MSO202M Mid semester exam

ARCHIT KUMAR

TOTAL POINTS

40 / 60

QUESTION 1

1 Question 1 **4 / 6**

+ **+ 0 pts** Incorrect/Unattempted
+ ✓ **+ 4 pts** *Part (a) Correct*
+ **+ 2 pts** (a) For showing $$T(z) = \frac{3z + 2i}{iz+6}$$
+ **+ 2 pts** (a) Two fixed points: $$z=i$$ and $$z=2i$$.
+ **+ 2 pts** Part (b) Correct
+ **+ 1 pts** (b) Yes, Mobius transformation exist.
+ **+ 1 pts** (b) For correct justification/example.

QUESTION 2

2 Question 2 **5 / 6**

+ **+ 0 pts** Incorrect/Unattempted
+ **+ 6 pts** Correct
+ ✓ **+ 1 pts** *For finding all three poles of $$f(z) = \frac{ze^{2zi}}{z^3+3}$$. $$z_1 = -3^{1/3}, z_2=z_1\omega, z_3 = z_1\omega^2$$ with $$\omega = \frac{-1+i\sqrt{3}}{2}$$.*
+ ✓ **+ 1 pts** *For contour $$C=\Gamma_R \cup [-R,-(z_1+r)] \cup \gamma_{z_1,r} \cup [-(z_1-r), R]$$.*
+ ✓ **+ 1 pts** *$$\text{Res}_{z = z_3} = \frac{z_3\,e^{2z_3i}}{(z_3-z_1)\,(z_3-z_2)}$$.*
+ ✓ **+ 1 pts** *For showing $$\int_{\Gamma_R} f(z) dz = 0$$.*
+ ✓ **+ 1 pts** *$$\int_{\gamma_{z_1,r}}\,f(z)\, dz \,=\, \,\frac{-\pi i\,z_1\,e^{2z_1i}}{(z_1-z_2)\,(z_1-z_3)}$$.*
+ **+ 1 pts** $$\int_{-\infty}^\infty \frac{x\sin{2x}}{x^3+3} dx = \text{Im} \left(\frac{2 \pi i\,z_3\,e^{2z_3i}}{(z_3-z_1)\,(z_3-z_2)} + \frac{\pi i\,z_1\,e^{2z_1i}}{(z_1-z_2)\,(z_1-z_3)}\right)$$.

QUESTION 3

3 Question 3 **6 / 6**

+ **+ 0 pts** Incorrect/Unattempted.
+ ✓ **+ 3 pts** *For showing $$p(z)$$ has no zero in the disc $$|z|<1$$.*
+ ✓ **+ 3 pts** *For showing all four zeros of $$p(z)$$ are in annulus $$1<|z|<3$$.*

QUESTION 4

4 Question 4 **5 / 6**

+ **+ 0 pts** Incorrect/Unattempted
+ **+ 6 pts** Correct
+ ✓ **+ 2 pts** *For correct C-R equations: $$u_r = \frac{1}{r}v_\theta$$ and $$v_r = \frac{-1}{r}u_\theta$$.*
+ ✓ **+ 2 pts** *For showing $$v(r,\theta) = \sqrt{r}\sin{(\theta/2)} + C,$$ where $$C$$ is constant.*
+ ✓ **+ 1 pts** *$$f(z) = \sqrt{r}\, e^{i \theta/2} = \sqrt{z}$$.*
+ **+ 1 pts** $$f'(z) = \frac{1}{2 \sqrt{r}}\, e^{-i \theta/2} = \frac{1}{2 \sqrt{z}}$$.

QUESTION 5

5 Question 5 0 / 6

✓ **+ 0 pts** *Incorrect/ Not Attempt*

+ 6 pts Correct

+ 2.5 pts $$S_1 = \sum_{n=0}^{\infty} \left(\frac{z}{5}-i\right)^{n^2}$$ converges for $$|z-5i| < 5$$

+ 1.5 pts For $$R_1 = \left|\frac{a_n}{a_{n+1}} \right|^{\frac{1}{\lambda_{n+1}-\lambda_{n}}} $$ where $$a_{n} = \frac{1}{5^{n^2}}$$ and $$\lambda_{n} = n^2$$

+ 1 pts For correctly calculating $$R_1=5$$.

+ 2.5 pts $$S_2=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n (z-5i)^n}$$ converges for $$|z-5i| > \frac{1}{2}$$

+ 1.5 pts For $$R_2 = \left|\frac{a_n}{a_{n+1}} \right|^{\frac{1}{\lambda_{n+1}-\lambda_{n}}} $$ where $$a_{n} = \frac{(-1)^n}{5^{n}}$$ and $$\lambda_{n} = -n$$

+ 1 pts For correctly calculating $$R_2=\frac{1}{2}$$

+ 1 pts For convergence of annulus $$\frac{1}{2} < |z-5i| < 5$$

QUESTION 6

6 Question 6 1 / 6

+ 6 pts Correct

+ 0 pts wrong

+ 3 pts part a correct

+ 3 pts part b correct

+ 2 pts correct statement of schwarz lemma

+ 1 pts equality condition in schwarz lemma

✓ **+ 1 pts** *assuming $g(z)=1/|f(z)|$ for $|z|\leq 1$*

+ 1 pts $|g(z)|\leq 1$ in $|z|\leq 1$ and $g(0)=1$

+ 1 pts concluding with max. modulus principle that $f(z)$ is constant.

QUESTION 7

7 Question 7 6 / 6

+ 0 pts Incorrect/ Not attempted

✓ **+ 6 pts** *Completely correct !*

+ 1 pts For writing $$f(z)$$ has simple pole at $$z=0$$, and poles of order 2 at $$z= \pm 2$$ (without justification)

+ 1 pts for writing the following formula correctly : At $$z=z_0$$ (where $$z_0$$ is a pole of order n)$$\text{Res}_{z=z_0}f(z) = \lim_{z\rightarrow z_0}\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}[(z-z_0)^{n}f(z)]$$

+ 1 pts **Residue at $$z=0\\$$** is the following: $$\lim _{z\rightarrow 0}\frac{z^3 +3}{(z^2-4)^2} = \frac{3}{16}$$

+ 1 pts **Residue at $$z=2\\$$** is the following: $$\lim _{z\rightarrow 2}\frac{d}{dz}\frac{z^3 +3}{z(z+2)^2} = \frac{1}{32}$$

+ 1 pts **Residue at $$z=-2\\$$** is the following: $$\lim _{z\rightarrow -2}\frac{d}{dz}\frac{z^3 +3}{z(z+2)^2} = \frac{-7}{32}$$

+ 1 pts **Sum of residue** $$\frac{3}{16}+\frac{1}{32}+\frac{-7}{32}=0$$

QUESTION 8

8 Question 8 6 / 6

+ 0 pts incorrect/not attempted

✓ **+ 4 pts** *Question (a) Correct*

✓ **+ 2 pts** *Question (b) correct*

+ 1 pts Let $$z=re^{i\theta}$$. Then $$z^3=r^3e^{i3\theta}$$ where $$-\pi<3\theta\le\pi$$. And $$Log(z^3)=3Log(r)+iArg(3\theta)$$

+ 1 pts $$z^{3/2}=e^{3/2Log(z)}=e^{3/2(ln|z|+i\theta)}$$, and $$|z^{3/2}|=e^{3/2(ln|z|}=r^{3/2}$$

+ 1 pts When $$|z|=R>1$$ then $$|\frac{Log(z^3)}{z^{3/2}}|\le \frac{3ln(R)+\pi}{R^{3/2}}$$

+ 1 pts Now length of the contour is $$2\pi R$$. Therefore using M-L inequality , we have $$|\int_{C_R}\frac{Log(z^3)}{z^{3/2}}dz|\le \frac{2\pi(3ln(R)+\pi)}{R^{1/2}}\to 0$$ as $$R\to \infty$$

+ 1 pts Let $$w=(2^i)^i$$ and $$z=2^i$$ then $$w=z^i$$. Now $$z=e^{ilog2}$$

+ 1 pts So $$w=e^{ilog(z)}=e^{iln|z|-argz}=1/2$$

QUESTION 9

9 Question 9 **4 / 6**

+ 0 pts Incorrect/Unattempted

+ 3 pts Part (a) correct

+ 1 pts (a) For $$(x,y) \neq (0,0)$$, $$f(x,y)$$ is continuous.

✓ **+ 2 pts** *(a) For showing continuity at $$(x,y) = (0,0)$$.*

✓ **+ 3 pts** *Part (b) correct*

+ 1 pts (b) $$h_{xx} = u_{xx}v + 2u_xv_x + uv_{xx}$$ and $$h_{yy} = u_{yy}v + 2u_yv_y + uv_{yy}$$.

+ 2 pts (b) $$h_{xx} + h_{yy} = (u_{xx} + u_{yy})v + 2(u_xv_x + u_yv_y) + u(v_{xx} + v_{yy}) = 0.$$

- 1 *Point adjustment*

Incorrect method of proving continuity at (0,0)

QUESTION 10

10 Question 10 **3 / 6**

+ 0 pts Incorrect/Unattempted.

+ 4 pts Part (a) correct

✓ **+ 1 pts** *(a) $$f(z) = \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} z^n.$$*

+ 1 pts (a) Using Identity theorem to show $$g(z) = f(z)$$ for all $$z\in \mathbb{C}$$.

+ 1 pts (a) For showing $$f^{(n)}(0) \in \mathbb{R}$$ for all $$n\geq 0$$.

✓ **+ 1 pts** *(a) For showing converse i.e., $$f(z) \in \mathbb{R}$$ when $$z \in \mathbb{R}$$.*

+ 2 pts Part (b) correct.

✓ **+ 1 pts** *For partially correct part (b).*

Mid Semester: MSO202M (2023-2024 I)

Date: 18 September 2023 Time: 13:00 pm-15:00 pm Maximum marks: 60

Name: Archit Kumar Roll No. 220196

Instructions: (Read carefully)

- Please enter your NAME and ROLL NUMBER in the space provided on EACH page.
- Only those booklets with name and roll number on every page will be graded. All other booklets will NOT be graded.
- This answer booklet has 12 pages. Check to see if the print is either faulty or missing on any of the pages. In such a case, ask for a replacement immediately.
- Please answer each question ONLY in the space provided. Answers written outside the space provided for it WILL NOT be considered for grading. So remember to use space judiciously.
- For rough work, separate sheets will be provided to you. Write your name and roll number on rough sheets as well. However, they WILL NOT be collected back along with the answer booklet.
- No calculators, mobile phones, smart watches, or other electronic gadgets are permitted in the exam hall.
- Notations: All notations used are as discussed in class.
- All questions are compulsory.
- Do NOT remove any of the sheets in this booklet.

Name: Archit Kumar　　　　Roll Number: 220196　　　　

Q 1. (a) Find Möbius transformation T that maps the points $z = i, 2, -2$ to $w = i, 1, -1$. Also, find the fixed points of T.

(b) Is there a Möbius transformation that has exactly one fixed point? Justify.

4+2 marks

→ (a) $w = T(z) = \dfrac{az+b}{cz+d}$, $z_1 = i$, $z_2 = 2$, $z_3 = -2$; $w_1 = i$, $w_2 = 1$, $w_3 = -1$

~~$(z-z_1)(z$~~

For $w_1 = i, z_1 = i$

$i = \dfrac{ai+b}{ci+d}$

$\Rightarrow ai+b = -c+di$ — (I)

For $w_2 = 1, z_2 = 2$

$1 = \dfrac{2a+b}{2c+d}$

$\Rightarrow 2a+b = 2c+d$ — (II)

For $w_3 = -1, z_3 = -2$

$-1 = \dfrac{-2a+b}{-2c+d}$

$\Rightarrow -2a+b = 2c-d$ — (III)

By (II) + (III), we get, $\cancel{2a}+b = 2c+\cancel{d}$, $\cancel{-2a}+b = 2c-\cancel{d}$

$2b = 4c \Rightarrow \underline{b = 2c}$

∴ (III) becomes ⇒ $-2a+b = 2c-d \Rightarrow -2a+\cancel{2c} = \cancel{2c}-d \Rightarrow d = 2a \Rightarrow \underline{a = \frac{d}{2}}$

∴ (I) becomes ⇒ $ai+b = -c+di \Rightarrow ai+2c = -c+2ai \Rightarrow ai = 3c \Rightarrow c = \frac{ai}{3} = \frac{id}{6}$

$b = 2c = \frac{id}{3}$

$$\therefore T(z) = \frac{az+b}{cz+d} = \frac{\frac{dz}{2}+\frac{id}{3}}{\frac{idz}{6}+d} = \frac{\cancel{d}\left(\frac{z}{2}+\frac{i}{3}\right)}{\cancel{d}\left(\frac{iz}{6}+1\right)} = \boxed{\frac{\frac{z}{2}+\frac{i}{3}}{\frac{iz}{6}+1} = T(z)}$$

For fixed point of $T(z) \Rightarrow T(z) = z$

$$\Rightarrow \frac{\frac{z}{2}+\frac{i}{3}}{\frac{iz}{6}+1} = z \Rightarrow \frac{z}{2}+\frac{i}{3} = \frac{iz^2}{6}+z \Rightarrow \frac{iz^2}{6}+\frac{z}{2}-\frac{i}{3} = 0$$

$$\Rightarrow z = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4}+4\cdot\frac{i}{6}\cdot\frac{i}{3}}}{\frac{i}{3}} \Rightarrow \boxed{z = i, 2i}$$

(b) For fixed points, $T(z) = \dfrac{az+b}{cz+d} = z \Rightarrow az+b = cz^2+dz$

$\Rightarrow cz^2-(a-d)z-b = 0$

~~This~~ According to Fundamental theorem of Algebra, this equation must have 2 complex roots. It means it must have 2 fixed points.

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Q 2. Evaluate the integral **6 marks**

$$\int_{-\infty}^{\infty} \frac{x \sin 2x}{x^3+3}\,dx.$$

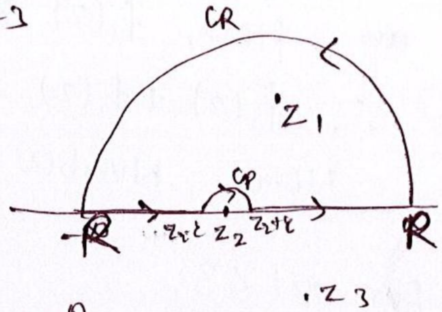
$$\rightarrow \int_{-\infty}^{\infty} \frac{x \sin 2x}{x^3+3} = -i\, \text{Im} \int_{-\infty}^{\infty} \frac{z e^{i2z}}{z^3+3}\, dz$$

$$z^3+3 = 0 \Rightarrow z = \underset{\displaystyle z_1}{\underset{\downarrow}{3^{1/3} e^{i\pi/3}}},\ \underset{\displaystyle -3^{1/3}}{\underset{\downarrow}{\underset{\displaystyle z_2}{\underset{\downarrow}{3^{1/3} e^{i\pi}}}}},\ \underset{\displaystyle z_3}{\underset{\downarrow}{3^{1/3} e^{-i\pi/3}}}$$

The f^n has ~~2~~ 3 poles of order 1 at

$$z = z_1, z_2, z_3$$

z_2 is at real axis.



$$\Rightarrow \int_{-R}^{z_2-\epsilon} f(z)\,dz + \int_{C_p} f(z)\,dz + \int_{z_2+\epsilon}^{R} f(z)\,dz + \underbrace{\cancel{\int_{C_R} f(z)\,dz}}_{R\to\infty}{}^{\to 0} = \int_C f(z)\,dz$$

$$= \int_{-\infty}^{\infty} \frac{z e^{i2z}}{z^3+3} - i\pi \operatorname*{Res}_{z=z_2} \frac{z e^{i2z}}{z^3+3} = 2\pi i \operatorname*{Res}_{z=z_1} \frac{z e^{i2z}}{z^3+3}$$

$$= \int_{-\infty}^{\infty} \frac{z e^{i2z}}{z^3+3} - i\pi \frac{\cancel{z_2} e^{i2z_2}}{3 z_2^{\cancel{2}}} = 2\pi i \frac{\cancel{z_1} e^{i2z_1}}{3 z_1^{\cancel{2}}}$$

$$= \int_{-\infty}^{\infty} \frac{z e^{i2z}}{z^3+3} - i\pi \frac{e^{i2z_2}}{3z_2} = 2\pi i \frac{e^{i2z_1}}{3z_1} \Rightarrow \int_{-\infty}^{\infty} \frac{z e^{i2z}}{z^3+3} = \pi i \left[\frac{2 e^{i2z_1}}{3z_1} + \frac{e^{i2z_2}}{3z_2} \right]$$

2

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Q 3 Find the number of roots of polynomial $p(z) = z^4 - 8z + 10$ in the unit disk $|z| < 1$ and in annulus $1 < |z| < 3$.

6 marks

→ $p(z) = z^4 - 8z + 10$

Let $f_1(z) = 10$, $f_2(z) = z^4 - 8z$

$p(z) = f_1(z) + f_2(z)$

According to Rouche's Theorem,

$|f_2(z)| = |z^4 - 8z| \leq |z|^4 + 8|z| \leq 9$ for $|z| < 1$

and $|f_1(z)| = 10 > 9$ for $|z| < 1$ $\Rightarrow |f_1(z)| > |f_2(z)|$ for $|z| < 1$

Here, $f_1(z)$ has no zero for $|z| < 1$

∴ $f_1(z) + f_2(z)$ will also have no zero for $|z| < 1$

Hence, Number of roots of $P(z)$ in the unit disk $|z| < 1$ $= 0$

For $|z| < 3$,

let $f_1(z) = z^4$, $f_2(z) = 10 - 8z$

$p(z) = f_1(z) + f_2(z)$

$|f_2(z)| = |10 - 8z| < \cancel{|10|}\,10 + |8z| < 10 + 8 \cdot 3 < 34$ for $|z| < 3$

$|f_1(z)| = |z|^4 < \cancel{81}\ |3|^4 < 81$ for $|z| < 3$

According to Rouche's Theorem ⇒ $|f_1(z)| > |f_2(z)|$ for $|z| < 3$.

Here $f_1(z) = z^4$ has 4 zeroes in $|z| < 3$.

∴ $f_1(z) + f_2(z)$ will also have 4 zeroes in $|z| < 3$

Hence, $P(z)$ will have 4 roots in $|z| < 3$.

and it has zero root in $|z| < 1$

∴ $P(z)$ will have 4 roots in annulus $1 < |z| < 3$

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Q 4 Write Cauchy Riemann equation in Polar coordinates. Let $u(r, \theta) = \sqrt{r}\cos(\theta/2)$. Find $f'(z)$ and $f(z)$. **6 marks**

→ ~~$ru_r = v_\theta$~~ In polar coordinates,

~~$rv_r = -u_\theta$~~ $f(r,\theta) = u(r,\theta) + iv(r,\theta)$

Cauchy Riemann equation,

$\Rightarrow \quad ru_r = v_\theta$

$rv_r = -u_\theta$

given $u(r,\theta) = \sqrt{r}\cos(\theta/2)$

$u_r = \frac{1}{2\sqrt{r}}\cos(\theta/2)$, $u_\theta = -\frac{\sqrt{r}}{2}\sin(\theta/2)$

$ru_r = v_\theta \Rightarrow \frac{\sqrt{r}}{2}\cos(\theta/2) = v_\theta$

$\Rightarrow v = \sqrt{r}\sin(\theta/2) + P(r)$

$rv_r = -u_\theta \Rightarrow v_r = \frac{1}{2\sqrt{r}}\sin(\theta/2)$

$v = \sqrt{r}\sin(\theta/2) + Q(\theta)$

$\Rightarrow P(r) = Q(\theta)$ → But $P(r)$ is a fn of 'r' only and $Q(\theta)$ is a fn of 'θ' only,

$\therefore P(r) = Q(\theta) = 0$.

$\therefore v = \sqrt{r}\sin(\theta/2)$

$f(r,\theta) = \sqrt{r}\left[\cos(\theta/2) + i\sin(\theta/2)\right] \Rightarrow f(z) = \sqrt{z}$, where $z = re^{i\theta}$

$= \sqrt{r}e^{i\theta/2} = (re^{i\theta})^{1/2}$

~~$f'(z) = \frac{1}{2\sqrt{z}}$~~

$f'(r,\theta) = u_r + iv_r = \frac{1}{2\sqrt{r}}\left(\cos\theta/2 + i\sin\theta/2\right) = \frac{1}{2\sqrt{r}}e^{i\theta/2}$

$= \frac{\sqrt{r}e^{i\theta/2}}{2r}$

$f(z) = \frac{\sqrt{z}}{2z} = \frac{1}{2\sqrt{z}}$

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Q 5. Find the annulus of convergence of the series **6 marks**

$$\sum_{n=0}^{\infty}\left(\frac{z}{5}-i\right)^{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n(z-5i)^n}.$$

→

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Q 6. (a) Write the statement of Schwarz lemma.

(b) Suppose $f(z)$ is analytic for $|z| \leqslant 1$ and $|f(z)| \geqslant 1$ for $|z| \leqslant 1$. If $f(0) = 1$, then show that f is constant.

3+3 marks

→ (a) Schwarz lemma

If $f(z)$ is an analytic function over $|z| \leq R$ such that $f(0)=0$

$|f(z)| \leq M$ for $|z| \leq R$.

then $|f(z)| \leq \frac{M|z|}{R}$ for $|z| < R$ where ~~$R = |z_0|$ and $z < z_0$~~

$f'(z) < \frac{M}{R}$

(b)

Let $g(z) = \frac{1}{f(z)}$ be another analytic function for $|z| \leq 1$

⇒ $|g(z)| \leq 1$ for $|z| \leq 1$

and $g(0) = 1$.

As we know that if a function is an entire fn^n and it is bounded, then it must be a constant fn^n.

∴ $g(z) = \frac{1}{f(z)}$ is a constant fn^n.

∴ $f(z)$ is a constant fn^n.

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Q 7. Find singularities and residues of the function $f(z)$ in extended complex plane, where

$$f(z) = \frac{z^3+3}{z(z^2-4)^2}.$$

Show that the sum of all residue is zero. **6 marks**

$\rightarrow \quad f(z) = \frac{z^3+3}{z(z^2-4)^2} = \frac{z^3+3}{z(z-2)^2(z+2)^2}$

It has pole of order = 1 at $z=0$

and pole of order = 2 at $z = \pm 2$

$\underset{z=0}{\text{Res}}\, f(z) = \lim_{z\to 0} z\cdot f(z) = \lim_{z\to 0} \frac{z^3+3}{(z-2)^2(z+2)^2} = \frac{3}{16}$

$\underset{z=2}{\text{Res}}\, f(z) = \lim_{z\to 2} \frac{d}{dz} (z-2)^2 f(z) = \frac{d}{dz} \frac{z^3+3}{z(z+2)^2}$

$= \lim_{z\to 2} \frac{z(z+2)^2\cdot 3z^2 - (z^3+3)\cdot(3z^2+8z+4)}{z^2(z+2)^4}$

$= 2/64 = 1/32$

$\underset{z=-2}{\text{Res}}\, f(z) = \lim_{z\to -2} \frac{d}{dz} (z+2)^2 f(z) = \frac{d}{dz} \frac{z^3+3}{z(z-2)^2}$

$\lim_{z\to -2} \frac{z(z-2)^2\cdot 3z^2 - (z^3+3)(3z^2+4-8z)}{z^2(z-2)^4}$

$= -14/64 = -7/32$

$\therefore$ Sum of all residues $= \frac{3}{16} + \frac{1}{32} - \frac{7}{32} = \frac{6+1-7}{32} = 0$

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Q 8. (a) Let C_R be the circle of radius R centered at origin and oriented clock-wise direction. Show that

$$\lim_{R\to\infty} \int_{C_R} \frac{Log(z^3)}{z^{3/2}} dz = 0. \quad \Big(z^\alpha := \exp(\alpha \text{ Log } z)\Big)$$

(b) The value of $(2^i)^i =$ **4 + 2 marks**

→ (a) $\cancel{Log(z^3) = Log(e^{3Log z}) =}$

$$|Log(z^3)| = |Log|z^3| + i\arg(z^3)|$$

$$\leq |Log|z^3|| + \underbrace{|i\arg(z^3)|}_{\text{max value} = i\pi}$$

$$\leq Log(z^3) + \pi$$

By ML estimator,

$$\lim_{R\to\infty}\int_{CR} \frac{Log\, z^3}{z^{3/2}} dz \leq \lim_{R\to\infty} \frac{(Log(R^3)+\pi)\cdot 2\pi R}{R^{3/2}} \leq \lim_{R\to\infty} \frac{(3Log\, R^3 + \pi)2\pi}{\sqrt{R}}$$

$$\frac{(3\log R + \pi)2\pi}{\sqrt{R}} \to 0 \quad \text{as } R\to\infty.$$

$$\therefore \lim_{R\to\infty}\int_{CR} \frac{Log\, z^3}{z^{3/2}} dz = 0.$$

(b)

$$(2^i)^i = e^{i\log 2^i} = \cancel{e^{i\cdot e^{i\log 2}}} = \cancel{e^{i\cdot}}$$

$$= e^{i\log e^{i\log 2}} = e^{i\cdot i\log 2} = e^{-\log 2}$$

$$= e^{\log 1/2}$$

$$= 1/2$$

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Q 9. (a) Discuss the continuity of $f(x, y)$, where

$$f(x,y) = \begin{cases} \frac{x^3-2y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(b) Let $f = u + iv$ be an analytic function on a domain D. Is uv a harmonic function on D? Justify.

3+3 marks

→ (a)

$$\lim_{\Delta x \to 0} f(\Delta x, 0) - f(0,0) = \frac{(\Delta x)^3}{(\Delta x)^2} = \Delta x \to 0$$

$$\lim_{\Delta y \to 0} f(0,\Delta y) - f(0,0) = \frac{-2(\Delta y)^3}{(\Delta y)^2} = -2\Delta y \to 0$$

$$\lim_{\Delta x = \Delta y}, \quad f(\Delta x, \Delta y) - f(0,0) = \frac{(\Delta x)^3 - 2(\Delta x)^3}{(\Delta x)^2 + (\Delta x)^2} = \frac{-(\Delta x)^3}{2(\Delta x)^2}$$

$$= -\frac{\Delta x}{2} \to 0$$

∴ $f(x,y)$ is continuous at $(x,y) = (0,0)$.

(b) $f = u + iv$ is an analytic fn^x.

⇒ $u_x = v_y$, $u_y = -v_x$, $u_{xx} + u_{yy} = 0$ or, $v_{xx} + v_{yy} = 0$

To check whether 'uv' is a harmonic fn^x or not.

$$\Rightarrow \frac{d^2(uv)}{dx^2} + \frac{d^2(uv)}{dy^2} = \frac{d}{dx}\left(u_x v + v_x u\right) + \frac{d}{dy}\left(u_y v + u v_y\right)$$

$$= u_{xx} v + 2u_x v_x + v_{xx} u + 2u_y v_y + u_{yy} v + u v_{yy}$$

$$= v(u_{xx} + u_{yy}) + u(v_{xx} + v_{yy}) + 2(u_x v_x + u_y v_y)$$

$$= 0 + 0 + 2(\cancel{-u_x u_y + u_y u_x})$$

$$= 0$$

∴ $(uv)_{xx} + (uv)_{yy} = 0$ ∴ 'uv' is a harmonic function on D.

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Q 10. (a) Suppose $f(z)$ is an entire function. Show that $f(\mathbb{R}) \subset (\mathbb{R})$ if and only if $f^{(n)}(0) \in \mathbb{R}$ $\forall\, n = 0, 1, 2 \cdots$

(b) Let $\omega \neq 1$ be an n-th root of unity. Show that

$$1 + 2\omega + 3\omega^2 + \cdots + n\omega^{n-1} = \frac{-n}{1-\omega}. \qquad 4 + 2\textbf{marks}$$

→ (b) $z^n - 1 = 0$. $\omega^n = 1$

$\Rightarrow (\cancel{z-1})(z-\omega)(z-\omega^2) \ldots (z-\omega^{n-1}) = (\cancel{z-1})(1+z+z^2+\ldots+z^{n-1})$ — 1

Differentiating both sides w.r.t. 'z'

$\Rightarrow (z-\omega^2)(z-\omega^3)\ldots(z-\omega^{n-1}) + (z-\omega)(z-\omega^3)\ldots(z-\omega^{n-1}) + \ldots$

$= 1 + 2z + 3z^2 + \ldots + (n-1)z^{n-2}$

$\Rightarrow (\omega-\omega^2)(\omega-\omega^3)\ldots(\omega-\omega^{n-1}) = 1 + 2\omega + 3\omega^2 + \ldots (n-1)\omega^{n-2} + n\omega^{n-1}$

$\Rightarrow$

As we know,

$(\omega-1)(\omega-\omega^2)(\omega-\omega^3)\ldots(\omega-\omega^{n-1}) = n$

$\Rightarrow \frac{n}{\omega-1} = 1 + 2\omega + 3\omega^2 + \ldots + n\omega^{n-1}$ Proved

(a) According to Taylor Series,

$f(z) = f(z_0) + \frac{f'(z_0)z}{1!} + \frac{f''(z_0)z^2}{2!} + \ldots$

$f(z) = \sum_{n=0} \frac{f^n(z_0)z^n}{n!}$

If $f^n(0) \in \mathbb{R}$ then $f(\mathbb{R}) \in \mathbb{R}$
and $z \in \mathbb{R}$

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Q 10 Use this space for the solution of question 10 only