Orthogonal family

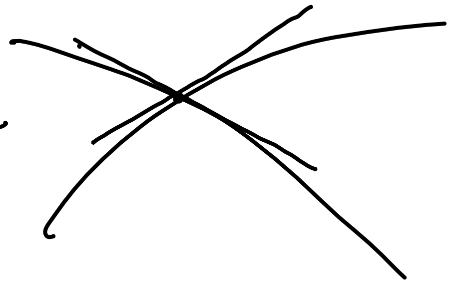
$f = u + iv$, $f \in H(\Omega)$

$f'(z) \neq 0 \ \forall z \in \Omega$.

Then the level curves $u(x,y) = \text{const}$ & $v(x,y) = \text{const}$ are orthogonal.

Proof Suppose $u(x,y) = c_1$, $v(x,y) = c_2$ intersect at a point $z_0 = x_0 + i y_0$.

$f'(z) = u_x + i v_x$



The normal vector to $u(x,y) = c_1$ at a point z_0 is.

$$\nabla u(z_0) = \left(u_x(x_0, y_0), u_y(x_0, y_0) \right)$$

provided $\nabla u(z_0) \neq 0$.

(Justification

Assume the curve $u(x,y) = c_1$ is given by parametric for $x = x(t)$, $y = y(t)$)

$u(x(t), y(t)) = c_1$

Differentiating w.r.t 't'

$u_x x'(t) + u_y y'(t) = 0$

$\nabla u \cdot (x', y') = 0$

$\Rightarrow \nabla u$ is the normal vector to $u(x,y) = c_1$)

Since $f'(z_0) \neq 0$, $\nabla u(z_0) \neq 0$, $\nabla v(z_0) \neq 0$.

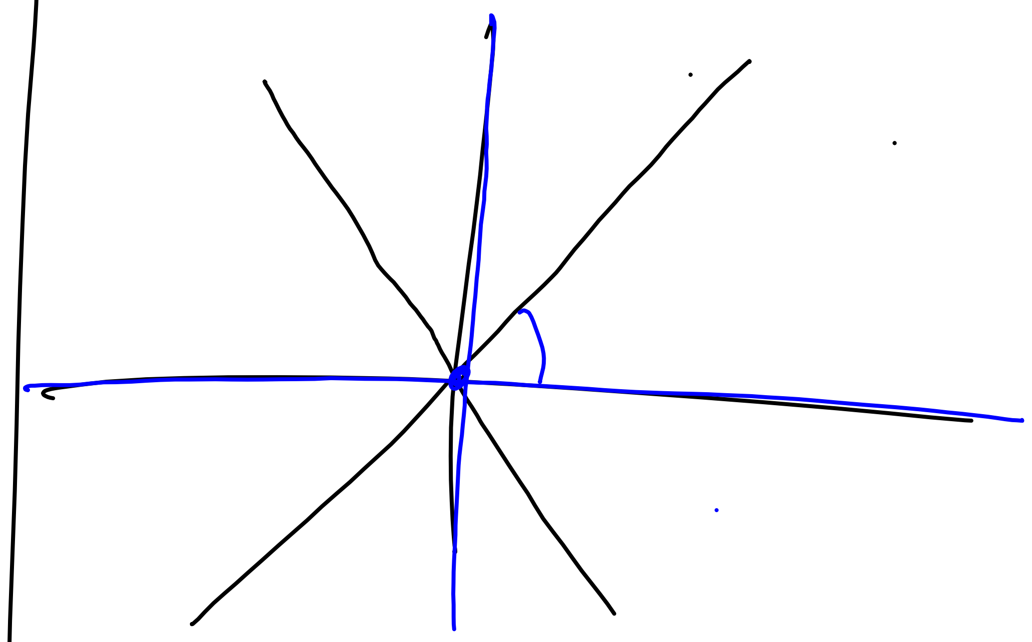
So $\nabla u(z_0)$ & $\nabla v(z_0)$ are normal vectors to the level curves. $u(x,y) = c_1$ & $v(x,y) = c_2$.

$$\nabla u(z_0) \cdot \nabla v(z_0)$$
$$= (u_x, u_y)(v_x, v_y)$$
$$= u_x v_x + u_y v_y$$
$$= u_x v_x + (-v_x)(u_x) = 0$$

$f(z) = z^2$

$u(x,y) = x^2 - y^2 = 0$

$v(x,y) = 2xy. = 0.$



$$u_x = v_y$$

$$u_y = -v_x.$$

$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = v_{yx}$$

$$\frac{\partial^2 u}{\partial y^2} = u_{yy} = -v_{xy}$$

$$u_{xx} + u_{yy} = v_{yx} - v_{xy}$$

$= 0$ (provided v has continuous second order partial derivatives.)

Definition

A function $u : \Omega \to \mathbb{R}$ with second order continuous partial derivatives is called **Harmonic** if

$$u_{xx} + u_{yy} = 0 \quad \text{on } \Omega$$

Laplace equation.

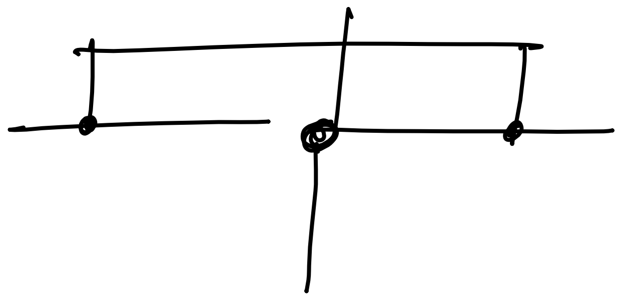
Theorem $f \in \mathcal{H}(\Omega)$, $f = u + iv$

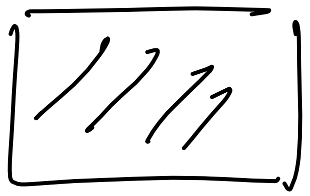
$\Rightarrow$ u & v are Harmonic functions on Ω.

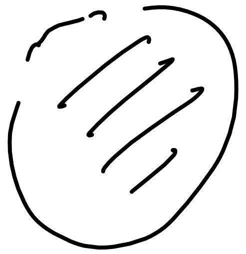
Definition Suppose $u : \Omega \longrightarrow \mathbb{R}$ is Harmonic. A **Harmonic conjugate** to u is a function $v : \Omega \to \mathbb{R}$ s.t. $f = u + iv$ is holomorphic on Ω.

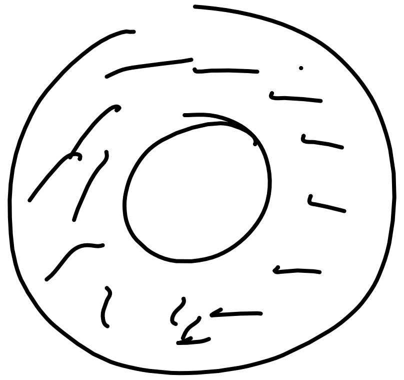
Theorem If $\Omega = \mathbb{C}$ or disc or a convex set, then any harmonic function on Ω has a Harmonic conjugate.

Example $\mathbb{C} \setminus \{0\} = \mathbb{C}^*$

$C^* =$  not convex

 convex

 convex.

 not convex

Example

① $u(x,y) = x^2 - y^2 + 1.$

$u_{xx} = 2 \qquad u_{yy} = -2$

$u_{xx} + u_{yy} = 0$ on $\mathbb{C}$.

$v_x = -u_y = 2y \Rightarrow v = 2xy + \varphi(y)$

$v_y = +u_x = 2x$

$2x + \varphi'(y) = 2x$

$\varphi'(y) = 0 \Rightarrow \varphi(y) = c.$

$u + iv = x^2 - y^2 + 1 + i(2xy + c)$

$= z^2 + 1 + c'$

② $u(x,y) = \frac{1}{2}\log(x^2+y^2).$

defined on $\mathbb{C}^*$

Calculate $u_{xx} + u_{yy} = 0$

on $\mathbb{C}^*$

This does not admit a Harmonic conjugate on $\mathbb{C}^*$.

③ $u(x,y) = \frac{x}{x^2+y^2}$ function on $\mathbb{C}^*$

$f(z) = \frac{1}{z} = \frac{x - iy}{x^2+y^2}$

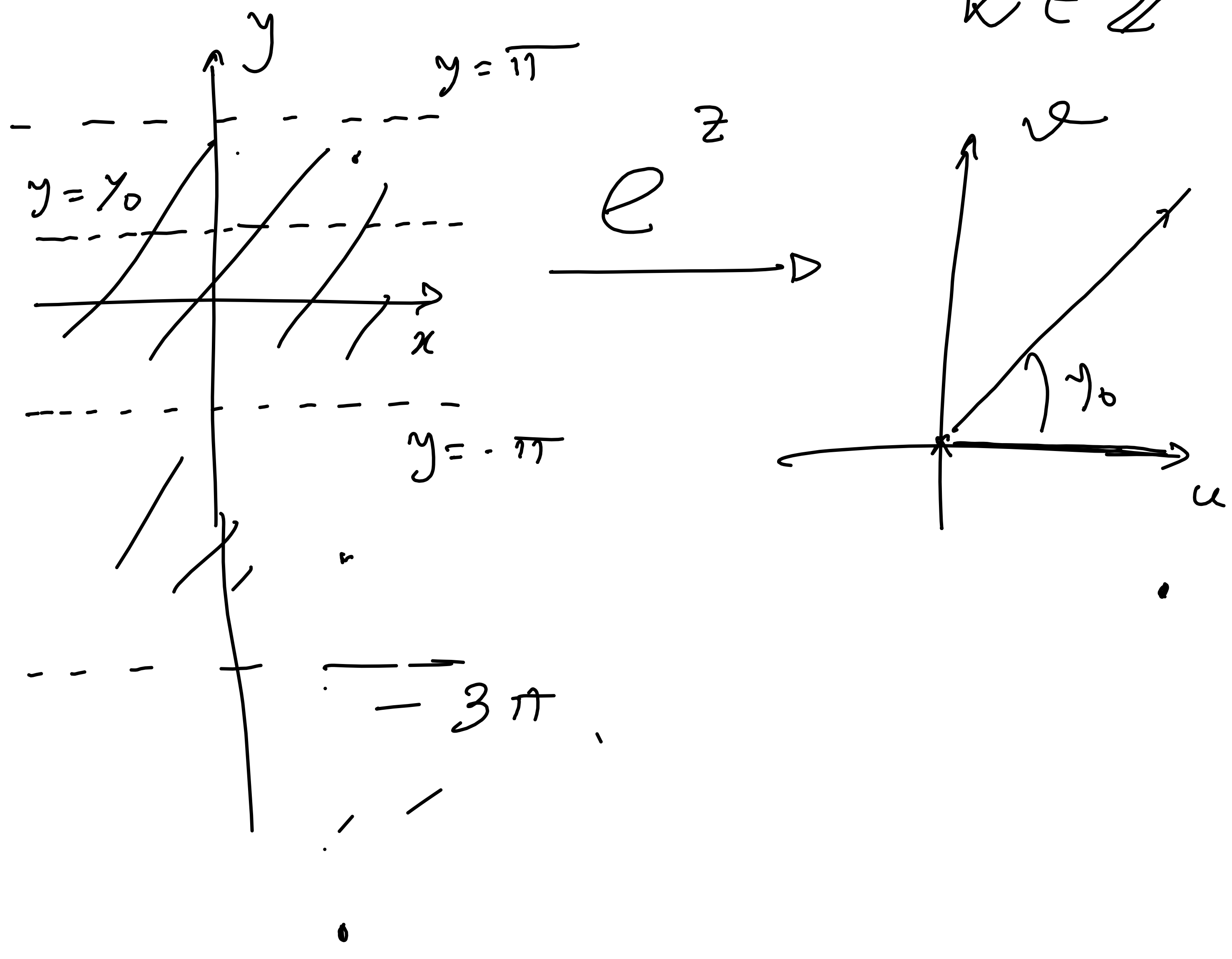
Recall

$$e^{z} = e^{x}(\cos y + i \sin y) \in \mathcal{H}(\mathbb{C})$$

- $\frac{d}{dz}(e^{z}) = e^{z}$
- $e^{z} \neq 0$
- $e^{z+w} = e^{z} e^{w}$ $\quad z, w \in \mathbb{C}$
- $e^{z} = e^{z + 2\pi i k}$ $\quad k \in \mathbb{Z}$
- $\frac{e^{z}}{e^{w}} = e^{z-w}$

$$e^{z} = e^{w} \iff z - w = 2k\pi i$$

for some $k \in \mathbb{Z}$



Definitn

$z \neq 0$

$$\log(z) := \log(|z|) + i \arg(z).$$

$$\mathrm{Log}(z) = \log(|z|) + i\, \mathrm{Arg}(z)$$

$$-\pi < \mathrm{Arg}(z) \leq \pi$$

$$\log(z) = \mathrm{Log}(z) + 2k\pi i \quad k \in \mathbb{Z}.$$

Note $-4 = 4e^{i\pi}$

$$\log(-4) = \log(4) + i(\pi + 2k\pi) \quad k \in \mathbb{Z}.$$

$$\mathrm{Log}(-4) = \log(4) + i\pi$$

- $e^{\omega} = z \iff \omega = \log z$
- $e^{\log z} = z$
- $\log e^{\omega} = \omega + 2k\pi i \quad k \in \mathbb{Z}.$

$\blacksquare$