Problem Set #1

(1)

$$\Omega = \{HH, TT, HTT, THH, HTHH, THTT, \dots\}.$$

$$P(HH) = \frac{1}{4} = P(TT)$$

$$P(HTT) = P(THH) = \frac{1}{2^3}.$$

(a) $$P(A) = \sum_{i=2}^{5} P(\text{exp ends in } i \text{ tosses})$$

$$= P(\text{exp ends in 2 tosses}) + P(\text{ends in 3}) + P(\text{ends in 4}) + P(\dots 5)$$

$$= 2 \times \frac{1}{2^2} + 2 \times \frac{1}{2^3} + \dots$$

(b) $$P(B) = 2 \sum_{i=1}^{\infty} \frac{1}{2^{2i}} = \dots$$

(c) $$P(A \cap B) = P(\text{exp ends in 2 tosses}) + P(\text{exp ends in 4 tosses})$$

$$= \dots$$

$$P(A^c \cap B) = 2 \sum_{i=3}^{\infty} \frac{1}{2^{2i}} = \dots$$

(2) Total # of cases : $\binom{100}{3}$

(i) Nos in AP

Common diff $\quad 1, 2, \ldots, 49$

# of cases $\quad 98, 96, \ldots 2$

$\Rightarrow$ total # of favorable cases $\quad 98+96+\ldots+2$

$$= 2\left(\frac{49\times 50}{2}\right) = 49\times 50$$

$$\text{reqd prob} = \frac{49\times 50}{\binom{100}{3}} = \frac{1}{66}$$

(ii) Nos in GP.

Common ratio can be integer or fraction

Case 1 : c.r. integer

c.r.	# of fav cases	total #
2	$\to (1,2,4), \ldots (25,50,100) \longrightarrow$	25
3	$\to (1,3,9), \ldots (11,33,99) \longrightarrow$	11
4	$\to (1,4,16), \ldots (6,24,96) \longrightarrow$	6
5	$\to (1,5,25), \ldots (4,20,100) \longrightarrow$	4
6	$\to (1,6,36), (2,12,72) \longrightarrow$	2
7	$\to (1,7,49), (2,14,98) \longrightarrow$	2
8	$\to (1,8,64) \longrightarrow$	1
9	$\to (1,9,81) \longrightarrow$	1
10	$\to (1,10,100) \longrightarrow$	1
		Total 53

Case 2 : c.r. fractional

east #	c.r.	fav cases	total #
4	$3/2$	$\to (4,6,9), (8,12,18), \ldots (44,66,99) \to$	11
4	$5/2$	$\to (4,10,25), (8,20,50), (12,30,75) (16,40,100) \to$	4
4	$7/2$	$\to (4,14,49), (8,28,98) \longrightarrow$	2
4	$9/2$	$\longrightarrow (4,18,81) \longrightarrow$	1

ly

$9 \to \left(\frac{4}{3}, \frac{5}{3}, \frac{7}{3}, \frac{8}{3}, \frac{10}{3}\right) \longrightarrow 6+4+2+1+1$

$16 \to \left(\frac{5}{4}, \frac{7}{4}, \frac{9}{4}\right) \longrightarrow 4+2+1$

$25 \to \left(\frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}\right) \longrightarrow 2+2+1+1$

$36 \to \frac{7}{6} \longrightarrow 2$

$49 \to \left(\frac{8}{7}, \frac{9}{7}, \frac{10}{7}\right) \longrightarrow 1+1+1$

$64 \to \frac{9}{8} \longrightarrow 1$

$81 \to \frac{10}{9} \longrightarrow 1$

Total 52

$\Rightarrow$ req'd prob $\dfrac{53+52}{\binom{100}{3}}$.

(3) C can win in exactly 4 or 5 or 6 or 7 additional games

Case 1: 4 additional games

C wins all → prob $\left(\frac{1}{3}\right)^4$. — (i)

Case 2: 5 additional games.

C wins 3 out of 1st 4 & the 5th game

prob → $\binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right) \times \frac{1}{3}$ — (ii)

Case 3: 6 additional games

C wins 3 out of 1st 5 & 6th game and

either (i) B wins 2 and A wins none

or (ii) B wins 1, A wins 1.

prob → $\binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{1}{3}\right)^2 \times \frac{1}{3} + \binom{5}{3}\binom{2}{1}\left(\frac{1}{3}\right)^3 \frac{1}{3} \cdot \frac{1}{3} \times \frac{1}{3}$ — (iii)

Case 4: 7 additional games

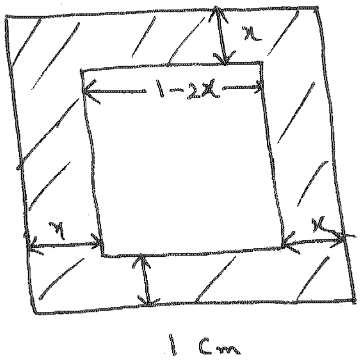
out of 1st 6 games $\begin{cases} A \text{ wins } 1 \\ B \ \ldots \ 2 \end{cases}$

C .. 3

prob $\binom{6}{3}\binom{3}{1}\left(\frac{1}{3}\right)^3 \frac{1}{3}\left(\frac{1}{3}\right)^2\left(\frac{1}{3}\right)$ — (iv)

reqd prob = (i) + (ii) + (iii) + (iv) ← m. e. ways.

(4)



The pt P must lie in the shaded region so that the distance from P to the nearest side does not exceed x cm

If $x \geq \frac{1}{2}$, then prob $= 1$

If $0 < x < \frac{1}{2}$, then area of the shaded region
$$= 1 - (1-2x)^2.$$

$\Rightarrow$ reqd prob $= 1 - (1-2x)^2$.

(5) reqd prob $= \dfrac{365 \times 364 \times \cdots \times (365-(n-1))}{365^n}$.

$$= \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{n-1}{365}\right) = p \text{ say}.$$

$$\log_e p = \sum_{K=1}^{n-1} \log_e\left(1 - \frac{K}{365}\right) \approx \sum_{K=1}^{n-1}\left(-\frac{K}{365}\right) = -\frac{1}{365} \cdot \frac{n(n-1)}{2}$$

for $n = 10$ $\quad \log_e p \approx -\frac{1}{365} \cdot \frac{10 \times 9}{2} = \ldots$

$\Rightarrow p \approx \ldots$

(6) Total # of possible outcomes : 3^3

favorable # of outcomes : $3! = 6$

reqd prob $= \frac{6}{27}$.

(7) Total # of ways in which n men can stand in a row $\rightarrow n!$

# of possible positions for A & B $\ni$ there are exactly r positions available betn them

$= \underbrace{2!}_{\text{permutation among A \& B}} \times \underbrace{(n-r-1)}_{\text{possible positions}}$

possible positions ($\{1, r+2\}, \{2, r+3\}, \ldots, \{(n-r-1), n\}$ for A & B).

Further # of ways that r persons can be chosen to stand betn A & B $= \binom{n-2}{r}$

Favorable # of cases

$\left(2! \times (n-r-1)\right) \times \binom{n-2}{r} \times r! \times (n-r-2)!$

$r!$: perm of r men betn A & B

$(n-r-2)!$: perm of $(n-r-2)$ men excluding A, B and r men in betn.

$\Rightarrow$ reqd prob

$$\frac{2! \times (n-r-1)! \times r! \times \binom{n-2}{r}}{n!}$$

(8)

(a) Total # of ways n^r

Originator $\rightarrow$ n ways

2^{nd} person $\rightarrow$ $(n-1)$ ways

$\vdots$

r^{th} person $\rightarrow$ $(n-1)$ ways.

$\rightarrow n(n-1)^{r-1}$

$$\text{reqd prob} = \frac{n(n-1)^{r-1}}{n^r}.$$

(b) Orginator $\rightarrow$ n options

2^{nd} person $\rightarrow$ $n-1$

3^{rd} person $\rightarrow$ $n-2$

$\vdots$

r^{th} person $\rightarrow$ $(n-r+1)$

$$\text{reqd prob} = \frac{n(n-1)\cdots(n-r+1)}{n^r}.$$

Second part: Total # of cases $\binom{n}{N}^r$.

Case favorable to 1^{st} event $\binom{n}{N}\binom{n-1}{N}^{r-1}$

$$\text{reqd prob} = \binom{n}{N}\binom{n-1}{N}^{r-1} \Big/ \binom{n}{N}^r.$$

Sly for 2^{nd} event

$$\text{reqd prob} = \frac{\binom{n}{N}\binom{n-N}{N}\binom{n-2N}{N}\cdots\binom{n-(r-1)N}{N}}{\binom{n}{N}^r}$$

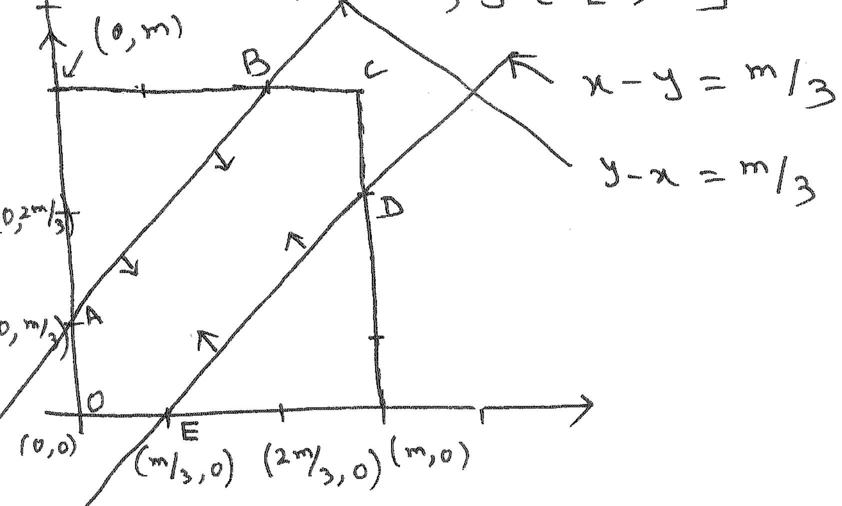
with obvious assumptions.

(9) Let the distance of 2 randomly chosen pts from a fixed pt A on the line segment be denoted as x & y

Reqd condition is $|x-y| < \frac{m}{3}$.

i.e. $-m/3 < x - y < m/3$

Note that $x, y \in [0, m]$



Inside the rect bounded by x axis, y axis, $x = m$ and $y = m$, the area favorable to $|x-y| < m/3$ is clearly the region OABCDE

Area of OABCDE $= \cancel{1}\, m^2 - \left(\frac{2}{3}\right)^2 m^2$.

$\left(= m^2 - 2\left(\frac{1}{2}\,\frac{2m}{3} \times \frac{2m}{3}\right)\right)$

$\Rightarrow$ reqd prob $= \left(m^2 - \frac{4}{9}m^2\right) / m^2 = 5/9$.

(10) n pts must lie on or outside a sphere of raidus r, having same centre as the orginal sphere of raidus R.

For any of the n pts,

$$P\left[\text{lie inside the smaller sphere}\right) = \frac{\text{Vol}^m \text{ of sph with rad } r}{\text{Vol}^m \text{ of sph with rad } R}$$

$$= \frac{r^3}{R^3}.$$

$$\Rightarrow P\left(\begin{array}{c}\text{lie on or outside the}\\ \text{smaller sphere}\end{array}\right) = \left(1 - \frac{r^3}{R^3}\right)$$

As the pts are taken independently, the reqd

$$\text{prob} = \left(1 - \frac{r^3}{R^3}\right)^n.$$

(ii) Owner's car can be in any of the $(N-2)$ places
(leaving 2 ends)

Remaining $(r-1)$ cars in $(N-1)$ remaining place

$\Rightarrow$ total # of cases $= (N-2)\binom{N-1}{r-1}$

Favorable # of cases:

Owner's car in any of the $(N-2)$ places and
2 neighboring places are empty

$\Rightarrow$ remaining $(r-1)$ cars can be in $(N-3)$ remaining.
places

$\Rightarrow$ favorable # of cases: $(N-2)\binom{N-3}{r-1}$.

$$\text{reqd prob} = \frac{\binom{N-3}{r-1}}{\binom{N-1}{r-1}}.$$

(12) Let x, y, z be the distances of X, Y, Z from a fixed pt P on the line segment

The six possibilities are

$x < y < z$; $x < z < y$; $z < x < y$;

$y < x < z$; $y < z < x$; $z < y < x$;

The above 6 possibilities are equally likely due to random draws

Y lie betn X & Z in 2 cases ($x < y < z$ & $z < y < x$).

reqd prob $= \frac{2}{6}$

(13) Coeffs a, b or c can take any of the values $1, 2, \dots 6$

Total # of (a, b, c) combinations $6 \times 6 \times 6 = 216$

Real roots → requirement $b^2 \geq 4ac$

Listing of favorable # of cases

ac	(a, c)		$4ac$	$b \ni b^2 \geq 4ac$	# of cases
1	$(1, 1)$		4	2, 3, 4, 5, 6	5
2	$(1, 2)$ $(2, 1)$	→	8	3, 4, 5, 6	$2 \times 4 = 8$
3	$(1, 3)$ $(3, 1)$	→	12	4, 5, 6	$2 \times 3 = 6$
4	$(1, 4)$ $(4, 1)$ $(2, 2)$	→	16	4, 5, 6	$3 \times 3 = 9$
5	$(1, 5)$ $(5, 1)$	→	20	5, 6	$2 \times 2 = 4$
6	$(1, 6)$ $(6, 1)$ $(2, 3)$ $(3, 2)$	→	24	5; 6	$4 \times 2 =$ [illegible]
7	— not possible to obtain $ac = 7$				
8	$(2, 4)$ $(4, 2)$	→	32	6	$2 \times 1 = 2$

ac values higher than 9 will not have any $b \ni$
$b^2 \geq 4ac$

$\Rightarrow$ # of favorable case for $b^2 \geq 4ac$ $= \begin{pmatrix} 5+8+6+9 \\ +4+8+2+1. \end{pmatrix}$

$= 43.$

req^d prob $= \dfrac{43}{216}$.

(14).

(i) $\phi^c = \Omega \in \mathcal{F}_1 \Rightarrow \mathcal{F}_1$ is not a σ-field

(ii) $\{1\} \cup \{3,4\} = \{1,3,4\} \notin \mathcal{F}_2$

$\mathcal{F}_2$ is not closed under union $\Rightarrow \mathcal{F}_2$ is not a σ-field.

or ~~...~~ $\{1,2\} \cap \{2,3,4\} = \{2\} \in \mathcal{F}_2$

$\Rightarrow \mathcal{F}_2$ is not a σ-field.

(iii) $\mathcal{F}_3$ contains Ω and is closed under complementation and ~~...~~ union $\Rightarrow \mathcal{F}_3$ is a σ-field.

(15) $\Omega \in \mathcal{F}_1, \mathcal{F}_2 \Rightarrow \Omega \in \mathcal{F}_1 \cap \mathcal{F}_2$ — (i)

let $A \in \mathcal{F}_1 \cap \mathcal{F}_2$, then $A \in \mathcal{F}_1$ & $A \in \mathcal{F}_2$

$\Rightarrow A^c \in \mathcal{F}_1$ & $A^c \in \mathcal{F}_2$

$\Rightarrow A^c \in \mathcal{F}_1 \cap \mathcal{F}_2$ — (ii).

If $A_1, A_2, \ldots \in \mathcal{F}_1 \cap \mathcal{F}_2$, then

$A_1, A_2, \ldots \in \mathcal{F}_1$ & $\Rightarrow \cup A_i \in \mathcal{F}_1$

$A_1, A_2, \ldots \in \mathcal{F}_2$ $\Rightarrow \cup A_i \in \mathcal{F}_2$

$\Rightarrow \bigcup_i A_i \in \mathcal{F}_1 \cap \mathcal{F}_2$ — (iii)

(i), (ii) & (iii) $\Rightarrow \mathcal{F}_1 \cap \mathcal{F}_2$ is a σ-field.

Counter example

$\Omega = \{1, 2, 3\}$

Take $\mathcal{F}_1 = \{\phi, \Omega, \{1\}, \{2,3\}\} \to \sigma$-field

$\mathcal{F}_2 = \{\phi, \Omega, \{2\}, \{1,3\}\} \to \sigma$-field.

$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\phi, \Omega, \{1\}, \{2\}, \{1,3\}, \{2,3\}\}$.

$\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ-field $(\{1\} \cup \{2\} \notin \mathcal{F}_1 \cup \mathcal{F}_2)$.

(16) (i) $A \in \mathcal{F}$ & $A \cap A = A \Rightarrow A \in \mathcal{F}_A$.

(ii) Let $C \in \mathcal{F}_A$ then $C = A \cap B$ for $B \in \mathcal{F}$

C^{c_A} (Complement w.r.t. A) $= A - C$

$= A - A \cap B$

$= A \cap B^c \in \mathcal{F}_A$ (as $B^c \in \mathcal{F}$)

(iii) Let $C_1, C_2, \ldots \in \mathcal{F}_A$, then

$C_i = A \cap B_i$; $i = 1, 2, \ldots$ for $B_i \in \mathcal{F}$

$\cup C_i = \bigcup_i (A \cap B_i) = A \cap (\bigcup_i B_i) \in \mathcal{F}_A$

(as $\cup B_i \in \mathcal{F}$)

$\Rightarrow \mathcal{F}_A$ is a σ-field of subsets of A.