Preliminar notes on maximum-flow problem solved in FCPP.
Notation:
• G will denote the directed weighted graph of capacities, s and t will denote respectively source and sink of our graph.
We will assume that capacities are nonnegative and symmetrical.
• For a path $\mathfrak{p}$ we will denote with $|\mathfrak{p}|$ its lenght.

When it will be convenient, we will treat a directed weighted graph G as a function $G : V \times V \to \mathbb{R}_+$, where V is its set of vertices. With a little abuse of notation we will say that the edge from a node δ to a node δ' is in G, or $(\delta, \delta') \in G$, in place of $G(\delta, \delta') \neq 0$.
Likewise we will say that a path $\mathfrak{p}$ is contained in G, or $\mathfrak{p} \subset G$, if every edg

• For a field Φ , $\mathcal{R}_\Phi$ will denote the directed weighted graph of residual capacities respect to Φ:

$$\mathcal{R} = G - \Phi \tag{1}$$

• We define admissible path respect to Φ a path γ in G from source to sink such that

$$\Phi|_\gamma < G|_\gamma$$

• We say that a flow Φ is maximal if there are no admissible paths in G respect to Φ.

Proposizione 0.1. *Φ is a maximum flow iff $\mathcal{R}_\Phi$ does not have paths from source to sink.*

For a field-value f in δ we define $|f| := \sum_{\delta \sim \delta'} f(\delta')$. We now formalize functions involved in the algorithm.
We define $\Phi^*(\delta, \delta') := \Phi(\delta', \delta)$.

Definition 1. *Let Φ be a field with values in $\mathbb{R}$ and let d be a locally constant field d with values in $\mathbb{N} \cup \{\infty\}$. We say that the couple (Φ, d) is admissible for G if, for all devices $\delta \sim \delta'$, the following are satisfied:*
(i) $\Phi(\delta, \delta) = 0 \ \wedge \ -G \leq \Phi \leq G \ \wedge \ (\Phi < 0 \to \Phi^* \geq 0)$;
(ii) $d(t) = 0$;
(iii) $(d(\delta') < \infty \wedge d(\delta) = d(\delta') + 1) \ \to \ (\Phi(\delta, \delta') \ < \ G(\delta, \delta'))$.

Example 1. *For a flow Φ, $(\Phi, \mathrm{dist}_{\mathcal{R}_\Phi}(\cdot, t))$ is admissible.*

Example 2. $(0 \ , \ \delta \neq t \mapsto \infty)$ *is admissible.*

Example 3. *Let (Φ, d) be admissible for G. If H is a subgraph of G then $(\Phi_{|H}, d_{|H})$ is admissible for H.*

Example 4. *Let (Φ, d) be admissible for G. If H is a supergraph of G then (Φ_H, d_H) is admissible for H, where Φ_H is the extension of Φ with default 0 and d_H is the extension of d with default ∞.*

Proposizione 0.2. *Let G and G' be two graphs on the set of devices. Suppose that G and G' are compatible, i.e. for all devices δ, δ' one has*
$(G \neq 0 \ \wedge \ G' \neq 0) \to (G = G')$.
If (Φ, d) is an admissible couple for G, then $((\Phi_{|G\cap G'})_{G'} \ , \ (\Phi_{|G\cap G'})_{G'})$ is compatible for G'.

Proof. (SKETCH) It's a consequence of the two previous examples. $\square$

Definition 2. *Given an admissible couple (Φ, d) for G, we define by co-induction:*

$$\begin{aligned}
\Phi_0 &:= \Phi \\
\mathcal{R}_n &:= \mathcal{R}_{\Phi_n} \\
e_n(\delta) &:= \begin{cases} \infty & \textit{if } \delta = s \\ -\infty & \textit{if } \delta = t \\ |\Phi_n(\delta)| & \textit{otherwise} \end{cases} \\
d_0 &:= d \\
d_n(\delta) &:= \begin{cases} 0 & \textit{if } \delta = t \\ \min\big\{d_{n-1}(\delta') + 1 \mid \mathcal{R}_{n-1}(\delta, \delta') > 0\big\} & \textit{otherwise} \end{cases} \\
I_n &:= \mathrm{trunc}((G + \Phi^*_{n-1}) \cdot (d^*_{n-1} < d_n), e(\Phi^*_{n-1})) \\
\Phi_n &:= -\Phi^*_{n-1} + I_n + \mathrm{trunc}(\Phi^*_{n-1} \ , \ e(\Phi^*_{n-1} - I_n)) \\
X_n &:= \{\delta \mid d_n(\delta) = \infty\}
\end{aligned}$$

Lemma 0.3. *Eventually $\{X_n\}_n$ stabilizes to a set of devices X.*

Proof. For every $n \in \mathbb{N}$ let's consider the set $T_n = \{\delta \mid \mathrm{dist}_G(\delta, t) \leq n\}$.
We prove by induction that for $\delta \notin T_n$ we have $d_n(\delta) = \infty$.
Base step: $T_0 = \{t\}$, and condition on d_0 holds by definition.
Inductive step: suppose that $\forall \delta' \notin T_{n-1}$ $d_{n-1}(\delta') = \infty$ and consider a $\delta \notin T_n$.
We have $d_n(\delta) = \min\big\{d_{n-1}(\delta') + 1 \mid \mathcal{R}_{n-1}(\delta, \delta') > 0\big\}$ and since no $\delta' \sim \delta$ is in T_{n-1}, condition on d_n holds by inductive hypothesis.

As a consequence, for a $\delta \in T_n$

$$d_n(\delta) = \min_{\delta' \in T_{n-1}} \big\{d_{n-1}(\delta') + 1 \mid \mathcal{R}_{n-1}(\delta, \delta') > 0\big\}$$

Now we want to show by induction that $\forall n$ $X_n \cap T_n \subseteq X_{n+1} \cap T_{n+1}$.
Base step: $X_0 \cap T_0 = \emptyset$ and we have done.
Inductive step: let δ be a device in $X_n \cap T_n$ and $\delta' \in T_{n-1}$ such that $\delta' \sim \delta$.
We have $d_{n-1}(\delta') = \infty \ \vee \ \mathcal{R}_{n-1}(\delta, \delta') = 0$. If $d_{n-1}(\delta') = \infty$ then $d_n(\delta') = \infty$ by inductive hypothesis. Otherwise let's suppose $d_{n-1}(\delta') < \infty$, $\mathcal{R}_{n-1}(\delta, \delta') = 0$ and $d_n(\delta') < \infty$. In this case $e(I_n(\delta')) = e(\Phi^*_{n-1}(\delta'))$, i.e.

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$$\forall \delta' \sim \delta \quad \mathcal{R}_{n-1}(\delta, \delta') \cdot (d_{n-1}(\delta') < d_n(\delta)) = 0$$

and thus $I_n(\delta)$ is a zero field-value. So we have

$\square$