Preliminar notes on maximum-flow problem solved in FCPP.
Notation:
• G will denote the directed weighted graph of capacities, s and t will denote respectively source and sink of our graph.
We will assume that capacities are nonnegative and symmetrical.
• For a path $\mathfrak{p}$ we will denote with $|\mathfrak{p}|$ its lenght.

When it will be convenient, we will treat a directed weighted graph G as a function $G : V \times V \to \mathbb{R}_+$, where V is its set of vertices. With a little abuse of notation we will say that the edge from a node δ to a node δ' is in G, or $(\delta, \delta') \in G$, in place of $G(\delta, \delta') \neq 0$.
Likewise we will say that a path $\mathfrak{p}$ is contained in G, or $\mathfrak{p} \subset G$, if every edg

• For a field Φ ,$\mathcal{R}_\Phi$ will denote the directed weighted graph of residual capacities respect to Φ:

$$\mathcal{R} = G - \Phi \tag{1}$$

• We define admissible path respect to Φ a path γ in G from source to sink such that

$$\Phi|_\gamma < G|_\gamma$$

• We say that a flow Φ is maximal if there are no admissible paths in G respect to Φ.

Proposizione 0.1. *Φ is a maximum flow iff $\mathcal{R}_\Phi$ does not have paths from source to sink.*

For a field-value f in δ we define $|f| := \sum_{\delta \sim \delta'} f(\delta')$. We now formalize functions involved in the algorithm. Let Φ be a field and $\delta \sim \delta'$ be devices, we define

$\Phi^*(\delta,\delta') := \Phi(\delta',\delta)$. By co-induction:

$$\begin{aligned}
\Phi_0 &:= \Phi \\
\mathcal{R}_n &:= \mathcal{R}_{\Phi_n} \\
e_n(\delta) &:= \begin{cases} \infty & \text{if } \delta = s \\ -\infty & \text{if } \delta = t \\ |\Phi_n(\delta)| & \text{otherwise} \end{cases} \\
d_0(\delta) &:= \begin{cases} 0 & \text{if } \delta = t \\ \infty & \text{otherwise} \end{cases} \\
d_n(\delta) &:= \begin{cases} 0 & \text{if } \delta = t \\ \min\big\{ d_{n-1}(\delta') + 1 \mid \mathcal{R}_{n-1}(\delta,\delta') > 0 \big\} & \text{otherwise} \end{cases} \\
I_{n+1} &:= \operatorname{trunc}(\mathcal{R}_n \cdot (d_n^* < d_{n+1}), e(\Phi_n^*)) \\
\Phi_{n+1} &:= -\Phi_n^* + I_{n+1} + \operatorname{trunc}(\Phi_n^* \ , \ e(\Phi_n^* - I_{n+1})) \\
X_n &:= \{\delta \mid d_n(\delta) = \infty\}
\end{aligned}$$

Lemma 0.2. *Eventually $\{X_n\}_n$ stabilizes to a set of devices X.*

Proof. For every $n \in \mathbb{N}$ let's consider the set $T_n = \{\delta \mid \operatorname{dist}_G(\delta, t) \leq n\}$.
We prove by induction that for $\delta \notin T_n$ we have $d_n(\delta) = \infty$.
Base step: $T_0 = \{t\}$, and condition on d_0 holds by definition.
Inductive step: suppose that $\forall \delta' \notin T_{n-1}\ d_{n-1}(\delta') = \infty$ and consider a $\delta \notin T_n$.
We have $d_n(\delta) = \min\big\{d_{n-1}(\delta') + 1 \mid \mathcal{R}_{n-1}(\delta,\delta') > 0\big\}$ and since no $\delta' \sim \delta$ is in T_{n-1}, condition on d_n holds by inductive hypothesis.

As a consequence, for a $\delta \in T_n$

$$d_n(\delta) = \min_{\delta' \in T_n} \big\{d_{n-1}(\delta') + 1 \mid \mathcal{R}_{n-1}(\delta,\delta') > 0\big\}$$

Now we want to show by induction that $\forall n\ X_n \cap T_n \subseteq X_{n+1} \cap T_{n+1}$.
Base step: $X_0 \cap T_0 = \emptyset$ and we're done.
Inductive step: let δ be a device in $X_n \cap T_n$ and $\delta' \in T_n$ such that $\delta' \sim \delta$.
We have $d_{n-1}(\delta') = \infty\ \vee\ \mathcal{R}_{n-1}(\delta,\delta') = 0$. If $d_{n-1}(\delta') = \infty$ then $d_n(\delta') = \infty$ by inductive hypothesis. Otherwise let's suppose $d_{n-1}(\delta') < \infty$ and $d_n(\delta') < \infty$

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$$\forall \delta' \sim \delta \quad \mathcal{R}_{n-1}(\delta,\delta') \cdot (d_{n-1}(\delta') < d_n(\delta)) = 0$$

and thus $I_n(\delta)$ is a zero field-value. So we have

$\square$