MSO 202A: Complex Variables
August-September 2022
Assignment-4

Throughout C_R will denote the circle of radius R around origin, oriented counterclockwise. and $C_1 = C$.

1. (**T**)Show that
$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx = \pi/2$$

2. Suppose $f(z)$ is defined by the integral
$$f(z) = \int_{C_3} \frac{2\xi^2 + 7\xi + 1}{\xi - z} d\xi.$$
Find $f'(1+i)$

3. Compute $\int_{C_4} \frac{z}{z^2+4} dz$ where C_4 is the circe $|z| = 4$ oriented anticlockwise.

4. (**T**)Suppose that $f = u + iv$ is an entire function and u is bounded (or v is bounded). Show that f is constant.

5. Using Liouville's theorem, conclude that $\sin z, \cos z$ are not bounded functions.

6. (**T**)Suppose that $f = u + iv$ is an entire function and $|f(z)| < |z|^n$ for some $n \geq 0$ and for all sufficiently large $|z|$. Show that f is a polynomial.

7. Suppose that $f = u + iv$ is an entire function and u(or v) is a polynomial. Then show that f is a polynomial.

8. Show that if u is a bounded harmonic function on $\mathbb{C}$ then u is constant.

9. (**T**)Let τ be a complex number which is not real. Suppose that f is an entire function such that $f(z+1) = f(z)$ and $f(z+\tau) = f(z)$. Then show that f is a constant. (This exercise says that a doubly periodic entire function is constant.)

10. Let f be an entire function satisfying $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. Show that f is constant.

11. (**T**)Suppose that $f\colon \mathbb{D} \to \mathbb{C}$ is anlytic on unit disc $\mathbb{D} = \{z : \ |z| < 1\}$. Show that $|f'(0)| \leq d/2$, where $d = \sup\limits_{z,w\in\mathbb{D}} |f(z) - f(w)|$ is the diameter of the image of f.

12. (**T**)Let Ω be a bounded open subset of $\mathbb{C}$ and $f\colon \Omega \to \Omega$ is a holomorphic function. Prove that if there exists a point $a \in \Omega$ such that $f(a) = a$ and $f'(a) = 1$ then f is linear.

13. Let f be anlytic on a region Ω and let C be a circle with interior containd in Ω. For any $a \in \Omega$ not on C show that

$$\int_C \frac{f'(\xi)}{(\xi - a)} d\xi = \int_C \frac{f(\xi)}{(\xi - a)^2} d\xi$$

14. (a) If $f(z)$ is a holomorphic inside and on a circle C containng a prove that

$$f(a)^n = \frac{1}{2\pi i} \int_C \frac{f(z)^n}{(z-a)} dz.$$

   (b) Use (a) to show that $|f(a)|^n \leq LM^n/(2\pi D)$ where D is the distance of a from C, L is the length of C and M is the maximum value of $|f(z)|$ on C.

   (c) Use (b) to show that $|f(a)| \leq M$. In other words, the maximum value of $|f(z)|$ is obtained on the boundary. This result is known as Maximum Modulus Principle.

   (d) The maximum modulus value of $f(z) = 1/z$ on unit circle is 1, yet $|f(1/2)| = 2$. Expalin why this does not contradict (c).

15. This exercise gives a generalization of Goursat's and Cauchy's theorem.

   Let T be a triangle whose interior is contained in an open set Ω of $\mathbb{C}$. Suppose that $f : \Omega \to \mathbb{C}$ is a continuous function which is holomorphic on Ω in except possibly at a point z_0. Prove that

$$\int_T f(z) dz = 0.$$

16. Let $\mathbb{D}$ be an open disc and $f \colon \mathbb{D} \to \mathbb{C}$ be a continuous function which is holomorphic on $\mathbb{D} \setminus \{z_0\}$ for some fixed $z_0 \in \mathbb{D}$. Then prove that f has a primitive on $\mathbb{D}$.

   (Remark: Hence we conclude that: Let $f : \Omega \to \mathbb{C}$ is a continuous function on an open set Ω and anlytic on $\Omega \setminus \{z_0\}$ where $z_0 \in \Omega$. Then show that f is analytic on Ω.)