ESc201 : Introduction to Electronics

Sinusoidal Steady state Analysis

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Recap

Canonical Form

$$x(t) = x_m \cos(\omega t + \theta)$$

Phase difference is usually considered between -180 to 180°

Average Power

$$p_{avg} = \frac{V_{rms}^2}{R}$$

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

$$p_{avg} = \frac{V_m^2}{2R}$$

$$I_{rms} = \frac{I_m}{\sqrt{2}}$$

$$p_{avg} = \frac{1}{2} I_m^2 R$$

$$z = x + jy \quad \Longleftrightarrow \quad z = |z|\angle\theta$$

$$? \quad \Longleftarrow \quad z_1 = 5\angle 30^\circ$$

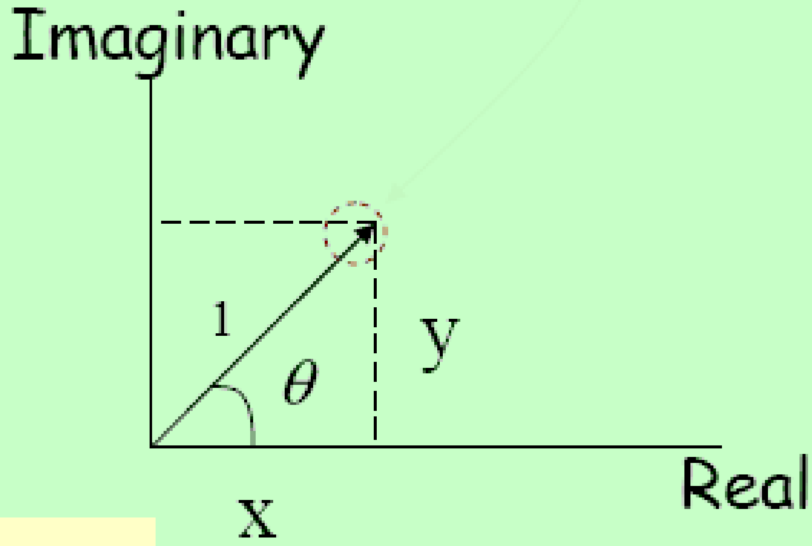
Euler's Identities

$$e^{j\theta} = 1\angle\theta = \cos\theta + j\sin\theta$$

$$\left|e^{j\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$e^{-j\theta} = 1\angle -\theta = \cos\theta - j\sin\theta$$

$$A\angle\theta \;=\; A\cos\theta + Aj\sin\theta$$



$$z_1 = 5\angle 30^\circ$$

$$z_1 = 5\cos(30^\circ) + j5\sin(30^\circ)$$

$$= 4.33 + j2.5 = x + jy$$

Forms of a Complex Number

$$z_2 = 10 + j5$$ ← Rectangular form

$$z_2 = \sqrt{(10)^2 + (5)^2} \angle \tan^{-1}\left(\frac{5}{10}\right)$$

$$= 11.18\angle 26.57^\circ$$ ← Polar form

$$= 11.18 e^{j26.57^\circ}$$ ← Exponential form

Important Note:

$$1\angle 90^\circ = \cos 90 + j\sin 90 = j$$

Arithmetic Operations in Polar and Complex Form

To **add** or **subtract** two complex numbers, convert them first into rectangular form and then perform the operation

- To multiply two complex numbers in polar form

$$z_1\, z_2 = |z_1| \angle \theta_1 \times |z_2| \angle \theta_2$$
$$= |z_1|\,|z_2| \angle (\theta_1 + \theta_2)$$

$$|z_1| \angle \theta_1 \quad = |z_1|\, e^{j\theta_1}$$

$$z_1\, z_2 = |z_1|\, e^{j\theta_1} \times |z_2|\, e^{j\theta_2}$$
$$= |z_1|\,|z_2|\, e^{j(\theta_1 + \theta_2)}$$

- To divide two complex numbers in polar form

$$\frac{z_1}{z_2} = \frac{|z_1|\angle\theta_1}{|z_2|\angle\theta_2} = \frac{|z_1|}{|z_2|}\angle(\theta_1 - \theta_2)$$

$$|z_1|\angle\theta_1 = |z_1|e^{j\theta_1}$$

$$\frac{z_1}{z_2} = \frac{|z_1|e^{j\theta_1}}{|z_2|e^{j\theta_2}} = \frac{|z_1|}{|z_2|}e^{j(\theta_1-\theta_2)}$$

$$j150 \times 0.707\angle -15^\circ = 106.1\angle 75^\circ$$

Important Note:

$$1\angle 90^\circ = \cos 90 + j\sin 90 = j$$

$$v(t) = V_m \cos(\omega t + \theta)$$

$$v(t) = \mathrm{Re}(V_m \times e^{j(\omega t + \theta)})$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$v(t) = \mathrm{Re}(V_m \cos(\omega t + \theta) + jV_m \sin(\omega t + \theta))$$

$$v(t) = V_m \cos(\omega t + \theta) \Rightarrow \mathrm{Re}(V_m \angle \omega t + \theta)$$

$$v(t) = V_m \cos(\omega t + \theta) \Leftrightarrow V_m \angle \theta$$

Phasor

$$v(t)=3\cos(\omega t+45) \Rightarrow 3\angle 45$$

$$\Rightarrow 3\cos(45)+j3\sin(45)$$

$$v(t)=5\cos(\omega t-60) \Leftarrow 5\angle -60$$

$$v_1(t)=20\cos\left(\omega t-45^\circ\right) \qquad v_2(t)=10\sin\left(\omega t+60^\circ\right)$$

$$v_1(t)+v_2(t)=?$$

$$v_1(t) = 20\cos(\omega t - 45^\circ) \qquad v_2(t) = 10\sin(\omega t + 60^\circ)$$

$$20\cos(\omega t - 45^\circ) \rightarrow 20\angle -45^\circ$$

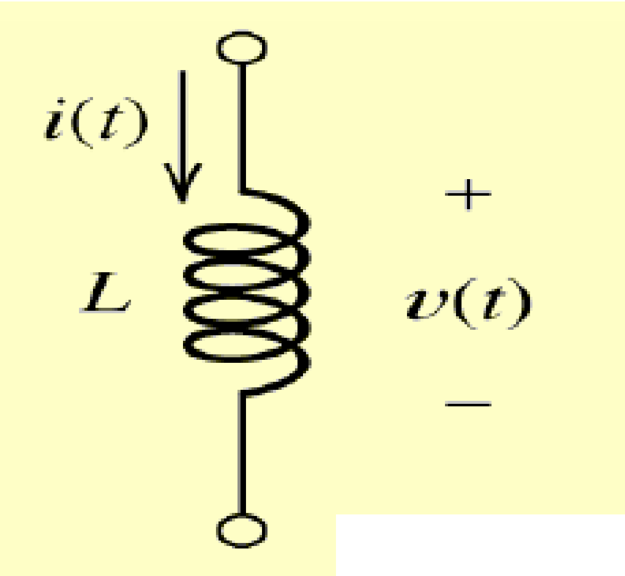
$$10\sin(\omega t + 60^\circ) = 10\cos(\omega t + 60^\circ - 90^\circ) \rightarrow 10\angle -30^\circ$$

$$\begin{aligned}
\mathbf{V}_s &= \mathbf{V}_1 + \mathbf{V}_2 \\
&= 20\angle -45^\circ + 10\angle -30^\circ \\
&= 14.14 - j14.14 + 8.660 - j5 \\
&= 23.06 - j19.14 \\
&= 29.97\angle -39.7^\circ
\end{aligned}$$

$$v_s(t) = 29.97\cos(\omega t - 39.7^\circ)$$

Complex Impedances

For the purpose of sinusoidal steady state analysis, inductors and capacitors can be represented as Complex Impedances

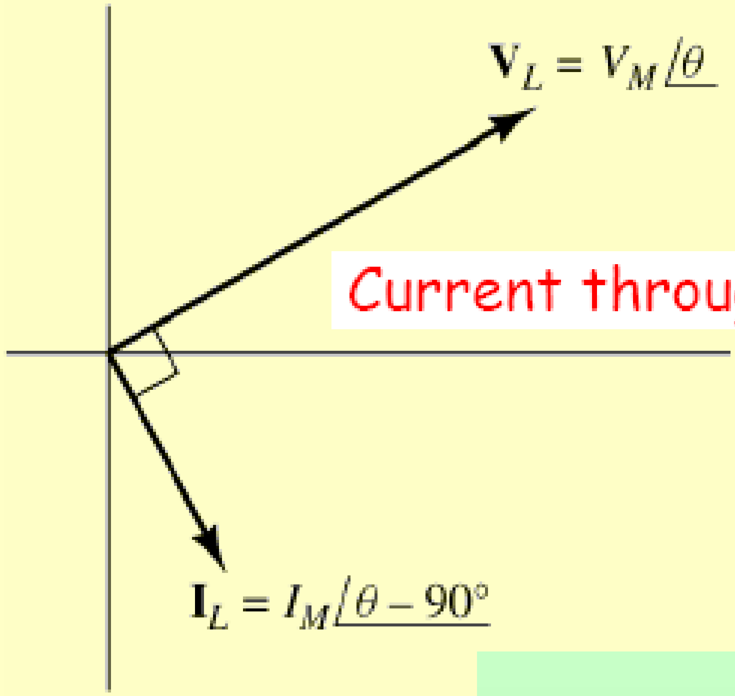


$$i_L = I_m \sin(\omega t + \theta) \longrightarrow \mathbf{I_L} = I_m \angle \theta - 90^\circ$$

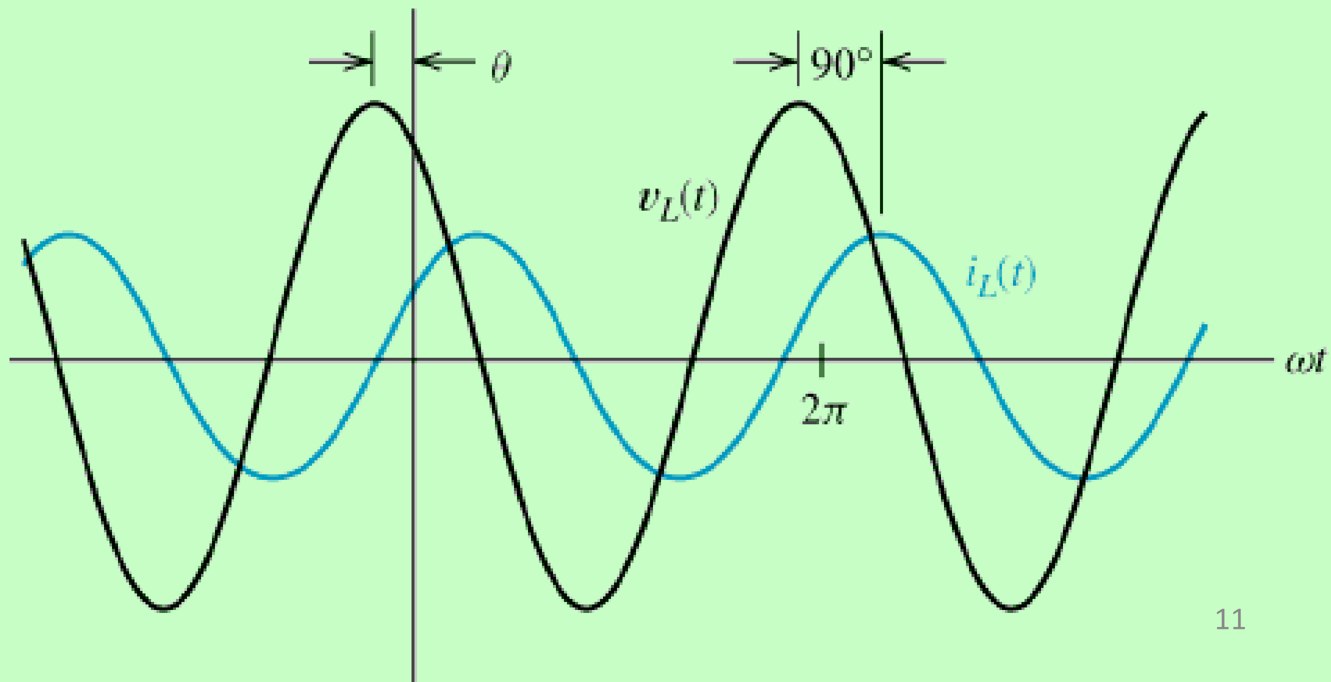
$$v_L = L\frac{di_L}{dt} \longrightarrow \mathbf{V_L} = \omega L I_m \angle \theta$$

$$= \omega L I_m \cos(\omega t + \theta)$$

Current through the inductor lags the voltage by 90°



Current through the inductor lags the voltage by 90°



$$I_L = I_M \angle \theta - 90$$

$$V_L = \omega L I_M \angle \theta$$

$$V_L = \omega L I_M \angle \theta - 90 + 90$$

$$V_L = I_M \angle \theta - 90 \times \omega L \angle 90$$

$$V_L = I_L \times \omega L \angle 90$$

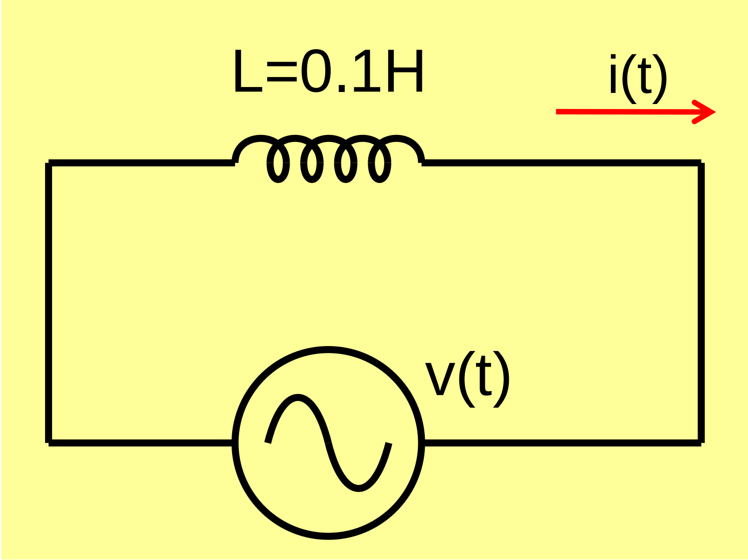
$$V_L = I_L \times j\omega L$$

$$V_L = I_L \times Z_L$$

$$Z_L = j\omega L$$

This is like ohms law relationship between phasor voltage and current

Example-4



$$v(t) = 2\cos(200t + 45)$$ V $\quad$ $$\omega = 200$$ rad/s

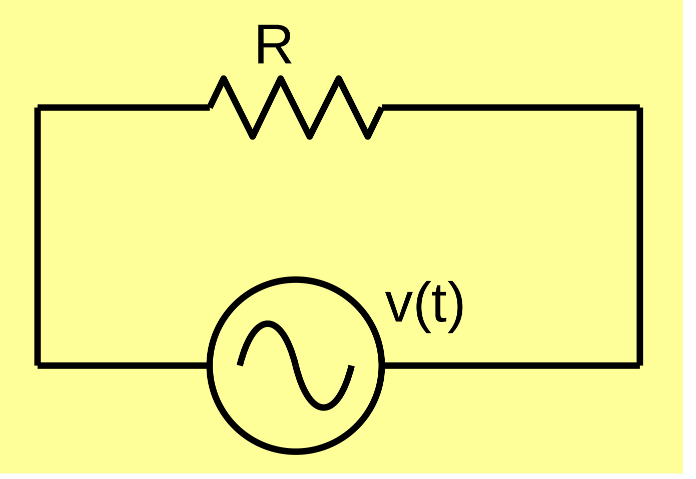
$$\mathbf{V}_L = 2\angle 45$$ V

$$\mathbf{V}_L = \mathbf{I}_L \times j\omega L \Rightarrow \mathbf{I}_L = \frac{\mathbf{V}_L}{j\omega L}$$

$$\mathbf{I}_L = \frac{2\angle 45}{j20} = \frac{2\angle 45}{20\angle 90} = 0.1\angle -45$$ A

$$i(t) = 0.1\cos(200t - 45)$$ A

Resistor



$$v(t) = V_M \cos(\omega t + \theta)$$

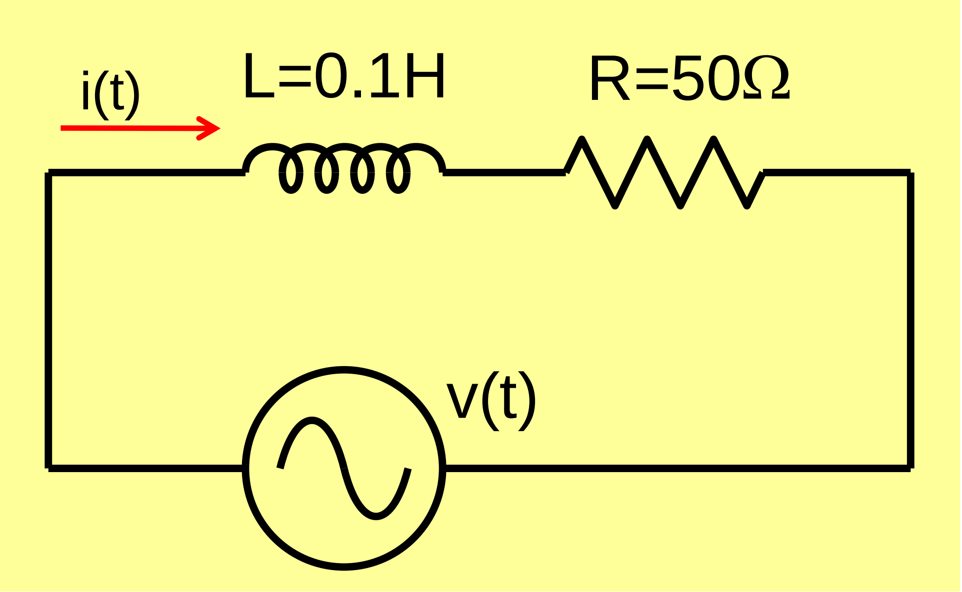
$$i(t) = \frac{V_M}{R} \cos(\omega t + \theta)$$

$$\mathbf{V}_R = V_M \angle \theta$$

$$\mathbf{I}_R = \frac{V_M}{R} \angle \theta$$

$$\mathbf{I}_R = \frac{\mathbf{V}_R}{R}$$

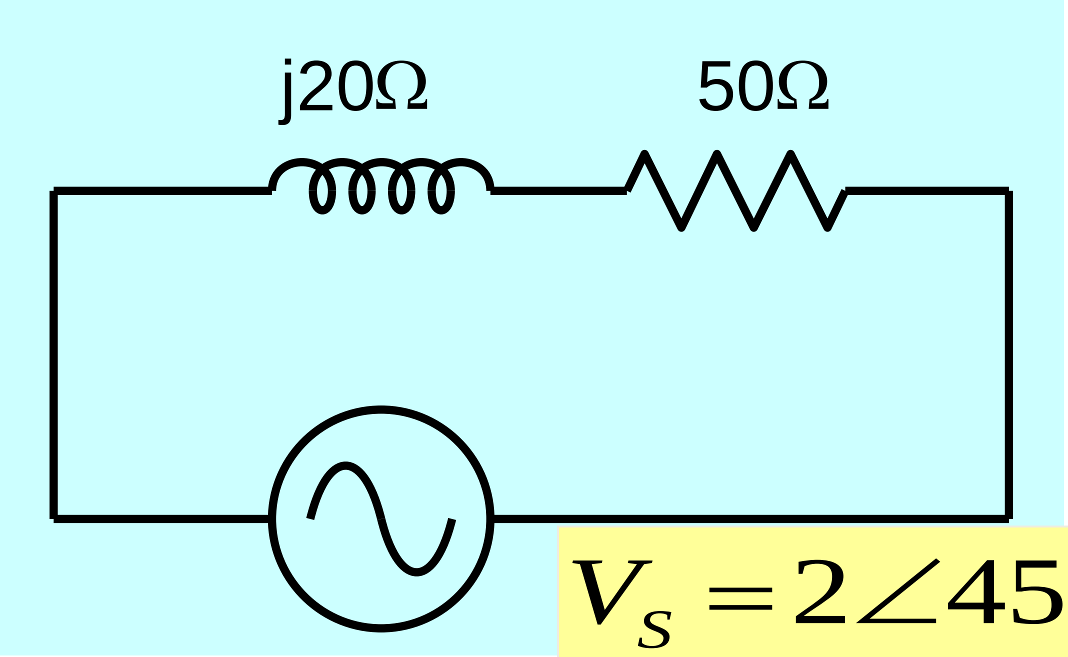
Example-5



$$v(t) = 2\cos(200t + 45)$$ V

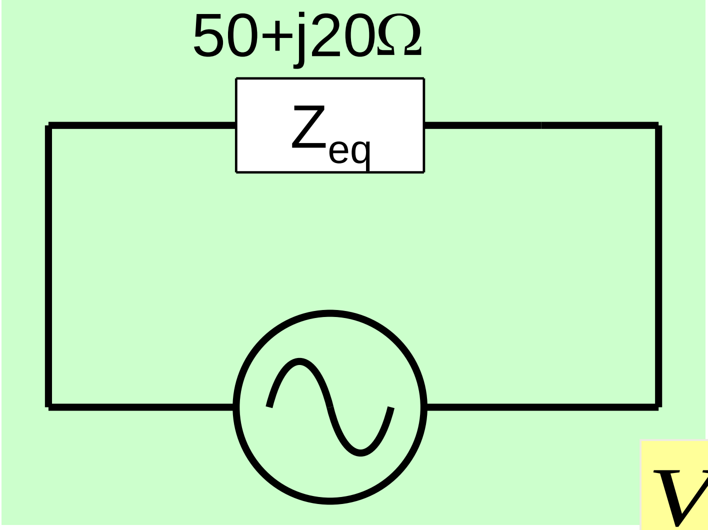
$$\omega = 200$$ rad/s

$$Z_L = j\omega L$$



$$V_S = 2\angle 45$$ V

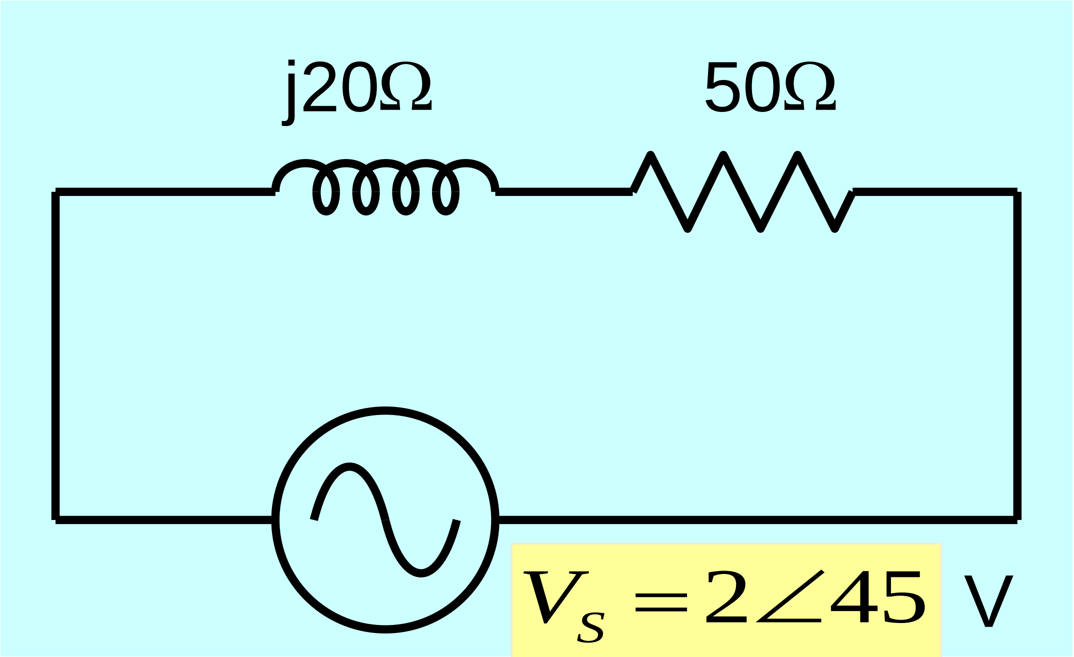
Example-5 contd.



$$V_s = 2\angle 45 \text{ V}$$

$$I = \frac{2\angle 45}{50 + j20} = \frac{2\angle 45}{53.85\angle 21.8} = 0.037\angle 23.2 \text{ A}$$

$$i(t) = 0.037 \cos(200t + 23.2) \text{ A}$$

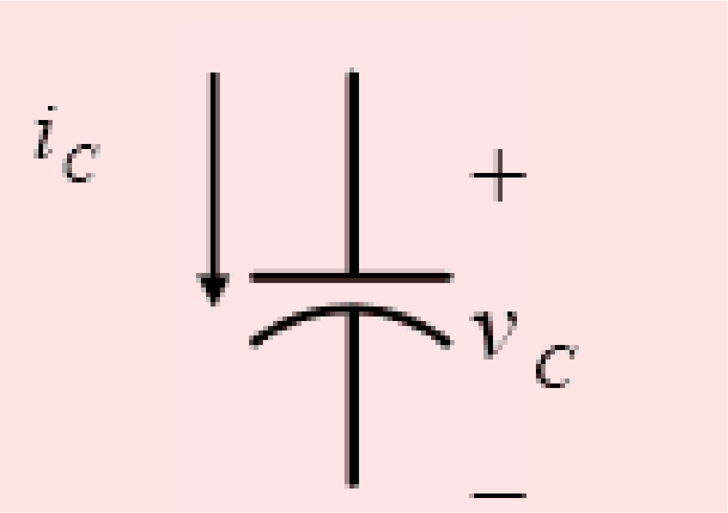


$V_s = 2\angle 45$ V

$$V_R = 2\angle 45 \times \frac{50}{50 + j20} \text{ V}$$

Concept of voltage or current division can be used as before

Capacitor



$$v(t) = V_M \cos(\omega t + \theta)$$

$$i_c = C\frac{dv_c}{dt}$$

$$i(t) = -\omega C V_M \sin(\omega t + \theta)$$

$$i(t) = \omega C V_M \cos(\omega t + \theta + 90^o)$$

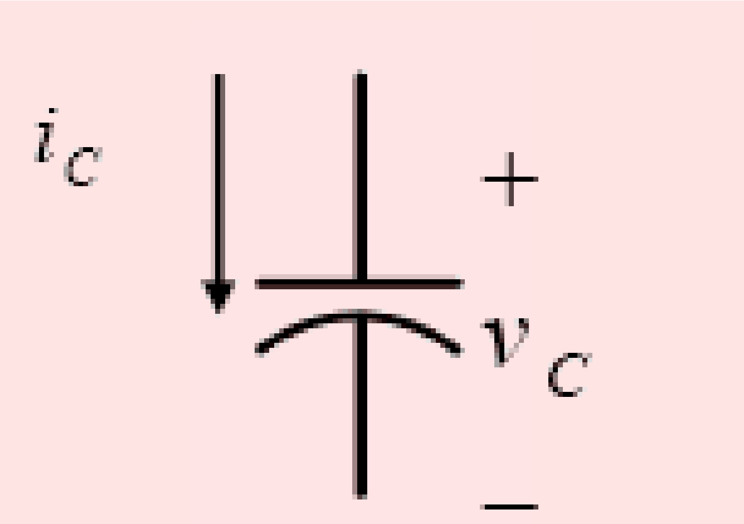
$$V_C = V_M \angle \theta$$

$$I_C = \omega C V_M \angle \theta + 90$$

In a capacitor, current leads voltage by 90^0

Capacitor

$$V_C = V_M \angle \theta$$



$$I_C = \omega C V_M \angle \theta + 90$$

$$I_C = \omega C \angle 90 \times V_M \angle \theta$$

$$I_C = j\omega C V_C$$

$$V_C = I_C \times Z_C$$

$$Z_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C}$$

Circuit Analysis Using Phasors and Impedances

1. Replace the time descriptions of the voltage and current sources with the corresponding phasors. (All of the sources must have the same frequency)

2. Express components by their complex impedances:

   Replace inductances by their complex impedances

$$Z_L = j\omega L$$

   Replace capacitances by their complex impedances

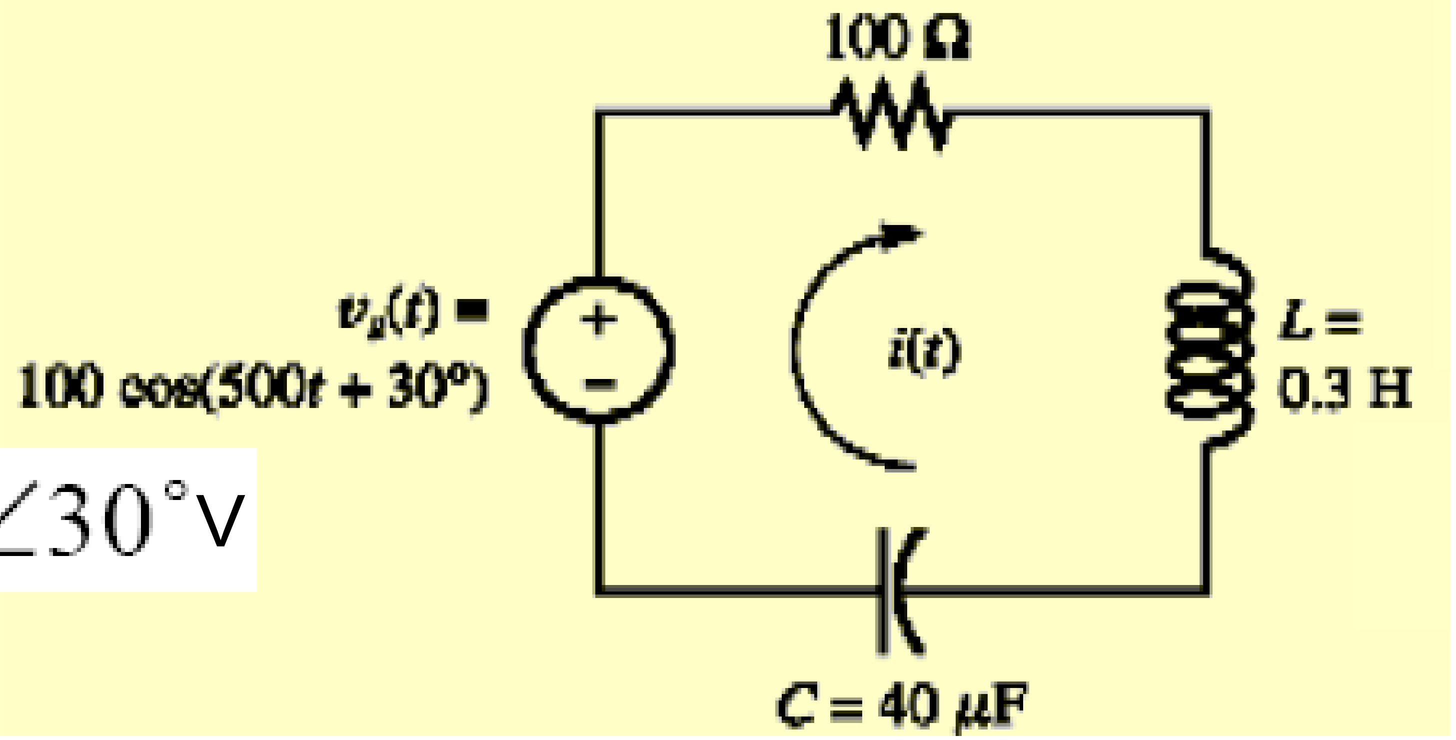
$$Z_C = 1/(j\omega C)$$

   Resistances have impedances equal to their resistances

$$Z_R = R$$

3 Analyze the circuit using any of the techniques studied earlier performing the calculations with complex arithmetic

Example-6

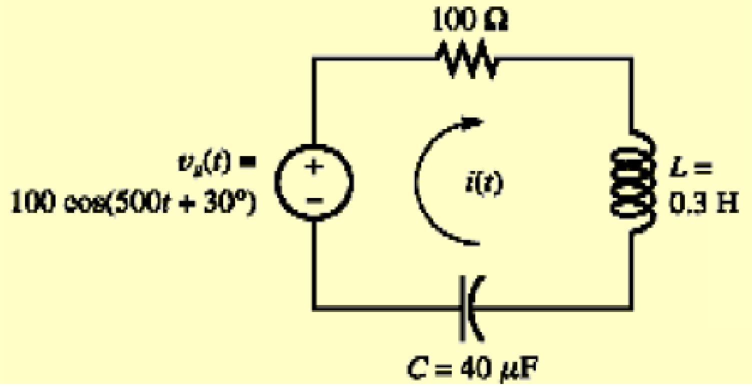


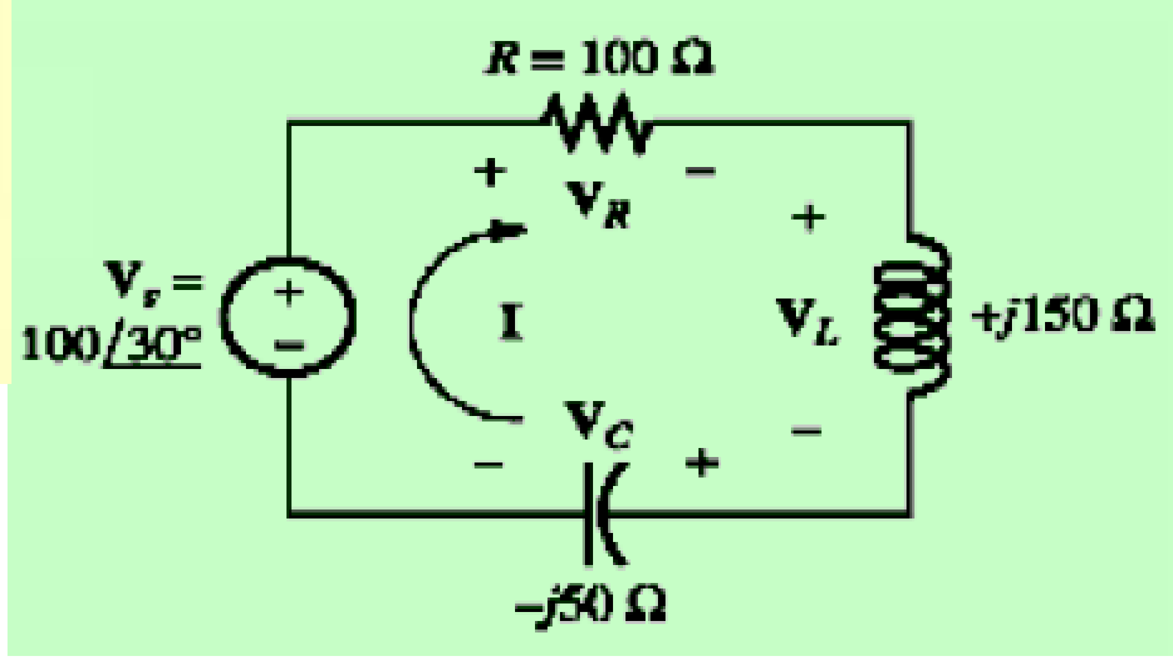
$$\mathbf{V}_s = 100\angle 30^\circ \text{V}$$

$$Z_L = j\omega L = j500 \times 0.3 = j150\,\Omega$$

$$Z_C = -j\frac{1}{\omega C} = -j\frac{1}{500 \times 40 \times 10^{-6}} = -j50\,\Omega$$

$$Z_{eq} = 100 + j150 - j50 = 100 + j100 = 141.4\angle 45^\circ\ \Omega$$





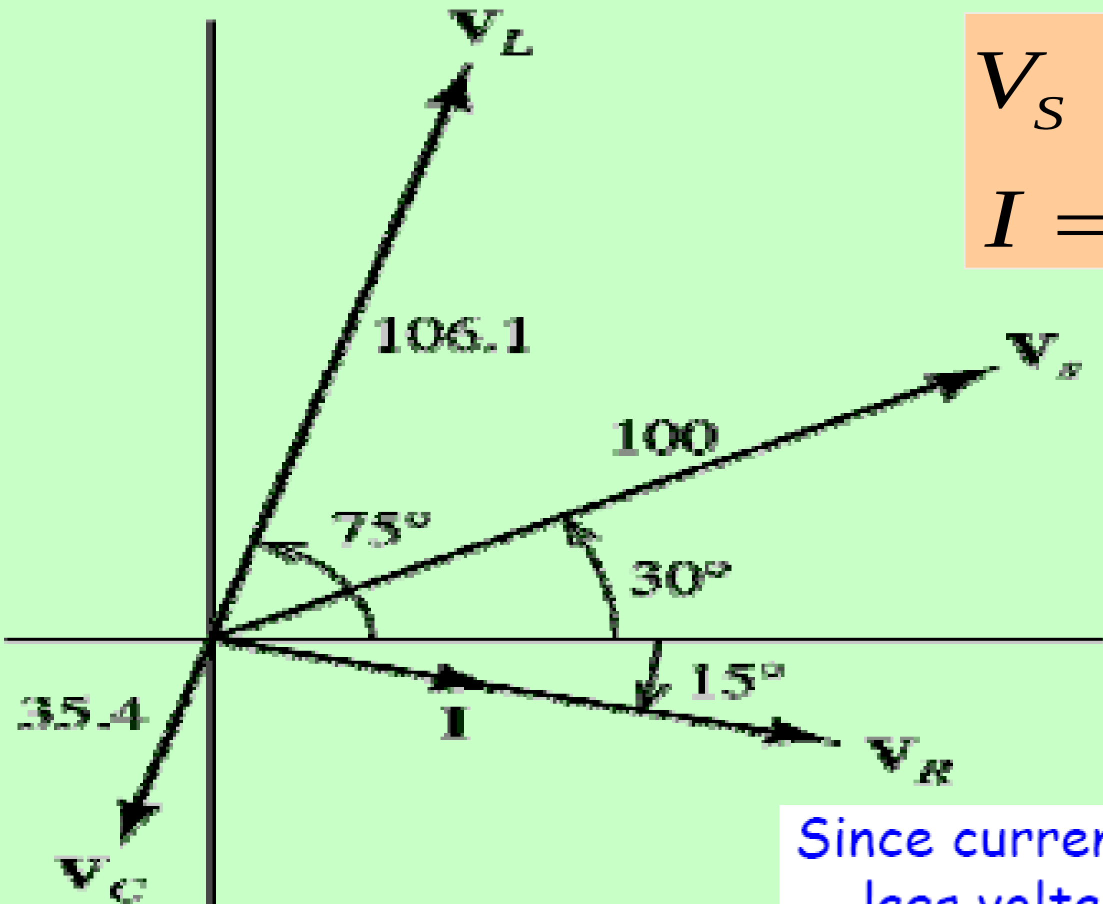
$$I = \frac{\mathbf{V}_s}{Z_{eq}} = \frac{100\angle 30^\circ}{141.4\angle 45^\circ} = 0.707\angle -15^\circ \text{ A}$$

$$i(t) = 0.707\cos(500t - 15^\circ) \text{ A}$$

$$\mathbf{V_R} = R\mathbf{I} = 100 \times 0.707\angle -15^\circ = 70.7\angle -15^\circ \text{ V}$$

$$\mathbf{V_L} = j\omega L\mathbf{I} = j150 \times 0.707\angle -15^\circ = 106.1\angle 75^\circ \text{ V}$$

$$\mathbf{V_C} = -j\frac{1}{\omega C}\mathbf{I} = -j50 \times 0.707\angle -15^\circ = 35.4\angle -105^\circ \text{ V}$$



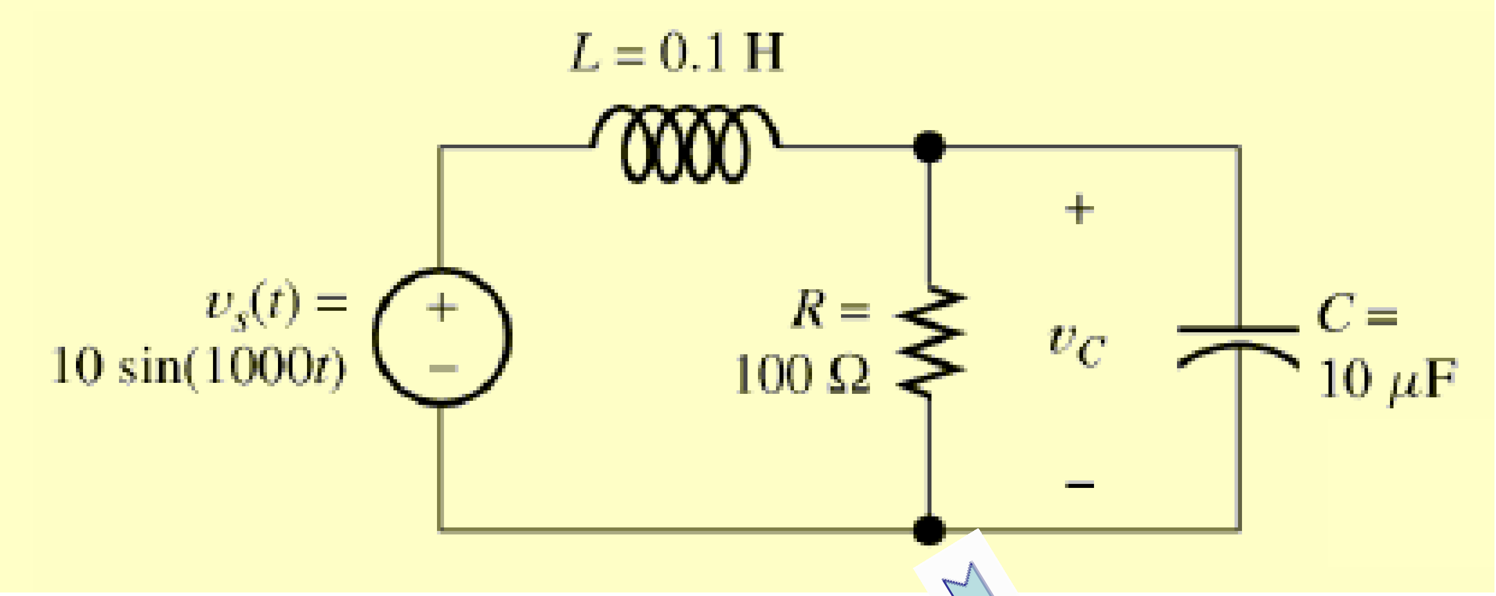
$$V_S = 100\angle 30^o$$

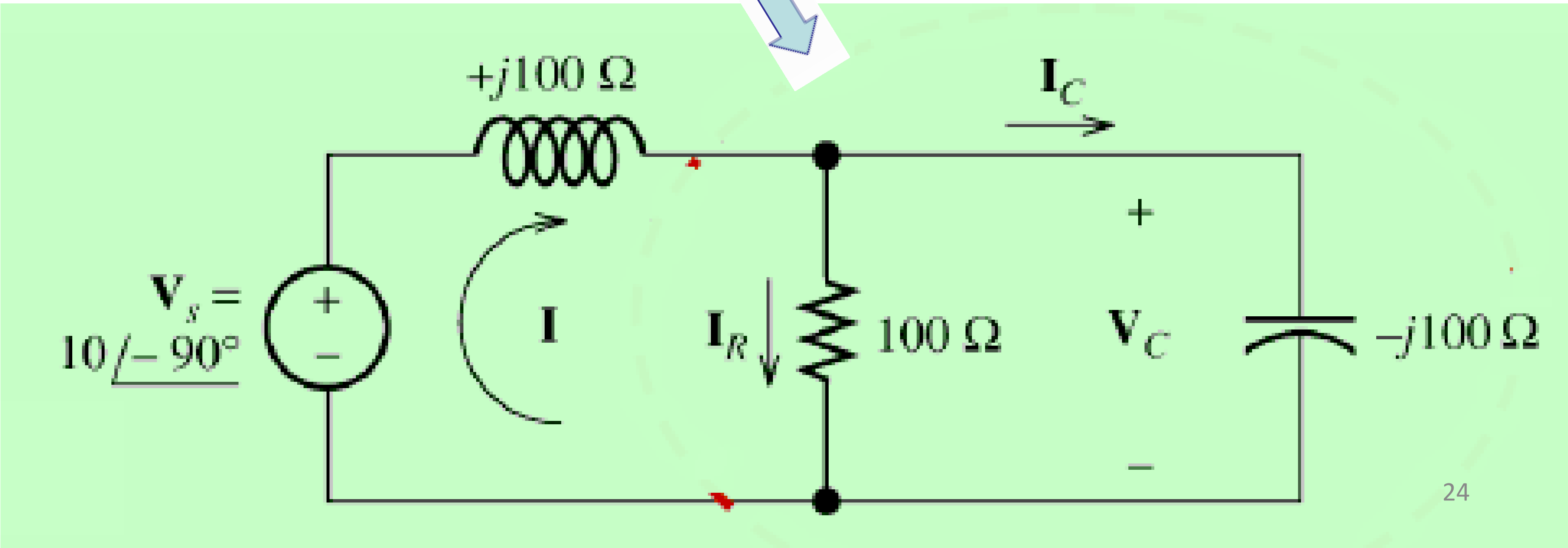
$$I = 0.707\angle -15^o$$

Since current drawn from the supply lags voltage → inductive circuit

Example-7

Find the voltage $v_c(t)$ in steady state





$$Z_{RC} = \frac{1}{1/R + 1/Z_c} =$$

$$= \frac{1}{0.01 + j0.01}$$

$$= \frac{1}{0.1414 \angle 45^\circ}$$

$$= 70.71 \angle -45^\circ$$

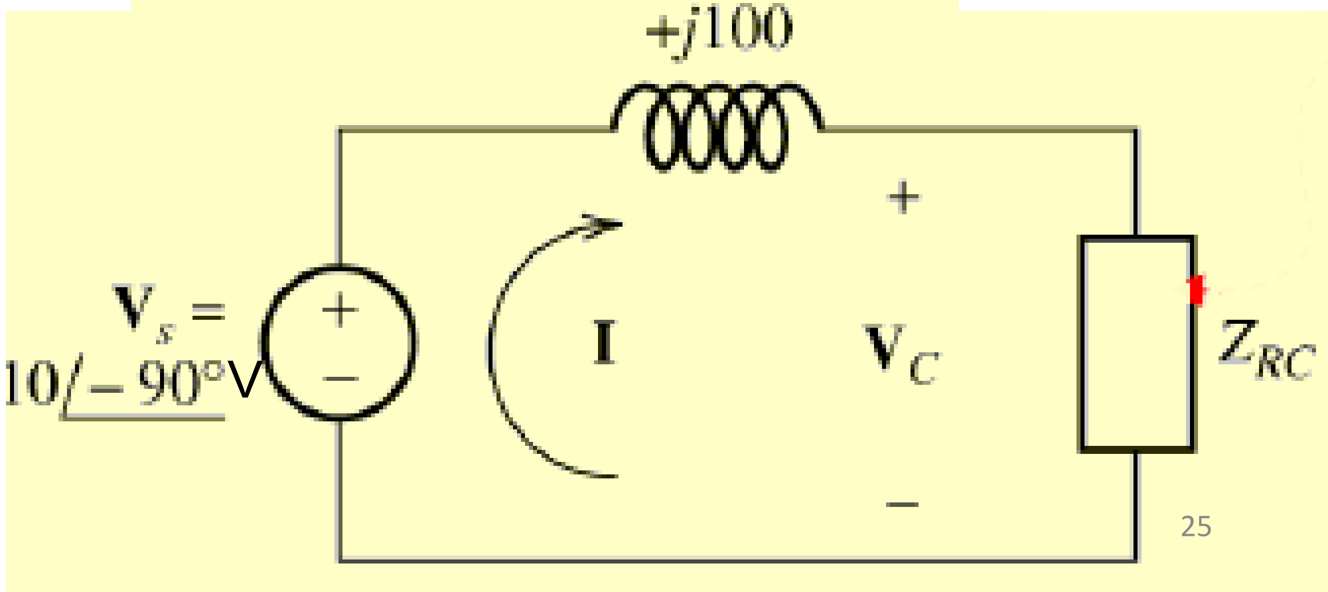
$$= 50 - j50\ \Omega$$

$$V_c = V_s \frac{Z_{RC}}{Z_L + R_{RC}}$$

$$= 10\angle -90^\circ \frac{70.71\angle -45^\circ}{j100 + 50 - j50}$$

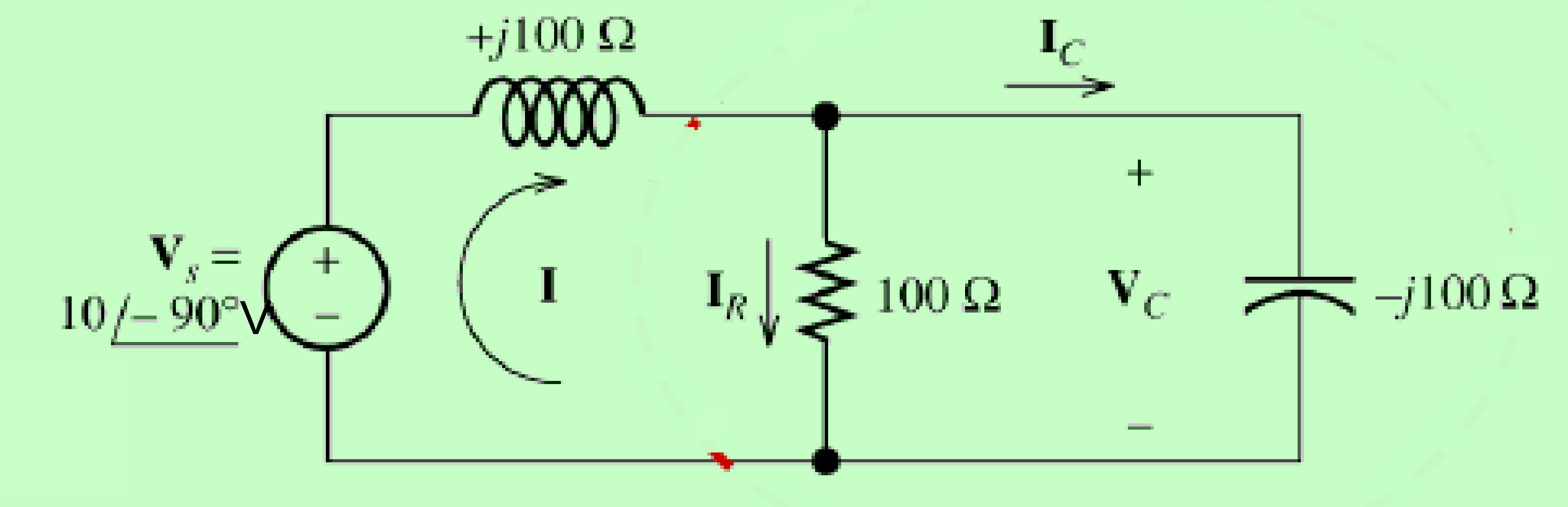
$$= 10\angle -180^\circ = -10 \text{ V}$$

$$v_c(t) = -10\cos(1000t)\text{ V}$$



Currents

$$Z_{RC} = 50 - j50 \quad \Omega$$



$$I = \frac{V_s}{Z_L + Z_{RC}}$$

$$= \frac{10\angle -90^\circ}{50 + j50}$$

$$= 0.141\angle -135^\circ \text{ A}$$

$$I_c = \frac{V_c}{Z_c} = \frac{10 \angle -180^\circ}{-j100} = -j0.1$$

$$I_R = V_c / R = \frac{10\angle -180^\circ}{100} = -0.1 \text{ A}$$