

Preliminar notes on maximum-flow problem solved in FCPP.

Notation:

- G will denote the directed weighted graph of capacities, s and t will denote respectively source and sink of our graph.
- We will assume that capacities are nonnegative and symmetrical.
- For a path \mathbf{p} we will denote with $|\mathbf{p}|$ its lenght.

When it will be convenient, we will treat a directed weighted graph G as a function $G : V \times V \rightarrow \mathbb{R}_+$, where V is its set of vertices. With a little abuse of notation we will say that the edge from a node δ to a node δ' is in G , or $(\delta, \delta') \in G$, in place of $G(\delta, \delta') \neq 0$.

Likewise we will say that a path \mathbf{p} is contained in G , or $\mathbf{p} \subset G$, if every edge

- For a field Φ , \mathcal{R}_Φ will denote the directed weighted graph of residual capacities respect to Φ :

$$\mathcal{R} = G - \Phi \quad (1)$$

- We define admissible path respect to Φ a path γ in G from source to sink such that

$$\Phi|_\gamma < G|_\gamma$$

- We say that a flow Φ is maximal if there are no admissible paths in G respect to Φ .

Proposizione 0.1. *Φ is a maximum flow iff \mathcal{R}_Φ does not have paths from source to sink.*

For a field-value f in δ we define $|f| := \sum_{\delta \sim \delta'} f(\delta')$. We now formalize functions involved in the algorithm. Let Φ be a field and $\delta \sim \delta'$ be devices, we define

$\Phi^*(\delta, \delta') := \Phi(\delta', \delta)$. By co-induction:

$$\begin{aligned}
\Phi_0 &:= \Phi \\
\mathcal{R}_n &:= \mathcal{R}_{\Phi_n} \\
e_n(\delta) &:= \begin{cases} \infty & \text{if } \delta = s \\ -\infty & \text{if } \delta = t \\ |\Phi_n(\delta)| & \text{otherwise} \end{cases} \\
d_0(\delta) &:= \begin{cases} 0 & \text{if } \delta = t \\ \infty & \text{otherwise} \end{cases} \\
d_n(\delta) &:= \begin{cases} 0 & \text{if } \delta = t \\ \min\{d_{n-1}(\delta') + 1 \mid \mathcal{R}_{n-1}(\delta, \delta') > 0\} & \text{otherwise} \end{cases} \\
I_{n+1} &:= \text{trunc}((G + \Phi_n^*) \cdot (d_{n-1}^* < d_n), e(\Phi_n^*)) \\
\Phi_{n+1} &:= -\Phi_n^* + I_{n+1} + \text{trunc}(\Phi_n^*, e(\Phi_n^*) - |I_{n+1}|) \\
X_n &:= \{\delta \mid d_n(\delta) = \infty\}
\end{aligned}$$

For a field Φ we define

$$\|\Phi\| := \sum_{\substack{\delta \text{ connected to} \\ t \text{ in } \mathcal{R}_\Phi}} |\mathcal{R}_\Phi(\delta)|$$

Lemma 0.2. *Let Φ be a pre-flow. Then $\|\mathbf{update}(\Phi)\| \leq \|\Phi\|$ with equality holding iff $\mathbf{update}(\Phi) = \Phi$.*