MSO201A: PROBABILITY & STATISTICS
Problem Set #7

[1] The joint p.d.f. of (X,Y) is given by $f(x,y)=\begin{cases}4xy & 0<x<1, 0<y<1\\ 0 & \text{otherwise.}\end{cases}$

Find the marginal p.d.f.s and verify whether the random variables are independent. Also find $P(0<X<1/2, 1/4<Y<1), P(X+Y<1)$

[2] If the joint p.d.f. of (X,Y) $f(x,y)=\begin{cases}e^{-(x+y)} & 0<x,y<\infty\\ 0 & \text{otherwise,}\end{cases}$

show that X and Y are independent.

[3] If the joint p.d.f. of (X,Y) is $f(x,y)=\begin{cases}2e^{-(x+y)} & 0<x<y<\infty\\ 0 & \text{otherwise,}\end{cases}$

show that X and Y are not independent.

[4] Show that the random variables X and Y with joint p.d.f.

$$f(x,y)=\begin{cases}12xy(1-y) & 0<x<1, 0<y<1\\ 0 & \text{otherwise.}\end{cases}$$

are independent.

[5] Suppose the joint p.d.f. of (X,Y) is $f(x,y)=\begin{cases}cx^2y & 0<x<y<1,\\ 0 & \text{otherwise.}\end{cases}$

Find (a) the value of the constant c, (b) the marginal p.d.f.s of X and Y and (c) $P(X+Y\le 1)$.

[6] The joint p.d.f. of (X,Y) is given by $f(x,y)=\begin{cases}6(1-x-y) & x>0, y>0, x+y<1\\ 0 & \text{otherwise.}\end{cases}$

Find the marginal p.d.f. s of X and Y and $P(2X+3Y<1)$.

[7] The joint p.d.f. of (X,Y) is $f(x,y)=\begin{cases}x+y & 0<x<1, 0<y<1\\ 0 & \text{otherwise.}\end{cases}$

Find the conditional distribution of Y given $X=x, 0<x<1;$ the conditional mean and conditional variance of the conditional distribution.

[8] Suppose the conditional p.d.f. of X given $Y=y$ is $f(x\mid y)=\begin{cases}cx/y^2 & 0<x<y\\ 0 & \text{otherwise.}\end{cases}$

Further, the marginal distribution of Y is $g(y)=\begin{cases}d\,y^4 & 0<y<1\\ 0 & \text{otherwise.}\end{cases}$

(a) Find the constants c and d.

(b) The joint p.d.f. of (X,Y).

(c) $P(0.25<X<0.5)$ and $P(0.25<X<0.5\mid Y=0.625)$

[9] Let $f(x)$ and $g(y)$ be two arbitrary p.d.f.s with corresponding distribution functions $F(x)$ and $G(y)$ respectively. Suppose the joint p.d.f. of X and Y is given by

$$h(x,y)=f(x)g(y)\left[1+\alpha\{2F(x)-1\}\{2G(y)-1\}\right],\ |\alpha|\leq 1$$

Show that the marginal p.d.f.s of X and Y are $f(x)$ and $g(y)$, respectively. Does there exist a value of α for which the random variables X and Y are independent?

[10] Suppose the marginal density of the random variable X is $f_X(x)=\begin{cases}4x\left(1-x^2\right), & 0<x<1,\\ 0, & \text{otherwise.}\end{cases}$

and the conditional density of the random variable Y given $X=x$ is

$$f_{Y|X=x}(y\,|\,x)=\begin{cases}2\,y/\left(1-x^2\right), & x<y<1, 0<x<1,\\ 0, & \text{otherwise.}\end{cases}$$

Find the conditional p.d.f. of X given $Y=y$, $E(X\,|\,Y=1/2)$ and $Var(X\,|\,Y=1/2)$.

[11] The joint p.d.f. of (X,Y) $f(x,y)=\begin{cases}e^{-(x+y)} & 0<x,y<\infty\\ 0 & \text{otherwise.}\end{cases}$

Find the joint m.g.f. of (X,Y) and the m.g.f. of $Z=X+Y$ and hence $V(Z)$.

[12] Derive the joint m.g.f. of $(X_1,X_2)\sim N_2\left(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho\right)$ and using the joint m.g.f find $\rho(X_1,X_2)$.

[13] Let the joint p.d.f. of (X,Y) be $f(x,y)=\begin{cases}2 & 0<x<y<1\\ 0 & \text{otherwise,}\end{cases}$

Find the conditional mean and conditional variance of X given $Y=y$ and that of Y given $X=x$. Compute further $\rho(X,Y)$.

[14] Let X,Y and Z be three random variables and a and b be two scalar constants. Prove that (a) $Cov(X,b)=Cov(Y,b)=Cov(Z,b)=0$; (b) $Cov(X,aY+b)=a\,Cov(X,Y)$;(c) $Cov(X,Y+Z)=Cov(X,Y)+Cov(X,Z)$;(d) $\rho(X,aY+b)=\rho(X,Y)$ for $a>0$.

[15] Let X_1,X_2 and X_3 be three independent random variables each with a variance σ^2. Define the new random variables

$$W_1=X_1,\ W_2=\frac{\sqrt{3}-1}{2}X_1+\frac{3-\sqrt{3}}{2}X_2 \text{ and } W_3=\left(\sqrt{2}-1\right)X_2+\left(2-\sqrt{2}\right)X_3.$$

Find $\rho(W_1,W_2)$, $\rho(W_1,W_3)$ and $\rho(W_2,W_3)$.

[16] Let $(X,Y)\sim N_2(3,1,16,25,0.6)$. Find (a) $P(3<Y<8)$;(b) $P(3<Y<8\,|\,X=7)$; (c) $P(-3<X<3)$ and (d) $P(-3<X<3\,|\,Y=4)$.

[17] Let $(X,Y)\sim N_2(5,10,1,25,\rho)$ with $\rho>0$. If it is given that $P(4<Y<16\,|\,X=5)=0.954$

and $\Phi(2)=0.977$, find the value of ρ.

[18] Let $X_1, X_2, \ldots, X_{20}$ be independent random variables with identical distributions, each with a mean 2 and variance 3. Define $Y=\sum_{i=1}^{15} X_i$ and $Z=\sum_{i=11}^{20} X_i$. Find $E(Y)$, $E(Z)$, $V(Y)$, $V(Z)$ and $\rho(Y,Z)$.

[19] Let X and Y be a jointly distributed random variables with $E(X)=15, E(Y)=20$, $V(X)=25, V(Y)=100$ and $\rho(X,Y)=-0.6$. Find $\rho(X-Y, 2X-3Y)$.

[20] Suppose that the lifetime of light bulbs of a certain kind follows exponential distribution with p.d.f.

$$f_X(x)=\begin{cases}\dfrac{1}{50}e^{-x/50} & x>0\\ 0 & \text{otherwise.}\end{cases}$$

Find the probability that among 8 such bulbs, 2 will last less that 40 hours, 3 will last anywhere between 40 and 60 hours, 2 will last anywhere between 60 and 80 hours and 1 will last for more than 80 hours. Find the expected number of bulbs in a lot of 8 bulbs with lifetime between 60 and 80 hours and also the expected number of bulbs in a lot of 8 with lifetime between 60 and 80 hours, given that the number of bulbs with lifetime anywhere between 40 and 60 hours is 2.

[21] Let the random variables X and Y have the following joint p.m.f.s

(a) $P(X=x, Y=y)=1/3$, if $(x,y)\in\{(0,0),(1,1),(2,2)\}$ and 0 otherwise.

(b) $P(X=x, Y=y)=1/3$, if $(x,y)\in\{(0,2),(1,1),(2,0)\}$ and 0 otherwise.

(c) $P(X=x, Y=y)=1/3$, if $(x,y)\in\{(0,0),(1,1),(2,0)\}$ and 0 otherwise.

In each of the above cases find the coefficient of correlation between X and Y.

[22] The joint p.m.f. of (X,Y) is

$$P(X=x, Y=y)=xy/10, \text{ if } (x,y)\in\{(1,1),(2,1),(2,2),(3,1)\} \text{ and 0 otherwise.}$$

Find the joint m.g.f. of X and Y and the coefficient of correlation between X and Y. Using the joint m.g.f., find the p.m.f. $Z=X+Y$.

[23] Let $M_{X,Y}(u,v)$ denote the joint m.g.f. (X,Y) and $\psi(u,v)=\log\left(M_{X,Y}(u,v)\right)$.

Show that

$$\left.\frac{\partial\psi(u,v)}{\partial u}\right|_{u=v=0}, \left.\frac{\partial\psi(u,v)}{\partial v}\right|_{u=v=0}, \left.\frac{\partial^2\psi(u,v)}{\partial u^2}\right|_{u=v=0}, \left.\frac{\partial^2\psi(u,v)}{\partial v^2}\right|_{u=v=0} \text{ and } \left.\frac{\partial^2\psi(u,v)}{\partial u\,\partial v}\right|_{u=v=0}$$ yields the means, the variances and the covariance of the two random variables.

[24] The joint probability density function of X and Y is given by

$$f_{X,Y}(x,y)=\frac{1}{2}\left(f_\rho(x,y)+f_{-\rho}(x,y)\right); \qquad -\infty<x,y<\infty$$

where, $f_\rho(x,y)$ is the probability density function of $N_2(0,0,1,1,\rho)$ and $f_{-\rho}(x,y)$ is the probability density function of $N_2(0,0,1,1,-\rho)$. Find the marginal p.d.f.s of X and Y, the correlation coefficient between X and Y. Are the 2 variables independent?

[25] Let the joint p.d.f. of X and Y be given by

$$f_{X,Y}(x,y)=\begin{cases}k, & \text{if } -x<y<x;\ 0<x<1,\\ 0, & \text{otherwise.}\end{cases}$$

Find the value of the constant k and obtain the conditional expectations $E(X\mid Y=y)$ and $E(Y\mid X=x)$. Verify whether the 2 random variables are independent and/or uncorrelated.

[26] The joint moment generating function of X and Y is given by

$$M_{X,Y}(s,t)=\left\{a\left(e^{s+t}+1\right)+b\left(e^{s}+e^{t}\right)\right\},\quad a,b>0;\quad a+b=1/2.$$

Find the correlation coefficient between X and Y.

[27] Let X and Y be jointly distributed random variables with

$$E(X)=E(Y)=0,\ E\left(X^2\right)=E\left(Y^2\right)=2 \text{ and } \rho(X,Y)=1/3$$

Find $\rho(X/3+2Y/3,\,2X/3+Y/3)$.

[28] Let $\underset{\sim}{X}=(X_1,X_2,X_3)'\sim N_3(\underset{\sim}{0},\Sigma);\ \Sigma=\begin{pmatrix}1 & -0.5 & 0\\ & 1 & -0.5\\ & & 1\end{pmatrix}$.

(a) Verify whether $X_1+X_2+X_3$ and $X_1-X_2-X_3$ are independent.

(b) Find the distribution of $(X_1-X_2-X_3)^2$.

[29] Let $\underset{\sim}{X}=(X_1,X_2,X_3)'$ be distributed as $N_3(\underset{\sim}{\mu},\Sigma)$,

$$\Sigma=\begin{pmatrix}1 & \rho & \rho\\ \rho & 1 & \rho\\ \rho & \rho & 1\end{pmatrix};\ -1/2<\rho<1.$$

Find the joint distribution of $(X_1+X_2,X_1-X_2)'$.

[30] Let the joint p.d.f. of $\underset{\sim}{X}=(X_1,X_2,X_3,X_4)^T$ be

$$f_{\underset{\sim}{X}}(\underset{\sim}{x})=\frac{\exp\left(-\underset{\sim}{x}^T\Sigma_1^{-1}\underset{\sim}{x}/2\right)}{8\pi^2|\Sigma_1|^{1/2}}+\frac{\exp\left(-\underset{\sim}{x}^T\Sigma_2^{-1}\underset{\sim}{x}/2\right)}{8\pi^2|\Sigma_2|^{1/2}};\quad \underset{\sim}{x}\in\Re^4,$$

where, $\Sigma_1=\begin{pmatrix}A_1 & 0\\ 0 & A_1\end{pmatrix}$, $A_1=\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}$; $\Sigma_2=\begin{pmatrix}A_1 & 0\\ 0 & A_3\end{pmatrix}$, $A_3=\begin{pmatrix}1 & -\rho\\ -\rho & 1\end{pmatrix}$; $|\rho|<1$.

(a) Find the joint p.d.f. of $\underset{\sim}{X}^{(1)}=(X_1,X_2)^T$.

(b) Find the joint p.d.f. of $\underset{\sim}{X}^{(2)}=(X_3,X_4)^T$.

(c) Prove or disprove "X_3 and X_4 are uncorrelated but not independent".