Preliminar notes on maximum-flow problem solved in FCPP.
Notation:
• $\mathcal{C}$ will denote the directed weighted graph of capacities.
s and t will denote respectively source and sink of our graph.
• For a path $\mathfrak{p}$ we will denote with $|\mathfrak{p}|$ its lenght.

When it will be convenient, we will treat a directed weighted graph G as a function $G : V \times V \rightarrow \mathbb{R}_+$, where V is its set of vertices. With a little abuse of notation we will say that the edge from a node δ to a node δ' is in G, or $(\delta, \delta') \in G$, in place of $G(\delta, \delta') \neq 0$.
Likewise we will say that a path $\mathfrak{p}$ is contained in G, or $\mathfrak{p} \subset G$, if every edge in $\mathfrak{p}$ is in G.

• For a pre-flow Φ , $\mathcal{R}_\Phi$ will denote the directed weighted graph of residual capacities respect to Φ:

$$\mathcal{R} = \begin{cases} G, & \text{where } \Phi < 0 \\ G - \Phi, & \text{otherwise} \end{cases} \tag{1}$$

• We define admissible path respect to Φ a path γ in G from source to sink such that

$$\Phi|_\gamma < G|_\gamma$$

• We say that a flow Φ is maximum if there are no admissible paths in G respect to Φ.

Proposizione 0.1. *Φ is a maximum flow iff $\mathcal{R}_\Phi$ does not have paths from source to sink.*

Proof. Let's show that a path γ in G is admissible iff γ is a path in $\mathcal{R}_\Phi$. On every edge e in G we have

$$0 < \mathcal{R}_\Phi(e) \Leftrightarrow \begin{cases} \Phi(e) \leq 0 < \mathcal{R}_\Phi(e) = G(e) & \text{or} \\ 0 < \mathcal{R}_\Phi(e) = G(e) - \Phi(e) \end{cases}$$
$$\Leftrightarrow \Phi(e) < G(e) \text{ and } 0 < G(e)$$

Hence

$$0 < \mathcal{R}_\Phi|_\gamma \Leftrightarrow \Phi|_\gamma < G|_\gamma \text{ and } 0 < G|_\gamma$$

This is what we wanted to prove. □

For a field-value f in δ we define $|f| := \sum_{\delta \mapsto \delta'} f(\delta')$.

For a pre-flow Φ we define

$$\|\Phi\| := \sum_{\substack{\delta \text{ connected to} \\ t \text{ in } \mathcal{R}_\Phi}} |\mathcal{R}_\Phi(\delta)|$$

Lemma 0.2. *Let Φ be a pre-flow. Then $\|\boldsymbol{update}(\Phi)\| \leq \|\Phi\|$ with equality holding iff $\boldsymbol{update}(\Phi) = \Phi$.*