riemmanian metric on M. The Steklov eigenvalue problem on M is

$$\begin{cases} \Delta_g u = 0 & \text{on } M \\ \frac{\partial u}{\partial \eta} = \sigma u & \text{on } \partial M \end{cases}$$

where η is the outward unit normal to ∂M. The Steklov problem admits a discrete spectrum

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_k \leq \ldots \to \infty,$$

which satisfies the variational characterization

$$\sigma_k(M) = \min_{E \in \mathcal{E}_k} \max_{0 \neq \tau \in E} \frac{\int_M |\nabla \tau|^2 \, dA_g}{\int_{\partial M} \tau^2 \, ds_g},$$

where $\mathcal{E}_k$ is the set of all k-dimensional subspaces of $H^1(M)$ that are orthogonal to the constant functions on the boundary.

Again, these eigenvalues are not scaling invariants. Therefore, it is useful to define their scaled invariant version

$$\overline{\sigma}_k(M, g) \coloneqq \sigma_k(M, g) \text{length}(\partial M).$$

The quantity $\sigma_k(M, g)\sqrt{\text{area}(M)}$ is also scale invariant, but the motivation for using the length comes from the denominator in the Rayleigh quotient, and the conformal invariance of the Dirichlet energy in the numerator.

Theorem 1.2 (cf. [FS13, Proposition 2.4])**.** *Let M be a smooth compact surface with boundary. Assume that g_0 is a metric on M such that*

$$\overline{\sigma}_k(M, g_0) = \max_g \overline{\sigma}_k(M, g).$$

Then there exist linearly independent eigenfunctions $u_1, \dots, u_n$ relative to the kth Steklov eigenvalue such that (after possibly rescaling the metric) the map $u = (u_1, \dots, u_n)\colon M \to \mathbb{R}^n$ gives a conformal free boundary minimal immersion into the unit ball B^n, which is an isometry when restricted to ∂M.

Remark: We emphasise the hypothesis – *if there exists a maximising metric*. It is known from the works of [KKMS] that a maximiser exists for the first normalised Steklov eigenvalue for surfaces with a given genus equal to 0 or 1 and any number of boundary components. Some explicit known examples include

- The Euclidean unit disk $\mathbb{B}^2$ among the topological balls. Minimally embedded in $\mathbb{B}^3$.
- The critical catenoid among annuli. Minimally embedded in $\mathbb{R}^3$.
- The critical Möbius band among Möbius bands. Minimally embedded in $\mathbb{B}^4$.

(see the website of Mario Schulz for more interesting visual examples, some of which are established rigorously). Likewise, the existence is known for any given closed surface, orientable or non-orientable, from the works of [KKMS, Pet, KPS], for the first normalised Laplace eigenvalue. However, the metric is only smooth up to finitely many conical singularities. Explicit examples include

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More on the existence results later in Giada's talk.

Why only for surfaces? In the proof of the theorem, the stress-energy tensor being zero implies that the constructed map is conformal, which is only true in dimension 2. Further, we use a 2D fact that a harmonic+conformal map is minimal.

2. Steklov eigenvalues and the topology of a surface

Abstract. In this talk, we will prove upper bounds for the first (renormalized) Steklov eigenvalue, and its multiplicity, in terms of the topology of the surface. This shows an interesting connection between analytic properties of a surface (i.e. the Steklov eigenvalues) and topological ones.

Literature. Section 2 in [FS16], in particular Theorems 2.2 and 2.3.

2.1. Topological upper bound on the first Steklov eigenvalue.

Theorem 2.1 (cf. [FS16, Theorem 2.2]). *Let M be a smooth compact oriented surface with genus γ and β boundary components. Then*

$$\overline{\sigma}_1(M,g) \leq \min\left\{2\pi(\gamma+\beta), 8\pi\left\lfloor\frac{\gamma+3}{2}\right\rfloor\right\}.$$

The main idea in this proof is to find good functions to test against the variational characterization of the first eigenvalue.

To find the first bound, we use the existence of a proper conformal branched cover $\varphi\colon M \to D$ of degree at most $\gamma+\beta$, where D is the disk. This is a result due to Ahlfors and Gabard. For the second bound, we use a version of the Riemann-Roch theorem, which gives the existence of a holomorphic map $\varphi\colon M \to S^2$ of degree at most $\left\lfloor\frac{\gamma+3}{2}\right\rfloor$. In both of these, we will use the Hersch lemma, which is phrased as follows in [Kok14]:

Lemma 2.2 (Hersch lemma [Her70]). *Let x^i, $i = 1,2,3$, be coordinate functions in $\mathbb{R}^3$ and $\varphi\colon M \to S^2 \subset \mathbb{R}^3$ be a conformal map to the unit sphere centred at the origin. Then for any continuous Radon measure μ on M there exists a conformal diffeomorphism $s\colon S^2 \to S^2$ such that*

$$\int_M (x^i \circ s \circ \varphi)d\mu = 0 \quad \textit{for any } i = 1,2,3.$$

Example 2.3. On the disk D, the previous bound is $\overline{\sigma}_1(D,g) \leq 2\pi$. This bound is sharp, as it is attained by the standard metric. For genus 0 surfaces, we get the bound $\overline{\sigma}_1(M,g) \leq 8\pi$, which turns out to be sharp in the asymptotic regime where the number of boundary components β goes to infinity, as we will see in Mateus' and Luca's talks.

2.2. Topological upper bound on the multiplicity of the eigenvalue.

Theorem 2.4 (cf. [FS16, Theorem 2.3]). *Let M be a smooth compact oriented surface with genus γ and β boundary components. Then the multiplicity of the eigenvalue σ_k is at most $4\gamma + 2k + 1$.*

The main ingredient in this proof is the Courant nodal domain theorem. We use that the nodal set of aany Steklov eigenfunction (namely the set where the eigenfunction vanishes) consists of C^2-immersed one dimensional submanifolds which meet only at a finite number of points. These nodal lines of an i-th eigenfunction separate M into at most $i+1$ connected components.

We will use this Courant nodal domain theorem to argue that the i-th eigenfunction cannot vanish to an order higher than $2\gamma+i$. The proof then concludes by contradiction: if the i-th eigenvalue has high multiplicity, we can construct an i-th eigenfunction with a high order of vanishing at some point of M.

3. Free boundary minimal surfaces in $\mathbb{B}^3$: area bounds, compactness, uniqueness

Abstract. In this third lecture, we will give an overview on properties of free boundary minimal surfaces in $\mathbb{B}^3$ whose proof relies on the study of the Steklov eigenvalue problem on the surfaces. In particular, we will discuss area bounds of a surface in terms of its topology, compactness of the space of free boundary minimal surfaces with a fixed topology, and uniqueness results for free boundary minimal annuli with reflection symmetries.

Literature. Proposition 3.4, Corollary 3.6, and Theorem 1.2 in [FL14]. Time permitting, proof of Theorem 3.1 (specialized to surfaces in $\mathbb{B}^3$). Theorem 6.6 in [FS16] and Corollary 1 in [McG18].

4. Relation between Steklov and Laplacian eigenvalue optimization

Abstract. In this talk, we will present a relation between Laplacian eigenvalue optimization on a closed surface and Steklov eigenvalue optimization on a subset with boundary of the initial surface. If time allows, we will see how this can be proven as a consequence of a min-max characterization of conformal eigenvalues by Karpukhin–Stern.

Literature. Theorem 1.6 in [KS24b], Corollary 1.6 in [GL21]. Time permitting, proof of Theorem 1.6 in [KS24b] in Section 5.3 of the same paper.

5. Behaviour of Steklov optimisers for large number of boundary components

Abstract. In this talk, we will show that free boundary minimal surfaces in the ball, obtained from maximisation of the first Steklov eigenvalue, converge to minimal surfaces in the sphere, as the number of boundary components converge to infinity and the genus is fixed. This gives an interesting relationship between free boundary minimal surfaces in balls and minimal surfaces in spheres. The most explicit example of this behavior are a sequence of free boundary minimal surfaces with genus zero in $\mathbb{B}^3$, which converge to the sphere $\mathbb{S}^2$.

Literature. Theorem 1.1 in [KS24a] (potentially with idea of the proof, Section 3 therein, or related results). Lemma 2.1 and Theorem 1.5 in [MZ24].

6. Recap, recent results and future directions

Abstract. In this final lecture, we will give a brief recap, and present recent results and future directions in the field. In particular, we will focus on recent papers about existence of maximising metrics for the first (renormalized) Laplace eigenvalue, and existence of many new examples of minimal surfaces in $\mathbb{S}^3$ and free boundary minimal surfaces in $\mathbb{B}^3$ by eigenvalue optimisation.

Literature. [KKMS], [Pet], [KPS].

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