Department of Mathematics and Statistics
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MSO202 Assignment 2 Solutions
Introduction To Complex Analysis

The problems marked (T) need an explicit discussion in the tutorial class. Other problems are for enhanced practice.

Note: For uniformity, use *ln x* for natural logarithm of real variable x, l*og z* for logarithmic function of complex variable z and *Log z* as the Principal Branch of *log z*.

1. **(T)** Show that if $\operatorname{Re} z_1 > 0$ *and* $\operatorname{Re} z_2 > 0$, *then* $Log(z_1 z_2) = Log(z_1) + Log(z_2)$.

Solution: $Log(z_1 z_2) = \ln(r_1 r_2) + i\, Arg(z_1 z_2)$.

Re z_1, Re $z_2 > 0$

$$\Rightarrow -\frac{\pi}{2} < Arg\, z_1,\ Arg\, z_2 < \frac{\pi}{2} \Rightarrow -\pi < Arg\, z_1 + Arg\, z_2 < \pi \Rightarrow Arg(z_1 z_2) = Arg\, z_1 + Arg\, z_2$$

$$\Rightarrow Log(z_1 z_2) = Log(z_1) + Log(z_2)$$

2. Express the following complex numbers in the form a + i b:

   $(i)\ \log(Log\, i)$ $\quad (ii)\ \sinh(e^i)$ $\quad$ **(T)** $(iii)\ (-3)^{\sqrt{2}}$ $\quad (iv)\ 1^{-i}$

Solution:

$$(i)\ \log(Log\, i) = \log(i\pi/2) = \log(\pi/2) + i(\frac{\pi}{2} + 2n\pi)$$

$$\begin{aligned}(ii) \sinh(e^i) &= \sinh(\cos 1 + i \sin 1) \\ &= \sinh(\cos 1)\cos(\sin 1) - i\cosh(\cos 1)\sin(\sin 1)\end{aligned}$$

$$\begin{aligned}(iii)(-3)^{\sqrt{2}} &= \exp(\sqrt{2}\log(-3)) \\ &= \exp\{\sqrt{2}(\ln 3 + i(2n+1)\pi)\} \\ &= 3^{\sqrt{2}}\{\cos(\sqrt{2}(2n+1)\pi) + i\sin(\sqrt{2}(2n+1)\pi)\}\end{aligned}$$

$$(iv)\ i^{-i} = \exp(\log(i^{-i})) = \exp\{-i(Log\, i + i\ 2n\pi)\} = e^{\frac{\pi}{2}+2n\pi},\ n = 0, \pm 1, \pm 2, \ldots$$

3. **(T)** Prove that $(a)|\sinh(\operatorname{Im} z)| \le |\sin(z)|$ (b) $|\cos(z)| \le \cosh(\operatorname{Im} z)$. Deduce that $|\sin z|$ *and* $|\cos z|$ tend to ∞ *as* $z \to \infty$ in either of the angles $\delta \le \arg z \le \pi - \delta,\ \pi + \delta < \arg z < 2\pi - \delta,$ *where* $0 < \delta < \pi/2$. (b) Find the points on the square region $-\pi \le \operatorname{Re} z \le \pi,\ -\pi \le \operatorname{Im} z \le \pi$ at which $|\cos z|$ takes its maximum value.

Solution: (a) sin z = sin x cosh y + i cos x sinh y, $z = x + iy$

$$\Rightarrow |\sin z|^2 = \sin^2 x + \sinh^2 y$$
$$\Rightarrow (\sinh y)^2 \le |\sin z|^2 \le 1 + \sinh^2 y = \cosh^2 y$$
$$\Rightarrow |\sinh y| \le |\sin z| \le \cosh y. \qquad (*)$$

Similarly, $|\cos z|^2 = \cos^2 x + \sinh^2 y \Rightarrow |\sinh y| \le |\cos z| \le \cosh y \qquad (**)$

Now, for $\delta \le \arg z \le \pi - \delta$ *or* $\pi + \delta < \arg z < 2\pi - \delta$, *with* $0 < \delta < \pi/2$,

$\Rightarrow y \to \infty$ *as* $z \to \infty$(since, $\arg z \ne 0, \pi$) $\Rightarrow |\sinh y|, |\cosh y| \to \infty$ as $z \to \infty$

Therefore, by (*) *and* (**), $|\sin z| \to \infty$ *and* $|\cos z| \to \infty$ *as* $z \to \infty$

(b) As in (a), $|\cos z|^2 = \cos^2 x + \sinh^2 y$. Now cos² x is maximum at $x = -\pi, 0, \pi$ for $-\pi \le x \le \pi$ and sinh² y is maximum at $y = \pi, -\pi$ for $-\pi \le y \le \pi$. Consequently, $|\cos z|$ takes its maximum value on $-\pi \le \operatorname{Re} z \le \pi, -\pi \le \operatorname{Im} z \le \pi$ at $z = \pm i\pi, z = \pm\pi(1 \pm i)$.

4. Find the values of z for which

$(i)\ \exp(\overline{z}) = \overline{\exp(z)}\quad (ii)\ \sinh z + \cosh z = i\quad (iii)\ \cos(i\,\overline{z}) = \overline{\cos i z}$ **(T)** $(iv)\ |\cot z| = 1$

Solution:

(i) satisfied for all z (use definition)

(ii) $\sinh z + \cosh z = i \Rightarrow \exp(z) = i \Rightarrow z = i(\frac{\pi}{2} + 2n\pi).$

(iii) satisfied for all z (use definition)

(iv) $|\cot z| = 1$

$$\Rightarrow \frac{\cos^2 x + \sinh^2 y}{\sin^2 x + \sinh^2 y} = 1 \Rightarrow |\cos x| = |\sin x| \Rightarrow x = n\pi \pm \frac{\pi}{4}$$

$$\Rightarrow z = (n\pi \pm \frac{\pi}{4},\ y),\ y\ \textit{arbitrary}$$

5. Prove that

(T) $(i)\ \sin^{-1} z = -i \log i(z + \sqrt{z^2 - 1})$ $\qquad (ii)\ \cos^{-1} z = -i \log(z + \sqrt{z^2 - 1})$

$(iii)\ \tan^{-1}(z) = \frac{i}{2}\log(\frac{i+z}{i-z}) = \frac{1}{2i}\log(\frac{1+iz}{1-iz})$ $\qquad (iv)\ \cot^{-1}(z) = \frac{i}{2}\log(\frac{z-i}{z+i})$

$(v)\ \sinh^{-1}(z) = \log(z + \sqrt{z^2 + 1})$ $\qquad$ **(T)** $(vi)\ \cosh^{-1}(z) = \log(z + \sqrt{z^2 - 1})$

$(vii)\ \tanh^{-1}(z) = \frac{1}{2}\log(\frac{1+z}{1-z})$ $\qquad (viii)\ \coth^{-1}(z) = \frac{1}{2}\log(\frac{z+1}{z-1})$

Solution:

(iv) $w = \cot^{-1} z$

$$\Rightarrow \cot w = z \Rightarrow \frac{i(e^{iw}+e^{-iw})}{e^{iw}-e^{-iw}} = z \Rightarrow e^{2iw} = \log\left(\frac{z+i}{z-i}\right)$$

$$\Rightarrow w = \frac{i}{2}\log(\frac{z-i}{z+i}).$$

(vi) $w = \cosh^{-1} z$

$$\Rightarrow \frac{e^{w}+e^{-w}}{2} = z \Rightarrow e^{2w} - 2ze^{w} + 1 = 0$$

$$\Rightarrow w = \log(z+\sqrt{z^2-1})$$

The other relations follow similarly.

6. Test whether the following functions are harmonic and find their harmonic conjugates:

(T) (i) $u = x^2 - y^2 + x + y - \dfrac{y}{x^2+y^2}$

(ii) $u = \sin x \cosh y + 2 \cos x \sinh y + x^2 - y^2 + 2xy$

Solution: The harmonicity of the functions is tested routinely using the definition of harmonic functions.

(i) Obtain the harmonic conjugate v by using $v(x,y) = \int_0^y u_x(x,t)\ dt - \int_0^x u_y(s,0)\ ds.$ Or, alternatively, by CR equations

$$v_y = u_x = 2x+1+\frac{2yx}{(x^2+y^2)^2} \ \Rightarrow v = 2xy + y - \frac{x}{x^2+y^2} + g(x).$$

Now, $v_x = -u_y \Rightarrow 2y - \dfrac{y^2-x^2}{(x^2+y^2)^2} + g'(x) = 2y - 1 - \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$g'(x) = -1 \Rightarrow g(x) = -x + c$, where c is a constant.

Consequently, the required harmonic conjugate is $v = 2xy + y - \dfrac{x}{x^2+y^2} - x + c$.

(ii) Obtain the harmonic conjugate v by using $v(x,y) = \int_0^y u_x(x,t)\ dt - \int_0^x u_y(s,0)\ ds.$ Or, alternatively, by CR equations

$$v_y = u_x = \cos x \cosh y - 2\sin x \sinh y + 2(x+y) \ \Rightarrow v = \cos x \sinh y - 2\sin x \cosh y + 2xy + y^2 + g(x).$$

Now, $v_x = -u_y \Rightarrow$

$$-\sin x \sinh y - 2\cos x \cosh y + 2y + g'(x)$$
$$= -(\sin x \sinh y - 2\cos x \cosh y - 2y + 2x)$$

$g'(x) = -x \Rightarrow g(x) = -x^2 + c.$

Consequently, the required harmonic conjugate is $v = \cos x \sinh y - 2\sin x \cosh y + 2xy + y^2 - x^2 + c$.

7. **(T)** Using that $u(x,y)=3x^3+3x^2y-9xy^2-y^3$ is a homogenous harmonic function, determine an analytic function, as a function of z, whose real part is $u(x,y)$.

Solution: The given u is a homogeneous harmonic function of degree 3. Therefore, it's conjugate harmonic function is given by

$$v=\frac{1}{m}(yu_x-xu_y)=\frac{1}{3}[y(9x^2+6xy-9y^2)-x(3x^2-18xy-3y^2)]$$
$$=[y(3x^2+2xy-3y^2)-x(x^2-6xy-y^2)] \;=-3y^3-x^3+9x^2y+3xy^2 \Rightarrow f(z)=u+i\,v=(3-i)z^3.$$

8. For each of the following functions find a function f(z) such that f(z) = $R\,e^{i\varphi}$ is analytic:
   (T) (i) $R=r^2e^{r\cos\theta}$ (ii) $\varphi=r^2\cos\theta\sin\theta$.

Solution: $f(z)=R\,e^{i\varphi}=R\cos\varphi+iR\sin\varphi=u+iv\;(say)$

(i) $R=r^2e^{r\cos\theta}=(x^2+y^2)e^x,\quad u^2+v^2=R^2$

$$\Rightarrow uu_x+vv_x=RR_x,\; vu_x-uv_x=RR_y \Rightarrow u_x=\frac{-uRR_x-vRR_y}{-R^2},\; v_x=\frac{uRR_y-vRR_x}{-R^2}$$

$$\Rightarrow f'(z)=u_x+iv_x=\frac{1}{R}R_x(u+iv)-\frac{i}{R}R_y(u+iv)$$

$$\Rightarrow \frac{f'(z)}{f(z)}=\frac{1}{R}(R_x-iR_y)$$

$$=\frac{1}{(x^2+y^2)e^x}(2xe^x+(x^2+y^2)e^x-i\;2ye^x)=1+\frac{2}{z}$$

$$\Rightarrow f(z)=z^2e^z$$

(ii) $\varphi=r^2\cos\theta\sin\theta=xy=\tan^{-1}(\frac{v}{u})$

$$\Rightarrow \varphi_x=\frac{-vu_x+uv_x}{u^2+v^2},\; \varphi_y=\frac{uu_x+vv_x}{u^2+v^2}$$

$$\Rightarrow u_x=\frac{(u^2+v^2)(v\varphi_x-u\varphi_y)}{-(v^2+u^2)},\; v_x=\frac{(u^2+v^2)(-v\varphi_y-u\varphi_x)}{-(v^2+u^2)}$$

$$\Rightarrow f'(z)=u_x+iv_x=(\varphi_y+i\varphi_x)(u+iv)$$

$$\Rightarrow \frac{f'(z)}{f(z)}=(\varphi_y+i\varphi_x)=x+iy=z\Rightarrow f(z)=c\exp(z^2/2)$$

9. **(T)** If f(z) is an analytic function, determine the domain, if any, in which the following functions are harmonic?:
   (i) arg f(z) (ii) $|f(z)|$ (iii) ln $|f(z)|$.

Solution: The functions in (i) and (iii) are imaginary and real parts of the function log f(z) analytic in the region $D=Complex\,Plane-\{suitable\;curves\;\;joining\;zeros\;of\;\;f(z)\;to\;\infty\}-\{z: f(z)=0\}$, therefore these functions are harmonic in *D*. The function $|f(z)|$ need not be harmonic in any domain, take for example f(z) = z^2.

10. If the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R ($0 < R < \infty$), find the radius of convergence of each of the following (k being a fixed natural number):

(T) $(i)\ \sum_{n=0}^{\infty} a_n z^{kn}$ $\qquad (ii)\ \sum_{n=0}^{\infty} n^k a_n z^n$ $\qquad (iii)\ \sum_{n=0}^{\infty} \frac{a_n}{\lfloor n} z^n.$

Solution: Let R* be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^{\lambda_n}$ Using $R^* = \frac{1}{L} = \frac{1}{L^*}$, provided the limits $L = \lim_{n\to\infty} |a_n|^{1/\lambda_n}$ and $L^* = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|^{1/(\lambda_{n+1}-\lambda_n)}$ exist, the radii of convergence of the given series are $(i)\ R^{1/k}$ $\qquad (ii)\ R$ $\qquad (iii)\ \infty$

11. **(T)** Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$, where $a_n = \begin{cases} 2^n & if\ \ n\ is\ even \\ 3^n & if\ \ n\ is\ odd. \end{cases}$

Solution: Use $R^* = \frac{1}{L},\ where\, L = \limsup_{n\to\infty} |a_n|^{1/n}$ and R* is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$. The sequence $|a_n|^{1/n} \to 2\ if\ \ n\ is\ even\ and\ tends\ to\ \infty$.
while $|a_n|^{1/n} \to 3\ if\ \ n\ is\ odd\ and\ tends\ to\ \infty$. Therefore, L = 3. Consequently, R* = 1/3.

12. Find the region of convergence for each of the following power series:

$(i)\sum_{n=0}^{\infty} \frac{z^{2n+1}}{\lfloor n}$ $\qquad$ **(T)** $(ii)\ \sum_{n=0}^{\infty} \frac{\lfloor 3n}{(\lfloor n)^3}(z+\pi i)^n$ $\ (iii)\ \sum_{n=0}^{\infty} (3z-2i)^{3n}$ **(T)** $(iv)\sum_{n=0}^{\infty} \frac{1}{\lfloor n} z^{n^2}$

Solution: (i) $L^* = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|^{1/(\lambda_{n+1}-\lambda_n)} = \lim_{n\to\infty} \left|\frac{1}{n+1}\right|^{1/2} = 0.$ Therefore, the radius of convergence of the given power series is ∞.Consequently, it converges in the whole complex plane

(ii) As in (i), L^* = 27. Therefore, the desired region of convergence is $|z+\pi i| < \frac{1}{27}$

(iii) $L = \lim_{n\to\infty} |a_n|^{1/\lambda_n} = \lim_{n\to\infty} \left|3^{3n}\right|^{1/3n}$ = 3. Consequently, the desired region of convergence is $\left|z - \frac{2i}{3}\right| < \frac{1}{3}$

(iv) $L^* = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|^{1/(\lambda_{n+1}-\lambda_n)} = \lim_{n\to\infty} \left|\frac{1}{n+1}\right|^{1/((n+1)^2-n^2)} = 1.$ Consequently, the desired region of convergence is $|z| < 1$.

222