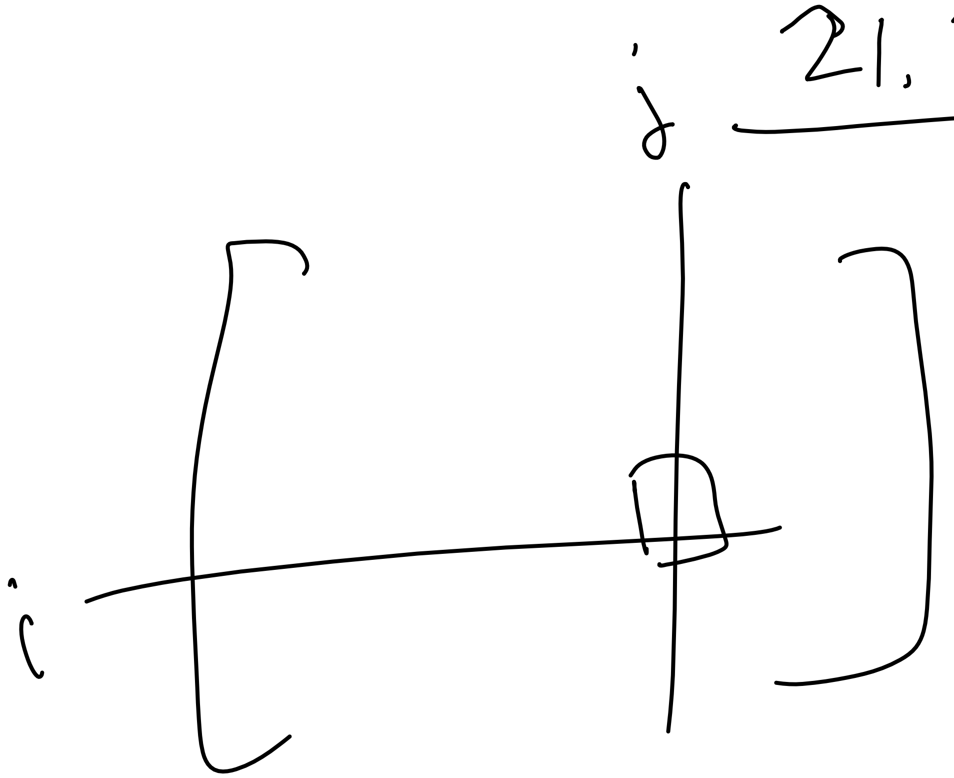
21.10.2024

wt:

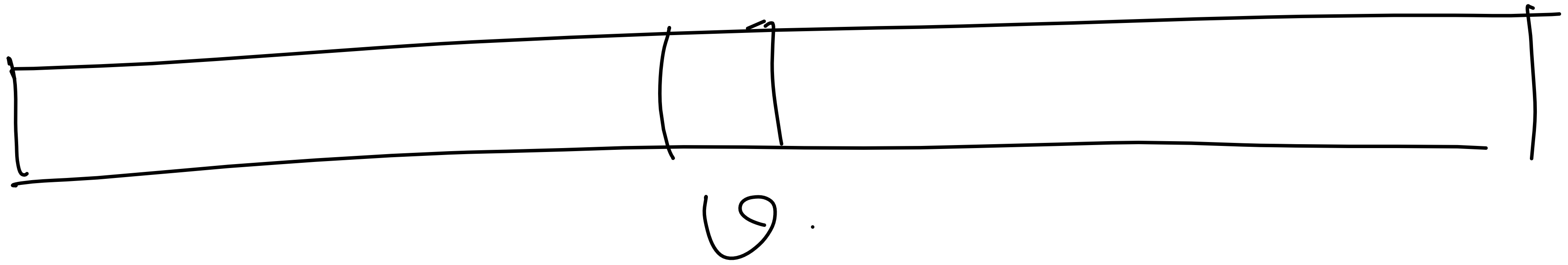


adj. matrix (non-boolean) integer: $0, 1, \ldots, \infty$.

$wt(i,j) \equiv$ count of parallel edges between i & j

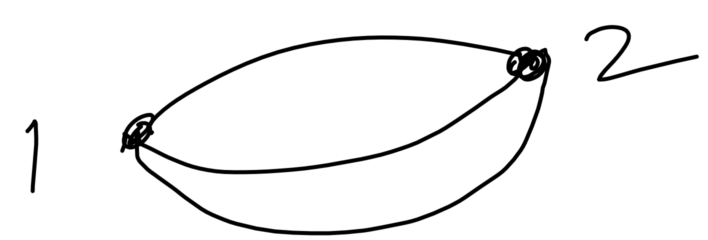
$= wt(j,i)$.

degree array:



$deg(v)$: includes parallel edges

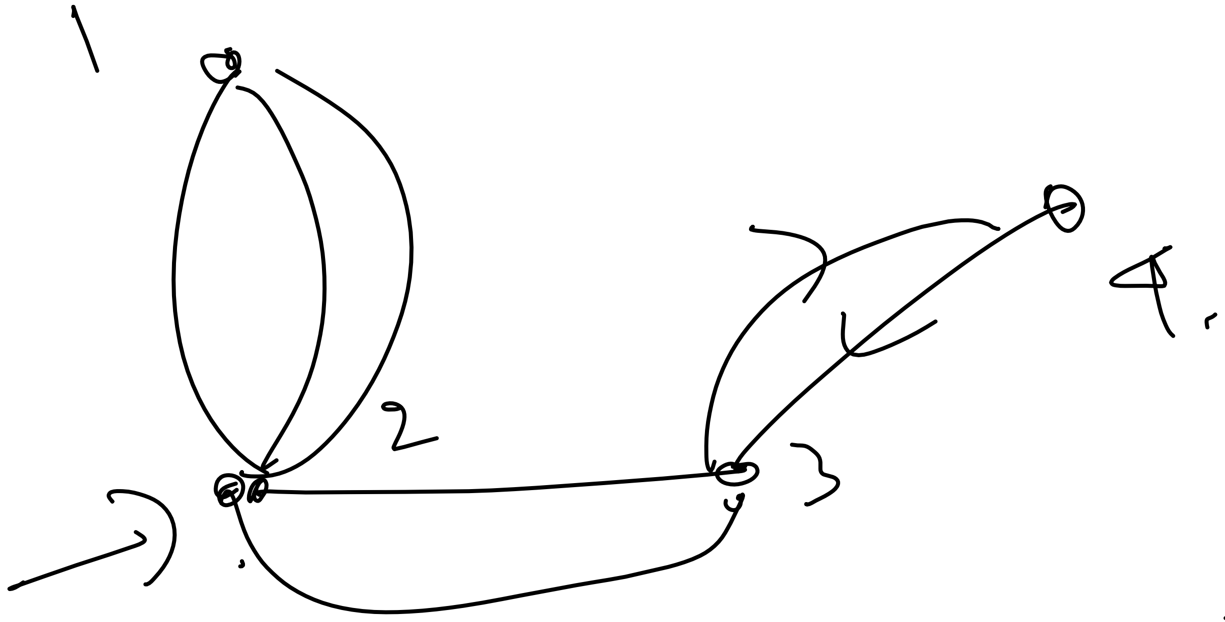
$$\text{wt}: \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 3 \\ 2 & 3 & 0 \end{array}$$



$$\text{deg} \quad \begin{array}{|c|c|} \hline 3 & 3. \\ \hline \end{array}$$
$\quad\quad\quad\quad 1 \quad 2$

$[\,1,\ 2,\ 3,\ 4,\ \ldots\ldots,\ 15,\ 7,\ 9,\,]$

$$\frac{15}{\left(\sum_v \deg_v\right)}$$



5, 3, 31

selected, 2 $\rightarrow$ $\frac{5}{12}$.

select 1 w.p. $\frac{3}{5}$

select 2 w.p. $\frac{2}{5}$

- Select a vertex v w.p. prop. to $\deg(v)$
- select a neighbor w of v w.p. prop. to $\frac{wt(v,w)}{\deg(v)}$.

$$\frac{2}{\sum \deg(v)} = \frac{2}{2 \cdot m} = \frac{1}{m}.$$

$\Rightarrow$ each edge is selected with uniform prob.

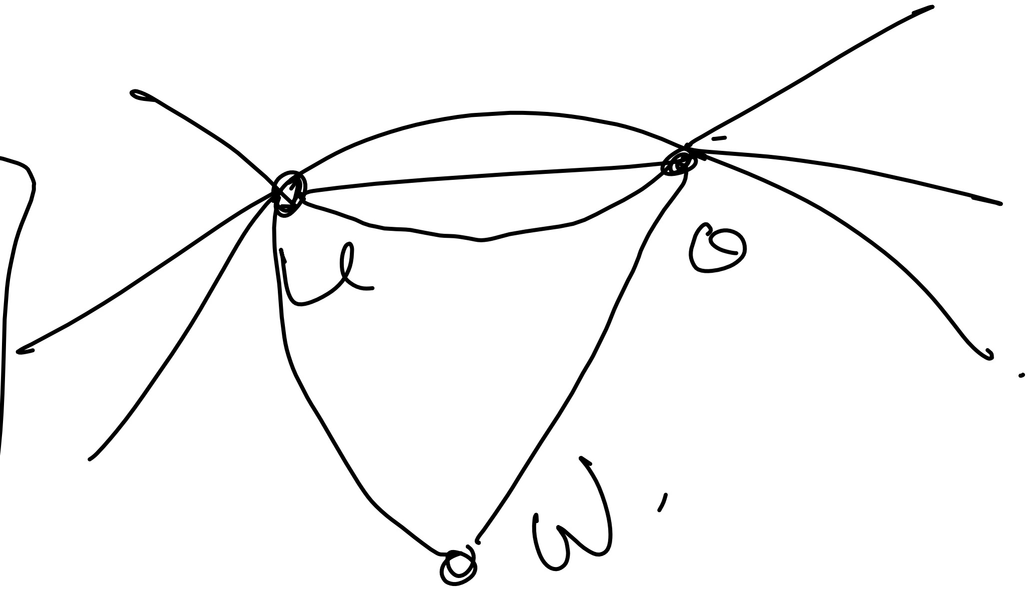
~~for~~ How to ~~perform~~ relax(u, v) ?

w.l.o.g. - u = min(u, v)

- ~~deg~~

$$\underbrace{\deg(u)}_{\text{new}} \leftarrow \deg(u) + \deg(v) - 2\,wt(u,v)$$

$$\deg(v) \leftarrow 0.$$



for every $w \in V$

$$wt(u,w) \leftarrow wt(u,w) + wt(v,w).$$

$$wt(w,u) \leftarrow wt(u,w).$$

$$wt(v,w) \leftarrow 0.$$

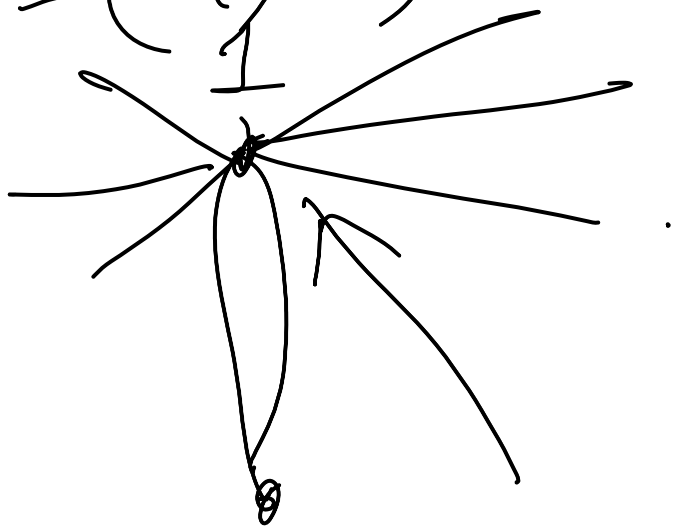
$$wt(u,v) \leftarrow 0.$$

(1, 5)

u or ~~v~~

~~(u, v)~~

1



rename this vertex as min(u, v).

Time Complexity

$n = \#$ vertices.

$$\underbrace{O(n)}_{\text{update}} \cdot \underbrace{O(n)}_{\#\text{ iterations}} \; O(\underbrace{n^2 \log 1/\delta}_{\text{repeating}}) \equiv O\left(n^4 \log 1/\delta\right).$$

Correctness

$(1-\delta)$ prob. you'll return the min-cut correctly.

David Karger.

Karger's min cut algorithm.

- Can we avoid randomness ?

Max flow min-cut :

Randomized Algorithms

1. output always correct
running time is a random variable

(Las-Vegas algorithm).

- quicksort.

2. running time is precise.
Output is a random variable.

(Monte Carlo algorithm).

- Karger's

Polynomial Identity Testing

$$P(x_1, x_2, x_3)$$

$$= -2x_3^2(x_1+x_2)(x_1-x_2) - x_3^2$$

$$+\left(x_1^2 + (1+x_2)(1-x_2)\right)x_3^2.$$

$$\equiv 0.$$

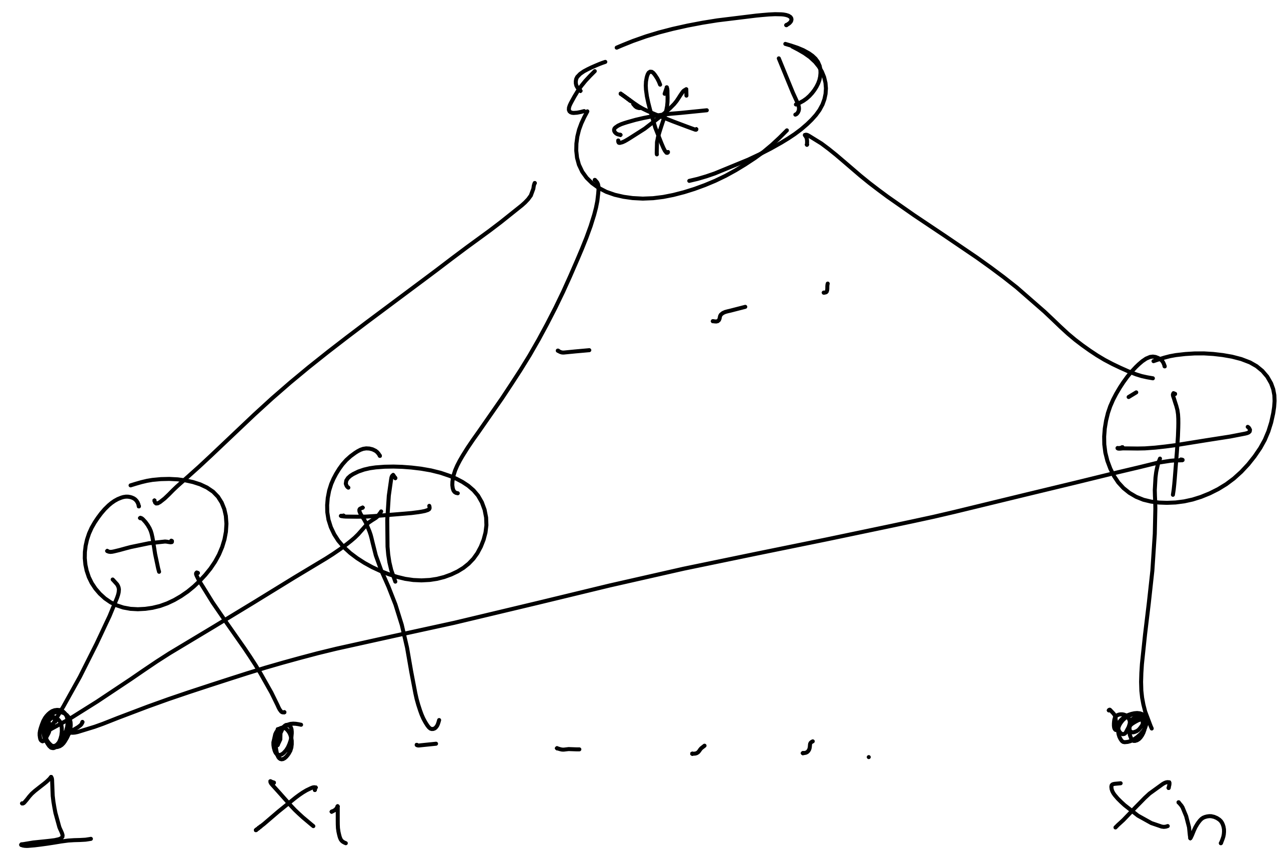
Input: - evaluation is easy
- expansion is difficult.

$(1+x_1)(1+x_2) \cdots (1+x_n).$

$\downarrow$

2^n terms in this expansion.

algebraic Circuit:



$O(n)$

$\underline{Deg(P)}$: the maximum degree of any monomial in the polynomial.

$$p(x) = \underbrace{x^3} + \underbrace{3x^2} + \underbrace{4x} + \underbrace{5}$$

$$\deg \Big(\underbrace{x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}}_{\text{monomial}} \Big) = d_1 + d_2 + \cdots + d_n.$$

$$\deg \Big(\underbrace{x_1^3 x_2}_{4} + \underbrace{x_2^2 x_1}_{3.} \Big) = 4.$$

Assumptions: $P(x_1 \cdots x_n)$ is of degree $\le d$.

How much will the trivial algorithm take?

$\equiv$ How many monomials can P have?

$$x_1^{d_1} \cdots x_n^{d_n} \; : \; d_1 + d_2 \cdots + d_n \le d.$$

$$* \mid * \mid \; * \mid\mid \; \cdots \mid\mid *$$

$\binom{d+n-1}{d}$ many monomials. $\gtrsim n^d$.

$d = n/2$ the ~~time~~ complexity can be as large as $\binom{2n}{n}$

Procedure PIT.

Input: $P(x_1, \ldots, x_n)$ given as a circuit

Output: yes: if $P \equiv 0$

no: o/w.

1. Choose a set S of inputs large enough.

2. Choose a random input from S:
$$(r_1, \ldots, r_n)$$

3. Evaluate $P(r_1, \ldots, r_n)$

- if $= 0$: yes.
- if $\neq 0$: $P \not\equiv 0$.

Obs: If $P \equiv 0$ then the algo. is always correct.

Remark: The problem is that $P(r_1, \ldots, r_n) = 0$ even if $P \not\equiv 0$.

- $(r_1, \ldots, r_n)$ is a root of this polynomial P.

We'll show if $|S|$ is large enough the above event happens with small prob.

Claim: Let $P \not\equiv 0$.

$$\Pr[P(r_1 \ldots r_n) = 0] \leq \frac{\deg(P)}{|S|}.$$

Proof: On Induction on n, deg. remains fixed.

Base: $(n=1)$.

roots P can have $\leq \deg(P)$.

Therefore, prob that we select a root $\leq \frac{\deg(P)}{|S|}$

$$\mathbb{R} \setminus \{\text{roots}\}.$$

I.H. This fact is true for $(n-1)$ variate polynomials.

I.S. We'll show Fact also hold for n variate.

w.l.o.g. let x_1 be a variable that contributes to a non-zero monomial. $\deg(P)$

$$P(x_1, \dots, x_n)$$

$$= \boxed{x_1^K Q(x_2, \dots, x_n)} + \underbrace{R(x_1, x_2, \dots, x_n)}_{x_1^{<K}}$$

$$(r_1, r_2, \dots, r_n) \xrightarrow[\text{at}]{\text{look}} (r_2, \dots, r_n)$$

$p(x_1, r_2, \ldots r_n) = \deg(p)$,

$= \boxed{x_1^k Q(r_2 \ldots r_n)} + R(x_1; r_2 \ldots, r_n)$.

Q is a polynomial : $(n-1)$ variables.

$\Rightarrow \Pr[Q(r_2 \ldots r_n) = 0] \leq \frac{\deg(Q)}{|S|}$.

Event : E

Condion on the event that $\overline{E}$ happens.

Therefore, $P(x_1, r_2, \ldots, r_n)$ is a univariate polynomial of degree $k \geq 1$ in x_1

Now plug back r_1

$$\underbrace{\Pr\left[P(r_1, r_2, \ldots, r_n) = 0\right]}_{F \mid E} \leq \frac{k}{|S|}$$

$$\leq \frac{\deg(P) - \deg(g)}{|S|}$$

$$K + \deg(Q) \leq \deg(P).$$

$$K \leq \deg(P) - \deg(Q).$$