

Problem Set #7 : solution

(1)

jt p.d.f. $f(x, y)$

$$f_{x,y}(x, y) = \begin{cases} 4xy, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

Marginal of x :

$$f_x(x) = \int_0^1 4xy dy = 2x \quad (0 < x < 1)$$

$$= 0 \quad \text{elsewhere.}$$

So $f_y(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$

observe that $f_{x,y}(x, y) = f_x(x) f_y(y)$

$\Rightarrow x \& y$ are indep.

$$P(0 < x < \frac{1}{2}, \frac{1}{4} < y < 1) = P(0 < x < \frac{1}{2}) P(\frac{1}{4} < y < 1)$$

$$= \left(\int_0^{\frac{1}{2}} 2x dx \right) \left(\int_{\frac{1}{4}}^1 2y dy \right) = \dots$$

$$P(x+y < 1) = \int_0^1 P(x < 1-y) f_y(y) dy \rightarrow x \& y \text{ are indep.}$$

$$= \int_0^1 \left[\int_0^{1-y} 2x dx \right] 2y dy$$

$$= \dots$$

$$(2) f_X(x) = \int_0^x e^{-x} e^{-y} dy = e^{-x} \quad x > 0$$

$$= 0 \quad \text{if } x \leq 0$$

$$f_Y(y) = e^{-y} \quad y > 0$$

$$= 0 \quad \text{if } y \leq 0$$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$\Rightarrow X \& Y$ are indep.

$$(3) f_X(x) = \int_x^\infty 2e^{-x} e^{-y} dy$$

$$= 2e^{-x} e^{-x} = 2e^{-2x} \quad x > 0$$

$$= 0 \quad \text{if } x \leq 0$$

$$\text{Sly } f_Y(y) = 2 \int_0^y e^{-y} e^{-x} dx = 2e^{-y} (1 - e^{-y}) \quad y > 0$$

$$= 0 \quad \text{if } y \leq 0$$

$$f(x,y) \neq f(x) f(y)$$

$\Rightarrow X \& Y$ are not indep.

$$(4) f_X(x) = 12x \int_0^1 (y-x^y) dy = 12x \left(\frac{y^2}{2} - \frac{y^3}{3} \right) \Big|_0^1$$

$$= 12x \cdot \frac{1}{6} = 2x$$

$$\Rightarrow f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{if } x \end{cases}$$

$$f_Y(y) = 12y(1-y) \int_0^1 x dx = 6y(1-y) \quad 0 < y < 1$$

$$= 0 \quad \text{if } y \leq 0$$

$$f(x,y) = f(x) f(y)$$

$\Rightarrow X \& Y$ are indep.

$$(5) \int_0^1 \int_x^1 f(x,y) dy dx = 1, \text{ i.e. } \int_0^1 x^2 \int_x^1 y dy dx = 1$$

$$\Rightarrow C \int_0^1 x^2 \frac{1}{2} (1-x^2) dx = 1$$

$$\Rightarrow \frac{C}{2} \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = 1 \Rightarrow C = 15$$

$$(b) f_X(x) = 15x^2 \int_x^1 y dy = \begin{cases} 15/2 x^2(1-x^2), & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$f_Y(y) = 15y \int_0^y x^2 dx = \begin{cases} 5y^4, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$(c) P(X+Y \leq 1) = \iint_{\substack{x+y \leq 1}} 15x^2 y dy dx$$

$$\begin{aligned} &= 15 \int_0^{1/2} x^2 \int_x^{1-x} y dy dx = 15 \int_0^{1/2} x^2 \left[\frac{y^2}{2} \right]_x^{1-x} dx \\ &= \dots = \frac{15}{192}. \end{aligned}$$

Alt

$$\begin{aligned} P(X+Y \leq 1) &= \iint_{\substack{x+y \leq 1}} 15x^2 y dx dy \\ &= 15 \int_0^{1/2} y \int_0^{\min(y, 1-y)} x^2 dx dy \\ &= 15 \int_0^{1/2} y \int_0^y x^2 dx dy + 15 \int_{1/2}^1 y \int_0^{1-y} x^2 dx dy \\ &= \dots = \frac{15}{15 \times 32} + \frac{15}{10 \times 32} = \frac{15}{192} \end{aligned}$$

$$(6) f_X(x) = \int_0^{1-x} f_{X,Y}(x,y) dy = 6 \int_0^{1-x} (1-x-y) dy = 6 \left[(1-x)y - \frac{y^2}{2} \right]_0^{1-x}$$

$$= \begin{cases} 3(1-x)^2, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$\int y$ (by symmetry)

$$f_Y(y) = \begin{cases} 3(1-y)^2, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$\begin{aligned}
 P(2x+3y < 1) &= 6 \int_0^{1/2} \int_0^{\frac{1-2x}{3}} (1-x-y) dy dx \\
 &= 6 \int_0^{1/2} \left[(1-x)y - \frac{y^2}{2} \right]_0^{\frac{1-2x}{3}} dx \\
 &= 6 \int_0^{1/2} \left\{ (1-x)\left(\frac{1-2x}{3}\right) - \frac{1}{2}\left(\frac{1-2x}{3}\right)^2 \right\} dx \\
 &= 6 \int_0^{1/2} \left(\frac{1+2x^2-3x}{3} - \frac{1+4x^2-4x}{18} \right) dx \\
 &= 6 \int_0^{1/2} \frac{8x^2-14x+5}{18} dx \\
 &= \frac{6}{18} \left(\frac{8x^3}{3} - 14 \frac{x^2}{2} + 5x \right) \Big|_0^{1/2} \\
 &= \frac{6}{18} \left(\frac{8}{3} \cdot \frac{1}{8} - 7 \cdot \frac{1}{4} + \frac{5}{2} \right) = \frac{13}{36}
 \end{aligned}$$

Alt

$$\begin{aligned}
 P(2x+3y < 1) &= 6 \int_0^{1/3} \int_0^{\frac{1-3y}{2}} (1-y-x) dx dy \\
 &= 6 \int_0^{1/3} \left((1-y)x - \frac{x^2}{2} \right) \Big|_0^{\frac{1-3y}{2}} dy \\
 &= 6 \int_0^{1/3} \left[(1-y)\left(\frac{1-3y}{2}\right) - \frac{1}{2}\left(\frac{1-3y}{2}\right)^2 \right] dy \\
 &\quad \cdots = \frac{13}{36}
 \end{aligned}$$

$$(7) f_x(x) = \int_0^1 f(x,y) dy = \int_0^1 (x+y) dy$$

$$= \begin{cases} x + \frac{1}{2} & 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

$$f_{y|x} = \frac{f(x,y)}{f_x(x)} = \frac{\cancel{x+y}}{\cancel{2x+1}} = \frac{x+y}{\frac{1}{2}(2x+1)} = \frac{2(x+y)}{(2x+1)} \quad 0 < y < 1$$

$$E(y|x) = \int_0^1 y \frac{2(x+y)}{2x+1} dy = \frac{2}{2x+1} \int_0^1 (xy + y^2) dy$$

$$= \frac{2}{2x+1} \left(\frac{x}{2} + \frac{1}{3} \right) = \frac{2(3x+2)}{6(2x+1)} = \frac{3x+2}{6x+3}$$

$$E(y^2|x) = \int_0^1 y^2 \frac{2(x+y)}{2x+1} dy = \frac{2}{2x+1} \int_0^1 (y^2 x + y^3) dy$$

$$= \frac{2}{2x+1} \left(\frac{x}{3} + \frac{1}{4} \right) = \frac{2(4x+3)}{12(2x+1)} = \frac{4x+3}{6(2x+1)}$$

$$\text{V}(y|x) = E(y^2|x) - E^2(y|x)$$

$$= \frac{4x+3}{6(2x+1)} - \left(\frac{3x+2}{3(2x+1)} \right)^2 = \dots$$

$$(8) f(x,y) = f(x|y) g(y) = c d x y^2 ; \quad 0 < x < y, \quad 0 < y < 1$$

$$\int_0^1 g(y) dy = 1 \Rightarrow d \int_0^1 y^4 dy = 1 \Rightarrow d = 5$$

$$\Rightarrow f(x,y) = 5c x y^2 ; \quad 0 < x < y < 1$$

$$\Rightarrow 5c \int_0^1 \int_0^y x dx dy = 1 \Rightarrow \frac{5c}{2} \int_0^1 y^4 dy = 1$$

$$\Rightarrow c = 2$$

$$\Rightarrow f(x,y) = 10xy^2 ; \quad 0 < x < y < 1 \\ = 0 \quad \text{if } w$$

$$f_x(x) = 10x \int_x^1 y^2 dy \quad 0 < x < 1$$

$$f_x(x) = \begin{cases} \frac{10}{3}x(1-x^3) & 0 < x < 1 \\ 0 & \text{if } w \end{cases}$$

$$P(0.25 < x < 0.5) = \frac{10}{3} \int_{\frac{1}{4}}^{\frac{1}{2}} (x-x^4) dx = - - -$$

$$P\left(\frac{1}{4} < x < \frac{1}{2} \mid y=0.625\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} f_{x|y=y} dx \\ = 2 \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{x}{(0.625)^2} dx = \frac{2}{(0.625)^2} \times \left(\frac{1}{2}^2 - \left(\frac{1}{4}\right)^2\right) \\ = - - -$$

(9) Marg of x from $h(x,y)$

$$f_x(x) = \int_{-\infty}^{+\infty} h(x,y) dy = \int_{-\infty}^{+\infty} f(x) g(y) \left\{ 1 + \alpha [2F(x)-1] [2G(y)-1] \right\} dy \\ = f(x) \int_{-\infty}^{+\infty} g(y) dy + f(x) \alpha [2F(x)-1] \int_{-\infty}^{+\infty} g(y) (2G(y)-1) dy \\ = f(x) \times 1 + f(x) \alpha [2F(x)-1] \int_{-\infty}^{+\infty} g(y) (2G(y)-1) dy \\ \int_{-\infty}^{+\infty} g(y) (2G(y)-1) dy \stackrel{G(y)=u}{=} \int_0^1 (2u-1) du = \left[\frac{u^2}{2} - u \right]_0^1 = 0$$

$$\Rightarrow f_x(x) = f(x) + 0$$

$$\text{Sly } f_y(y) = \int_{-\infty}^{+\infty} h(x,y) dx = g(y)$$

$$h(x,y) = f_x(x) \cdot f_y(y) = f(x)g(y) \text{ iff } \alpha = 0$$

$$(10) \quad f_{x,y}(x,y) = f_{y|x=x}^{(y|x)} f_x(x)$$

$$= \begin{cases} 8xy, & 0 < x < y < 1 \\ 0 & \text{else} \end{cases}$$

Marginal p.d.f. of y

$$f_y(y) = \begin{cases} 8y \int_0^y x dx = 4y^3, & 0 < y < 1 \\ 0, & \text{else} \end{cases}$$

Conditional p.d.f. of x given y

$$f_{x|y=y}^{(x|y)} = \begin{cases} \frac{8xy}{4y^3} = \frac{2x}{y^2}, & y^2 \\ 0, & \text{else} \end{cases} \quad 0 < x < y \Rightarrow 0 < y < 1$$

$$E(x|y=y) = \frac{2}{y^2} \int_0^y x^2 dx = \frac{2}{y^2} \cdot \frac{y^3}{3} = \frac{2y}{3}$$

$$\Rightarrow E(x|y=\frac{1}{2}) = \frac{1}{3}$$

$$E(x^2|y=y) = \frac{2}{y^2} \int_0^y x^3 dx = \frac{2}{y^2} \cdot \frac{y^4}{4} = \frac{y^2}{2}$$

$$\Rightarrow E(x^2|y=\frac{1}{2}) = \frac{1}{8}$$

$$V(x|y=y) = E(x^2|y=y) - E^2(x|y=y)$$

$$= \frac{1}{8} - \frac{1}{9} = \frac{1}{72}$$

(11) M.g.f.

$$\begin{aligned}
 M_{X_1, X_2}(t_1, t_2) &= E(e^{t_1 x_1 + t_2 x_2}) \\
 &= \int_0^\infty \int_0^\infty e^{t_1 x_1 + t_2 x_2} e^{-(x_1 + x_2)} dx_2 dx_1 \\
 &= \int_0^\infty e^{-x_2(1-t_1)} dx_1 \quad \int_0^\infty e^{-x_2(1-t_2)} dt_2 \\
 &= (1-t_1)^{-1} (1-t_2)^{-1} \quad \text{if } t_1, t_2 < 1
 \end{aligned}$$

Note : Since X_1, X_2 are indep, we can write
 $M_{X_1, X_2}(t_1, t_2) = M_{X_1}(t_1) M_{X_2}(t_2)$

m.g.f. if $Z = X_1 + X_2$
 $M_Z(t) = E(e^{t(x_1 + x_2)}) = (1-t)^{-2}, t < 1$

$$E(Z) = \frac{\partial M_Z(t)}{\partial t} \Big|_{t=0} = 2(1-t)^{-3} \Big|_{t=0} = 2$$

$$E(Z^2) = \frac{\partial^2 M_Z(t)}{\partial t^2} \Big|_{t=0} = 6(1-t)^{-4} \Big|_{t=0} = 6 \Rightarrow V(Z) = 2$$

(12) $M_{X_1, X_2}(t_1, t_2) = E(e^{t_1 x_1 + t_2 x_2})$?
 $= E(E(e^{t_1 x_1 + t_2 x_2} | x_1))$
 $= E\left(E^{t_1 x_1} E(e^{t_2 x_2} | x_1)\right)$

Since $x_2 | x_1 \sim N(\mu_2 + \rho \frac{\sigma_2}{\sigma_1} (x_1 - \mu_1), \sigma_2^2(1-\rho^2))$

$E(e^{t_2 x_2} | x_1) \rightarrow \text{const.} \text{ m.g.f. of } x_2 \text{ given } x_1$

$$M_{X_1, X_2}(t_1, t_2) = E \left(e^{t_1 X_1} \left(e^{t_2 (u_2 + p \frac{\sigma_2}{\sigma_1} (X_1 - u_1)) + \frac{t_2^2}{2} \sigma_2^2 (1-p^2)} \right) \right)$$

$$= e^{t_2 u_2 + \frac{t_2^2}{2} \sigma_2^2 (1-p^2)} E \left(e^{t_1 X_1 + t_2 p \frac{\sigma_2}{\sigma_1} X_1} \right) e^{-t_2 p \frac{\sigma_2}{\sigma_1} u_1}$$

$$= e^{t_2 u_2 + \frac{t_2^2}{2} \sigma_2^2 (1-p^2) - t_2 p \frac{\sigma_2}{\sigma_1} u_1} \times E \left(e^{(t_1 + t_2 p \frac{\sigma_2}{\sigma_1}) X_1} \right)$$

$$= e^{t_2 u_2 + \frac{t_2^2}{2} \sigma_2^2 (1-p^2) - t_2 p \frac{\sigma_2}{\sigma_1} u_1} e^{(t_1 + t_2 p \frac{\sigma_2}{\sigma_1}) u_1 + \frac{\sigma_1^2}{2} (t_1 + t_2 p \frac{\sigma_2}{\sigma_1})}$$

$$\begin{aligned} &= \exp \left(t_2 u_2 + \frac{t_2^2}{2} \sigma_2^2 (1-p^2) - t_2 p \cancel{\frac{\sigma_2}{\sigma_1}} u_1 + t_1 u_1 + t_2 p \cancel{\frac{\sigma_2}{\sigma_1}} u_1 \right. \\ &\quad \left. + \frac{\sigma_1^2}{2} (t_1^2 + t_2^2 p^2 \cancel{\frac{\sigma_2^2}{\sigma_1^2}} + 2 t_1 t_2 p \frac{\sigma_2}{\sigma_1}) \right) \end{aligned}$$

$$= \exp \left(t_2 u_2 + \frac{t_2^2}{2} \sigma_2^2 + t_1 u_1 + \frac{t_1^2}{2} \sigma_1^2 + t_1 t_2 p \sigma_1 \sigma_2 \right)$$

$$= \exp \left(t_1 u_1 + t_2 u_2 + \frac{1}{2} (t_1^2 \sigma_1^2 + t_2^2 \sigma_2^2 + 2 t_1 t_2 p \sigma_1 \sigma_2) \right)$$

$$\frac{\partial M_{X_1, X_2}(t_1, t_2)}{\partial t_1} \Big|_{t_1=0, t_2=0} = u_1 \quad \text{by } \frac{\partial M_{X_1, X_2}}{\partial t_2} \Big|_{t_1=0, t_2=0} = u_2$$

$\& V(X_1) = \sigma_1^2, V(X_2) = \sigma_2^2$

$$\text{E} \cancel{M_{X_1, X_2}} E(X_1, X_2) = \frac{\partial^2 M_{X_1, X_2}(t_1, t_2)}{\partial t_1 \partial t_2} \Big|_{t_1=0, t_2=0} = p \sigma_1 \sigma_2 + u_1 u_2$$

$$\Rightarrow \text{Cov}(X_1, X_2) = (p \sigma_1 \sigma_2 + u_1 u_2) - u_1 u_2 = p \sigma_1 \sigma_2$$

$$\Rightarrow \text{Corr}(X_1, X_2) = p.$$

$$(13) \quad f(x,y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$f_x(x) = \int_x^1 2 dy = 2(1-x), \quad 0 < x < 1$$

$$= 0 \quad \text{elsewhere}$$

$$f_y(y) = \int_0^y 2 dx = 2y, \quad 0 < y < 1$$

$$= 0 \quad \text{elsewhere}$$

$$f_{y|x=x} = \begin{cases} \frac{2}{2(1-x)}, & x < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$f_{x|y=y} = \begin{cases} \frac{2}{2y}, & 0 < x < y \\ 0, & \text{elsewhere} \end{cases}$$

$$E(y|x) = \int_x^1 y \cdot \frac{1}{1-x} dy = \frac{1-x^2}{2(1-x)} = \frac{1+x}{2}$$

$$E(y^2|x) = \int_x^1 y^2 \cdot \frac{1}{1-x} dy = \frac{1-x^3}{3(1-x)}.$$

$$\Rightarrow V(y|x) = E(y^2|x) - E(y|x)^2 = \frac{1-x^3}{3(1-x)} - \frac{1+x}{2}$$

Similarly $E(x|y)$, $E(x^2|y)$ and hence $V(x|y)$.

$$(14)(a) \quad \text{Cov}(X, b) = E(X - E(X))(b - E(b)) = 0 = \text{Cov}(X, b) = \text{Cov}(Z, b)$$

$$\begin{aligned} (b) \quad \text{Cov}(X, \alpha Y + b) &= E(X - E(X))(\alpha Y + b - E(\alpha Y + b)) \\ &= E(X - E(X))(\alpha Y + b - \alpha E(Y) - b) \\ &= \alpha \text{Cov}(X, Y) \end{aligned}$$

$$\begin{aligned} (c) \quad \text{Cov}(X, Y + Z) &= E(X - E(X))(Y + Z - E(Y) - E(Z)) \\ &= E(X - E(X))[(Y - E(Y)) + (Z - E(Z))] \\ &= \text{Cov}(X, Y) + \text{Cov}(X, Z) \end{aligned}$$

$$(d) \quad \text{Cov}(X, \alpha Y + b) = \alpha \text{Cov}(X, Y)$$

$$\begin{aligned} \text{Corr}^n(X, \alpha Y + b) &= \frac{\text{Cov}(X, \alpha Y + b)}{[V(X) V(\alpha Y + b)]^{1/2}} = \frac{\alpha \text{Cov}(X, Y)}{[V(X) \alpha^2 V(Y)]^{1/2}} \\ &= \text{Corr}(X, Y) \end{aligned}$$

$$\begin{aligned} (15) \quad \text{Cov}(W_1, W_2) &= \text{Cov}\left(X_1, \frac{\sqrt{3}-1}{2} X_1 + \frac{3-\sqrt{3}}{2} X_2\right) \\ &= \frac{\sqrt{3}-1}{2} V(X_1) + \frac{3-\sqrt{3}}{2} \text{Cov}(X_1, X_2) = \frac{\sqrt{3}-1}{2} \sigma_x^2 \end{aligned}$$

$$\begin{aligned} V(W_1) &= \sigma_x^2 \quad \& V(W_2) = \left(\frac{\sqrt{3}-1}{2}\right)^2 \sigma_x^2 + \left(\frac{3-\sqrt{3}}{2}\right)^2 \sigma_x^2 = (\sqrt{3}-1)^2 \sigma_x^2 \\ \Rightarrow \rho_{W_1, W_2} &= \frac{1}{2} \end{aligned}$$

$$\text{S} \quad \rho_{W_1, W_3} \quad \& \quad \rho_{W_2, W_3}$$

$$\begin{aligned} \downarrow \\ \text{Cov}(W_1, W_3) &= \text{Cov}\left(X_1, (\sqrt{2}-1) X_2 + (2-\sqrt{2}) X_3\right) = 0 \\ \Rightarrow \rho_{W_1, W_3} &= 0 \end{aligned}$$

$$(16) \quad (a) \quad P(3 < Y < 8) \quad Y \sim N(1, 25)$$

$$= P\left(\frac{3-1}{5} < \frac{Y-1}{5} < \frac{8-1}{5}\right) = \Phi\left(\frac{7}{5}\right) - \Phi\left(\frac{2}{5}\right) \\ = \dots \quad (\text{from table})$$

$$(b) \quad P(3 < Y < 8 \mid X=7) \quad \left[Y \mid X \sim N\left(1 + \rho \frac{5}{4}(x-3), 25(1-\rho^2)\right) \right]$$

$$= P\left(\frac{3-4}{4} < \frac{Y-4}{4} < \frac{8-4}{4} \mid X=7\right) \quad \text{i.e. } Y \mid X=7 \sim N(4, 16)$$

$$= \Phi(1) - \Phi(-0.25) = \dots$$

$$(c) \quad P(-3 < X < 3) \quad X \sim N(3, 16)$$

$$= P\left(\frac{-3-3}{4} < \frac{X-3}{4} < \frac{3-3}{4}\right) = \Phi(0) - \Phi\left(-\frac{6}{4}\right) = \dots$$

$$(d) \quad P(-3 < X < 3 \mid Y=4) \quad \left[X \mid Y \sim N\left(3 + \rho \frac{4}{5}(y-1), 16(1-\rho^2)\right) \right]$$

$$= P\left(\frac{-3-4.44}{3.2} < \frac{X-4.44}{3.2} < \frac{3-4.44}{3.2} \mid Y=4\right) \quad \text{i.e. } X \mid Y=4 \sim N(4.44, (3.2)^2)$$

$$= \Phi\left(-\frac{1.44}{3.2}\right) - \Phi\left(-\frac{7.44}{3.2}\right) = \dots$$

$$(17) \quad (X, Y) \sim N_2(5, 10, 1, 25, \rho) \quad ; \rho > 0$$

$$Y \mid X=5 \sim N_1(10, 25(1-\rho^2))$$

$$P(4 < Y < 16 \mid X=5) = P\left(\frac{4-10}{5\sqrt{1-\rho^2}} < \frac{Y-10}{5\sqrt{1-\rho^2}} < \frac{16-10}{5\sqrt{1-\rho^2}} \mid X=5\right)$$

$$= \Phi\left(\frac{6}{5\sqrt{1-p^2}}\right) - \Phi\left(-\frac{6}{5\sqrt{1-p^2}}\right)$$

$$= 2 \Phi\left(\frac{6}{5\sqrt{1-p^2}}\right) - 1 = 0.954 \quad (\text{given condition})$$

$$\Rightarrow \Phi\left(\frac{6}{5\sqrt{1-p^2}}\right) = 0.977 = \Phi(2)$$

$$\Rightarrow \frac{6}{5\sqrt{1-p^2}} = 2 \Rightarrow 1-p^2 = 0.36 \Rightarrow p = 0.8 \quad (\text{as } p > 0).$$

$$(18) \quad E(y) = \sum_1^{15} E(x_i) = 30$$

$$V(y) = 15 V(x_i) = 45 ; V(z) = 10 \times 3 = 30$$

$$\text{Cov}(y, z) = \text{Cov}\left(\sum_1^{15} x_i, \sum_{ii}^{20} x_i\right) = 5 V(x_i) = 15$$

$$\rho_{y,z} = \frac{15}{[45 \times 30]} V_z$$

$$(19) \quad u = x - y ; v = 2x - 3y$$

$$E(u) = -5 \quad E(v) = 2 \times 15 - 3 \times 20 = -30$$

$$V(u) = V(x) + V(y) - 2 \text{Cov}(x, y) \quad V(v) = 4V(x) + 9V(y) - 12 \text{Cov}(x, y)$$

Now,

$$\rho_{x,y} = -0.6 = \frac{\text{Cov}(x, y)}{\sqrt{5 \times 100}} = \frac{\text{Cov}(x, y)}{50} \Rightarrow \text{Cov}(x, y) = -30$$

$$\Rightarrow V(u) = 185 \quad \& V(v) = 100 + 900 + 360 = 1360$$

$$\Rightarrow \text{Cov}(u, v) = \text{Cov}(x-y, 2x-3y) = \rightarrow$$

$$= 2V(x) - 3W(x,y) - 2W(y,x) + 3V(y)$$

$$= 2 \times 25 - 5(-30) + 3 \times 100 = 500$$

$$\Rightarrow P_{V,W} = \frac{500}{\sqrt{185 \times 1360}}.$$

(20) X : r.v. denoting life time

$$X \sim \text{Exp}(50) \quad \text{p.d.f.} \quad f(x) = \begin{cases} \frac{1}{50} e^{-x/50}, & x > 0 \\ 0 & \text{else.} \end{cases}$$

$$F_X(x) = 1 - e^{-x/50}, \quad x > 0$$

Y_1 : # of bulbs out of 8 to have lifetime < 40

Y_2 : $\dots - - - - - - - -$ $40 \leq x \leq 60$

Y_3 : $\dots - - - - - - - -$ $\geq 60 \quad \& \quad \leq 80$

Y_4 : $\dots - - - - - - - -$ > 80

$$P(X < 40) = F_X(40) = 1 - e^{-40/50} = p_1, \text{ say}$$

$$P(40 \leq X < 60) = F_X(60) - F_X(40) = e^{-40/50} - e^{-60/50} = p_2, \text{ say}$$

$$P(60 \leq X \leq 80) = F_X(80) - F_X(60) = e^{-60/50} - e^{-80/50} = p_3, \text{ say}$$

$$P(X > 80) = 1 - p_1 - p_2 - p_3 = e^{-80/50}.$$

It. p.m.f. if Y_1, Y_2, Y_3 is multinomial. $(8, p_1, p_2, p_3)$.

$$\Rightarrow P(Y_1=2, Y_2=3, Y_3=2) = \frac{8!}{2! 3! 2! 1!} p_1^2 p_2^3 p_3^2 (1-p_1-p_2-p_3).$$

Marginal dist^r $y_3 \sim \text{Bin}(8, p_3 = e^{-60/50} - e^{-80/50})$

$$E(y_3) = 8(e^{-60/50} - e^{-80/50})$$

Condⁿ $y_3 | y_2 = y_2 \sim \text{Bin}(8-y_2, \frac{p_3}{1-p_2})$

$$\Rightarrow E(y_3 | y_2 = 1) = (8-1) \left(\frac{e^{-60/50} - e^{-80/50}}{1 - e^{-40/50} + e^{-60/50}} \right).$$

$x \setminus y$	0	1	2	
0	$\frac{1}{3}$	0	0	$\frac{1}{3}$
1	0	$\frac{1}{3}$	0	$\frac{1}{3}$
2	0	0	$\frac{1}{3}$	$\frac{1}{3}$
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

$$E(x) = 1 = E(y)$$

$$V(x) = E(x^2) - 1$$

$$= \frac{5}{3} - 1 = \frac{2}{3} = V(y)$$

$$E(xy) = (0 \times 0) \frac{1}{3} + (1 \times 1) \frac{1}{3} \neq (2 \times 2) \times \frac{1}{3}$$

$$= \frac{2}{3}$$

$$Cov(x, y) = \frac{5}{3} - 1 = \frac{2}{3}$$

$$\rho_{x,y} = 1$$

$x \setminus y$	0	1	2	
0	0	0	$\frac{1}{3}$	
1	0	$\frac{1}{3}$	0	
2	$\frac{1}{3}$	0	0	
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

sl_y

$$\Rightarrow \rho_{x,y} = -1$$

$x \setminus y$	0	1	2	
0	$\frac{1}{3}$	0	0	
1	0	$\frac{1}{3}$	0	
2	$\frac{1}{3}$	0	0	
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

$$\rho_{x,y} = 0$$

x\y	1	2	
1	$\frac{1}{10}$	0	$\frac{1}{10}$
2	$\frac{2}{10}$	$\frac{4}{10}$	$\frac{6}{10}$
3	$\frac{3}{10}$	0	$\frac{3}{10}$
	$\frac{6}{10}$	$\frac{4}{10}$	

$$E(X) = \frac{1}{10} + 2 \cdot \frac{6}{10} + 3 \cdot \frac{3}{10} = \frac{22}{10}$$

$$E(Y) = \frac{6}{10} + 2 \cdot \frac{4}{10} = \frac{14}{10}$$

$$E(X^2) = \frac{1}{10} + 4 \cdot \frac{6}{10} + 9 \cdot \frac{3}{10} = \frac{52}{10}$$

$$E(Y^2) = \frac{6}{10} + 4 \cdot \frac{4}{10} = \frac{22}{10}$$

$$V(X) = \frac{52}{10} - \left(\frac{22}{10} \right)^2 = \dots$$

$$V(Y) = \left(\frac{14}{10} \right)^2 - \left(\frac{6}{10} \right)^2 = \dots$$

$$E(XY) = (1 \times 1) \frac{1}{10} + (2 \times 1) \frac{2}{10} + (2 \times 2) \frac{4}{10} + (3 \times 1) \frac{3}{10} = \frac{30}{10} = 3$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 3 - \frac{22}{10} \cdot \frac{14}{10} = \dots$$

$$\text{Correl}^2(X, Y) = \frac{\text{Cov}(X, Y)}{[V(X)V(Y)]^{1/2}} = \dots$$

Joint m.g.f. of (X, Y)

$$M_{X,Y}(t_1, t_2) = \sum_{x,y} e^{t_1 x + t_2 y} P(X=x, Y=y)$$

$$= e^{t_1 + t_2} \times \frac{1}{10} + e^{2t_1 + t_2} \frac{2}{10} + e^{2(t_1 + t_2)} \frac{4}{10}$$

$$+ e^{3t_1 + t_2} \cdot \frac{3}{10}$$

$$(23) M_{X,Y}(u, v) = E(e^{uX+vY})$$

$$\Psi(u, v) = \log M_{X,Y}(u, v)$$

$$\frac{\partial \Psi(u, v)}{\partial u} = \frac{1}{M_{X,Y}(u, v)} \cdot \frac{\partial M_{X,Y}(u, v)}{\partial u}$$

$$\left. \frac{\partial \Psi(u, v)}{\partial u} \right|_{u=0, v=0} = \frac{1}{M(0,0)} \cdot E(X) = E(X)$$

$$\text{Sly} \quad \frac{\partial \Psi(0,0)}{\partial v} = \frac{\partial \Psi(u,v)}{\partial v} \Big|_{u=0, v=0} = E(y)$$

$$\begin{aligned} \frac{\partial^2 \Psi(u,v)}{\partial u^2} &= \frac{1}{M(u,v)} \frac{\partial^2 M(u,v)}{\partial u^2} + \left[\frac{-1}{(M(u,v))^2} \frac{\partial M(u,v)}{\partial u} \right] \left(\frac{\partial M(u,v)}{\partial u} \right) \\ &= \frac{1}{M(u,v)} \cdot \frac{\partial^2 M(u,v)}{\partial u^2} - \left(\frac{\partial M(u,v)}{\partial u} \cdot \frac{1}{M(u,v)} \right)^2 \end{aligned}$$

$$\frac{\partial^2 \Psi(u,v)}{\partial u^2} \Big|_{u=0, v=0} = E(x^y) - E(x) = V(x) = \frac{\partial \Psi(0,0)}{\partial u^2}$$

$$\text{Sly} \quad \frac{\partial^2 \Psi(u,v)}{\partial v^2} \Big|_{u=0, v=0} = V(y)$$

$$\frac{\partial^2 \Psi(u,v)}{\partial v \partial u} = \frac{1}{M(u,v)} \cdot \frac{\partial^2 M(u,v)}{\partial v \partial u} - \frac{1}{(M(u,v))^2} \cdot \frac{\partial M(u,v)}{\partial v} \cdot \frac{\partial M(u,v)}{\partial u}$$

$$\frac{\partial^2 \Psi(u,v)}{\partial v \partial u} \Big|_{u=0, v=0} = E(xy) - E(x)E(y)$$

$$\text{i.e. } \frac{\partial^2 \Psi(u,v)}{\partial v \partial u} \Big|_{u=0, v=0} = \text{cov}(x,y)$$

(24) Marginal p.d.f. of X .

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy \\ &= \frac{1}{2} \int_{-\infty}^{\infty} f_p(x,y) dy + \frac{1}{2} \int_{-\infty}^{\infty} f_{-p}(x,y) dy \\ &= \frac{1}{2} \phi(x) + \frac{1}{2} \phi(x) \quad \left[\phi(x) \text{ p.d.f. of } N(0,1) \right] \\ &= \phi(x) \Rightarrow X \sim N(0,1) \end{aligned}$$

$$\text{Sly} \quad f_Y(y) = \phi(y) \Rightarrow Y \sim N(0,1)$$

$$\begin{aligned}
 E(XY) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x,y) dx dy \\
 &= \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_p(x,y) dx dy + \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{-p}(x,y) dx dy \\
 &= \frac{1}{2} (\rho) + \frac{1}{2} (-\rho) = 0
 \end{aligned}$$

$$\begin{aligned}
 \text{Cov}(X,Y) &= E(XY) - E(X)E(Y) \\
 &= 0 - 0 \cdot 0 = 0
 \end{aligned}$$

$\rho(X,Y) = 0 \Rightarrow X \text{ & } Y \text{ are uncorrelated}$

Since, $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$.

$X \text{ & } Y \text{ are not independent}$

$$(25) \quad \int_0^1 \int_{-x}^x k dy dx = 1 \Rightarrow k \int_0^1 2x dx = 1 \Rightarrow k = 1$$

$$\text{Marginal of } X, f_X(x) = \int_{-x}^x k dy = \begin{cases} 2x, & 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

$$\text{Marginal of } Y, f_Y(y) = \int_{|y|}^1 dx = \begin{cases} 1-|y|, & -1 < y < 1 \\ 0 & \text{else} \end{cases}$$

$$\text{Conditional distn of } Y | X=x ; f_{Y|X=x} = \begin{cases} \frac{1}{2x}, & -x < y < x \\ 0, & \text{else} \end{cases}$$

$$E(Y | X=x) = \int_{-x}^x y \frac{1}{2x} dy = 0$$

$$\begin{aligned}
 \text{Sly } E(X | Y=y) &= \int_{|y|}^1 x \frac{1}{1-|y|} dx = \frac{1-y^2}{2(1-|y|)} \\
 f_{X|Y=y} &= \begin{cases} (1-|y|)^{-1}, & |y| < x < 1 \\ 0 & \text{else} \end{cases}
 \end{aligned}$$

$$f_{x,y}(x,y) = 1 \neq f_x(x) f_y(y)$$

$\Rightarrow x \text{ & } y$ are not indep.

$$E(xy) = \int_0^1 \int_{-\infty}^{\infty} xy dy dx = 0$$

$$E(y) = E[E(y|x)] = 0$$

$$\Rightarrow \text{Cov}(x,y) = \rho_{x,y} = 0$$

$\Rightarrow x \text{ & } y$ are uncorrelated

$$(26) H_{x,y}(s,t) = \{a(e^{s+t}+1) + b(e^s + e^t)\}, \left(a, b > 0, a+b = \frac{1}{2} \right)$$

$$E(x) = \frac{\partial}{\partial s} (a(e^{s+t}+1) + b(e^s + e^t)) \Big|_{s=t=0}$$

$$= a e^t e^s + b e^s \Big|_{t=s=0} = a+b = \frac{1}{2} = E(y)$$

$$E(x^2) = \frac{\partial^2}{\partial s^2} (a(e^{s+t}+1) + b(e^s + e^t)) \Big|_{s=t=0}$$

$$= a e^t e^s + b e^s \Big|_{s=t=0} = a+b = \frac{1}{2} = E(y^2)$$

$$V(x) = V(y) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$E(xy) = \frac{\partial^2}{\partial t \partial s} (a(e^{s+t}+1) + b(e^s + e^t)) \Big|_{s=t=0}$$

$$= a e^t e^s \Big|_{s=t=0} = a$$

$$\therefore \text{Cov}(x,y) = a - \frac{1}{4} \Rightarrow \rho_{x,y} = \frac{a - \frac{1}{4}}{\frac{1}{4}} = 4a - 1$$

$$\begin{aligned}
 (27) \quad & \text{Var} \left(\frac{x}{3} + \frac{2y}{3} \right) \left(= \text{Var} \left(\frac{2x}{3} + \frac{y}{3} \right) \right) \quad \because V(x) = V(y) \\
 & = \frac{1}{9} V(x) + \frac{4}{9} V(y) + 2 \text{Cov} \left(\frac{x}{3}, \frac{2y}{3} \right) \\
 & = \frac{2}{9} + \frac{8}{9} + \frac{4}{9} \times \frac{2}{3} = \frac{2}{9} + \frac{8}{9} + \frac{8}{27} = \frac{38}{27} \\
 & \text{Cov} \left(\frac{x}{3} + \frac{2y}{3}, \frac{2x}{3} + \frac{y}{3} \right) \\
 & = \frac{2}{9} V(x) + \frac{1}{9} \text{Cov}(x, y) + \frac{4}{9} \text{Cov}(x, y) + \frac{2}{9} V(y) \\
 & = \frac{4}{9} + \frac{2}{27} + \frac{8}{27} + \frac{4}{9} = \frac{34}{27} \\
 & \text{Correl}^2 \left(\frac{x}{3} + \frac{2y}{3}, \frac{2x}{3} + \frac{y}{3} \right) = \frac{\frac{34}{27}}{\frac{38}{27}} = \underline{\underline{\frac{34}{38}}}
 \end{aligned}$$

$$(28) \quad \tilde{x} \sim N_3(\underline{0}, \Sigma) ; \quad \Sigma = \begin{pmatrix} 1 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 1 \end{pmatrix}$$

$$\text{(a)} \quad \tilde{y} = \begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 - x_2 - x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A \tilde{x}$$

$\forall \underline{\alpha} \in \mathbb{R}^2 ; \quad \underline{\alpha}' \tilde{y} = \underline{\alpha}' A \tilde{x} = \underline{\beta}' \tilde{x} ; \quad \underline{\beta} \in \mathbb{R}^3$
 $\sim N_1 \quad (\text{as } \tilde{x} \sim N_3).$

$$\Rightarrow \tilde{y} \sim N_2$$

$$E(\tilde{y}) = \underline{0}$$

$$\text{Cov}(\tilde{y}) = A \text{Cov}(\tilde{x}) A'$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\Rightarrow \tilde{y} \sim N_2(\underline{0}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix})$$

$\Rightarrow (x_1 + x_2 + x_3) \text{ & } (x_1 - x_2 - x_3)$ is N_2 with

$\text{cov}(\cdot, \cdot) = 0 \Rightarrow x_1 + x_2 + x_3 \text{ & } x_1 - x_2 - x_3$ are
 indep.

$$(b) \quad X_1 - X_2 - X_3 = \begin{pmatrix} 1 & -1 & -1 \end{pmatrix} \underline{\underline{X}} \sim N_1 \quad (\text{as } \underline{\underline{X}} \sim N_3)$$

$$E(X_1 - X_2 - X_3) = 0$$

$$V(X_1 - X_2 - X_3) = V(\underline{\alpha}' \underline{\underline{X}}) = \underline{\alpha}' \sum \underline{\alpha}$$

$$= 3 \quad \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore X_1 - X_2 - X_3 \sim N_1(0, 3)$$

$$\Rightarrow \frac{X_1 - X_2 - X_3}{\sqrt{3}} \sim N_1(0, 1)$$

$$\Rightarrow \left(\frac{X_1 - X_2 - X_3}{\sqrt{3}} \right)^2 \sim \chi^2_1$$

$$\Rightarrow (X_1 - X_2 - X_3)^2 \sim 3 \chi^2_1$$

$$(29) \quad \underline{\underline{X}} \sim N_3(\underline{\underline{\mu}}, \Sigma); \quad \Sigma = \begin{pmatrix} 1 & p & p \\ p & 1 & p \\ p & p & 1 \end{pmatrix} \quad -\frac{1}{2} < p < 1$$

$$\underline{\underline{Y}} = \begin{pmatrix} X_1 + X_2 \\ X_1 - X_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = A \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

Subvector

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2 \left(\begin{pmatrix} E X_1 \\ E X_2 \end{pmatrix} = \underline{\underline{\mu}}^* \quad , \quad \begin{pmatrix} 1 & p \\ p & 1 \end{pmatrix} \rightarrow \Sigma^* \right)$$

$$\Rightarrow \underline{\underline{Y}} \sim N_2(A \underline{\underline{\mu}}^*, A \Sigma^* A')$$

$$A \Sigma^* A' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & p \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1+p & 1+p \\ 1-p & p-1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2(1+p) & 0 \\ 0 & 2(1-p) \end{pmatrix}$$

$$(30) \quad \underline{x} = (x_1, x_2, x_3, x_4)'$$

$$f_{\underline{x}}(\underline{x}) = \frac{1}{2} \left(\frac{1}{(2\pi)^{4/2} |\Sigma_1|^{1/2}} \exp\left(-\frac{1}{2} \underline{x}' \Sigma_1^{-1} \underline{x}\right) \right) \\ + \frac{1}{2} \left(\frac{1}{(2\pi)^{4/2} |\Sigma_2|^{1/2}} \exp\left(-\frac{1}{2} \underline{x}' \Sigma_2^{-1} \underline{x}\right) \right)$$

$$\text{i.e. } f_{\underline{x}}(\underline{x}) = \frac{1}{2} \left(\text{p.d.f. of } N_4(\underline{0}, \Sigma_1) \right) \\ + \frac{1}{2} \left(\text{p.d.f. of } N_4(\underline{0}, \Sigma_2) \right)$$

$$(a) \quad \underline{x}^{(1)} = (x_1, x_2)'$$

$$f_{\underline{x}^{(1)}}(x_1, x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\underline{x}}(\underline{x}) dx_3 dx_4 \\ = \frac{1}{2} \left(\text{p.d.f. of } N_2(\underline{0}, A_1) \right) \\ + \frac{1}{2} \left(\text{p.d.f. of } N_2(\underline{0}, A_1) \right) \\ = \text{p.d.f. of } N_2(\underline{0}, A_1)$$

$$\Rightarrow \underline{x}^{(1)} \sim N_2(\underline{0}, A_1)$$

$$(b) \quad \underline{x}^{(2)} = (x_3, x_4)'$$

$$f_{\underline{x}^{(2)}}(x_3, x_4) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\underline{x}}(\underline{x}) dx_1 dx_2 \\ = \frac{1}{2} \left(\text{p.d.f. of } N_2(\underline{0}, A_1) \right) \\ + \frac{1}{2} \left(\text{p.d.f. of } N_2(\underline{0}, A_3) \right)$$

Note that $\tilde{X}^{(2)} \not\sim N_2$; it's is ~~a~~ mixture Gaussian

$$(c) f_{\tilde{X}^{(2)}}(x_3, x_4) = \frac{1}{2} f_1(x_3, x_4) + \frac{1}{2} f_2(x_3, x_4)$$

$\xrightarrow{\quad}$

\uparrow

p.d.f. of $N_2(0, A_1)$

p.d.f. of $N_2(0, A_3)$

$$E(x_3 x_4) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_3 x_4 f_{\tilde{X}^{(2)}}(x_3, x_4) dx_3 dx_4$$

$$= \frac{1}{2} (\text{cov}(x_3, x_4) \text{ for } N_2(0, A_1))$$

$$+ \frac{1}{2} (\text{cov}(x_3, x_4) \text{ for } N_2(0, A_3))$$

$$= \frac{1}{2} (P + (-P)) = 0$$

$$\Rightarrow \text{cov}(x_3, x_4) = E(x_3 x_4) - 0 = 0$$

x_3 & x_4 are uncorrelated.

Marginal of X_3 :

$$\int_{-\infty}^{\infty} f_{\tilde{X}^{(2)}}(x_3, x_4) dx_4$$

$$= \frac{1}{2} (\text{p.d.f. of } N_1(0, 1))$$

$$+ \frac{1}{2} (\text{p.d.f. of } N_1(0, 1))$$

$$= \text{p.d.f. of } N_1(0, 1)$$

$$\Rightarrow x_3 \sim N(0, 1) \Rightarrow f(x_3, x_4) \neq f_{x_3}(x_3) f_{x_4}(x_4)$$

Since $x_4 \sim N(0, 1)$ $\Rightarrow x_3$ & x_4 are not indep