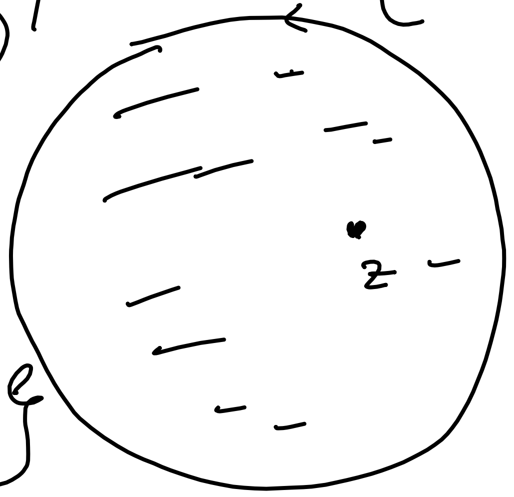
Cauchy's Integral formula

Suppose f is holomorphic on an open set containing a disc $\mathbb{D}$ with C as boundary circle.



$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}\,d\xi$$

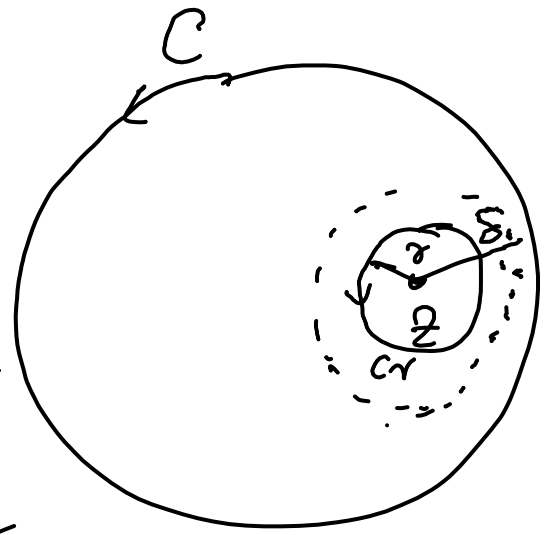
$z \in \mathbb{D}$.

Proof

$$\int_C \frac{f(\xi)}{\xi - z}\,d\xi = \int_{C_r} \frac{f(\xi)}{\xi - z}\,d\xi$$

(By Cauchy's Thm)

$$= \int_{C_r} \frac{f(\xi) - f(z)}{\xi - z}\,d\xi + \underbrace{\int_{C_r} \frac{f(z)}{\xi - z}\,d\xi}_{f(z)\,2\pi i}$$



First integral

$$\left|\int_{C_r} \frac{f(\xi) - f(z)}{(\xi - z)}\,d\xi\right| \leq 2\pi r \sup_{\xi \in C_r} \frac{|f(\xi) - f(z)|}{\underbrace{|\xi - z|}_{r}}$$

$$= 2\pi \sup_{\xi \in C_r} \underbrace{|f(\xi) - f(z)|} \leq 2\pi\varepsilon$$

given ε, $\exists\ \delta > 0$ s.t

$$|f(\xi) - f(z)| < \varepsilon \quad \forall\ \xi \text{ s.t } |\xi - z| < \delta.$$

(f is continuous at the point z.)

choose r small enough so that

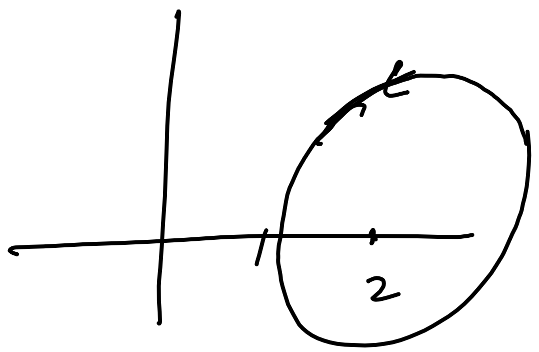
$0 < r < \delta$.

Hence $$\int_{C_r} \frac{f(\xi) - f(z)}{\xi - z}\, d\xi = 0.$$

$$\int_C \frac{f(\xi)}{\xi - z}\, d\xi = \int_{C_r} \frac{f(\xi)}{\xi - z}\, d\xi$$

Examples

① $$\int_C \frac{e^{z^2}}{z-2}\, dz$$

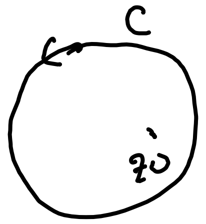


$= 2\pi i\, e^4$

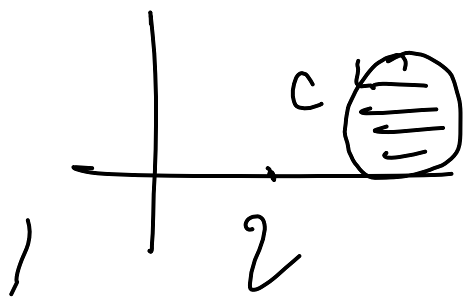
$\left(f(z) = e^{z^2},\ z_0 = 2 \right)$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}\, d\xi$$

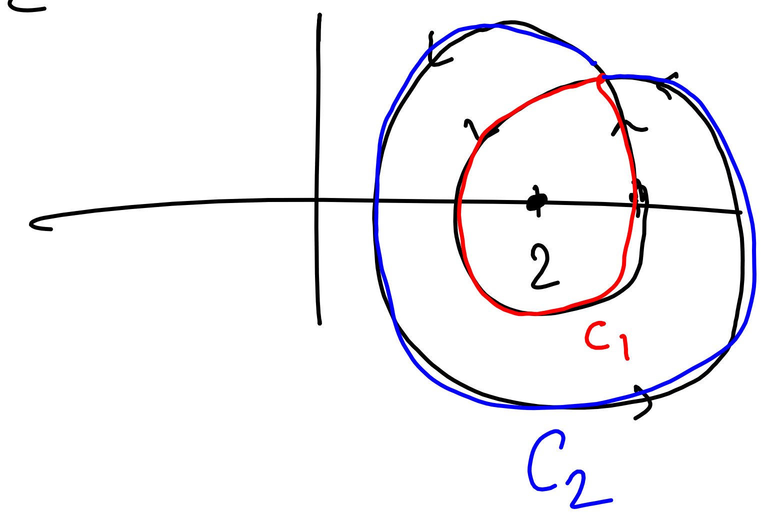
$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\, dz$$



② $$\int_C \frac{e^{z^2}}{z-2}\, dz = 0$$



(3) $$\int_C \frac{e^{z^2}}{z-2}\, dz$$



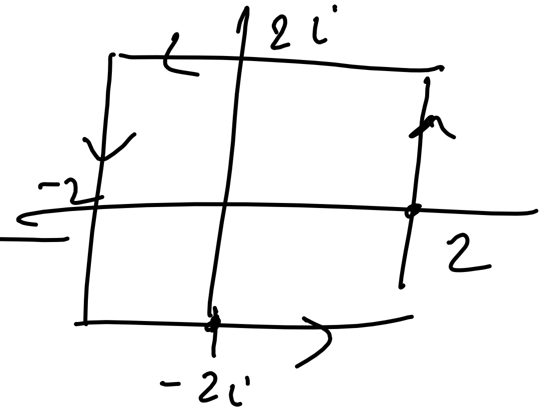
$$= \int_{C_1} + \int_{C_2}$$

$$= 2\pi i\, e^4 + 2\pi i\, e^4$$

$$= 4\pi i\, e^4$$

(4) $$\int_C \frac{\cos z}{z(z^2+8)}\, dz$$

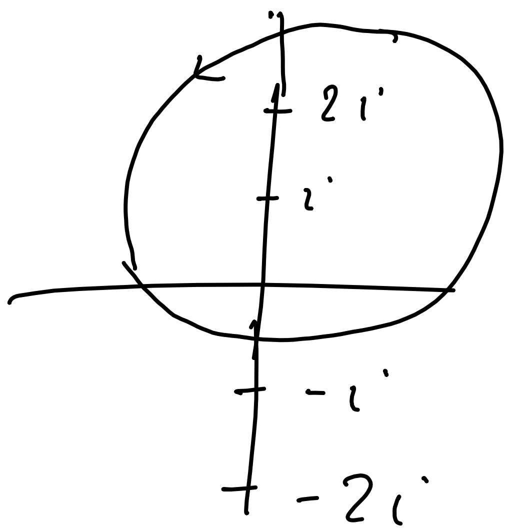
$$= \int_C \frac{\overbrace{\cos z / z^2+8}^{f}}{z}\, dz$$



$$= 2\pi i\, f(0)$$

$$= 2\pi i \cdot \frac{1}{8} = \frac{\pi i}{4}$$

(5) $$\int_C \frac{dz}{(z^2+4)^2}$$



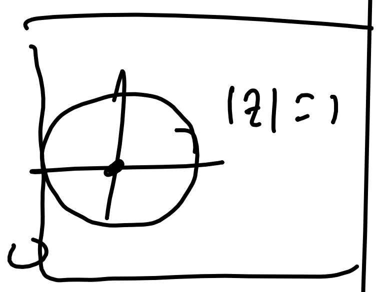
$$= \int_C \frac{dz}{(z+2i)^2(z-2i)^2}$$

$$= \int_C \frac{1/(z+2i)^2}{(z-2i)^2}\, dz \qquad 1/(z+2i)^2 = f$$

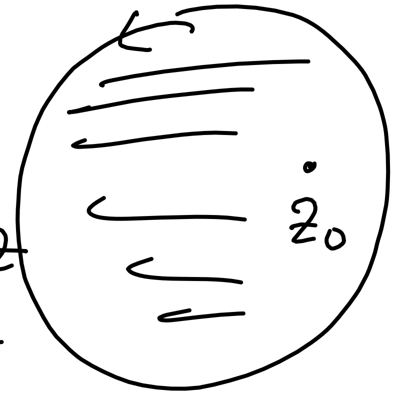
$z_0 = 2i$

$$= 2\pi i\, f'(2i)$$

(6) $$\int_{|z|=1} \frac{e^{2z}}{z^4} = \frac{2\pi i}{3!}\frac{d^3}{dz^3}\left(e^{2z}\right)\Big|_{z=0}$$



Cauchy's Integral formula for derivatives



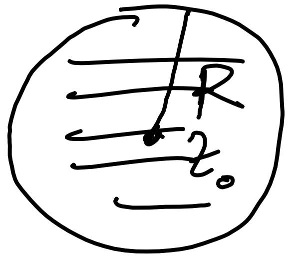
$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\, dz$$

$$f''(z_0) = .$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

Corollary. If z_0 is at the centre.

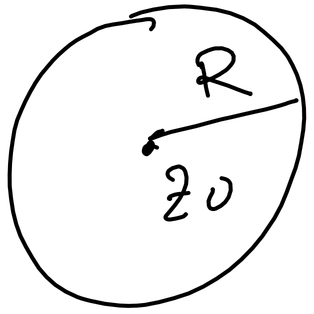
$$|f^{(n)}(z_0)| = \left|\frac{n!}{2\pi i}\int_{C_R}\frac{f(z)}{(z-z_0)^{n+1}}\, dz\right|$$



$$\le \frac{n!}{\cancel{2\pi}}\,\frac{\cancel{2\pi} R}{R^{n+1}}\,\|f\|_{C_R} \quad \left(\|f\|_{C_R} = \sup_{C_R}|f|\right)$$

$$\boxed{\left|f^{(n)}(z_0)\right| \le \frac{n!}{R^n}\,\|f\|_{C_R}}$$

Cauchy's inequality.



(Liouville's Th)

Corollary: f is entire ($\in H(\mathbb{C})$) + f bounded $|f(z)|<M$ $\forall z \in \mathbb{C}$

$\Rightarrow f$ is constant.

Prf: Applying Cauchy's inequality for ~~derivative~~ f'

$$|f'(z_0)| \le \frac{1}{R} M$$

— true for any $R>0$.

Making R large, we get that $f'(z_0) = 0$ $\forall z_0 \in \mathbb{C}$.

$\Rightarrow f = \text{constant}.$ ∎

Corolly (Fundaml Th of algebra)

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n \quad a_n \neq 0$$

$a_i \in \mathbb{C}$.

Any polynomil has a root in $\mathbb{C}$.

Prf

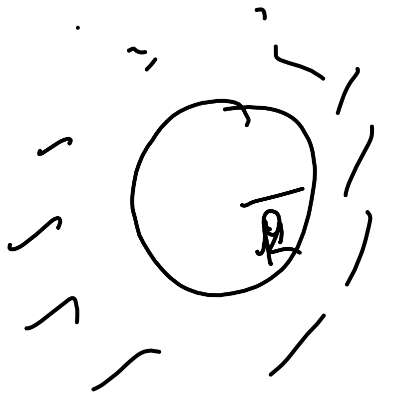
- $p(z)$ - entire.
- $|p(z)| \longrightarrow \infty$ as $z \to \infty$.

If $p(z)$ has no root in $\mathbb{C}$.

$\Rightarrow g(z) = \dfrac{1}{p(z)}$ - entire

$$|g(z)| = \frac{1}{|p(z)|} \longrightarrow 0 \quad \text{as } z \to \infty$$

$$\boxed{\begin{array}{l} \varepsilon > 0 \ \exists R \\ |g(z)| < \varepsilon \\ \quad \forall |z| > R \end{array}}$$



$\Rightarrow g(z)$ is bondd

$\Rightarrow g = $ cnst

$\Rightarrow f = $ cnst

which is a contradictn