Mid Sem: MSO202M (2024-2025 I)

Date: 18 September 2024 Time: 18:00 -20:00 hr Maximum marks: 80

Name: **Roll No.**

Instructions: (Read carefully)

- Please enter your NAME and ROLL NUMBER in the space provided on EACH page.
- Only those booklets with name and roll number on every page will be graded. All other booklets will NOT be graded.
- This answer booklet has 5 pages. Check to see if the print is either faulty or missing on any of the pages. In such a case, ask for a replacement immediately.
- Please answer each question ONLY in the space provided. Answers written outside the space provided for it WILL NOT be considered for grading. So remember to use the space judiciously.
- For rough work, separate sheets will be provided to you. Write your name and roll number on rough sheets as well. However, they WILL NOT be collected back along with the answer booklet.
- No calculators, mobile phones, smart watches, or other electronic gadgets are permitted in the exam hall.
- Notations: All notations used are as discussed in class.
- All questions are compulsory.
- Do NOT remove any of the sheets in this booklet.

Q 1. Using complex integration method evaluate **10 marks**

$$\int_{\mathbb{R}} \frac{dx}{(x-2)(x^2+9)}.$$

Sol: Consider the contour $C := C_R(0) \cup [-R, 2-\rho] \cup C_\rho(2) \cup [2+\rho, R]$, where $C_r(z_0)$ represents the circle with center at z_0 and radius r. By Cauchy residue theorem, **2 marks for writing correct contour**

$$\int_C f(z)dz = 2\pi i \times \textit{ sum of residue.} \tag{1}$$

Only pole at $z = 3i$ lies inside the curve C. We get

$$2\pi i \times Res_{z=3i} \frac{1}{(z-2)(z^2+9)} = \frac{\pi}{3(3i-2)} = \frac{\pi(-2-3i)}{39}. \quad \textbf{2 marks}$$

As done in the class

$$\int_{C_R(0)} \frac{dz}{z-2(z^2+9)} \leqslant \frac{2\pi R}{(R-2)(R^2-9)} \to 0 \text{ as } R \to \infty. \quad \textbf{1 marks}$$

Using formula done in the class

$$\frac{1}{2\pi i} \lim_{\rho \to 0} \int_{\theta_1}^{\theta_2} \frac{f(z)dz}{z-z_0} = \frac{(\theta_1-\theta_2)}{2\pi} f(z_0), \quad \textbf{2 marks}$$

where curve is arc of circle with center at z_0 and radius ρ, traversing angle θ_1 and θ_2. We obtain

$$\lim_{\rho \to 0} \int_{C_\rho(2)} \frac{1}{(z-2)(z^2+9)} = \frac{-\pi i}{13}. \quad \textbf{1 marks}$$

Taking limit $R \to \infty$ and $\rho \to 0$ in equation (1), we obtain

$$0 + \int_{-\infty}^{2} \frac{dx}{(x-2)(x^2+9)} + \frac{-\pi i}{13} + \int_{2}^{\infty} \frac{dx}{(x-2)(x^2+9)} = \frac{\pi(-2-3i)}{39}$$
$$\Rightarrow \int_{\mathbb{R}} \frac{dx}{(x-2)(x^2+9)} = -\frac{2\pi}{39}. \quad \textbf{2 marks}$$

Q 2. State Rouche's theorem. Show that the polynomial $p(z) = z^{10} + 13z^4 + 2z^2 + 3$ does not vanishes on $|z| = 1$. Find the number of roots of $p(z)$ in (a) $|z| < 1$ and (b) $1 < |z| < 2$.

10 marks

Ans: *(Rouche's Theorem)* Let f and g be analytic inside and on a simple, closed, piece-wise smooth curve C. If $|f(z)| > |g(z)|$ for all points z on C, then $f(z)$ and $f(z)+g(z)$ have same number of zeros inside C. Here C is oriented in counter clockwise direction. **2 marks**

If $f(z) = 0$ for $|z| = 1$, we obtain that

$$z^{10} + 13z^4 + 2z^2 + 3 = 0 \Rightarrow 13z^4 = -z^{10} - 2z^2 - 3,$$

which is a contradiction as LHS has modulus 13 while modulus of right hand side is $\leqslant 6$.

2 marks

(a) Take $f(z) = 13z^4$ and $g(z) = z^{10} + 2z^2 + 3$. For $|z| = 1$, $|f(z)| = 13$ and $|g(z)| \leqslant 6$. Thus, $|f(z)| > |g(z)|$ on $|z| = 1$. Therefore, $p(z)$ has 4 roots inside $|z| < 1$. **3 marks**

(b) Take $f(z) = z^{10}$ and $g(z) = 13z^4 + 2z^2 + 3$. On $|z| = 2$, $|f(z)| = 2^{10} = 1024$ and $|g(z)| \leqslant 13 \times 16 + 8 + 3 = 219$. Thus, $|f(z)| > |g(z)|$ on $|z| = 2$. Therefore, $p(z)$ has 10 roots inside $|z| < 2$ out of which 4 are inside $|z| < 1$. Hence, there are 6 roots of $p(z)$ inside $1 < |z| < 2$. **3 marks**

Name:...   **Roll Number: ....**

Q 3. Define cross ratio for the points z_1, z_2, z_3, z_4. Find a Möbius transformation T such that $T(0) = -1$, $T(i) = 1$ and $T(\infty) = 3$. Find the fixed points of T. **10 marks**

Ans: Cross Ratio for the points z_1, z_2, z_3, z_4 is given by

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}. \qquad \textbf{2 marks}$$

Let

$$T(z) = \frac{az + b}{cz + d} \quad \text{such that} \quad ad - bc \neq 0.$$

Sol 1 Since cross ratios are invariant under Möbius transformation, we have

$$\frac{(z - z_3)(z_2 - z_4)}{(z - z_4)(z_2 - z_3)} = \frac{(w - w_3)(w_2 - w_4)}{(w - w_4)(w_2 - w_3)} \Leftrightarrow \frac{(z - i)(0 - \infty)}{(z - \infty)(0 - i)} = \frac{(w - 1)(-1 - 3)}{(-1 - 1)(w_2 - w_3)}.$$

Solving for w, we get

$$T(z) = \frac{3z - i}{z + i} \qquad \textbf{6 marks}$$

Sol 2

Since

$$\begin{aligned} T(0) = -1 \quad &\implies b = -d \\ T(i) = 1 \quad &\implies \frac{ai + b}{ci - b} = 1 \Rightarrow 2b = (c - a)i \qquad (2) \\ T(\infty) = 3 \quad &\implies a = 3c. \qquad (3) \end{aligned}$$

From equations (2) and (3), we have $b = -ci$. Thus,

$$T(z) = \frac{3cz - ci}{cz + ci} = \frac{3z - i}{z + i}$$

as $c \neq 0$.

For fixed points put $T(z) = z$:

$$\frac{3z - i}{z + i} = z \implies z^2 + z(i - 3) + i = 0,$$

The fixed points are given by

$$z = \frac{-(i - 3) \pm \sqrt{(i - 3)^2 - 4i}}{2} = \frac{-(i - 3) \pm \sqrt{8 - 10i}}{2}.$$

2 marks

Name:.. **Roll Number: ....**

Q 4. (a) Expand **6 + 4 marks**

$$\frac{z}{z^2 - 11z + 28} \quad \text{in} \quad (i) \quad |z-4| < 1 \quad (ii) \quad 1 < |z-7| < 2.$$

(b) Find all analytic functions in $D := \{z \in \mathbb{C} : |z| \leqslant 1\}$ such that $|f(z)| \geqslant 2$ on $|z| = 1$, $f(z) \neq 0$ for $z \in D$, and $f(0.5) = 2$.

Ans: (a)

$$\frac{z}{z^2 - 11z + 28} = \frac{z}{(z-4)(z-7)}.$$

(i) When $|z-4| < 1$.

$$\frac{z}{(z-4)(z-7)} = \frac{z}{(z-4)(z-4-3)} = \frac{-z}{3(z-4)} \frac{1}{\left(1 - \frac{z-4}{3}\right)}.$$

As $|z-4|/3 < 1$, we have

$$\begin{aligned}
&= \frac{-z}{3} \sum_{n=0}^{\infty} \left(\frac{z-4}{3}\right)^{n-1} = -\sum_{n=0}^{\infty} \frac{(z-4)^n}{3^{n+1}} - 4\sum_{n=0}^{\infty} \frac{(z-4)^{n-1}}{3^{n+1}} \\
&= -\frac{4}{3(z-4)} - \sum_{n=0}^{\infty} \left(\frac{7}{3^{n+2}}\right)(z-4)^n
\end{aligned}$$

3 marks

(ii) When $1 < |z-7| < 2$.

$$\frac{z}{(z-4)(z-7)} = \frac{z}{(z-7)(z-7+3)} = \frac{z}{3(z-7)} \frac{1}{\left(1 + \frac{z-7}{3}\right)}.$$

As $|z-7| < 2 \implies |z-7|/3 < 2/3$, we have

$$= \frac{z}{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n}(z-7)^{n-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}(z-7)^n + 7\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}(z-7)^{n-1}.$$

3 marks

(b) Define $g(z) = 1/f(z)$ then $|g(z)| \leq 1/2$ on $|z| = 1$. Also, if f is analytic then so is g as $f(z) \neq 0$ on D.

Maximum Modulus Theorem states that if f is an analytic function in a domain D and if there is a point $a \in D$ such that $|f(a)| \geqslant |f(z)|$ for all $z \in D$, then f is a constant function.

By using maximum modulus theorem for g we have g is a constant function. As $g(0.5) = 1/2 \Rightarrow g(z) = 1/2$. Therefore, $f(z) = 2$. **4 marks**

Name:.. **Roll Number: ....**

Q 5. (a) Find domain and radius of convergence of **2 + 4 marks**

$$S_1 := \sum_{n=1}^{\infty}(7z-3)^n \quad \text{and} \quad S_2 := \sum_{n\in\mathbb{Z}}(27)^{-|n|}z^{3n}.$$

(b) Let $f = u + iv$ be an entire function such that $5u + 18v$ is bounded. If $f(3) = 10$ then find f. **4 marks**

Ans:

$$\frac{1}{R} = \limsup_{n\to\infty}|a_n|^{1/n}.$$

If it is power series with gaps, the radius of convergence is given by

$$\frac{1}{R} = \lim\left|\frac{a_n}{a_{n-1}}\right|^{\frac{1}{\lambda(n)-\lambda(n-1)}}$$

(a)

$$S_1 = \sum_{n=1}^{\infty}7^n\left(z-\frac{3}{7}\right)^n.$$

Then $1/R_1 = \limsup_{n\to\infty}|7^n|^{1/n} = 7 \Rightarrow R_1 = 1/7$. Domain of convergence is $|z-3/7| < 1/7$ or $|7z-3| < 1$.

2 marks

(b)

$$S_2 = \sum_{n=-\infty}^{-1}27^n z^{3n} + \sum_{n=0}^{\infty}(27)^{-n}z^{3n} = \sum_{n=1}^{\infty}(3z)^{-3n} + \sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^{3n}.$$

Both the series are geometric series. The first series converges if $|3z| > 1$ and the second series converges for $|z/3| < 1$. Hence the domain of the convergence for S_2 is

$$\{z \in \mathbb{C} : \frac{1}{3} < |z| < 3\} \qquad \textbf{4 marks}$$

Sol 2:

$$S_2 = \sum_{n=-\infty}^{-1}27^n z^{3n} + \sum_{n=0}^{\infty}(27)^{-n}z^{3n} = \sum_{n=1}^{\infty}27^{-n}z^{-3n} + \sum_{n=0}^{\infty}(27)^{-n}z^{3n}.$$

For first series: $1/R_2 = \limsup_{n\to\infty}27^{-1} = 1/27 \Rightarrow R_2 = 27$. If $w = z^{-3}$, we have D.O.C. $|w| < 27 \iff |z^{-3}| < 27 \iff |z| > 1/3$. For second series: $1/R_3 = \limsup_{n\to\infty}27^{-1} = 1/27 \Rightarrow R_3 = 27$. D.O.C. is $|z^3| < 27 \iff |z| < 3$. Therefore, Radiuc of convergence is 27 and Domain of convergence of S_2 is $1/3 < |z| < 3$.

(b) Define

$$g(z) = e^{(5-i18)f(z)} = e^{(5-i18)(u+iv)} = e^{5u+18v+i(5v-18u)},$$

then $|g(z)| = e^{5u+18v}$. If $5u + 18v$ is bounded say by M, then $|g(z)| \leqslant e^M$.

Now, $g(z)$ is entire because $f(z)$ is entire and exponential function is entire.

By Liouville Theorem, $g(z)$ is constant function as it is entire and bounded. This implies $f(z)$ is also a constant function. Since $f(3) = 10 \Rightarrow f(z) = 10$. **4 marks**

Name:.. **Roll Number:** ....

Q 6. (a) Let $f = u + iv$ be an entire function such that $3u + 8v = x^3 - 3xy^2 + 3x^2 - 3y^2$. Find f by determining the harmonic conjugate. **6 marks**

(b) Find all analytic function in $\{z \in \mathbb{C} : |z| < 1\}$ such that $f(1/n) = 1/(n+3)\ \forall\ n \in \mathbb{N}$. **4 marks**

Ans: (a)

Let $f(z) = u+iv$. Define $F(z) = (3-8i)f := U+iV$. Then $U = 3u+8v = x^3-3xy^2+3x^2-3y^2$. We get

$$U = \Re F = x^3 - 3xy^2 + 3x^2 - 3y^2 = \Re(z^3 + 3z^2) \quad \textbf{2 marks}$$

(2 marks for finding suitable F).

Sol 1: from here we obtain that

$$F = z^3 + 3z^2 + ic \Rightarrow f = \frac{z^3 + 3z^2 + ic}{3 - 8i} \text{ and } V = 3x^2y - y^3 + 6xy + c,$$

where c is a real constant. **4 marks**

Sol 2: From $C - R$ equation $U_x = V_y$, we get

$$V = \int (3x^2 - 3y^2 + 6x)dy + g(x) = 3x^2y - y^3 + 6xy + g(x).$$

To determine $g(x)$, using $U_y = -V_x$ we obtain that g is a constant. Hence we obtain that

$$F = U + iV = x^3 - 3xy^2 + 3x^2 - 3y^2 + i(3x^2y - y^3 + 6xy + c) = z^3 + 3z^2 + ic$$
$$\Rightarrow f = \frac{F}{3-8i} = \frac{z^3 + 3z^2 + ic}{3-8i}.$$

part (b) Consider the function

$$g(z) = \frac{z}{3z+1}.$$

We have that

$$f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right) \quad \text{and } f(0) = g(0),$$

here the last equality comes from taking limit $n \to \infty$. Using identity theorem we obtain that

$$f(z) = g(z) \quad \forall \ \{z \in \mathbb{C} : |z| < 1\}.$$

4 marks

Name:.. **Roll Number: ....**

Q 7. Find zeros and poles (all finite places and at infinity) of the function

$$f(z) = \frac{z^5 - 32}{z(z-1)^2}.$$

Calculate its residues and show that the sum of all residues in 0. **10 marks**

Solution: $f(z)$ have zeros at the point

$$z_k = 2\omega_5^k, \quad k = 0, 1, 2, 3, 4 \quad \text{where} \quad \omega_f = e^{\frac{2\pi i}{5}}.$$

2 marks

$f(z)$ has simple pole at $z = 0$ with residue

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{z^5 - 32}{(z-1)^2} = -32. \qquad \textbf{2 marks}$$

It has a double pole at $z = 1$ with residue

$$\frac{d}{dz}(z-1)^2 f(z)\Big|_{z=1} = \frac{d}{dz}\frac{z^5 - 32}{z}\Big|_{z=1} = 36.$$

2 marks

Since degree on numerator is 2 more that degree of the denominator, $f(z)$ has a pole of order 2 at $z = \infty$.

Residue at $z = \infty$ is negative of coefficient of z in the expansion of $f(1/z)$ in a neighbourhood of $z = 0$.

$$\begin{aligned}
&= -\text{ coeff. of } z \text{ in } \quad \frac{z^{-5} - 32}{z^{-1}(z^{-1} - 1)^2} \\
&= -\text{ coeff. of } z \text{ in } \quad \frac{1 - 32z^5}{z^2(1-z)^2} \\
&= -\text{ coeff. of } z^3 \text{ in } \quad \frac{1 - 32z^5}{(1-z)^2} \\
&= -4
\end{aligned}$$

4 marks

Hence we obtain

$$\text{Res}_{z=0} f(z) + \text{Res}_{z=1} f(z) + \text{Res}_{z=\infty} f(z) = -32 + 36 - 4 = 0$$

Q 8. (a) Determine the singularities and find the order of all zeros and poles of the function.

$$f(z) = \frac{\sin^3(\pi z)}{z(z-1)^2(z-2)^3(z-8)^9}. \qquad \textbf{7 marks}$$

(b) Let $f(z)$ be an entire function. Calculate the pole of $f(z)\cot(\pi z)$ and $f(z)\csc(\pi z)$ at $z = n$, where n is an integer. **3 marks**

Solution (a) For any integer k, we have

$$\lim_{z\to k} \frac{\sin(\pi(z-k))}{z-k} = \pi.$$

We have $\sin(\pi(z-k)) = \sin(\pi z)\cos(\pi k) - \cos(\pi z)\sin(\pi k) = \pm\sin(\pi z)$ (depending on the parity of k). From this we obtain $\sin(\pi z) = \pm\sin(\pi(z-k))$.

At $z = 0$, we have

$$\frac{\sin^3(\pi z)}{z(z-1)^2(z-2)^3(z-8)^9} = \pi\frac{\sin(\pi z)}{\pi z(z-1)^2(z-2)^3(z-8)^9} \times \sin^2(\pi z) := \sin^2(\pi z)g(z),$$

with $g(0) \neq 0$. It has zero of order 2 at $z = 0$. **1 marks**

At $z = 1$, we have

$$\begin{aligned}\frac{\sin^3(\pi z)}{z(z-1)^2(z-2)^3(z-8)^9} &= \frac{\sin^2(\pi(z-1))}{\pi z(z-1)^2(z-2)^3(z-8)^9} \times \sin(\pi(z-1)) \\ &:= \sin(\pi(z-1))h(z),\end{aligned}$$

With $h(1) \neq 0$. It has a zero of order 1 at $z = 1$. **1 marks**

Similarly

$$\frac{\sin^3(\pi z)}{z(z-1)^2(z-2)^3(z-8)^9} = \frac{}{\pi z(z-1)^2(z-8)^9} \times \frac{\sin^3(\pi(z-2))}{(z-2)^3}.$$

It has removal singularity at $z = 2$.

2 marks (Deduct 1 marks if removal singularity is not mentioned).

At $z = 8$

$$\frac{\sin^3(\pi z)}{z(z-1)^2(z-2)^3(z-8)^9} = \frac{}{\pi z(z-1)^2(z-2)^3} \times \frac{\sin^3(\pi(z-8))}{(z-8)^3} \times \frac{1}{(z-8)^6}.$$

It has a pole of order 6 at $z = 8$. **2 marks**

Function $f(z)$ has a zero of order z if z is an integer and $z \neq 0, 1, 2, 8$. **1 marks**

Name:.. **Roll Number: ....**

Space for question 8

Solution (b)

We have $f(z)\cot(\pi z)$ has simple pole at $z = n$ with residue

$$\lim_{z\to n}(z-n)f(z)\frac{\cos\pi z}{\sin\pi z} = \lim_{z\to n}\frac{\pi(z-n)}{\sin\pi z}\lim_{z\to n}\frac{f(z)\cos\pi z}{\pi} = \frac{1}{\pi}f(n).$$

1.5 marks

Similarly, $f(z)\csc(\pi z)$ has simple pole at $z = n$ with residue

$$\lim_{z\to n}(z-n)\frac{f(z)}{\sin\pi z} = \lim_{z\to n}\frac{\pi(z-n)}{\sin\pi z}\ \lim_{z\to n}\frac{f(z)}{\pi} = \frac{(-1)^n}{\pi}f(n).$$

1.5 marks