Holomorphic function

$\Omega \subseteq \mathbb{C}$ - open set

$f : \Omega \longrightarrow \mathbb{C}$

f is called complex diff at $z_0 \in \Omega$ if

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} = f'(z_0)$$

- The function f is called holomorphic on Ω if it is complex differentiable $\forall z_0 \in \Omega$.
- A function is called holomorphic at z_0 the function is holomorphic on $D_r(z_0)$ for some $r > 0$.

$H(\Omega)$ = set of all holomorphic fnctn on Ω.

Property $f, g \in H(\Omega)$.

(1) $f+g,\ f\cdot g,\ \alpha f \in H(\Omega)$

$\alpha \in \mathbb{C}$.

$\left((f\cdot g)(z) = f(z)\, g(z) \right)$

$(f+g)' = f' + g'$

$(fg)' = f g' + f' g$

$(\alpha f)' = \alpha f' \quad (\alpha \in \mathbb{C})$.

Proof

$$\lim_{h\to 0} \frac{(fg)(z_0+h) - (fg)(z_0)}{h}$$

$$= \lim_{h\to 0} \frac{f(z_0+h)\, g(z_0+h) - f(z_0) g(z_0)}{h}$$

$$= \lim_{h\to 0} \frac{f(z_0+h)\left[g(z_0+h) - g(z_0)\right] + g(z_0)\left[f(z_0+h) - f(z_0)\right]}{h}$$

$$= f(z_0)\, g'(z_0) + f'(z_0)\, g(z_0)$$

(ii) $f \in \mathcal{H}(\Omega)$

$\Rightarrow f$ continuous on Ω

$\left(\lim_{h \to 0} f(z_0+h) = f(z_0) \quad \forall z_0 \in \Omega \right)$

Proof

$$\lim_{h \to 0} \left[f(z_0+h) - f(z_0) \right]$$

$$= \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \cdot h$$

$$= f'(z_0) \cdot 0 = 0$$

(iii) $f, g \in \mathcal{H}(\Omega)$

$+ \; g(z) \neq 0 \;\; \forall z \in \Omega$

$\Rightarrow \frac{f}{g} \in \mathcal{H}(\Omega)$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

(iv) $\Omega \xrightarrow{f} U \xrightarrow{g} \mathbb{C}$

$(\subset \mathbb{C})$

$g \circ f$ is holomorphic & the derivative

$$(g \circ f)'(z) = g'(f(z)) f'(z).$$

Examples

① $f(z) = z^n$ $\quad n \in \mathbb{N}$, $n \geq 1$.

holomphic on $\mathbb{C} = \Omega$

$$f'(z) = n z^{n-1}$$

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

$a_i \in \mathbb{C}$

- polynomials are holomphic on $\mathbb{C}$.

② $f(z) = \frac{1}{z^n} \in \mathcal{H}(\mathbb{C} \setminus \{0\})$

$n \geq 1$, $n \in \mathbb{N}$

$$f'(z) = -\frac{n}{z^{n+1}}.$$

③ $f(z) = \bar{z}$ – not complex diff at any point

$= x - iy$

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} = \lim_{h \to 0} \frac{\bar{z}_0 + \bar{h} - \bar{z}_0}{h}$$

$$= \lim_{h \to 0} \frac{\bar{h}}{h}$$ – does not exist

④ $f(z) = |z|^2 = (x^2+y^2)$

$= z\bar{z}$

$z_0 \in \mathbb{C}$

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

$$= \lim_{h\to 0} \frac{(\bar{z}_0+\bar{h})(z_0+h) - z_0\bar{z}_0}{h}$$

$$= \lim_{h\to 0} \frac{\bar{z}_0 h + \bar{h} z_0 + h\bar{h}}{h}$$

– exists only for $\underline{z_0 = 0}$

⑤ $f(z) = \dfrac{z-1}{(z+2)(2z+i)}$

$\Omega \quad \in H(\Omega)$

$\Omega = \mathbb{C} \setminus \left\{-2, -\frac{i}{2}\right\}$

Necesary condition for a function to be holomphic

$$f(z) : \Omega \longrightarrow \mathbb{C}$$

Sxmple $f(z) = z^2$

$= (x+iy)^2$

$= (x^2-y^2) + 2ixy$

$= \underline{u(x,y)} + i\,\underline{v(x,y)}$

$$f(z) = u(x,y) + i\,v(x,y)$$

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$

$$= \lim_{\substack{h \to 0 \\ \shortparallel \\ h_1 + ih_2}} \frac{u(x+h_1, y+h_2) + i\,v(x+h_1, y+h_2) - u(x,y) - i\,v(x,y)}{h_1 + ih_2}$$

letting $h \to 0$ along x-axis.

$$f'(z) = \lim_{h_1 \to 0} \frac{u(x+h_1, y) + i\,v(x+h_1, y) - u(x,y) - i\,v(x,y)}{h_1}$$

$$= u_x(x,y) + i\,v_x(x,y)$$

$h \to 0$ along y-axis

$$f'(z) = \frac{1}{i}(u_y + i\,v_y) = v_y - i\,u_y$$

Company

$u_x = v_y$

$u_y = -v_x$

Cauchy Riemann equations