Preliminar notes on maximum-flow problem solved in FCPP.

Notation:

• G will denote the directed weighted graph of capacities, s and t will denote respectively source and sink of our graph.

We will assume that capacities are symmetrical.

• For a path $\mathfrak{p}$ we will denote with $|\mathfrak{p}|$ its lenght.

When it will be convenient, we will treat a directed weighted graph G as a function $G : V \times V \to \mathbb{R}_+$, where V is its set of vertices. With a little abuse of notation we will say that the edge from a node δ to a node δ' is in G, or $(\delta, \delta') \in G$, in place of $G(\delta, \delta') \neq 0$.

Likewise we will say that a path $\mathfrak{p}$ is contained in G, or $\mathfrak{p} \subset G$, if every edg

• For a field Φ , $\mathcal{R}_\Phi$ will denote the directed weighted graph of residual capacities respect to Φ:

$$\mathcal{R} = G - \Phi \tag{1}$$

• We define admissible path respect to Φ a path γ in G from source to sink such that

$$\Phi|_\gamma < G|_\gamma$$

• We say that a flow Φ is maximal if there are no admissible paths in G respect to Φ.

Proposizione 0.1. Φ *is a maximum flow iff* $\mathcal{R}_\Phi$ *does not have paths from source to sink.*

For a field-value f in δ we define $|f| := \sum_{\delta \sim \delta'} f(\delta')$. We now formalize functions involved in the algorithm. Let Φ be a field and $\delta \sim \delta'$ be devices, we define $\Phi^*(\delta, \delta') := \Phi(\delta', \delta)$. By co-induction:

$$
\begin{aligned}
\Phi_0 &:= \Phi \\
\mathcal{R}_n &:= \mathcal{R}_{\Phi_n} \\
e_n(\delta) &:= \begin{cases} \infty & \text{if } \delta = s \\ -\infty & \text{if } \delta = t \\ |\Phi_n(\delta)| & \text{otherwise} \end{cases} \\
d_0(\delta) &:= \begin{cases} 0 & \text{if } \delta = t \\ \infty & \text{otherwise} \end{cases} \\
d_n(\delta) &:= \begin{cases} 0 & \text{if } \delta = t \\ \min\{ d_{n-1}(\delta') + 1 \mid \mathcal{R}_{n-1}(\delta, \delta') > 0 \} & \text{otherwise} \end{cases} \\
I_{n+1} &:= \operatorname{trunc}((G + \Phi_n^*)(d_{n-1}^* < d_n), e(\Phi_n^*)) \\
\Phi_{n+1} &:= -\Phi_n^* + I_{n+1} + \operatorname{trunc}(\Phi_n^*, e(\Phi_n^*) - |I_{n+1}|)
\end{aligned}
$$

For a field Φ we define

$$\|\Phi\| := \sum_{\substack{\delta \text{ connected to} \\ t \text{ in } \mathcal{R}_\Phi}} |\mathcal{R}_\Phi(\delta)|$$

Lemma 0.2. *Let Φ be a pre-flow. Then $\|\boldsymbol{update}(\Phi)\| \leq \|\Phi\|$ with equality holding iff $\boldsymbol{update}(\Phi) = \Phi$.*