MSO 202A: Complex Variables
August-September 2022
Assignment-1

1. Verify Cauchy-Riemann equation for $z^2,\ z^3$.

   Solution: $f(z) = z^2 = (x^2 - y^2) + 2\iota xy$. So $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. Clearly CR equations are satisfied.

   $f(z) = z^3 = (x + \iota y)^3 = (x^3 - 3xy^2) + \iota(3x^2y - y^3)$. So $u(x,y) = x^3 - 3xy^2$ and $v(x,y) = 3x^2y - y^3$. Clearly CR equations are satisfied.

2. Which of the following maps are holomorphic? If so then write as a function of z.

   (a) **(T)**$P(x + \iota y) = x^3 - 3xy^2 - x + \iota(3x^2y - y^3 - y)$

   (b) $P(x + \iota y) = x^2 + \iota y^2$

   (c) $P(x + \iota y) = 2xy + \iota(y^2 - x^2)$

   Solution: In each cases u, v has continuous partial derivatives. So to check analyticity, we only need to check the CR equations.

   (a) CR equations satisfied. $f(z) = z^3 - z$

   (b) Not analytic CR not satisfied.

   (c) CR equations satisfieid. $f'(z) = u_x + iv_x = 2y - 2ix = -2i(x + iy) = -2iz$. So $f(z) = -iz^2$

3. Suppose that $f = u + \iota v$ is analytic on region Ω and $f^{'}(z) \neq 0$ for all $z \in \Omega$. Show that the family of level curves $u(x,y) = c_1, v(x,y) = c_2$ are orthogonal to each other. Verify it for the example of $f(z) = z^2$ by drawing pictures. What happens in this case to the level curves $u(x,y) = 0, v(x,y) = 0$?

   Solution:

   Suppose that $u(x,y) = c_1, v(x,y) = c_2$ intersect at $z_0 = (x_0, y_0)$ and $f'(z_0) \neq 0$. Since$f = u + \iota v$ is analytic on region Ω, we have $f' = u_x + \iota v_x$. Since $f'(z_0) \neq 0$, assume without loss of generality, that $u_x(x_0, y_0) \neq 0$. By CR equation $v_y(x_0, y_0) \neq 0$. Then $\nabla(u)(z_0) = (u_x, u_y) \neq 0$ and $\nabla(v)(z_0) = (v_x, v_y) \neq 0$. Therefore these are normal vectors to the level curves $u = c_1$ and $v = c_2$ respectively. Using CR equations $\nabla(u)(z_0).\nabla(v)(z_0) = 0$.

4. (a) **(T)**Let z, w be two complex numbers such that $\overline{z}w \neq 1$. Prove that

   $$|\frac{w - z}{1 - \overline{w}z}| < 1 \text{ if } |z| < 1 \text{ and } |w| < 1,$$

   and also that

   $$|\frac{w - z}{1 - \overline{w}z}| = 1 \text{ if } |z| = 1 \text{ or } |w| = 1,$$

(b) (**T**)Prove that for a fixed $w \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the mapping

$$F : z \mapsto \frac{w - z}{1 - \overline{w}z}$$

satisfy the following conditions:

- F maps $\mathbb{D}$ to itself and $F : \mathbb{D} \to \mathbb{D}$ is holomorphic.
- F interchanges 0 and w, i.e., $F(0) = w$ and $F(w) = 0$.
- $|F(z)| = 1$ if $|z| = 1$.
- F is bijective.

Solution:

(a) $|w - z|^2 - |1 - \overline{w}z|^2$
$= (w - z)(\overline{w} - \overline{z}) - (1 - \overline{w}z)(1 - w\overline{z})$
$= w\overline{w} + z\overline{z} - w\overline{z} - z\overline{w} - 1 + w\overline{z} + \overline{w}z - zw\overline{z}\overline{w}$
$= (1 - z\overline{z})(1 - w\overline{w})$.

Thus if $|z| < 1$ and $|w| < 1$, then $|w - z|^2 - |1 - \overline{w}z|^2 < 0$ which implies $|\frac{w-z}{1-\overline{w}z}| < 1$.

Also clearly, if $|z| = 1$ or $|w| = 1$, then $|w - z|^2 - |1 - \overline{w}z|^2 = 0$ that is $|\frac{w-z}{1-\overline{w}z}| = 1$.

(b) The first and third property follows from (a). The function F is complex differentiable everywhere except $z = \frac{1}{\overline{w}}$ which lies outside the unit disc $\mathbb{D}$. This F is holomorphic on $\mathbb{D}$. The second property is easy to verify.

For the last property, it is easy to check check that $F \circ F(z) = z$. Thus F is bijective.

5. Suppose that $f = u + \iota v$ is holomorphic on an open connected set Ω. Prove that in each one of the following cases f is constant.

   (a) u is constant.

   (b) v is constant.

   (c) (**T**)$|f|$ is constant.

   Solution:

   (a) Sine $u =$ constant, we have $v_x = v_y = 0$ by CR equations. Since Ω is connected we get $v =$ constant. Hence f is constant.

   (c) Here $u^2 + v^2 = c$. If $c = 0$, then $|f|$ is zero and so $f = 0$. When $c \neq 0$, differentiating partially w.r.t x and y we get $uu_x + vv_x = 0$ $uu_y + vv_y = 0$. Using CR equations, the second relation becomes $vu_x - uv_x = 0$. Then solving for u_x and v_x, we get $u_x = v_x = 0$. Using CR equations $u_x = u_y = v_x = v_y = 0$. So u, v are constants. Hence f is constant.

6. Suppose $f = u + \iota v \in \mathcal{H}(\mathbb{C})$ satisfy $u(x, y) = u(-y, x)$. Show that $f(z) = f(\iota z)$ for all $z \in \mathbb{C}$.

   Solution:

Let $g(z) = f(z) - f(\iota z)$. Then $g \in \mathcal{H}(\mathbb{C})$. Note that $\mathrm{Re}(f(\iota z)) = u(-y, x)$. Thus $\mathrm{Re}(g) = 0$ by the given condition. Hence $g(z) =$ constant by the previous exercise. But $g(0) = 0$. So $g(z) = 0$.

7. For $\Omega \subseteq \mathbb{C}$ define $\tilde{\Omega} = \{z \in \mathbb{C} : \overline{z} \in \Omega\}$ (It is the reflection of Ω about x-axis).

   (a) For $\Omega = \{z \in \mathbb{C} : |z - i| < 1\}$, draw $\tilde{\Omega}$.

   (b) If Ω is open and connected then so is $\tilde{\Omega}$.

   (c) If Ω is open and $f \in \mathcal{H}(\Omega)$ then show that $g \in \mathcal{H}(\tilde{\Omega})$ where $g(z) = \overline{f(\overline{z})}$. Find $g'(z)$.

   Solution:

   (b) The reflection of an open nbd in Ω will give open nbd in $\tilde{\Omega}$. Similarly for path. The reflection map is actually a homeomorphism.

   (c) $f(z) = f(x + \iota y) = u(x, y) + \iota v(x, y)$. So $f(\overline{z}) = f(x - \iota y) = u(x, -y) + \iota v(x, -y)$. Hence $g(z) = u(x, -y) - \iota v(x, -y)$. Let $U(x, y) = u(x, -y)$ and $V(x, y) = -v(x, -y)$. Then $U_x(x, y) = u_x(x, -y)$ and $V_y(x, y) = v_y(x, -y)$. By CR equations of f we see that U, V satisfies CR equations. Also they have continuous partial derivatives. Hence g is holomorphic.

8. **(T)**Define differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ by setting:

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}); \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}).$$

   Show that $f = u + \iota v$ satisfy CR-equations if and only if $\partial f/\partial \overline{z} = 0$. Moreover, if f is holomorphic, then $f'(z) = \partial f/\partial z$. Further show that for a real valued function $u(x, y)$ with continuous second order partial derivatives,

$$4\frac{\partial^2 u}{\partial z \partial \overline{z}} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

   Solution:

   $\frac{\partial}{\partial \overline{z}} f = \frac{1}{2}(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y})(u + \iota v) = \frac{1}{2}\{(u_x - v_y) + \iota(v_x + u_y)\}$. Hence $\frac{\partial}{\partial \overline{z}} f = 0$ if and only if f satisfies CR equations.

   $\frac{\partial}{\partial \overline{z}} f = \frac{1}{2}(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y})(u + \iota v) = \frac{1}{2}\{(u_x + v_y) + \iota(v_x - u_y)\}$ Using CR equations we get $= u_x + \iota v_x = f'(z)$.

   $\frac{\partial^2}{\partial z \partial \overline{z}} u = \frac{\partial}{\partial z}\frac{1}{2}\{u_x + \iota u_y\} = \frac{1}{2}(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y})\frac{1}{2}\{u_x + \iota u_y\} = \frac{1}{4}(u_{xx} + u_{yy} + \iota(u_{yx} - u_{xy})) = \frac{1}{4}(u_{xx} + u_{yy})$.

   Since u has continuous second order partial derivatives, $u_{yx} = u_{xy}$

9. **(T)**Let $f(x + \iota y)$ be a polynomial with complex coefficients in x and y. Show that f is holomorphic if and only if it can be expressed as a polynomial in the single variable z.

   Solution:

Use the substitutions $z = x + iy$ and $\overline{z} = x - iy$ to express f as polynomial in z and $\overline{z}$. We write $f(x + \iota y) = p_0(z) + p_1(z)\overline{z} + p_2(z)\overline{z}^2 + \cdots p_n(z)\overline{z}^n$, where p_i are polynomials in z. Now, by the previous exercise polynomial f is holomorphic if and only if $\frac{\partial}{\partial \overline{z}} f = 0$. So $\frac{\partial}{\partial \overline{z}} f = p_1(z) + 2p_2(z)\overline{z} + \cdots np_n(z)\overline{z}^{n-1}$ must be a zero polynomial in $\overline{z}$. Hence the coefficients are zero: $p_1 = p_2 = \cdots = p_n = 0$. Hence $f(z) = p_0(z)$.

10. Consider the function

$$f(z) = \begin{cases} \frac{xy(x+\iota y)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Show that f satisfies the Cauchy-Riemann equations at the origin $z = 0$, yet f is not complex differentiable at the origin.

Solution:

Here $u(x,y) = \frac{x^2 y}{x^2+y^2}$ and $v(x,y) = \frac{xy^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $u(0,0) = v(0,0) = 0$. Let us now calculate the partial derivatives at the origin:

$u_x(0,0) = \lim\limits_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = \lim\limits_{h\to 0} \frac{0-0}{h} = 0$. Similarly, $u_y(0,0) = v_x(0,0) = v_y(0,0) = 0$.

But $\lim\limits_{z\to 0} \frac{f(z)-f(0)}{z} = \lim\limits_{z\to 0} \frac{xyz}{z|z|^2} = \lim\limits_{z\to 0} \frac{xy}{x^2+y^2}$ – which does not exist, since along $y = mx$, the limit is $m/(1+m^2)$.

11. **(T)**Show that the set of natural numbers $\mathbb{N}$ can not be partitioned into finite number of subsets that are in arithmetic progression with distinct common difference.

Solution:

Suppose $\mathbb{N}$ is partitioned into n disjoint arithmetic progressions $\{(a_i, d_i),\ \ 1 \leq i \leq n)\}$, where a_i is the first element and d_i is the common difference. Then the power series $\frac{z}{1-z} = \sum\limits_{n=1}^{\infty} z^n$ converges uniformly in $|z| < 1$, it can be rearranged as:

$$\frac{z}{1-z} = \sum_{n=1}^{\infty} z^n = (z^{a_1} + z^{a_1+d_1} + z^{a_1+2d_1} \cdots) + \cdots + (z^{a_n} + z^{a_n+d_n} + z^{a_n+2d_n} \cdots), \quad |z| < 1.$$

Summing up the different series on the right hand side,

$$\frac{z}{1-z} = \frac{z^{a_1}}{1-z^{d_1}} + \frac{z^{a_2}}{1-z^{d_2}} + \cdots + \frac{z^{a_n}}{1-z^{d_n}}, \quad |z| < 1.$$

Since the common differences are distinct, there is UNIQUE k such that $d_k = \max\{d_1, \cdots, d_n\}$. Let ω be the primitive d_k-th root of unity. Then if we take the limit as $z \to \omega$ from inside the unit disk, the left hand side is finite. But on the right hand side, one term goes to infity and other terms are finite so overall the right side goes to infinity. This is a contradiction.

12. **(T)**Show that it is impossible to define a total ordering on $\mathbb{C}$. In other words, there does not exist a relation $\succ$ between complex numbers so that:

- For any two $z, w \in \mathbb{C}$ one and only one of the following is true: $z \succ w,\ w \succ z,\ z = w$.
- For all $z_1, z_2, z_3 \in \mathbb{C}$ the relation $z_1 \succ z_2$ implies $z_1 + z_3 \succ z_2 + z_3$.
- For all $z_1, z_2, z_3 \in \mathbb{C}$ with $z_3 \succ 0$ the relation $z_1 \succ z_2$ implies $z_1 z_3 \succ z_2 z_3$.

Solution: If possible suppose we can define a total ordering on $\mathbb{C}$.

CaseI: $\iota \succ 0$. Using the third property, we multiply both side by ι and obtain $-1 = \iota^2 \succ 0.\iota = 0$. We again multiply both side by $\iota \succ 0$. We get $-\iota \succ 0.\iota = 0$. Adding both side ι we get $0 \succ \iota$. This is a contradiction to our assumption $\iota \succ 0$.

CaseII: $0 \succ \iota$. Proceeding as above, we obtain a contradicition.

Case III: $\iota = 0$. Then $z.\iota = z.0 = 0$. Repeating we have $z = 0$ for any $z \in \mathbb{C}$. This is a contradiction.

13. Determine if there exist an analytic function with u as real part. (a)(**T**)$u = x^2y^2$. (b) $u = \sin x \cosh y$. (c) $u = x/(x^2+y^2)$ (d)(**T**)$u = xy + 3x^2y - y^3$

Solution:

(a) Not harmonic

(b) Harmonic on $\mathbb{C}$. So harmonic conjugate exists. $v_x = -u_y = -\sin x \sinh y \implies v = \cos x \sinh y + \phi(y)$

$v_y = u_x = \cos x \cosh y \implies \phi'(y) = 0 \implies \phi = c$

Thus $v = \cos x \sinh y + c$

$f = u + \iota v = \sinh(z) + c$

(c) u is harmoninc on $\mathbb{C}^* = \mathbb{C} - 0$. So the theorem does not tell us if harmoninc conjugate exists or not. By inspection $1/z = \frac{x - \iota y}{x^2+y^2}$ is analytic on $\mathbb{C}^*$. Thus u has harmonic conjugate on $\mathbb{C}^*$.

(d) $u_{xx} = 6y,\quad u_{yy} = -6y$. So u harmonic on $\mathbb{C}$ and hence harmoninc conjugate exists.

$v_x = -u_y = -x - 3x^2 + 3y^2 \implies v = -x^2/2 - x^3 + 3y^2x + \phi(y)$

$v_y = u_x = y + 6xy \implies \phi'(y) = y \implies \phi(y) = y^2/2 + c$

Thus $f = xy + 3x^2y - y^3 + i(-x^2/2 - x^3 + 3y^2x + y^2/2 + c) = (\iota z)^3 + \iota z^2/2$.