① jt p.m.f. of Y_1, Y_2

$$P(Y_1 = y_1, Y_2 = y_2) = P(X_1 - X_2 = y_1, X_1 + X_2 = y_2)$$

$$= P\left(X_1 = \frac{y_1+y_2}{2}, X_2 = \frac{y_2-y_1}{2}\right)$$

$$= \begin{cases} \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{2-0} & \text{if } \frac{y_1+y_2}{2} = 0 \ \& \ \frac{y_2-y_1}{2} = 0, \text{ i.e. } y_1 = 0, y_2 = 0 \\ \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^{2-1} & \text{if } \frac{y_1+y_2}{2} = 1 \ \& \ \frac{y_2-y_1}{2} = 0, \text{ i.e. } y_1 = 1, y_2 = 1 \\ \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^{2-1} & \text{if } \frac{y_1+y_2}{2} = 0, \frac{y_2-y_1}{2} = 1, \text{ i.e. } y_1 = -1, y_2 = 1 \\ \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^{2-2} & \text{if } \frac{y_1+y_2}{2} = 1, \frac{y_2-y_1}{2} = 1, \text{ i.e. } y_1 = 0, y_2 = 2 \end{cases}$$

i.e.

$$P(Y_1 = y_1, Y_2 = y_2) = \begin{cases} 1/9 & \text{if } (y_1, y_2) = (0,0) \\ 2/9 & \text{if } (y_1, y_2) = (-1,1), (1,1) \\ 4/9 & \text{if } (y_1, y_2) = (0,2) \\ 0 & \text{o/w.} \end{cases}$$

$Y_1 \backslash Y_2$	0	1	2
0	1/9	0	4/9
-1	0	2/9	0
1	0	2/9	0

$$P(Y_1 = y_1) = \begin{cases} 5/9 & y_1 = 0 \\ 2/9 & y_1 = -1 \\ 2/9 & y_1 = 1 \\ 0 & \text{o/w} \end{cases} \qquad P(Y_2 = y_1) = \begin{cases} 1/9 & y_2 = 0 \\ 4/9 & y_2 = 1 \\ 4/9 & y_2 = 2 \\ 0 & \text{o/w} \end{cases}$$

Since $P(Y_1 = y_1, Y_2 = y_2) \neq P(Y_1 = y_1) P(Y_2 = y_2) \ \forall (y_1, y_2)$

Y_1 & Y_2 are not indep.

② $Y_1 = X_1 X_2$; $Y_2 = X_2$

jt. p.m.f.

$$P(Y_1 = y_1, Y_2 = y_2) = P(X_1 X_2 = y_1, X_2 = y_2) = P\left(X_1 = \frac{y_1}{y_2}, X_2 = y_2\right)$$

$$= \begin{cases} \frac{y_1}{36} & \text{if } \frac{y_1}{y_2} = 1, 2, 3 ; \ y_2 = 1, 2, 3 \\ 0 & \text{o/w.} \end{cases}$$

Possible values of Y_1 in $\{1,2,3,4,6,9\}$.

$$P(Y_1 = y_1) = \begin{cases} P(X_1=1, X_2=1) = \frac{1}{36} & y_1 = 1 \\ P(X_1=1, X_2=2) + P(X_1=2, X_2=1) = \frac{2}{36} + \frac{2}{36} = \frac{4}{36} & y_1 = 2 \\ P(X_1=1, X_2=3) + P(X_1=3, X_2=1) = \frac{3}{36} + \frac{3}{36} = \frac{6}{36} & y_1 = 3 \\ P(X_1=2, X_2=2) = \frac{4}{36} & y_1 = 4 \\ P(X_1=2, X_2=3) + P(X_1=3, X_2=2) = \frac{6}{36} + \frac{6}{36} = \frac{12}{36} & y_1 = 6 \\ P(X_1=3, X_2=3) = \frac{9}{36} & y_1 = 9 \\ 0 & \text{o/w} \end{cases}$$

$Z = X_1 + X_2 \rightarrow$ possible values of Z in $\{2,3,4,5,6\}$.

$P(Z = z)$

$$= P(X_1 + X_2 = z) = \begin{cases} P(X_1=1, X_2=1) = \frac{1}{36} & z = 2 \\ P(X_1=1, X_2=2) + P(X_1=2, X_2=1) = \frac{4}{36} & z = 3 \\ P(X_1=1, X_2=3) + P(X_1=2, X_2=2) + P(X_1=3, X_2=1) = \frac{10}{36} & z = 4 \\ P(X_1=2, X_2=3) + P(X_1=3, X_2=2) = \frac{12}{36} & z = 5 \\ P(X_1=3, X_2=3) = \frac{9}{36} & z = 6 \\ 0 & \text{o/w.} \end{cases}$$

③ $$P(X=x \mid X+Y=t) = \frac{P(X=x, X+Y=t)}{P(X+Y=t)} = \frac{P(X=x, Y=t-x)}{P(X+Y=t)}$$

$$= \frac{P(X=x)\, P(Y=t-x)}{P(X+Y=t)} = \frac{\binom{n_1}{x} p^x (1-p)^{n_1-x} \binom{n_2}{t-x} p^{t-x} (1-p)^{n_2-(t-x)}}{\binom{n_1+n_2}{t} p^t (1-p)^{n_1+n_2-t}}$$

$[X+Y \sim B(n_1+n_2, p)]$

$$= \frac{\binom{n_1}{x}\binom{n_2}{t-x}}{\binom{n_1+n_2}{t}} ; \quad 0 \le x \le n_1, \quad 0 \le t-x \le n_2$$

Hypergeometric (n_1, n_2)

④ $$P(X=x \mid X+Y=t) = \frac{P(X=x, Y=t-x)}{P(X+Y=t)}$$

$$\left[\begin{array}{l} X \sim P(\lambda_1) \\ Y \sim P(\lambda_2) \\ X+Y \sim P(\lambda_1+\lambda_2) \end{array}\right)$$

$$= \frac{P(X=x)\, P(Y=t-x)}{P(X+Y=t)} = \frac{\left(\frac{e^{-\lambda_1}\lambda_1^x}{x!}\right)\left(\frac{e^{-\lambda_2}\lambda_2^{t-x}}{(t-x)!}\right)}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^t}{t!}} = \binom{t}{x}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x \left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{t-x}$$

i.e. $X \mid X+Y=t \sim \text{Bin}\left(t, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$.

5. $$f_X(x) = \begin{cases} 3(1-x)^2 & 0<x<1 \\ 0 & \text{o/w.} \end{cases} \qquad F_X(x) = \begin{cases} 0 & x<0 \\ 3\int_0^x (1-t)^2\,dt = 1-(1-x)^3 & 0\le x<1 \\ 1 & x\ge 1 \end{cases}$$

$Y = \text{Min}(X_1, X_2, X_3, X_4)$; $Z = \text{Max}(X_1, X_2, X_3, X_4)$

X_1, X_2, X_3, X_4 i.i.d. from $f_X(x)$

d.f. of Y ; $F_Y(y) = P(Y \le y) = 1 - P(Y > y)$

$$= 1 - \prod_1^4 P(X_i > y)$$

$$= 1 - (1 - P(X \le y))^4$$

$$= 1 - (1 - \{1-(1-y)^3\})^4$$

$$= 1 - (1-y)^{12} \qquad 0<y<1$$

$$f_Y(y) = 12(1-y)^{11} \qquad 0<y<1$$

$$= 0 \qquad \text{o/w.}$$

d.f. of Z : $F_Z(z) = P(Z \le z) = \prod_1^4 P(X_i \le z) = [P(X \le z)]^4$

$$= (1-(1-z)^3)^4 \qquad 0<z<1$$

$$f_Z(z) = \begin{cases} 12(1-z)^2(1-(1-z)^3)^3 & 0<z<1 \\ 0 & \text{o/w.} \end{cases}$$

6. Similar to 5

(7) X : arrival time of A

Y : — — — B

X & Y i.i.d $Exp(\lambda)$:- p.d.f $f(x) = \lambda e^{-\lambda x}$ $x > 0$

reqd prob $= P(X < Y,\ Y - X \le t) + P(Y < X,\ X - Y \le t)$

$= P(Y - t \le X \le Y) + P(X - t \le Y \le X)$

$= P(X \le Y \le X + t) + P(Y \le X \le Y + t)$

[Jt p.d.f of X, Y $\rightarrow \lambda^2 e^{-\lambda(x+y)}$ $x > 0, y > 0$

$= \int_0^\infty \int_x^{x+t} \lambda^2 e^{-\lambda(x+y)}\, dy\, dx + \int_0^\infty \int_y^{y+t} \lambda^2 e^{-\lambda(x+y)}\, dx\, dy$

$= 2\lambda^2 \int_0^\infty e^{-\lambda x} \int_x^{x+t} e^{-\lambda y}\, dy\, dx$

$= 2\lambda^2 \frac{1}{\lambda}(1 - e^{-\lambda t}) \int_0^\infty e^{-2\lambda x}\, dx = (1 - e^{-\lambda t})$.

(8) $X_1, X_2 \sim U(0,1)$

$Y_1 = X_1 + X_2$

$Y_2 = X_2 - X_1$ $\Rightarrow \frac{1}{|J|} = \left| \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} \right| = \left| \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} \right| = 2$

$|J| = \frac{1}{2}$

$f_{X_1, X_2}(x_1, x_2) = 1$; $0 < x_1 < 1$, $0 < x_2 < 1$

$\Rightarrow f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{2}$; $0 < y_1 + y_2 < 2$, $0 < y_1 - y_2 < 2$

<u>Range</u> unconditionally $0 < y_1 < 2$ & $-1 < y_2 < 1$

$x_2 = \frac{y_1 + y_2}{2}$

$x_1 = \frac{y_1 - y_2}{2}$

Also $0 < x_1 < 1$; $\Rightarrow 0 < \frac{y_1 - y_2}{2} < 1$

$\Rightarrow 0 < y_1 - y_2 < 2$

$\left.\begin{array}{l} y_2 < y_1 < 2 + y_2 \\ \& \ y_1 - 2 < y_2 < y_1 \end{array}\right\} \quad -(1)$

Also $0 < x_2 < 1$; $0 < \frac{y_1 + y_2}{2} < 1$

$$0 < y_1 + y_2 < 2$$

$$\left.\begin{array}{l} -y_2 < y_1 < 2 - y_2 \\ \& \quad -y_1 < y_2 < 2 - y_1 \end{array}\right\} - (2)$$

Combining (1) & (2)

$$\left.\begin{array}{l} \max(y_2, -y_2) < y_1 < \min(2 + y_2, 2 - y_2) \\ \& \quad \max(y_1 - 2, -y_1) < y_2 < \min(y_1, 2 - y_1) \end{array}\right) \quad (3)$$

If $-1 < y_2 < 0$ then from (3) $-y_2 < y_1 < 2 + y_2$

& if $0 < y_2 < 1$ then from (3) $y_2 < y_1 < 2 - y_2$ $\Big\} - (4)$

Alternatively, if $0 < y_1 < 1$ then from (3) $-y_1 < y_2 < y_1$

& if $1 < y_1 < 2$ then from (3) $y_1 - 2 < y_2 < 2 - y_1$ $\Big\} - (5)$

$\Rightarrow$ Marg of Y_1

$$f_{Y_1}(y_1) = \frac{1}{2} \int_{-y_1}^{y_1} dy_2 = y_1 \qquad \text{if } 0 < y_1 < 1$$

(using (5)) $\longrightarrow$

$$= \frac{1}{2} \int_{y_1 - 2}^{2 - y_1} dy_2 = 2 - y_1 \qquad \text{if } 1 < y_1 < 2$$

& Marg of Y_2

$$f_{Y_2}(y_2) = \frac{1}{2} \int_{-y_2}^{2 + y_2} dy_1 = (1 + y_2) \qquad \text{if } -1 < y_2 < 0$$

(using (4)) $\longrightarrow$

$$= \frac{1}{2} \int_{y_2}^{2 - y_2} dy_1 = (1 - y_2) \quad \text{if } 0 < y_2 < 1$$

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$$\left.\begin{array}{l} X \sim N(0,1) \\ Y \sim N(0,1) \end{array}\right\rangle \text{ ind.}$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x^2+y^2)\right)$$

$$\left.\begin{array}{l} U = Y \\ Z = X/Y \end{array}\right\} \quad \begin{array}{l} X = UZ \\ Y = U \end{array} \qquad |J| = \left| \begin{vmatrix} z & u \\ 1 & 0 \end{vmatrix} \right| = |u|$$

$$f_{U,Z}(u,z) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\left(u^2 z^2 + u^2\right)\right\} |u| \quad ; \begin{array}{l} -\infty < u < \infty \\ -\infty < z < \infty \end{array}$$

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |u| \exp\left(-\frac{1}{2} u^2 (1+z^2)\right) du$$

$$= \frac{1}{\pi} \int_0^{\infty} u \exp\left(-\frac{u^2}{2}(1+z^2)\right) du$$

$$= \frac{1}{\pi} \cdot \frac{1}{1+z^2} \quad ; \quad -\infty < z < \infty$$

(i.e. $Z \sim$ Cauchy distr^n $(0,1)$

In gen. $X \sim$ Cauchy $(\mu, \theta) \to f_X(x) = \frac{\theta}{\pi} \frac{1}{1+(x-\mu)^2}$; $-\infty < x < \infty$).

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$$f_{X,Y}(x,y) = \frac{1}{\Gamma\alpha_1 \Gamma\alpha_2 \, \theta^{\alpha_1+\alpha_2}} x^{\alpha_1 - 1} y^{\alpha_2 - 1} e^{-\frac{x+y}{\theta}} \; ; \; x>0, y>0$$

$$= 0 \quad \text{o/w}.$$

$$\left.\begin{array}{l} U = X+Y \\ V = \dfrac{X}{X+Y} \end{array}\right\} \quad \begin{array}{l} X = UV \\ Y = U(1-V) \end{array} \qquad J = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u$$

<u>Range</u> $u > 0$, $0 < v < 1$

$$f_{U,V}(u,v) = \frac{1}{\Gamma\alpha_1 \Gamma\alpha_2 \, \theta^{\alpha_1+\alpha_2}} (uv)^{\alpha_1 - 1} (u(1-v))^{\alpha_2 - 1} e^{-\frac{u}{\theta}} \cdot u \qquad u>0, 0<v<1$$

$$= 0 \quad \text{o/w}$$

i.e.

$$f_{U,V}(u,v) = \begin{cases} \dfrac{1}{\Gamma\alpha_1+\alpha_2 \; \theta^{\alpha_1+\alpha_2}} u^{\alpha_1+\alpha_2-1} e^{-u/\theta} \times \dfrac{1}{B(\alpha_1,\alpha_2)} v^{\alpha_1-1}(1-v)^{\alpha_2-1} & u>0,\ 0<v<1 \\ 0 & o/w. \end{cases}$$

$$\Rightarrow \quad f_U(u) = \frac{1}{\Gamma\alpha_1+\alpha_2 \; \theta^{\alpha_1+\alpha_2}} u^{\alpha_1+\alpha_2-1} e^{-u/\theta} \qquad u>0$$

$U \sim$ Gamma.

$$= 0 \qquad o/w$$

$$f_V(v) = \frac{1}{B(\alpha_1,\alpha_2)} v^{\alpha_1-1}(1-v)^{\alpha_2-1} \qquad 0<v<1$$

$V \sim$ Beta.

$$= 0 \qquad o/w$$

$\Rightarrow$ U & V are indep

(11)

$$f_{X,Y} = \frac{c^2}{(1+x^4)(1+y^4)} \qquad -\infty<x<\infty,\ -\infty<y<\infty$$

$$U_1 = \frac{X}{Y},\ U_2 = Y \quad \Big\}\quad \begin{matrix} X = U_1U_2 \\ Y = U_2 \end{matrix} \quad \Big\}\quad J = \begin{vmatrix} u_2 & u_1 \\ 0 & 1 \end{vmatrix} = u_2$$

Range $-\infty < u_1 < \infty,\ -\infty < u_2 < \infty$

$$f_{U_1,U_2}(u_1,u_2) = \frac{c^2|u_2|}{(1+u_1^4u_2^4)(1+u_2^4)} \qquad -\infty<u_1<\infty,\ -\infty<u_2<\infty$$

$$f_{U_1}(u_1) = \int_{-\infty}^{\infty} f_{U_1,U_2}(u_1,u_2)\,du_2 = 2c\int_0^{\infty} \frac{u_2}{(1+u_1^4u_2^4)(1+u_2^4)}\,du_2$$

$$= \frac{c\pi}{2}\cdot\frac{1}{1+u_1^2} \quad \text{(on integration)}.$$

$$\int_{-\infty}^{\infty} f_{U_1}(u_1)\,du_1 = 1 \Rightarrow c = \frac{2}{\pi^2}$$

$$\Rightarrow f_{U_1}(u_1) = \frac{1}{\pi}\cdot\frac{1}{1+u_1^2} \qquad -\infty<u_1<\infty.$$

Cauchy distn.

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(12) $f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$ $\quad -\infty < x < \infty, \ -\infty < y < \infty$

$X = R\cos\Theta$

$Y = R\sin\Theta$

$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

Range $\quad r \geq 0, \quad 0 < \theta < 2\pi$

$f_{R,\Theta}(r,\theta) = \frac{1}{2\pi} e^{-r^2/2}\, r \quad , \quad r > 0, \ 0 < \theta < 2\pi$

$= 0 \quad$ o/w

$f_R(r) = r e^{-r^2/2} \quad r > 0$

$0 \quad$ o/w

$f_\Theta(\theta) = \frac{1}{2\pi} \quad 0 < \theta < 2\pi$

$= 0 \quad$ o/w. $\quad \Theta \sim U(0, 2\pi)$.

$\Rightarrow$ R & Θ are indep.

Define $Y = \frac{R^2}{2} \quad y > 0$

$R = \sqrt{2}\sqrt{y} \quad \frac{dr}{dy} = \frac{1}{\sqrt{2y}}$

$f_Y(y) = \frac{1}{\sqrt{2y}} \sqrt{2y}\, e^{-y} \quad y > 0$

$= 0 \quad$ o/w

i.e $f_Y(y) = e^{-y} \quad y > 0$

$= 0 \quad$ o/w

$\Rightarrow \frac{R^2}{2} \sim Exp(1)$.

$$U = X^2 + Y^2 = R^2 \quad - f^n \text{ of r.v. } R$$

$$V = \frac{X}{Y} = \cot \Theta \quad - f^n \text{ of r.v. } \Theta$$

Since R & Θ are indep, U & V are also indep

i.e. X^2+Y^2 & $\frac{X}{Y}$ are indep

(13) $U_1 \sim U(0,1)$

$-\ln U_1 \sim Exp(1)$ - straight forward

$U_2 \sim U(0,1)$

$2\pi U_2 \sim U(0, 2\pi)$ - straight forward.

$\Rightarrow -\ln U_1 \sim Exp(1)$ & $2\pi U_2 \sim U(0, 2\pi)$ and are indep

By problem # (12)

jt dist^n of $(-\ln U_1, 2\pi U_2)$ is same as jt dist of $\left(\frac{R^2}{2}, \Theta\right)$

i.e. $(-\ln U_1, 2\pi U_2) \stackrel{d}{=} \left(\frac{R^2}{2}, \Theta\right)$

i.e. $(-2\ln U_1, 2\pi U_2) \stackrel{d}{=} (R^2, \Theta)$

i.e $\left(\sqrt{-2\ln U_1}\cos(2\pi U_2), \sqrt{-2\ln U_1}\sin(2\pi U_2)\right) \stackrel{d}{=} (R\cos\Theta, R\sin\Theta)$

i.e. $(X_1, X_2) \stackrel{d}{=} (R\cos\Theta, R\sin\Theta)$

$\Rightarrow X_1$ and X_2 are i.i.d $N(0,1)$ r.v-s.

<u>Alt sol^n</u>

<u>Direct method</u> U_1, U_2 i.i.d $U(0,1)$.

$$f_{U_1,U_2}(u_1,u_2) = 1 \;; \quad 0<u_1<1,\; 0<u_2<1$$
$$= 0 \quad \text{o/w}$$

$$X_1 = \sqrt{-2\ln U_1}\cos(2\pi U_2)$$

$$X_2 = \sqrt{-2\ln U_2}\sin(2\pi U_2)$$

Range of X_1 ; $-\infty < x_1 < \infty$, Sly $-\infty < x_2 < \infty$

$$x_1^2 + x_2^2 = -2\ln U_1$$

$$\frac{x_2}{x_1} = \tan(2\pi U_2)$$

$$U_1 = \exp\left(-\tfrac{1}{2}(x_1^2 + x_2^2)\right)$$

$$U_2 = \frac{1}{2\pi}\tan^{-1}\left(\frac{x_2}{x_1}\right)$$

$$J = \begin{vmatrix} \exp\left(-\frac{1}{2}(x_1^2+x_2^2)\right)(-x_1) & \exp\left(-\frac{1}{2}(x_1^2+x_2^2)\right)(-x_2) \\ -\dfrac{x_2}{2\pi(x_1^2+x_2^2)} & \dfrac{x_1}{2\pi(x_1^2+x_2^2)} \end{vmatrix}$$

$$J = \exp\left(-\tfrac{1}{2}(x_1^2+x_2^2)\right)\left(-\frac{1}{2\pi}\right)$$

$$|J| = \frac{\exp\left(-\frac{1}{2}(x_1^2+x_2^2)\right)}{2\pi}$$

$$\Rightarrow f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi}\exp\left(-\tfrac{1}{2}(x_1^2+x_2^2)\right) \; ; \; \begin{array}{l} -\infty < x_1 < \infty \\ -\infty < x_2 < \infty \end{array}$$

$$= \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_1^2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_2^2}\right)$$

$\Rightarrow X_1$ & X_2 are indep $N(0,1)$ r.v.s.

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$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = e^{-(x_1+x_2+x_3)} \; ; \; x_1>0,\ x_2>0,\ x_3>0$$

$$Y_1 = \frac{X_1}{X_1+X_2} \; ; \; Y_2 = \frac{X_1+X_2}{X_1+X_2+X_3} \; ; \; Y_3 = X_1+X_2+X_3$$

i.e.

$$X_1 = Y_1 Y_2 Y_3$$
$$X_2 = Y_2 Y_3 (1-Y_1)$$
$$X_3 = Y_3(1-Y_2)$$

$x_1+x_2 = y_2y_3$

$x_1 = y_1y_2y_3$

$x_2 = y_2y_3$

(11)

$$J = \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ -y_2 y_3 & y_3(1-y_1) & y_2(1-y_1) \\ 0 & -y_3 & (1-y_2) \end{vmatrix} = y_2 y_3^2$$

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = y_2 y_3^2 e^{-y_3} \quad ; \quad 0<y_1<1 ; 0<y_2<1 , y_3>0$$

$$f_{Y_1}(y_1) = \int_0^1 y_2 \, dy_2 \int_0^\infty y_3^2 e^{-y_3} dy_3 = 1 \qquad 0<y_1<1$$

i.e. $Y_1 \sim U(0,1)$

$$f_{Y_2}(y_2) = y_2 \int_0^1 dy_1 \int_0^\infty y_3^2 e^{-y_3} dy_3$$

$$f_{Y_2}(y_2) = y_2 \times 1 \times 2 \qquad 0<y_2<1$$

i.e. $Y_2 \sim \text{Beta}(2,1)$

$$\left[\begin{array}{l} X \sim \text{Beta}(m,n) \\ f_X(x) = \dfrac{\Gamma m+n}{\Gamma m \, \Gamma n} x^{m-1}(1-x)^{n-1} \end{array} \right]$$

& $$f_{Y_3}(y_3) = \left(\int_0^1 dy_1 \int_0^1 y_2 \, dy_2 \right) y_3^2 e^{-y_3}$$

$$= \frac{1}{2} e^{-y_3} y_3^2 \qquad 0<y_3<\infty$$

$y_2 y_3 \left(y_3(1-y_1)(1-y_2) + y_2 y_3 (1-y_1) \right)$

$y_1 y_2$

y_2

$y_1 y_3$

$1-y_2$

$y_2 y_3$ $\quad y_3$

$-y_3$

0

$y_1 y_3$ $\quad 0$

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(a) $f_{X_1,X_2}(x_1, x_2) = \prod_{i=1}^{2} \frac{1}{2^{n_i/2}\Gamma\frac{n_i}{2}} e^{-x_i/2} x_i^{\frac{n_i}{2}-1}$; $x_i > 0$

$= c \prod_{i=1}^{2} e^{-x_i/2} x_i^{n_i/2-1}$; $x_i > 0$

$Y_1 = \frac{X_1}{X_2}$; $Y_2 = X_1 + X_2$ | $X_1 = \frac{Y_1 Y_2}{Y_1+1}$, $X_2 = \frac{Y_2}{Y_1+1}$

$$\frac{1}{J} = \begin{vmatrix} \frac{1}{x_2} & -\frac{x_1}{x_2^2} \\ 1 & 1 \end{vmatrix} = \frac{1}{x_2} + \frac{x_1}{x_2^2} = \frac{x_1+x_2}{x_2^2} = \frac{y_2}{\left(\frac{y_2}{1+y_1}\right)^2} = \frac{(1+y_1)^2}{y_2}$$

$$|J| = \frac{y_2}{(1+y_1)^2}$$

$$f_{Y_1,Y_2}(y_1, y_2) = c\, e^{-y_2/2} \left(\frac{y_1 y_2}{y_1+1}\right)^{n_1/2-1} \left(\frac{y_2}{1+y_1}\right)^{n_2/2-1} \frac{y_2}{(y_1+1)^2}$$

$y_1 > 0,\ y_2 > 0$

i.e. $f_{Y_1,Y_2}(y_1,y_2) = \underbrace{\left(c_1 e^{-y_2/2} y_2^{\frac{n_1+n_2}{2}-1}\right)}_{f_{Y_2}} \times \underbrace{\left(c_2 \frac{y_1^{n_1/2-1}}{(1+y_1)^{\frac{n_1+n_2}{2}}}\right)}_{f_{Y_1}}$ $y_1 > 0,\ y_2 > 0$

$\Rightarrow Y_1$ & Y_2 are indep

& $f_{Y_2}(y_2) = c_1 e^{-y_2/2} y_2^{\frac{n_1+n_2}{2}-1}$ $y_2 > 0$

$\int_0^\infty f_{Y_2}(y_2)\, dy_2 = 1 \Rightarrow c_1 = \left(\Gamma\frac{n_1+n_2}{2} \cdot 2^{\frac{n_1+n_2}{2}}\right)^{-1}$

$\Rightarrow Y_2 \sim \chi^2$ with (n_1+n_2) d.f.

(b) Similar to (a)

$$Z_1 = \frac{X_1/n_1}{X_2/n_2} \sim F_{n_1, n_2} \to \text{F dist}^{n} \text{ with the } (n_1, n_2) \text{ d.f}$$

& $$Z_2 = \frac{X_3/n_3}{X_1+X_2/n_1+n_2} \sim F_{n_3, n_1+n_2}$$

16.

$X \sim N(0,1)$ $\quad$ $Y \sim \chi^2_n$ $\quad$ – indep.

$$f_{X,Y} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{2^{n/2}\Gamma n/2} e^{-y/2} y^{n/2-1}$$

$$T = \frac{X}{\sqrt{Y/n}} \quad \text{define dummy } U = Y \quad \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} T \\ U = y \end{pmatrix}$$

$$\Rightarrow \quad X = T\sqrt{\frac{U}{n}} \qquad J = \begin{vmatrix} \sqrt{\frac{u}{n}} & \frac{t}{2\sqrt{n}\sqrt{u}} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{u}{n}}$$

$$Y = U$$

$$\Rightarrow \quad f_{T,U}(t,u) = \frac{1}{\sqrt{2\pi}\, 2^{n/2}\,\Gamma n/2\, \sqrt{n}} \exp\left(-\frac{1}{2}\frac{t^2 u}{n}\right) \exp\left(-\frac{u}{2}\right) u^{n/2 - \frac{1}{2}}$$

$-\infty < t < \infty$

$u > 0$.

$$f_T(t) = \int_0^\infty f_{T,U}(t,u)\, du =$$

$$= \frac{1}{\sqrt{2\pi}\, 2^{n/2}\,\Gamma n/2\, \sqrt{n}} \int_0^\infty u^{n/2-1/2} \exp\left(-\frac{u}{2}\left(1+\frac{t^2}{n}\right)\right) du$$

$$= \frac{\Gamma \frac{n+1}{2}}{\sqrt{2\pi}\, 2^{n/2}\,\Gamma n/2\, \sqrt{n}} \cdot \frac{1}{\left(\frac{1}{2}\left(1+\frac{t^2}{n}\right)\right)^{\frac{n+1}{2}}} \qquad -\infty < t < \infty$$

$$= \_\_\_\_\_\_$$

(17) $M_Y(t) = E(e^{tY}) = E\left(e^{t\sum_1^n X_i^2}\right)$

$$= \prod_{i=1}^{n} E\left(e^{tX_i^2}\right)$$

$$= \prod_{i=1}^{n} M_{X_i^2}(t)$$

$X_i^2 \sim \chi_1^2 \rightarrow \quad = \prod_{i=1}^{n} (1-2t)^{-1/2} = (1-2t)^{-n/2}$

$\Rightarrow Y \sim \chi_n^2$.

$\left.\begin{array}{l} X_{n+1} \sim N(0,1) \\ Y \sim \chi^2_n \end{array}\right\}$ indep.

jt p.d.f. of Y & X_{n+1}

$$f_{Y, X_{n+1}}(y, x) = \left(\frac{1}{2^{n/2}\,\Gamma n/2} e^{-y/2}\, y^{n/2-1}\right) \times \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right)$$

$$\left.\begin{array}{l} T = \dfrac{X_{n+1}}{\sqrt{Y/n}} \\ U = Y \end{array}\right\} \Rightarrow \begin{array}{l} X_{n+1} = T\sqrt{\dfrac{U}{n}} \\ Y = U \end{array}$$

$$J = \begin{vmatrix} \sqrt{\frac{u}{n}} & \frac{t}{2\sqrt{n}\sqrt{u}} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{u}{n}}.$$

jt p.d.f. of T & U

$$f_{T,U}(t, u) = \left(2^{n/2}\,\Gamma n/2\, \sqrt{2\pi}\, \sqrt{n}\right)^{-1} \exp\left(-\frac{1}{2}\frac{t^2 u}{n}\right) \exp\left(-\frac{u}{2}\right) u^{n/2-1};$$

$-\infty < t < \infty$

$u > 0$

$$f_T(t) = \left(2^{n/2}\,\Gamma_{n/2}\,\sqrt{2\pi}\,\sqrt{n}\right)^{-1} \int_0^{\infty} u^{\frac{n-1}{2}} \exp\left(-\frac{u}{2}\left(1+\frac{t^2}{n}\right)\right) du$$

$$= \left(2^{n/2}\,\Gamma_{n/2}\,\sqrt{2\pi}\,\sqrt{n}\right)^{-1} \frac{\Gamma_{\frac{n+1}{2}}}{\left(\frac{1}{2}\left(1+\frac{t^2}{n}\right)\right)^{n+1/2}} ; \quad -\infty < t < \infty .$$

$$= \frac{\Gamma_{\frac{n+1}{2}}}{\sqrt{\pi}\,\Gamma_{n/2}\,\sqrt{n}} \left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}} ; \quad -\infty < t < \infty$$

(18) $Z = X+Y \quad ; \quad z \in \{0,1,\ldots\}.$

$$P(Z=z) = P(X+Y=z) = P\left(\bigcup_{x=0}^{z}\left(X=x \cap Y=z-x\right)\right)$$

$$= \sum_{x=0}^{z} P(X=x \cap Y=z-x)$$

$$= \sum_{x=0}^{z} P(X=x)\, P(Y=z-x)$$

$$= \sum_{x=0}^{z} q^x p \; q^{z-x} p = p^2 \sum_{x=0}^{z} q^z$$

i.e. $P(Z=z) = \begin{cases} p^2 q^z (z+1), & z=0,1,\ldots \\ 0, & o/w. \end{cases}$

$$P(X=x, Z=z) = P(X=x, Y=z-x)$$

$$= \begin{cases} p^2 q^z & ; \quad x=0,1,\ldots z; \; z=0,1,\ldots \\ 0 & o/w. \end{cases}$$

Conditional p.m.f.

$$P(X=x \mid Z=z) = \frac{P(X=x, Z=z)}{P(Z=z)}$$

$$= \begin{cases} \frac{1}{z+1}, & x=0,\dots,z; \\ 0, & o/w. \end{cases}$$

$\Rightarrow X \mid Z=z \sim$ discrete uniform on $(0, 1, \dots z)$.

(b):

$Z = \min(X, Y)$

$$P(Z=z) = P(X=z, Y=z) + P(X=z, Y>z) + P(X>z, Y=z)$$

$$= (pq^z)^2 + \sum_{y=z+1}^{\infty} P(X=z, Y=y) + \sum_{x=z+1}^{\infty} P(X=x, Y=z)$$

$$= (pq^z)^2 + \sum_{y=z+1}^{\infty} pq^z\, pq^y + \sum_{x=z+1}^{\infty} pq^x\, pq^z$$

$$= (pq^z)^2 + 2p^2 q^z \sum_{x=z+1}^{\infty} q^x$$

$$= (pq^z)^2 + 2p^{\cancel{2}} q^z q^{z+1} \cancel{(1-q)^{-1}}$$