Cauchy's Theorem for tringle / Goursat Theorem

Theorem

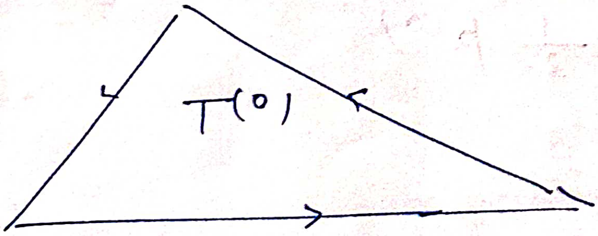
Let Ω be an open set in $\mathbb{C}$ & $T \subset \Omega$ a triangle whose interior is also contained in Ω.

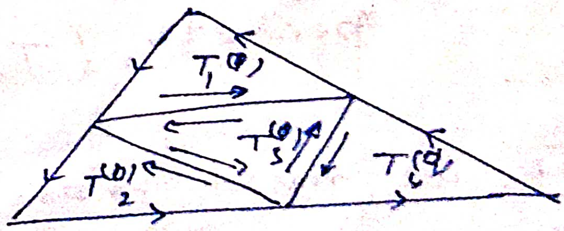
If $f \in H(\Omega)$, then $\int_{\partial T} f(z)\,dz = 0$.

(The statement of the theorem says that f is analytic on a trigle & its interior, then $\int_T f(z)\,dz = 0$).

Proof $T^{(0)} = T$ with anticlockwise orientation.

Let $d^{(0)}$ = diameter of $T^{(0)}$
$p^{(0)}$ = perimeter of $T^{(0)}$.





The first step in our construction is to bisect each side of the triangle & connecting the middle points. This creates four new smaller trigles, denoted by $T_1^{(1)}$, $T_2^{(1)}$, $T_3^{(1)}$, $T_4^{(1)}$.

The orientation is chosen to be consistent with that of the original triangle & so after cancelling arising from integrating

$$\cancel{\int_{T^{(0)}} f(z)\,dz =}$$

$$\eta(T^{(0)}) = \int_{T^{(0)}} f(z)\,dz$$

$$\eta(T^{(0)}) = \sum_{i=1}^{4} \eta(T_i^{(1)})$$

$$|\eta(T^{(0)})| \leq \sum_{i=1}^{4} |\eta(T_i^{(1)})|$$

note

$$\text{diam}(T_i^{(1)}) = \frac{1}{2} d^{(0)}$$

$$\text{perimeter}(T_i^{(1)}) = \frac{1}{2} p^{(0)}$$

Then we must have

$$|\eta(T^{(0)})| \leq 4|\eta(T_i^{(1)})| \text{ for some } i = 1, 2, 3, 4$$

Rename the triangle as $T^{(1)}$.

$$|\eta(T_i^{(1)})| = \max_{j=1,2,3,4} |\eta(T_j^{(1)})|$$

Thus starting with $T^{(0)}$ we have obtained a triangle $T^{(1)}$ s.t

$d^{(1)} = \text{diam}\,(T^{(1)}) = \frac{1}{2} d^{(0)}$ & $|\eta(T^{(0)})| \leq 4|\eta(T^{(1)})|$

$p^{(1)} = \text{perimeter}\,(T^{(1)}) = \frac{1}{2} p^{(0)}$.

Continuing the process, we obtain a sequence of triangles

$$T^{(0)}, T^{(1)}, T^{(2)}, \ldots$$

$$|\eta(T^{(0)})| \leq 4^n |\eta(T^{(n)})|.$$

$$d^{(n)} = \frac{1}{2^n} d^{(0)}$$

$$p^{(n)} = \frac{1}{2^n} p^{(0)}.$$

Let K_n = union of T_n with its interior.

Then K_n are compact subsets with

$$K_0 \supseteq K_1 \supset K_2 \ldots$$

with $\text{diam}\,(K_n) = \text{diam}\,(T^{(n)}) = \frac{1}{2^n} d^{(0)} \to 0$ as $n \to \infty$.

Therefore $\bigcap_{n=0}^{\infty} K_n = \{z_0\} \in \Omega$ by Cantor's Theorem

Since f is holomorphic at z_0 on Ω we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$$

Where $\psi(z) \to 0$ as $z \to z_0$.

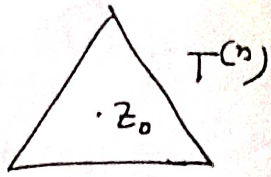
Then integrating

$$\int_{T^{(n)}} f(z)\,dz = \int_{T^{(n)}} \psi(z)\,(z-z_0)\,dz$$

($\because$ other terms has primitive)

Now $z_0 \in K_n$ & $z \in \partial K_n = T^{(n)}$

$$\Rightarrow |z-z_0| \leq d^{(n)}.$$



$$\left|\int_{T^{(n)}} f(z)\,dz\right| \leq \varepsilon_n\, d^{(n)}\, p^{(n)}$$ (using ML estimate)

where $\varepsilon_n = \sup_{z \in T^{(n)}} |\psi(z)| \longrightarrow 0$ as $n \to \infty$

($\because \psi(z) \to 0$ as $z \to z_0$)

$$\left|\int_{T^{(n)}} f(z)\,dz\right| \leq \varepsilon_n\, 4^{-n}\, d^{(0)}\, p^{(0)}.$$

$$\Rightarrow \left|\int_{T^{(0)}} f(z)\,dz\right| \leq \varepsilon_n\, d^{(0)}\, p^{(0)} \longrightarrow 0$$ as $n \to \infty$

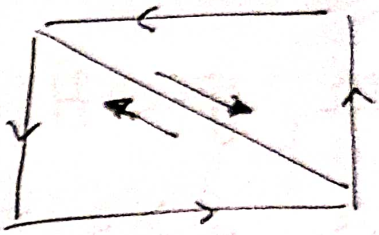
$$\therefore \int_T f(z)\,dz = 0 \qquad \blacksquare$$

Cantor's Theorem

A metric space (X, d) is complete iff for any sequence $\{F_n\}$ of non-empty closed sets with $F_1 \supset F_2 \supset \cdots$ & $\text{diameter}(F_n) \to 0$ as $n \to \infty$, then $\bigcap_{n=1}^{\infty} F_n = \{x^*\}$

Corollary If f is holomorphic in a open set Ω that contains a rectangle R & its interior, then

$$\int_R f(z)\,dz = 0.$$



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