MSO 203 m.

The problem

$$\begin{cases} \underline{a(x,y,u)}\, u_x + b(x,y,u)\, u_y = c(x,y,u) \\ \underline{u=g} \quad \text{on } \underline{\Gamma} - \text{IC} \end{cases} \longrightarrow (*)$$

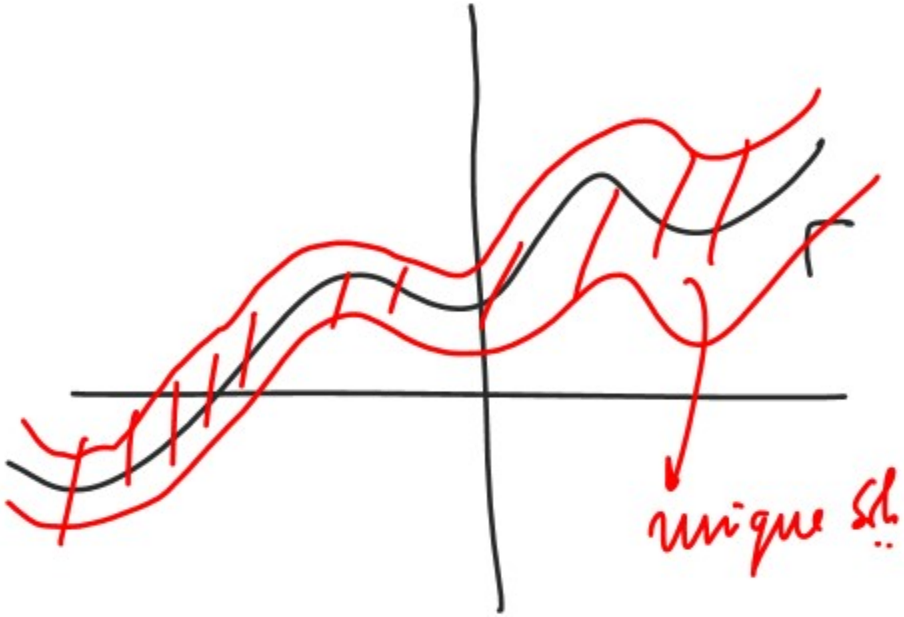
Defn The problem $(*)$ is called of "non-characteristics type" if

$$\boxed{\begin{aligned} & b(f(s), h(s), \underbrace{g(f(s),h(s))}_{\text{or, } g(s)})\, f'(s) \\ & \quad - a(f(s), h(s), g(f(s),h(s)))\, h'(s) \neq 0 \end{aligned}}$$

where $\Gamma = \{(f(s), h(s))\}$

Theorem:

If the problem $(*)$ is of "non characteristics type" then there exist an unique soln in some neighborhood of Γ.



If "non characteristics "type" cond: is not satisfied then all possibilities are open.

— α —

Ex: $u_y = 0$

$u(x,0) = \tilde{g}(x)$

Recall: $u(x,y) = \tilde{g}(x)$ was the uniqu sln.

$a \equiv 0$, $b \equiv 1$, $c \equiv 0$ | $f(s) = s$, $h(s) = 0$, $g(s) = \tilde{g}(s)$

$$b f'(s) - a h'(s)$$

$$= 1 \cdot 1 - 0 \cdot 0 = 1 \neq 0.$$

(2) $u_y = 0$

$u(0,y) = \sin y.$ $\rightarrow$ No sln

$a \equiv 0$, $b \equiv 1$, $c = 0$ | $f(s) = 0$, $h(s) = s$, $g(s) = \sin s$

$\overset{b}{1} \cdot \overset{f'(s)}{0} - 0 \cdot \cdot = 0$

If you change $u(0,y) = 5$ $\Rightarrow$ Inf-many sln.

Classification of Second order linear PDEs.

The general form of 2nd order linear PDE is:

$$\boxed{A(x,y)\,u_{xx} + B(x,y)\,u_{xy} + C(x,y)\,u_{yy} + D(x,y)\,u_x + E(x,y)\,u_y + F(x,y)\,u = G(x,y)} \longrightarrow (*)$$

∴ Note: No u_{yx} term ??

Ans as we assume u is twice diff and the second derivative continuous

Mixed derivative thm ⇒ $u_{xy} = u_{yx}$

— Shankara Rao
Introduction to Partial differential Eqn.

— x —

(*) is called Elliptic if

$(B^2 - 4AC)(x,y) \le 0, \ \forall (x,y)$

$\boxed{\text{Parabolic}}$ $(B^2 - 4AC)(x,y) = 0$

$\boxed{\text{Hyperbolic}}$ $(B^2 - 4AC)(x,y) > 0$

$A u_{xx} + B u_{xy} + C u_{yy} + \dots\dots$
$= 0$

$ax^2 + bxy + cy^2 + dx + ey = f$
conic section

$b^2 - 4ac < 0$, $x^2 + y^2 = 5$
Elliptic

$b^2 - 4ac = 0$ $y^2 = 4ax$

$b^2 - 4ac > 0$, $x^2 - y^2 = a^2$ $a = 1$ $b = 0$ $c = 1$
$xy = 5$

Ex: 1. $\underline{\Delta u}$
$u_{xx} + u_{yy}$ $+$ lower order $= 0$ liner

Type - Laplace Eq.

Elliptic type | $\Delta u = f$

Ex:2 $u_t - u_{xx} = 0$ Parabolic
(Heat Eq.)

Ex-3 wave Eq.
$u_{tt} - u_{xx} = f(x,t)$ (Hyperbolic)

$$A u_{xx} + B u_{xy} + C u_{yy} + \cdots = G$$

Reduction to cannonical form

Idea is to reduce any general 2nd order linear PDE to Easy form.

Idea: $\boxed{\text{CHANGE OF VARIABLE}}$

Let us Introduce a "general" change of Variable

$$\xi = \xi(x, y)$$

$$\eta = \eta(x, y)$$

$$\boxed{A u_{xx}(x,y) + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G} \quad (A)$$

$$v(\xi, \eta) = u(x, y)$$

$$\boxed{\frac{\partial u}{\partial x}} = v_{\xi}\,\xi_x + v_{\eta}\,\eta_x \rightarrow \text{chain rule}$$

$$u_{xx} = \xi_x\left(v_{\xi\xi}\,\xi_x + v_{\xi\eta}\,\eta_x\right) + v_{\xi}\,\xi_{xx} + \eta_x\left(v_{\eta\eta}\,\eta_x + v_{\eta\xi}\,\xi_x\right) + v_{\eta}\,\eta_{xx}.$$

use $\boxed{v_{\xi\eta} = v_{\eta\xi}}$

M.D. [illegible]

Similarly

$$u_{xy} = v_{\xi\xi}\,\xi_x\xi_y + v_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + v_{\eta\eta}\,\eta_x\eta_y + v_{\xi}\,\xi_{xy} + v_{\eta}\,\eta_{xy}$$

$$u_{yy} = v_{\xi\xi}\,\xi_y^2 + 2v_{\xi\eta}\,\xi_y\eta_y + v_{\eta\eta}\,\eta_y^2 + v_{\xi}\,\xi_{yy} + v_{\eta}\,\eta_{yy}.$$

Substitute all this in A

$$\bar{A} v_{\xi\xi} + \bar{B} v_{\xi\eta} + \bar{C} v_{\eta\eta} + \bar{D} v_{\xi} + \bar{E} v_{\eta} + \bar{F} v = \bar{G}$$

where

$$\bar{A} = A \xi_x^2 + B \xi_x \xi_y + C \xi_y^2$$

$$\bar{B} = 2A \xi_x \eta_x + B(\xi_x \eta_y + \xi_y \eta_x) + 2C \xi_y \eta_y$$

$$\bar{C} = A \eta_x^2 + B \eta_x \eta_y + C \eta_y^2$$

$$\bar{D} = A \xi_{xx} + B \xi_{xy} + C \xi_{yy} + D \xi_x + E \xi_y$$

$$\bar{E} = A \eta_{xx} + B \eta_{xy} + C \eta_{xy} + D \eta_x + E \eta_y$$

$$\bar{F} = F, \quad \bar{G} = G.$$