MSO201A: PROBABILITY & STATISTICS
Problem Set #5

Useful data

$\Phi(1/3)=0.6293, \Phi(5/6)=0.7967, \Phi(1)=0.8413, \Phi(4/3)=0.918, \Phi(1.96)=0.975$

[1] A machine contains two belts of different lengths. These have times to failure which are exponentially distributed, with means α and 2α. The machine will stop if either belt fails. The failures of the belts are assumed to be independent. What is the probability that the system performs after time α from the start?

[2] Let X be a normal random variable with parameters $\mu=10$ and $\sigma^2=36$. Compute (a) $P(X>5)$, (b) $P(4<X<16)$, (c) $P(X<8)$.

[3] Let $X \sim N(\mu, \sigma^2)$. If $P(X \le 0)=0.5$ and $P(-1.96 \le X \le 1.96)=0.95$, find μ and σ^2.

[4] It is assumed that the lifetime of computer chips produced by a certain semiconductor manufacturer are normally distributed with parameters $\mu=1.4\times10^6$ and $\sigma = 3 \times 10^5$ hours. What is the approximate probability that a batch of 10 chips will contain at least 2 chips whose lifetime are less than 1.8×10^6 hours?

[5] Let X be a Normal random variable with mean 0 and variance 1, i.e. $N(0,1)$. Prove that

$$P(|X|\ge t) \le \sqrt{\frac{2}{\pi}}\frac{e^{-t^2/2}}{t}; \forall\, t>0.$$

[6] Show that if X is a discrete random variable with values $0,1,2,....$ then $E(X)=\sum_{k=0}^{\infty}(1-F(k))$, where $F(x)$ is the distribution function of the random variable X.

[7] The cumulative distribution function of a random variable X defined over $0 \le x < \infty$ is $F(x)=1-e^{-\beta x^2}, x \ge 0;\ = 0, x<0.$ $\beta>0$. Find the mean, median and variance of X.

[8] Show that for any $x>0$, $1-\Phi(x) \le \frac{\phi(x)}{x}$, where $\Phi(x)$ is the c.d.f. and $\phi(x)$ is the p.d.f. of standard normal distribution.

[9] A point m_0 is said to be mode of a random variable X, if the p.m.f. or the p.d.f. of X has a maximum at m_0. For the distribution given in problem [7], if m_0 denotes the mode; μ, the mean and σ^2, the variance of the corresponding random variable, then show that $m_0=\mu\sqrt{2/\pi}$ and $2m_0^2-\mu^2=\sigma^2$.

[10] A random variable X is said to be symmetric about a point θ if $X-\theta$ has an identical distribution with $\theta - X$. Suppose X is a continuous random variable, which is symmetric about θ, with distribution function $F_X(.)$ and p.d.f $f_X(.)$.

(a) Prove that distribution of X is symmetric about θ iff $f_X(\theta-x)=f_X(\theta+x)$, for all x.

(b) Prove that distribution of X is symmetric about θ iff $F_X(\theta + x) + F_X(\theta - x) = 1$, for all x.

(c) Prove that $E(X) = median(X) = \theta$.

[11] Let X be a random variable having a 2-parameter exponential distribution with p.d.f.

$$f(x) = \begin{cases} \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}}, & \text{if } x \geq \mu \\ 0, & \text{otherwise.} \end{cases} \quad \mu \in \mathcal{R}, \sigma > 0.$$

(a) Find $E(X-\mu)^r, r = 1,2, \ldots$

(b) Find $E(X)^r, r = 1,2$

(c) Find the median of X.

[12] Let $X \sim Bin(n,p)$ and $X \sim NB(r,p)$; $0 < p < 1$ and $r \in \{1,2, \ldots, n\}$. Prove that $P(X \geq r) = P(Y \leq n - r)$.

[13] Let $X \sim P(\lambda)$. Find $E((2+X)^{-1})$.

[14] Let $\theta > 0$ and $t > 0$ and $n \in \{1,2, \ldots\}$. Prove that if $X \sim G(n, \theta)$ and $Y \sim P(t/\theta)$, then $P(X \geq t) = P(Y \leq n-1)$.

[15] Let X be a Poisson random variable with parameter λ. Find the probability mass function of $Y = X^2 - 5$.

[16] Let X be Binomial random variable with parameters n and p. Find the probability mass function of $Y = n - X$.

[17] Consider the discrete random variable X with the probability mass function

$$P(X=-2)=\frac{1}{5}, \quad P(X=-1)=\frac{1}{6}, \quad P(X=0)=\frac{1}{5},$$
$$P(X=1)=\frac{1}{15}, \quad P(X=2)=\frac{10}{30}, \quad P(X=3)=\frac{1}{30}.$$

Find the probability mass function of $Y = X^2$.

[18] The probability mass function of the random variable X is given by

$$P(X=x)=\begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^x & x = 0,1,2,\ldots \\ 0 & \text{otherwise.} \end{cases}$$

Find the p.m.f. of $Y = X/(X+1)$.

[19] Let X be a discrete random variable with probability mass function

$$f_X(x) = P(X=x) = \begin{cases} e^{-1}, & x = 0 \\ \frac{e^{-1}}{2\,(|x|)!}, & x = \pm 1, \pm 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Find the p.m.f. of $Y = |X|$ and identify the distribution.