MSO 202A: Complex Variables
August-September 2022
Assignment-1

1. Verify Cauchy-Riemann equation for $z^2,\ z^3$.

2. Which of the following maps are holomorphic? If so then write as a function of z.

   (a) **(T)**$P(x+\iota y) = x^3 - 3xy^2 - x + \iota(3x^2y - y^3 - y)$

   (b) $P(x+\iota y) = x^2 + \iota y^2$

   (c) $P(x+\iota y) = 2xy + \iota(y^2 - x^2)$

3. Suppose that $f = u + \iota v$ is analytic on region Ω and $f^{'}(z) \neq 0$ for all $z \in \Omega$. Show that the family of level curves $u(x,y) = c_1, v(x,y) = c_2$ are orthogonal to each other. Verify it for the example of $f(z) = z^2$ by drawing pictures. What happens in this case to the level curves $u(x,y) = 0, v(x,y) = 0$?

4. (a) **(T)**Let z, w be two complex numbers such that $\overline{z}w \neq 1$. Prove that

   $$|\frac{w-z}{1-\overline{w}z}| < 1 \text{ if } |z| < 1 \text{ and } |w| < 1,$$

   and also that

   $$|\frac{w-z}{1-\overline{w}z}| = 1 \text{ if } |z| = 1 \text{ or } |w| = 1,$$

   (b) **(T)**Prove that for a fixed $w \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the mapping

   $$F : z \mapsto \frac{w-z}{1-\overline{w}z}$$

   satisfy the following conditions:

   - F maps $\mathbb{D}$ to itself and $F : \mathbb{D} \to \mathbb{D}$ is holomorphic.
   - F interchanges 0 and w, i.e., $F(0) = w$ and $F(w) = 0$.
   - $|F(z)| = 1$ if $|z| = 1$.
   - F is bijective.

5. Suppose that $f = u + \iota v$ is holomorphic on an open connected set Ω. Prove that in each one of the following cases f is constant.

   (a) u is constant.

   (b) v is constant.

   (c) **(T)**$|f|$ is constant.

6. Suppose $f = u + \iota v \in \mathcal{H}(\mathbb{C})$ satisfy $u(x,y) = u(-y,x)$. Show that $f(z) = f(\iota z)$ for all $z \in \mathbb{C}$.

7. For $\Omega \subseteq \mathbb{C}$ define $\tilde{\Omega} = \{z \in \mathbb{C} : \overline{z} \in \Omega\}$ (It is the reflection of Ω about x-axis).

   (a) For $\Omega = \{z \in \mathbb{C} : |z - i| < 1\}$, draw $\tilde{\Omega}$.

   (b) If Ω is open and connected then so is $\tilde{\Omega}$.

   (c) If Ω is open and $f \in \mathcal{H}(\Omega)$ then show that $g \in \mathcal{H}(\tilde{\Omega})$ where $g(z) = \overline{f(\overline{z})}$. Find $g'(z)$.

8. **(T)** Define differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ by setting:

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - \iota\frac{\partial}{\partial y}); \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + \iota\frac{\partial}{\partial y}).$$

   Show that $f = u + \iota v$ satisfy CR-equations if and only if $\partial f/\partial \overline{z} = 0$. Moreover, if f is holomorphic, then $f'(z) = \partial f/\partial z$. Further show that for a real valued function $u(x, y)$ with continuous second order partial derivatives,

$$4\frac{\partial^2 u}{\partial z \partial \overline{z}} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

9. **(T)** Let $f(x + \iota y)$ be a polynomial with complex coefficients in x and y. Show that f is holomorphic if and only if it can be expressed as a polynomial in the single variable z.

10. Consider the function

$$f(z) = \begin{cases} \frac{xy(x+\iota y)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

   Show that f satisfies the Cauchy-Riemann equations at the origin $z = 0$, yet f is not complex differentiable at the origin.

11. **(T)** Show that the set of natural numbers $\mathbb{N}$ can not be partitioned into finite number of subsets that are in arithmetic progression with distinct common difference.

12. **(T)** Show that it is impossible to define a total ordering on $\mathbb{C}$. In other words, there does not exist a relation $\succ$ between complex numbers so that:

   - For any two $z, w \in \mathbb{C}$ one and only one of the following is true: $z \succ w,\ w \succ z,\ z = w$.
   - For all $z_1, z_2, z_3 \in \mathbb{C}$ the relation $z_1 \succ z_2$ implies $z_1 + z_3 \succ z_2 + z_3$.
   - For all $z_1, z_2, z_3 \in \mathbb{C}$ with $z_3 \succ 0$ the relation $z_1 \succ z_2$ implies $z_1 z_3 \succ z_2 z_3$.

13. Determine if there exist an analytic function with u as real part. (a)**(T)** $u = x^2y^2$. (b) $u = \sin x \cosh y$. (c) $u = x/(x^2 + y^2)$ (d)**(T)** $u = xy + 3x^2y - y^3$