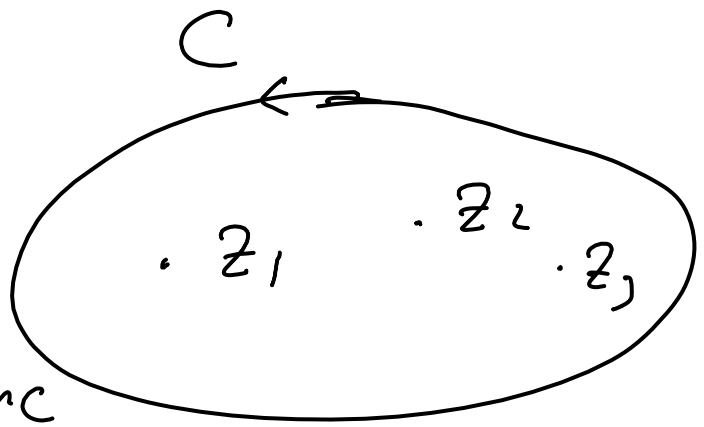
Recall Residue Theorem

$$\int_C f(z)\,dz = 2\pi i \left(\text{Sum of residues of } f \text{ inside } C\right)$$



C - simple closed curve

f is holomorphic on & inside C, except for finitely many isolated singularities.

C - oriented counterclockwise.

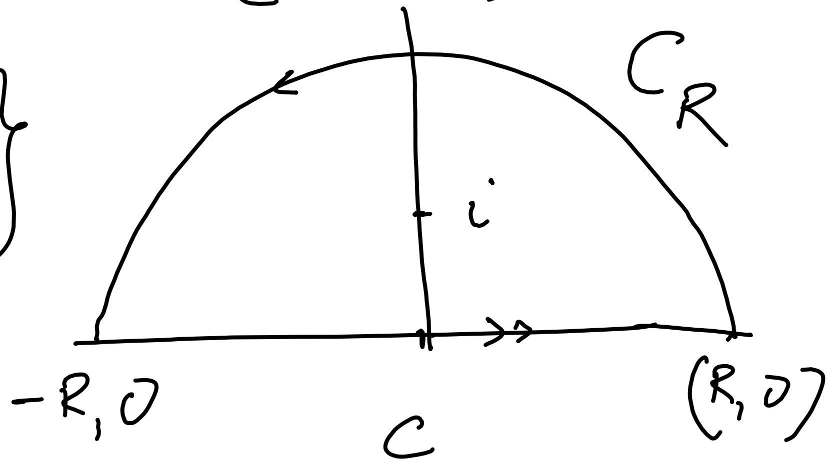
Example - 1

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx.$$

Sol

$$f(z) = \frac{1}{(1+z^2)^2}$$

Singularities of $f = \{\pm i\}$



$$C = C_R + [-R \quad R]$$

$$\int_C f(z)\,dz = 2\pi i \;\operatorname{Res}(f; i)$$

$$\operatorname{Res}(f, i) = \lim_{z \to i} \frac{d}{dz}(z-i)^2 f(z) = \lim_{z\to i} \frac{d}{dz}\frac{1}{(z+i)^2} = \frac{-2}{(2i)^3} = +\frac{1}{4i}.$$

$$\int_C f(z)\,dz = 2\pi i \frac{1}{4i} = \frac{\pi}{2}.$$

$$\int_{-R}^{R} \frac{1}{(1+x^2)^2}\,dx + \int_{C_R} f(z)\,dz = \pi/2 \quad \text{--- } (*)$$

$\forall R > 1$

$$\left| \int_{C_R} f(z)\,dz \right| = \left| \int_{C_R} \frac{1}{(1+z^2)^2}\,dz \right|$$

$$\leq \pi R \cdot \frac{1}{(R^2-1)^2}$$

$|1+z^2| \geqslant (|z|^2-1) = R^2-1$ on C_R

On C_R

$$\left| \frac{1}{(1+z^2)^2} \right| \leq \frac{1}{(R^2-1)^2}$$

$$\left| \int_{C_R} f(z)\,dz \right| \leq \frac{\pi R}{(R^2-1)^2}$$

$\longrightarrow 0$ as $R \to \infty$.

Hence letting $R \to \infty$ in $(*)$

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \pi/2$$

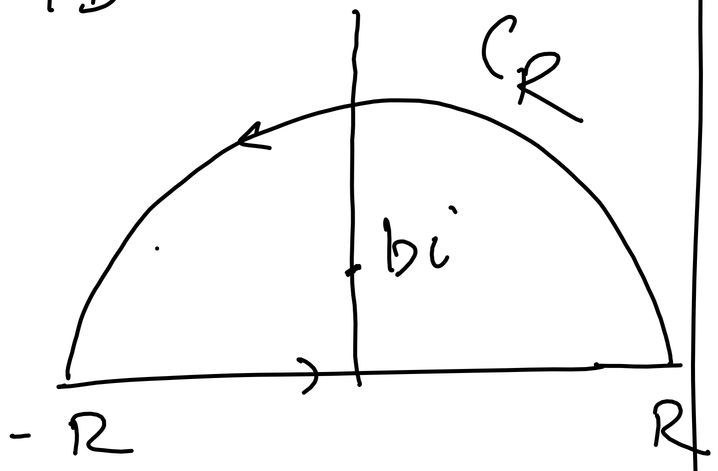
$\blacksquare$

(2) $\displaystyle\int_0^\infty \frac{\cos x}{x^2+b^2}\,dx = \frac{\pi e^{-b}}{2b}$ $\quad b>0.$

Sol

$$f(z) = \frac{e^{iz}}{z^2+b^2}$$

$C = C_R + [-R \;\; R]$



$$\int_C f(z) = 2\pi i\,\big(\mathrm{Res}(f;\, bi)\big)$$

$$= \cancel{2}\pi \cancel{i} \cdot e^{-b} / \cancel{2} b \cancel{i} = \frac{\pi e^{-b}}{b}$$

$$\mathrm{Res}(f;\, bi) = \lim_{z\to bi} (z-bi)\frac{e^{iz}}{(z^2+b^2)}$$

$$= \lim_{z\to bi} \frac{e^{iz}}{z+bi} = \frac{e^{-b}}{2bi}$$

$$\int_C f(z)\,dz = \int_{-R}^{R} \frac{e^{ix}}{x^2+b^2}\,dx + \int_{C_R} f(z)\,dz$$

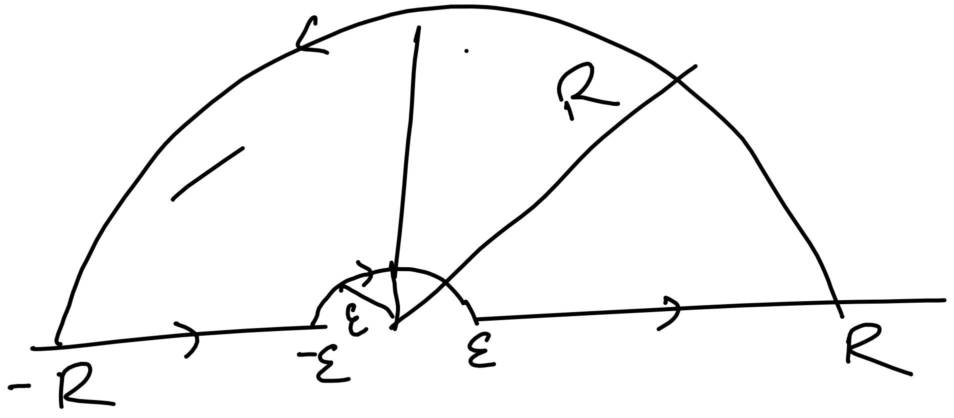
$$\left|\int_{C_R} \frac{e^{iz}}{z^2+b^2}\,dz\right| \le \pi R \cdot \frac{1}{R^2-b^2} \longrightarrow 0 \text{ as } R\to\infty$$

$$|e^{iz}| = e^{-y} \le 1 \quad (y \geq 0)$$

Thm $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+b^2}\,dx = \pi e^{-b}/b$

③ $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \pi/2.$

Sol $\quad f(z) = e^{iz}/z.$



$$\int_C f(z)\,dz = 0.$$

$$z = Re^{i\theta}$$

$$\left|\int_{C_R} \frac{e^{iz}}{z}\,dz\right| = \left|\int_0^\pi \frac{e^{iRe^{i\theta}}}{\cancel{Re^{i\theta}}}\,\cancel{Re^{i\theta}}\,i\,d\theta\right|$$

$$\cancel{|e^{iz}| = e^{-y}}$$

$$= \left|\int_0^\pi e^{iR(\cos\theta + i\sin\theta)}\,i\,d\theta\right|$$

$$\le \int_0^\pi e^{-R\sin\theta}\,d\theta$$

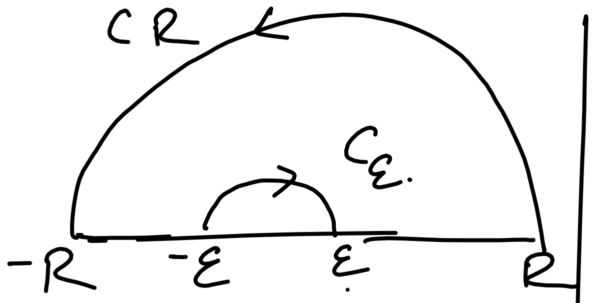
$$= 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta$$

$$\le 2\int_0^{\pi/2} e^{-R\frac{2\theta}{\pi}}\,d\theta \qquad \boxed{\frac{\sin\theta}{\theta} \ge \frac{2}{\pi} \text{ for } 0<\theta<\pi/2}$$

$$= \frac{\pi}{R}\left(1 - e^{-R}\right) \longrightarrow 0 \text{ as } R\to\infty.$$

$$\int_{C_\varepsilon} f(z)\,dz$$

$$= \int_{C_\varepsilon} \frac{e^{iz}}{z}\,dz$$



$$E(z) = i + z(\ ) + z^2(\ ) \cdots$$

$$= \int_{C_\varepsilon} \left(\frac{1}{z} + E(z) \right)$$

$$= -\pi i + \int_{C_\varepsilon} E(z)\,dz$$

$$\left| \int_{C_\varepsilon} E(z)\,dz \right| \le \text{length}(C_\varepsilon)\cdot M$$

$$\longrightarrow 0 \text{ as } \varepsilon \to 0.$$

Thn $\displaystyle\lim_{\varepsilon\to 0} \int_{C_\varepsilon} f(z)\,dz = -\pi i$

Since $E(z)$ is a power series in z, it is holomorphic near origin

$\Rightarrow$ $E(z)$ is continuous at origin.

$\Rightarrow$ $E(z)$ bounded near origin.

$\left(E(z) \longrightarrow \overset{\overset{i}{\shortparallel}}{E(0)} \text{ as } z \to 0 \right)$

$\Big[$ Choose $\varepsilon > 0$. Thn $\exists\, \delta > 0$ s.t. $|E(z) - i| < \varepsilon \ \forall\ |z - 0| < \delta$.

$$|E(z)| \le |E(z) - i| + |i|$$

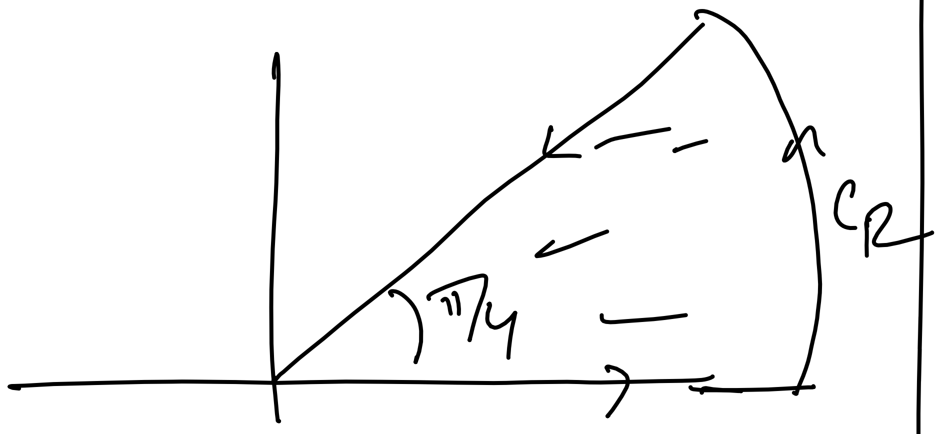
$$< \underbrace{\varepsilon + 1}_{M} \quad \forall\ \underline{|z| < \delta} \Big]$$

$$\text{Th } \underbrace{\int_{C_R}}_{\downarrow\, 0} + \underbrace{\int_{C_\varepsilon}}_{\downarrow\, -\pi i} + \underbrace{\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}}_{\downarrow\, \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx} = 0$$

(3) $\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx$

$= \frac{\sqrt{2\pi}}{4}$.

Sol. $f(z) = e^{iz^2}$



$$\int_C f(z)\,dz = 0.$$

On x-axis

$\int_0^R e^{ix^2}\,dx \to \int_0^\infty e^{ix^2}\,dx$ as $R \to \infty$

On C_R

$\left|\int_{C_R} f(z)\,dz\right| \to 0$ as $R \to \infty$

On line $\theta = \pi/4$ Put $z = re^{i\pi/4}$.

& The integral

$= -\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\frac{\sqrt{\pi}}{2}$.