

MSO201A: PROBABILITY & STATISTICS

Problem Set #9

- [1] Let $\{X_n\}$ be a sequence of random variables with $E(X_n) \rightarrow c$ and $V(X_n) \rightarrow 0$ as $n \rightarrow \infty$.

Show that $X_n \xrightarrow{p} c$.

- [2] Let $\{X_n\}$ be a sequence of random variables with $E(X_n) = \mu_n$ and finite variance such that

$$\frac{1}{n^2} V\left(\sum_{i=1}^n X_i\right) \rightarrow 0 \text{ as } n \rightarrow \infty. \text{ Show that WLLN holds and } \bar{X}_n \xrightarrow{p} \bar{\mu}_n = \frac{1}{n} \sum_{i=1}^n \mu_i.$$

- [3] Let X_1, X_2, \dots, X_n be a random sample from $U(0,1)$. Let $Y_n = \min(X_1, \dots, X_n)$ and

$$Z_n = \max(X_1, \dots, X_n). \text{ Show that (a) } \sqrt{Y_n} \xrightarrow{p} 0, \text{ (b) } Z_n^2 \xrightarrow{p} 1 \text{ and (c) } Y_n^2 Z_n^2 \xrightarrow{p} 0.$$

- [4] Let X_1, X_2, \dots, X_n be i.i.d. $N(0,1)$. Show that $\bar{X}_n/S_n \xrightarrow{p} 0$, where $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ and

$$S_n^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2.$$

- [5] Suppose $Y_n \sim \text{Bin}(n, p)$, show that $(1 - Y_n/n) \xrightarrow{p} 1 - p$.

- [6] Let $\{X_n\}$ be a sequence of independent random variables with

$$P(X_n = x) = \begin{cases} 1/2, & x = -n^{1/4}, n^{1/4} \\ 0, & \text{otherwise.} \end{cases}$$

Show that $\bar{X}_n \xrightarrow{p} 0$.

- [7] Let $\{X_n\}$ be a sequence of i.i.d. random variables with mean μ and finite variance. Show that

$$(a) \quad \frac{2}{n(n+1)} \sum_{i=1}^n i X_i \xrightarrow{p} \mu$$

$$(b) \quad \frac{6}{n(n+1)(2n+1)} \sum_{i=1}^n i^2 X_i \xrightarrow{p} \mu$$

- [8] Let $\{X_n\}$ be a sequence of i.i.d. random variables with $U(0,1)$ distribution and

$$Z_n = \left(\prod_{i=1}^n X_i \right)^{1/n}. \text{ Show that } Z_n \xrightarrow{p} e^{-1}.$$

- [9] Let $\{X_n\}$ be a sequence of uncorrelated random variables with $E(X_n) = \mu_n$ and

$$V(X_n) = \sigma_n^2. \text{ Show that if } \sum_{i=1}^n \sigma_i^2 \rightarrow \infty \text{ as } n \rightarrow \infty, \text{ then WLLN holds for } \{X_n\}.$$

- [10] Let $\{X_n\}$ be a sequence of i.i.d. random variables with $U(0,1)$ distribution. Find c such that $\bar{X}_n \xrightarrow{p} c$.

- [11] Let $\{X_n\}$ be a sequence of $N\left(\frac{1}{n}, 1 - \frac{1}{n}\right)$. Show that $X_n \xrightarrow{L} Z$, where $Z \sim N(0,1)$.

[12] Let $\{X_n\}$ be a sequence of i.i.d. random variables with $E(X_i) = \mu$, $Var(X_i) = \sigma^2$ and

$$E(X_i - \mu)^4 = \sigma^4 + 1. \text{ Find } \lim_{n \rightarrow \infty} P\left[\sigma^2 - \frac{1}{\sqrt{n}} \leq \frac{(X_1 - \mu)^2 + \dots + (X_n - \mu)^2}{n} \leq \sigma^2 + \frac{1}{\sqrt{n}}\right].$$

[13] Let X_1, X_2, \dots, X_n be i.i.d. $B(1, p)$, $S_n = \sum_{i=1}^n X_i$. Find n which would guarantee

$$P\left(\left|\frac{S_n}{n} - p\right| \geq 0.01\right) \leq 0.01, \text{ no matter whatever the unknown } p \text{ may be.}$$

[14] Let X_1, \dots, X_n be i.i.d. from a distribution with mean μ and finite variance σ^2 . Prove that

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{S_n} \xrightarrow{L} Z, \text{ where } Z \sim N(0, 1).$$

[15] The p.d.f. of a random variable X is

$$f(x) = \begin{cases} 1/x^2 & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Consider a random sample of size 72 from the distribution having the above p.d.f. Compute, approximately, the probability that more than 50 of these observations are less than 3.

[16] Let X_1, \dots, X_{100} be i.i.d. from Poisson(3) distribution and let $Y = \sum_{i=1}^{100} X_i$. Using CLT, find

an approximate value of $P(100 \leq Y \leq 200)$.

[17] Let $X \sim Bin(100, 0.6)$. Find an approximate value of $P(10 \leq X \leq 16)$.

[18] The p.d.f. of X_n is given by

$$f_n(x) = \begin{cases} \frac{1}{n} e^{-x} x^{n-1} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the limiting distribution of $Y_n = X_n / n$.

[19] Let \bar{X} denote the mean of a random sample of size 64 from the Gamma distribution with density

$$f_n(x) = \begin{cases} \frac{1}{\Gamma(p)\alpha^p} e^{-x/\alpha} x^{p-1} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

With $\alpha = 2, p = 4$. Compute the approximate value of $P(7 < \bar{X} < 9)$.

[20] X_1, \dots, X_n is a random sample from $U(0, 2)$. Let $Y_n = \bar{X}_n$, show that

$$\sqrt{n}(Y_n - 1) \xrightarrow{L} N(0, 1/3).$$