MSO201A: PROBABILITY & STATISTICS
Problem Set #3

[1] Let X be a random variable defined on $(\Omega, \mathcal{F}, \mathcal{P})$. Show that the following are also random variables; (a) $|X|$, (b) X^2 and (c) $\sqrt{X}$, given that $\{X < 0\} = \phi$.

[2] Let $\Omega = [0,1]$ and $\mathcal{F}$ be the Borel $\sigma-$field of subsets of Ω. Define X on Ω as follows:

$$X(\omega) = \begin{cases} \omega & \text{if } 0 \le \omega \le 1/2 \\ \omega - 1/2 & \text{if } 1/2 < \omega \le 1 \end{cases}$$

Show that X defined above is a random variable.

[3] Let $\Omega = \{1,2,3,4\}$ and $\mathcal{F} = \{\phi, \Omega, \{1\}, \{2,3,4\}\}$ be a $\sigma-$field of subsets of Ω. Verify whether $X(\omega) = \omega + 1;\ \forall \omega \in \Omega$, is a random variable with respect to $\mathcal{F}$.

[4] Let a card be selected from an ordinary pack of playing cards. The outcome ω is one of these 52 cards. Define X on Ω as:

$$X(\omega) = \begin{cases} 4 & \text{if } \omega \text{ is an ace} \\ 3 & \text{if } \omega \text{ is a king} \\ 2 & \text{if } \omega \text{ is a queen} \\ 1 & \text{if } \omega \text{ is a jack} \\ 0 & \text{otherwise.} \end{cases}$$

Show that X is a random variable. Further, suppose that $P(.)$ assigns a probability of $1/52$ to each outcome ω. Derive the distribution function of X.

[5] Let $F(x) = \begin{cases} 0 & \text{if } x < -1 \\ (x+2)/4 & \text{if } -1 \le x < 1 \\ 1 & \text{if } x \ge 1. \end{cases}$

Show that $F(.)$ is a distribution function. Sketch the graph of $F(x)$ and compute the probabilities $P(-1/2 < X \le 1/2)$, $P(X = 0), P(X = 1)$ and $P(-1 \le X < 1)$. Further, obtain the decomposition $F(x) = \alpha F_d(x) + (1-\alpha) F_c(x)$; where, $F_d(x)$ and $F_c(x)$ are purely discrete and purely continuous distribution functions, respectively.

[6] Which of the following functions is(are) distribution functions?

(a) $F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1/2 \\ 1, & x > 1/2. \end{cases}$; (b) $F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x}, & x \ge 0 \end{cases}$; (c) $F(x) = \begin{cases} 0 & x \le 1 \\ 1 - 1/x & x > 1. \end{cases}$

[7] Let $F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 - \dfrac{2}{3} e^{-x/3} - \dfrac{1}{3} e^{-[x/3]} & \text{if } x > 0 \end{cases}$

where, $[x]$ is the largest integer $\le x$. Show that $F(.)$ is a distribution function and compute $P(X > 6)$, $P(X = 5)$ and $P(5 \le X \le 8)$.

[8] The distribution function of a random variable X is given by

$$F(x)=\begin{cases}0, & x<-2,\\ 1/3, & -2\le x<0,\\ 1/2, & 0\le x<5,\\ 1/2+(x-5)^2/2, & 5\le x<6,\\ 1, & x\ge 6.\end{cases}$$

Find $P(-2\le X<5)$, $P(0<X<5.5)$ and $P(1.5<X\le 5.5\,|\,X>2)$.

[9] Prove that if $F_1(.),....,F_n(.)$ are n distribution functions, then $F(x)=\sum_{i=1}^{n}\alpha_i F_i(x)$ is also a distribution function for any $(\alpha_1,...,\alpha_n)$, such that $\alpha_i\ge 0$ and $\sum_{i=1}^{n}\alpha_i=1$.

[10] Suppose F_1 and F_2 are distribution functions. Verify whether $G(x)=F_1(x)+F_2(x)$ is also a distribution function.

[11] Find the value of α and k so that F given by

$$F(x)=\begin{cases}0 & \text{if } x\le 0\\ \alpha+k e^{-x^2/2} & \text{if } x>0\end{cases}$$

is distribution function of a continuous random variable.

[12] Let $F(x)=\begin{cases}0 & \text{if } x<0\\ (x+2)/8 & \text{if } 0\le x<1\\ (x^2+2)/8 & \text{if } 1\le x<2\\ (2x+c)/8 & \text{if } 2\le x\le 3\\ 1 & \text{if } x>3.\end{cases}$

Find the value of c such that F is a distribution function. Using the obtained value of c, find the decomposition $F(x)=\alpha F_d(x)+(1-\alpha)F_c(x)$; where, $F_d(x)$ and $F_c(x)$ are purely discrete and purely continuous distribution functions, respectively.

[13] Suppose F_X is the distribution function of a random variable X. Determine the distribution function of (a) X^+ and (b) $|X|$. Where,

$$X^+=\begin{cases}X & \text{if } X\ge 0\\ 0 & \text{if } X<0\end{cases}$$

[14] The convolution F of two distribution functions F_1 and F_2 is defined as follows;

$$F(x)=\int_{-\infty}^{\infty}F_1(x-y)\,dF_2(y)\,;\ x\in\mathcal{R},$$

and is denoted by $F=F_1\star F_2$. Show that F is also a distribution function.

[15] Which of the following functions are probability mass functions?

(a) $f(x)=\begin{cases}(x-2)/2 & \text{if } x=1,2,3,4\\ 0 & \text{otherwise.}\end{cases}$; (b) $f(x)=\begin{cases}\left(e^{-\lambda}\lambda^x\right)/x! & \text{if } x=0,1,2,3,4,...\\ 0 & \text{otherwise.}\end{cases}$

where, $\lambda > 0$.

(c) $f(x) = \begin{cases} \left(e^{-\lambda}\lambda^{x}\right)/x! & \text{if } x = 1,2,3,4,... \\ 0 & \text{otherwise.} \end{cases}$

where, $\lambda > 0$.

[16] Find the value of the constant c such that $f(x) = (1-c)c^{x};\ x = 0,1,2,3...$ defines a probability mass function.

[17] Let X be a discrete random variable taking values in $\mathcal{X} = \{-3,-2,-1,0,1,2,3\}$ such that $P(X=-3) = P(X=-2) = P(X=-1) = P(X=1) = P(X=2) = P(X=3)$ and $P(X<0) = P(X=0) = P(X>0)$ Find the distribution function of X.

[18] A battery cell is labeled as good if it works for at least 300 days in a clock, otherwise it is labeled as bad. Three manufacturers, A, B and C make cells with probability of making good cells as 0.95, 0.90 and 0.80 respectively. Three identical clocks are selected and cells made by A, B and C are used in clock numbers 1, 2 and 3 respectively. Let X be the total number of clocks working after 300 days. Find the probability mass function of X and plot the corresponding distribution function.

[19] Prove that the function $f_{\theta}(x) = \begin{cases} \theta^{2}\, x e^{-\theta x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$

defines a probability density function for $\theta > 0$. Find the corresponding distribution function and hence compute $P(2 < X < 3)$ and $P(X > 5)$.

[20] Find the value of the constant c such that the following function is a probability density function.

$$f_{\lambda}(x) = \begin{cases} c(x+1)e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

where, $\lambda > 0$. Obtain the distribution function of the random variable associated with probability density function $f_{\lambda}(x)$.

[21] Show that $f(x) = \begin{cases} x^{2}/18 & \text{if } -3 < x < 3 \\ 0 & \text{otherwise} \end{cases}$

defines a probability density function. Find the corresponding distribution function and hence find $P(|X| < 1)$ and $P(X^{2} < 9)$