Problem set # 5

(1)

Belt 1

$X \sim$ Exp with mean $\alpha \sim \frac{1}{\alpha} e^{-x/\alpha}$; $x > 0$

Belt 2 $Y \sim$ Exp with mean $2\alpha \sim \frac{1}{2\alpha} e^{-x/2\alpha}$; $x > 0$

P(system works beyond α)

$$P(X > \alpha \cap Y > \alpha) = P(X > \alpha)\, P(Y > \alpha)$$

$$= \left(\int_\alpha^\infty \frac{1}{\alpha} e^{-x/\alpha} dx \right) * \left(\int_\alpha^\infty \frac{1}{2\alpha} e^{-y/2\alpha} dx \right)$$

$$= e^{-1} \times e^{-1/2} = e^{-3/2}$$

(2) (a) $P(X > 5) = P\left(\frac{X-10}{6} > \frac{5-10}{6} \right) = P\left(Z > -\frac{5}{6}\right)$; $Z \sim N(0,1)$

$$= 1 - \Phi(-5/6)$$

$$= 1 - (1 - \Phi(5/6))$$

$$= \Phi(5/6) = .7967$$

(b) $P(4 < X < 16) = P\left(\frac{4-10}{6} < Z < \frac{16-10}{6} \right) = P(-1 < Z < 1)$

$$= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1$$

$$= \ldots$$

(c) $P(X < 8) = P\left(Z < \frac{8-10}{6} \right) = \Phi(-\frac{1}{3}) = 1 - \Phi(1/3) = \ldots$

(3) $P(X \le 0) = \frac{1}{2} = P(X \ge 0) \Rightarrow \mu = 0$

$$P(-1.96 \le X \le 1.96) = 0.95$$

$$P\left(-\frac{1.96}{\sigma} \le \frac{X}{\sigma} \le \frac{1.96}{\sigma} \right) = 0.95$$

$$P\left(-\frac{1.96}{\sigma} \le Z \le \frac{1.96}{\sigma} \right) = 0.95 \; ; \; Z \sim N(0,1)$$

$$2\Phi\left(\frac{1.96}{\sigma}\right) - 1 = 0.95$$

$$\Phi\left(\frac{1.96}{\sigma}\right) = 0.975$$

$$\Rightarrow \frac{1.96}{\sigma} = \Phi^{-1}(0.975) = 1.96$$

$$\Rightarrow \sigma = 1$$

(4) X : lifetime r.v.

$X \sim N(\mu, \sigma^2)$

$\mu = 1.4 \times 10^6$ hrs

$\sigma = 3 \times 10^5$ hrs

$P(X < 1.8 \times 10^6)$

$= P\left(\frac{X - 1.4\times 10^6}{3\times 10^5} < \frac{0.4 \times 10^6}{3 \times 10^5}\right)$

$= \cancel{\Phi(Z < \frac{4}{3})} = P(Z < \frac{4}{3})$ $[Z \sim N(0,1)]$

$= \Phi(\frac{4}{3}) = 0.918$

Y : r.v. denoting # of chips that have lifetime $< 1.8 \times 10^6$ hr

$Y \sim Bin(10, 0.918)$

$\Rightarrow P(Y \geq 2) = 1 - P(Y < 2)$

$= 1 - P(Y=0) - P(Y=1)$

$= 1 - \binom{10}{0}(.918)^0(1-.918)^{10} - \binom{10}{1}(0.918)^1(1-.918)^9$

$= \ldots$

(5) $X \sim N(0,1)$

$\forall\, t > 0 \quad P(|X| \geq t) = 1 - P(|X| < t)$

$= 1 - P(-t < X < t)$

$= 1 - [\Phi(t) - \Phi(-t)]$

$= 1 - [2\Phi(t) - 1]$

or $= 1 - [2(1 - P(X > t)) - 1]$

$= 2 - 2 + 2P(X > t) = 2P(X > t)$

$$P(X > t) = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-x^2/2}\, dx$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_t^{\infty} \frac{x}{t} e^{-x^2/2}\, dx \quad [t < x < \infty]$$

$y = x^2/2$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{t} \int_{t^2/2}^{\infty} e^{-y}\, dy = \frac{1}{\sqrt{2\pi}} \frac{e^{-t^2/2}}{t}$$

$$\Rightarrow P(|X| \geq t) \leq 2 \frac{1}{\sqrt{2\pi}} \frac{e^{-t^2/2}}{t} = \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}$$

(6)

$X = x$	0	1	2	......
$P(X=x)$	p_0	p_1	p_2	......

$$\sum_{k=0}^{\infty} (1 - F(k)) = \sum_{k=0}^{\infty} P(X > k) = P(X > 0) + P(X > 1) + P(X > 2) + \cdots$$

$= (p_1 + p_2 + p_3 + \cdots)$

$+ (p_2 + p_3 + \cdots)$

$+ (p_3 + p_4 + \cdots)$

$= p_1 + 2p_2 + 3p_3 + \cdots$

$$= \sum_{i=1}^{\infty} i\, p_i = \sum_{i=0}^{\infty} i\, P(X=i) = E(X)$$

(7) d.f. $F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\beta x^2}, & x \geq 0. \end{cases}$

$\beta > 0$

p.d.f. $f(x) = \begin{cases} 2\beta x e^{-\beta x^2}, & x \geq 0 \\ 0 & \text{o/w.} \end{cases}$

$$E(X) = 2\beta \int_0^{\infty} x^2 e^{-\beta x^2} dx$$

$y = x^2$

$$= \beta \int_0^{\infty} y^{1/2} e^{-\beta y} = \beta \cdot \frac{\Gamma 3/2}{\beta^{3/2}} = \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{\beta}} = \mu.$$

$$E X^2 = 2\beta \int_0^{\infty} x^3 e^{-\beta x^2} dx = \beta \int_0^{\infty} y e^{-\beta y} dy = \beta \frac{\Gamma 2}{\beta^2} = \frac{1}{\beta}$$

$y = x^2$

$$V(X) = E(X^2) - (E X)^2 = \frac{1}{\beta} - \mu^2 = \frac{1}{\beta} - \frac{\pi}{4\beta}$$

median : m_0

$$m_0 \ni F(m_0) = \frac{1}{2} = 1 - F(m_0)$$

i.e. $2\beta \int_0^{m_0} x e^{-\beta x^2} dx = 2\beta \int_{m_0}^{\infty} x e^{-\beta x^2} dx = \frac{1}{2}$

i.e. $1 - e^{-\beta m_0^2} = \frac{1}{2}$

$\Rightarrow m_0 = \ldots$

(8) $$1-\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_x^{\infty} \frac{1}{y} \left(y e^{-y^2/2} \right) dy \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{y} \cdot \left(-e^{-y^2/2} \right) \Big|_x^{\infty} - \int_x^{\infty} \left(-\frac{1}{y^2} \right) \left(-e^{-y^2/2} \right) dy \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{x} e^{-x^2/2} - \underbrace{\int_x^{\infty} \frac{1}{y^2} e^{-y^2/2}\, dy}_{\geq 0.} \right]$$

$$\Rightarrow \quad 1-\Phi(x) \leq \frac{1}{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = \frac{\phi(x)}{x}.$$

(9) p.d.f. of (7)

Mode – pt at which $f(x)$ is max^m.

$$f'(x) = 2\beta\left(x e^{-\beta x^2}(-2\beta x) + e^{-\beta x^2}\right)$$

$$f'(x) = 0 \Rightarrow 2\beta x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{2\beta}}.$$

$$f''(x) = 2\beta \frac{d}{dx}\left(e^{-\beta x^2}(1 - 2\beta x^2)\right)$$

$$= 2\beta\left(e^{-\beta x^2}(-4\beta x) + (1-2\beta x^2) e^{-\beta x^2}(-2\beta x)\right).$$

$$f''(x)\Big|_{x=\frac{1}{\sqrt{2\beta}}} = 2\beta\left(e^{-\frac{1}{2}}\left(-4\sqrt{\beta/2}\right)\right) < 0.$$

$\beta > 0$

$\Rightarrow$ m^*, the mode of the dist^n is at $\frac{1}{\sqrt{2\beta}}$.

$$m^* = \frac{1}{\sqrt{2\beta}}$$

$$\mu = E(X) = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{\beta}} = \left(\frac{\sqrt{\pi}}{2}\right)\sqrt{2}\, m^*$$

i.e. $\mu = \sqrt{\frac{\pi}{2}}\, m^*$.

& $2m^{*2} - \mu^2 = 2\left(\frac{2}{\pi}\mu^2\right) - \mu^2$

$$= \frac{4}{\pi}\mu^2 - \mu^2 = \frac{\cancel{4}}{\cancel{\pi}} \cdot \frac{\cancel{\pi}}{\cancel{4}} \cdot \frac{1}{\beta} - \mu^2$$

i.e.

$2m^{*2} - \mu^2 = \sigma^2$

$$\cancel{= \left(\frac{4}{\pi} - 1\right)\mu^2} = \frac{1}{\beta} - \mu^2$$

$$= E X^2 - \mu^2$$

$$= V(X).$$

(10) Let $Y = X - \theta$ & $Z = \theta - X$

(a) then p.d.f. of Y is $f_Y(y) = f_X(\theta + y) \quad \forall y \in \mathbb{R}$

- - . Z is $f_Z(z) = f_X(\theta - z) \quad \forall z \in \mathbb{R}$

Now $X - \theta \stackrel{d}{=} \theta - X$ i.e. $Y \stackrel{d}{=} Z$

$$\Leftrightarrow f_Y(x) = f_Z(x) \quad \forall x \in \mathbb{R}$$

$$\Leftrightarrow f_X(\theta + x) = f_X(\theta - x) \quad \forall x \in \mathbb{R}$$

(b) $Y = X - \theta \qquad Z = \theta - X$

$$F_Y(x) = P(X \leq x + \theta) = F_X(x + \theta) \quad \forall x \in \mathbb{R}$$

$$F_Z(x) = P(\theta - X \leq x) = P(X \geq \theta - x) = 1 - F_X(\theta - x) \quad \forall x \in \mathbb{R}$$

$$Y \stackrel{d}{=} Z \Leftrightarrow F_Y(x) = F_Z(x) \quad \forall x \in \mathbb{R}$$

$$\Leftrightarrow F_X(x + \theta) = 1 - F_X(\theta - x) \quad \forall x \in \mathbb{R}$$

$$\Leftrightarrow F_X(x + \theta) + F_X(\theta - x) = 1$$

(c). $X - \theta \stackrel{d}{=} \theta - X$

$$\Rightarrow E(X - \theta) = E(\theta - X)$$

$$\Rightarrow E(X) = \theta$$

since $X - \theta \stackrel{d}{=} \theta - X$

$$P(X - \theta \leq 0) = P(\theta - X \leq 0)$$

$$\Leftrightarrow F_X(\theta) = 1 - F_X(\theta)$$

i.e. $F_X(\theta) = \frac{1}{2} \Rightarrow \theta$ is the median of X

(11)

(a) $E\left(\frac{X-\mu}{\sigma}\right)^r = \int_{\mu}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^r \frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)} dx$

$= \int_0^{\infty} t^r e^{-t} dt = \Gamma(r+1) = r!$

$r = 1, 2, \ldots$

$\Rightarrow E(X-\mu)^r = \sigma^r \Gamma(r+1)$

(b) $E(X-\mu) = \sigma \Rightarrow E(X) = \mu + \sigma$

$E(X-\mu)^2 = 2\sigma^2$

i.e. $E(X^2 + \mu^2 - 2X\mu) = 2\sigma^2$

i.e. $E(X^2) = 2\sigma^2 - \mu^2 + 2(\mu+\sigma)\mu$

$= 2\sigma^2 - \mu^2 + 2\mu^2 + 2\mu\sigma$

$= 2\sigma^2 + 2\mu\sigma + \mu^2$

(c) let med $= \eta_{1/2}$

$F(\eta_{1/2}) = \frac{1}{2}$

i.e. $1 - e^{-(\eta_{1/2} - \mu)/\sigma} = \frac{1}{2}$

i.e. $\eta_{1/2} = \mu - \sigma \ln\left(\frac{1}{2}\right)$

(12) Let n be the number of indep Bernoulli trials with prob of success p $(0<p<1)$

X : number of successes in n trials

$X \sim B(n,p)$

$$P(X \geq r) = P[\text{at least } r \text{ successes in } n \text{ trials}$$

$$= P\left(\bigcup_{l=0}^{n-r} \{ r^{th} \text{ success in } (r+l)^{th} \text{ trial} \} \right)$$

$$= \sum_{l=0}^{n-r} P(r^{th} \text{ success in } (r+l)^{th} \text{ trial})$$

$$= \sum_{l=0}^{n-r} \underbrace{\binom{r+l-1}{r-1} p^{r-1} (1-p)^{l}}_{(r-1) \text{ successes in first } r+l-1 \text{ trials}} \underbrace{p}_{\text{success in } r^{th} \text{ trial}}$$

$$= \sum_{l=0}^{n-r} \binom{r+l-1}{r-1} p^{r} (1-p)^{l}$$

$$= P(Y \leq n-r) \quad ; Y \sim NB(r,p)$$

(13) $X \sim P(\lambda)$

$$E \frac{1}{2+X} = \sum_{x=0}^{\infty} (x+2)^{-1} \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(x+1)\lambda^{x}}{(x+2)!} = e^{-\lambda} \sum_{x=2}^{\infty} (x-1) \frac{\lambda^{x-2}}{x!}$$

$$= \frac{1}{\lambda^2} \left(\sum_{x=2}^{\infty} (x-1) \frac{e^{-\lambda} \lambda^{x}}{x!} \right)$$

$$= \frac{1}{\lambda^2} \left[\sum_{x=0}^{\infty} (x-1) \frac{e^{-\lambda} \lambda^{x}}{x!} + e^{-\lambda} \right]$$

$$= \frac{1}{\lambda^2} \left(E(X-1) + e^{-\lambda} \right)$$

$$= \frac{1}{\lambda^2} \left(\lambda - 1 + e^{-\lambda} \right)$$

(14) $X \sim G(n, \theta) \quad ; \quad Y \sim P(t/\theta)$

To prove that

$$P(X \geq t) = P(Y \leq n-1)$$

i.e.
$$\frac{1}{\Gamma n \, \theta^n} \int_t^\infty e^{-x/\theta} x^{n-1} \, dx = \sum_{k=0}^{n-1} \frac{e^{-t/\theta} (t/\theta)^k}{k!}$$

$$\text{l.h.s} = \frac{1}{\Gamma n \, \theta^n} \int_t^\infty e^{-x/\theta} x^{n-1} \, dx \quad (= I_n \text{ (say)})$$

$\left(z = \frac{x}{\theta}\right) \rightarrow$
$$= \frac{1}{(n-1)!} \int_{t/\theta}^\infty e^{-z} z^{n-1} \, dz$$

Integration by parts
$$= \frac{e^{-t/\theta} (t/\theta)^{n-1}}{(n-1)!} + \frac{1}{(n-2)!} \int_{t/\theta}^\infty e^{-z} z^{n-2} \, dz$$

$$= \frac{e^{-t/\theta} (t/\theta)^{n-1}}{(n-1)!} + I_{n-2}$$

$$= P(Y = n-1) + I_{n-2} = P(Y = n-1) + P(Y = n-2) + I_{n-3}$$

$\vdots$ continue integration by parts to get I_1

$$= P(Y = n-1) + P(Y = n-2) + \cdots + \underbrace{I_2}_{P(Y=1)} + \underbrace{I_1}_{P(Y=0)}$$

$$= P(Y \leq n-1)$$

(15) $X \sim P(\lambda)$

p.m.f. $P(X=x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!}, & x = 0, 1, 2, \ldots \\ 0, & o/w. \end{cases}$

$Y = X^2 - 5 \Rightarrow$ Range space of $Y = \{-5, -4, -1, 4, 11, \ldots\} = \mathcal{Y}$

$P(Y=y) = P(X^2 - 5 = y) = P(X^2 = y+5)$

p.m.f. of Y : $P(Y=y) = P(X = \sqrt{y+5}) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{\sqrt{y+5}}}{(\sqrt{y+5})!}, & y \in \mathcal{Y} \\ 0, & o/w. \end{cases}$

(16) $X \sim B(n, p)$

p.m.f. $P(X=x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, \ldots n \\ 0, & o/w. \end{cases}$

$Y = n - x \Rightarrow \mathcal{Y} = \{0, 1, \ldots n\}$.

$P(Y=y) = P(n - X = y) = P(X = n-y)$

$\Rightarrow$ p.m.f. of Y

$P(Y=y) = \begin{cases} \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}; & y = 0, 1, \ldots n \\ 0, & o/w \end{cases}$

$= \begin{cases} \binom{n}{y} (1-p)^y p^{n-y}, & y = 0, 1, \ldots n \\ 0, & o/w. \end{cases}$

$\Rightarrow Y \sim B(n, 1-p)$.

$\mathcal{X} = \{-2, -1, 0, 1, 2, 3\}$

(17) $Y = X^2 \rightarrow$ range sp $= \{0, 1, 4, 9\}$

p.m.f. $P(Y=y) = \begin{cases} P(X=0) & y=0 \\ P(X=-1)+P(X=1) & y=1 \\ P(X=-2)+P(X=2) & y=4 \\ P(X=3) & y=9 \end{cases}$

$= \begin{cases} \frac{1}{5}, & y=0 \\ \frac{1}{6}+\frac{1}{15}, & y=1 \\ \frac{1}{5}+\frac{1}{3}, & y=4 \\ \frac{1}{30} & y=9 \end{cases}$

(18) $P(X=x) = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^x, & x=0,1,2,\ldots \\ 0, & o/w \end{cases}$

$Y = \frac{X}{X+1} \Rightarrow X = \frac{Y}{1-Y}$

range space of $Y = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$

$P(Y=y) = P\left(\frac{X}{X+1}=y\right) = P\left(X=\frac{y}{1-y}\right)$

$= \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}, & y=0,\frac{1}{2},\frac{2}{3},\ldots \\ 0 & o/w. \end{cases}$

(19) p.m.f. of X

$$P(X=x) = \begin{cases} e^{-1}, & x=0 \\ \dfrac{e^{-1}}{2(|x|)!}, & x \in \{\pm 1, \pm 2, \dots\} \\ 0, & \text{o/w}. \end{cases}$$

$Y = |X| \qquad \mathcal{Y} = \{0, 1, 2, \dots\}.$

$$P(Y=0) = P(X=0) = e^{-1}$$

$$P(Y=1) = P(X=-1) + P(X=1)$$

$$= \frac{e^{-1}}{2} + \frac{e^{-1}}{2} = e^{-1}$$

$$P(Y=2) = P(X=-2) + P(X=2)$$

$$= \frac{e^{-1}}{2.2!} + \frac{e^{-1}}{2.2!} = \frac{e^{-1}}{2!}$$

Sly for $k = 1, 2, \dots$

$$P(Y=k) = P(X=-k) + P(X=k)$$

$$= \frac{e^{-1}}{2.k!} + \frac{e^{-1}}{2.k!} = \frac{e^{-1}}{k!}$$

p.m.f. of Y

$$P(Y=y) = \begin{cases} \dfrac{e^{-1}}{y!}, & y = 0, 1, 2, \dots \\ 0, & \text{o/w} \end{cases}$$

$$Y \sim P(1)$$