MSO201A: PROBABILITY & STATISTICS
Problem Set #6

[1] The probability density function of the random variable X is

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

i.e. $X \sim U(0,1)$. Find the distribution of the following functions of X

(a) $Y = \sqrt{X}$; (b) $Y = X^2$; (c) $Y = 2X + 3$; (d) $Y = -\lambda \log X; \lambda > 0$.

[2] Let X be a random variable with $U(0,\theta),\ \theta > 0$ distribution. Find the distribution of $Y = \min(X, \theta/2)$.

[3] The probability density function of X is given by

$$f_X(x) = \begin{cases} \frac{1}{2}, & -\frac{1}{2} < x < \frac{3}{2} \\ 0, & \text{otherwise} \end{cases}$$

Find the distribution of $Y = X^2$.

[4] The probability density function of X is given by

$$f_X(x) = \begin{cases} k \dfrac{x^{p-1}}{(1+x)^{p+q}} & x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

$p, q > 0$. Derive the distribution of $Y = (1+X)^{-1}$.

[5] The probability density function of X is given by

$$f_X(x) = \begin{cases} k\,x^{\beta-1} \exp\left(-\alpha x^{\beta}\right) & x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

$\alpha, \beta > 0$. Derive the distribution of $Y = X^{\beta}$

[6] According to the Maxwell-Boltzmann law of theoretical physics, the probability density function of V, the velocity of a gas molecule, is

$$f_V(v) = \begin{cases} k\,v^2 \exp\left(-\beta v^2\right) & v > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where, $\beta > 0$ is a constant which depends on the mass and absolute temperature of the molecule and $k > 0$ is a normalizing constant. Derive the distribution of the kinetic energy $E = mV^2/2$.

[7] The probability density function of the random variable X is

$$f_X(x)=\begin{cases}\frac{3}{8}(x+1)^2 & -1<x<1\\ 0 & \text{otherwise.}\end{cases}$$

Find the distribution of the following functions of $Y=1-X^2$.

[8] Let X be a random variable with $U(0,1)$ distribution. Find the distribution function of $Y=\min(X,1-X)$ and the probability density function of $Z=(1-Y)/Y$.

[9] Suppose $X\sim N(\mu,\sigma^2)$, $\mu\in\Re,\sigma\in\Re^+$. Find the distribution of $2X-6$.

[10] Let X be a continuous random variable on (a,b) with p.d.f f and c.d.f. F. Find the p.d.f. of $Z=-\log(F(X))$.

[11] Let X be a continuous r.v. having the following p.d.f.

$$f(x)=\begin{cases}6x(1-x) & \text{if } 0\le x\le 1\\ 0 & \text{if otherwise}\end{cases}.$$

Derive the distribution function of X and hence find the p.d.f. of $Y=X^2(3-2X)$.

[12] Let X be distributed as double exponential with p.d.f. $f(x)=\frac{1}{2}e^{-|x|}; x\in\Re$. Find the p.d.f. of $Y=|X|$

[13] 3 balls are placed randomly in 3 boxes B_1,B_2 and B_3. Let N be the total number of boxes which are occupied and X_i be the total number of balls in the box B_i, $i=1,2,3$. Find the joint p.m.f. of (N,X_1) and (X_1,X_2). Obtain the marginal distributions of N,X_1 and X_2 from the joint p.m.f.s.

[14] The joint p.m.f. of X and Y is given by

$$p(x,y)=\begin{cases}c\,xy & \text{if } (x,y)\in\{(1,1),(2,1),(2,2),(3,1)\}\\ 0 & \text{otherwise.}\end{cases}$$

Find the constant c, the marginal p.m.f. of X and Y and the conditional p.m.f. of X given $Y=2$.

[15] The joint p.m.f. of X and Y is given by

$$p(x,y)=\begin{cases}(x+2y)/18 & \text{if } (x,y)\in\{(1,1),(1,2),(2,1),(2,2)\}\\ 0 & \text{otherwise.}\end{cases}$$

(a) Find the marginal distributions.
(b) Verify whether X and Y are independent random variables.
(c) Find $P(X<Y),P(X+Y>2)$.
(d) Find the conditional p.m.f. of Y given $X=x,x=1,2$.

[16] 5 cards are drawn at random without replacement from a deck of 52 playing cards. Let the random variables X_1,X_2,X_3 denote the number of spades, the number of hearts, the number of diamonds, respectively, that appear among the five cards. Find the joint p.m.f. of X_1, X_2, X_3. Also determine whether the 3 random variables are independent.

[17] Consider a sample of size 3 drawn with replacement from an urn containing 3 white, 2 black and 3 red balls. Let the random variables X_1 and X_2 denote the number of white balls and number of black balls in the sample, respectively. Determine whether the two random variables are independent.

[18] Let $\underset{\sim}{X} = \left(X_1, X_2, X_3\right)^T$ be a random vector with joint p.m.f.

$$f_{X_1,X_2,X_3}\left(x_1,x_2,x_3\right) = \begin{cases} 1/4 & \left(x_1,x_2,x_3\right) \in \mathcal{X} \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{X} = \left\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\right\}$. Show that X_1, X_2, X_3 are pairwise independent but are not mutually independent.

Note: The above problem shows that random variables may be pairwise independent without being mutually independent.