Cauchy's Theorem

Ω - convex domain.

f is holomorphic on Ω

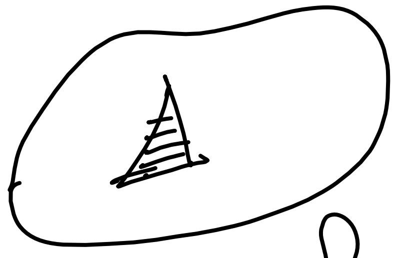
$\Rightarrow$ f admits a primitive on Ω.

$\Rightarrow$ $\int_\gamma f\,dz = 0$ for any closed curve γ in Ω.

$\Uparrow$

Goursat Thm

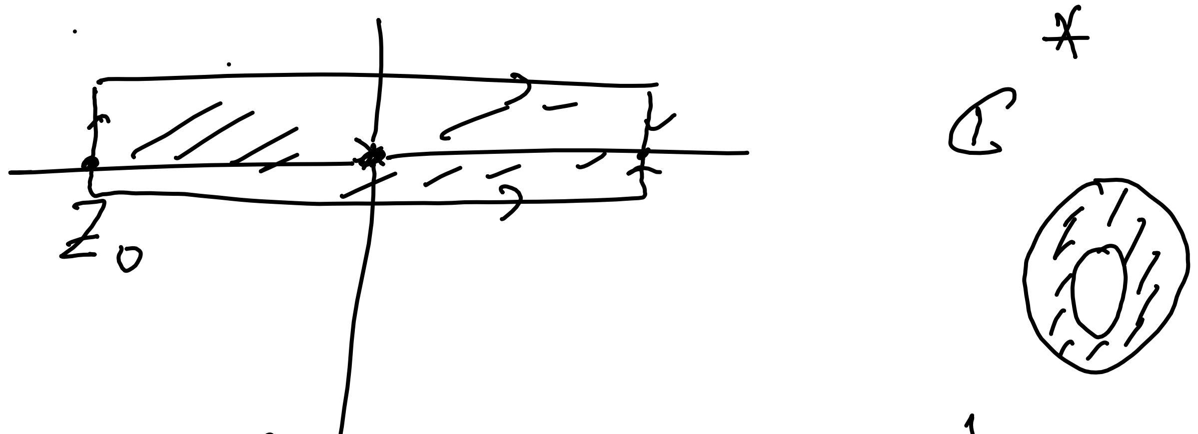
$$\int_T f(z)\,dz = 0 \qquad f \in H(\Omega).$$



Ω

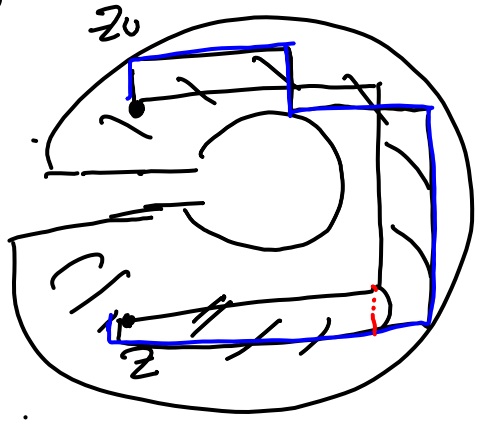
note Cauchy's Th does not hold for $\mathbb{C}^* = \Omega$

$$f(z) = \frac{1}{z} \in H(\Omega)$$

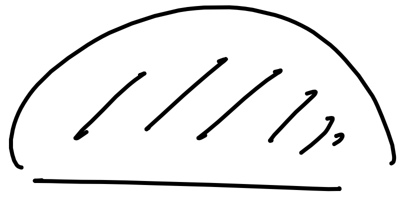


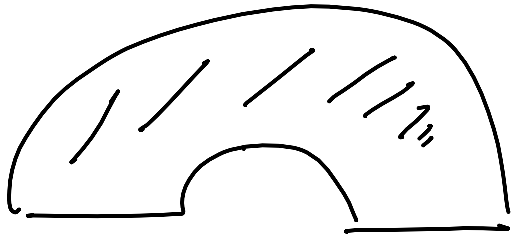
- $F(z) = \int_{\gamma_z} f(z)\,dz$ — not well defined
- since $\mathbb{C}^*$ has a 'hole'
- If Ω 'does not has a hole' then Cauchy's Th holds on Ω.

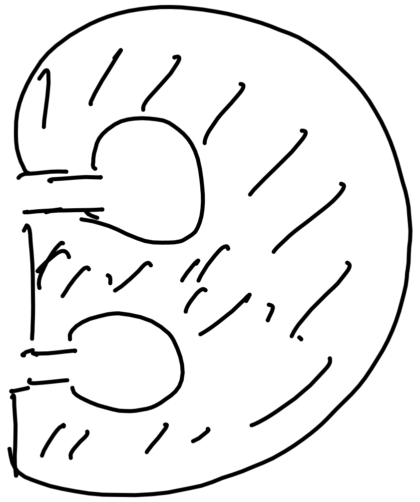
Examples

(1) 

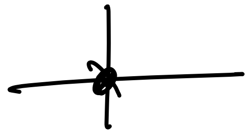
Simply connected = domain without hole

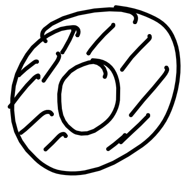
(2) 

(3) 

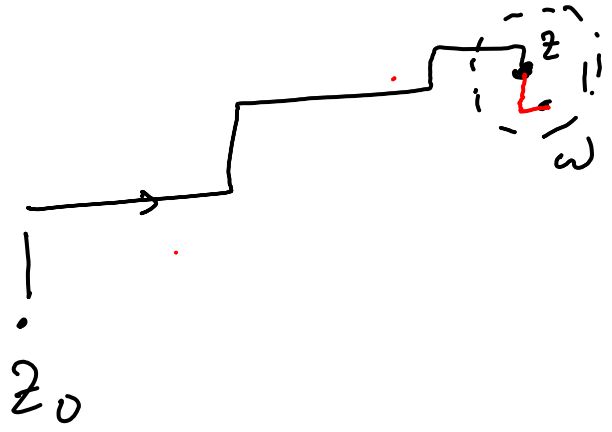
(4) 

not simply connected

 $= \mathbb{C}^*$

 annulus

$\mathbb{C} \setminus \{p, q\}$ not simply connected



$$\lim_{w \to z} \frac{F(w) - F(z)}{w - z} \to f(z)$$

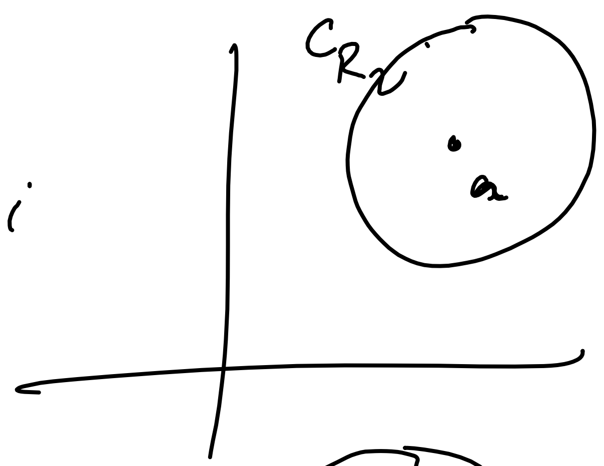
upshot

Cauchy's Theorem holds for simply connected domains.

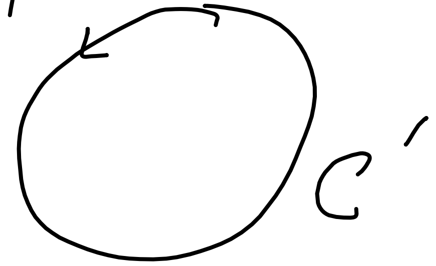
note A harmonic function on a simply connected domain admits a harmonic conjugate. ∎

Example

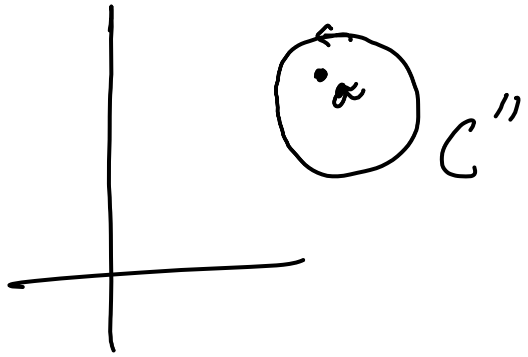
$$\int_{C_R} \frac{1}{z-a}\,dz = 2\pi i$$

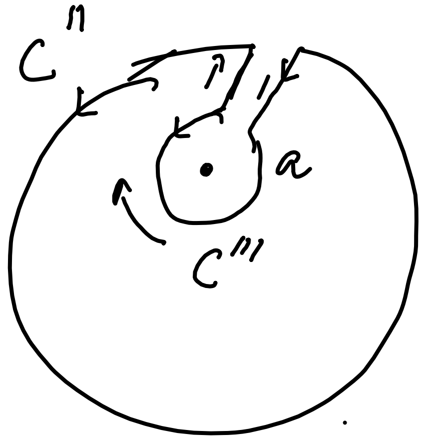


$$\int_{C'} \frac{1}{z-a}\,dz = 0$$



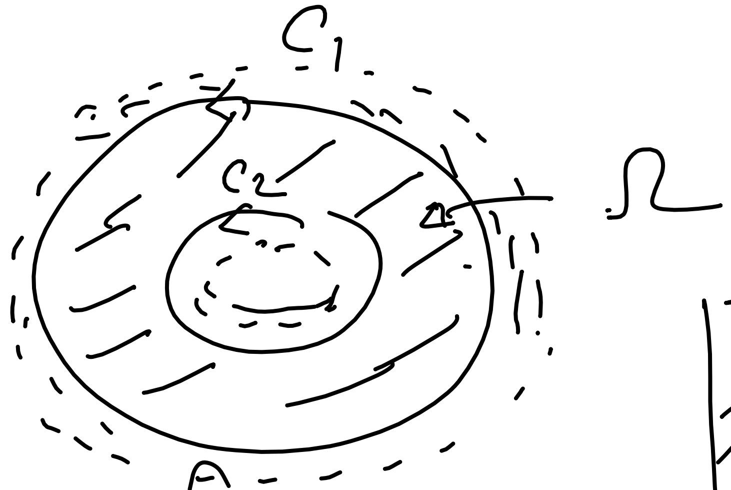
$$\int_{C''} \frac{1}{z-a}\,dz \;?\; = 2\pi i$$





$$\int_{C''} \frac{dz}{z-a} = \int_{C'''} \frac{1}{z-a}\,dz = \underline{2\pi i}$$

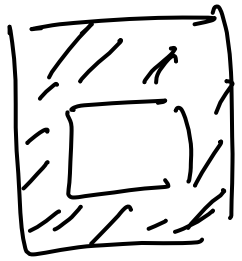
Summary



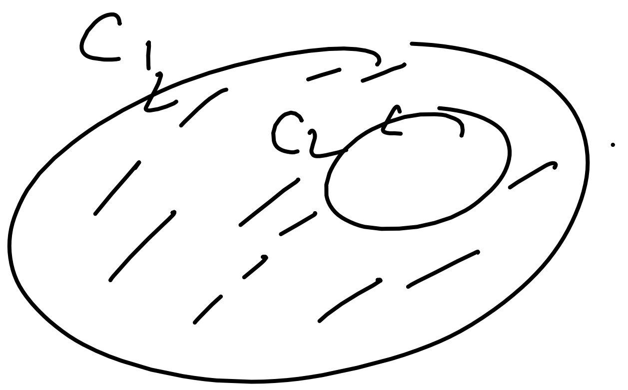
$f \in H(\Omega)$

Then

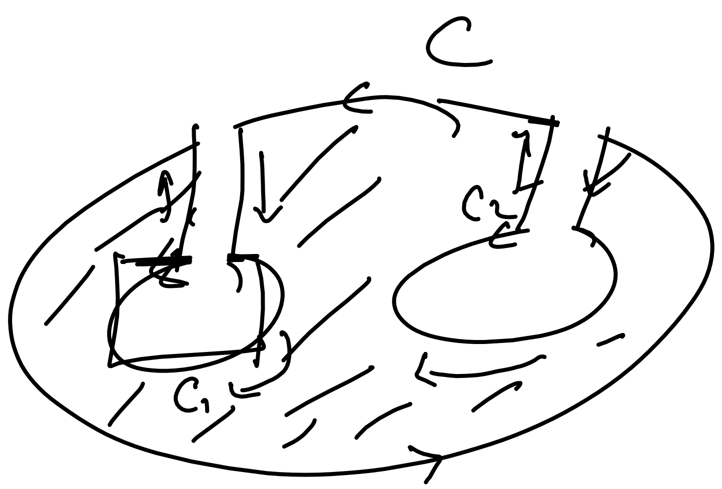
$$\int_{C_1} f = \int_{C_2} f$$

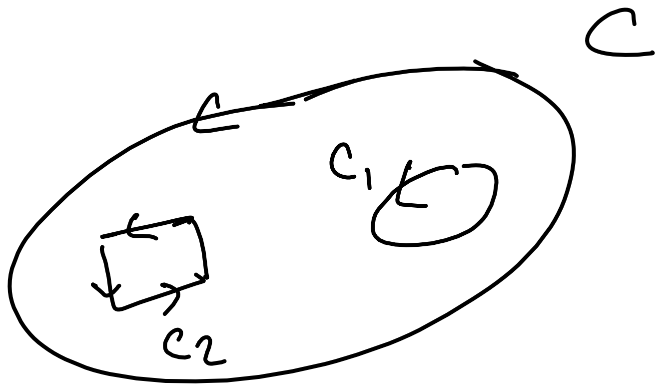


$$\int_{C_1} f = \int_{C_2} f$$



$$\int_{C} f = \int_{C_1} f + \int_{C_2} f$$





Remark $f \in H(\Omega \setminus \{p\})$

$+ \; f \in C(\Omega)$

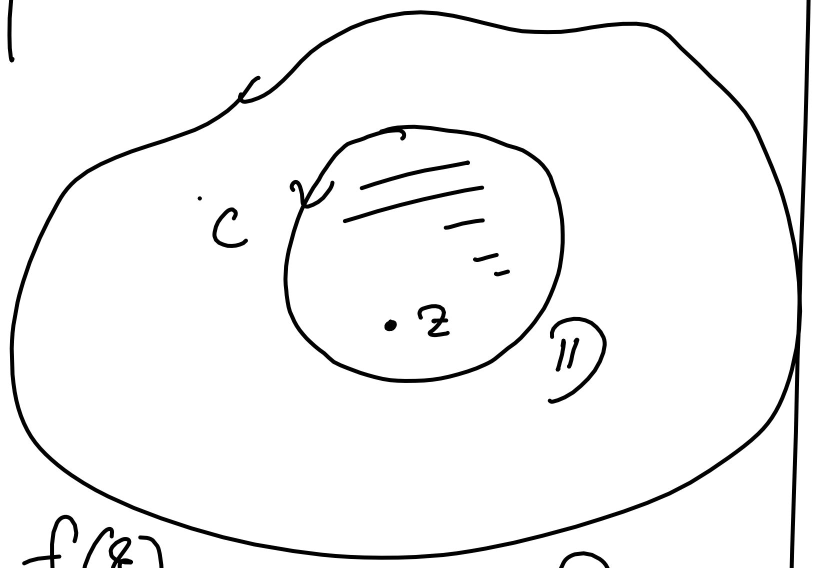
, Ω simply connected.

$$\Rightarrow \int_C f\,dz = 0$$

C - closed curve in Ω.

Cauchy's integral formula

f is holomorphic on an open set Ω.



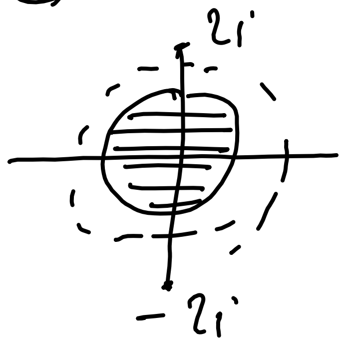
$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z}\, d\xi \qquad \Omega$$

$z \in D$.

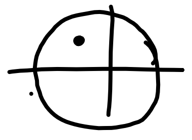
$D =$ disc

Examples

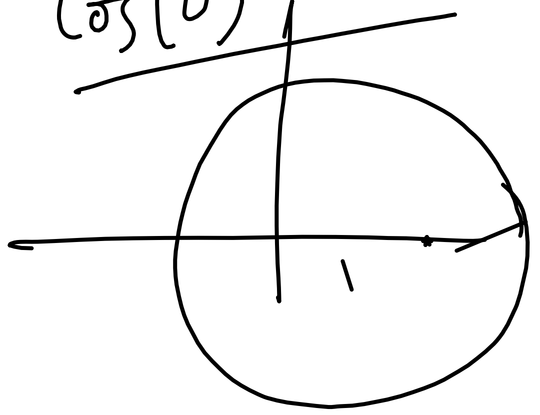
① $$\int_{|z|=1} \frac{e^{iz^2}}{z^2+4}\, dz = 0$$



② $$\int_{|z|=1} \frac{1}{z-\alpha}\, dz = \begin{cases} 2\pi i & |\alpha|<1 \\ 0 & |\alpha|>1 \end{cases}$$



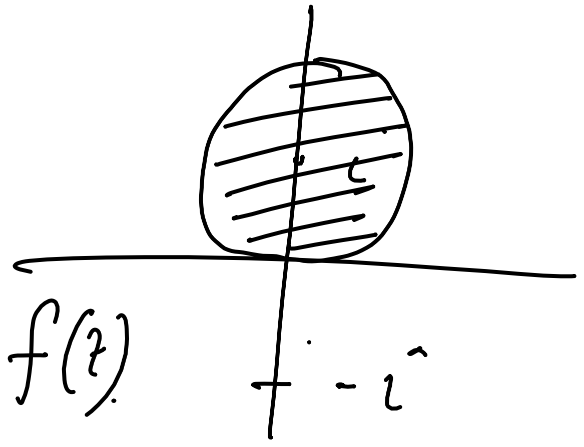
③ $$\int_{|z-4|=5} \frac{\cos \xi}{\xi - 0}\, d\xi = \cos(0)\cdot 2\pi i$$



$$\int \frac{z^2}{z^2+1}\, dz$$

$|z-i|=1$

$$= \int \frac{\boxed{z^2 / z+i}}{(z-i)} \qquad = f(z)$$



$|z-i|=1$

$$= 2\pi i\, f(i) = 2\pi i \frac{i^2}{2i}$$

$$= -\pi \qquad \blacksquare$$