ESc201 : Introduction to Electronics

Sinusoidal Steady state Analysis

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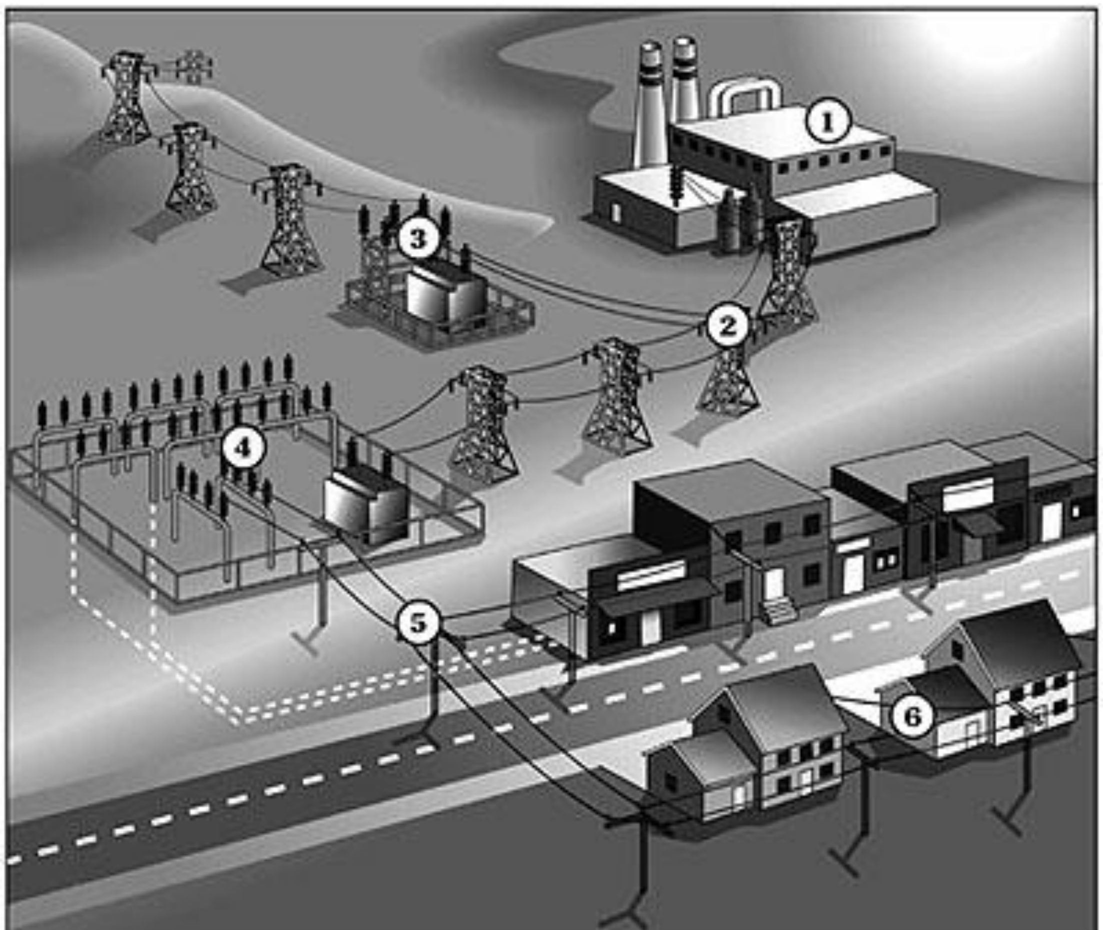
Importance of Sinusoidal Sources

- Appear in many practical applications
  - Electric power is distributed by sinusoidal currents and voltages
  - Sinusoidal signals are used widely in radio communications

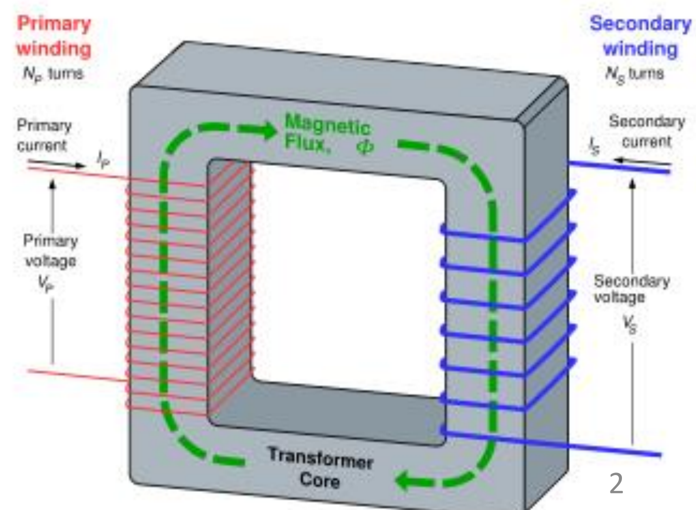


$$Loss = i^2 R_{wire}$$

$$p = v \times i$$

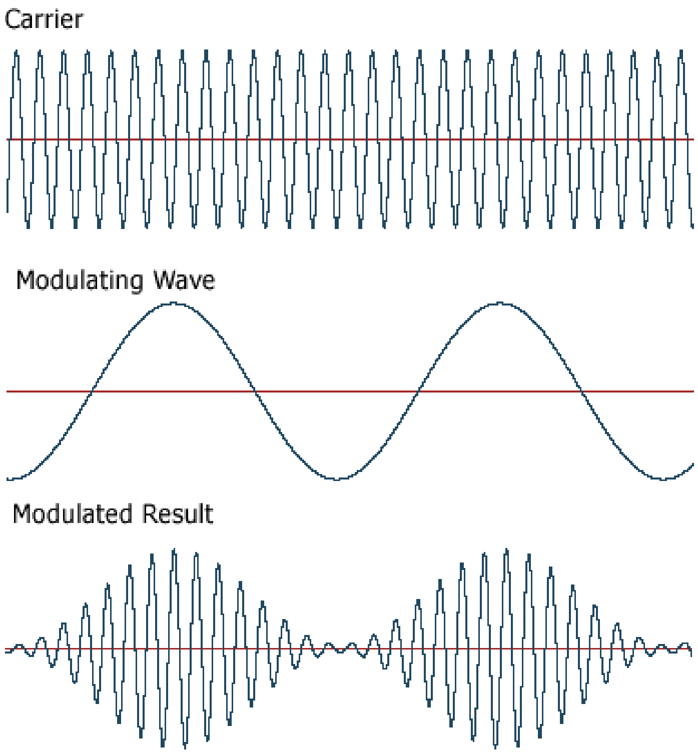
$$2.2KW = 2.2KV \times 1A$$

$$2.2KW = 220V \times 10A$$

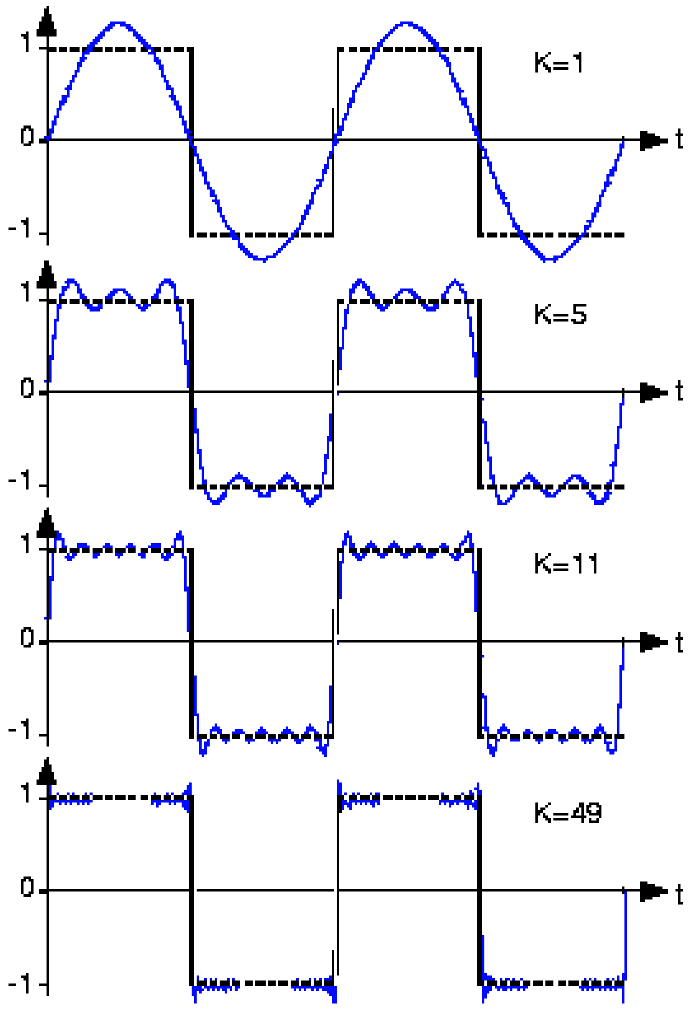


Communication

20 Hz -20KHz



- Any signal can be represented by a sum of sinusoidal components (Fourier Analysis)



$$f(t) = \frac{4}{\pi} \sum_{1,3,5}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi t}{T}\right)$$

- Sinusoids have good mathematical properties
  - Derivative is a sinusoid
  - Integral is a sinusoid

$$\frac{d(Sin\,x)}{dx} = Cos\,x = Sin(90 - x)$$

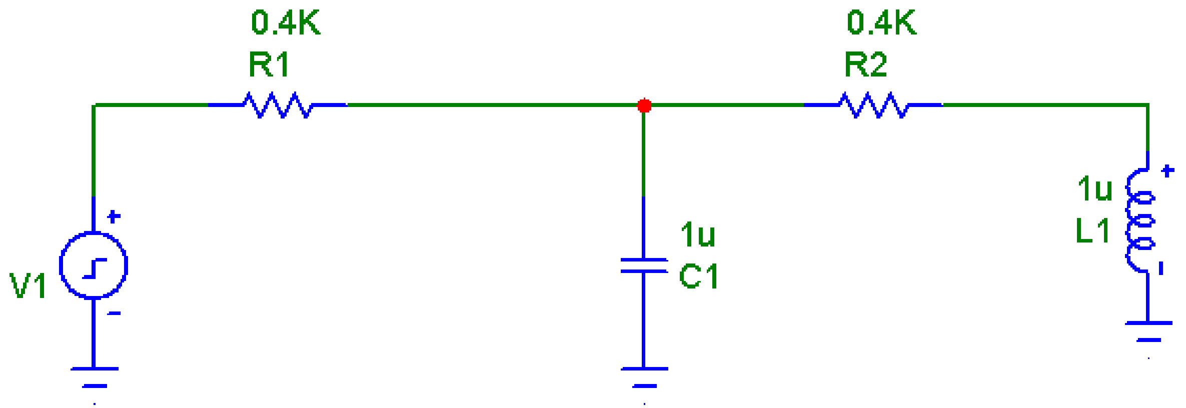
$$i_c = C\frac{dv_c}{dt}$$

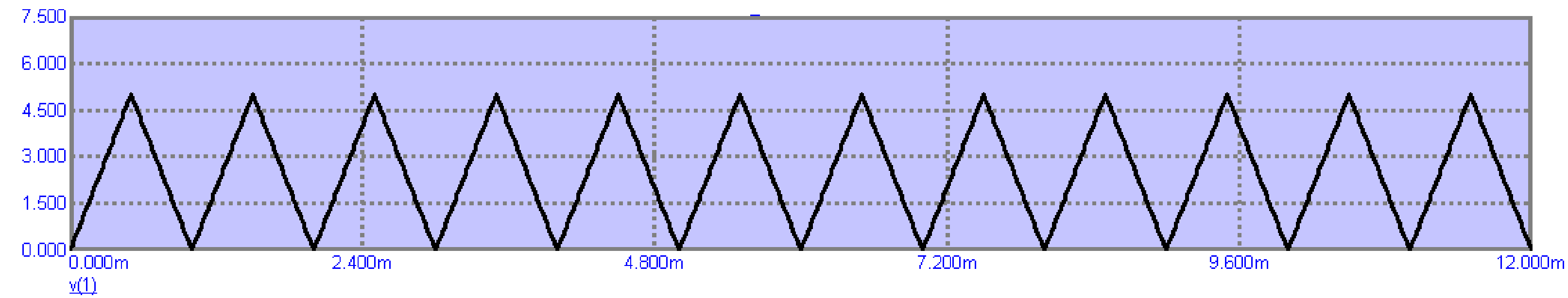
$$\int Sin\,x\,dx = -Cos\,x = Sin(x - 90)$$

$$v = L\frac{di}{dt}$$

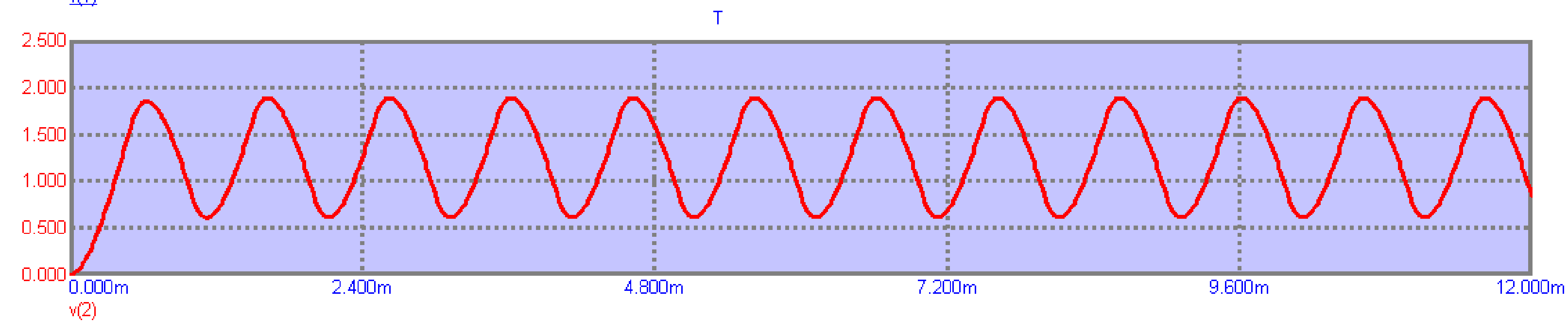
So as a sinusoidal signal goes through a circuit, it remains a sinusoid

This makes analysis easier

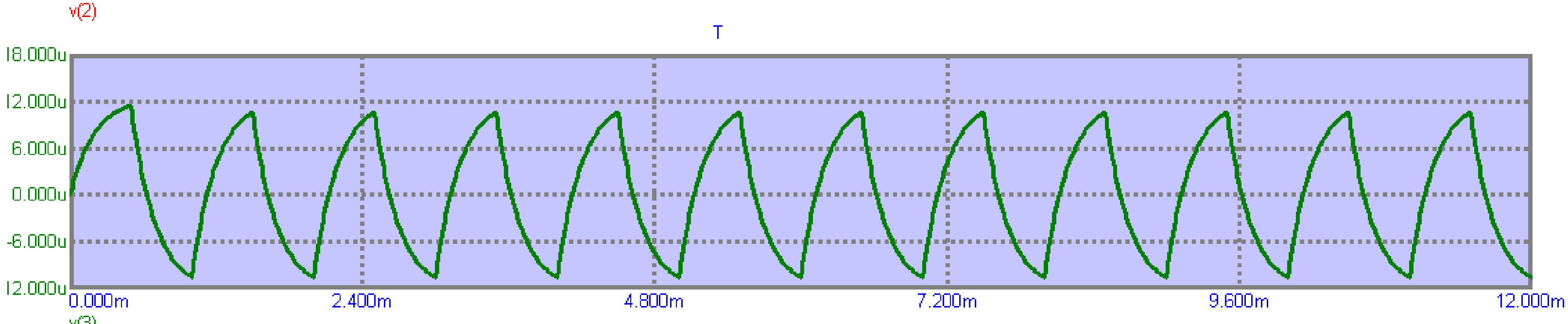

0.4K
R1
0.4K
R2
V1
1u
C1
1u
L1


7.500
6.000
4.500
3.000
1.500
0.000
0.000m
2.400m
4.800m
7.200m
9.600m
12.000m
v(1)

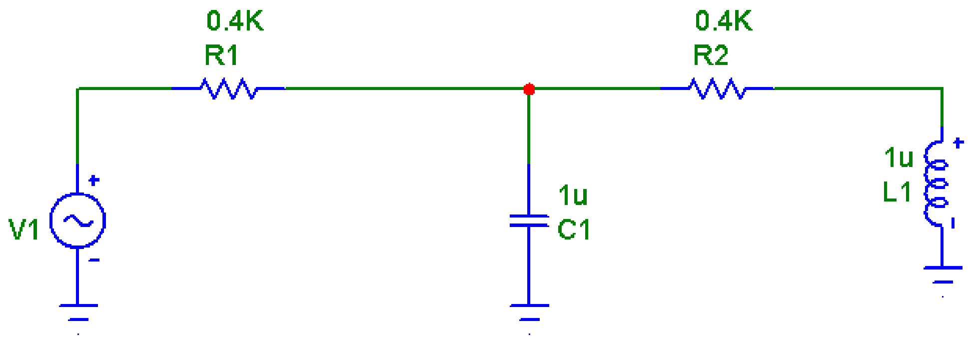
V_{IN}

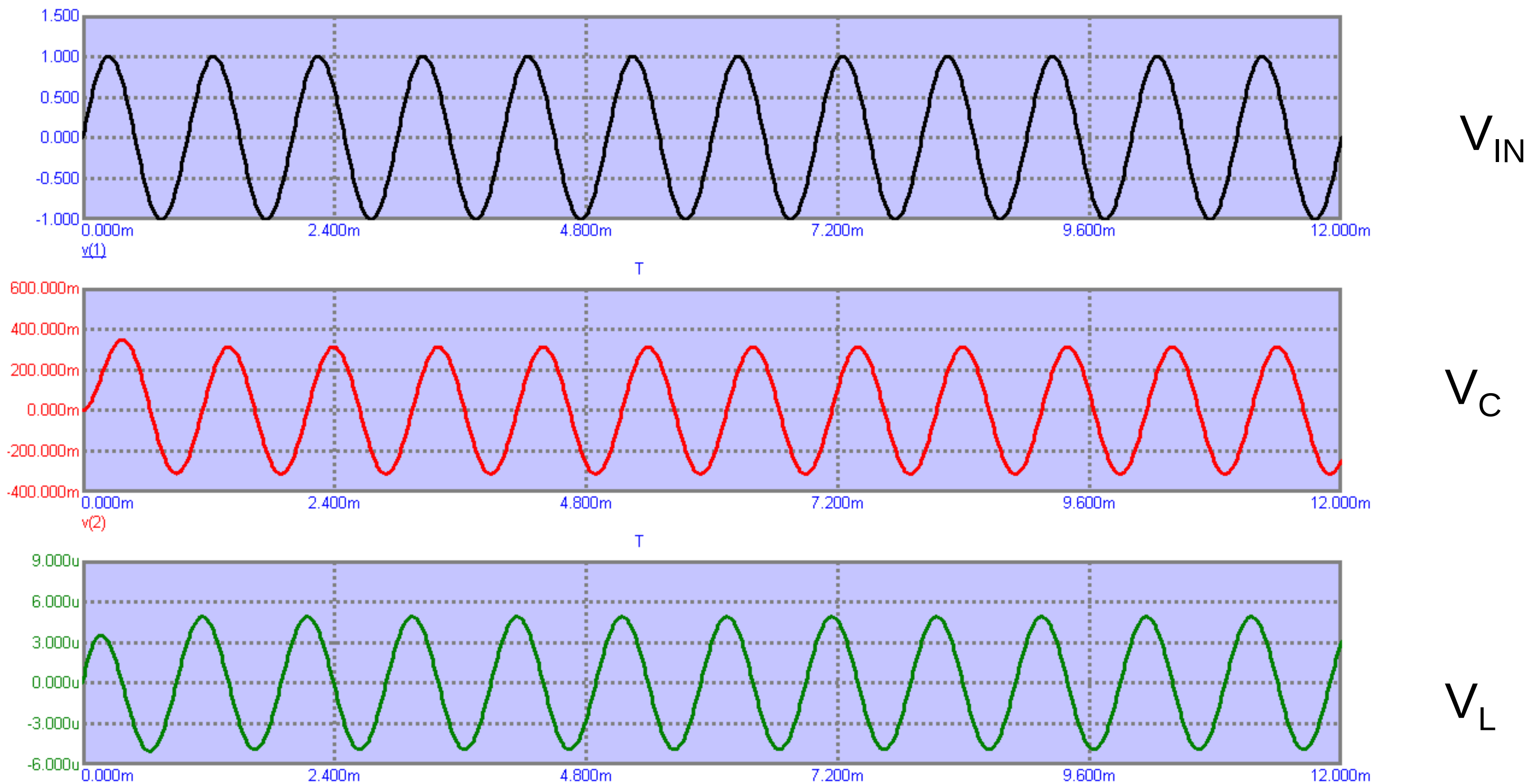

T
2.500
2.000
1.500
1.000
0.500
0.000
0.000m
2.400m
4.800m
7.200m
9.600m
12.000m
v(2)

V_C


T
18.000u
12.000u
6.000u
0.000u
-6.000u
12.000u
0.000m
2.400m
4.800m
7.200m
9.600m
12.000m

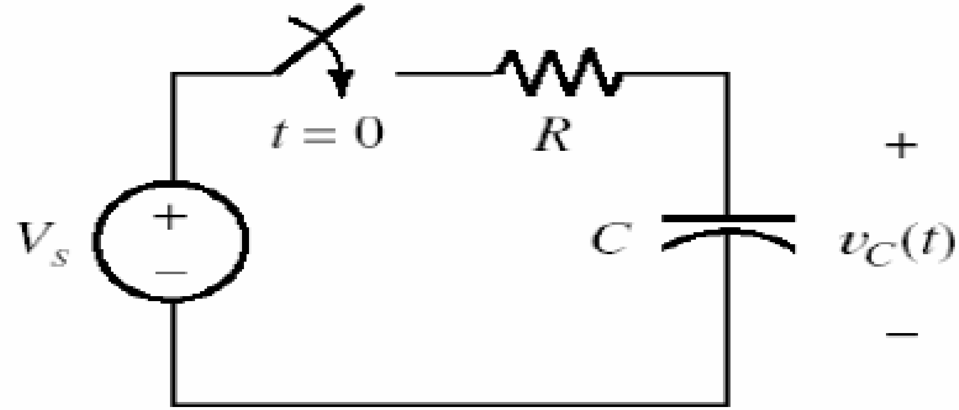
V_L



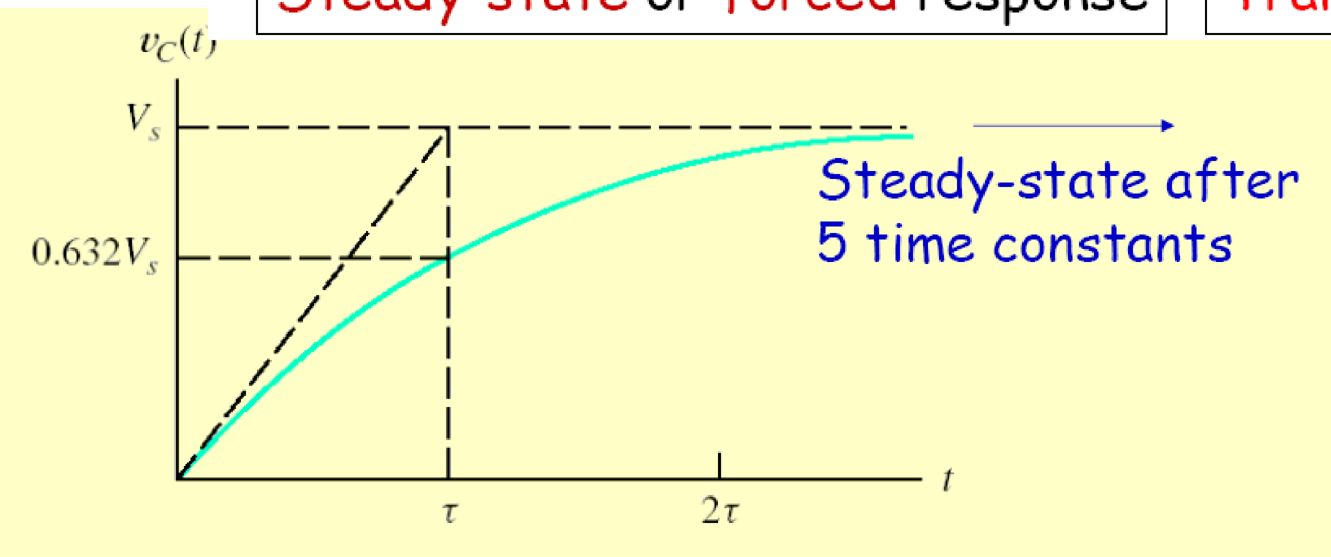


Voltage everywhere in the circuit is sinusoidal

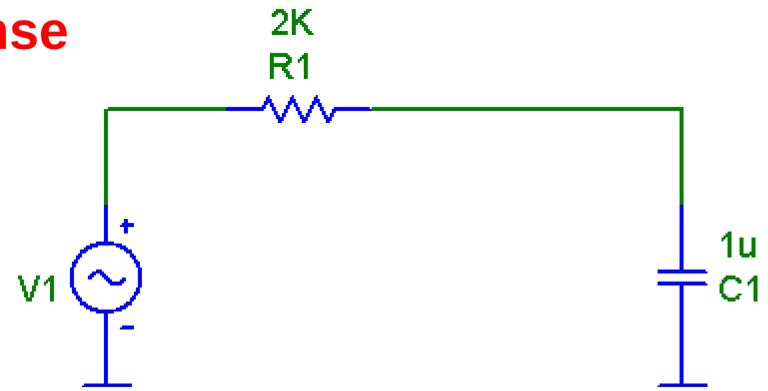
Transient and Forced Response

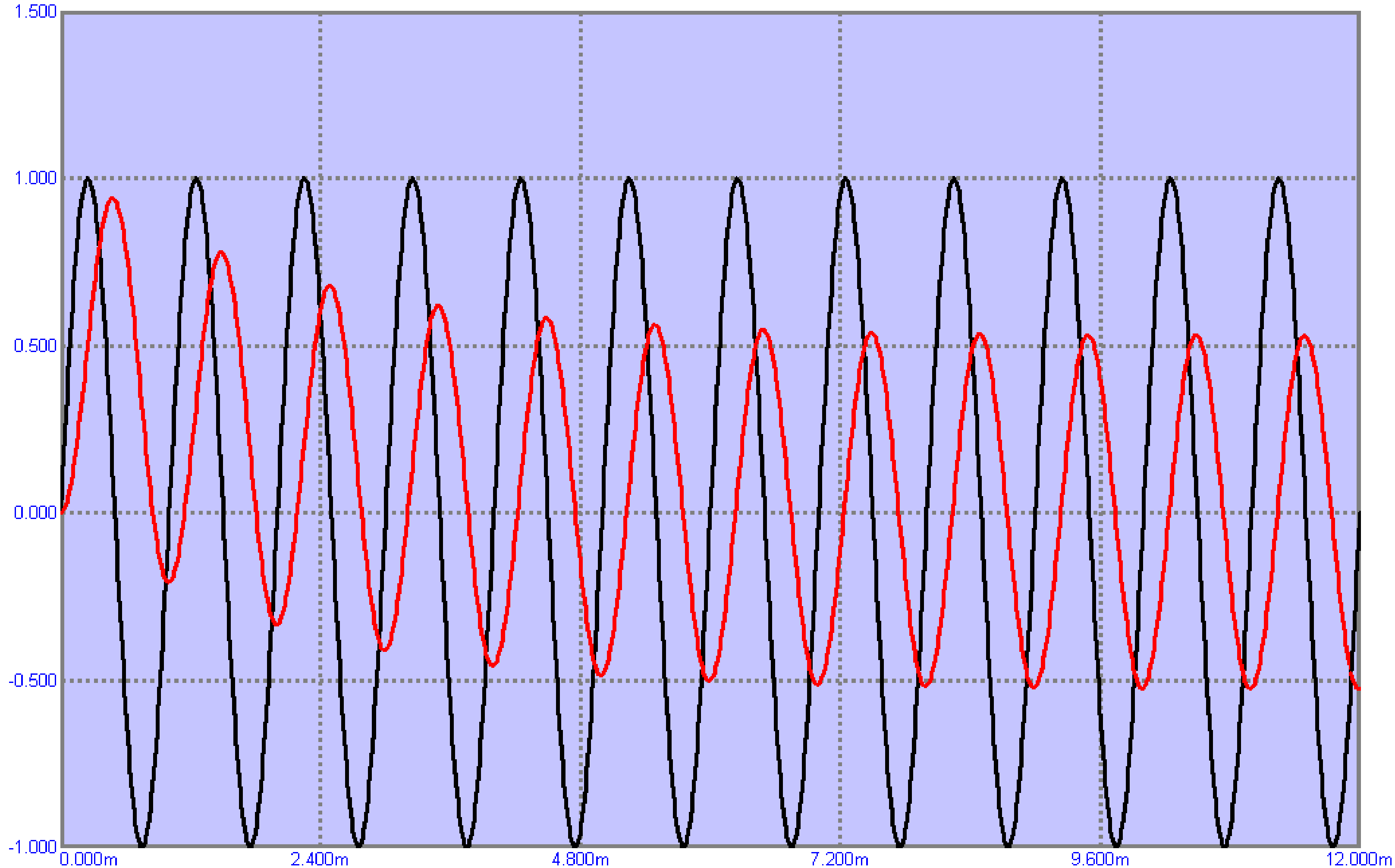


$$v_C(t) = \underbrace{V_s}_{\text{Steady-state or forced response}} - \underbrace{V_s e^{-t/\tau}}_{\text{Transient response}}$$



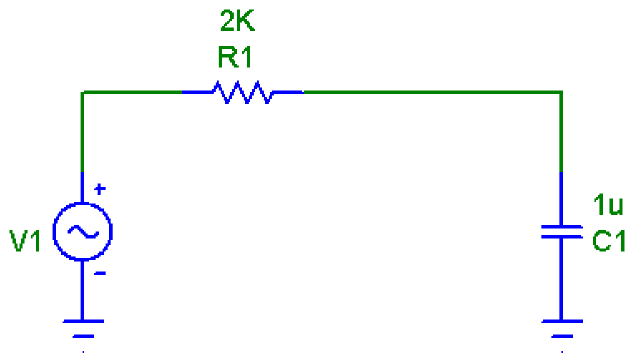
Transient and Forced Response

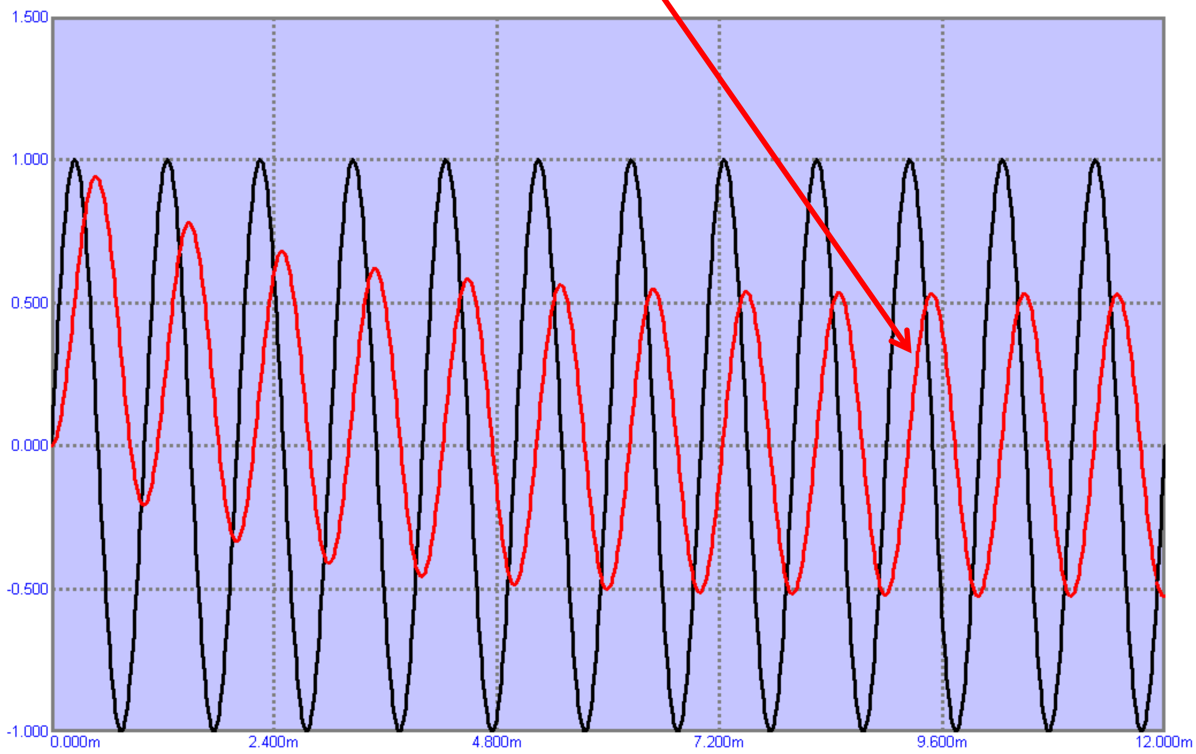




Sinusoidal Steady-State

- Whenever the forced input to the circuit is sinusoidal the response will be sinusoidal
- If the input persists, the response will persist and we call it steady-state response



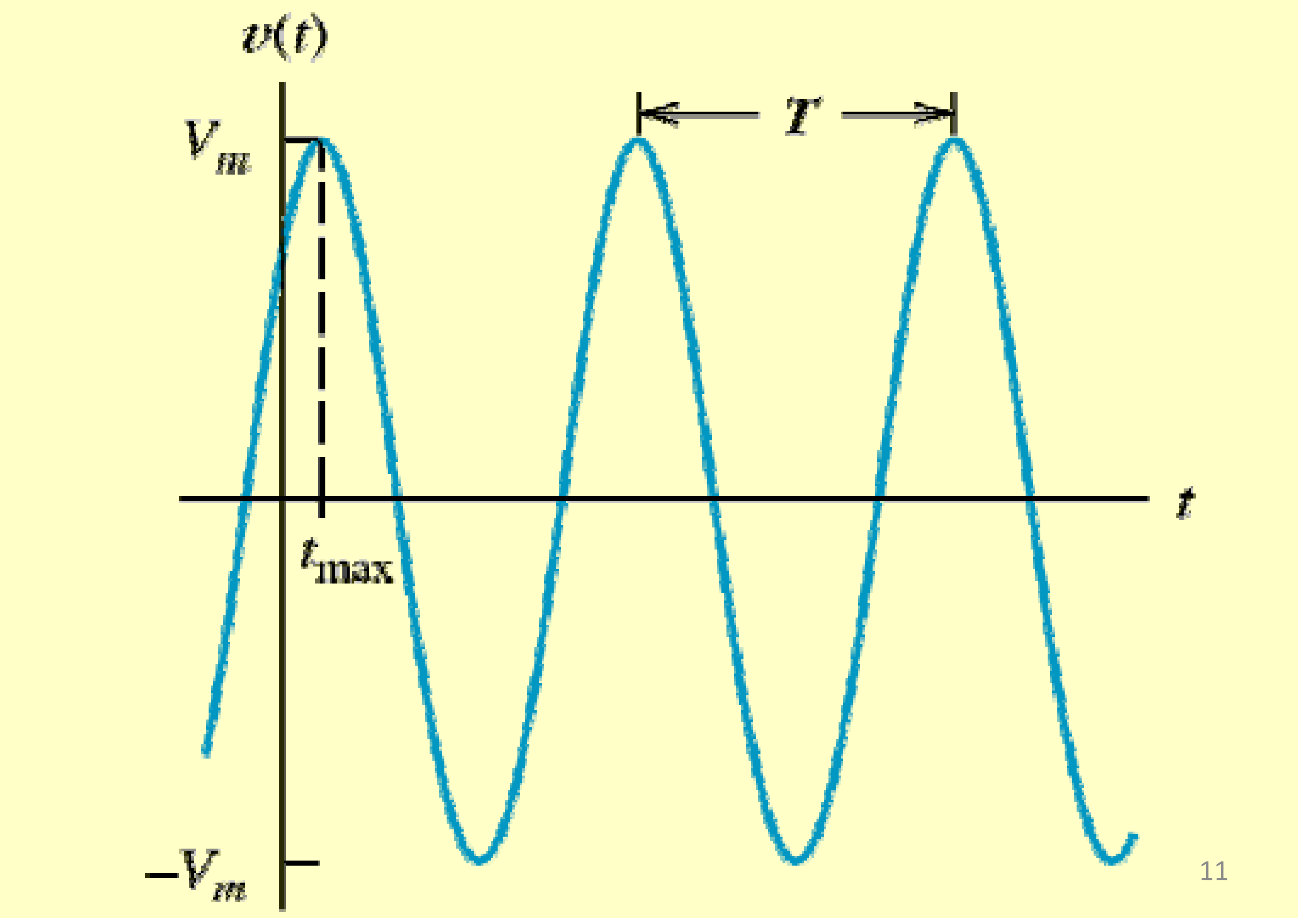


Sinusoidal Currents and Voltages

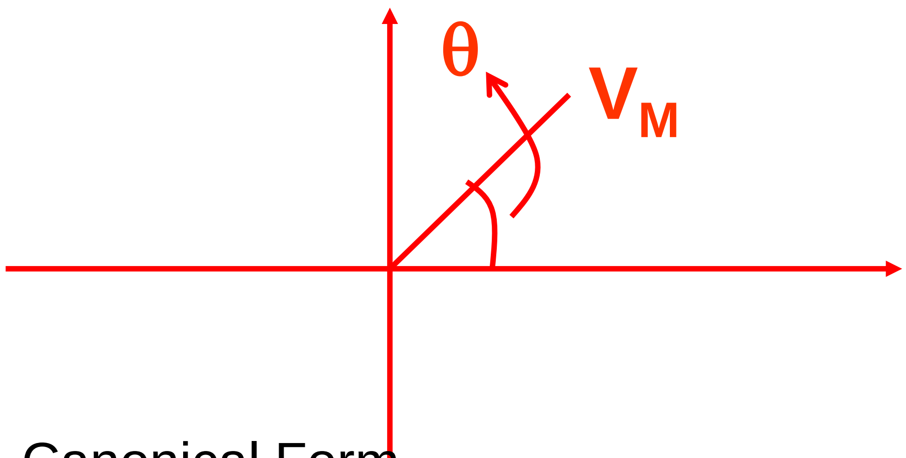
Canonical Form

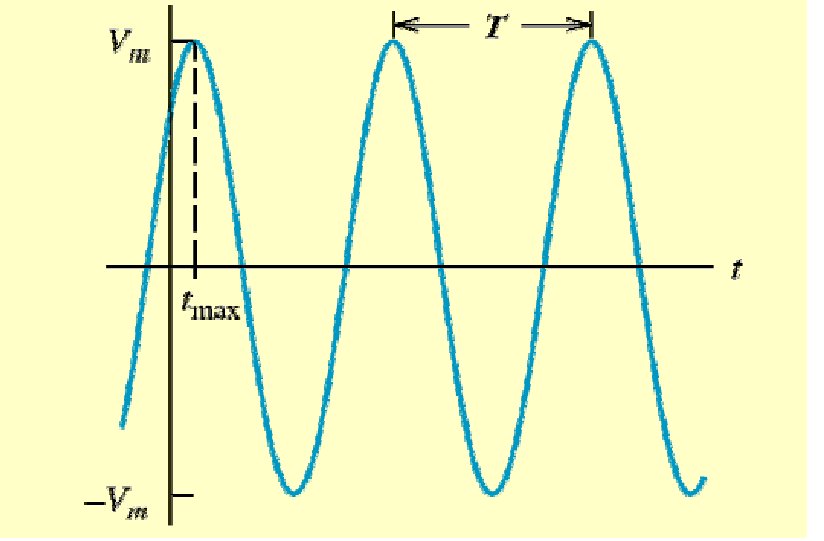
$$v(t) = V_m \cos(\omega t + \theta)$$

V_m is the peak value



Sinusoidal Currents and Voltages





Canonical Form

$$v(t) = V_m \cos(\omega t + \theta)$$

ω is the angular frequency in radians per second

T is the period , where $f = \dfrac{1}{T}$ is the frequency

$\omega = \dfrac{2\pi}{T}$ $\omega = 2\pi f$ θ is the phase angle

Example-1

$$5\sin(4\pi t - 60^o)$$

What is the amplitude, phase, angular frequency, time period, frequency?

$$v(t) = V_m \cos(\omega t + \theta)$$

$$\sin(z) = \cos(z - 90^\circ)$$

$$v(t) = 5\cos(4\pi t - 60^o - 90^o)$$

Amplitude = 5 ; Phase = -150º

Phase in radians:

$360^o = 2\pi$

$$\theta = \frac{-150}{360} \times 2\pi = -2.618 \quad \text{radians}$$

$$\omega = 4\pi \quad r/s$$

$$\omega = \frac{2\pi}{T} = 4\pi \;\Rightarrow T = 0.5s$$

$$f = \frac{1}{T} = 2Hz$$

Example-2 Find the phase difference between the two currents

$$i_1 = 4\sin(377t + 25^o)$$

$$i_2 = -5\cos(377t - 40^o)$$

$$x(t) = x_m \cos(\omega t + \theta)$$ Canonical Form

$$i_1 = 4\cos(377t + 25^o - 90^o)$$

$$\theta_1 = -65^o$$

$$i_2 = 5\cos(377t - 40^o + 180^o)$$

$$\theta_2 = 140^o$$

$$\theta_1 - \theta_2 = -205^o$$

Which signal leads and by how much?

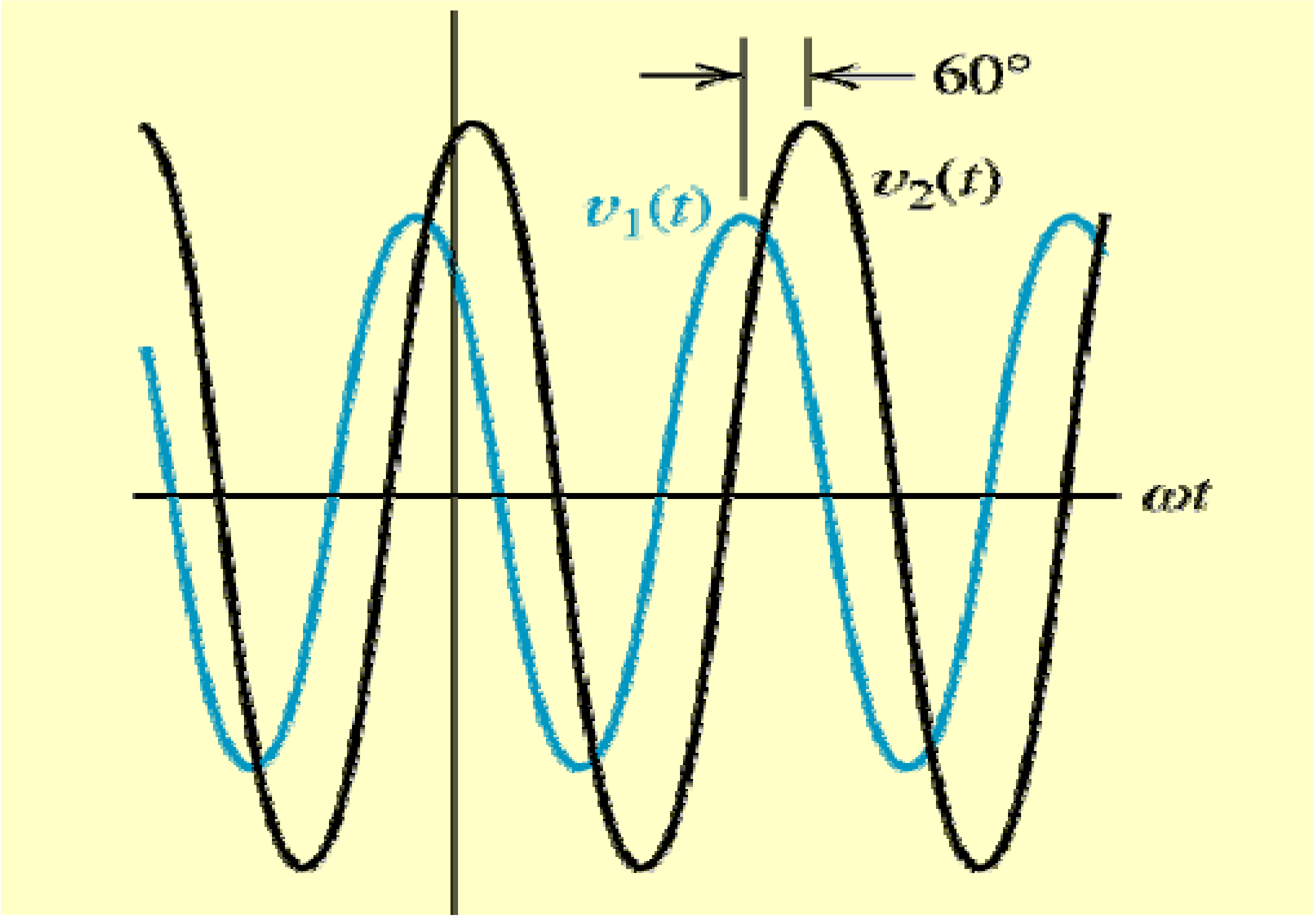
$$\sin(\omega t \pm 180^o) = -\sin \omega t$$

$$\cos(\omega t \pm 180^o) = -\cos \omega t$$

$$\sin(\omega t \pm 90^o) = \pm \cos \omega t$$

$$\cos(\omega t \pm 90^o) = \mp \sin \omega t$$

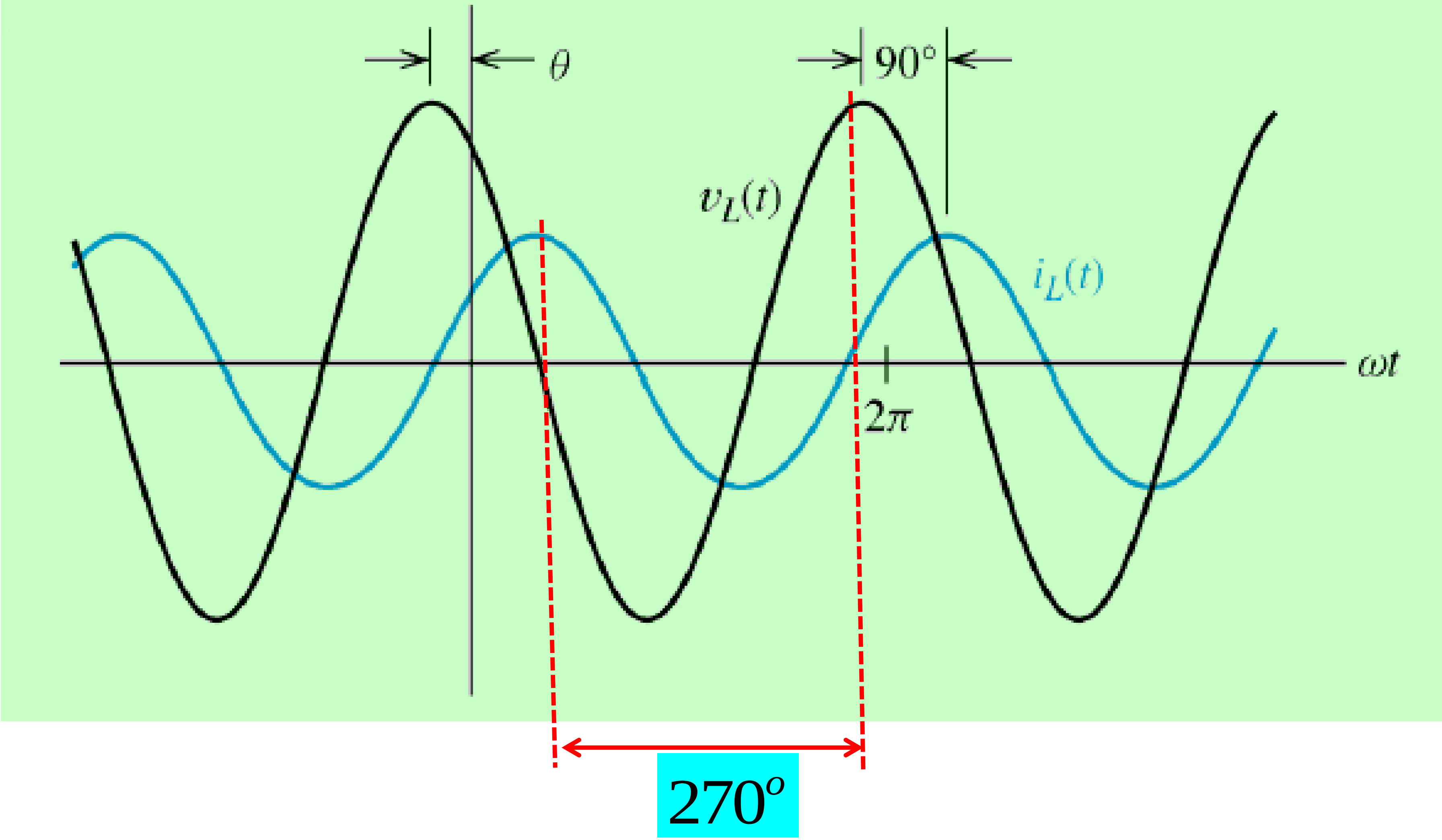
Phase Relationships



$$v_2(t) = v_{2m} \cos(\omega t)$$

$$v_1(t) = v_{1m} \cos(\omega t + 60^o)$$

The peaks of $v_1(t)$ occur 60° before the peaks of $v_2(t)$. In other words, $v_1(t)$ leads $v_2(t)$ by 60°.



Voltage leads current by 90° or lags current by 270° ?

Phase difference is usually considered between -180 to 180°

Add or subtract 360° to bring the phase between -180 to 180°

$$i_1 = 4\cos(377t - 65^o)$$

$$i_2 = 5\cos(377t + 140^o)$$

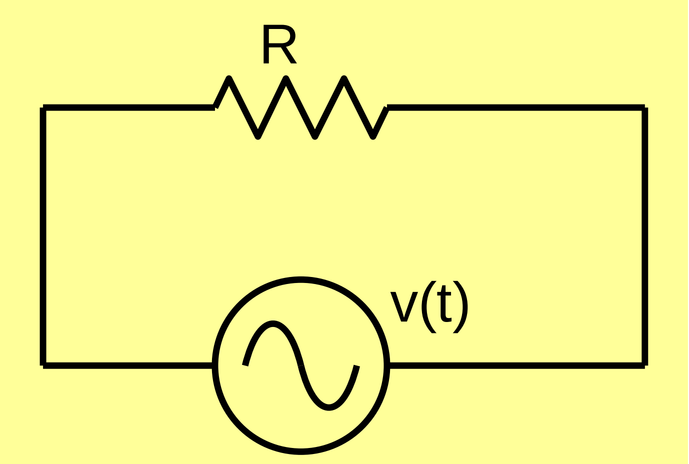
Does i_2 lead i_1 ?

$$\theta_1 - \theta_2 = -205^o$$

$$\theta_1 - \theta_2 = -205^o + 360^o = 155^o$$

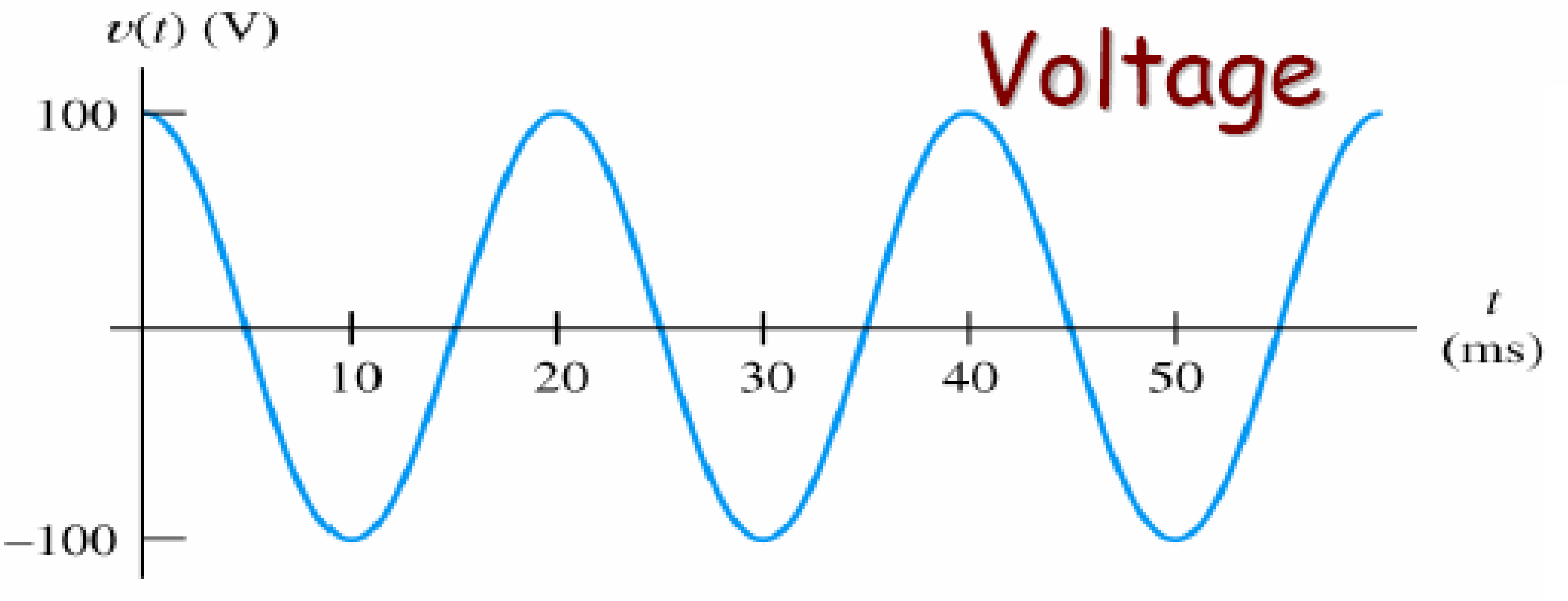
i_1 leads i_2 by 155^o

Power dissipation with sinusoidal Voltage

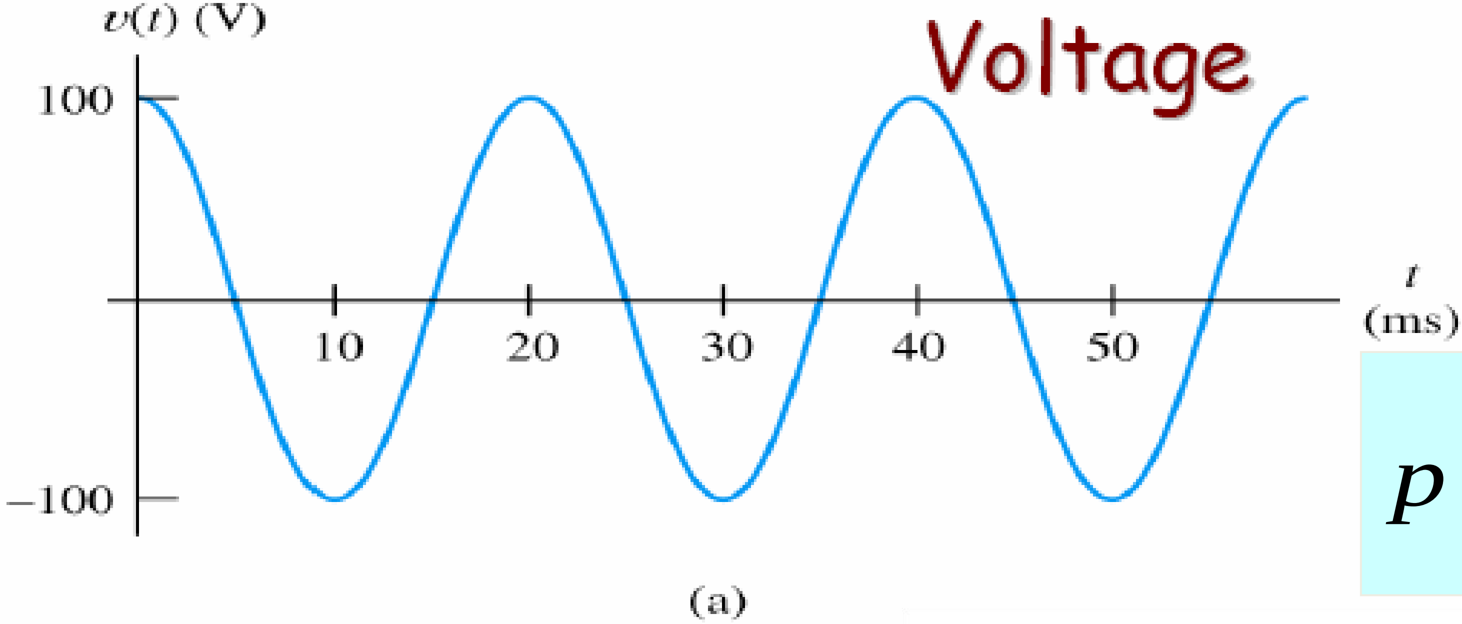


$$p = \frac{v(t)^2}{R}$$

Voltage applied to a 50-Ω resistance

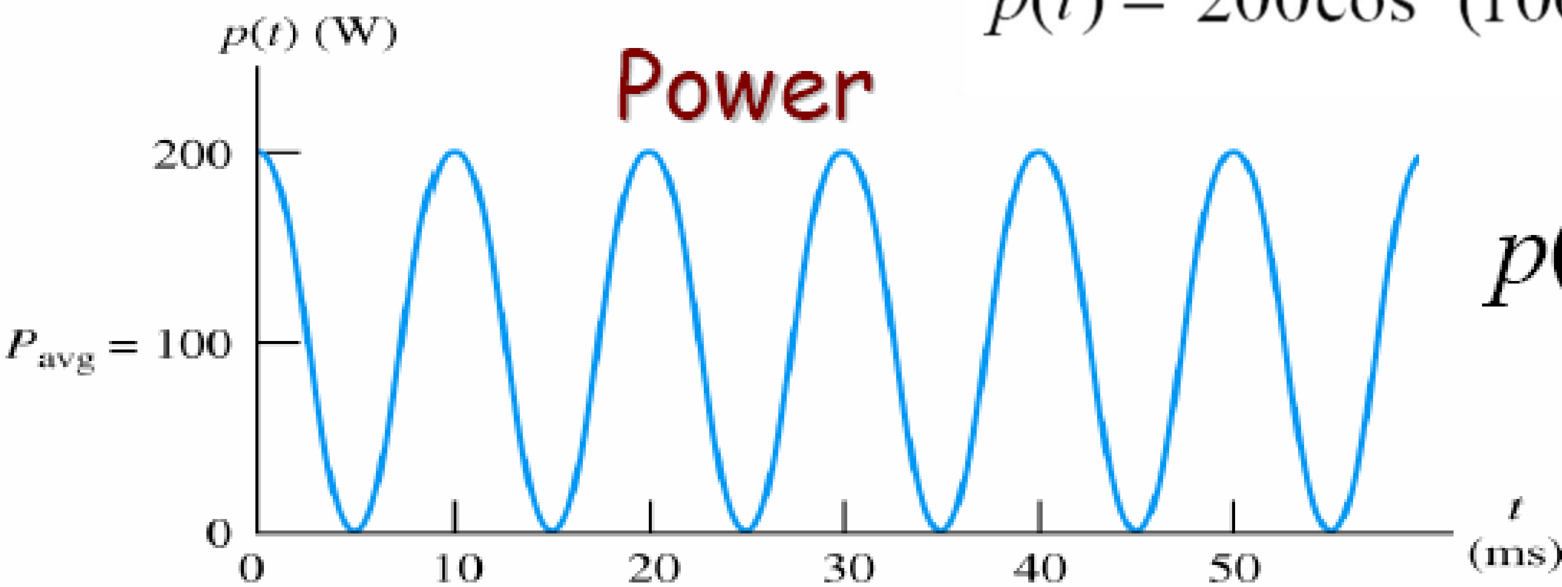


(a)



$$p = \frac{v(t)^2}{R}$$

$$p(t) = 200\cos^2(100\pi t)\ \mathrm{W}$$



$$p(t)$$

Average

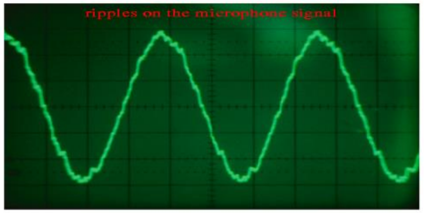
X: $x_1, x_2, x_3, \ldots\ldots\ldots\ldots x_N$

$$x_{avg} = \frac{1}{N}\sum x_i$$

If X is continuous, its average over a time t_1

$$x_{avg} = \frac{1}{t_1}\int_0^{t_1} x(t)\,dt$$

For periodic signals



$$x_{avg} = \frac{1}{T}\int_0^{T} x(t)\,dt$$

Average Power

$$p_{avg} = \frac{1}{T}\int_0^T \frac{v(t)^2}{R}\,dt$$

We would like to express it like the dc power dissipated in a resistor

$$p_{avg} = \frac{\left[\sqrt{\frac{1}{T}\int_0^T v(t)^2\,dt}\right]^2}{R}$$

$$p = \frac{V^2}{R}$$

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v(t)^2\,dt}$$

$$p_{avg} = \frac{V_{rms}^2}{R}$$

This is true for any periodic waveform

RMS Value of a Sinusoid

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v(t)^2\,dt}$$

$$v(t) = V_m \cos(\omega t + \theta)$$

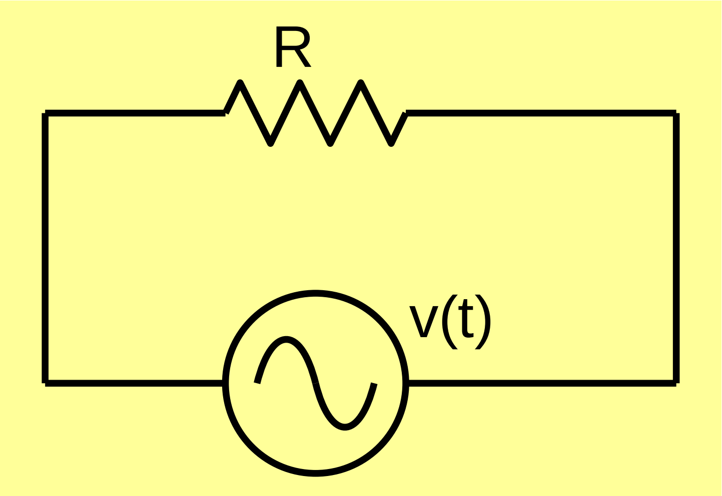
$$\int_0^T \cos^2(\omega t + \theta)\,dt = \int_0^T \frac{1 - \cos(2\omega t + 2\theta)}{2}\,dt$$

$$= 0.5T - \frac{1}{4\omega}\sin(2\omega t + 2\theta)\Big|_0^T = 0.5T$$

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

The RMS value for a sinusoid is the peak value divided by the square root of 2

Power dissipation with sinusoidal Voltage



$$v(t) = V_m \cos(\omega t + \theta)$$

$$p_{avg} = \frac{V_{rms}^2}{R}$$

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

$$p_{avg} = \frac{V_m^2}{2R}$$

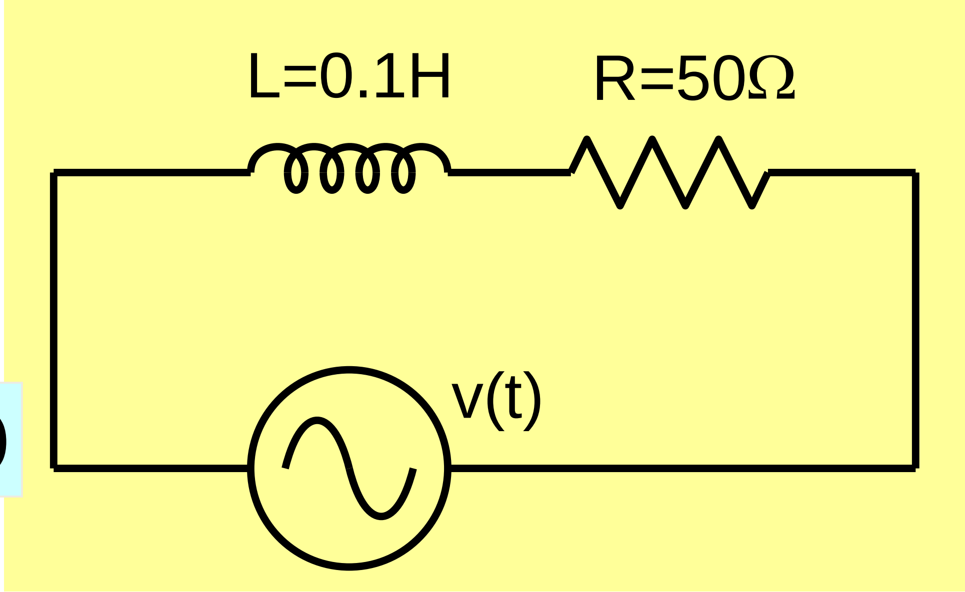
$$i(t) = I_m \cos(\omega t + \theta)$$

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T i(t)^2\, dt}$$

$$I_{rms} = \frac{I_m}{\sqrt{2}}$$

$$p_{avg} = \frac{1}{2} I_m^2 R$$

Example-3



$$v(t) = 2\cos(200t + 45)$$

$$v_R(t) = 1.85\cos(200t + 23.2)$$

$$v_L(t) = v(t) - v_R(t)$$
$$= 2\cos(200t + 45) - 1.85\cos(200t + 23.2)$$

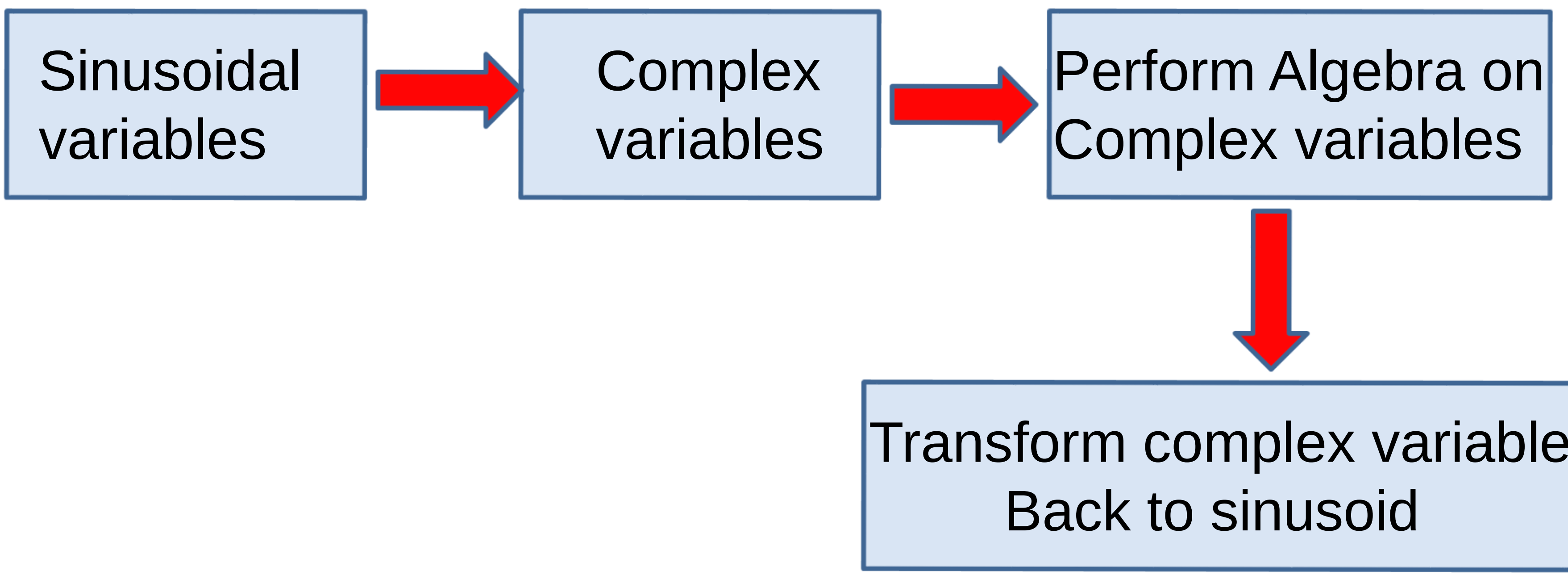
Solving such circuits requires us to add/subtract sinusoids !

Performing algebra on sinusoids by representing them as complex numbers

$$v_1(t) = 20\cos(\omega t - 45^\circ) \qquad v_2(t) = 10\sin(\omega t + 60^\circ)$$

$$v_1(t) + v_2(t) = ?$$

Strategy



$$20\cos(\omega t-45^\circ) \longrightarrow \mathbf{V}_1 = 20\angle -45^\circ$$

$$14.14 - j14.14$$

$$10\sin(\omega t+60^\circ) \longrightarrow \mathbf{V}_2 = 10\angle -30^\circ$$

$$8.660 - j5$$

$$\begin{aligned}
\mathbf{V}_s &= \mathbf{V}_1 + \mathbf{V}_2 \\
&= 20\angle -45^\circ + 10\angle -30^\circ \\
&= 14.14 - j14.14 + 8.660 - j5 \\
&= 23.06 - j19.14 \\
&= 29.97\angle -39.7^\circ
\end{aligned}$$

$$v_s(t) = 29.97\cos(\omega t - 39.7^\circ)$$

Complex Numbers

$$z = x + jy$$

Real part: x

$j = \sqrt{-1}$

Imaginary part: y

$$z_1 = 5 + j5 \qquad z_2 = 3 - j4$$

$$z_1 + z_2 = (5 + j5) + (3 - j4) = 8 + j1$$

$$z_1 - z_2 = (5 + j5) - (3 - j4) = 2 + j9$$

Complex conjugate of z is:

$$z^* = x - jy$$

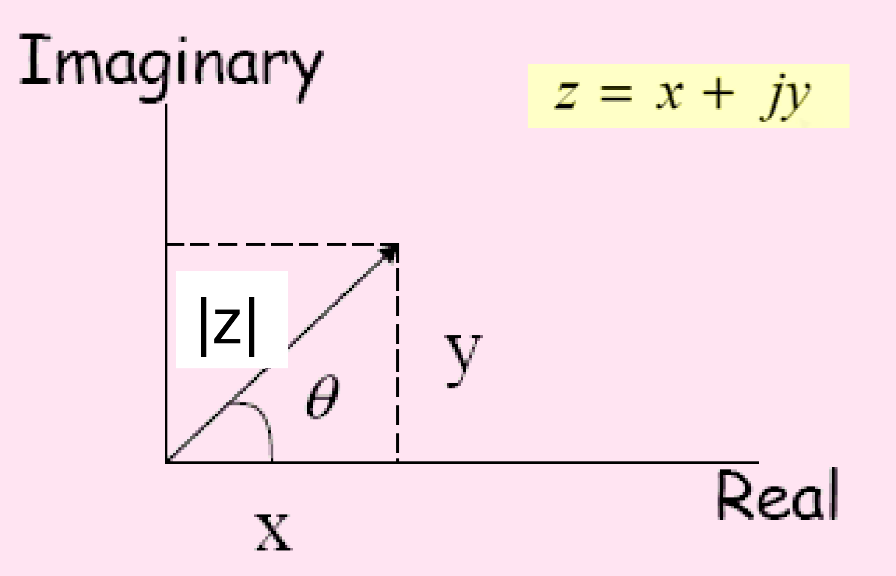
Complex Numbers $z_1 = 5 + j5$ $z_2 = 3 - j4$

$$z_1 z_2 = (5 + j5)(3 - j4)$$
$$= 15 - j20 + j15 - j^2 20$$
$$= 15 - j20 + j15 + 20$$
$$= 35 - j5$$

$$\frac{z_1}{z_2} = \frac{5 + j5}{3 - j4} \times \frac{z_2^*}{z_2^*}$$
$$= \frac{5 + j5}{3 - j4} \times \frac{3 + j4}{3 + j4}$$

$$= \frac{15 + j20 + j15 + j^2 20}{9 + j12 - j12 - j^2 16}$$
$$= \frac{15 + j20 + j15 - 20}{9 + j12 - j12 + 16}$$
$$= \frac{-5 + j35}{25}$$
$$= -\frac{5}{25} + j\frac{35}{25}$$
$$= 0.2 + j1.4$$

A complex number can be represented as a point in the complex Plane



Represent the complex number by the length of the arrow and the angle between the arrow and the positive real axis

$$|z| = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x} \quad \text{or} \quad \theta = \tan^{-1}\frac{y}{x}$$

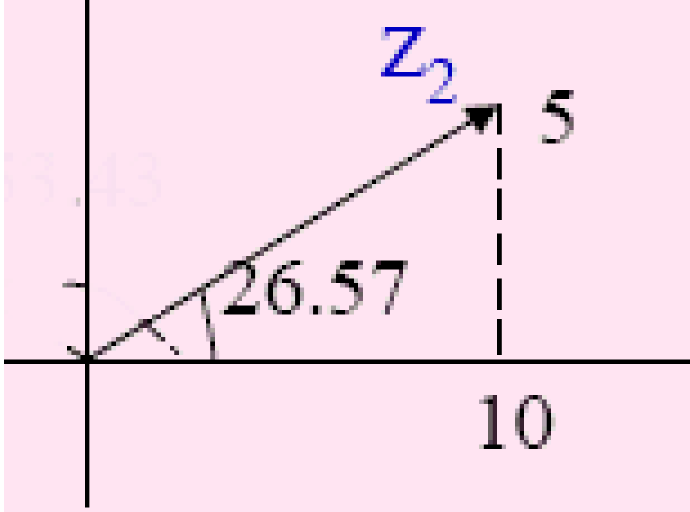
Polar form: $z = |z|\angle\theta$

Rectangular → Polar form

$$z_2 = 10 + j5$$

$$z_2 = \sqrt{(10)^2 + (5)^2} \angle \tan^{-1}\left(\frac{5}{10}\right)$$

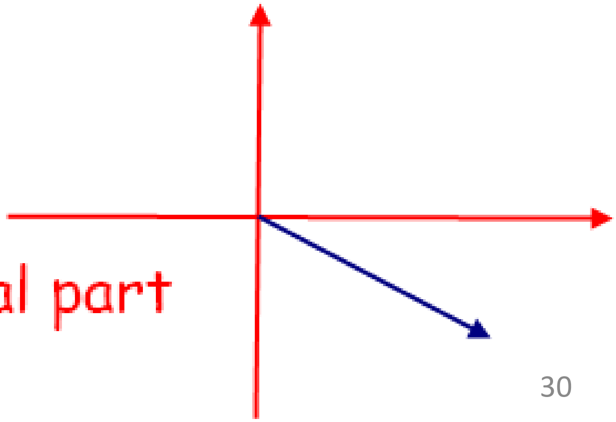
$$= 11.18 \angle 26.57^\circ$$



$$z_3 = -10 + j5$$

$$z_3 = \sqrt{(10)^2 + (5)^2} \angle \tan^{-1}\left(\frac{5}{-10}\right)$$

$$= 11.18 \angle -26.57^\circ$$



Wrong angle since real part is negative;

Rectangular → Polar form:

$$z_3 = -10 + j5$$

$$z_3 = \sqrt{(10)^2 + (5)^2} \angle \tan^{-1}\left(\frac{5}{-10}\right)$$

the true angle is:

$$\theta = \tan^{-1}(y/x) \pm 180^\circ$$

$$= -26.57 + 180 = 153.43^\circ$$

Be careful while determining the phase angle