Preliminar notes on maximum-flow problem solved in FCPP.

Notation:

- $\mathcal{C}$ will denote the directed weighted graph of capacities.

s and t will denote respectively source and sink of our graph.

- $\mathcal{R}$ will denote the directed weighted graph of residual capacities.
- For a directed graph G, $d_G(\delta, \delta')$ will be the distance from node δ to node δ'. It takes values in $\mathbb{N} \cup \{\infty\}$.
- For a acyclic directed graph G, $D_G(\delta, \delta')$ will be the maximum of the lenghts of all paths from node δ to node δ'.
- For a path $\mathfrak{p}$ we will denote with $|\mathfrak{p}|$ its lenght.

When it will be convenient, we will treat a directed weighted graph G as a function $G : V \times V \ \to \mathbb{R}_+$, where V is its set of vertices. With a little abuse of notation we will say that the edge from a node δ to a node δ' is in G, or $(\delta, \delta') \in G$, in place of $G(\delta, \delta') \neq 0$.

Likewise we will say that a path $\mathfrak{p}$ is contained in G, or $\mathfrak{p} \subset G$, if every edge in $\mathfrak{p}$ is in G.

Observation 1. $\mathcal{C}$ is acyclic and

$$\forall \delta, \delta' : \ \mathcal{C}(\delta, \delta') \neq 0 \Rightarrow \mathcal{C}(\delta', \delta) \ = \ 0 \tag{1}$$

i.e. the capacity between any two nodes can be nonzero in almost one direction. $\mathcal{R}$, has the same set of vertices than $\mathcal{C}$, it can have cycles, and has the property that

$$\forall \delta, \delta' : \ \mathcal{R}(\delta, \delta') + \mathcal{R}(\delta', \delta) \ = \ \mathcal{C}(\delta, \delta') + \mathcal{C}(\delta', \delta) \ . \tag{2}$$

Lemma 0.1. *With the above notation, we have*

$$\forall \delta \ d_{\mathcal{R}}(s, \delta) < \infty \ \Rightarrow \ d_{\mathcal{R}}(s, \delta) \leq D_{\mathcal{C}}(s, \delta)$$

Proof. By contradiction. Let's suppose that exists $\bar{\delta}$ such that $d_{\mathcal{R}}(s, \delta) < \infty$ and $d_{\mathcal{R}}(s, \bar{\delta}) > D_{\mathcal{C}}(s, \bar{\delta})$. Moreover suppose that $\bar{\delta}$ has minimal $d_{\mathcal{R}}(s, \bar{\delta})$ among nodes with this property.

Let $\mathfrak{p}$ be a minimal path from s to $\bar{\delta}$. From $|\mathfrak{p}| > D_{\mathcal{C}}(s, \bar{\delta})$ it follows that there are two nodes δ' and δ'' such that $(\delta', \delta'') \in \mathfrak{p}$ and $(\delta', \delta'') \notin \mathcal{C}$.

Between edges with this property, we can always pick the edge that is the closest to $\bar{\delta}$, so we have $d_{\mathcal{R}}(\delta'', \bar{\delta}) = d_{\mathcal{C}}(\delta'', \bar{\delta})$.

By minimality of $|\mathfrak{p}|$ we have

$d_{\mathcal{R}}(s, \delta') \leq D_{\mathcal{C}}(s, \delta')$ and $d_{\mathcal{R}}(s, \delta'') \leq D_{\mathcal{C}}(s, \delta'')$

and so

$$\begin{aligned} d_{\mathcal{R}}(s, \delta') + 1 + d_{\mathcal{R}}(\delta'', \bar{\delta}) &= d_{\mathcal{R}}(s, \delta'') + d_{\mathcal{R}}(\delta'', \bar{\delta}) \\ &= d_{\mathcal{R}}(s, \bar{\delta}) \\ &\geq D_{\mathcal{C}}(s, \bar{\delta}) + 1 \\ &\geq D_{\mathcal{C}}(s, \delta'') + d_{\mathcal{C}}(\delta'', \bar{\delta}) + 1 \\ &= D_{\mathcal{C}}(s, \delta'') + d_{\mathcal{R}}(\delta'', \bar{\delta}) + 1 \end{aligned}$$

Thus

$$d_{\mathcal{R}}(s,\delta') \geq D_{\mathcal{C}}(s,\delta'') \geq d_{\mathcal{R}}(s,\delta'') = d_{\mathcal{R}}(s,\delta') + 1$$

Absurd. $\square$