Mid semester exam

Graded

Student

JIYANSHU DHAKA

Total Points

27 / 80 pts

Question 1

Question 1

2 / 10 pts

- **+ 10 pts** Correct.
- **+ 0 pts** Wrong/ not answered
- **+ 1 pt** If partially correct.
- **+ 2 pts** Writing correct contour.
- ✔ **+ 2 pts** Finding correct residue $\frac{\pi(-2-3i)}{39}$ with proper justification.
- **+ 1 pt** Calculation for $R \to \infty$
- **+ 2 pts** For writing the correct formula with $lim(\rho \to 0)$
- **+ 1 pt** Finding $\frac{-\pi i}{13}$
- **+ 2 pts** Finding the limit $R \to \infty, \rho \to 0$

Question 2

Question 2

4 / 10 pts

- ✔ **+ 2 pts** Rouche theorem
- ✔ **+ 2 pts** |Z|=1
- **+ 3 pts** number of root in |Z|<1
- **+ 3 pts** number of root in 1<|Z|<2
- **+ 0 pts** No score
- **+ 10 pts** Full score
- **+ 8 pts** Without the second part

Question 3

Question 3

8 / 10 pts

+ **2 pts** If definition of cross ratio is correct

✔ + **6 pts** If correct transformation has been found.

✔ + **2 pts** If fixed points are correct.

+ **1 pt** If partially correct

+ **2 pts** if transformation is partially correct

+ **3 pts** half correct.

+ **4 pts** procedure is correct but final answer is not correct.

+ **0 pts** Incorrect/Blank

+ **0 pts** if definition of cross ratio is incorrect.

+ **3 pts** If transformation process is attempted but not got final answer.

Question 4

Question 4

4 / 10 pts

+ **3 pts** if first expansion (a)(i) is correct

+ **3 pts** if second expansion (a)(ii) is correct

✔ + **4 pts** if part (b) is correct

+ **1 pt** part (a)(i) partially Correct

+ **1 pt** part (a)(ii) partially correct

+ **2 pts** if a(i) partially correct

+ **2 pts** if a(ii) partially correct

✔ + **0 pts** incorrect or unattempted a(i)

✔ + **0 pts** incorrect or unattempted a(ii)

+ **0 pts** incorrect or unattempted (b)

− **0.5 pts** sign error

− **5 pts** penalty

+ **0.5 pts** no steps but answer to (b) is correct

− **0.5 pts** (b) proof is correct, final answer is incomplete

Question 5

Question 5

5 / 10 pts

+ 10 pts Correct

+ 2 pts 5(a) first one is correct

✔ **+ 4 pts** 5(a) second one is correct

+ 4 pts 5(b) is correct

+ 0 pts Wrong or unattended

+ 2 pts 5(a) second one is partially correct

+ 2 pts 5(b) is partially correct

✔ **+ 1 pt** Only radius of convergence found but no domain for 5(a) first part

Question 6

Question 6

0 / 10 pts

✔ **+ 0 pts** Part(a): Incorrect/Unattempted.

+ 6 pts Part(a): correct.

+ 2 pts Part(a): For finding suitable F.

+ 2 pts Part(a): For finding imaginary part of F.

+ 3 pts Part(a): For finding u which is real part of f.

+ 3 pts Part(a): For finding v which is imaginary part of f.

+ 0 pts Part(b): Incorrect/Unattempted

+ 4 pts Part(b): Correct

+ 1 pt Part(b): For Identity Theorem.

+ 1 pt Part(b): For finding g.

+ 2 pts Part(b): Partial correct answer.

− 0.5 pts Part(b): For not showing that g is not analytic/ No analytic function exists.

Question 7

Question 7

4 / 10 pts

+ 0 pts no answer or completely wrong answer

+ 2 pts Zeors are correct

✔ **+ 2 pts** Resudue at 0 is correct

+ 2 pts Resudue at 1 is correct

+ 4 pts Residue at infinity is correct

✔ **+ 1 pt** partially correct.

✔ **+ 1 pt** partial residue calculations

wrong residue calculation at 1

Question 8

Question 8

0 / 10 pts

✔ **+ 0 pts** Part a: InCorrect

+ 1 pt Proper calculation to show f(z) has zero of order 2 at z=0

+ 0.5 pts For only writing zero of order 2 at z=0

+ 1 pt Proper calculation to show f(z) has zero of order 1 at z=1

+ 0.5 pts For only writing zero of order 1 at z=1

+ 2 pts Proper calculation to show f has removal singularity at z=2

+ 1 pt For only writing removal singularity at z=2

+ 2 pts Proper calculation to show f has pole of order 6 at z=8

+ 1 pt For only writing pole of order 6 at z=8

+ 1 pt Just write $f(z)$ has zero of order 3 if z is an integer and $z\neq 0,1,2,8$

✔ **+ 0 pts** Part b::: Incorrect

+ 3 pts Part b: Totally correct

− 1 pt For not writing residue formula

+ 1.5 pts Show that $f(z)cot(\pi z)$ has simple pole at z=n with residue $\frac{f(n)}{\pi}$

+ 1.5 pts Show that $f(z)csc(\pi z)$ has simple pole at z=n with residue $(-1)^n\frac{f(n)}{\pi}$

Mid Sem: MSO202M (2024-2025 I)

Date: 18 September 2024 Time: 18:00 -20:00 hr Maximum marks: 80

Name: Jiyanshu Dhaka Roll No. 220481

Instructions: (Read carefully)

- Please enter your NAME and ROLL NUMBER in the space provided on EACH page.
- Only those booklets with name and roll number on every page will be graded. All other booklets will NOT be graded.
- This answer booklet has 5 pages. Check to see if the print is either faulty or missing on any of the pages. In such a case, ask for a replacement immediately.
- Please answer each question ONLY in the space provided. Answers written outside the space provided for it WILL NOT be considered for grading. So remember to use the space judiciously.
- For rough work, separate sheets will be provided to you. Write your name and roll number on rough sheets as well. However, they WILL NOT be collected back along with the answer booklet.
- No calculators, mobile phones, smart watches, or other electronic gadgets are permitted in the exam hall.
- Notations: All notations used are as discussed in class.
- All questions are compulsory.
- Do NOT remove any of the sheets in this booklet.

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Q 1. Using complex integration method evaluate **10 marks**

$$\int_{\mathbb{R}} \frac{dx}{(x-2)(x^2+9)}.$$

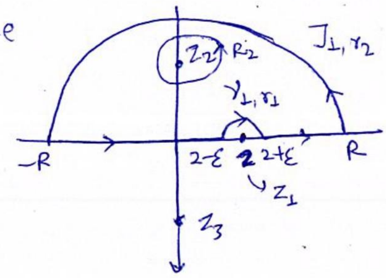
Ans

$$I = \int_{\mathbb{R}} \frac{dx}{(x-2)(x^2+9)} = \operatorname{Re} \int_{-\infty}^{\infty} \frac{dz}{(z-2)(z^2+9)} \Rightarrow f(z) = \frac{1}{(z-2)(z^2+9)}$$

Poles of $f(z)$ are $(z-2)(z^2+9)=0$

$\Rightarrow z=2,\ 3e^{i\pi/2},\ 3e^{i3\pi/2}$ with order 1.

$z=2$ lies on real line, make a countor like



now

$$\int_{C_R} f(z)\,dz = \int_{-R}^{2-\varepsilon} f(z)dz + \int_{\gamma_{1,r_1}} f(z)\,dz + \int_{2+\varepsilon}^{R} f(z)dz + \int_{\Gamma_{1,r_2}} f(z)dz$$

now according to Cauchy Residual Theorem for an analytic function f and in a compact, simply connected Domain the Integration becomes sum of residuals that are inside the domain.
(Here only $z_2 = 3e^{i\pi/2}$ is inside the domain.)

By Theorem $\Rightarrow$

$$\int_{C_R} f(z)dz = 2\pi i \left(\sum_{i=1}^{n} R_i\right)$$

R_i is sum of residuals that are inside.

$$R_2 = \lim_{z\to 3e^{i\pi/2}} (z-3e^{i\pi/2}) \frac{1}{(z-2)(z-3e^{i\pi/2})(z-3e^{i3\pi/2})}$$

$$= \frac{1}{(3i-2)(3i+3i)} = \frac{1}{(3i-2)\,6i}$$

$$\Rightarrow \int_{C_R} f(z)dz = \frac{2\pi i}{(3i-2)\,6i} = \frac{\pi(3i+2)}{3(-9-4)} = \frac{\pi(3i+2)}{-39}$$

$$I = \operatorname{Re}\left(\frac{(3i+2)\pi}{-39}\right) = -\frac{2\pi}{39}$$

?

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Q 2. State Rouche's theorem. Show that the polynomial $p(z) = z^{10} + 13z^4 + 2z^2 + 3$ does not vanishes on $|z| = 1$. Find the number of roots of $p(z)$ in (a) $|z| < 1$ and (b) $1 < |z| < 2$.

10 marks

Ans

Theorem :- Let $f(z)$ and $g(z)$ are two analytic function in a simply connected domain D. If $|f(z)| > |g(z)|$ for all $z \in D$ then $f(z)$ and $f(z)+g(z)$ will have same number of roots for $z \in D$.

now $p(z)-3 = z^{10} + 13z^4 + 2z^2$

let $h(z) = |z|^{10} + 13|z|^4 + 2|z|^2 \Rightarrow h(z) = 16$ for $|z|=1$

now we know $h(z) \geq p(z)-3 > 3$ for $|z|=1$

As $h(z) = 16$ does not any root for $|z|=1 \Rightarrow p(z)-3$ does not have any root for $|z|=1$ and using Rouche's theorem $p(z)-3+3 = p(z)$ does not vanish.

(a) $|z|<1$ $\quad p(z) = \underbrace{z^{10} + 13z^4 + 2z^2 - 13}_{g(z)} + \underbrace{13-3}_{f(z)}$

for $|z|<1$ $\quad |f(z)| = 10 > |g(z)|$ and $f(z)$ have no root.

$\Rightarrow f(z) + g(z) = p(z)$ have no root.

(b) for $1<|z|<2$

$$|\underbrace{z^{10} + 13z^4}_{f(z)}| > |\underbrace{2z^2+3}_{g(z)}|$$

and for $|z|<2$ $\quad z^{10} + 13z^4$ have 4 roots.

$\Rightarrow f(z) + g(z) = p(z)$ have 4 roots.

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Q 3. Define cross ratio for the points z_1, z_2, z_3, z_4. Find a Möbius transformation T such that $T(0) = -1$, $T(i) = 1$ and $T(\infty) = 3$. Find the fixed points of T. **10 marks**

Ans for 4 points z_1, z_2, z_3, z_4

cross ratio will become $\frac{(z-z_1)(z_1-z_2)(z_3-z_4)}{(z_1-z_2)(z_2-z_3)(z_4-z)} = \frac{(w-w_1)(w_2-w_3)(w_3-w_4)}{(w_1-w_2)(w_3-w_4)(w_4-w)}$

for 3 points :- $\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$

let $T(z) = \frac{az+b}{cz+d}$ now

~~$T(0) = -1$~~	~~$T(i) = 1$~~	~~$T(\infty) = 3$~~
~~$\frac{b}{d} = -1$~~	~~$\frac{ai+b}{ci+d} = 1$~~	~~$\lim_{z\to\infty} \frac{a+b/z}{c+d/z} = 3 \Rightarrow a = 3$~~
~~$b = -d$~~	~~$ai+b = ci+d$~~	~~$\Rightarrow a = 3c$~~
	~~$\Rightarrow a = c, \; b = d$~~	

$$\Rightarrow \frac{(z-0)(i-\infty)}{(0-i)(\infty-z)} = \frac{(w+1)(1-3)}{(-1-1)(3-w)} \Rightarrow \frac{-2(w+1)}{+2(w-3)} = \frac{z(i-\infty)}{(z-\infty)i}$$

$$= \frac{z\left(\frac{i}{\infty}-1\right)}{\left(\frac{z}{\infty}-1\right)i}$$

$$\Rightarrow \frac{-2(w+1)}{2(w-3)} = \frac{+z}{+i}$$

$$\Rightarrow -2iw-2i = 2zw-6z \Rightarrow (2z+2i)w - 6z+2i = 0$$

$$\Rightarrow w = \frac{6z-2i}{2z+2i} = \boxed{\frac{3z-i}{z+i} = T(z)}$$

for fixed pt. $\frac{3z-i}{z+i} = z \Rightarrow z^2+iz-3z+i=0$

$$z^2+(i-3)z+i=0$$

$$\frac{-(i-3) \pm \sqrt{(i-3)^2-4i}}{2} = \boxed{\frac{(3-i) \pm \sqrt{8-10i}}{2}}$$

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Q 4. (a) Expand **6 + 4 marks**

$$\frac{z}{z^2 - 11z + 28} \quad \text{in } (i)\ |z-4| < 1 \quad (ii)\ 1 < |z-7| < 2.$$

(b) Find all analytic functions in $D := \{z \in \mathbb{C} : |z| \leqslant 1\}$ such that $|f(z)| \geqslant 2$ on $|z| = 1$, $f(z) \neq 0$ for $z \in D$, and $f(0.5) = 2$.

(a) By using laurent series we can write formula.

$f(z) = \frac{z}{(z-4)(z-7)}$ as . $\boxed{f(z) = \sum_{n=0}^{\infty} c_n (w-a)^n + \sum_{n=1}^{\infty} \frac{d_n}{(w-a)^n}}$

(i) $|z-4| < 1$ can be consider as annulus $\boxed{0 < |z-4| < 1 = C_1}$

$f(z) = \frac{z}{(z-7)} \sum_{n=1}^{\infty} \frac{d_n}{(z-4)}$, where d_n are coeff.

$d_n = \frac{1}{2\pi i} \oint_{C_1} (w-4)^{n-1} f(w)\, dw$

(ii) for annulus $\boxed{1 < |z-7| < 2 = C_2}$

$f(z) = \frac{z}{(z-4)} \sum_{n=1}^{\infty} \frac{d_n'}{(z-7)}$; $d_n' = \frac{1}{2\pi i} \oint_{C_2} f(w) (w-7)^{n-1} dw$

(b) Using Minimum Modulus Theorem which says that if $f(z)$ is analytic function in simply connected domain D and $f(z) \neq 0$ for $z \in D$. and $|f(z)| \geq M$ on Boundary then minima of function lies on boundary unless it is constant function. So As $f(0.5) = 2$ which is not at boundary $\Rightarrow$ f is const.

$\boxed{f(z) = 2}$

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Q 5. (a) Find domain and radius of convergence of **2 + 4 marks**

$$S_1 := \sum_{n=1}^{\infty}(7z-3)^n \quad \text{and} \quad S_2 := \sum_{n\in\mathbb{Z}}(27)^{-|n|}z^{3n}.$$

(b) Let $f = u + iv$ be an entire function such that $5u + 18v$ is bounded. If $f(3) = 10$ then find f. **4 marks**

5 (a) S_1 : $\sum_{n=1}^{\infty} 7^n (z-3/7)^n$. By root test

$R = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |7^n|^{1/n} = 7$

Domain : $|z - 3/7| < 7$

S_2 : $\underbrace{\sum_{n\in Z-N} (27)^n z^{3n}}_{\text{Primary}} + \underbrace{\sum_{n\in N} (27)^{-n} z^{3n}}_{\text{general}}$

$\boxed{d(n) = 3n}$

$\frac{1}{R_2} = \left|\frac{a_n}{a_{n-1}}\right|^{\frac{1}{d(n)-d(n-1)}}$

$R_2 = 3$

$\frac{1}{R_1} = \left|\frac{a_n}{a_{n-1}}\right|^{\frac{1}{d(n)-d(n-1)}}$

$\Rightarrow R_1 = 1/3$

Domain : $1/3 < |z| < 3$

(b) Using Louiville Theorem for bounded entire function

f is constant function

$\Rightarrow \boxed{f = 10}$

(-5) Why..

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Q 6. (a) Let $f = u + iv$ be an entire function such that $3u + 8v = x^3 - 3xy^2 + 3x^2 - 3y^2$. Find f by determining the harmonic conjugate. **6 marks**

(b) Find all analytic function in $\{z \in \mathbb{C} : |z| < 1\}$ such that $f(1/n) = 1/(n+3)\ \forall\, n \in \mathbb{N}$. **4 marks**

(a)

$3u_x + 8v_x = 3x^2 - 3y^2 + 6x$

$3u_v + 8v_y = -6xy - 6y$

solve to get $u_x \; v_x$, $u_y \; v_y$

CR eqn: $u_x = v_y$

$u_y = -v_x$

$3u_x + 8$

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Q 7. Find zeros and poles (all finite places and at infinity) of the function

$$f(z) = \frac{z^5 - 32}{z(z-1)^2}.$$

Calculate its residues and show that the sum of all residues in 0. **10 marks**

for zeros: $f(z) = 0 \Rightarrow \frac{z^5-32}{z(z-1)^2} = 0$

$\Rightarrow z^5 = 32$

$z = 2$ is zero of order 5

Poles: $\rightarrow$

$z(z-1)^2 = 0$

$z = 0$, 1 order

$z = 1$ 2 order

$\lim_{z \to \infty} f(z) \to \infty \Rightarrow z = \infty$ is also pole

$R_1 = \lim_{z \to 0} \cancel{z} \cdot \frac{z^5 - 32}{\cancel{z}(z-1)^2} = -32$

$R_2 = \frac{1}{1} \lim_{z \to 1} \frac{d}{dz}\left(\frac{(z^5-32)}{z(z-1)^2}\right)$

$= 32$

$R_1 + R_2 = 0$ (Sum of residuals)

Hence proved

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Q 8. (a) Determine the singularities and find the order of all zeros and poles of the function.

$$f(z) = \frac{\sin^3(\pi z)}{z(z-1)^2(z-2)^3(z-8)^9}.$$ **7 marks**

(b) Let $f(z)$ be an entire function. Calculate the residue of $f(z)\cot(\pi z)$ and $f(z)\csc(\pi z)$ at $z = n$, where n is an integer. **3 marks**

8a

roots :-

$\sin^3(\pi z) = 0$

$z \in \mathbb{Z}$ for order 3 for all $\mathbb{Z}$

poles :-

$z(z-1)^2 (z-2)^3 (z-8)^9 = 0$

$\Rightarrow z = 0$	1 order	removable singularity
$z = 1$	2 order	non-remo.
$z = 2$	3 order	non-remov.
$z = 8$	9 order	non-remo.

removable singularity if $\lim_{z \to a} (z-a)\, f(a)$ is finite

b

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Space for question 8