MSO201A: PROBABILITY & STATISTICS
Problem Set #11

[1] Find minimal sufficient statistic based on a random sample $X_1,...,X_n$ in each of the following cases

(a) $f_\alpha(x) = \begin{cases} \dfrac{1}{\alpha}\exp\left(-\dfrac{x}{\alpha}\right) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$ $\quad \alpha > 0.$

(b) $f_\beta(x) = \begin{cases} \exp\left(-(x-\beta)\right) & \text{if } x > \beta \\ 0 & \text{otherwise.} \end{cases}$ $\quad \beta \in \mathbb{R}.$

(c) $f_{\alpha,\beta}(x) = \begin{cases} \dfrac{1}{\alpha}\exp\left(-\dfrac{(x-\beta)}{\alpha}\right) & \text{if } x > \beta \\ 0 & \text{otherwise.} \end{cases}$ $\quad \alpha > 0, \beta \in \mathbb{R}.$

(d) $f_{\mu,\sigma}(x) = \begin{cases} \dfrac{1}{x\,\sigma\,\sqrt{2\pi}}\exp\left(-\dfrac{(\log x_i - \mu)^2}{2\sigma^2}\right) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$ $\quad \mu \in \mathbb{R}; \sigma > 0.$

(e) $f_\theta(x) = \begin{cases} \dfrac{1}{\theta} & -\theta/2 \le x \le \theta/2 \\ 0 & \text{otherwise} \end{cases}$ $\quad \theta > 0.$

(f) $f(x) = \begin{cases} \dfrac{\overline{|\alpha+\beta}}{\overline{|\alpha}\,\overline{|\beta}}\, x^{\alpha-1}(1-x)^{\beta-1} & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$ $\quad \alpha > 0, \beta > 0.$

[2] Let $X_1,...,X_n$ be a random sample from $P(\theta), \theta \in (0,\infty)$. Show that $T = \sum_{i=1}^{n} X_i$ is complete sufficient statistic. Find the Uniformly Minimum Variance Unbiased Estimator (UMVUE) of the following parametric functions: (a) $g(\theta) = \theta$, (b) $g(\theta) = e^{-\theta}$ and (c) $g(\theta) = e^{-\theta}(1+\theta)$.

[3] Suppose $X_1,...,X_n$ be a random sample from $B(1,\theta), \theta \in (0,1)$. Show that $T = \sum_{i=1}^{n} X_i$ is complete sufficient statistic and hence find the UMVUE for each of the following parametric functions: (a) $g(\theta) = \theta$, (b) $g(\theta) = \theta^4$ and (c) $g(\theta) = \theta(1-\theta)^2$.

[4] Let $X_1,...,X_n$ be a random sample from $Exp(\theta,1)$, i.e.

$$f(x \mid \theta)=\begin{cases} e^{-(x-\theta)} & \text{if } x>\theta \\ 0 & \text{otherwise} \end{cases}$$

Show that $T=X_{(1)}=\min\{X_1,\dots,X_n\}$ is a complete sufficient statistic and hence find the UMVUE of $g(\theta)=\theta$.

[5] $X_1,\dots,X_n$ is a random sample from $U(0,\theta),\theta>0$. Show that $T=X_{(n)}=\max\{X_1,\dots,X_n\}$ is a complete sufficient statistic and find the UMVUE of $g(\theta)=\theta^2$.

[6] $X_1,\dots,X_n$ is a random sample from $Gamma(2,\theta),\theta>0$, i.e.

$$f(x \mid \theta)=\begin{cases} \dfrac{1}{\Gamma 2\,\theta^2} e^{-x/\theta} x & \text{if } x>0 \\ 0 & \text{otherwise} \end{cases}$$

Show that $T=\sum_{i=1}^{n} X_i$ is complete sufficient statistic and find the UMVUE of θ.

[7] Let $X_1,\dots,X_n$ be a random sample from $U(\theta-1/2,\,\theta+1/2)$. Show that the minimal sufficient statistic is not complete.

[8] Let $X_1,\dots,X_n$ be a random sample from $N(0,\,\theta)$. Find the UMVUE of θ^2.

[9] Let $X_1,\dots,X_n$ be a random sample from $N(\mu,\,\theta)$. Find the UMVUE of (a) θ when μ is known, (b) θ when μ is not known and (c) δ such that $P(X\le\delta)=p;\ p$ is a known fixed constant, both μ and θ are unknown parameters.

[10] Let $X_1,\dots,X_n$ be a random sample from $U(0,\,\theta),\theta>0$. Of the following three estimators given below, which one would you prefer and why?

$$T_1(\underset{\sim}{X})=\frac{n+1}{n}X_{(n)},\ T_2(\underset{\sim}{X})=2\bar{X} \text{ and } T_3(\underset{\sim}{X})=X_{(1)}+X_{(n)}.$$

[11] $X_1,\dots,X_n$ be a random sample from $N(\mu,\sigma^2),\mu\in\mathfrak{R},\sigma\in\mathfrak{R}^+$. Assuming completeness of the associated minimal sufficient statistic find the UMVUE of μ^2 and $\mu+\sigma$.