MSO 202A: Complex Variables
August-September 2022
Assignment-0

Exercises marked (**T**) are to be discussed in the tutorials.

1. (**T**)Let $P(z)$ be a polynomial with real coefficients. Show that if z_0 is a root of P then so is $\overline{z}_0$.

   Solution: Since the coefficinets are real, we have $\overline{P(z)} = P(\overline{z})$.

2. Solve the following equations in polar form and locate the roots in the complex plane:
   (a) $z^4 = -1$

   (b) (**T**)$z^4 = -1 + \sqrt{3}\iota$

   Solution:

   (a) Write $z^4 = -1 = e^{\iota\pi + 2\iota k\pi}, \quad k \in \mathbb{Z}$.

   Roots are

   $e^{\iota\pi/4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\iota,$
   $e^{\iota 3\pi/4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\iota,$
   $e^{\iota 5\pi/4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\iota,$
   $e^{\iota 7\pi/4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\iota.$
   (b) $z^4 = -1 = 2e^{2\iota\pi/3} = 2e^{2\iota\pi/3 + 2\iota k\pi}, \; k \in \mathbb{Z}$.

   Roots are

   $\sqrt[4]{2}e^{\iota\pi/6} = \sqrt[4]{2}(\frac{\sqrt{3}}{2} + \frac{1}{2}\iota),$
   $\sqrt[4]{2}e^{\iota 2\pi/3} = \sqrt[4]{2}((\iota\frac{\sqrt{3}}{2} - \frac{1}{2})),$
   $\sqrt[4]{2}e^{\iota 7\pi/6} = \sqrt[4]{2}(-\frac{\sqrt{3}}{2} - \frac{1}{2}\iota),$
   $\sqrt[4]{2}e^{\iota 5\pi/3} = \sqrt[4]{2}(-\iota\frac{\sqrt{3}}{2} + \frac{1}{2})).$

3. Simplify $(1+\iota)^{17}$ into the form $a + b\iota$.

   Solution: $1 + \iota = \sqrt{2}(\cos(\pi/4) + \iota\sin(\pi/4))$.

   Thus $(1+\iota)^{17} = (\sqrt{2})^{17}(\cos(17\pi/4) + \iota\sin(17\pi/4)) = 256(1+\iota)$

4. Show that if two integers can be expressed as the sum of two squares, then so can their product.

   Solution: Let $M = a^2 + b^2$ and $N = c^2 + d^2$ where $a, b, c, d \in \mathbb{Z}$. Take $z = a + \iota b$ and $w = c + \iota d$. Then $MN = |z|^2|w|^2 = |zw|^2 = \text{Re}(zw)^2 + \text{Im}(zw)^2$.

5. (**T**)Show that the n-th roots of 1 (aside from 1) satisfy the cyclotomic equation $z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$

   Solution:
$$z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + z + 1).$$
   Let ω be a n-th roots of 1 (aside from 1). Putting $z = \omega$ in the above identity, the left hand side becomes 0. But $\omega \neq 1$. So $\omega^{n-1} + \omega^{n-2} + \cdots + \omega + 1 = 0$

6. (**T**)Consider the $n-1$ diagonals of a regular n-gon inscribed in a unit circle obtained by connecting one vertex with all the others. Show that the product of their lengths is n.

   Solution: Let the vertices of the regular n-gon be $1, a_1, a_2, \cdots, a_{n-1}$. The the required product of their lengths of the diagonals is $|1-a_1| \cdots |1-a_{n-1}|$. By the previous problem, $a_1, a_2, \cdots, a_{n-1}$ are the roots of the equation $z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$. So $z^{n-1} + z^{n-2} + \cdots + z + 1 = (z-a_1)\cdots(z-a_{n-1})$. Putting $z = 1$ and taking modulus, we have the desired result.

7. Let ω be a p-th root of unity. Define
$$\chi(p) = \sum_{n=0}^{p-1} \omega^{n^2}.$$
   Verify that $\chi(3)^2 = -3,\ \chi(5)^2 = 5,\ \chi(7)^2 = -7.$

   (Remark: The expression $\chi(p)$ is known as Gauss Sum. For odd prime p it can be shown that $\chi(p)^2 = (-1)^{\frac{p-1}{2}} p$.)

   Solution:

   $\chi(3)^2 = (1+2\omega)^2 = 1 + 4\omega + 4\omega^2 = -3.$

   $\chi(5)^2 = (1+2\omega+2\omega^4)^2 = 1+4\omega^2+4\omega^8+4\omega+4\omega^4+8\omega^5 = 5+4(1+\omega+\omega^2+\omega^3+\omega^4) = 5.$

   $\chi(7)^2 = (1+2\omega+2\omega^2+2\omega^4)^2 = 1+8\omega+8\omega^2+8\omega^3+8\omega^4+8\omega^5+8\omega^6 = -7.$

8. For each of the following equations, give a geometric description of the set of complex numbers. (a) (**T**)$|z-z_1| = |z-z_2|$ (b) $|z-z_1| + |z-z_2| = c$ (c) $|z-2+3\imath| < 1$ (d) (**T**)$0 \leq z < \pi/4$ (e) $|z-4| \geq |z|$ (f) $|\operatorname{Re} z| \geq a > 0$

   Solution:

   (a) The equation $|z-z_1| = |z-z_2|$ exactly expresses the fact that z is the same distance to z_1 as it is to z_2. From geometry, this set of points is just the perpendicular bisector of the segment connecting z_1 and z_2 (the line which is perpendicular to this segment and passes through the midpoint of this segment).

   (b) The triangle inequality can also be expressed in the form $|z_1|+|z_2| \geq |z_1-z_2|$. (Take the usual form of the triangle inequality and replace z_2 with -z_2). Then applying this form of the triangle inequality gives $|z-z_1| + |z-z_2| \geq |z_1-z_2|$.

Therefore, if $c < |z_1 - z_2|$, it is impossible for $|z - z_1| + |z - z_2| = c$ to have any solutions, so the set in question is the empty set.

Suppose $c = |z_1 - z_2|$. Then z must lie on the line segment connecting z_1 and z_2. Indeed, if z is not on the line connecting z_1, z_2, then z, z_1, z_2 are not collinear, and since z, z_1, z_2 form the vertices of an actual triangle, the triangle inequality yields $|z-z_1|+|z-z_2| > |z_1-z_2|$. Also, if z is on the line connecting z_1, z_2 but not on the segment between them, then one of $|z - z_1|, |z - z_2|$ is greater than $|z_1 - z_2$, so $|z - z_1| + |z - z_2| > c$ would be impossible.

Finally, if $c > |z_1 - z_2|$, then the set of points in question form an ellipse. This is actually one of the possible definitions of an ellipse: as the set of points whose sum of distances from two fixed points is constant. Also, z_1, z_2 are the foci of this ellipse.

(c) Open disc with center at $2 - 3\iota$ with radius 1.

(d) region between two rays $\theta = 0$ and $\theta = \pi/4$.

(e) Note that $|z - 4| = |z|$ represents the line perpendicularly bisecting $z = 4$ and $z = 0$, which is the line $x = 2$. Thus the inequality represents the half plane, given by $x \leq 2$.

(f). $x \geq a$ or $x \leq -a$

9. In each following functions $f(z)$, compute the limit $\lim\limits_{z\to 0} f(z)$. Hence conclude whether the functions can be defined at $z = 0$ to become continuous.

   (**T**)(a) $2z\frac{\operatorname{Re} z}{|z|}$ (**T**)(b) $\frac{\iota z}{|z|}$ (c) $3\frac{\operatorname{Re} z}{z}$

   Solution:

   (a) $|f(z)| = |2z\frac{\operatorname{Re} z}{|z|}| = |\operatorname{Re} z| \to 0$ as $z \to 0$. Thus the limit is 0. So if we define $f(0) = 0$, then the function is continuous at 0.

   (b) $f(z) = \frac{-y+\iota x}{\sqrt{x^2+y^2}}$. If $z \to 0$ along positive side of x axis then $f(z) \to \iota$. If $z \to 0$ along positive side of y axis then $f(z) \to -1$. Thus $\lim\limits_{z\to 0} \frac{-y+\iota x}{\sqrt{x^2+y^2}}$ does not exist and hence the function can not be made continuous at $z = 0$.

   (c). $f(z) = \frac{3x}{x+\iota y}$. If $z \to 0$ along x axis then $f(z) \to 3$. If $z \to 0$ along y axis then $f(z) \to 0$. Thus $\lim\limits_{z\to 0} \frac{3x}{x+\iota y}$ does not exist and hence the function can not be made continuous at $z = 0$.

10. (**T**)Let

$$f(z) = \frac{\{(1-\iota)z + (1+\iota)\bar{z}\}^2}{z\overline{z}}.$$

   Show that $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(z) = \lim\limits_{y\to 0}\lim\limits_{x\to 0} f(z)$ but $\lim\limits_{z\to 0} f(z)$ does not exist.

   Solution:

   $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(z) = \lim\limits_{x\to 0} \frac{\{(1-\iota)x+(1+\iota)x\}^2}{x^2} = 4$. Similarly $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(z) = 4$. But along the line $y = -x$, the limit is 0. So $\lim\limits_{z\to 0} f(z)$ does not exist.