1.a. Let X be a random variable with pdf

$$f(x) = \begin{cases} c(x+4) & \text{if} -4 < x < 5, \\ 0 & \text{otherwise.} \end{cases}$$

Find c.

Define $Y = X|X|$. Find the df of Y, and its pdf. [$1 + 4 = 5$]

1.b. (i) Let $\beta > 0$. Find the median and the mode of the following pdf:

$$f(x) = \begin{cases} 2\beta x e^{-\beta x^2} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

(ii) Let X be a random variable with mgf

$$M_X(t) = \frac{e^{-t}}{8} + \frac{1}{4} + \frac{5e^{2t}}{8} \text{ for } t \in \mathbb{R}.$$

Find the distribution of X. Use it to compute $Var(X)$. [$4 + 2 = 6$]

2.a. (i) Let $X_1, \ldots, X_n$ be i.i.d. $N(0,1)$, $\bar{X}_n$ is the sample mean and S_n^2 is the sample variance. Fix $n = 5$. Find the value of $E(\bar{X}_5/S_5)$ and $Var(\bar{X}_5/S_5)$.

(ii) Let $X_1, \ldots, X_n$ be i.i.d. Unif$[0,1]$. Prove that $X_{(1)}X_{(n)} \xrightarrow{P} c$ as $n \to \infty$. Find the limit c. $[3 + 3 = 6]$

2.b. Let $(Z_1, Z_2) \sim N_2((0,0), \Sigma)$ with $\Sigma = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$. Find

(i) $E(\max\{Z_1, Z_2\})$.
(ii) $E(e^{tZ_1Z_2})$ with $t \in \mathbb{R}$.

Hint for (ii): $Z_2|Z_1 \sim$? $[2 + 3 = 5]$

3.a. Let $X_1, \ldots, X_n$ be i.i.d. Unif$[\theta - a, \theta + a]$, where $\theta \in \mathbb{R}$ and $a > 0$ are unknown parameters.
(i) Find the MLE $(\hat{\theta}, \hat{a})$ of (θ, a).
(ii) Is $\hat{\theta}$ an unbiased estimate of θ?
(iii) Is $\hat{a}$ an unbiased estimate of a? [$2 + 2 + 2 = 6$]

3.b. If $X_1, \ldots, X_n$ be i.i.d. f, where $f(x) = \theta x^{\theta - 1}$, $0 < x < 1$, zero elsewhere. Using the NP lemma, find the critical region for testing

$$H_0 : \theta = 1 \text{ against } H_1 : \theta = 2$$

at $\alpha \in (0, 1)$ level of significance.
Hint: Under H_0, $-2 \log_e X_1 \sim$?
Compute the cut-off for $n = 10$ and $\alpha = 0.10$. [$3 + 1 = 4$]