MSO201A: PROBABILITY & STATISTICS
Problem Set # 10

[1] Let $X_1, X_2, ... X_n$ be a random sample from an exponential distribution with p.d.f.

$$f_X(x) = \frac{1}{\beta} \exp\left(-\frac{x}{\beta}\right); x > 0$$

Show that $\bar{X} = \sum_{i=1}^{n} X_i \Big/ n$ is an unbiased estimator of β.

[2] Let $X_1, X_2, ... X_n$ be a random sample from $U(0, \theta)$; $\theta > 0$. Show that $\frac{n+1}{n} X_{(n)}$ and $2\bar{X}$ are both unbiased estimators of θ.

[3] Let $X_1, X_2, ... X_n$ be a random sample from an exponential distribution with p.d.f.

$$f(x) = \beta \exp(-\beta x); x > 0$$

Show that $\bar{X}$ is an unbiased estimator of $1/\beta$.

[4] Let $X_1, X_2, ... X_n$ be a random sample from $N(\theta, \theta^2)$, $\theta > 0$. Show that $\left(\sum_{i=1}^{n} X_i\right)^2 \Big/ n(n+1)$ and $\sum_{i=1}^{n} X_i^2 \Big/ 2n$ are both unbiased estimators of θ^2.

[5] Let $X_1, X_2, ... X_n$ be a random sample from $P(\theta)$; $\theta > 0$. Find an unbiased estimator of $\theta e^{-2\theta}$.

[6] Let $X_1, X_2, ... X_n$ be a random sample from $B(1, \theta); 0 \le \theta \le 1$.

(a) Show that the estimator $T(\underset{\sim}{X}) = \dfrac{\frac{1}{2}\sqrt{n} + \sum_{i=1}^{n} X_i}{n + \sqrt{n}}$ is not unbiased θ?

(b) Show that $\lim_{n \to \infty} E\left(T(\underset{\sim}{X})\right) = \theta$.

(An estimator satisfying the condition in (b) is said to be unbiased in the limit)

[7] $X_1, ..., X_n$ be a random sample from $N(\mu, \sigma^2), \mu \in \Re, \sigma \in \Re^+$. Find unbiased estimators of μ/σ^2 and μ/σ.

[8] Let $X_1, X_2, ... X_n$ be a random sample from $B(1, \theta); 0 \le \theta \le 1$. Find an unbiased estimator of $\theta^2(1-\theta)$.

[9] Using Neyman Fisher Factorization Theorem, find a sufficient based on a random sample $X_1, X_2, ... X_n$ from each of the following distributions

(a) $f_\alpha(x)=\begin{cases}\dfrac{1}{\alpha}\exp\left(-\dfrac{x}{\alpha}\right) & \text{if } x>0\\ 0 & \text{otherwise.}\end{cases}$

(b) $f_\beta(x)=\begin{cases}\exp\left(-\left(x-\beta\right)\right) & \text{if } x>\beta\\ 0 & \text{otherwise.}\end{cases}$

(c) $f_{\alpha,\beta}(x)=\begin{cases}\dfrac{1}{\alpha}\exp\left(-\dfrac{(x-\beta)}{\alpha}\right) & \text{if } x>\beta\\ 0 & \text{otherwise.}\end{cases}$

(d) $f_{\mu,\sigma}(x)=\begin{cases}\dfrac{1}{x\,\sigma\,\sqrt{2\pi}}\ \exp\left(-\dfrac{(\log x_i-\mu)^2}{2\sigma^2}\right) & \text{if } x>0\\ 0 & \text{otherwise.}\end{cases}$

(e) $f_\theta(x)=\begin{cases}\dfrac{1}{\theta} & -\theta/2\le x\le\theta/2\\ 0 & \text{otherwise}\end{cases}$

[10] Let X_1 and X_2 be independent random samples with densities $f_1(x_1)=\theta\, e^{-\theta x_1}$ and $f_2(x_2)=2\,\theta\, e^{-2\theta x_2}$ as the respective p.d.f.s where $\theta>0$ is an unknown parameter and $0<x_1,x_2<\infty$. Using Neyman Fisher Factorization Theorem find a sufficient statistic for θ.

[11] Let $X_1,\dots,X_n$ be a random sample with densities

$$f_{X_i}(x)=\begin{cases}\exp(i\theta-x) & \text{if } x\ge i\theta\\ 0 & \text{otherwise.}\end{cases}$$

Using Neyman Fisher Factorization Theorem find a sufficient statistic for θ.

[12] Let $X_1,X_2,\dots X_n$ be a random sample from a $Beta\left(\alpha,\beta\right)$ distribution $\left(\alpha>0,\beta>0\right)$ with p.d.f.

$$f(x)=\begin{cases}\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,x^{\alpha-1}\left(1-x\right)^{\beta-1} & 0<x<1\\ 0 & \text{otherwise.}\end{cases}$$

Show that

(a) $\prod_{i=1}^{n}X_i$ is sufficient for α if β is known to be a given constant.

(b) $\prod_{i=1}^{n}\left(1-X_i\right)$ is sufficient for β if α is known to be a given constant.

(c) $\left(\prod_{i=1}^{n}X_i,\ \prod_{i=1}^{n}\left(1-X_i\right)\right)$ is jointly sufficient for $\left(\alpha,\beta\right)$ if both the parameters are unknown.

[13] Let T and T^* be two statistic such that $T = \psi\left(T^*\right)$. Show that if T is sufficient then T^* is also sufficient.

[14] $X_1,...,X_n$ be a random sample from $U\left(\theta - 1/2, \theta + 1/2\right), \theta \in \Re$. Find a sufficient statistic for θ.

[15] Let $X_1,...,X_n$ be independent random variables with $X_i \left(i = 1,2,...,n\right)$ having the probability density function

$$f_i\left(x_i\right) = \begin{cases} i\theta e^{-i\theta x_i} & x_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find a sufficient statistic for θ.