(1)

(a) $$L(\theta) = \frac{e^{-n\theta}\,\theta^{\Sigma x_i}}{\Pi x_i!}$$

log likelihood: $$l(\theta) = -n\theta + \Sigma x_i \log\theta - \log(\Pi x_i!)$$

$$\frac{\partial l(\theta)}{\partial\theta} = -n + \frac{\Sigma x_i}{\theta} = 0$$

$$\Rightarrow \hat{\theta} = \bar{x}$$

$$\left.\frac{\partial^2 l(\theta)}{\partial\theta^2}\right|_{\hat{\theta}} = -\left.\frac{\Sigma x_i}{\theta^2}\right|_{\hat{\theta}} < 0$$

$$\Rightarrow \hat{\theta}_{MLE} = \bar{X}$$

(b) $$L(\theta) = \theta^n\left(\frac{1}{\Pi x_i}\right)(\Pi x_i)^\theta$$ ← ind of range (indep of θ)

$$l(\theta) = n\log\theta + \theta\,\Sigma\log x_i + K$$ ← indep of θ

$$\frac{\partial l(\theta)}{\partial\theta} = \frac{n}{\theta} + \Sigma\log x_i = 0$$

$$\Rightarrow \hat{\theta} = -\frac{n}{\Sigma\log x_i}$$

$$\left.\frac{\partial^2 l(\theta)}{\partial\theta^2}\right|_{\hat{\theta}} = -\left.\frac{n}{\theta^2}\right|_{\hat{\theta}} < 0$$

$$\Rightarrow \hat{\theta}_{MLE} = -\frac{n}{\Sigma\log X_i}$$

(c) $$L(\theta) = \frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum x_i} \quad c \leftarrow \text{inf on range (indep of } \theta)$$

$$l(\theta) = -n\log\theta - \frac{1}{\theta}\sum x_i + k \leftarrow \text{indep of } \theta$$

$$\frac{\partial l(\theta)}{\partial\theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum x_i = 0$$

$$\hat{\theta} = \bar{x}$$

$$\left.\frac{\partial^2 l(\theta)}{\partial\theta^2}\right|_{\hat{\theta}} = \left.\frac{n}{\theta^2} - 2\frac{1}{\theta^3}\sum x_i\right|_{\hat{\theta}}$$

$$= \frac{n}{\hat{\theta}^2} - \frac{2}{\hat{\theta}^3} n\hat{\theta} = \frac{n}{\hat{\theta}^2} - \frac{2n}{\hat{\theta}^2} = -\frac{n}{\hat{\theta}^2} < 0$$

$$\Rightarrow \hat{\theta}_{MLE} = \bar{X}$$

(d) $$L(\theta) = \frac{1}{2^n} e^{-\sum |x_i - \theta|}$$

$$l(\theta) = K - \sum_1^n |x_i - \theta|$$

maximization of $l(\theta)$ (or $L(\theta)$) w.r.t. θ is equiv to

minimization of $\sum_1^n |x_i - \theta|$

$$\Rightarrow \hat{\theta} = \text{median}(x_1, \ldots, x_n)$$

$$\Rightarrow \hat{\theta}_{MLE} = \text{median}(X_1, \ldots, X_n)$$

(2)

$$L(\underset{\sim}{\theta}) = \frac{1}{\theta_2^n} e^{-\frac{1}{\theta_2}\Sigma(x_i-\theta_1)} I(\theta_1, x_{(1)})$$

$\underset{\sim}{\theta} = (\theta_1, \theta_2)'$

$$I(a,b) = \begin{cases} 1, & a \le b \\ 0, & o/w \end{cases}$$

$$L(\underset{\sim}{\theta}) = \frac{1}{\theta_2^n} e^{-\frac{1}{\theta_2}\Sigma x_i} e^{\frac{n\theta_1}{\theta_2}} I(\theta_1, x_{(1)})$$

This is a 2-parameter MLE setup.

Realize that for a fixed θ_2, $L(\theta_1, \theta_2)$ is maximized at $\hat{\theta}_1 = x_{(1)}$ (as for fixed θ_2, $L(\theta_1,\theta_2)$ is $\uparrow \theta_1$)

Note that $\hat{\theta}_1 = x_{(1)}$ is indep of fixed level of θ_2

and $\hat{\theta}_1 = x_{(1)}$ would maximise $L(\underset{\sim}{\theta})$ $\forall\ \theta_2$

$$\Rightarrow \hat{\theta}_{1(MLE)} = X_{(1)}$$

Consider now,

$$L(\hat{\theta}_1, \theta_2) = \frac{1}{\theta_2^n} e^{-\frac{1}{\theta_2}\Sigma(x_i - x_{(1)})}$$

$$l(\hat{\theta}_1, \theta_2) = -n\log\theta_2 - \frac{1}{\theta_2}\Sigma(x_i - x_{(1)})$$

$$\frac{\partial l(\hat{\theta}_1,\theta_2)}{\partial\theta_2} = -\frac{n}{\theta_2} + \frac{1}{\theta_2^2}\Sigma(x_i - x_{(1)}) = 0$$

$$\Rightarrow \hat{\theta}_2 = \frac{1}{n}\Sigma(x_i - x_{(1)})$$

$$\left.\frac{\partial^2 l}{\partial\theta_2^2}\right|_{\hat{\theta}} < 0 \Rightarrow \hat{\theta}_{2(MLE)} = \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)})$$

(3)

$$L(\underline{\theta}) = \frac{\lambda^{n\alpha}}{(\Gamma\alpha)^n} e^{-\lambda \Sigma x_i} (\Pi x_i)^{\alpha-1}$$

$$\underset{\sim}{\theta} = (\alpha, \lambda)$$

$$l(\underline{\theta}) = n\alpha \log \lambda - n \log \Gamma\alpha + (\alpha-1) \Sigma \log x_i - \lambda \Sigma x_i$$

likelihood eq^{n}s :

$$\frac{\partial \log L}{\partial \lambda} = \frac{n\alpha}{\lambda} - \Sigma x_i$$

$$\frac{\partial \log L}{\partial \alpha} = n \log \lambda - n \frac{(\Gamma\alpha)'}{(\Gamma\alpha)} + \Sigma \log x_i$$

$$\frac{\partial \log L}{\partial \lambda} = 0 \quad \Rightarrow \quad \lambda = \frac{n\alpha}{\Sigma x_i} = \frac{\alpha}{\bar{x}}$$

$\dfrac{\partial \log L}{\partial \alpha} = 0$ gives

$$n \log\left(\frac{\alpha}{\bar{x}}\right) - n \frac{(\Gamma\alpha)'}{(\Gamma\alpha)} + \Sigma \log x_i = 0 \quad -(*)$$

Solve (*) by numerical method to get $\hat{\alpha}_{MLE}$

$$\& \quad \hat{\lambda}_{MLE} = \frac{\hat{\alpha}_{MLE}}{\overline{X}}$$

(4) $\underset{\sim}{\theta} = (\mu, \sigma)'$

$$L(\underset{\sim}{\theta}) = \left(\frac{1}{2\sqrt{3}\sigma}\right)^n I_{(\mu-\sqrt{3}\sigma,\, x_{(1)})} \, I_{(x_{(n)},\, \mu+\sqrt{3}\sigma)}$$

Note that $L(\underset{\sim}{\theta}) = \left(\frac{1}{2\sqrt{3}\sigma}\right)^n$ if $\mu-\sqrt{3}\sigma \le x_{(1)}$ and $x_{(n)} \le \mu+\sqrt{3}\sigma$

$= 0$ o/w

Now $\mu-\sqrt{3}\sigma \le x_{(1)}$ & $x_{(n)} \le \mu+\sqrt{3}\sigma$

$\Rightarrow$ $\mu \le x_{(1)}+\sqrt{3}\sigma$ & $x_{(n)}-\sqrt{3}\sigma \le \mu$

i.e. $x_{(n)}-\sqrt{3}\sigma \le \mu \le x_{(1)}+\sqrt{3}\sigma$

Thus, for a given σ, $L(\underset{\sim}{\theta})$ is maximized w.r.t μ if

$$\mu \in [x_{(n)}-\sqrt{3}\sigma,\; x_{(1)}+\sqrt{3}\sigma]$$

$\Rightarrow$ Any value of μ in the above interval is an MLE of μ

In particular $\dfrac{(X_{(n)}-\sqrt{3}\sigma)+(X_{(1)}+\sqrt{3}\sigma)}{2} = \dfrac{X_{(n)}+X_{(1)}}{2}$

Choice is not unique clearly

is MLE of μ

Since the MLE of μ is indep of σ, it's MLE of μ $\forall\sigma$

$$\Rightarrow \hat{\mu}_{MLE} = \frac{X_{(n)}+X_{(1)}}{2}$$

Further, $L(\hat{\mu}, \sigma)$ is maximized w.r.t. σ if σ is minimum.

Observe that $\sqrt{3}\sigma \ge \mu - x_{(1)}$ & $\sqrt{3}\sigma \ge x_{(n)}-\mu$

and at the above MLE of μ

$$\sqrt{3}\sigma \ge \frac{x_{(n)}-x_{(1)}}{2}$$

$$\Rightarrow \hat{\sigma}_{MLE} = \frac{X_{(n)} - X_{(1)}}{2\sqrt{3}}$$

(5) $L(\theta) = 1$ if $\theta - \frac{1}{2} \le x_{(1)}$ and $x_{(n)} \le \theta + \frac{1}{2}$

$= 0$ o/w

$\Rightarrow L(\theta)$ is maximized w.r.t. θ if

$$\theta - \frac{1}{2} \le x_{(1)} \text{ \& } x_{(n)} \le \theta + \frac{1}{2}$$

i.e. if $x_{(n)} - \frac{1}{2} \le \theta \le x_{(1)} + \frac{1}{2}$

$\Rightarrow L(\theta)$ is maximized w.r.t. θ $\forall$ values of θ satisfying

$$x_{(n)} - \frac{1}{2} \le \theta \le x_{(1)} + \frac{1}{2}$$

$\Rightarrow$ Any statistic $U(X_1, \ldots, X_n) \ni$

$x_{(n)} - \frac{1}{2} \le u(x_1, \ldots, x_n) \le x_{(1)} + \frac{1}{2}$ is an MLE of θ

In particular $\frac{X_{(1)} + X_{(n)}}{2}$ (mid pt) is an MLE of θ

In general,

$\alpha\left(X_{(1)} + \frac{1}{2}\right) + (1-\alpha)\left(X_{(n)} - \frac{1}{2}\right)$ $\forall$ $0 \le \alpha \le 1$

is an MLE of θ

With $\alpha = \frac{3}{4}$, we get

$\frac{3}{4}\left(X_{(1)} + \frac{1}{2}\right) + \frac{1}{4}\left(X_{(n)} - \frac{1}{2}\right)$ as MLE of θ

(6)

X : r.v. denoting lifetime of the component

p.d.f. $f_X(x) = \begin{cases} \frac{1}{\lambda} e^{-x/\lambda}, & x > 0 \\ 0, & o/w \end{cases}$

Define the r.v.

$$Y_i = \begin{cases} 1, & \text{if } i^{th} \text{ component has lifetime} < 100 \text{ hrs} \\ 0, & o/w. \end{cases}$$

$$P(Y_i = 1) = P(X < 100) = \frac{1}{\lambda} \int_0^{100} e^{-x/\lambda} \, dx = (1 - e^{-100/\lambda})$$

$X_1, \ldots, X_n$ are i.i.d.

$\Rightarrow Y_1, \ldots, Y_n$ are i.i.d.

$$Y_i \sim B(1, \underbrace{(1 - e^{-100/\lambda})}_{= \theta, \text{ say}})$$

$$\hat{\theta}_{MLE} = \bar{Y}$$

Note that $\theta = 1 - e^{-100/\lambda}$

$$\Rightarrow e^{-100/\lambda} = 1 - \theta$$

$$\Rightarrow \lambda = -\frac{100}{\log(1-\theta)} = g(\theta)$$

Inv of MLE: MLE of $g(\theta)$ is $g(\hat{\theta}_{MLE})$

$$\Rightarrow \hat{\lambda}_{MLE} = -\frac{100}{\log(1 - \hat{\theta}_{MLE})}.$$

From data $\bar{x} = \frac{3}{10}$ $\Rightarrow$ ML estimate of λ is $\left(-\frac{100}{\log(7/10)}\right)$

(7) Let X denote the r.v. denoting number of sales in a day

$$X \sim P(\mu) \qquad \mu > 0$$

Define

$$Y_i = \begin{cases} 1, & \text{if 0 sales on day } i \\ 0, & \text{o/w} \end{cases}$$

$$P(Y_i = 1) = P(X = 0) = e^{-\mu}$$

$X_1, \ldots, X_{30}$ i.i.d. $P(\mu)$

$\Rightarrow Y_1, \ldots, Y_{30}$ are i.i.d. $B(1, \underset{\substack{\parallel \\ \theta \text{ say}}}{e^{-\mu}})$

$$\hat{\theta}_{MLE} = \bar{Y}$$

Note that $\theta = e^{-\mu} \Rightarrow \mu = -\log \theta = g(\theta)$

$$\Rightarrow \quad \hat{\mu}_{MLE} = -\log \hat{\theta}_{MLE}.$$

$\Rightarrow$ ML estimate of μ from the data : $\left(-\log(20/30)\right)$

— x —