MSO202- INTRODUCTION TO COMPLEX ANALYSIS

1. Assignment 6

(1) Find residues of the following functions at all its poles:
$$(i)\cot(\pi z) \quad (ii)\frac{z}{z^n-1} \quad (iii)\frac{z^2(z-1)^3(z-2)}{\sin^2(\pi z)}$$

(2) Evaluate $(a, b > 0)$
$$(i)\int_{\mathbb{R}}\frac{x\sin(ax)}{x^2+b^2}dx \quad (ii)\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}(|a|>|b|) \quad (iii)\int_0^{2\pi}\frac{d\theta}{1+a\sin\theta}(|a|<1)$$
$$(iv)\int_0^{2\pi}\frac{\cos(n\theta)d\theta}{a+\cos\theta}\ (a>1, n\in\mathbb{N}) \quad (v)\int_{\mathbb{R}}\frac{1}{(x-1)(x^2+4)}dx \quad (vi)\int_{\mathbb{R}}\frac{\sin^2 x}{x^2+4}dx$$

(3) Let $\phi \in (0, \pi)$ and $n \in \mathbb{N}$. Prove that
$$\int_{|z|=2}\frac{z^n dz}{1-2z\cos\phi+z^2} = \frac{\sin(n\phi)}{\sin\phi}.$$

(4) Use Argument principle to evaluate
$$(i)\int_{|z+1+i|=2}\frac{z+i}{z^2+2iz-4}dz \quad (ii)\int_{|z|=2}\frac{z+2}{z(z+1)}dz$$

(5) Use Rouche's Theorem to determine the roots of polynomial
$$(i) p(z) = z^{10}-6z^9-3z+1 \quad in \quad |z|<1 \quad (ii) z^5+6z^3+2z+10 \quad in \quad 1<|z|<3.$$

(6) Use Rouche's Theorem to prove fundamental theorem of algebra.

(7) Let g be analytic for $|z| \leqslant 1$ and $|g(z)| < 1$ for $|z| = 1$. Then prove that g has a unique fixed point in $|z| < 1$. What happens if we replace $|g(z)| < 1$ with $|g(z)| \leqslant 1$ in above condition.

(8) If $f(z)$ is analytic at a with $f'(a) \neq 0$, then f(z) is one-to-one in some neighborhood of a. Conversely, if $f(z)$ is analytic and one-to-one in a domain D, then $f'(z) \neq 0$, in D.

(9) Find the linear fractional transformation that maps the points $z_1 = -i$, $z_2 = 0$ $z_3 = i$ to $w_1 = -1$, $w_2 = i$, $w_3 = 1$.

(10) Let f be analytic on an open set D, and $f'(a) \neq 0$ for some $a \in D$. Evaluate
$$\frac{1}{2\pi i}\int_C \frac{dz}{f(z)-f(a)},$$
where C is a sufficiently small circle centered at a.

Preliminary version – September 9, 2023