MSO 201a: Probability and Statistics

2015-2016: II Semester

End Semester Examination

Time Allowed: 3 Hrs. Maximum Marks: 100

NOTE: (i) Attempt all the parts of a question at one place.

(ii) For each question, provide details of all the steps involved in arriving at the final answer(s)/conclusion(s) and box your final answer(s)/conclusion(s).

(iii) Write legibly. Answers in illegible handwriting will not be graded.

1. Let $\{E_n\}_{n\geq 1}$ and $\{F_n\}_{n\geq 1}$ be two sequences of events in a probability space with probability function $P(\cdot)$. Suppose that $P(E_n) = 0$ and $P(F_n) = 1$, $n = 1, 2, \ldots$. Show that $P(\bigcup_{n=1}^{\infty} E_n) = 0$. Hence show that $P(\bigcap_{n=1}^{\infty} F_n) = 1$. $\boxed{\text{10 MARKS}}$

2. Suppose that the distribution of marks of students in a class test has mean 70 and variance 10. Show that the probability of a student having marks between 60 and 80 is atleast 0.9. $\boxed{\text{10 MARKS}}$

3. Let $X \sim \mathrm{N}(0,1)$ and let Y be a r.v. such that $Y = X$, if $|X| \leq 1$, $= -X$, if $|X| > 1$. Find the distribution function of Y and hence find the m.g.f. of Y. $\boxed{\text{10 MARKS}}$

4. Let X and Y, respectively, denote the proportions of iron and bronze present in a specific metal. Suppose that the joint p.d.f. of (X, Y) is given by:

$$f_{X,Y}(x,y) = \begin{cases} 2, & \text{if } x + y \leq 1,\ x \geq 0,\ y \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Find $P(|X - Y| \leq \frac{1}{4})$. $\boxed{\text{10 MARKS}}$

5. Let $\underline{X} = (X_1, X_2)' \sim N_2(0, 0, 1, 1, \frac{1}{2})$ and, let $Y = 2X_1 + X_2$ and $Z = 4X_1 - 5X_2$. Find the joint p.d.f. of (Y, Z) and hence find the p.d.f. of $T = \frac{Y^2}{7} + \frac{Z^2}{21}$. $\boxed{\text{10 MARKS}}$

6. Let $X_1, X_2, \ldots, X_8$ be a random sample from $\mathrm{U}(0,4)$ distribution and let $Y_1 \leq Y_2 \leq \cdots \leq Y_8$ be the corresponding order statistics (i.e., $Y_r = r$-th smallest of $X_1, \ldots, X_8$, $r = 1, \ldots, 8$). Find the probability that $P(Y_3 < 1 < Y_5 < Y_6 < Y_7 < 3 < Y_8)$. $\boxed{\text{10 MARKS}}$

7. Let Z_1 and Z_2 be i.i.d. $\mathrm{N}(0,1)$ r.v.s. Show that $\frac{Z_1}{Z_2} \stackrel{d}{=} \frac{Z_1}{|Z_2|}$. Hence show that $Z = \frac{Z_1}{Z_2}$ follows Cauchy distribution (i.e., $Z \sim t_1$). $\boxed{\text{10 MARKS}}$

8. Let X_1, X_2 and X_3 be i.i.d. random variables with $P(\{X_1 = X_2\}) = \frac{1}{4}$, $P(\{X_1 = X_2 = X_3\}) = \frac{1}{12}$ and $P(\{X_1 = X_2 < X_3\}) = \frac{1}{12}$. Find $P(\{X_3 < X_1 < X_2\})$. 10 MARKS

9. Let $Y_n \sim \chi_n^2$, $n = 1, 2, \ldots$, and let

$$Z_n = \frac{Y_n - n}{\sqrt{n}}, \quad n = 1, 2, \ldots.$$

(a) Using WLLN, show that $\frac{Y_n}{n} \xrightarrow{p} 1$, as $n \to \infty$; 5 MARKS

(b) Find the limiting distribution of Z_n, as $n \to \infty$. 5 MARKS

(Hint: Use reproductive property of chi-squared distribution.)

10. Let $X_1, \ldots, X_n$ be a random sample from a $\mathrm{U}(0, \theta)$ distribution, where $\theta \in (0, \infty) = \Theta$ is unknown, and let the estimand be $g(\theta) = \theta$.

(a) Find the M.M.E. and the M.L.E. of $g(\theta)$; 5 MARKS

(b) Find an unbiased estimator of $g(\theta)$ that is based on M.L.E. 5 MARKS

Some Useful Information

Standard Distributions

- **Binomial Distribution**: The r.v. $X \sim \mathrm{B}(n, p)$ has p.m.f. $f_X(x) = \binom{n}{x} p^x (1-p)^{n-x} I_{\{0,1,\ldots,n\}}(x)$, m.g.f. $M_X(t) = (q + pe^t)^n$, $t \in \mathbb{R}$, $E(X) = np$, and $\mathrm{Var}(X) = npq$; here $q = 1 - p$.

- **Bernoulli distribution**: A $B(1, p)$ distribution is called a Bernoulli distribution.

- **Negative Binomial Distribution**: The r.v. $X \sim \mathrm{NB}(r, p)$ has p.m.f. $f_X(x) = \binom{x+r-1}{r-1} p^r q^x I_{\{0,1,2,\ldots\}}(x)$, m.g.f. $M_X(t) = p^r (1 - qe^t)^{-r}$, $t < -\ln q$, $E(X) = \frac{rq}{p}$, and $\mathrm{Var}(X) = \frac{rq}{p^2}$; here $q = 1 - p$.

- **Hypergeometric Distribution**: The r.v. $X \sim \mathrm{Hyp}(a, n, N)$ has p.m.f. $f_X(x) = \frac{\binom{a}{x}\binom{N-a}{n-x}}{\binom{N}{n}} I_{S_X}(x)$, where $S_X = \{x \in \mathbb{N} : \max\{0, n - N + a\} \leq x \leq \min\{n, a\}\}$, $E(X) = n\frac{a}{N}$, and $\mathrm{Var}(X) = n\left(\frac{a}{N}\right)\left(1 - \frac{a}{N}\right)\frac{N-n}{N-1}$. A closed form expression for m.g.f. of X cannot be obtained.

- **Poisson Distribution**: The r.v. $X \sim \mathrm{P}(\lambda)$ has p.m.f. $f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} I_{\{0,1,\ldots\}}(x)$, m.g.f. $M_X(t) = e^{\lambda(e^t - 1)}$, $t \in \mathbb{R}$, $E(X) = \mathrm{Var}(X) = \lambda$.

- **Discrete Uniform Distribution:** The r.v. $X \sim \mathrm{U}(1-N)$ has p.m.f. $f_X(x) = \frac{1}{N} I_{\{1,2,\ldots,N\}}(x)$, m.g.f. $M_X(t) = \frac{1}{N}\sum\limits_{x=1}^{N} e^{tx}$, $E(X) = \frac{N+1}{2}$, and $\mathrm{Var}(X) = \frac{N^2-1}{12}$.

- **Uniform or Rectangular Distribution:** The r.v. $X \sim \mathrm{U}(\alpha,\beta)$ has p.d.f. $f_X(x) = \frac{1}{\beta-\alpha} I_{(\alpha,\beta)}(x)$, m.g.f. $M_X(t) = \frac{e^{t\beta}-e^{t\alpha}}{t(\beta-\alpha)}$, if $t \neq 0, = 1$, if $t = 0$, $E(X) = \frac{\alpha+\beta}{2}$, and $\mathrm{Var}(X) = \frac{(\beta-\alpha)^2}{12}$.

- **Gamma Distribution:** The r.v. $X \sim \mathrm{G}(\alpha,\beta)$ has p.d.f. $f_X(x) = \frac{1}{\theta^\alpha \Gamma(\alpha)} e^{-\frac{x}{\theta}} x^{\alpha-1} I_{(0,\infty)}(x)$, m.g.f. $M_X(t) = (1-t\theta)^{-\alpha}$, $t < \frac{1}{\theta}$, $E(X) = \alpha\theta$ and $\mathrm{Var}(X) = \alpha\theta^2$.

- **Exponential Distribution:** A $\mathrm{G}(1,\theta)$ distribution, $\theta > 0$, is called $\mathrm{Exp}(\theta)$ distribution.

- **Chi-Squared Distribution:** A $\mathrm{G}(\frac{n}{2},2)$ distribution is called χ^2_n distribution (chi-squared distribution with n degrees of freedom), $n = 1,2,\ldots$.

- **Beta Distribution:** The r.v. $X \sim \mathrm{B}(a,b)$ has p.d.f. $f_X(x) = \frac{1}{\beta(a,b)} x^{a-1}\,(1-x)^{b-1} I_{(0,1)}(x)$, m.g.f. $M_X(t) = \sum\limits_{r=0}^{\infty} \frac{\Gamma(a+r)\,\Gamma(a+b)}{\Gamma(a)\,\Gamma(a+b+r)} \frac{t^r}{r!}$, $t \in \mathbb{R}$, $E(X) = \frac{a}{a+b}$, $\mathrm{Var}(X) = \frac{ab}{(a+b)^2\,(a+b+1)}$.

- **Normal or Gaussian Distribution:** The r.v. $X \sim \mathrm{N}(\mu,\sigma^2)$ has p.d.f. $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} I_{(-\infty,\infty)}(x)$, m.g.f. $M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$, $t \in \mathbb{R}$, $E(X) = \mu$, and $\mathrm{Var}(X) = \sigma^2$.

- **Student t Distribution:** The r.v. $X \sim \mathrm{t}_m$ has p.d.f. $f_X(x) = \frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{m\pi}\,\Gamma(\frac{m}{2})} \frac{1}{\left(1+\frac{x^2}{m}\right)^{\frac{m+1}{2}}} I_{(-\infty,\infty)}(x)$, $E(X) = 0$, for $m \in \{2,3,\ldots\}$ and $\mathrm{Var}(X) = \frac{m}{m-2}$, for $m \in \{3,4,\ldots\}$. A closed form expression for m.g.f. of X cannot be obtained.

- **Snedecor F Distribution:** The r.v. $X \sim \mathrm{F}(n_1,n_2))$ has p.d.f. $f_X(x) = \frac{(\frac{n_1}{n_2})}{\beta\left(\frac{n_1}{2},\frac{n_2}{2}\right)} \frac{(\frac{n_1}{n_2}x)^{\frac{n_1}{2}-1}}{\left(1+\frac{n_1}{n_2}x\right)^{\frac{n_1+n_2}{2}}} I_{(0,\infty)}(x)$; $E(X) = \frac{n_2}{n_2-2}$, if $n_2 \in \{3,4,\ldots\}$ and $\mathrm{Var}(X) = \frac{2n_2^2(n_1+n_2-2)}{n_1(n_2-2)^2(n_2-4)}$, if $n_2 \in \{5,6,\ldots\}$.

- **Cauchy Distribution:** The r.v. $X \sim \mathrm{t}_1$ distribution is called standard Cauchy distribution and it has p.d.f. $f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2} I_{(-\infty,\infty)}(x)$, $E(X)$ does not exist.

- **Multinomial Distribution:** The r.v. $\underline{X} = (X_1,\ldots,X_p) \sim \mathrm{Mult}(n,\theta_1,\ldots,\theta_p)$ has joint p.m.f. $f_{\underline{X}}(x_1,\ldots,x_p) = \frac{n!}{x_1!\cdots x_p!\left(n-\sum\limits_{i=1}^{p} x_i\right)!} \theta_1^{x_1}\ldots\theta_p^{x_p}\left(1-\sum\limits_{i=1}^{p}\theta_i\right)^{(n-\sum_{i=1}^{p} x_i)} I_{S_{\underline{X}}}(\underline{x})$, where $S_{\underline{X}} = \{\underline{x} = (x_1,x_2,\ldots,x_p) : x_i \in \{0,1,\ldots,n\}, i = 1,\ldots,p, \sum_{i=1}^{p} x_i \leq n\}$, joint m.g.f. $M_{\underline{X}}(\underline{t}) = \left(\theta_1 e^{t_1} + \cdots + \theta_p e^{t_p} + 1 - \sum_{i=1}^{p}\theta_i\right)^n$, $\underline{t} \in \mathbb{R}^p$, $E(X_i) = n\theta_i$, $\mathrm{Var}(X_i) = n\theta_i(1-\theta_i)$, $i = 1,\ldots,p$, $\mathrm{Cov}(X_i,X_j) = -n\theta_i\theta_j$, $i \neq j$.

- **Bivariate Normal Distribution**: The joint p.d.f. of $\underline{X} = (X_1, X_2) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ is $f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}$, $\underline{x} = (x_1, x_2) \in \mathbb{R}^2$, m.g.f. $M_{X_1,X_2}(t_1,t_2) = e^{\mu_1 t_1 + \mu_2 t_2 + \frac{\sigma_1^2 t_1^2}{2} + \frac{\sigma_2^2 t_2^2}{2} + \rho\sigma_1\sigma_2 t_1 t_2}$, $\underline{t} = (t_1, t_2) \in \mathbb{R}^2$, $E(X_i) = \mu_i$, $\mathrm{Var}(X_i) = \sigma_i^2$, $i = 1, 2$, $\mathrm{Cov}(X_1, X_2) = \rho\sigma_1\sigma_2$, $\mathrm{Corr}(X_1, X_2) = \rho$.

Some important results

- **Boole's Inequaltiy**: For events $E_1, E_2, \ldots, E_n$, $\mathrm{P}\left(\bigcup_{i=1}^n E_i\right) \le \sum_{i=1}^n \mathrm{P}\left(E_i\right)$.

- **Bonferroni's Inequality**: For events $E_1, E_2, \ldots, E_n$, $P\left(\bigcap_{i=1}^n E_i\right) \ge \sum_{i=1}^n P\left(E_i\right) - (n-1)$.

- **Markov Inequality**: Suppose that $E(|X|) < \infty$. Then, for any $c > 0$, $P(\{|X| > c\}) \le \frac{E(|X|)}{c}$.

- **Chebychev's Inequality**: Let X be a r.v. with finite mean $\mu = E(X)$ and finite variance $\sigma^2 = E((X-\mu)^2)$. Then, for any $\epsilon > 0$, $P\left(\{|X-\mu| \ge \epsilon\sigma\}\right) \le \frac{1}{\epsilon^2}$, or, equivalently, $P\left(\{|X-\mu| < \epsilon\sigma\}\right) \ge 1 - \frac{1}{\epsilon^2}$.

- **Jensen's Inequality**: Let X be a r.v. with support $S_X \subseteq (a,b)$ and let $\phi : (a,b) \to \mathbb{R}$ be a convex (concave) function; here $-\infty \le a < b \le \infty$. Then $E(\phi(X)) \ \ge \ (\le) \ \phi(E(X))$, provided the expectations exist.

- $X \sim \mathrm{F}_{n_1,n_2}, \Rightarrow Y = \frac{\frac{n_1}{n_2}X}{1+\frac{n_1}{n_2}X} \sim \mathrm{Beta}(\frac{n_1}{2}, \frac{n_2}{2})$, the beta distribution with shape parameter $(\frac{n_1}{2}, \frac{n_2}{2})$.

- Let $Z \sim \mathrm{N}(0,1)$ and $Y \sim \chi^2_m$ (where $m \in \{1, 2, \ldots\}$) be independent random variables. Then $T = \frac{Z}{\sqrt{\frac{Y}{m}}} \sim \mathrm{t}_m$.

- For positive integers n_1 and n_2, let $X_1 \sim \chi^2_{n_1}$ and $X_2 \sim \chi^2_{n_2}$ be independent random variables. Then $U = \frac{X_1/n_1}{X_2/n_2} \sim \mathrm{F}_{n_1,n_2}$.

- **Weak Law of Large Numbers (WLLN)**: For a random sample $X_1, \ldots, X_n$, suppose that $E(X_1) = \mu$ is finite. Then $\bar{X}_n \xrightarrow{P} \mu$, as $n \to \infty$.

- **Central Limit Theorem (CLT)**: For a random sample $X_1, \ldots, X_n$, suppose that $E(X_1) = \mu$ and $0 < \mathrm{Var}(X_1) = \sigma^2 < \infty$. Then $Z_n \stackrel{\text{def}}{=} \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0,1)$, as $n \to \infty$.

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Problem No. 1 By Boole's inequality

$$0 \le P\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} P(E_n) = 0$$

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} E_n\right) = 0. \quad \cdots \quad \boxed{4 \text{ MARKS}}$$

$$P(F_n) = 1 \Rightarrow P(F_n^c) = 1 - P(F_n) = 0, \quad n = 1, 2, \ldots$$

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} F_n^c\right) = 0 \quad \text{(by first assertion)}$$

$$\Rightarrow P\left(\left(\bigcup_{n=1}^{\infty} F_n^c\right)^c\right) = 1 - P\left(\bigcup_{n=1}^{\infty} F_n^c\right) = 1$$

$$\Rightarrow P\left(\bigcap_{n=1}^{\infty} F_n\right) = 1 \quad \text{(De-Morgan's Law)}$$

$$\cdots \quad \boxed{6 \text{ MARKS}}$$

Problem No. 2 Let X = distribution of marks of students.

Then $\mu = E(X) = 70$ and $\sigma^2 = Var(X) = 10$.

$$\begin{aligned}
\text{Required probability} &= P(60 < X < 80) \\
&= P(-10 < X - \mu < 10) \\
&= P(|X - \mu| < 10) \quad \cdots \quad \boxed{4 \text{ MARKS}} \\
&\ge 1 - \frac{\sigma^2}{10^2} \quad \text{(Chebychev Inequality)} \\
&= 1 - \frac{10}{100} = 0.9 \quad \cdots \quad \boxed{6 \text{ MARKS}}
\end{aligned}$$

Problem No. 3 For $y \in \mathbb{R}$

$$\begin{aligned}
F_Y(y) &= P(Y \le y) \\
&= P(Y \le y, |X| \le 1) + P(Y \le y, |X| > 1) \\
&= P(X \le y, -1 \le X \le 1) + P(-X \le y, |X| > 1) \\
&= P(X \le y, -1 \le X \le 1) + P(-X \le y, X < -1) + P(-X \le y, X > 1) \\
&= P(-1 \le X \le \min\{1, y\}) + P(-y \le X < -1) + P(X > \max\{1, -y\}) \\
&= \begin{cases} P(X > -y), & \text{if } y < -1 \\ P(-1 \le X \le y) + P(X > 1), & \text{if } -1 \le y < 1 \\ P(-1 \le X \le 1) + P(-y \le X \le -1) + P(X \ge 1), & \text{if } y \ge 1 \end{cases} \quad \cdots \quad \boxed{3 \text{ MARKS}}
\end{aligned}$$

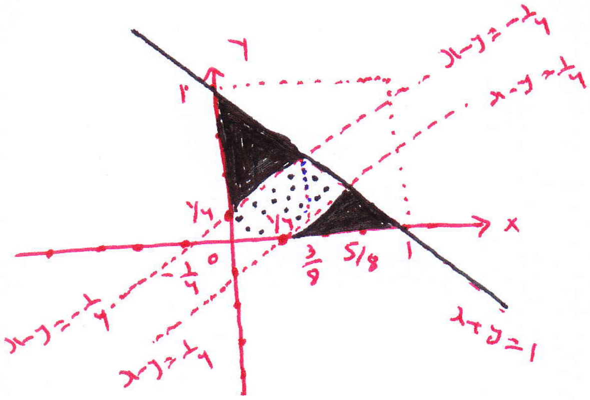
$$= \begin{cases} 1-\Phi(-y), & \text{if } y<-1 \\ \Phi(y)-\Phi(-1)+1-\Phi(1), & \text{if } -1\le y<1 \\ \Phi(1)-\Phi(-1)+\Phi(-1)-\Phi(-y) + 1-\Phi(1), & \text{if } y\ge 1 \end{cases}$$

$= \Phi(y)$ (since $\Phi(t)+\Phi(-t)=1 \ \forall t\in\mathbb{R}$) .... $\boxed{3 \text{ MARKS}}$

$\Rightarrow Y \sim N(0,1)$

$\Rightarrow M_Y(t) = e^{\frac{t^2}{2}}, \ t\in\mathbb{R}$. ... $\boxed{4 \text{ MARKS}}$

Problem No. 4



$P(|X-Y|\le \frac{1}{4}) = P(X-\frac{1}{4} \le Y \le X+\frac{1}{4})$

$= P(X-\frac{1}{4}\le Y\le X+\frac{1}{4},\ Y\le 1-X,\ X\ge 0,\ Y\ge 0)$

$= P(\max\{X-\frac{1}{4}, 0\} \le Y \le \min\{X+\frac{1}{4}, 1-X\},\ X\ge 0)$

$= P(0\le X\le \frac{1}{4},\ 0\le Y\le X+\frac{1}{4}) + P(\frac{1}{4}\le X\le \frac{3}{8},\ X-\frac{1}{4}\le Y\le X+\frac{1}{4})$
$+ P(\frac{3}{8}\le X,\ X-\frac{1}{4}\le Y\le 1-X)$

$= P(0\le X\le \frac{1}{4},\ 0\le Y\le X+\frac{1}{4}) + P(\frac{1}{4}\le X\le \frac{3}{8},\ X-\frac{1}{4}\le Y\le X+\frac{1}{4})$
$+ P(\frac{3}{8}\le X\le \frac{5}{8},\ X-\frac{1}{4}\le Y\le 1-X)$

$= \int_0^{1/4}\int_0^{x+\frac{1}{4}} 2\,dy\,dx + \int_{1/4}^{3/8}\int_{x-\frac{1}{4}}^{x+\frac{1}{4}} 2\,dy\,dx + \int_{3/8}^{5/8}\int_{x-\frac{1}{4}}^{1-x} 2\,dy\,dx$ .... $\boxed{5 \text{ MARKS}}$

$= \frac{7}{16}$. .... $\boxed{5 \text{ MARKS}}$

Alt: Required prob $= 2\times$ area of dotted region
$= 1 - 2\times$ area of shaded region ... $\boxed{5 \text{ MARKS}}$
$= 1 - 2\left[\frac{1}{2}\times\frac{3}{4}\times\frac{3}{8} + \frac{1}{2}\times\frac{3}{4}\times\frac{3}{8}\right] = \frac{7}{16}$
..... $\boxed{5 \text{ MARKS}}$

Problem No 5

For $a, b \in \mathbb{R}$, consider linear combination

$$aY + bZ = a(2X_1 + X_2) + b(4X_1 - 5X_2)$$
$$= (2a+4b)X_1 + (a-5b)X_2$$
$$= \text{a linear combination of } X_1 \text{ and } X_2$$
$$\sim N_1(\cdot,\cdot) \quad (\text{since } (X_1, X_2) \sim N_2(\cdot,\cdot,\cdot,\cdot,\cdot))$$

$E(Y) = E(2X_1 + X_2) = 0$; $\quad Var(Y) = Var(2X_1 + X_2)$
$$= 4Var(X_1) + Var(X_2) + 4Cov(X_1, X_2)$$
$$= 4 + 1 + 2 = 7$$

$E(Z) = E(4X_1 - 5X_2) = 0$

$Var(Z) = Var(4X_1 - 5X_2) = 16Var(X_1) + 25Var(X_2) - 40Cov(X_1, X_2)$
$$= 16 + 25 - 20 = 21$$

$Cov(Y, Z) = Cov(2X_1 + X_2, 4X_1 - 5X_2) = 8Var(X_1) - 6Cov(X_1, X_2) - 5Var(X_2)$
$$= 8 - 3 - 5 = 0$$

$(Y, Z) \sim N_2(0, 0, 7, 21, 0)$ ... $\boxed{5 \text{ MARKS}}$

$\Rightarrow f_{Y,Z}(y, z) = \frac{1}{2\pi} e^{-\frac{1}{2}\left(\frac{y^2}{7} + \frac{z^2}{21}\right)}, \quad -\infty < y, z < \infty$

$\Rightarrow Y \sim N(0, 7)$ and $Z \sim N(0, 21)$ are independent

$\Rightarrow \frac{Y}{\sqrt{7}}$ and $\frac{Z}{\sqrt{21}}$ are i.i.d. $N(0,1)$

$\Rightarrow \frac{Y^2}{7}$ & $\frac{Z^2}{21}$ are i.i.d. χ^2_1

$\Rightarrow \frac{Y^2}{7} + \frac{Z^2}{21} \sim \chi^2_2 = Exp(2)$

$$f_T(t) = \begin{cases} \frac{1}{2} e^{-\frac{t}{2}}, & t > 0 \\ 0, & \text{otherwise.} \end{cases}$$ ... $\boxed{5 \text{ MARKS}}$

Problem No. 6

$P(Y_3 < 1 < Y_5 < Y_6 < Y_7 < 3 < Y_8)$

$= P(Y_4 < 1 < Y_5 < Y_6 < Y_7 < 3 < Y_8) + P(Y_3 < 1 < Y_4 < Y_5 < Y_6 < Y_7 < 3 < Y_8)$

$P(Y_4 < 1 < Y_5 < Y_6 < Y_7 < 3 < Y_8)$

$= P(4\ X_i'\text{s in } (0,1),\ 3\ X_i'\text{s in } (1,3) \text{ and } 1\ X_i \text{ in } (3,4))$

$= \dfrac{\lfloor 8}{\lfloor 4\ \lfloor 3\ \lfloor 1} \left(\frac{1}{4}\right)^4 \left(\frac{2}{4}\right)^3 \left(\frac{1}{4}\right)^1 = \dfrac{35}{1024}$ ... $\boxed{\text{4 MARKS}}$

$P(Y_3 < 1 < Y_4 < Y_5 < Y_6 < Y_7 < 3 < Y_8)$

$= P(3\ X_i'\text{s in } (0,1),\ 4\ X_i'\text{s in } (1,3) \text{ and } 1\ X_i \text{ in } (3,4))$

$= \dfrac{\lfloor 8}{\lfloor 3\ \lfloor 4\ \lfloor 1} \left(\frac{1}{4}\right)^3 \left(\frac{2}{4}\right)^4 \left(\frac{1}{4}\right)^1 = \dfrac{35}{512}$ ... $\boxed{\text{4 MARKS}}$

Thus

$P(Y_3 < 1 < Y_5 < Y_6 < Y_7 < 3 < Y_8) = \dfrac{35}{1024} + \dfrac{35}{512} = \dfrac{105}{1024}$

... $\boxed{\text{2 MARKS}}$

$\boxed{\text{Problem No. 7}}$ Let $Z = \dfrac{Z_1}{Z_2}$ and $Y = \dfrac{Z_1}{|Z_2|}$.

For $y \in \mathbb{R}$

$F_Y(y) = P\left(\frac{Z_1}{|Z_2|} \le y\right)$

$= P(Z_1 \le |Z_2| y)$

$= P(Z_1 \le -Z_2 y,\ Z_2 < 0) + P(Z_1 \le Z_2 y,\ Z_2 > 0)$

Since Z_1, Z_2 are i.i.d. $N(0,1)$, $(Z_1, Z_2) \stackrel{d}{=} (-Z_1, Z_2)$.

Thus

$P(Z_1 \le -Z_2 y,\ Z_2 < 0) = P(-Z_1 \le -Z_2 y,\ Z_2 < 0)$

$= P(Z_1 \ge Z_2 y,\ Z_2 < 0)$.

Thus, for $y \in \mathbb{R}$,

$F_Y(y) = P(Z_1 \ge Z_2 y,\ Z_2 < 0) + P(Z_1 \le Z_2 y,\ Z_2 > 0)$

$= P\left(\frac{Z_1}{Z_2} \le y,\ Z_2 < 0\right) + P\left(\frac{Z_1}{Z_2} \le y,\ Z_2 > 0\right)$

$= P\left(\frac{Z_1}{Z_2} \le y\right) = F_Z(y)$

$\Rightarrow \quad Y \stackrel{d}{=} Z$

i.e. $\dfrac{Z_1}{|Z_2|} \stackrel{d}{=} \dfrac{Z_1}{Z_2} = Z$ .... $\boxed{\text{5 MARKS}}$

Z_1 and Z_2 are i.i.d. $N(0,1)$

$\Rightarrow Z_1 \sim N(0,1)$ and $Z_2^2 \sim \chi^2_1$ are independent

$$\Rightarrow \frac{Z_1}{\sqrt{Z_2^2/1}} \sim t_1 \Rightarrow \frac{Z_1}{|Z_2|} \sim t_1 \Rightarrow \frac{Z_1}{Z_2} \sim t_1$$

.... $\boxed{\text{5 MARKS}}$

$\boxed{\text{Problem No. 8}}$ X_1, X_2, X_3 are i.i.d.

$\Rightarrow (X_1, X_2, X_3) \stackrel{d}{=} (X_{\beta_1}, X_{\beta_2}, X_{\beta_3})$, $\forall$ permutation $\underline{\beta} = (\beta_1, \beta_2, \beta_3)$ of $(1, 2, 3)$.

Thus

P(all possible orderings between X_1, X_2 and X_3) = 1

$$\Rightarrow 6P(X_1 < X_2 < X_3) + 3P(X_1 = X_2 < X_3) + 3P(X_1 < X_2 = X_3) + P(X_1 = X_2 = X_3) = 1$$

.... $\boxed{\text{4 MARKS}}$

$$P(X_1 < X_2 = X_3) = P(X_2 = X_3) - P(X_1 = X_2 = X_3) - P(X_2 = X_3 < X_1)$$
$$= P(X_1 = X_2) - P(X_1 = X_2 = X_3) - P(X_1 = X_2 < X_3)$$
$$= \frac{1}{4} - \frac{1}{12} - \frac{1}{12} = \frac{1}{12}$$

.... $\boxed{\text{3 MARKS}}$

$$6P(X_1 < X_2 < X_3) + \frac{3}{12} + \frac{3}{12} + \frac{1}{12} = 1$$

$$\Rightarrow P(X_1 < X_2 < X_3) = \frac{5}{72}$$

$$\Rightarrow P(X_3 < X_1 < X_2) = \frac{5}{72}.$$

.... $\boxed{\text{3 MARKS}}$

$\boxed{\text{Problem No. 9}}$ (a) Let $X_1, X_2, \ldots$ be a sequence of i.i.d. χ^2_1 r.v.s. Then, by reproductive property of chi-squared distribution $Y_n \stackrel{d}{=} \sum_{i=1}^{n} X_i$. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{Y_n}{n}$.

... $\boxed{\text{2 MARKS}}$

By WLLN

$$\bar{X}_n \xrightarrow{p} \mu = E(X_1) = 1, \text{ as } n \to \infty$$

$$\Rightarrow \frac{Y_n}{n} \xrightarrow{p} 1, \text{ as } n \to \infty$$

$\boxed{\text{3 MARKS}}$

(b) Let $X_1, X_2, \ldots$ be a sequence of i.i.d. χ^2_1 r.v.s. Then

$$\frac{Y_n}{n} \stackrel{d}{=} \bar{X}_n.$$

$\mu = E(X_1) = 1$, $\sigma^2 = Var(X_1) = 2$. By CLT

$$\frac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma} \xrightarrow{d} Z \sim N(0,1), \text{ as } n\to\infty$$

$$\Rightarrow \frac{\sqrt{n}\left(\frac{Y_n}{n}-1\right)}{2} \xrightarrow{d} Z\sim N(0,1), \text{ as } n\to\infty \ldots \boxed{2 \text{ MARKS}}$$

$$\Rightarrow \frac{Y_n-n}{2\sqrt{n}} \xrightarrow{d} Z \sim N(0,1), \text{ as } n\to\infty$$

$$\Rightarrow \frac{Y_n-n}{\sqrt{n}} \xrightarrow{d} 2Z \sim N(0,4), \text{ as } n\to\infty.$$

$$\Rightarrow Z_n \xrightarrow{d} T\sim N(0,4) \text{ as } n\to\infty \ldots \boxed{3 \text{ MARKS}}$$

$\boxed{\text{Problem No. 10}}$ (a) $E(X_1)=\frac{\theta}{2}$. Thus

$$\frac{\hat{\theta}_{MME}}{2} = \bar{X} \Rightarrow \hat{\theta}_{MME} = 2\bar{X} \ldots \boxed{2 \text{ MARKS}}$$

For a fixed sample realization $\underline{x}\in(0,\infty)^n$,

$$L_{\underline{x}}(\theta) = \prod_{i=1}^n f(x_i\mid\theta) = \prod_{i=1}^n \frac{1}{\theta} I_{(0,\theta)}(x_i) = \begin{cases} \frac{1}{\theta^n}, & \text{if } \theta > x_{(n)} \\ 0, & \text{if } 0<\theta\le x_{(n)} \end{cases}$$

here $x_{(n)} = \max\{x_1,\ldots,x_n\}$.

Clearly, for fixed $\underline{x}\in(0,\infty)^n$, $L_{\underline{x}}(\theta)$ is maximized at $\hat{\theta}=x_{(n)}$. Thus

$$\hat{\theta}_{MLE} = X_{(n)} = \max\{X_1,\ldots,X_n\} \ldots \boxed{3 \text{ MARKS}}$$

(b) We have to find an unbiased estimator of $g(\theta)=\theta$ based on $X_{(n)}$. We have

$$F_{X_{(n)}}(x) = P(X_{(n)}\le x) = P(X_i\le x, \ i=1,\ldots,n) = \prod_{i=1}^n P(X_i\le x) = [F(x)]^n$$

$$= \begin{cases} 0, & \text{if } x<0 \\ \left(\frac{x}{\theta}\right)^n, & \text{if } 0\le x<\theta \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow f_{X_{(n)}}(x) = \begin{cases} \frac{n x^{n-1}}{\theta^n}, & \text{if } 0<x<\theta \quad (\text{p.d.f. of } X_{(n)}) \\ 0, & \text{otherwise} \end{cases} \ldots \boxed{2 \text{ MARKS}}$$

$$E(X_{(n)}) = \int_0^\theta x \frac{n x^{n-1}}{\theta^n}\,dx = \frac{n}{n+1}\theta, \quad \forall \theta>0$$

$$\Rightarrow E\left(\frac{n+1}{n}X_{(n)}\right) = \theta, \quad \forall\theta>0$$

Thus the unbiased estimator of $g(\theta)=\theta$ based on MLE is

$$\hat{\delta}(\underline{X}) = \frac{n+1}{n}X_{(n)}. \ldots \boxed{3 \text{ MARKS}}$$

$\boxed{\text{Problem NO. 3}}$ Aliter

The p.d.f. of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in S_X = (-\infty, \infty)$$

$$Y = \begin{cases} X, & \text{if } |X| \le 1 \\ -X, & \text{if } |X| > 1 \end{cases} = \begin{cases} -X, & \text{if } X < -1 \\ X, & \text{if } -1 \le X \le 1 \\ -X, & \text{if } X > 1 \end{cases}$$

$= g(X)$ (say).

$g(S_X) = (-\infty, \infty)$. The transformation $g: \mathbb{R} \to \mathbb{R}$ is 1-1 with inverse transformation

$$g^{-1}(y) = \begin{cases} -y, & \text{if } y < -1 \\ y, & \text{if } -1 \le y \le 1 \\ -y, & \text{if } y > 1 \end{cases}$$

$$\frac{d}{dy} g^{-1}(y) = \pm 1.$$

Thus the p.d.f. of Y is

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| I_{g(S_X)}(y) \quad \ldots \boxed{\text{3 MARKS}}$$

$$= \begin{cases} f_X(-y), & \text{if } y < -1 \text{ or } y > 1 \\ f_X(y), & \text{if } -1 \le y \le 1 \end{cases}$$

$$= f_X(y), \quad \forall -\infty < y < \infty \quad (\text{since } f_X(y) = f_X(-y) \ \forall y)$$

$\ldots \boxed{\text{3 MARKS}}$

$$\Rightarrow Y \overset{d}{=} X \sim N(0, 1)$$

$$\Rightarrow M_Y(t) = e^{\frac{t^2}{2}}, \quad -\infty < t < \infty$$

$\ldots \boxed{\text{4 MARKS}}$

Problem 6 (Alternate Solution)

The joint pdf of $(Y_1, \ldots, Y_8)$ is

$$g(y_1, \ldots, y_8) = \frac{\lfloor 8}{4^8}, \quad 0 < y_1 < y_2 < \cdots < y_8 < 4$$

$$\text{Required probability} = \int \cdots \int_{\substack{0 < y_1 < \cdots < y_8 < 4 \\ y_3 < 1 < y_5 < y_7 < 3 < y_8}} \frac{\lfloor 8}{4^8} \, dy_1 \ldots dy_8$$

.... 4 MARKS

$$= \frac{105}{1024}$$

.... 6 MARKS

Problem No 8 (Alternate Solution)

$$2P(X_1 < X_2) + P(X_1 = X_2) = 1$$

$$\Rightarrow P(X_1 < X_2) = \frac{3}{8}$$

.... 3 MARKS

$$\Rightarrow 3P(X_3 < X_1 < X_2) + P(X_3 = X_1 < X_2) + P(X_1 < X_2 = X_3) = \frac{3}{8}$$

$$\Rightarrow P(X_3 < X_1 < X_2) = \frac{1}{3}\left[\frac{3}{8} - P(X_1 = X_2 < X_3) - P(X_1 < X_2 = X_3)\right]$$

$$= \frac{1}{3}\left[\frac{7}{24} - P(X_1 < X_2 = X_3)\right]$$

.... 4 MARKS

$$P(X_1 < X_2 = X_3) = P(X_2 = X_3) - P(X_1 = X_2 = X_3) - P(X_2 = X_3 < X_1)$$

$$= \frac{1}{4} - \frac{1}{12} - \frac{1}{12} = \frac{1}{12}$$

$$\Rightarrow P(X_3 < X_1 < X_2) = \frac{5}{72}$$

.... 3 MARKS