(1) $x_1, \ldots, x_n$ i.i.d. $N(\mu, \sigma^2)$

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$\log f = K - \frac{1}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$\frac{\partial \log f}{\partial \mu} = \frac{(x-\mu)}{\sigma^2} \; ; \quad \frac{\partial^2 \log f}{\partial \mu^2} = -\frac{1}{\sigma^2}.$$

$$-E\left(\frac{\partial^2 \log f}{\partial \mu^2}\right) = \frac{1}{\sigma^2} = I(\mu)$$

CRLB for an u.e. for $\mu = \frac{\sigma^2}{n}$.

Since $V(\bar{X}) = \frac{\sigma^2}{n}$; $\bar{X}$ attains CRLB.

$$\frac{\partial \log f}{\partial \sigma^2} = -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}(x-\mu)^2$$

$$\frac{\partial^2 \log f}{\partial (\sigma^2)^2} = \frac{1}{2\sigma^4} - \frac{(x-\mu)^2}{\sigma^6}.$$

$$I(\sigma^2) = -E\left(\frac{\partial^2 \log f}{\partial (\sigma^2)^2}\right) = -\frac{1}{2\sigma^4} + \frac{1}{\sigma^4} = \frac{1}{2\sigma^4}.$$

CRLB for an u.e. for $\sigma^2 = \frac{2\sigma^4}{n}$.

Now $S^2 = \frac{1}{n-1}\sum_1^n (X_i - \bar{X})^2$ is UMVUE for σ^2 with

$$V(S^2) = \frac{2\sigma^4}{n-1} > \text{CRLB}$$

Since UMVUE is the unbiased est with lowest variance in the class of all unbiased estimators, CRLB can't be attained by any unbiased estimator of σ^2.

(2) $f(x|\beta) = \dfrac{1}{\Gamma\alpha \; \beta^{\alpha}} e^{-x/\beta} x^{\alpha-1}$; $x > 0$

$$\log f = -\log\Gamma\alpha - \alpha\log\beta - \frac{x}{\beta} + (\alpha-1)\log x$$

$$\frac{\partial \log f}{\partial \beta} = -\frac{\alpha}{\beta} + \frac{x}{\beta^2}$$

$$\frac{\partial^2 \log f}{\partial \beta^2} = \frac{\alpha}{\beta^2} - 2\frac{x}{\beta^3}.$$

$$I(\beta) = -E\left(\frac{\partial^2 \log f}{\partial \beta^2}\right) = -\frac{\alpha}{\beta^2} + 2\frac{\alpha\beta}{\beta^3} = \frac{\alpha}{\beta^2}.$$

$\Rightarrow$ CRLB for u.e. of β : $\dfrac{1}{n.\frac{\alpha}{\beta^2}} = \dfrac{\beta^2}{n\alpha}$.

(3) $X_1, \ldots X_n$ i.i.d. $P(\theta)$

$$f(x|\theta) = \frac{e^{-\theta}\theta^{x}}{x!}$$

$$\log f(x|\theta) = -\theta + x\log\theta - \log x!$$

$$\frac{\partial \log f}{\partial \theta} = -1 + \frac{x}{\theta} \; ; \; \frac{\partial^2 \log f}{\partial \theta^2} = -\frac{x}{\theta^2}.$$

$$I(\theta) = -E\left(\frac{\partial^2 \log f}{\partial \theta^2}\right) = \frac{1}{\theta}$$

CRLB for any u.e. of θ : $\dfrac{1}{n\frac{1}{\theta}} = \dfrac{\theta}{n}$

CRLB for any u.e. of $g(\theta) = \theta^2$: $\dfrac{(2\theta)^2}{\frac{n}{\theta}} = \dfrac{4\theta^3}{n}$

CRLB for any u.e. of $g(\theta) = e^{-\theta}$: $\dfrac{(-e^{-\theta})^2}{\frac{n}{\theta}} = \dfrac{\theta e^{-2\theta}}{n}$.

(4) $X_1, \ldots, X_n$ i.i.d $B(1,\theta)$

$$f(x|\theta) = \theta^x (1-\theta)^{1-x}$$

$$\log f(x|\theta) = x \log\theta + (1-x)\log(1-\theta)$$

$$\frac{\partial \log f}{\partial \theta} = \frac{x}{\theta} + \frac{(1-x)}{1-\theta}(-1) = \frac{x}{\theta(1-\theta)} - \frac{1}{1-\theta}.$$

$$I(\theta) = E\left(\frac{\partial \log f}{\partial \theta}\right)^2 = V\left(\frac{\partial \log f}{\partial \theta}\right) = \frac{\theta(1-\theta)}{(\theta(1-\theta))^2} = \frac{1}{\theta(1-\theta)}.$$

CRLB for u.e. of θ^4: $$\frac{(4\theta^3)^2}{n \cdot \frac{1}{\theta(1-\theta)}} = \frac{16\,\theta^7(1-\theta)}{n}$$

CRLB for u.e. of $\theta(1-\theta)$: $$\frac{(1-2\theta)^2}{n \cdot \frac{1}{\theta(1-\theta)}} = \frac{(1-2\theta)^2\,\theta(1-\theta)}{n}.$$

(5) $X_1, \ldots, X_n$ i.i.d. $U(0,\theta)$

$$P[|X_{(n)} - \theta| \geq \epsilon] \leq \frac{E(X_{(n)}-\theta)^2}{\epsilon^2} = \frac{E X_{(n)}^2 + \theta^2 - 2\theta E X_{(n)}}{\epsilon^2}$$

$$f_{X_{(n)}}(x) = \begin{cases} \frac{n}{\theta^n} x^{n-1}, & 0 < x < \theta \\ 0, & \text{o/w.} \end{cases}$$

$$E X_{(n)} = \frac{n}{\theta^n}\int_0^\theta x^n\,dx = \frac{n}{n+1}\cdot\theta$$

$$E X_{(n)}^2 = \frac{n}{\theta^n}\int_0^\theta x^{n+1}\,dx = \frac{n}{n+2}\,\theta^2$$

$$\Rightarrow P[|X_{(n)}-\theta| \geq \epsilon] \leq \frac{1}{\epsilon^2}\left[\frac{n}{n+2}\theta^2 + \theta^2 - 2\theta\frac{n}{n+1}\theta\right] \to 0 \text{ as } n \to \infty$$

$$\Rightarrow X_{(n)} \xrightarrow{P} \theta$$

$\Rightarrow \frac{n}{n+1} X_{(n)} \xrightarrow{p} \theta$

$\Rightarrow \frac{n}{n+1} X_{(n)}$ is a consistent estimator for θ

Further since $X_{(n)} \xrightarrow{p} \theta$

$$e^{X_{(n)}} = g\,(X_{(n)}) \xrightarrow{p} g(\theta) = e^{\theta}$$

$\Rightarrow e^{X_{(n)}}$ is a consistent estimator for e^{θ}

(6) $X_1 \ldots X_n$ i.i.d $U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$

$$F_X(x) = \int_{\theta-\frac{1}{2}}^{x} dx = (x - \theta + \tfrac{1}{2})$$

$$f_{X_{(1)}}(x) = n\,(1 - F_X(x))^{n-1} f(x)$$

i.e $$f_{X_{(1)}}(x) = \begin{cases} n\,(\theta - x + \frac{1}{2})^{n-1}, & \theta - \frac{1}{2} \le x \le \theta + \frac{1}{2} \\ 0, & o/w. \end{cases}$$

$$E\,X_{(1)} = n \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} x\,(\theta - x + \tfrac{1}{2})^{n-1} dx = \theta + \tfrac{1}{2} - \frac{n}{n+1}$$

$$E\,X_{(1)}^2 = n \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} x^2 (\theta - x + \tfrac{1}{2})^{n-1} dx$$

$$= (\theta + \tfrac{1}{2})^2 + \frac{n}{n+2} - \frac{n}{n+1}(2\theta + 1)$$

$$P\left[\left|X_{(1)} - \left(\theta - \tfrac{1}{2}\right)\right| \geq \epsilon\right] \leq \frac{E\left(X_{(1)} - \left(\theta - \frac{1}{2}\right)\right)^2}{\epsilon^2}$$

r.h.s $= \frac{1}{\epsilon^2}\left[E(X_{(1)}^2) + \left(\theta - \tfrac{1}{2}\right)^2 - 2\left(\theta - \tfrac{1}{2}\right)E(X_{(1)})\right]$

$$= \frac{1}{\epsilon^2}\left[\left\{\left(\theta + \tfrac{1}{2}\right)^2 + \frac{n}{n+2} - \frac{n}{n+1}(2\theta + 1)\right\} + \left(\theta - \tfrac{1}{2}\right)^2 - 2\left(\theta - \tfrac{1}{2}\right)\left(\theta + \tfrac{1}{2} - \frac{n}{n+1}\right)\right]$$

$$\rightarrow \frac{1}{\epsilon^2}\left[\left\{\left(\theta + \tfrac{1}{2}\right)^2 + 1 - (2\theta + 1)\right\} + \left(\theta - \tfrac{1}{2}\right)^2 - 2\left(\theta - \tfrac{1}{2}\right)\left(\theta - \tfrac{1}{2}\right)\right]$$

as $n \to \infty$

$= 0$

$\Rightarrow P\left[\left|X_{(1)} - \left(\theta - \tfrac{1}{2}\right)\right| \geq \epsilon\right] \longrightarrow 0$ as $n \to \infty$

$\Rightarrow X_{(1)} \xrightarrow{p} \theta - \frac{1}{2}$ — (1)

We can similarly prove that

$X_{(n)} \xrightarrow{p} \theta + \frac{1}{2}$ — (2)

Combining (1) & (2), we get.

$$\frac{X_{(1)} + X_{(n)}}{2} \xrightarrow{p} \theta$$

$\Rightarrow \frac{X_{(1)} + X_{(n)}}{2}$ is a consistent estimator for θ

Also,

$X_{(1)} + \frac{1}{2}$ is a consistent estimator for θ (from (1))

& $X_{(n)} - \frac{1}{2}$ is a consistent estimator for θ (from (2)).

(7) $X_1, \ldots X_n$ i.i.d. $f_X(x) = \begin{cases} \frac{1}{2}(1+\theta x), & -1 < x < 1 \\ 0, & o/w. \end{cases}$

$$E(X) = \frac{1}{2}\int_{-1}^{1}(1+\theta x)\,dx = \frac{\theta}{3}$$

$\Rightarrow X_1, \ldots X_n$ are i.i.d. with $E(X_1) = \frac{\theta}{3}$

By Khintchine's WLLN

$$\frac{1}{n}\sum_{1}^{n} X_i \xrightarrow{p} E(X_1)$$

i.e. $\bar{X} \xrightarrow{p} \frac{\theta}{3} \Rightarrow 3\bar{X} \xrightarrow{p} \theta$

$\Rightarrow 3\bar{X}$ is a consistent estimator for θ.

(8) $X_1, \ldots X_n$ are i.i.d. $P(\theta)$

$E(X_i) = \theta \quad \forall i = 1(1)n$

By WLLN $\bar{X}_n \xrightarrow{p} \theta$

$$\Rightarrow g(\bar{X}_n) \xrightarrow{p} g(\theta)$$

$$\Rightarrow \bar{X}_n^3(3\sqrt{\bar{X}_n} + \bar{X}_n + 12) \xrightarrow{p} \theta^3(3\sqrt{\theta} + \theta + 12)$$

$\Rightarrow \bar{X}_n^3(3\sqrt{\bar{X}_n} + \bar{X}_n + 12)$ is a consistent estimator for $\theta^3(3\sqrt{\theta} + \theta + 12)$.

(9) $X_1, \ldots, X_n$ are i.i.d. $G(\alpha, \beta)$

α is known constant and $\beta > 0$ is unknown

By WLLN

$$\frac{1}{n}\sum_{1}^{n} X_i \xrightarrow{p} E(X_1) \left(= \alpha\beta\right)$$

i.e. $\frac{1}{n}\sum_{1}^{n} X_i \xrightarrow{p} \alpha\beta$

i.e. $\frac{1}{n\alpha}\sum_{1}^{n} X_i \xrightarrow{p} \beta$

$\Rightarrow \frac{1}{n\alpha}\sum_{i=1}^{n} X_i$ is a consistent estimator for β.