14.10.2014

Randomized algorithm

algorithm + (toss random coin)

quick sort:

$f($ input, random bits $)$

random variable.

15, 12, 3, 9, 112, 7, 6, 52.

9,
3, 6, 7, | 12 | 15, 52, 112

pivot

3, 52

Sort low to high

3, 9, 6, 7, 12, 15, 52, 112

$$\boxed{T(n) = T(n-1) + \underline{O(n)}}$$

$$\boxed{O(n^2)}$$

Worst case behavior of quicksort :

$$T(n) = 2T(n/2) + \boxed{O(n)}$$

$$\boxed{O(n \log n)}$$

$x_1, x_2, \dots, x_n.$

random permutation of this array.

$z_1, z_2, \dots, z_n.$

How to generate random permutations?

$x_1, \ldots, x_n$

$\underbrace{\qquad\qquad\qquad\qquad}_{n! \text{ options}}$ (each equally likely) (Sampling without replacement)

$i \in \{1, \ldots, \underline{n}\} \to 2^k$

$\to$ k random bits.

x_j

$\underline{\qquad}, j, \underline{\qquad},$

$2^k \leq n < 2^{k+1}$

$[1, 2^k]$

$n < 2^k$

toss until you get a number $[0, 2^k - 1]$.

$> \frac{1}{2}$.

p

$(k+1)$

$\leftarrow \{0, 2^{k+1} - 1\}$

12.

4.

1, ..., 16,

$\{13, 14, 15, 16\}$

$\underbrace{2^k+1, 2^k+2, \ldots, 2^{k+1}-1}$

$> \frac{1}{2}$ fraction $\{0, 2^{k+1} - 1\}$.

Coin with prob $= p$ of seeing a head.

How many coin tosses do I need to see the first head. ?

(Geometric random variable).

$1/p$ samples in expectations?

$\Rightarrow$ roughly ≤ 2 tosses are needed in expectation to get a good number.

~~How~~ to get the second number of permutation?

$$\underbrace{X_1, \ldots, X_{j-1}, \cancel{X_{j+1}}, X_{j+2}, X_{j+3}, X_n}$$

Choose one index from here without replacement.

$$\{1, \ldots, j-1, j, \ldots, n-1\}$$

$j+2$

$$\cancel{\{X_j, X_i, \ldots, X_{n-1}\}}$$

$$\{ z_1, z_2, \cdots, z_{n-1}, z_n \}$$
$$\;\; \| \quad \|$$
$$\{ x_j, x_i, \cdots, \}$$

$\{z_1, \ldots, z_n\}$

very first pivot : z_1

$, z_{i_j}$

$\underbrace{z_{i_1}, z_{i_2}, \ldots, z_{i_j}}_{\leq z_1},\ z_1\ \underbrace{z_{i_{j+2}}, \ldots, z_{i_n}}_{> z_1}$

j

z_{i_1} is the pivot for left part

$z_{i_{j+2}}$ is the pivot for the right part.

$T(n)$: a random variable.

$O(n \log n)$

$O(n^2)$

$E[T(n)] = ?$

$\simeq$ average behavior of a random variable in the limit of infinity.

$\Theta(n \log n)$.

$T(n)$: # comparisons.

Z_{ij} : 1 if i is ever compared with j in the algorithm.

: 0 o/w.

Observations: What pairs are compared?

- either i or j is chosen as a pivot.

Assumptions: no two numbers are same ?

true sorting order: $x_1, \ldots x_i \ldots x_j \ldots, x_n$

smallest ... largest

input order after sorting: $y_1, \ldots \ldots, y_n$.

$$\{x_i, \ldots \ldots, x_j\}$$

What is the probability that x_i & x_j will ever be compared during the algorithm ?

x_k

$$\underbrace{x_1, \ldots\ldots}, \underbrace{x_i, \ldots\ldots, x_j}, \underbrace{\ldots\ldots x_n}$$

<u>Case 1</u>: pivot is either left of x_i or right of x_j.

<u>Case 2</u>: pivot is in between x_i & x_j

<u>Case 3</u>: only case when x_i & x_j will be compared.

Case 3 happens iff

" x_i or x_j is chosen to be the first pivot within the subgroup

$$\{x_i, x_{i+1}, \ldots, x_{j-1}, x_j\}$$

$$\frac{2}{(j-i+1)}$$

$$\{ y_1, \ldots, \bigcirc \ldots \bigcirc, y_n \}$$

$$\qquad \downarrow \qquad\qquad \downarrow$$

$$\qquad x_j \qquad\qquad x_i$$

$Z_{ij} = 1$ with prob. $\dfrac{2}{(j - i + 1)}$

$= 0$ with remaining prob.

$\underline{T(n)}$ $Z = \sum_{i=1}^{n}\sum_{j=i+1}^{n} Z_{ij}$ $\underline{i < j}$

$$\mathbb{E}[Z] = \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=i+1}^{n} Z_{ij}\right]$$

$$= \sum_{i=1}^{n}\sum_{j=i+1}^{n} \mathbb{E}[Z_{ij}]$$

$$= \sum_{i=1}^{n}\sum_{j=i+1}^{n} \frac{2}{(j-i+1)} \rightarrow \underline{n\log n}$$

$$= \sum_{i=1}^{n} \sum_{t=1}^{n-i} \frac{2}{t+1}$$

$j - i = t \quad , \quad r = t+1$

$$\leq 2 \sum_{i=1}^{n} \underbrace{\sum_{t+1=1}^{n} \frac{1}{t+1}}_{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}} \cancel{\leq 2 \sum_{i=1}^{n} \sum_{r=1}^{n}}$$

$$\underbrace{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}}_{\log n +}$$

$$\leq 2 (n \log n)$$