MSO 202 A : Complex Variables

Instructor: DEBASIS SEN

Ref: Churchil & Brown

- complex Variables & applications.

Quiz - 30 (23 Aug 6-7 pm)

Final Exam - 70 -

Complex Numbers

$z = x + iy \quad x, y \in \mathbb{R}$

$i^2 = -1$

$\mathbb{R}$ = Set of real numbers

$\mathbb{C}$ = - - - - complex - - - -

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3 \cdots\}$

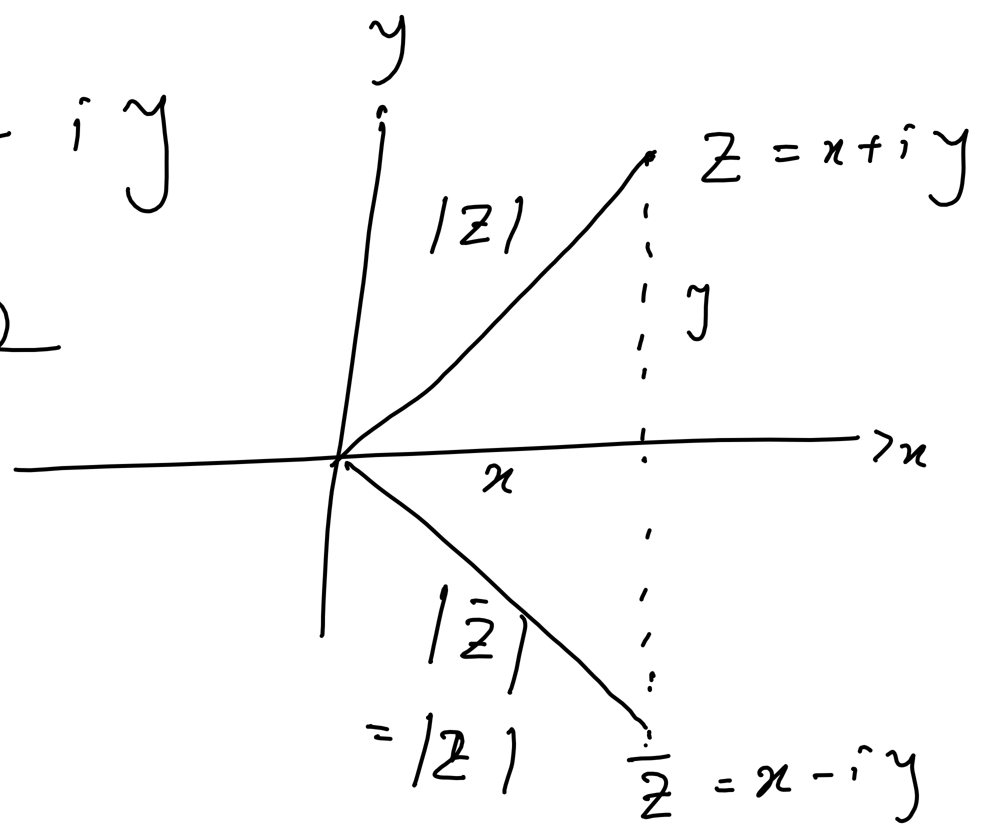
$\mathbb{N} = \{1, 2, 3, 4, \cdots\}$

x = real part of z = $\mathrm{Re}(z)$

y = Imaginary part of z
$= \mathrm{Im}(z)$

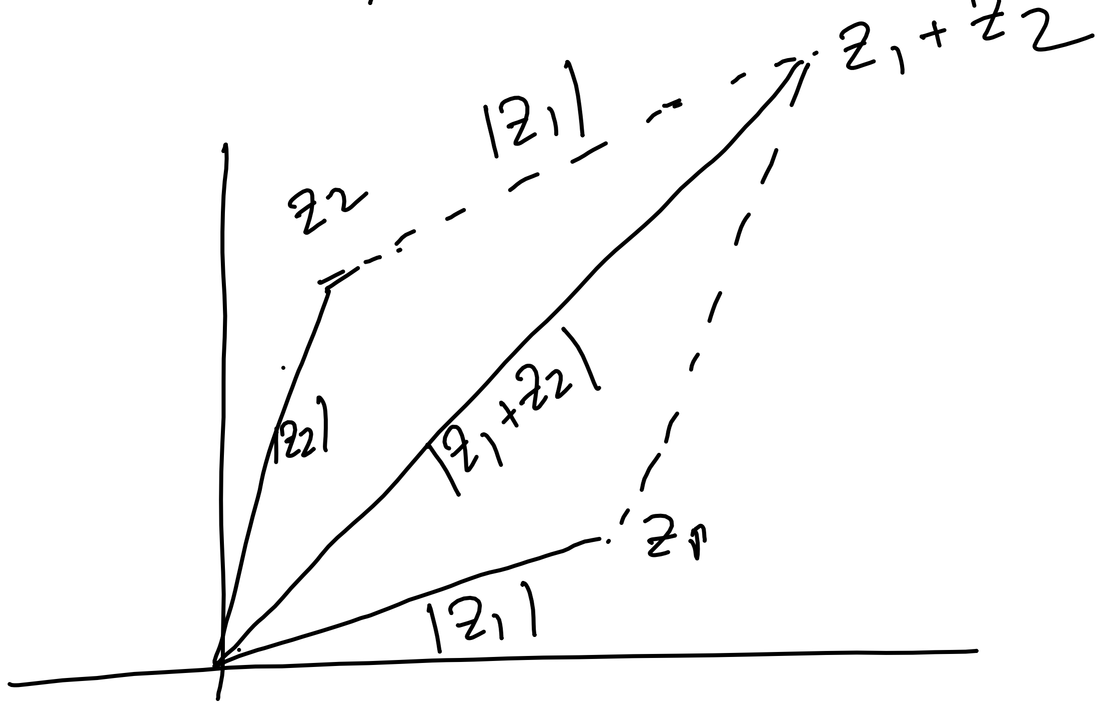
$\bar{z} = x - iy$

$|z|^2 = x^2 + y^2$
$= z \cdot \bar{z}$



$z_1, z_2 \in \mathbb{C}$

$$|z_1 + z_2| \le |z_1| + |z_2|$$



$$|z_1 - z_2| \le |z_1| + |z_2|$$

Polar form

$z \neq 0$

$$z = r e^{i\theta} = r(\cos\theta + i\sin\theta)$$

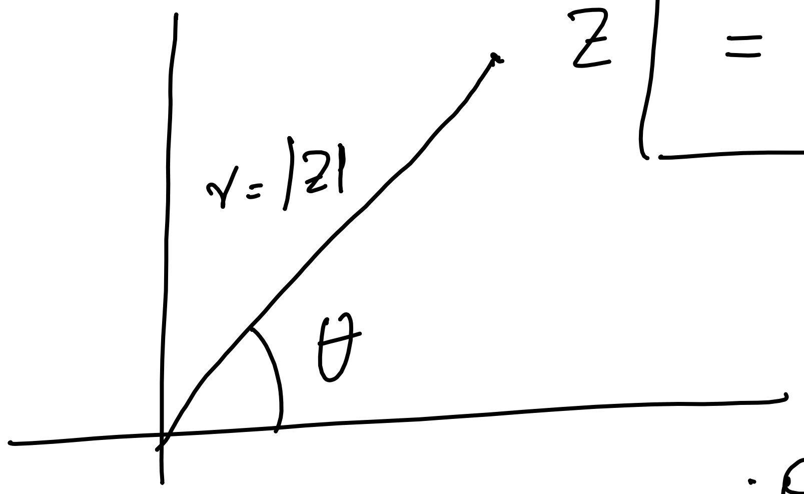
$$e^{i\theta} := \cos\theta + i\sin\theta$$

$r > 0;\quad \theta \in \mathbb{R}$

$r = |z|$

$$e^{2\pi i k} = 1$$

$$e^{i\theta} e^{i\varphi} = e^{i(\theta+\varphi)}$$



$$z = r e^{i\theta} = r e^{i\theta + 2\pi i k} \qquad k \in \mathbb{Z}$$

$\theta = \arg(z)$

$\theta = \arg(z)$ = argument of z

— which is not unique.

But if restrict θ in $-\pi < \theta \le \pi$, then the unique θ is called the Principal value of argument

$$-\pi < \operatorname{Arg}(z) \le \pi$$

Example

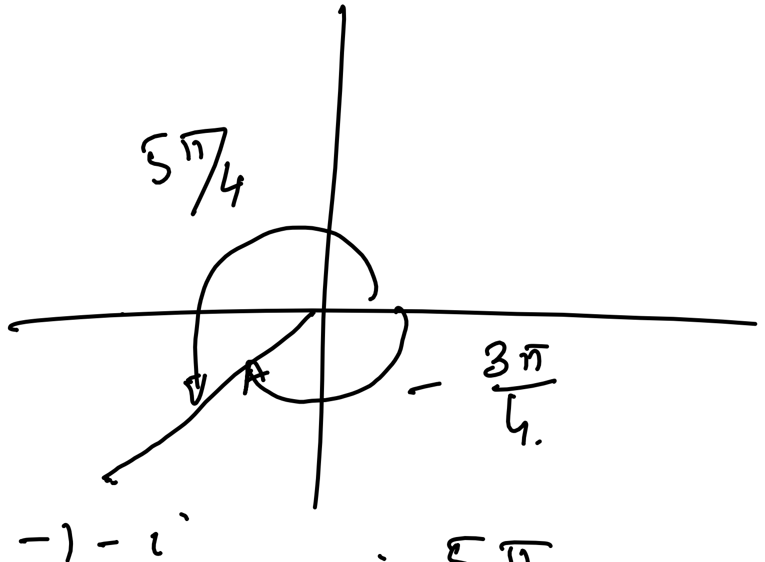
$\cancel{\arg(z)}$

$$\arg(-1) = \{\pi + 2k\pi \,/\, k \in \mathbb{Z}\}.$$

$$\operatorname{Arg}(-1) = \pi$$

(2) $Z = -1 - i$

write in polar form & find $Arg(Z)$.



$$Z = -1 - i = \sqrt{2}\, e^{i\frac{5\pi}{4}}$$

$$= \sqrt{2}\, e^{-i\frac{3\pi}{4}}$$

$$Arg(Z) = -\frac{3\pi}{4}.$$

Roots of complex numbers

$$Z = r e^{i\theta} = r e^{i(\theta + 2k\pi)}$$

$k \in \mathbb{Z}$.

$$\sqrt[n]{Z} = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + \frac{k}{n}\cdot 2\pi\right)}$$

$k \in \mathbb{Z}$

$$= \begin{cases} \sqrt[n]{r}\, e^{i\frac{\theta}{n}} & k = 0 \\ \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}\right)} & k = 1 \\ \vdots & \\ \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}\cdot n-1\right)} & k = n-1 \end{cases}$$

$$\sqrt[n]{z} = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + 2\pi\frac{k}{n}\right)}$$

$k = 0, \dots, n-1$

Example $\sqrt[n]{1}$

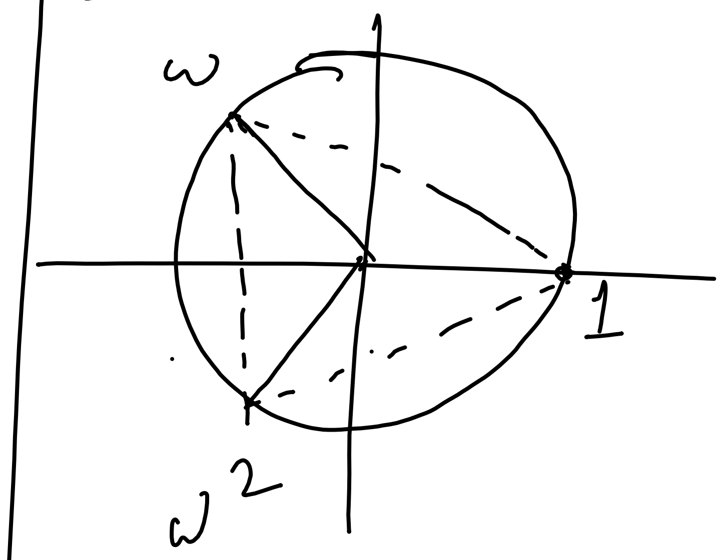
$$1 = 1\, e^{i0} = e^{2k\pi i} \qquad k \in \mathbb{Z}$$

$$\sqrt[n]{1} = e^{\frac{2k\pi i}{n}} \qquad k = 0, \dots, n-1$$

$$= 1, \underbrace{e^{\frac{2\pi}{n}i}}_{\omega}, \underbrace{e^{\frac{4\pi i}{n}}}_{\omega^2}, \dots, \underbrace{e^{\frac{2\pi i}{n}(n-1)}}_{\omega^{n-1}}$$

$$= 1, \omega, \omega^2, \dots \omega^{n-1}$$

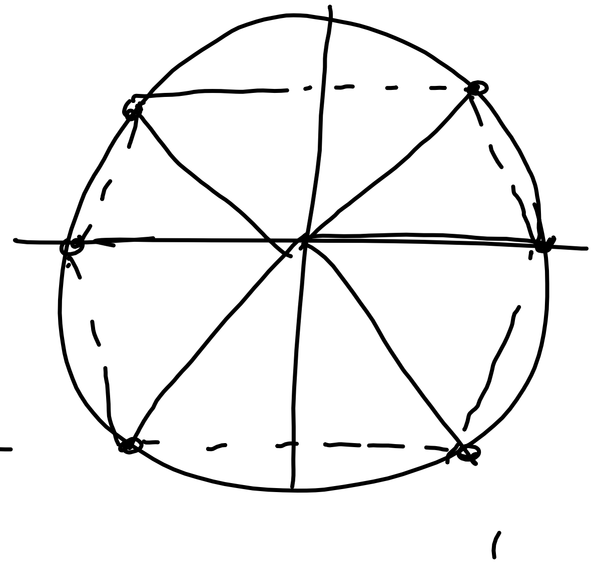
$n = 3$



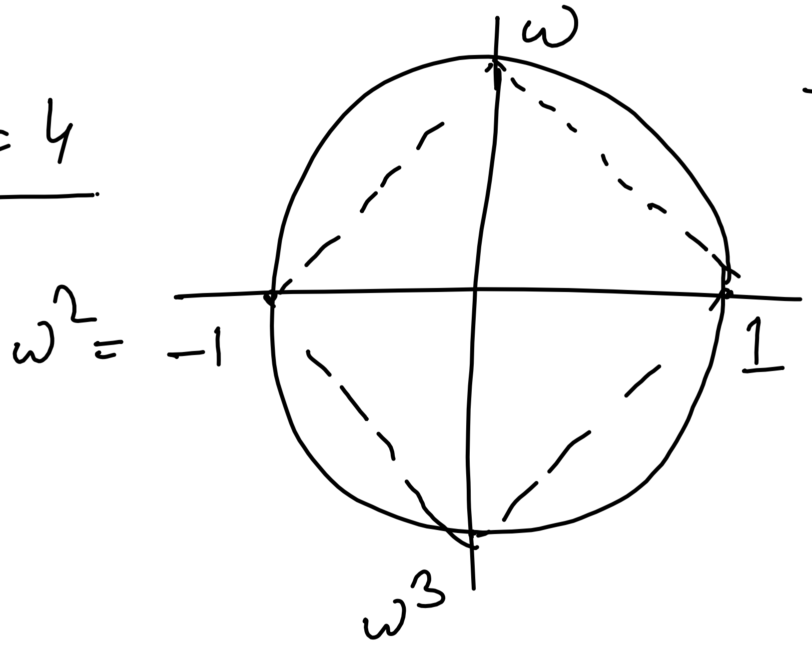
$|z| = 1$

unit circle

$n = 6$



$n = 4$



(2) Find $\sqrt{\sqrt{3}+i}$

$$z = \sqrt{3}+i = 2e^{i\pi/6}$$

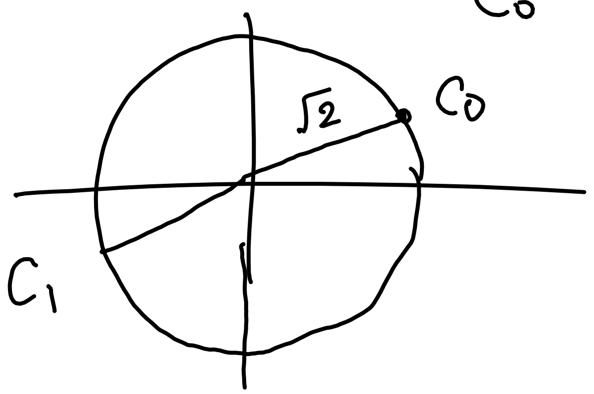
$$= 2e^{i\frac{\pi}{6} + 2k\pi i}$$

$k \in \mathbb{Z}$.

$$\sqrt{z} = \sqrt{2}\, e^{i\frac{\pi}{12} + k\pi i}$$

$k = 0, 1$

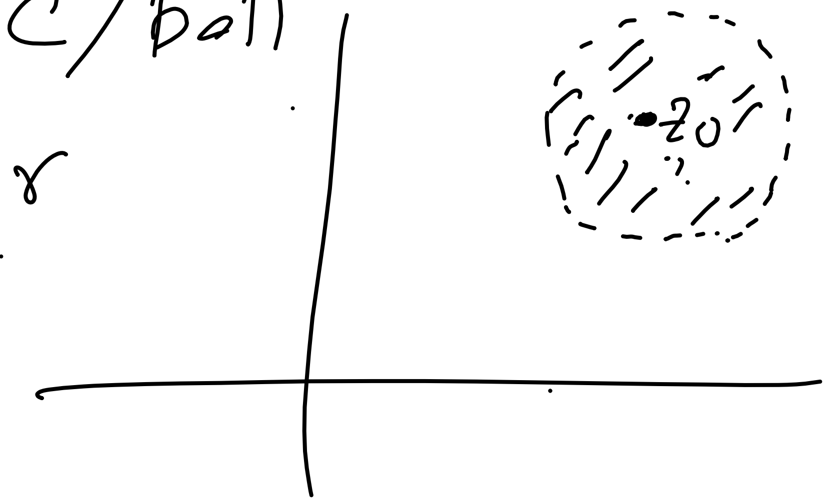
$$= \underbrace{\sqrt{2}\, e^{i\frac{\pi}{12}}}_{C_0},\ \underbrace{\sqrt{2}\, e^{i\frac{\pi}{12} + \pi i}}_{C_1 = -C_0}$$



Domains in $\mathbb{C}$

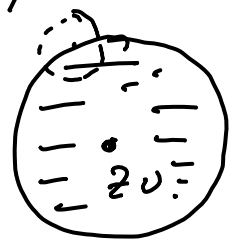
$$\mathbb{D}_r(z_0) = \{ z \in \mathbb{C} \,/\, |z - z_0| < r \}$$

- open disc/ball of radius r around z_0



$$\overline{\mathbb{D}_r}(z_0) = \{ z \in \mathbb{C} \,/\, |z - z_0| \leq r \}$$

- closed disc/ball

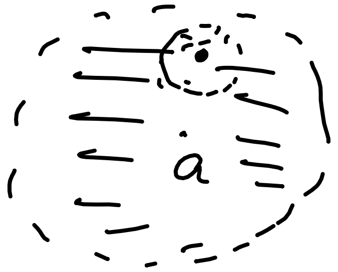


Open Set

$\Omega \subseteq \mathbb{C}$ is called open if any $z_0 \in \Omega$, then $\exists\ r > 0$ s.t $\mathbb{D}_r(z_0) \subseteq \Omega$.

Example

① $\mathbb{D}_r(a) = \Omega$

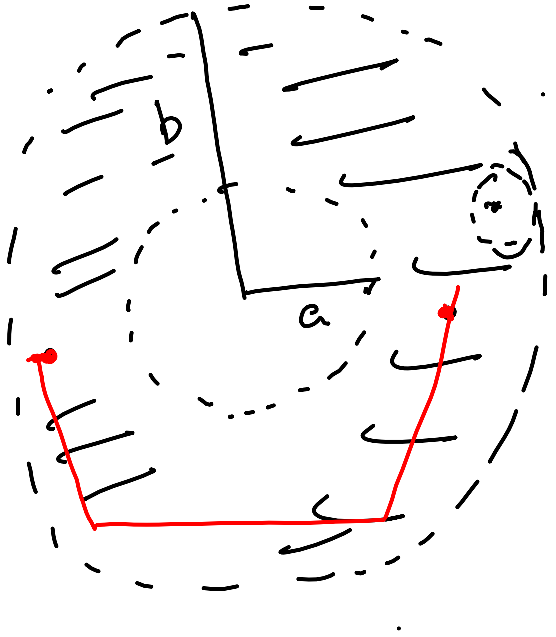


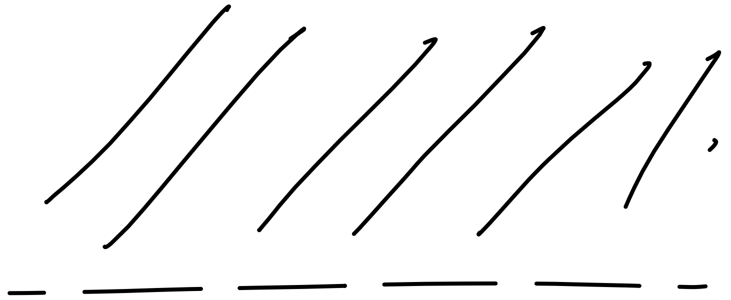
- is an open set.

② $\overline{\mathbb{D}_r}(a)$ - not an open set.

② $0 < a < |z| < b$ $\quad$ $0 < a < b \in \mathbb{R}$

- open set
- annulus



③  $= \mathbb{H}^2$

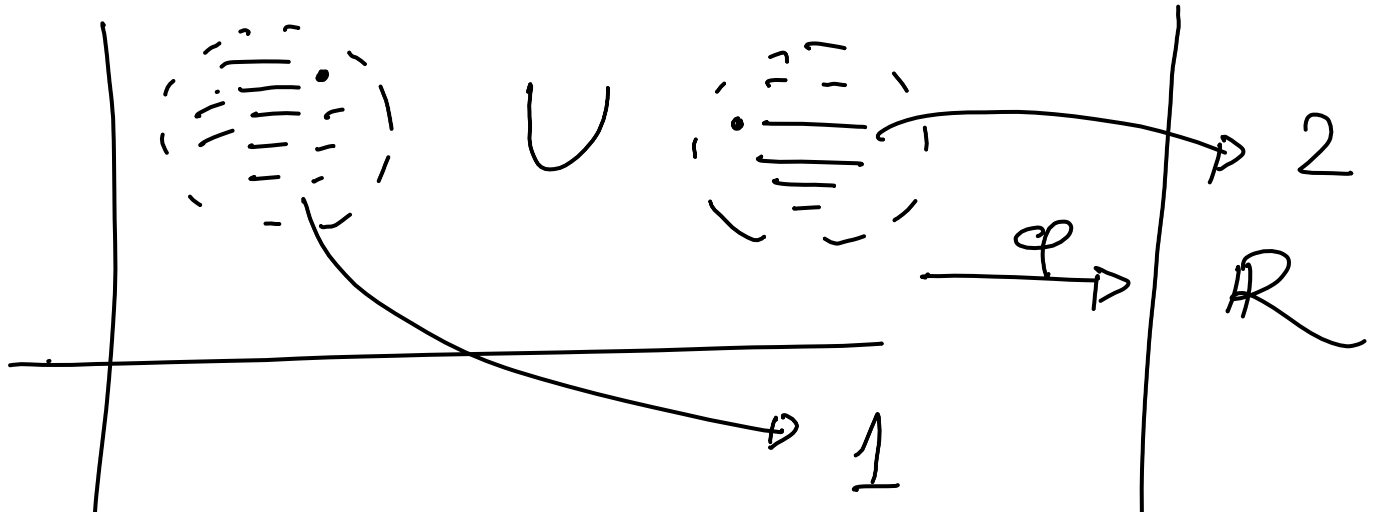
$\mathbb{H}^2 = \{ z \,/\, Im(z) > 0 \}$ upper half plane

- open set

Definition An open set $\Omega \subseteq \mathbb{C}$ is called connected if any two points in Ω can be joined by a polygonal path (i.e finite no. of line segments)

region / domain = open connected set

Ω =



This is open, but not connected.

<u>Thm</u> $\varphi : \Omega \longrightarrow \mathbb{R}$

$\Omega \subseteq \mathbb{C}$ domain.

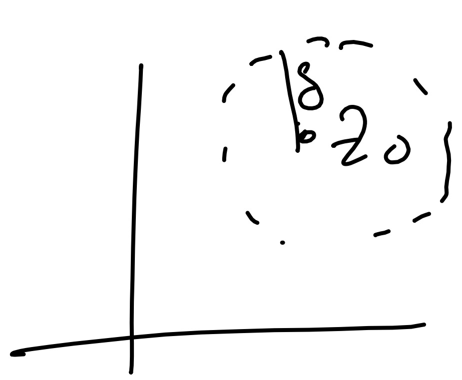
If $\varphi_x = 0 = \varphi_y$. Then

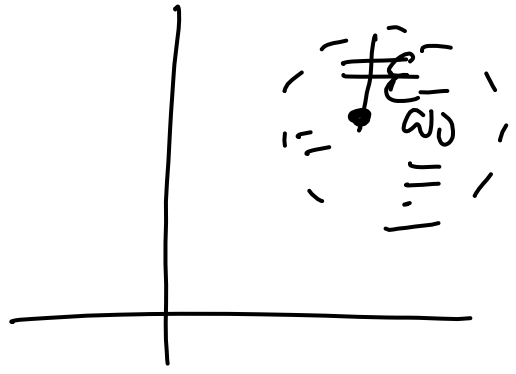
φ is const on Ω.

limit $\lim_{z \to z_0} f(z) = w_0 \in \mathbb{C}$

ie for any $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon$





Definition f is called **continuous** at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$

Example

(1) $f(z) = \dfrac{i\bar{z}}{2}$

$\lim_{z \to 1} f(z) = \dfrac{i}{2}$

$\left| f(z) - \dfrac{i}{2} \right|$

$= \left| \dfrac{i\bar{z}}{2} - \dfrac{i}{2} \right| = \dfrac{1}{2} |\bar{z} - 1|$

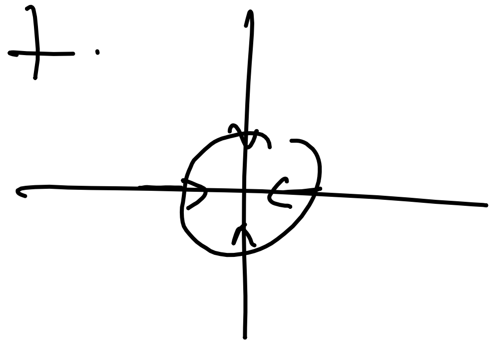
$= \dfrac{1}{2} |\overline{z-1}|$

$= \dfrac{1}{2} |z-1|$

$< \varepsilon$

if $|z-1| < \underbrace{2\varepsilon}_{\delta}$

② $\lim_{z \to 0} \frac{z}{\bar{z}}$ – does not exist.



Along x-axis

$$\lim_{x \to 0} \frac{x}{x} = 1.$$

Along y-axis

$$\lim_{y \to 0} \frac{iy}{-iy} = -1$$