

Problem Set # 11

(1)

(a) It p.d.f  $f_{\alpha}(\underline{x}) = \frac{1}{\alpha^n} e^{-\frac{1}{\alpha} \sum_{i=1}^n x_i}$

$\forall \underline{x}, \underline{y} \in \mathbb{X}$

$$\frac{f_{\alpha}(\underline{x})}{f_{\alpha}(\underline{y})} = e^{-\frac{1}{\alpha} \left( \sum_{i=1}^n x_i - \sum_{i=1}^n y_i \right)}$$

indep of  $\alpha$  iff  $\sum x_i = \sum y_i$

$\Rightarrow T(\underline{x}) = \sum_{i=1}^n x_i$  is minimal suff. stat.

(b) It p.d.f

$$f_{\beta}(\underline{x}) = \begin{cases} e^{-\sum_{i=1}^n x_i + n\beta}, & x_1, \dots, x_n > \beta \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} e^{-\sum_{i=1}^n x_i + n\beta}, & x_{(1)} > \beta \\ 0 & \text{otherwise} \end{cases}$$

i.e.  $f_{\beta}(\underline{x}) = e^{-\sum_{i=1}^n x_i} e^{n\beta} I(\beta, x_{(1)})$

$\forall \underline{x}, \underline{y} \in \mathbb{X}$

$$\frac{f_{\beta}(\underline{x})}{f_{\beta}(\underline{y})} = \frac{e^{-\sum_{i=1}^n x_i + \sum_{i=1}^n y_i}}{\frac{I(\beta, x_{(1)})}{I(\beta, y_{(1)})}}$$

indep of  $\beta$ . iff  $x_{(1)} = y_{(1)}$

$\Rightarrow T(\underline{x}) = X_{(1)}$  is minimal sufficient statistic

(c) If p.d.f is

$$f_{\alpha, \beta}(\underline{x}) = \frac{1}{\alpha^n} \exp\left(-\frac{1}{\alpha} \sum_i^n x_i + \frac{n\beta}{\alpha}\right) I(\beta, x_{(1)})$$

$\forall \underline{x}, \underline{y} \in \mathbb{X}$

$$\begin{aligned} \frac{f_{\alpha, \beta}(\underline{x})}{f_{\alpha, \beta}(\underline{y})} &= \frac{\exp\left(-\frac{1}{\alpha} \sum_i^n x_i + \frac{n\beta}{\alpha}\right) I(\beta, x_{(1)})}{\exp\left(-\frac{1}{\alpha} \sum_i^n y_i + \frac{n\beta}{\alpha}\right) I(\beta, y_{(1)})} \\ &= e^{-\frac{1}{\alpha} (\sum_i^n x_i - \sum_i^n y_i)} \frac{I(\beta, x_{(1)})}{I(\beta, y_{(1)})} \end{aligned}$$

indep. of  $(\alpha, \beta)$  iff  $\sum_i^n x_i = \sum_i^n y_i$  and  $x_{(1)} = y_{(1)}$

$\Rightarrow T(\underline{x}) = (\sum x_i, x_{(1)})$  is jointly minimal suff for  $(\alpha, \beta)$

(d) If p.d.f

$$f_{\mu, \sigma}(\underline{x}) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \left(\prod_{i=1}^n \frac{1}{x_i}\right) \exp\left(-\frac{1}{2\sigma^2} \left(\sum_1^n (\log x_i)^2 + n\mu^2 - 2\mu \sum_1^n \log x_i\right)\right)$$

$\forall \underline{x}, \underline{y} \in \mathbb{X}$

$$\begin{aligned} \frac{f_{\mu, \sigma}(\underline{x})}{f_{\mu, \sigma}(\underline{y})} &= \left(\prod_{i=1}^n \left(\frac{y_i}{x_i}\right)\right) \exp\left(-\frac{1}{2\sigma^2} \left(\sum (\log \tilde{x}_i) - \sum (\log \tilde{y}_i)^2\right)\right. \\ &\quad \left. + \frac{\mu}{\sigma^2} \left(\sum \log x_i - \sum \log y_i\right)\right) \end{aligned}$$

$\Rightarrow \frac{f_{\mu, \sigma}(\underline{x})}{f_{\mu, \sigma}(\underline{y})}$  is indep of  $(\mu, \sigma)$  iff  
 $\sum_1^n \log x_i = \sum_1^n \log y_i$  and  $\sum_1^n (\log x_i)^2 = \sum_1^n (\log y_i)^2$   
 $\Rightarrow T(\underline{x}) = \left( \sum_1^n \log x_i, \sum_1^n (\log x_i)^2 \right)$  is minimal suff for  $(\mu, \sigma)$

$$(e) f_\theta(\underline{x}) = \begin{cases} \frac{1}{\theta^n}, & |\underline{x}| < \theta/2 \\ 0, & \text{otherwise} \end{cases}$$

Jt p.d.f.

$$f_\theta(\underline{x}) = \begin{cases} \frac{1}{\theta^n}, & |\underline{x}_i| < \theta/2 \text{ for } i=1, \dots, n \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{\theta^n}, & \max_i |\underline{x}_i| < \theta/2 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{i.e. } f_\theta(\underline{x}) = \frac{1}{\theta^n} I\left(\max_i |\underline{x}_i|, \theta/2\right)$$

$\forall \underline{x}, \underline{y} \in \mathbb{X}$

$$\frac{f_\theta(\underline{x})}{f_\theta(\underline{y})} = \frac{I\left(\max_i |\underline{x}_i|, \theta/2\right)}{I\left(\max_i |\underline{y}_i|, \theta/2\right)}$$

$\max_i |\underline{x}_i| = \max_i |\underline{y}_i|$

$$\Rightarrow T(\underline{x}) = \max_i |\underline{x}_i| \text{ is minimal suff statistic}$$

(f) jst p.d.f.

$$f_{\alpha, \beta}(x) = \left( \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} \right)^n \left( \prod_{i=1}^n x_i \right)^{\alpha-1} \left( \prod_{i=1}^n (1-x_i) \right)^{\beta-1}$$

$\neq x, y \in \mathbb{R}$

$$\frac{f_{\alpha, \beta}(x)}{f_{\alpha, \beta}(y)} = \left( \frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n y_i} \right)^{\alpha-1} \left( \frac{\prod_{i=1}^n (1-x_i)}{\prod_{i=1}^n (1-y_i)} \right)^{\beta-1}$$

$f_{\alpha, \beta}(y)$  indep of  $(\alpha, \beta)$  iff  $\prod_{i=1}^n x_i = \prod_{i=1}^n y_i$   
and  $\prod_{i=1}^n (1-x_i) = \prod_{i=1}^n (1-y_i)$

$\Rightarrow T(x) = (\prod_{i=1}^n x_i, \prod_{i=1}^n (1-x_i))$  is jointly minimal suff for  $(\alpha, \beta)$

(2) completeness of  $T = \sum_{i=1}^n x_i$

p.d.f. of  $P(\theta)$   $f_{\theta}(x) = \frac{e^{-\theta}}{x!} \theta^x$

i.e.  $f_{\theta}(x) = \frac{e^{-\theta}}{x!} e^{\log \theta x} \cdot \frac{1}{x!}$   
 $= \left( \frac{1}{x!} \right) \exp \left( x \log \theta - \theta \right)$   
 $\uparrow \quad \uparrow \quad \uparrow$   
 $h(x) \quad T(n) \quad B(\theta)$

$\Rightarrow P(\theta)$  is 1-parameter exponential family

Natural parameter space:  $\{\eta : \eta \in \mathbb{R}\}$

So the natural parameter space associated with the 1-param exponential family representation contains open-intervals

$\Rightarrow$  The 1-parameter exponential family is of full rank

$\Rightarrow \sum_{i=1}^n X_i$  is complete sufficient

$$(a) g(\theta) = \theta$$

$$E(\bar{X}) = \theta$$

$\bar{X}$  is a function of complete sufficient statistic

$\Rightarrow \bar{X}$  is UMVUE of  $\theta$

$$(b) g(\theta) = e^{-\theta}$$

(difficult to guess u.e. of  $g(\theta)$  which is a f' of complete suff stat, hence we use Rao-Blackwell approach)

$$\delta_0(x) = \begin{cases} 1, & x_1 = 0 \\ 0, & \text{o/w} \end{cases} \quad E \delta_0(\bar{x}) = e^{-\theta} \quad \text{i.e. doing u.e. of } e^{-\theta}$$

Rao-Blackwellize  $\delta_0$ :

$$\eta(T) = E(\delta_0 | T)$$

$$\begin{aligned} E(\delta_0 | T=t) &= P(X_1=0 | T=t) \\ &= \frac{P(X_1=0, \sum_{i=2}^n X_i=t)}{P(T=t)} \\ &= \frac{(e^{-\theta}) \left( \frac{e^{-(n-1)\theta} ((n-1)\theta)^t}{t!} \right)}{e^{-nt} (n\theta)^t / t!} = \left( \frac{n-1}{n} \right)^t \end{aligned}$$

$\eta(\tau) = \left(\frac{n-1}{n}\right)^T$  is u.e. based on complete suff stat  $T$

$\Rightarrow \eta(\tau)$  is the UMVUE

$$(c) g(\theta) = e^{-\theta}(1+\theta) = e^{-\theta} + \theta e^{-\theta}$$

$$\begin{aligned} \delta_0(x) &= \begin{cases} 1, & x_1 \leq 1 \\ 0, & \text{otherwise} \end{cases} & E \delta_0(x) &= P(X_1 \leq 1) \\ &&&= P(X_1 = 0 \cup X_1 = 1) \\ &&&= e^{-\theta} + \theta e^{-\theta} = g(\theta) \end{aligned}$$

$\Rightarrow \delta_0$  is u.e. of  $e^{-\theta}(1+\theta)$

Rao-Blackwellization of  $\delta_0$ :

$$\eta(\tau) = E(\delta_0 | T) = P(X_1 \leq 1 | T)$$

$$\text{Now } P(X_1 \leq 1 | T=t) = \frac{P(X_1 \leq 1, \sum_i^n X_i = t)}{P(T=t)}$$

$$= \frac{P(\{X_1 = 0 \cup X_1 = 1\} \cap \sum_i^n X_i = t)}{P(T=t)}$$

$$= \frac{P(X_1 = 0, \sum_i^n X_i = t) + P(X_1 = 1, \sum_i^n X_i = t)}{P(T=t)}$$

$$= \frac{P(X_1 = 0, \sum_i^n X_i = t) + P(X_1 = 1, \sum_i^n X_i = t-1)}{P(T=t)}$$

$$= \frac{P(X_1 = 0) P(\sum_i^n X_i = t) + P(X_1 = 1) P(\sum_i^n X_i = t-1)}{P(T=t)}$$

$$= \frac{\cancel{e^{-\theta}} \left( \cancel{e^{-\frac{(n-1)\theta}{2}} \frac{((n-1)\theta)^t}{t!}} \right) + \theta \cancel{e^{-\theta}} \left( \cancel{e^{-\frac{(n-1)\theta}{2}} \frac{((n-1)\theta)^{t-1}}{(t-1)!}} \right)}{\cancel{e^{-n\theta}} \frac{(n\theta)^t}{t!}}$$

$$= \left(\frac{n-1}{n}\right)^T \left(1 + \frac{T}{n-1}\right)$$

$\Rightarrow \eta(T) = \left(\frac{n-1}{n}\right)^T \left(1 + \frac{T}{n-1}\right)$  is u.e. of  $g(\theta)$  based on complete suff stat

$\Rightarrow \left(\frac{n-1}{n}\right)^T \left(1 + \frac{T}{n-1}\right)$  is UMVUE of  $g(\theta) = e^\theta (1+\theta)$

(3)  $x_1, \dots, x_n$  r.s. from  $B(1, \theta)$ ;  $\theta \in (0, 1) = \mathbb{R}$

$T = \sum_{i=1}^n x_i$  is minimal suff;  $T \sim B(n, \theta)$

p.m.f. if  $x_i \sim B(1, \theta)$

$$\begin{aligned} f(x) &= \theta^x (1-\theta)^{1-x} \\ &= \exp\left(x \log \frac{\theta}{1-\theta} + \log(1-\theta)\right) \end{aligned}$$

With  $n(x)=1$ ,  $\eta(\theta) = \log \frac{\theta}{1-\theta}$ ,  $T(x)=x$  &  $\beta(\theta) = -\log(1-\theta)$

This is 1-parameter exponential family dist

Natural parameter space:  $\{\eta : \eta \in \mathbb{R}\}$

Natural parameter space contains open intervals

$\Rightarrow$  The 1-parameter expo family dist is of full rank (complete)

$\Rightarrow T(X) = \sum_i x_i$  is complete sufficient statistic

(a)  $g(\theta) = \theta$

$$E\left(\frac{T}{n}\right) = \theta$$

$\frac{T}{n}$  is an u.e. based on complete suff stat

$\Rightarrow \frac{T}{n}$  is UMVUE of  $\theta$ .

$$(b) g(\theta) = \theta^4$$

$$f_0(x) = \begin{cases} 1, & x_1=1, x_2=1, x_3=1, x_4=1 \\ 0, & \text{otherwise} \end{cases}$$

$$E(\delta_0(x)) = P(x_1=1, x_2=1, x_3=1, x_4=1) = \theta^4$$

$\Rightarrow \delta_0(x)$  is u.e. of  $\theta^4$

Re-Blackwellization of  $\delta_0$ :  $\eta(T) = E(\delta_0|T)$

$$E(\delta_0|T=t) = P(x_1=1, x_2=1, x_3=1, x_4=1 | T=t)$$

$$= \frac{P(x_1=1, x_2=1, x_3=1, x_4=1, \sum_{i=1}^n x_i = t-4)}{P(T=t)}$$

$$= \frac{P(x_1=1) P(x_2=1) P(x_3=1) P(x_4=1)}{P(T=t)} P\left(\sum_{i=1}^n x_i = t-4\right)$$

$$= \frac{\theta \cdot \theta \cdot \theta \cdot \theta \left( \binom{n-4}{t-4} \theta^{t-4} (1-\theta)^{n-4-(t-4)} \right)}{P(T=t)}$$

$$= \frac{\binom{n-4}{t-4}}{\binom{n}{t}} = \frac{\frac{t(t-1)(t-2)(t-3)}{n(n-1)(n-2)(n-3)}}{\frac{(n-t)}{(n-t)(n-t-1)\dots(n-t-n+1)}}$$

$$\eta(T) = \frac{T(T-1)(T-2)(T-3)}{n(n-1)(n-2)(n-3)} \text{ is an u.e. based on}$$

complete suff stat and hence is UMVUE

$$\delta_0 \theta^4$$

$$(c) g(\theta) = \theta(1-\theta)^2$$

$$\delta_0(\underline{x}) = \begin{cases} 1, & x_1=1, x_2=0, x_3=0 \\ 0, & \text{otherwise} \end{cases}$$

$$E \delta_0(\underline{x}) = \theta(1-\theta)^2$$

i.e.  $\delta_0(\underline{x})$  is u.e. based on  $(x_1, \dots, x_n)$

Rao-Blackwellization of  $\delta_0(\underline{x})$ :  $\gamma(T) = E(\delta_0 | T)$

$$E(\delta_0(\underline{x}) | T=t) = P(x_1=1, x_2=0, x_3=0 | T=t)$$

$$= \frac{P(x_1=1, x_2=0, x_3=0, \sum_i^n x_i = t-1)}{P(T=t)}$$

$$= \frac{\theta(1-\theta)(1-\theta) \left( \binom{n-3}{t-1} \theta^{t-1} (1-\theta)^{n-3-(t-1)} \right)}{\binom{n}{t} \theta^t (1-\theta)^{n-t}}$$

$$= \frac{\binom{n-3}{t-1}}{\binom{n}{t}} = \frac{t(n-t)(n-t-1)}{n(n-1)(n-2)}$$

$\Rightarrow \gamma(T) = \frac{T(n-T)(n-T-1)}{n(n-1)(n-2)}$  is u.e. based on

complete suff stat and hence is UMVUE

for  $\theta(1-\theta)^2$ .

$$(4) x_1, \dots, x_n \text{ r.s. from } f_\theta(x) = \begin{cases} e^{-x+\theta}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{i.e. } f_\theta(x) = e^{-x+\theta} I(\theta, x)$$

Can not be expressed in 1-parameter exponential dist form

Proof of completeness of minimal suff statistic using direct approach.

$T = X_{(1)}$  is minimal suff statistic.

p.d.f. of  $T$

$$f_T(t) = \begin{cases} n e^{-n(t-\theta)} & t > \theta \\ 0 & \text{otherwise} \end{cases}$$

Now  $E g(T) = 0 \quad \forall \theta \in \mathbb{R}$

$$\Rightarrow \int_{\theta}^{\infty} g(t) n e^{-nt} e^{n\theta} dt = 0 \quad \forall \theta \in \mathbb{R}$$

$$\text{i.e. } \int_{\theta}^{\infty} g(t) e^{-nt} dt = 0 \quad \forall \theta \in \mathbb{R}$$

Differentiating w.r.t.  $\theta$ , we get

$$g(\theta) e^{-n\theta} = 0 \quad \forall \theta \in \mathbb{R}$$

$$\Rightarrow g(\theta) = 0 \quad \forall \theta$$

$$\text{i.e. } g(\theta) = 0 \quad \forall 0 < \theta < t$$

$$\Rightarrow g(t) = 0 \quad \forall \theta < t$$

$$\text{i.e. } g(t) = 0 \quad \text{w.p. 1 (a.e.)}$$

$$\text{i.e. } g(t) = 0 \quad (\text{range of } t \text{ is } (\theta, \infty))$$

$\Rightarrow T = X_{(1)}$  is complete suff statistic

$$g(\theta) = \theta^2$$

$$E X_{(1)} = \int_{-\theta}^{\infty} t^n e^{-n(t-\theta)} dt$$

$\approx$

$$= n \int_0^{\infty} (y+\theta) e^{-ny} dy$$

$$= n \left[ \frac{1}{n^2} + \theta \cdot \frac{1}{n} \right] = \theta + \frac{1}{n}$$

$$\Rightarrow E \left( X_{(1)} - \frac{1}{n} \right) = \theta$$

$\Rightarrow X_{(1)} - \frac{1}{n}$  is u.e. of  $\theta$  based on complete suff stat

$\Rightarrow X_{(1)} - \frac{1}{n}$  is UMVUE for  $\theta$

Further,

$$E X_{(1)}^2 = n \int_{-\theta}^{\infty} t^2 e^{-n(t-\theta)} dt$$

$\approx$

$$= n \int_0^{\infty} (y+\theta)^2 e^{-ny} dy$$

$$= n \int_0^{\infty} (y^2 + \theta^2 + 2\theta y) e^{-ny} dy$$

$$= n \left[ \frac{1}{n^3} + \theta^2 \frac{1}{n} + 2\theta \frac{1}{n^2} \right]$$

$$= \frac{2}{n^2} + \theta^2 + \frac{2\theta}{n}$$

$$\Rightarrow E (X_{(1)}^2) = \frac{2}{n^2} + \theta^2 + \frac{2}{n} E (X_{(1)} - \frac{1}{n})$$

$$\Rightarrow E \left( X_{(1)}^2 - \frac{2}{n^2} - \frac{2}{n} (X_{(1)} - \frac{1}{n}) \right) = \theta^2$$

$$\text{i.e. } E (X_{(1)}^2 - \frac{2}{n} X_{(1)}) = \theta^2$$

$X_{(1)}^2 - \frac{2}{n} X_{(1)}$  is u.e. of  $\theta^2$  based on complete suff.

stat  $X_{(1)}$

$\Rightarrow X_{(1)}^2 - \frac{2}{n} X_{(1)}$  is UMVUE of  $\theta^2$

(5)  $X_1, \dots, X_n$  r.s. from  $U(0, \theta)$ ;  $\theta > 0$

$T = X_{(n)}$  is minimal suff statistic

$$f_T(t) = \begin{cases} \frac{n}{\theta^n} t^{n-1}, & 0 < t < \theta \\ 0, & \text{else} \end{cases}$$

Note that  $U(0, \theta)$  does not conform to exponential family dist<sup>n</sup> representation.

Direct approach to prove completeness:

$$E(g(T)) = 0 \quad \forall \theta \in \mathbb{R}$$

$$\Rightarrow \int_0^\theta g(t) \cdot \frac{n}{\theta^n} t^{n-1} dt = 0 \quad \forall \theta \in \mathbb{R}$$

$$\text{i.e. } \int_0^\theta g(t) t^{n-1} dt = 0 \quad \forall \theta \in \mathbb{R}$$

Differentiating both sides w.r.t.  $\theta$ , we get

$$g(\theta) \theta^{n-1} = 0 \quad \forall \theta \in \mathbb{R}$$

$$\text{i.e. } g(\theta) = 0 \quad \forall \theta > 0$$

$$\text{i.e. } g(t) = 0 \quad \forall 0 < t < \theta$$

$$\Rightarrow g(t) = 0 \quad \text{u.p.l (a.e.)}$$

$\Rightarrow T = X_{(n)}$  is complete sufficient statistic

$$g(\theta) = \theta^2$$

$$\begin{aligned} E(T^2) &= \frac{n}{\theta^n} \int_0^\theta t^2 t^{n-1} dt \\ &= \frac{n}{\theta^n} \int_0^\theta t^{n+1} dt = \frac{n}{\theta^n} \cdot \frac{\theta^{n+2}}{n+2} \\ &= \frac{n}{n+2} \theta^2 \end{aligned}$$

$$\Rightarrow E\left(\frac{n+2}{n} T^2\right) = \theta^2$$

$\frac{n+2}{n} T^2$  is u.e. of  $\theta^2$  based on complete suff stat

$\Rightarrow \frac{n+2}{n} X_{(n)}$  is UMVUE for  $\theta^2$ .

(6)  $x_1, \dots, x_n$  r.s. from  $G(2, \theta)$   $\theta > 0$

p.d.f.

$$f_\theta(x) = \begin{cases} \frac{1}{\Gamma(2)} \frac{e^{-x/\theta}}{\theta^2} x, & x > 0 \\ 0, & \text{o/w.} \end{cases}$$

Note that

$$f_\theta(x) = x \exp\left(-\frac{1}{\theta} x - 2 \log \theta\right)$$

$$\text{With } h(x) = x; \eta(\theta) = -\frac{1}{\theta}; T(x) = x, \beta(\theta) = 2 \log \theta$$

The above is 1-parameter exponential family dist<sup>n</sup>

Natural parameter space is

$$\{ \gamma : \gamma < 0 \} \quad (\{\gamma(\theta) : \theta \in (0, \infty)\})$$

The above obviously contains open intervals

$\Rightarrow$  The above is 1-parameter exponential family with full

rank

$$\Rightarrow T(\underline{x}) = \sum_{i=1}^n x_i \text{ is complete}$$

Thus  $\sum_{i=1}^n x_i$  is complete suff stat

Note that

$$E x_i = \frac{1}{\theta^2} \int_0^\infty x^2 e^{-x/\theta} dx = \frac{\Gamma_3 \theta^3}{\theta^2} = 2\theta$$

$$\Rightarrow E \left( \sum_{i=1}^n x_i \right) = 2n\theta$$

$\Rightarrow \frac{\sum x_i}{2n}$  is u.e. of  $\theta$  based on complete suff stat

and hence is UMVUE of  $\theta$ .

(7)  $x_1, \dots, x_n$  r.s. from  $U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$

$$f_\theta(\underline{x}) = I(\theta - \frac{1}{2}, x_{(1)}) I(x_{(n)}, \theta + \frac{1}{2})$$

$T(\underline{x}) = (x_{(1)}, x_{(n)})$  is minimal sufficient stat

Note that  $\text{dist}^n$  of  $x_{(n)} - x_{(1)}$  is indep of  $\theta$ . as

$$P(x_{(n)} - x_{(1)} \leq x) = P((x_{(n)} - \theta) - (x_{(1)} - \theta) \leq x)$$

$$= P \left( \max_i (x_i - \theta) - \min_i (x_i - \theta) \leq x \right)$$

$$x_i - \theta \sim U\left(-\frac{1}{2}, \frac{1}{2}\right)$$

$\text{dist}^n$  indep of  $\theta$

$$\Rightarrow \min_i (x_i - \theta): \text{dist}^n \text{ is indep of } \theta$$

$$\& \max_i (x_i - \theta): \text{dist}^n \text{ is indep of } \theta$$

$$\Rightarrow \text{dist}^n \text{ of } \max_i (x_i - \theta) - \min_i (x_i - \theta) \text{ is indep of } \theta$$

i.e. dist<sup>n</sup> of  $x_{(n)} - x_{(1)}$  is indep of  $\theta$

$$\Rightarrow E(x_{(n)} - x_{(1)}) = a \leftarrow \text{indep of } \theta$$

$$\Rightarrow E(x_{(n)} - x_{(1)} - a) = 0 \quad \forall \theta \in \mathbb{R}$$

$$\not\Rightarrow x_{(n)} - x_{(1)} = a \quad \text{u.p.1}$$

$$(\text{in fact } P(\underbrace{x_{(n)} - x_{(1)}}_{\text{const. v.v.}} = a) = 0 \neq a)$$

$$\Rightarrow T(\underline{x}) = (x_{(1)}, x_{(n)}) \text{ is NOT complete}$$

Alt approach:

One can show that

$$E(x_{(n)} - x_{(1)}) = \frac{n-1}{n+1} \quad \begin{array}{l} \text{by direct calculation} \\ \text{using p.d.f of} \\ x_{(1)} \& x_{(n)} \end{array}$$

$$\Rightarrow E(x_{(n)} - x_{(1)} - \frac{n-1}{n+1}) = 0 \quad \forall \theta \in \mathbb{R}$$

$$\not\Rightarrow x_{(n)} - x_{(1)} = \frac{n-1}{n+1} \quad \text{a.e.} \Rightarrow (x_{(1)}, x_{(n)}) \text{ is NOT complete}$$

(8)  $X_1, \dots, X_n$  r.s. from  $N(\theta, \theta)$

$$\text{p.d.f. } f_{\theta}(x) = \frac{1}{\sqrt{2\pi} \sqrt{\theta}} e^{-\frac{1}{2\theta} x^2}$$

The above is 1-parameter exponential family of full rank

$$\Rightarrow T = \sum_{i=1}^n X_i^2 \text{ is complete suff stat}$$

$$\text{Note that } \frac{T}{\theta} \sim \chi_n^2$$

$$\Rightarrow E\left(\frac{T}{\theta}\right) = n \quad V\left(\frac{T}{\theta}\right) = 2n$$

$$E(T) = n\theta \quad V(T) = 2n\theta^2$$

$$\Rightarrow E T^2 = V(T) + (ET)^2 = 2n\theta^2 + n^2\theta^2 \\ = \theta^2 n(n+2).$$

$$\Rightarrow E\left(\frac{T^2}{n(n+2)}\right) = \theta^2$$

$\frac{T^2}{n(n+2)}$  is u.e. based on complete suff stat and hence  $\theta^2$ .

hence is UMVUE for  $\theta^2$ .

(9)  $X_1, \dots, X_n$  r.s. from  $N(\mu, \theta)$

(9)  $\mu$  is known

$$\text{p.d.f. } \frac{1}{\sqrt{2\pi} \sqrt{\theta}} e^{-\frac{1}{2\theta} (x-\mu)^2}$$

The above is 1-parameter exp family dist of full rank

$$\Rightarrow T = \sum_{i=1}^n (X_i - \mu)^2 \text{ is complete suff stat}$$

$$\frac{T}{\theta} \sim \chi_n^2$$

$$E\left(\frac{T}{\theta}\right) = n$$

$\Rightarrow \frac{\sum (x_i - \mu)^2}{n}$  is u.e. f  $\theta$  based on complete suff stat and

hence is UMVUE for  $\theta$ , when  $\mu$  is known

(b)  $\mu$  &  $\theta$  both unknown : estimand  $\Theta$

p.d.f.  $\frac{1}{\sqrt{2\pi}\sqrt{\theta}} \exp\left(-\frac{1}{2\theta}(x^2 + \mu^2 - 2\mu x)\right)$

$$f_{\mu, \theta}(x) = \frac{1}{\sqrt{2\pi}} \frac{\exp\left(-\frac{1}{2\theta}x^2 + \frac{\mu}{\theta}x - \frac{\mu^2}{2\theta} - \frac{1}{2}\log\theta\right)}{h(x)}$$

The above is a 2-parameter exponential family dist of full rank

(Natural parameter space :  $\{(n_1, n_2) : n_1 < 0, n_2 \in \mathbb{R}\}$ )

The natural parameter space contains 2-dimensional open rectangles and hence the 2-param expo

family is of full rank.)

$$\Rightarrow T(\underline{x}) = (\sum x_i, \sum x_i^2)$$

$$\Leftrightarrow (\bar{x}, s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2)$$

Complete suff stat for  $(\mu, \theta)$

$$\text{Now } \frac{(n-1) s^2}{\theta} \sim \chi_{n-1}^2$$

$$\Rightarrow E\left(\frac{\sum(x_i - \bar{x})^2}{\theta}\right) = n-1$$

$$\text{i.e. } E\left(\frac{\sum(x_i - \bar{x})^2}{n-1}\right) = \theta$$

$\frac{\sum(x_i - \bar{x})^2}{n-1}$  is u.e. of  $\theta$  based on complete suff stat

and hence is UMVUE of  $\theta$

(c)  $\mu$  &  $\theta$  both unknown: estimand is  $\delta \rightarrow$   
 $p = P(X \leq \delta)$   
 given const

Note that

$$p = P(X \leq \delta) = P\left(\frac{X-\mu}{\sqrt{\theta}} \leq \frac{\delta-\mu}{\sqrt{\theta}}\right) = \Phi\left(\frac{\delta-\mu}{\sqrt{\theta}}\right)$$

$$\Rightarrow \frac{\delta-\mu}{\sqrt{\theta}} = \Phi^{-1}(p)$$

$$\Rightarrow \delta = \mu + \sqrt{\theta} \cdot \underbrace{\Phi^{-1}(p)}_{\text{known fixed const}} = f(\mu, \theta) \leftarrow \text{estimand}$$

$$(\sum X_i, \sum X_i^2) \Leftrightarrow (\bar{x}, \frac{1}{n-1} \sum (x_i - \bar{x})^2 = s^2) \text{ is}$$

complete suff statistic

$\Rightarrow \bar{x}$  is UMVUE of  $\mu$

$$Y = \frac{(n-1)S^2}{\theta} \sim \chi_{n-1}^2$$

For  $Y \sim \chi_{n-1}^2$  ( $r = n-1$  say)

$$E Y^{\frac{1}{2}} = \frac{1}{2^{\frac{r}{2}} \Gamma_{\frac{r}{2}}} \int_0^\infty y^{\frac{r}{2}} e^{-y/2} y^{\frac{r}{2}-1} dy$$

$$= \frac{1}{2^{\frac{r}{2}} \Gamma_{\frac{r}{2}}} \int_0^\infty e^{-y/2} y^{\frac{r+1}{2}-1} dy$$

$$= \frac{\frac{\Gamma_{\frac{r+1}{2}}}{2} \cdot 2^{\frac{r+1}{2}}}{2^{\frac{r}{2}} \Gamma_{\frac{r}{2}}} = \frac{\frac{\Gamma_{\frac{r+1}{2}}}{2} 2^{\frac{r}{2}}}{\Gamma_{\frac{r}{2}}}$$

i.e.  $E \left( \frac{\sqrt{n-1}}{\sqrt{\theta}} S \right) = \frac{\frac{\Gamma_{\frac{n}{2}}}{2} \sqrt{2}}{\sqrt{\frac{n-1}{2}}}$

$$\Rightarrow E \left( \frac{\frac{\Gamma_{\frac{n}{2}}}{2}}{\sqrt{\frac{n}{2}}} \cdot \sqrt{\frac{n-1}{2}} S \right) = \sqrt{\theta}$$

$\Rightarrow \frac{\frac{\Gamma_{\frac{n}{2}}}{2}}{\sqrt{\frac{n}{2}}} \cdot \sqrt{\frac{n-1}{2}} S$  is UMVUE of  $\sqrt{\theta}$

$\Rightarrow \bar{X} + \frac{\frac{\Gamma_{\frac{n}{2}}}{2}}{\sqrt{\frac{n}{2}}} \sqrt{\frac{n-1}{2}} S \hat{\phi}^{-1}(\beta)$  is UMVUE of  $\delta$   
 $(\delta = \mu + \sqrt{\theta} \hat{\phi}^{-1}(\beta))$

(10)  $T_1, T_2, T_3$  all are u.e. of  $\theta$  (trivial to verify)

Among the 3,  $T_1$  is UMVUE and hence is the preferred estimator

(11)  $X_1, \dots, X_n$  r.s. from  $N(\mu, \sigma^2)$

$$\mu \in \mathbb{R}, \sigma > 0$$

$(\sum_i^n X_i, \sum X_i^2) \Leftrightarrow (\bar{X}, S^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2)$  is

complete sufficient statistic

$$\bar{X} \sim N(\mu, \sigma^2/n)$$

$$\left( \frac{(n-1)S^2}{\sigma^2} \right) \sim \chi_{n-1}^2 \quad \text{indep}$$

$$\frac{g(\mu, \sigma^2) = \mu^2}{E(\bar{X}^2)} = V(\bar{X}) + (E\bar{X})^2$$

$$E(\bar{X}^2) = \frac{\sigma^2}{n} + \mu^2$$

$$\text{i.e. } E(\bar{X}^2) = \frac{1}{n} E(S^2) + \mu^2$$

$$\Rightarrow E(\bar{X}^2 - \frac{1}{n} S^2) = \mu^2$$

$(\bar{X}^2 - \frac{1}{n} S^2)$  is u.e. of  $\mu^2$  based on complete sufficient statistic and hence is UMVUE for  $\mu^2$

$$g(\mu, \sigma^2) = \mu + \sigma$$

$$E(\bar{X}) = \mu$$

$$E\left(\frac{\sqrt{\frac{n-1}{2}}}{\sqrt{n}} \frac{1}{\sqrt{2}} \left(\sum (x_i - \bar{x})^2\right)^{1/2}\right) = \sigma \quad (\text{from } g(c))$$

$$\Rightarrow E\left(\bar{X} + \underbrace{\frac{\sqrt{\frac{n-1}{2}}}{\sqrt{n}} \frac{1}{\sqrt{2}} \left(\sum (x_i - \bar{x})^2\right)^{1/2}}_{\sim} \right) = \mu + \sigma$$



u.e. of  $(\mu + \sigma)$  based on complete suff stat and  
hence is UMVUE of  $\mu + \sigma$ .