MSO 202A: Complex Variables
August-September 2022
Assignment-2

1. Show that absolute convergence implies convergence of a series.

   Solution:

   Assume that $\sum |z_n|$ is convergent. Let $s_n = z_1 + z_2 + \cdots z_n$ and $s_n' = |z_1| + |z_2| + \cdots |z_n|$. To show the sequence s_n is convergent, we show that it satisfies the Cauchy criterion.

   By using triangle inequality, $|s_{n+p} - s_n| \leq s'_{n+p} - s'_n = |s'_{n+p} - s'_n| < \epsilon$ for $n \geq n_0$ and for all $p \geq 1$. Hence s_n is a Cauchy sequence and so convergent.

2. Assume $a_n \neq 0$ except finitely many terms. Show that $1/R = \lim \frac{|a_{n+1}|}{|a_n|}$, provided the limit exists.

3. Let $z_n = x_n + \iota y_n$, where $x_n, y_n \in \mathbb{R}$. Show that $\sum z_n$ is convergent if and only if $\sum x_n$ and $\sum y_n$ are convergent. Moreover $\sum z_n = \sum x_n + \iota \sum y_n$. Use this to conclude that $e^{iy} = \cos y + \iota \sin y, \quad y \in \mathbb{R}$.

   Solution:

   Let $\sum z_n = a + \iota b$

   $s_n = \sum_{k \leq n} z_k, \quad s_n' = \sum_{k \leq n} x_k, \quad s_n'' = \sum_{k \leq n} y_k.$

   Clearly, $s_n = s_n' + \iota s_n''$. Now

   $|s_n - a - \iota b| \leq |s_n' - a| + |s_n'' - b|$. Then $\sum x_n$ converges to a and $\sum y_n$ converges ro b implies $\sum z_n$ converges to $a + \iota b$.

   On the other hand, assume $\sum z_n$ converges to $a + \iota b$.

   Then $|s_n' - a| \leq |s_n - a - \iota b| \ \ (|\operatorname{Re}(z)| \leq |z|)$. This shows $\sum x_n$ converges to a. Similarly, $\sum y_n$ converges to b.

   Put $z = \iota y$ in the power series of e^z and apply the above.

4. Given an example to show that $\operatorname{Log}(z^n) \neq n \operatorname{Log}(z)$.

5. Assume that we choose the branch $\sqrt{z} = e^{1/2 \operatorname{Log} z}$. Given an example to show that $\sqrt{z_1 z_2} \neq \sqrt{z_1}\sqrt{z_2}$.

   Solution:

   Let $z_1 = e^{\iota\pi/2}$ and $z_2 = e^{\iota 2\pi/3}$. Then $z_1 z_2 = e^{\iota 7\pi/6}$. So $\operatorname{Log}(z_1) = \iota\pi/2$, $\operatorname{Log}(z_2) = \iota 2\pi/3$, $\operatorname{Log}(z_1 z_2) = \iota(7\pi/6 - 2\pi) = -\iota 5\pi/6$,

   $\sqrt{z_1}\sqrt{z_2} = e^{1/2 \operatorname{Log} z_1} e^{1/2 \operatorname{Log} z_2} = e^{1/2 \operatorname{Log} z_1 + 1/2 \operatorname{Log} z_2} = e^{\iota 7\pi/12}$

   $\sqrt{z_1 z_2} = e^{1/2 \operatorname{Log}(z_1 z_2)} = e^{-\iota 5\pi/12} = -e^{\iota\pi} e^{-\iota 5\pi/12} = -e^{\iota 7\pi/12}$

6. Draw the domain and range of the complex log branches $\log_0, \log_{2\pi}, \log_{-2\pi}, \log_{-\pi}, \log_{\pi}, \log_{\pi/4}$. Calculate complex logrithm of -1 using the first 3 branches.

   Calculate complex logrithm of 1 using the last 3 branches. Can you do it using the first three branch?

   Solution:

   $\log_0(-1) = \log(1) + \pi\iota = \pi\iota$

   $\log_{2\pi}(-1) = \log(1) + 3\pi\iota = 3\pi\iota$

   $\log_{-2\pi}(-1) = \log(1) - \pi\iota = -\pi\iota$

   $\log_{\pi/4}(-1) = \log(1) + 3\pi\iota = \pi\iota$

   $\log_{-\pi}(1) = \log(1) + 0\iota = 0$

   $\log_{\pi}(1) = \log(1) + 2\pi\iota = 2\pi\iota$

   $\log_{\pi/4}(1) = \log(1) + 2\pi\iota = 2\pi\iota$

   No, it can not be done since 1 is not included in the domain of the first 3 branches.

   Remark: Note that first three and branches on the same domain $\mathbb{C}^*$ minus positive x-axis. Similarly, $\log_{-\pi} = \text{Log}$ and $\log_{\pi}$ are branches on same domain $\mathbb{C}^*$ minus the negative x-axis.

7. Where is the function $f(z) = \log_{3\pi/2}(3 - 5z)$ analytic? What is $f(1)$ and $f(0)$.

   Solution:

   We can write the given function as composition of two function $f(z) = \log_{3\pi/2}(3-5z) = \log_{3\pi/2}(g(z))$ where $g(z) = 3 - 5z$. The complex logarithm branch is $\log_{3\pi/2}$ defined on the $\mathbb{C}$ minus the negative y-axis. Now the $g(z)$ maps the vertical line $x = 3/5, x \geq 0$ onto the negative y-axis. Thus f is analytic on $\mathbb{C}$ minus the vertical line $x = 3/5, x \geq 0$.

   $f(1) = \log_{3\pi/2}(-2) = \log(2) + 3\pi\iota$

   $f(0) = \log_{3\pi/2}(3) = \log(3) + 2\pi\iota$

8. Let $\Omega, U \subseteq \mathbb{C}$ be open sets such that $f : \Omega \to U$ f is bijective analytic with $f'(z) \neq 0$ and $f^{-1} = g$ is continuous. Then show that g is analytic.

   (Remark: Think of the situation $U = \mathbb{C}^*$ and $\Omega = \{z| \quad -\pi < y < \pi\}, \quad f(z) = e^z, \quad g = \text{Log}$. We used this result to prove that Log or any other branch $\log_\alpha$ is analytic.)

   Solution:

   Let $w_0 \in U$. Then there exist $z_0 \in \Omega$ such that $f(z_0) = w_0$. Now,

$$\lim_{h\to 0} \frac{g(w_0+h) - g(w_0)}{h} = \lim_{h\to 0} \frac{g(f(z_0)+h) - g(f(z_0))}{h} = \lim_{h\to 0} \frac{g(f(z_0+h')) - z_0}{h}$$

$$= \lim_{h\to 0} \frac{z_0 + h' - z_0}{h} = \lim_{h\to 0} \frac{h'}{f(z_0+h') - f(z_0)} = \lim_{h'\to 0} \frac{1}{(f(z_0+h') - f(z_0))/h'} = 1/f'(z_0).$$

Here h' is such that $f(z_0)+h = f(z_0+h')$. Applying g on both side $g(f(z_0)+h) = z_0+h'$. Since g is continuous $h \to 0 \implies h' \to 0$. Continuity of f implies the other implication.

9. Write the following in the form $a + \iota b$.

   (a) $\log(\text{Log } \iota)$ (b) $(\iota)^{-\iota}$

   Solution:

   (a) $\log(\text{Log } \iota) = \log(\iota\pi/2) = \log(\pi/2) + \iota(\pi/2 + 2k\pi), \quad k \in \mathbb{Z}$.

   (b) $(\iota)^{-\iota} = \exp(-\iota \log \iota) = \exp(-\iota(\iota\pi/2 + \iota 2k\pi)) = \exp(\pi/2 + 2k\pi), \quad k \in \mathbb{Z}$.

   Remark: $(\iota)^{\iota} = \exp(\iota \log \iota) = \exp(\iota(\iota\pi/2 + \iota 2l\pi)) = \exp(-\pi/2 - 2l\pi), \quad l \in \mathbb{Z}$. If we multiply ι^{ι} with $\iota^{-\iota}$ we obtain infinitely many values $e^{2k\pi}$. So

   $$\iota^{\iota}\iota^{-\iota} \neq \iota^0 = 1.$$

10. Prove or disprove:

    $$\lim_{z\to 0} z \sin\frac{1}{z} = 0$$

    Solution: We know that if $z \to 0$ along x-axis then the limit is

    $$\lim_{z\to 0} z \sin\frac{1}{z} = \lim_{x\to 0} x \sin\frac{1}{x} = 0,$$

    since $\sin(y)$ is bounded for $y \in \mathbb{R}$.

    Let us now calculate the limit for $z \to 0$ along y-axis. In this case,

    $$\begin{aligned}\lim_{z\to 0} z \sin\frac{1}{z} &= \lim_{a\to 0} a\iota \sin\frac{1}{a\iota} \\ &= \lim_{a\to 0} a\iota \frac{e^{1/a} - e^{-1/a}}{2\iota} \\ &= \lim_{a\to 0} a \frac{e^{1/a} - e^{-1/a}}{2}.\end{aligned}$$

    Now the last expression is a limit of real variables. We can see that $\lim\limits_{a\to 0+} a\frac{e^{1/a}-e^{-1/a}}{2} = \lim\limits_{a\to 0-} a\frac{e^{1/a}-e^{-1/a}}{2} = \infty$.

    Thus $\lim\limits_{z\to 0} z \sin\frac{1}{z}$ does not exists.

11. Prove that the each of the three series the radius of convergence is 1. Further show the following:

    (a) Show that $\sum\limits_{n=1}^{\infty} nz^n$ does not converge at any point on the unit circle.

(b) Show that $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges at all points on the unit circle except.

(c) Show that $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges at all points on the unit circle except at $z = 1$.

Solution:

(a) By the Root test, $R = 1$. On the circle $|z| = 1$, the n-th term of the series $|nz^n| = n$ which does not go to zero as $n \to \infty$. So the series does not converge on the unit circle.

(b) By the Root test, $R = 1$. On the circle $|z| = 1$, the n-th term of the series $|\frac{z^n}{n^2}| = \frac{1}{n^2}$. We know $\sum \frac{1}{n^2}$ is convergent. Hence the series converges absolutely on each point of unit circle.

(c)By the Root test of power series

$$\frac{1}{R} = \lim \sum_{n \to \infty} \sqrt[n]{a_n} = \lim \sum_{n \to \infty} \sqrt[n]{n} = 1.$$

Thus $R = 1$.

Clearly for $z = 1$ the series diverges, being the Harmonic series. For other points on the circle, we apply Dirichlet test: which states that if b_n is a sequence (of real or complex number) such that the partial sums are bounded, and a_n is a decreasing sequence (of real numbers), which goes to zero as n goes to infinity, then the infinite series $\sum a_n b_n$ converges. We apply this to our sum with $a_n = 1/n$, and $b_n = z^n$.

Clearly a_n meets the conditions. To show that the partial sums of the b_n are bounded, $|z + z^2 + \cdots + z^n| = |z\frac{z^n - 1}{z-1}| \leq \frac{2|z|}{|z-1|}$– which is independent of n. So the partial sums are bounded except for the point $z = 1$. Hence completes the proof.

12. Consider $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = e^{-\frac{1}{x^2}}$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. Then:

(a) Calculate $f', \;\; f'', \;\; f'''$.

(b) Prove derivative of $\frac{c}{x^p}e^{-1/x^2}$ consists of sum of terms of similar form. Hence deduce that $f^{(n)}(x)$ consists of sum terms of the form $\frac{c}{x^p}e^{-1/x^2}$ for different $c, p \in \mathbb{N}$.

(c) Prove that

$$\lim_{x \to 0} \frac{c}{x^p} e^{-1/x^2} = 0, \quad c, p \in \mathbb{N}.$$

(d) Deduce that $f^{(n)}(0) = 0$ for all n.

(e) Thus conclude that f is infinitely differentiable but f can not be represented by a power series..

[Recall: A real function $f : \mathbb{R} \to \mathbb{R}$ is said to be *real analytic* at x_0 if $f(x)$ can be written as a convergent power series $\sum a_n(x - x_0)^n$. We know that any (complex) analytic function is infinitely differentiable BUT there exists infinitely differentiable real valued function which is NOT real analytic.]

Solution:

(a)

$$f'(x) = \frac{2}{x^3}e^{-1/x^2},\ f''(x) = \frac{4}{x^6}e^{-1/x^2} - \frac{6}{x^4}e^{-1/x^2},\ f'''(x) = \frac{8}{x^9}e^{-1/x^2} - \frac{36}{x^7}e^{-1/x^2} + \frac{24}{x^5}e^{-1/x^2}.$$

(b)

$$\frac{d}{dx}\left(\frac{c}{x^p}e^{-1/x^2}\right) = -\frac{pc}{x^{p+1}}e^{-1/x^2} + \frac{2c}{x^{p+3}}e^{-1/x^2}.$$

Clearly, by induction, $f^{(n)}(x)$ consists of sum terms of the form $\frac{c}{x^p}e^{-1/x^2}$ for different $c, p \in \mathbb{N}$.

(c)

$$\lim_{x\to 0} \frac{c}{x^p}e^{-1/x^2} = \lim_{u\to\infty} cu^p e^{-u^2} = \lim_{u\to\infty} \frac{cu^p}{e^{u^2}} = 0. \quad c, p \in \mathbb{N}.$$

(d) Combining (b) and (c) we conclude that $f^{(n)}(0) = 0$ for all n.

(e) If $f(x) = \sum a_n x^n$ on a nbd of 0, then $a_n = f^{(n)}(0)/n! = 0$. Hence $f = 0$ on a nbd of 0. This is a contradiction.

13. Prove that if p is a polynomial then

$$\lim_{z\to\infty} |p(z)| = \infty$$

. However,

$$\lim_{z\to\infty} |e^z| \neq \infty.$$

Solution: Let $p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ with $a_0 \neq 0$. $p(z) = z^n(a_0 + a_1/z + \cdots + a_n/z^n) = z^n g(z)$. Now $\lim\limits_{z\to\infty} g(z) = a_0 \neq 0$. Therefore $|g(z)|$ is bounded away from 0 for large z: i.e.. there exist $r, R > 0$ such that $|g(z)| > r$ for all $|z| > R$. To see that take $\epsilon = |a_0|/2$ and applly the definition of limit. So $|z^n g(z)| > r|z|^n$ for $|z| > R$. Hence $\lim\limits_{z\to\infty} z^n g(z) = \infty$.

Along y-axis, $|e^z| = |e^{\iota\theta}| = 1$. So. $\lim_{|z|\to\infty} |e^z| \neq \infty$.

14. Identify the following series as a holomorphic function f(z):

$$(a)\sum_{n=1}^{\infty} nz^n,\quad (b)\sum_{n=1}^{\infty} n^2 z^n,\quad (c)\sum_{n=1}^{\infty}(-1)^n\frac{z^n}{(2n+1)!},\quad (d)\sum_{n=1}^{\infty}(-1)^n\frac{z^{2n}}{2^n n!}$$

Solution:

(a) $1/(1-z) = \sum z^n \implies 1/(1-z)^2 = \sum nz^{n-1} \implies z/(1-z)^2 = \sum nz^n$.

(b) $\sum_{n=1}^{\infty} n^2 z^n = z\frac{d}{dz}(z/(1-z)^2)$

(c) $\sum_{n=1}^{\infty}(-1)^n \frac{z^n}{(2n+1)!} = \frac{\sin\sqrt{z}}{\sqrt{z}}$ ***

(d) $\sum_{n=1}^{\infty}(-1)^n \frac{z^{2n}}{2^n n!} = e^{-z^2/2}$

*** Remark: The power seires is analytic on $\mathbb{C}$ where as $\sqrt{z}$ (and so $\sin\sqrt{z}$) is analytic on $\mathbb{C}$ minus negative x -axis. Around the branch cut $\sqrt{z}$ changes to -$\sqrt{z}$ and $\sin\sqrt{z}$ chnages to $-\sin\sqrt{z}$. Thus the ratio $\sin(\sqrt{z})/\sqrt{z}$ is single valued anlytic on $\mathbb{C}$.

15. Let $f(z) = \frac{1}{z(z-1)}$. Where is the function analytic? Can you write f as a power series around $z = 2\iota$? What is the radius of convergence of this power series?

Solution:

The function is analytic in $\Omega = \mathbb{C} \setminus \{0, 1\}$.

Let $w = z - 2\iota$. Then we can write $f = \frac{1}{(w+2\iota)(w+2\iota-1)} = \frac{1}{w+2\iota-1} - \frac{1}{w+2\iota}$.

Note that

$$\frac{1}{w+\alpha} = \frac{1}{\alpha}\frac{1}{1+(w/\alpha)} = \frac{1}{\alpha}\sum(-w/\alpha)^n = \sum\frac{(-1)^n}{\alpha^{n+1}}w^n \quad for \quad |w| < |\alpha|.$$

Therefore $f(z) = \sum a_n(z - 2\iota)^n$ where $a_n = (-1)^n[\frac{1}{(2\iota-1)^{n+1}} - \frac{1}{(2\iota)^{n+1}}]$.

The radius of convergence of this power series is given by the minimum of the distance of 2ι from the singularity points $0, 1$ that is $R = |2\iota - 0| = 2$.

(The radius of convergence of a series represents the distance in the complex plane from the expansion point to the nearest singularity of the function expanded.)