Power Series

- $$e^z = 1 + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \cdots + \frac{z^n}{\lfloor n} + \cdots$$
$$= \sum_{n \geq 0} \frac{z^n}{\lfloor n} \qquad \forall z \in \mathbb{C}$$

- $$\sin z := \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} - \cdots$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{\lfloor 2} + \frac{z^4}{\lfloor 4} - \frac{z^6}{\lfloor 6} \cdots$$
$$\forall z \in \mathbb{C}$$

- $$\frac{d}{dz}(\sin z) = \cos z$$

$$\frac{d}{dz}(\cos z) = -\sin z$$

$$\sin^2 z + \cos^2 z = 1$$

- $\sin z$ & $\cos z$ are unbounded.
In fact, for any $w \in \mathbb{C}$, then $\exists z$ s.t $\sin z = w$

$$\frac{e^{iz} - e^{-iz}}{2i} = w \qquad e^{iz} = X$$

$$X - \frac{1}{X} = 2iw \qquad X = \frac{2iw \pm \sqrt{4 - 4w^2}}{2}$$

$$X^2 - 2iwX - 1 = 0 \qquad e^{iz} = w \pm \sqrt{1-w^2}$$

$$iz = \log(iw \pm \sqrt{1-w^2})$$

• ~~Ø~~ Similarly for Cos z also.

• Zeros of Sin z are same as of Sin x. ($x \in \mathbb{R}$)

$$\sin z = 0$$
$$e^{iz} = e^{-iz}$$
$$e^{2iz} = 1 = e^{0}$$

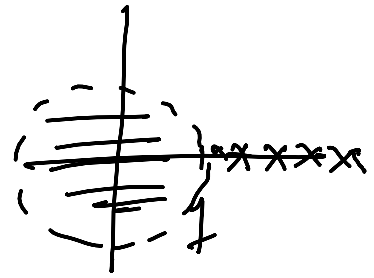
$$\boxed{e^{z} = e^{w} \Rightarrow z = w + 2k\pi i}$$

$$i2z = 2k\pi i \qquad k \in \mathbb{Z}$$
$$z = k\pi$$

Entire function = holomorphic on $\mathbb{C}$

• e^{z}, Sin z, Cos z are entire.

$$f(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \qquad |z| < 1$$
$$f'(z) = 1 + z + z^2 + z^3 + \cdots \qquad |z| < 1$$
$$= \frac{1}{1-z}$$
$$\frac{d}{dz}(\log z) = \frac{1}{z}$$
$$\frac{d}{dz}\left(\mathrm{Log}(1-z)\right) = \frac{-1}{1-z}$$



$|z| < 1$

$$\frac{d}{dz}\left(f(z) + \mathrm{Log}(1-z)\right) = 0$$
$$\mathrm{Log}(1-z) = -f(z) + C$$
$$C = 0 \quad (\because \mathrm{Log}(1) = 0)$$

Putting $z = 0$

$$\mathrm{Log}(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} \cdots \qquad |z| < 1$$

Theorem $f(z) = \sum a_n z^n \quad |z| < R$

R = radius of convergence of $\sum a_n z^n$.

Then $f(z)$ is analytic on $|z| < R$

& $f'(z) = \sum n a_n z^{n-1}$

Lemma. $\sum a_n z^n$ & $\sum n a_n z^{n-1}$

have same radius of convergence

Proof ~~Assume~~ radius of conv of ~~a_n~~

$\sum a_n z^n = R$.

Take $|z_0| < R$. $\quad z_0 \neq 0$

$|n a_n z_0^{n-1}|$

$= |n a_n z_0^n| \frac{1}{|z_0|}$

$= n |a_n z_0^n| \cdot \frac{1}{|z_0|}$

Choose r such that

$|z_0| < r < R$

$\sum a_n r^n$ is conv.

$\Rightarrow |a_n r^n| < M \quad \forall n$.

$|n a_n z_0^{n-1}| = \frac{n}{|z_0|} |a_n z_0^n|$

$= \frac{n}{|z_0|} \cdot \underbrace{|a_n r^n|} \cdot \frac{|z_0|^n}{r^n}$

$\le \frac{n}{|z_0|} M \cdot \underbrace{\left|\frac{z_0}{r}\right|^n}_{\rho < 1} = \frac{M}{|z_0|} n \rho^n$

$$|n a_n z_0^{n-1}| \le \underbrace{K \cdot n \rho^n}_{= M_n} \quad 0 < \rho < 1.$$

Then $\sum M_n$ is conv.

($\because$ by ratio test

$$\lim \frac{M_{n+1}}{M_n} = \rho < 1.$$)

Hence by comparison test

$\sum n a_n z_0^{n-1}$ is conv.

$$\lim \sqrt[n-1]{n |a_{n-1}|}$$

$$= \lim \sqrt[n-1]{n} \lim \sqrt[n-1]{|a_{n-1}|}$$

$$= 1 \quad \lim_n \sqrt[n]{|a_n|} = \frac{1}{R}$$

($\lim \sqrt[n]{n} = 1$)

Note $R = \lim_n \frac{|a_n|}{|a_{n+1}|}$

provided the limit exist

$$= \frac{1}{\lim \sqrt[n]{|a_n|}}.$$

[($a_n = 2^n$ for odd n

$a_n = 3^n$ for even n)

($R = \frac{1}{\lim\sup \sqrt[n]{|a_n|}}$ ✓)

$a_n = (-1)^n$

$\lim\sup a_n = 1 \qquad \lim\inf a_n = -1$]

Proof of the theorem

$$f(z) = \sum a_n z^n$$

$$g(z) = \sum n a_n z^{n-1} \qquad |z| < R$$

Take $|z_0| < R$.

we have to show that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - g(z_0) \right| \to 0 \text{ as } z \to z_0.$$

$$S_n(z) = \sum_{k \leq n} a_k z^k$$

$$R_n(z) = \sum_{k > n} a_k z^k$$

$$\frac{f(z) - f(z_0)}{z - z_0} - g(z_0)$$

$$= \frac{S_n(z) - S_n(z_0)}{z - z_0} + \frac{R_n(z) - R_n(z_0)}{z - z_0} - g(z_0)$$

$$= \left[\frac{S_n(z) - S_n(z_0)}{z - z_0} - S_n'(z_0) \right] = A$$

$$+ \left[\boxed{S_n'(z_0)} - g(z_0) \right] = B$$

$$+ \frac{R_n(z) - R_n(z_0)}{(z - z_0)} = C$$

- $|A| \to 0$ as $z \to z_0$
- $|B| \to 0$ ~~as~~ for large n.
- $|C| \to 0$ as $z \to z_0$