①
$$E(X) = \frac{1}{\beta}\int_0^{\infty} x e^{-\beta/x}\,dx = \beta$$

$$\Rightarrow E(\bar{X}) = E\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n}\sum E(X_i) = \beta$$

$\Rightarrow$ $\bar{X}$ is u.e. of β

②
$$f_{X_{(n)}}(x) = \begin{cases} \frac{n}{\theta^n} x^{n-1} & 0<x<\theta \\ 0 & o/w \end{cases}$$

$$E(X_{(n)}) = \frac{n}{\theta^n}\int_0^{\theta} x^n\,dx = \frac{n}{n+1}\theta$$

$$\Rightarrow E\left(\frac{n+1}{n} X_{(n)}\right) = \theta$$

$\Rightarrow$ $\frac{n+1}{n} X_{(n)}$ is u.e. of θ

Also
$$f_X(x) = \begin{cases} \frac{1}{\theta} & 0<x<\theta \\ 0 & o/w \end{cases}$$

$$E(X) = \frac{\theta}{2}$$

$$\Rightarrow E(2\bar{X}) = E\left(\frac{2}{n}\sum X_i\right)$$
$$= \frac{2}{n}\sum E(X_i) = \theta$$

$\Rightarrow$ $2\bar{X}$ is u.e for θ

③
$$E(X) = \beta\int_0^{\infty} x e^{-\beta x}\,dx = \frac{1}{\beta}$$

$E(\bar{X}) = \beta \Rightarrow \bar{X}$ is u.e of β.

$$\left(E(\bar{X}) = E\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n}\sum E(X_i)\right)$$

(4) $T_1 = \sum X_i \quad , \quad T_2 = \sum X_i^2$

$$E(T_1^2) = V(T_1) + E^2(T_1)$$

$$= n\theta^2 + n^2\theta^2 = \theta^2 n(n+1)$$

$$\Rightarrow E\left(T_1^2 / n(n+1) \right) = \theta^2$$

$$\Rightarrow \frac{T_1^2}{n(n+1)} \text{ is u.e. of } \theta^2$$

$$E(T_2) = E\left(\sum X_i^2\right) = \sum E(X_i^2)$$

$$= \sum \left(V(X_i) + E^2(X_i) \right)$$

$$= \sum (\theta^2 + \theta^2) = 2n\theta^2$$

$$\Rightarrow E\left(\frac{T_2}{2n}\right) = \theta^2$$

$$\Rightarrow \frac{T_2}{2n} \text{ is u.e. of } \theta^2$$

(5) . $g(\theta) = \theta e^{-2\theta}$

$$\delta_0(\underline{X}) = \begin{cases} 1 & \text{if } x_1 = 0,\ x_2 = 1 \\ 0 & \text{o/w} \end{cases}$$

$$E(\delta_0(\underline{x})) = 1 .\ P(x_1 = 0,\ x_2 = 1)$$

$$= P(x_1 = 0)\ P(x_2 = 1)$$

$$= e^{-\theta} . \frac{e^{-\theta}\theta^1}{1!} = \theta e^{-2\theta}$$

$$\Rightarrow \delta_0(\underline{x}) \text{ is u.e. of } \theta e^{-2\theta}.$$

(6) $x_1 \ldots x_n$ i.i.d $B(1, \theta)$

$$\sum X_i \sim B(n, \theta)$$

$$E(T(\underline{x})) = \frac{\frac{1}{2}\sqrt{n} + E(\sum X_i)}{n + \sqrt{n}} = \frac{\frac{1}{2}\sqrt{n} + n\theta}{n + \sqrt{n}} \neq \theta$$

$$\Rightarrow T(\underline{X}) \text{ is not u.e. of } \theta.$$

$$\lim_{n\to\infty} E(T(\underline{x})) = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{2} + n\theta}{n+\sqrt{n}} = \theta$$

$\Rightarrow T(\underline{x})$ is unbiased in the limit for θ

⑦ $X_1, \ldots X_n$ r.s. from $N(\mu, \sigma^2)$

$$\Rightarrow \quad \left.\begin{array}{l} \bar{X} \sim N(\mu, \sigma^2/n) \\ Y = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \end{array}\right\rangle \text{indep}$$

If $Z \sim \chi^2_m$, then

$$E\left(\frac{1}{Z}\right) = \frac{1}{2^{m/2}\Gamma m/2} \int_0^\infty z^{-1} e^{-z/2} z^{m/2-1}\, dz$$

$$= \frac{1}{2^{m/2}\Gamma m/2} \int_0^\infty e^{-z/2} z^{\frac{m}{2}-1-1}\, dz$$

$$= \frac{\Gamma\left(\frac{m}{2}-1\right) \cancel{2}^{m/2-1}}{\cancel{2^{m/2}}\,\Gamma m/2} = \frac{1}{m-2}$$

& $$E\left(\frac{1}{\sqrt{Z}}\right) = \frac{1}{2^{m/2}\Gamma m/2} \int_0^\infty e^{-z/2} z^{\frac{m}{2}-\frac{1}{2}-1}\, dz$$

$$= \frac{\Gamma\frac{m-1}{2}\; \cancel{2}^{m/2-1/2}}{\cancel{2^{m/2}}\,\Gamma m/2} = \frac{\Gamma\frac{m-1}{2}}{\sqrt{2}\,\Gamma\frac{m}{2}}$$

$$\Rightarrow E\left(\frac{1}{Y}\right) = E\left(\frac{\sigma^2}{(n-1)S^2}\right) = \frac{1}{(n-1)-2} = \frac{1}{n-3}$$

$$\Rightarrow E\left(\frac{1}{S^2}\right) = \frac{n-1}{n-3} \cdot \frac{1}{\sigma^2}$$

& $E\left(\frac{1}{\sqrt{Y}}\right) = E\left(\frac{\sigma}{\sqrt{n-1}\; s}\right) = \frac{\Gamma\frac{n-2}{2}}{\sqrt{2}\;\Gamma\frac{n-1}{2}}$

$\Rightarrow E\left(\frac{1}{s}\right) = \frac{\sqrt{n-1}\;\Gamma\frac{n-2}{2}}{\sqrt{2}\;\Gamma\frac{n-1}{2}} \cdot \frac{1}{\sigma}$

Since $\bar{x}$ & s^2 are indep.

$$E\left(\frac{\bar{x}}{s^2}\right) = E(\bar{x}) \cdot E\left(\frac{1}{s^2}\right)$$

$$= \mu \cdot \frac{n-1}{n-3} \cdot \frac{1}{\sigma^2}.$$

$$\Rightarrow E\left(\frac{n-3}{n-1} \cdot \frac{\bar{x}}{s^2}\right) = \frac{\mu}{\sigma^2}$$

$\Rightarrow \frac{n-3}{n-1} \frac{\bar{x}}{s^2}$ is an unbiased estimator of $\frac{\mu}{\sigma^2}$.

Further $E\left(\frac{\bar{x}}{s}\right) = E(\bar{x}) \cdot E\left(\frac{1}{s}\right)$

$$= \mu \cdot \frac{\sqrt{n-1}\;\Gamma\frac{n-2}{2}}{\sqrt{2}\;\Gamma\frac{n-1}{2}} \cdot \frac{1}{\sigma}$$

$$\Rightarrow E\left(\sqrt{\frac{2}{n-1}}\;\frac{\Gamma\frac{n-1}{2}}{\Gamma\frac{n-2}{2}}\;\frac{\bar{x}}{s}\right) = \frac{\mu}{\sigma}$$

$\Rightarrow \sqrt{\frac{2}{n-1}} \cdot \frac{\Gamma\frac{n-1}{2}}{\Gamma\frac{n-2}{2}} \cdot \frac{\bar{x}}{s}$ is an unbiased estimator

of $\frac{\mu}{\sigma}$.

8

$X_1, \ldots X_n$ are i.i.d $B(1, \theta)$

$$g(\theta) = \theta^2 (1-\theta)$$

Define $\delta(\underline{x}) = \begin{cases} 1 & \text{if } x_1 = 1,\ x_2 = 1,\ x_3 = 0 \\ 0 & \text{o/w} \end{cases}$

$$E_\theta\, \delta(\underline{x}) = P(x_1 = 1, x_2 = 1, x_3 = 0)$$
$$= P(x_1 = 1)\, P(x_2 = 1)\, P(x_3 = 0)$$
$$= \theta^2 (1-\theta)$$

$\Rightarrow \delta(\underline{x})$ is an u.e. of $g(\theta) = \theta^2 (1-\theta)$.

9.

(a) $f(x|\alpha) = \frac{1}{\alpha} e^{-\frac{x}{\alpha}}$; $\alpha > 0$

Jt p.d.f

$$f(\underline{x}|\alpha) = \frac{1}{\alpha^n} e^{-\frac{1}{\alpha}\sum x_i} \qquad x_1, \ldots x_n > 0$$
$$= \left(\frac{1}{\alpha^n} e^{-\frac{1}{\alpha}\sum x_i}\right) \cdot 1.$$

i.e. $f(\underline{x}|\alpha) = g\left(\alpha, \sum_1^n x_i\right) \cdot h(\underline{x}) \quad (h(\underline{x}) = 1)$.

By NFFT, $T(\underline{x}) = \sum_{i=1}^n x_i$ is suff for α.

9 (b) $f(x|\beta) = e^{-(x-\beta)} \qquad x > \beta$

$$f(\underset{\sim}{x}|\beta) = \begin{cases} e^{-\Sigma(x_i-\beta)}, & x_1, \ldots x_n > \beta \\ 0 & o/w \end{cases}$$

$$\text{i.e. } f(\underset{\sim}{x}|\beta) = \begin{cases} e^{-\Sigma x_i + n\beta}, & x_{(1)} > \beta \\ 0 & o/w \end{cases}$$

$$\text{i.e. } f(\underset{\sim}{x}|\beta) = e^{n\beta - \Sigma x_i} \, I_{(\beta, x_{(1)})} \qquad \left[I_{(a,b)} = \begin{cases} 1 & a<b \\ 0 & o/w \end{cases} \right.$$

$$= \left(e^{-\Sigma x_i}\right)\left(e^{n\beta} I_{(\beta, x_{(1)})}\right)$$

$$= h(\underline{x}) \; g(\beta, x_{(1)})$$

By NFFT, $T(\underline{X}) = X_{(1)}$ is a suff statistic

9 (c) $$f(\underset{\sim}{x}|\alpha,\beta) = \frac{1}{\alpha^n}\exp\left(-\frac{\Sigma x_i}{\alpha} + \frac{n\beta}{\alpha}\right) I_{(\beta, x_{(1)})}$$

$$= \left(\frac{1}{\alpha^n}\exp\left(-\frac{\Sigma x_i}{\alpha} + \frac{n\beta}{\alpha}\right) . I_{(\beta, x_{(1)})}\right) . 1.$$

$$= g\left((\alpha,\beta); (\Sigma x_i, x_{(1)})\right) . h(\underline{x})$$

By NFFT, $T(\underline{X}) = (\Sigma X_i, X_{(1)})$ is jointly sufficient for (α, β).

9 (d) $$f(\underset{\sim}{x} \mid \mu, \sigma) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \left[\prod_{i=1}^{n}\left(\frac{1}{x_i}\right)\right] \cdot \exp\left(-\frac{1}{2\sigma^2}\Sigma(\log x_i - \mu)^2\right)$$ ⑦

$$= \left(\frac{1}{\sigma^n} \exp\left(-\frac{n\mu^2}{2\sigma^2} - \frac{1}{2\sigma^2}\Sigma(\log x_i)^2 + \frac{\mu}{\sigma^2}\Sigma \log x_i\right)\right)$$

$$\times \left(\left(\frac{1}{\sqrt{2\pi}}\right)^n \prod_{i=1}^{n} x_i^{-1}\right).$$

$$= g\left((\mu, \sigma); \left(\Sigma \log x_i, \Sigma(\log x_i)^2\right)\right) \cdot h(\underset{\sim}{x})$$

$$h(\underset{\sim}{x}) = \left(\left(\frac{1}{\sqrt{2\pi}}\right)^n \prod_{1}^{n} x_i^{-1}\right).$$

By NFFT $\left(\Sigma \log X_i, \Sigma(\log X_i)^2\right)$ is jointly sufficient for (μ, σ)

⑩ jt p.d.f of X_1 & X_2

$$f(x_1, x_2) = \theta e^{-\theta x_1} . 2\theta e^{-2\theta x_2}$$

$$f(x_1, x_2) = 2\theta^2 e^{-\theta(x_1 + 2x_2)}$$

$$f(x_1, x_2) = \left(\theta^2 e^{-\theta(x_1+2x_2)}\right)(2)$$

By NFFT $T(X_1, X_2) = X_1 + 2X_2$ is suff for θ

9 (e) $$f(\underset{\sim}{x} \mid \theta) = \begin{cases} \frac{1}{\theta^n} & ; -\theta/2 < x_1 \ldots x_n < \frac{\theta}{2} \\ 0 & \text{o/w} \end{cases}$$

$$= \begin{cases} \frac{1}{\theta^n} & ; |x_i| < \theta/2 \;; i = 1, 2 \ldots n \\ 0 & \text{o/w.} \end{cases}$$

i.e. $f(\underset{\sim}{x}|\theta) = \begin{cases} \frac{1}{\theta^n} & ; \quad \underset{i}{\text{Max}} |x_i| < \theta/2 \\ 0 & \quad \text{o/w} \end{cases}$

$\Rightarrow \quad f(\underset{\sim}{x}|\theta) = \frac{1}{\theta^n} \, I\left(\underset{i}{\text{Max}} |x_i|, \theta/2\right)$

$\Rightarrow$ By NFFT $T(\underset{\sim}{X}) = \underset{i}{\text{Max}} |x_i|$ is suff for θ.

(11) jt p.d.f

$$f(\underset{\sim}{x}|\theta) = \begin{cases} \prod_{i=1}^{n} \exp(i\theta - x_i) & \text{if } \frac{x_1}{1}, \frac{x_2}{2}, \ldots \frac{x_n}{n} \geq \theta \\ 0 & \text{o/w} \end{cases}$$

$$= \begin{cases} e^{-\sum_1^n x_i} \, e^{\theta \frac{n(n+1)}{2}} & \text{if } \underset{i}{\text{Min}} \frac{x_i}{i} \geq \theta \\ 0 & \text{o/w} \end{cases}$$

i.e. $f(\underset{\sim}{x}|\theta) = \left(e^{\theta \frac{n(n+1)}{2}} \, I\left(\theta, \underset{i}{\text{Min}} \frac{x_i}{i}\right)\right)\left(e^{-\sum_{i=1}^{n} x_i}\right)$

$= g\left(\theta, \underset{i}{\text{Min}} \frac{x_i}{i}\right) \times h(\underset{\sim}{x})$

$\Rightarrow \quad T(\underset{\sim}{X}) = \underset{i}{\text{Min}} \frac{X_i}{i}$ is suff

(12) $X_1 \ldots X_n$ i.i.d Beta(α, β)

(a) β is known – α is the unknown parameter

$$f(\underset{\sim}{x} \mid \alpha) = \begin{cases} \left[\left(\frac{\Gamma\alpha+\beta}{\Gamma\alpha}\right)^n \left(\Pi\, x_i\right)^{\alpha-1}\right] \underbrace{\left[\left(\Pi(1-x_i)\right)^{\beta-1} \left(\frac{1}{\Gamma\beta}\right)^n\right]}_{h(\underset{\sim}{x})} & 0 < x_1, \ldots x_n < 1 \\ 0 & \text{o/w.} \end{cases}$$

By NFFT $\left(\prod_{i=1}^{n} X_i\right)$ is suff for α

(b) α is known – β is the unknown parameter

$$f(\underset{\sim}{x} \mid \beta) = \begin{cases} \left[\left(\frac{\Gamma\alpha+\beta}{\Gamma\beta}\right)^n \left(\Pi(1-x_i)\right)^{\beta-1}\right] \underbrace{\left[\left(\Pi\, x_i\right)^{\alpha-1} \left(\frac{1}{\Gamma\alpha}\right)^n\right]}_{h(\underset{\sim}{x})} ; & 0 < x_1, \ldots x_n < 1 \\ 0 & \text{o/w} \end{cases}$$

By NFFT $\prod_{i=1}^{n}(1-X_i)$ is suff for β

(c) α, β both unknown

$$f(\underset{\sim}{x} \mid \alpha, \beta) = \begin{cases} \left[\left(\frac{\Gamma\alpha+\beta}{\Gamma\alpha\,\Gamma\beta}\right)^n \left(\Pi\, x_i\right)^{\alpha-1} \left(\Pi(1-x_i)\right)^{\beta-1}\right] \cdot 1 & 0 < x_1 \ldots x_n < 1 \\ 0 & \text{o/w.} \end{cases}$$

By NFFT $(\Pi X_i, \Pi(1-X_i))$ is jointly sufficient for (α, β).

(13) T is suff for $\theta \in \Theta$ & $T = \psi(T^*)$

By NFFT T is suff for θ iff.

$$f(\underset{\sim}{x}|\theta) = g(\theta, t(\underset{\sim}{x}))\, h(\underset{\sim}{x})$$

$$\text{i.e. } f(\underset{\sim}{x}|\theta) = g(\theta, \psi(t^*(\underset{\sim}{x}))) \cdot h(\underset{\sim}{x})$$

$$f(\underset{\sim}{x}|\theta) = g'(\theta, t^*(\underset{\sim}{x})) \cdot h(\underset{\sim}{x})$$

$\Rightarrow$ $T^*(\underset{\sim}{X})$ is suff for θ.

(14)
$$f(\underset{\sim}{x}|\theta) = \begin{cases} 1, & \theta - \frac{1}{2} < x_{(1)}, \dots < x_{(n)} < \theta + \frac{1}{2} \\ 0, & \text{o/w} \end{cases}$$

$$\text{i.e. } f(\underset{\sim}{x}|\theta) = I_{(\theta - \frac{1}{2}, x_{(1)})}\, I_{(x_{(n)}, \theta + \frac{1}{2})}$$

$$= g(\theta, (x_{(1)}, x_{(n)}))\, \underset{\overset{\shortparallel}{1}}{h(\underset{\sim}{x})}$$

By NFFT, $T(\underset{\sim}{X}) = (X_{(1)}, X_{(n)})$ is jointly suff for θ

(15)
$$f(\underset{\sim}{x}|\theta) = (\theta e^{-\theta x_1})(2\theta e^{-2\theta x_2}) \dots (n\theta e^{-n\theta x_n})$$

$$\text{i.e. } f(\underset{\sim}{x}|\theta) = \theta^n \left(\prod_{i=1}^{n} i\right) e^{-\theta \sum_{i=1}^{n} i x_i}$$

$$= \left(\prod_{i=1}^{n} i\right) \left(\theta^n e^{-\theta \sum_{i=1}^{n} i x_i}\right)$$

$$= h(\underset{\sim}{x})\, g\left(\theta, \sum_{i=1}^{n} i x_i\right)$$

By NFFT, $T(\underset{\sim}{X}) = \sum_{i=1}^{n} i X_i$ is sufficient for θ.