

Problem Set #3

(1) (a)

$$\frac{y=|x|}{\forall x \in \mathbb{R}}$$

$$\begin{aligned} y^{-1}(-\infty, x] &= \{\omega : y(\omega) \leq x\} = \{\omega : |x(\omega)| \leq x\} \\ &= \{\omega : -x \leq x(\omega) \leq x\} \\ &= x^{-1}[-x, x] \cap x^{-1}(-\infty, x] \end{aligned}$$

$\Rightarrow x$ is a r.v. $x^{-1}[-x, x] \cap x^{-1}(-\infty, x] \in \mathcal{F}_c$

$[x \text{ is a r.v. } \Rightarrow x^{-1}(B) \in \mathcal{F}_c \quad \forall B \in \mathcal{B}]$

$$\begin{aligned} &\Rightarrow x^{-1}[-x, x] \in \mathcal{F}_c \quad \& x^{-1}(-\infty, x] \in \mathcal{F}_c \\ &\Rightarrow x^{-1}[-x, x] \cap x^{-1}(-\infty, x] \in \mathcal{F}_c \end{aligned}$$

$$\Rightarrow y^{-1}(-\infty, x] \in \mathcal{F}_c \quad \forall x \in \mathbb{R}$$

$\Rightarrow y=|x|$ is a r.v.

$$(b) \underline{y=x^2}; \quad \forall x \in \mathbb{R} \quad y^{-1}(-\infty, x] = \{\omega : y(\omega) \leq x\}$$

$$\begin{aligned} &= \{\omega : x^2(\omega) \leq x\} \\ &= \{\omega : -\sqrt{x} \leq x(\omega) \leq \sqrt{x}\} \\ &= x^{-1}[-\sqrt{x}, x] \cap x^{-1}(-\infty, \sqrt{x}] \in \mathcal{F}_c \\ &\Rightarrow y=x^2 \text{ is a r.v.} \end{aligned}$$

Sly part (c).

(2) $\Omega = [0, 1]$ \mathcal{F}_c : Borel σ -field of subsets of Ω

$$x(\omega) = \begin{cases} \omega, & 0 \leq \omega \leq \frac{1}{2} \\ \omega - \frac{1}{2}, & \frac{1}{2} < \omega \leq 1 \end{cases}$$

$$\begin{aligned} x^{-1}(-\infty, x] &= \begin{cases} \emptyset \in \mathcal{F}_c, & x < 0 \\ [0, x] \cup (\frac{1}{2}, \frac{1}{2}+x] \in \mathcal{F}_c, & 0 \leq x < \frac{1}{2} \\ \Omega \in \mathcal{F}_c, & x \geq \frac{1}{2} \end{cases} \end{aligned}$$

$\Rightarrow x^{-1}(-\infty, x] \in \mathcal{F}_c \quad \forall x \in \mathbb{R} \Rightarrow x \text{ is a r.v.}$

$$\textcircled{3} \quad \Omega = \{1, 2, 3, 4\}$$

$$\mathcal{F} = \{\emptyset, \Omega, \{1\}, \{2, 3, 4\}\}$$

$$X(\omega) = \omega + 1 \rightarrow 2, 3, 4, 5$$

$$X^{-1}(-\infty, x] = \begin{cases} \emptyset & x < 2 \\ \{1\} & 2 \leq x < 3 \\ \{1, 2\} \notin \mathcal{F} & 3 \leq x < 4 \end{cases}$$

$\Rightarrow X$ is not a r.v.

Ace

King

1S, ... KS

Spades

1D, ... KD

1C, ... KC

Diamond

Club

(4)

$$\Omega = \left\{ \begin{matrix} \downarrow & \downarrow \\ 1H, \dots, KH, & 1S, \dots, KS, \\ \xleftarrow{\text{all hearts}} & \xleftarrow{\text{Spades}} \end{matrix} \right. \xleftarrow{\text{Diamond}} \xleftarrow{\text{Club}} \left. \begin{matrix} 1D, \dots, KD, & 1C, \dots, KC \\ \xleftarrow{\text{Diamond}} & \xleftarrow{\text{Club}} \end{matrix} \right\}$$

$$X(\omega) = \begin{cases} 4 & \text{ace} \\ 3 & \text{King} \\ 2 & \text{Queen} \\ 1 & \text{Jack} \\ 0 & \text{9W} \end{cases} \quad \mathcal{F} = \text{powerset}$$

$$X^{-1}(-\infty, x] = \begin{cases} \emptyset & \text{if } x < 0 \\ \{2H, \dots, 10H, 2S, \dots, 10S, 2D, \dots, 10D, 2C, \dots, 10C\} & \text{if } 0 \leq x < 1 \\ \{2H, \dots, JH, 2S, \dots, JS, 2D, \dots, JD, 2C, \dots, JC\} & \text{if } 1 \leq x < 2 \\ \{2H, \dots, 8H, 2S, \dots, 8S, 2D, \dots, 8D, 2C, \dots, 8C\} & \text{if } 2 \leq x < 3 \\ \{2H, \dots, KH, 2S, \dots, KS, 2D, \dots, KD, 2C, \dots, KC\} & \text{if } 3 \leq x < 4 \\ \Omega & \text{if } x \geq 4 \end{cases}$$

$$\Rightarrow X^{-1}(-\infty, x] \in \mathcal{F} \quad \forall x \in \mathbb{R}$$

$\Rightarrow X$ is a r.v.

d. f

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{36}{52} & 0 \leq x < 1 \\ \frac{40}{52} & 1 \leq x < 2 \\ \frac{44}{52} & 2 \leq x < 3 \\ \frac{48}{52} & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

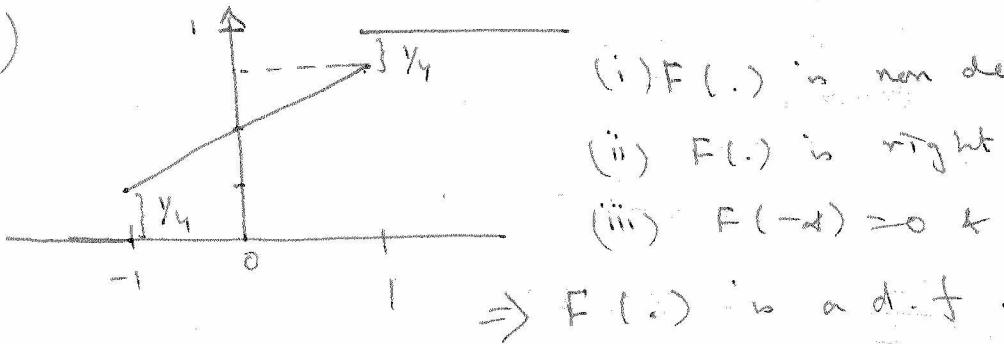
Jumps at $x=0 \rightarrow \frac{9}{13}$

$x=1 \rightarrow \frac{1}{13}$

$x=2 \rightarrow \frac{1}{13}$

$x=3 \text{ or } x=4 \rightarrow \frac{1}{13}$

(5)

(i) $F(\cdot)$ is non-decreasing ✓(ii) $F(\cdot)$ is right contin everywhere(iii) $F(-\infty) = 0$ & $F(\infty) = 1$ $\Rightarrow F(\cdot)$ is a d.f.

$P\left(-\frac{1}{2} < x \leq \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right)$

$= \frac{5}{8} - \frac{3}{8} = \frac{2}{8}$

$P(x=0) = F(0) - F(0-) = 0$

$P(x=1) = F(1) - F(1-) = 1 - \frac{3}{4} = \frac{1}{4}$

$P(-1 \leq x < 1) = F(1) - F(-1-)$

$= \frac{3}{4} - 0 = \frac{3}{4}$

(4)

Discrete part

$$\alpha F_d(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{4} & -1 \leq x < 1 \\ \frac{1}{2} & x \geq 1 \end{cases}$$

Continuous part

$$(1-\alpha) F_c(x) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{4} & -1 \leq x < 1 \\ \frac{1}{2} & x \geq 1 \end{cases}$$

$$\Rightarrow x = \frac{1}{2} + F_d(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

↓
discrete
d.f.

$$F_c(x) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{2} & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

continuous d.f.

(5) (a) $F(x)$ is not right continuous at $x = \frac{1}{2}$ $\Rightarrow F(\cdot)$ is NOT a.d.f.(b) & (c) The n.s.c. holds for these 2 and
hence they are a.d.f.s.

(7) $F(\cdot)$ is non-decreasing

$$F(x) = 1, F(-x) \Rightarrow$$

$F(x)$ is right cont $\forall x \in \mathbb{R}$

$\Rightarrow F(\cdot)$ is a d.f.

$$\begin{aligned} P(x > 6) &= 1 - P(x \leq 6) = 1 - F(6) = 1 - \left(1 - \frac{2}{3} e^{-6/3} - \frac{1}{3} e^{-[6/3]}\right) \\ &= 1 - \left(1 - \frac{2}{3} e^{-2} - \frac{1}{3} e^{-2}\right) = e^{-2}. \end{aligned}$$

$$P(x=5) = F(5) - F(5^-)$$

$$\begin{aligned} &= \left(1 - \frac{2}{3} e^{-5/3} - \frac{1}{3} e^{-[5/3]}\right) - \left(1 - \frac{2}{3} e^{-5/3} - \frac{1}{3} e^{-[\frac{5}{3}]}\right) \\ &= \left(1 - \frac{2}{3} e^{-5/3} - \frac{1}{3} e^{-1}\right) - \left(1 - \frac{2}{3} e^{-5/3} - \frac{1}{3} e^{-1}\right) = 0 \end{aligned}$$

$$P(5 \leq x \leq 8) = F(8) - F(5^-)$$

$$(8) \quad P(-2 \leq x < 5) = F(5^-) - F(-2^-)$$

$$= \frac{1}{2} - 0 = \frac{1}{2}$$

$$P(0 < x < 5.5) = F(5.5^-) - F(0)$$

$$= \left(\frac{1}{2} + \frac{1}{8}\right) - \frac{1}{2} = \frac{1}{8}$$

$$P(1.5 < x \leq 5.5 | x > 2) = \frac{P(2 < x \leq 5.5)}{P(x > 2)}$$

$$= \frac{F(5.5) - F(2)}{1 - F(2)} = \frac{\left(\frac{1}{2} + \frac{1}{8}\right) - \frac{1}{2}}{1 - \frac{1}{2}} = \frac{1}{4}$$

(9) For $x_1 < x_2$

$$F(x_2) - F(x_1) = \sum_{i=1}^n \alpha_i (\underbrace{F_i(x_2) - F_i(x_1)}_{\geq 0 \forall i}) \geq 0$$

$\Rightarrow F(\cdot)$ is non-decreasing.

$$F(-\infty) = \sum \alpha_i F_i(-\infty) = 0 \quad \& \quad F(\infty) = 1$$

$$F(x+) = \lim_{z \downarrow x} F(z) = \lim_{z \downarrow x} \sum \alpha_i F_i(z)$$

$$= \sum \alpha_i F_i(x+) = \sum \alpha_i F_i(x) = F(x)$$

$\Rightarrow F(\cdot)$ is a d.f.

$$(10) \quad G(\infty) = F_1(\infty) + F_2(\infty) = 2 \neq 1$$

$\Rightarrow G(\cdot)$ is NOT a d.f.

$$(11) \quad \text{Right cont at } x=0 \Rightarrow F(0) = F(0+)$$

$$\Rightarrow 0 = \alpha + k \quad \text{--- (i)}$$

$$F(\infty) = 1 \Rightarrow \alpha = 1 \Rightarrow k = -1$$

$$(12) \quad \text{Right cont at } x=3 \Rightarrow f(3) = F(3+)$$

$$\Rightarrow \frac{6+c}{8} = 1 \Rightarrow c = 2$$

$\therefore F(\cdot)$ is having jump at $x=0$ (magnitude $\frac{1}{8}$)
only

discrete part of d.f

$$\propto F_d(x) = \begin{cases} 0 & x < 0 \\ \frac{2}{8} & x \geq 0 \end{cases}$$

cont part

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$$(1-\alpha) F_C(x) = \begin{cases} 0 & \text{if } x < 0 \\ x/8 & 0 \leq x < 1 \\ x/8 & 1 \leq x < 2 \\ \frac{2x}{8} & 2 \leq x < 3 \\ \frac{6}{8} & x > 3 \end{cases}$$

$$\Rightarrow \alpha = \frac{2}{8} = \frac{1}{4} \quad \& \quad (1-\alpha) = \frac{3}{4}$$

\checkmark

$$F_D(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \quad \& \quad F_C(x) = \begin{cases} 0 & x < 0 \\ x/6 & 0 \leq x < 1 \\ x/6 & 1 \leq x < 2 \\ x/3 & 2 \leq x < 3 \\ 1 & x > 3 \end{cases}$$

$$(13) \quad Y = X^+$$

$$P(Y \leq y) = 0 \quad \text{if } y < 0$$

$$\begin{aligned} \text{if } y = 0 \quad P(Y \leq 0) &= P(X^+ \leq 0) = P(X^+ = 0) \\ &= P(X \leq 0) = F(0) \end{aligned}$$

$$\begin{aligned} \text{if } y > 0 \quad P(Y \leq y) &= P(X^+ \leq y) \\ &= P(X \leq y) = F(y) \end{aligned}$$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ F(y) & y \geq 0 \end{cases}$$

$$Z = |X|$$

$$\begin{aligned}
 F_Z(z) &= P(|X| \leq z) \\
 &= P(-z \leq X \leq z) \\
 &= \begin{cases} F(z) - F(-z) & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases}
 \end{aligned}$$

(14)

For $x_1 < x_2$

$$\begin{aligned}
 F(x_2) - F(x_1) &= \int_{-\infty}^{x_2} F_1(x_2-y) dF_2(y) - \int_{-\infty}^{x_1} F_1(x_1-y) dF_2(y) \\
 &= \int_{-\infty}^{x_2} \underbrace{(F_1(x_2-y) - F_1(x_1-y))}_{\geq 0} dF_2(y) \\
 &\geq 0
 \end{aligned}$$

 $\geq 0 \quad \forall x_1 < x_2$

$$F(\infty) = \int_{-\infty}^{\infty} F_1(\infty) dF_2(y) = F_2(\infty) - F_2(-\infty) = 1$$

$$F(-\infty) = 0$$

$$\begin{aligned}
 F(x+) &= \int_{-\infty}^{x+} F_1((x+)-y) dF_2(y) \\
 &= \int_{-\infty}^x F_1(x-y) dF_2(y) = F(x)
 \end{aligned}$$

$\Rightarrow F(\cdot)$ as defined is a d.f.

(15)

(a) $f(1) < 0 \Rightarrow f(\cdot)$ is not a p.m.f.

(b) $f(x) \geq 0 \forall x$

$$\sum_x f(x) = e^{-x} \sum_x \frac{x^x}{x!} = 1$$

$\Rightarrow f(\cdot)$ is a p.m.f.

(c) $\sum_x f(x) \neq 1 \Rightarrow f(\cdot)$ is not a p.m.f.

(16) $f(x) \geq 0 \quad \forall c \in [0, 1]$

$$\sum_x f(x) = 1 \quad \text{also } \forall c \in [0, 1]$$

$\Rightarrow \forall c \quad 0 \leq c \leq 1 \quad f(\cdot)$ is p.m.f.

(17). Suppose

$$P(X=-3) = P(X=-2) = P(X=-1) = P(X=1) = P(X=2) = P(X=3) = p$$

$$\Rightarrow P(X < 0) = 3p = P(X > 0) = P(X=0)$$

$$P(X < 0) + P(X=0) + P(X > 0) = 1$$

$$\Rightarrow p = \frac{1}{9}$$

x	-3	-2	-1	0	1	2	3
$P(X=x)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

$$F_X(x) = \begin{cases} 0 & x < -3 \\ \frac{1}{9} & -3 \leq x < -2 \\ \frac{2}{9} & -2 \leq x < -1 \\ \frac{3}{9} & -1 \leq x < 0 \\ \frac{6}{9} & 0 \leq x < 1 \\ \frac{7}{9} & 1 \leq x < 2 \\ \frac{8}{9} & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

(18) X : r.v. denoting total # of clocks working after 300 days.

X can take values in $\{0, 1, 2, 3\}$

$$\begin{aligned} P(X=0) &= P(A^c \cdot B^c \cdot C^c) = P(A^c) P(B^c) P(C^c) \\ &= (1-0.95) \cdot (1-0.9) \cdot (1-0.8) = p_0, \text{ say.} \end{aligned}$$

$$\begin{aligned} P(X=1) &= P(A \cdot B^c \cdot C^c \cup A^c \cdot B \cdot C^c \cup A^c \cdot B^c \cdot C) \\ &= P(A) P(B^c) P(C^c) + P(A^c) P(B) P(C^c) + P(A^c) P(B^c) P(C) \\ &= \dots \\ &= p_1, \text{ say} \end{aligned}$$

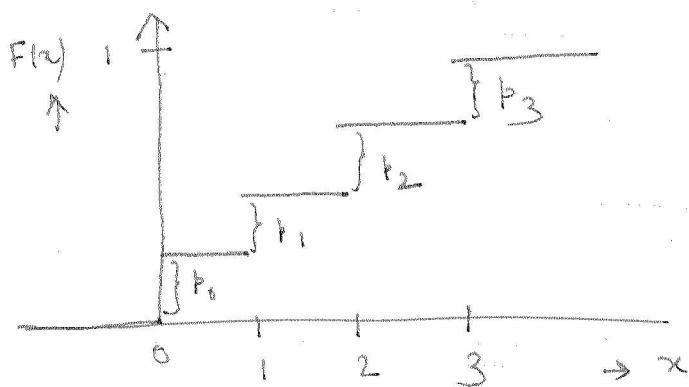
$$\begin{aligned} P(X=2) &= P(A \cdot B \cdot C^c \cup A \cdot B^c \cdot C \cup A^c \cdot B \cdot C) \\ &= P(A) P(B) P(C^c) + P(A) P(B^c) P(C) + P(A^c) P(B) P(C) \\ &= \dots = p_2 \text{ say} \end{aligned}$$

$$\begin{aligned} P(X=3) &= P(ABC) = P(A) P(B) P(C) = .95 \times .9 \times .8 \\ &= p_3 \text{ say} \end{aligned}$$

d. f. p.m.f

$x = x$	0	1	2	3
$P(X=x)$	p_0	p_1	p_2	p_3

d. f.



$$(19) \quad f(x) \geq 0 \quad \forall x$$

$$\int_{-\infty}^{\infty} f(x) dx = \theta \int_0^{\infty} x e^{-\theta x} dx = \theta^2 \cdot \frac{\Gamma(2)}{\theta^2} = 1$$

$\Rightarrow f(\cdot)$ is a p.d.f.

$$P(2 < X < 3) = F(3) - F(2)$$

$$\& P(X > 5) = 1 - F(5)$$

$$\text{where, } F(x) = \int_{-\infty}^x f(x) dx = \theta \int_0^x x e^{-\theta x} dx \quad \begin{cases} x > 0 \\ = 0 \quad \text{if } x \leq 0. \end{cases}$$

integration by parts

(20) If $c \geq 0$ then $f(x) \geq 0 \quad \forall x$

$$\int_0^{\infty} f(x) dx = 1$$

$$\Rightarrow c \int_0^{\infty} (x+1) e^{-\lambda x} dx = 1$$

$$\text{i.e. } c \left[\frac{1}{\lambda^2} + \frac{1}{\lambda} \right] = 1$$

$$\Rightarrow c = \frac{\lambda^2}{1+\lambda}$$

d.f.: $F(x) = \int_0^x f(x) dx = \frac{\lambda^2}{1+\lambda} \left[\int_0^x (1+x) e^{-\lambda x} dx \right] \quad \forall x \geq 0$

$= 0 \quad \uparrow \quad \text{if } x < 0$

by parts.

(21)

$f(x) \geq 0 \quad \forall x$

$$\int_{-3}^3 f(x) dx = \int_{-3}^3 \frac{x^2}{18} dx = \frac{1}{18} \cdot \frac{x^3}{3} \Big|_{-3}^3 = \frac{1}{18} \cdot 18 = 1$$

$\Rightarrow f(\cdot) \text{ is a p.d.f.}$

$F_X(x) = 0 \quad \text{if } x < -3$

$$\begin{cases} 0 & \text{if } x < -3 \\ \frac{x^3 + 27}{54} & -3 \leq x \leq 3 \end{cases}$$

$x > 3$

$$P(|X| < 1) = F(1) - F(-1)$$

$$= \frac{28}{54} - \frac{26}{54} = \frac{2}{54} = \frac{1}{27}$$

$$P(X^2 < 9) = 1$$