Assignment 10

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(1) $\left\{\begin{bmatrix}3\\-1\\4\end{bmatrix},\begin{bmatrix}-2\\3\\1\end{bmatrix}\right\}$, $\left\{\begin{bmatrix}5\\6\\-5\end{bmatrix},\begin{bmatrix}6\\2\\-5\end{bmatrix}\right\}$, and $\left\{\begin{bmatrix}-3\\1\\7\end{bmatrix},\begin{bmatrix}7\\-4\\7\end{bmatrix}\right\}$

Set 1: $\vec{u}=\begin{bmatrix}3\\-1\\4\end{bmatrix}$, $\vec{v}=\begin{bmatrix}-2\\3\\1\end{bmatrix}$ $\to$ $\vec{u}\cdot\vec{v}=(3)(-2)+(-1)(3)+(4)(1)=-5$

$\|u\|=\sqrt{3^2+(-1)^2+4^2}=\sqrt{26}$; $\|v\|=\sqrt{(-2)^2+3^2+1^2}=\sqrt{14}$

$\to \cos(\theta)=\dfrac{u\cdot v}{\|u\|\cdot\|v\|}=\dfrac{-5}{(\sqrt{26})(\sqrt{14})}=\dfrac{-5}{2\sqrt{91}}\approx -0.26207$

Set 2: $\vec{u}=\begin{bmatrix}5\\6\\-5\end{bmatrix}$, $\vec{v}=\begin{bmatrix}6\\2\\-5\end{bmatrix}$ $\to$ $\vec{u}\cdot\vec{v}=(5)(6)+(6)(2)+(-5)(-5)=67$

$\|u\|=\sqrt{5^2+6^2+(-5)^2}=\sqrt{86}$, $\|v\|=\sqrt{6^2+2^2+(-5)^2}=\sqrt{65}$

$\cos\theta=\dfrac{\vec{u}\cdot\vec{v}}{\|u\|\cdot\|v\|}=\dfrac{67}{\sqrt{86}\cdot\sqrt{65}}\approx 0.89612$

Set 3: $\vec{u}=\begin{bmatrix}-3\\1\\7\end{bmatrix}$, $\vec{v}=\begin{bmatrix}7\\-4\\7\end{bmatrix}$ $\to$ $\vec{u}\cdot\vec{v}=(-3)(7)+(1)(-4)+(7)(7)=24$

$\|u\|=\sqrt{59}$, $\|v\|=\sqrt{114}$ $\to$ $\cos\theta=\dfrac{\vec{u}\cdot\vec{v}}{\|u\|\cdot\|v\|}=\dfrac{24}{\sqrt{59}\cdot\sqrt{114}}\approx 0.2926$

(3) a) $\text{Proj}_v u=\left(\dfrac{u\cdot v}{v\cdot v}\right)\cdot\vec{v}$

$\vec{u}\cdot\vec{v}=(5,1,4,-2)\cdot(0,5,4,5)=(5)(0)+(1)(5)+(4)(4)+(-2)(5)$
$=11$

$\vec{v}\cdot\vec{v}=(0,5,4,5)\cdot(0,5,4,5)=(0)(0)+(5)(5)+(4)(4)+(5)(5)=66$

$\to \text{Proj}_v u=\left(\dfrac{11}{66}\right)(0,5,4,6)=\left(0,\dfrac{5}{6},\dfrac{2}{3},\dfrac{5}{6}\right)$

(4) The closest point in subspace spanned by v_1, v_2 to the point x

$\hat{x}=\left(\dfrac{X\cdot V_1}{V_1\cdot V_1}\right)\cdot\vec{V_1}+\left(\dfrac{X\cdot V_2}{V_2\cdot V_2}\right)\vec{v_2}$, $\vec{V_1}=\begin{bmatrix}3\\-6\\5\end{bmatrix}$, $\vec{v_2}=\begin{bmatrix}7\\3\\4\end{bmatrix}$, $\vec{X}=\begin{bmatrix}4\\-1\\3\end{bmatrix}$

$\vec{X}\cdot\vec{v_1}=(4)(3)+(-1)(-6)+(3)(5)=33$

$\vec{V_1}\cdot\vec{V_1}=(3)(3)+(-6)(-6)+(5)(5)=70$

$\vec{x}\cdot\vec{v_2}=(4)(7)+(-1)(3)+(3)(4)=37$

$\vec{V_2}\cdot\vec{V_2}=(7)(7)+(3)(3)+(4)(4)=74$

$\to \hat{x}=\left(\dfrac{33}{70}\right)(3,-6,5)+\left(\dfrac{37}{74}\right)(7,3,4)$

$=\left(\dfrac{172}{35},\dfrac{-93}{70},\dfrac{61}{14}\right)$

(5) $V_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}$, $V_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}$, $Z = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \end{bmatrix}$, $\hat{Z} = C_1 V_1 + C_2 V_2$

$$C_1 = \frac{Z \cdot V_1}{V_1 \cdot V_1}, \quad C_2 = \frac{Z \cdot V_2}{V_2 \cdot V_2}$$

$Z \cdot V_1 = (2)(2) + (4)(0) + (0)(-1) + (1)(-3) = 1$

$V_1 \cdot V_1 = (2)(2) + (0)(0) + (-1)(-1) + (-3)(-3) = 14$ $\rightarrow C_1 = \frac{1}{14}$

$Z \cdot V_2 = (2)(5) + (4)(-2) + (0)(4) + (1)(2) = 4$

$V_2 \cdot V_2 = (5)(5) + (-2)(-2) + (4)(4) + (2)(2) = 49$ $\rightarrow C_2 = \frac{4}{49}$

$\Rightarrow \hat{Z} = \frac{1}{14}(2, 0, -1, -3) + \frac{4}{49}(5, -2, 4, 2) = \left(\frac{27}{49}, \frac{-8}{49}, \frac{25}{98}, \frac{-5}{98}\right)$

Best approximation

$\rightarrow Z - \hat{Z} = (2, 4, 0, 1) - \left(\frac{27}{49}, \frac{-8}{49}, \frac{25}{98}, \frac{-5}{98}\right) = \left(\frac{71}{49}, \frac{204}{49}, \frac{-25}{98}, \frac{103}{98}\right)$

(6) $u_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$; $\vec{u_1} \cdot \vec{u_2} = (3)(-4) + (4)(3) + (0)(0) = 0$

$\Rightarrow \{u_1, u_2\}$ is an orthogonal set ; $\vec{y} = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$

$$\hat{y} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) \cdot \vec{u_1} + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) \vec{u_2}$$

$y \cdot u_1 = (4)(3) + (3)(4) + (-2)(0) = 24$

$u_1 \cdot u_1 = (3)(3) + (4)(4) + (0)(0) = 25$ $\rightarrow \hat{y} = \frac{24}{25}(3, 4, 0) + \left(\frac{-7}{25}\right)(-4, 3, 0)$

$y \cdot u_2 = (4)(-4) + (3)(3) + (-2)(0) = -7$ $= (4, 3, 0)$

$u_2 \cdot u_2 = (-4)(-4) + (3)(3) + (0)(0) = 25$

(7): $U = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} \rightarrow U^T = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix} \rightarrow U^T U = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} = \left(\frac{\sqrt{10} + 9}{10\sqrt{10}}\right)$

$UU^T = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix} \rightarrow \begin{bmatrix} 1/10 & -3/10 \\ \frac{-3}{10} & \frac{9}{10} \end{bmatrix}$

$\text{proj}_W y = (y \cdot u_1) u_1 = \left(\begin{bmatrix} 7 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}\right) \cdot \left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\right)$

$= -2\sqrt{10}\left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\right) = (-2, 6)$

$(UU^T) y = \begin{bmatrix} 1/10 & -3/10 \\ \frac{-3}{10} & 9/10 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} = (-2, 6)$

$\rightarrow \text{proj}_W y = (UU^T) y$

$\rightarrow$ as theorem 10 stated