Recall (Order of Zero)

$z = z_0$ is a zero of order $k$ of $f(z)$;

$\Leftrightarrow \quad f(z) = (z - z_0)^k g(z)$

$g(z_0) \neq 0$.

$\Leftrightarrow$ The Taylor series of $f$

$f(z) = (z - z_0)^k a_k + (z - z_0)^{k+1} a_{k+1}$

$- - \cdots$

$\Leftrightarrow \quad f(z_0) = f'(z_0) = \cdots = f^{(k-1)}(z_0) = 0$

$f^{(k)}(z_0) \neq 0$

Example

(1) $f(z) = z^8 (e^z - 1)$

$Z_f = \{0, 2k\pi i\}$

Order of $z_0 = 0$ is 9.

$f(z) = z^8 \left(\cancel{1} + z + \frac{z^2}{2!} + \cdots \cancel{-1}\right)$

$= z^9 + \frac{z^{10}}{2!} + \cdots$

Order of $z_0 = 2\pi i$ is 1

$f(z_0) = 0 \qquad f'(z_0)$

$= \cancel{8 z_0^7 (e^{z_0} - 1)} + z_0^8 e^{z_0}$

$0 \qquad \neq 0$.

- $f(z) = (e^z - 1)$

Order of all zeros of $(e^z - 1)$ are 1. $(2k\pi i)$

(2) $\cos(z) - e^z + z$. $z_0 = 0$

$= -z^2 + \frac{z^3}{3} + \cdots$

Order of $z_0 = 0$ is 2.

## Singularity

- A function $f(z)$ is called singular at $z_0$ if $f$ is not holomorphic at $z_0$.

Example
- $f(z) = \frac{z+1}{z^3(z^2+1)}$

  singularities are at $z = 0, \pm i$
  Isolated.

- $\frac{1}{\sin z}$ singularities are at $z = n\pi$, $n \in \mathbb{Z}$
  Isolated

- $\operatorname{Log} z$ singularities are $z \leq 0$
  Non-Isolated.

- $\frac{1}{\sin(\frac{\pi}{z})}$ singularitie are $0, \frac{1}{n}, n \in \mathbb{Z}$

  non-isolated sing. Isolated

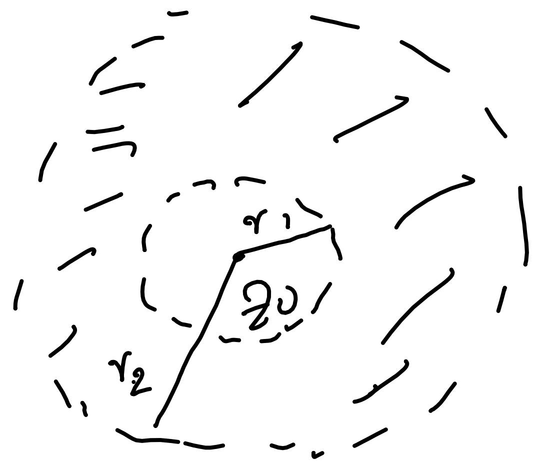
• A function $f(z)$ is said to have isolated singularity at $z = z_0$ iff $\exists$ $r>0$ s.t. $f(z)$ is holomorphic on

$$0 < |z - z_0| < r$$


## Annulus

• $$r_1 < |z - z_0| < r_2$$

$$0 \le r_1 < r_2 \le \infty$$




• $\mathbb{C}^*$ $\quad 0 < |z| < \infty$

• $1 < |z| < \infty$


**Theorem** Suppose $f$ is holomorphic on an Annulus $(r_1 < |z - z_0| < r_2)$. Then $f$ can be expressed uniquely as

$$f(z) = \underbrace{\sum_{n \ge 1} \frac{b_n}{(z - z_0)^n}}_{\text{Principal part}} + \underbrace{\sum_{n \ge 0} a_n (z - z_0)^n}_{\text{Analytic}} \quad (*)$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi$$

$$b_n = \frac{1}{2\pi i} \int_C f(\xi)(\xi - z_0)^{n-1} \, d\xi$$

$C : |z - z_0| = r.$

$r_1 < r < r_2$

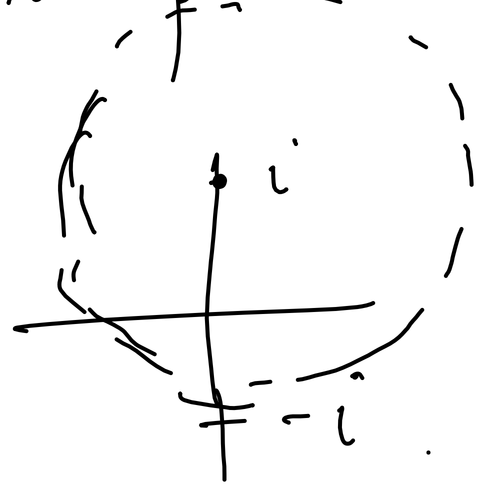

The series (*) is called the <u>Laurent Series</u> of $f$ in the given annulus.

Example

① $f(z) = \dfrac{z+1}{z}$ holomorphic on $\mathbb{C}^*$

$z = 0$ pole of order 1 $= \underbrace{\boxed{\dfrac{1}{z}}}_{\text{principal}} + \underbrace{1}_{\text{holomorphic}}.$

② $f(z) = \dfrac{z}{z^2+1}$




Take $\boxed{0 < |z - i| < 2}$

$$f(z) = \left[\frac{1}{z-i} + \frac{1}{z+i}\right]\frac{1}{2}$$

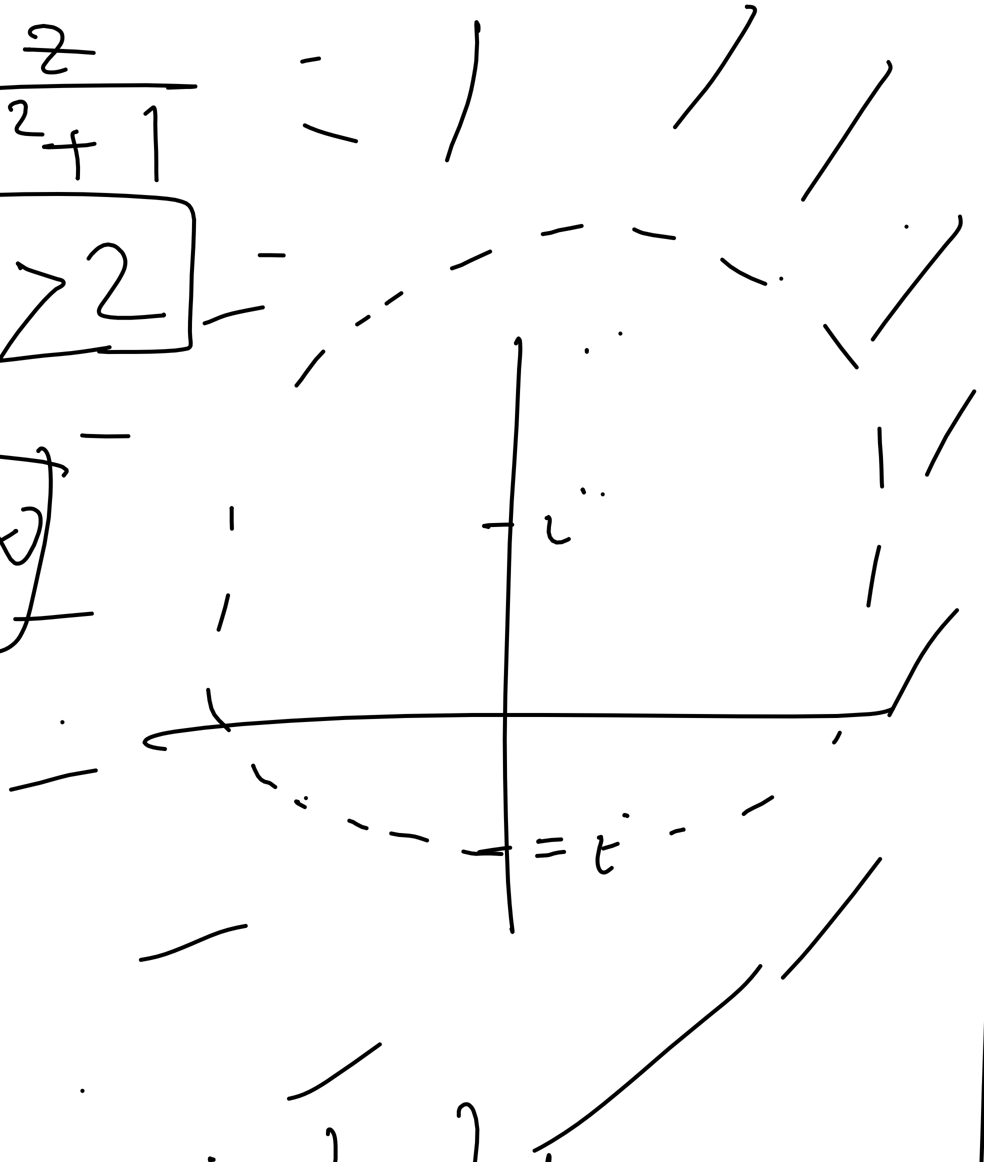
$$\frac{1}{z+i} = \frac{1}{(z-i)+2i} = \frac{1}{2i\left[1 + \frac{z-i}{2i}\right]}$$

$$= \frac{1}{2i} \sum_{n \geq 0} \left(\frac{z-i}{2i}\right)^n \times (-1)^n$$

$$f(z) = \frac{z}{z^2+1}$$

$$\boxed{|z-i| > 2}$$

$$\boxed{2 < |z-i| < \infty}$$




$$f(z) = \left[\frac{1}{z-i} + \frac{1}{z+i}\right]\frac{1}{2}$$

$$\frac{1}{z+i}$$

$$= \frac{1}{(z-i)+2i}$$

$$= \frac{1}{(z-i)\left(1+\frac{2i}{z-i}\right)}$$

$$= \frac{1}{(z-i)}\sum_{n\geq 0}\left(-\frac{2i}{z-i}\right)^n$$

$$= \sum_{n\geq 0}\frac{(-2i)^n}{(z-\cancel{2}\,i)^{n+1}}$$

$$f(z) = \frac{1}{z-i} + \frac{1}{2}\sum_{n\geq 0}\frac{(-2i)^n}{(z-i)^{n+1}}$$

4) $\frac{\sin z}{z}$ $z = 0$ removeable

$$= \frac{z - z^3/\lfloor 3 + \frac{z^5}{\lfloor 5} - \cdots}{z}$$

$$= 1 - \frac{z^2}{\lfloor 3} + \frac{z^4}{\lfloor 5} - \cdots$$

5) $e^{\frac{1}{z}}$ holomph. on $\mathbb{C}^*$

$$= 1 + \frac{1}{z} + \frac{1}{z^2}\frac{1}{\lfloor 2} + \frac{1}{z^3}\frac{1}{\lfloor 3} + \cdots$$

$z = 0$ is an essential singularity

Definitn

Assme $z = z_0$ is an isolated singularity of $f(z)$. Expand $f$ as a Laurent series in $0 < |z - z_0| < r$.

$$f(z) = \sum_{n \geq 1} \frac{b_n}{(z - z_0)^n} + \sum_{n \geq 0} a_n (z - z_0)^n$$

- If $b_n = 0$ $\forall n \geq 1$, then $z = z_0$ is called removable singularity.

- $b_n = 0$ $\forall n > k$
  $b_k \neq 0$. Then $z = z_0$ is called a pole of order $k$

$$f(z) = \frac{b_k}{(z - z_0)^k} + \cdots + \frac{b_1}{(z - z_0)} + \sum_{n \geq 1} a_n (z - z_0)^n$$

- $b_n \neq 0$ for infinitely many $n$.

In this case $z = z_0$ is called an essential singularity of $f(z)$.

Example ① $f(z) = \left[\frac{1}{z^3} + 0\right]$ $z = 0$ pole of order 3.

② $\frac{1}{z(z^2+1)(z-2)^2} = f(z)$

- Isolated singularities.

0 — pole of order 1

$i$ — pole of order 1

$-i$ — — — — —

2 — — — — 2.

- $z = z_0$ is a pole of order $k$

$$\iff f(z) = \frac{g(z)}{(z-z_0)^k}$$

$$g(z_0) \neq 0$$