Singularities

- $z = z_0$ is an isolated singularity of $f(z)$

- Expand f as Laurent Series in the region.

$$0 < |z - z_0| < r.$$

$$f(z) = \underbrace{\sum_{n \geq 1} \frac{b_n}{(z - z_0)^n}}_{\text{Principal part}} + \underbrace{\sum_{n \geq 0} a_n (z - z_0)^n}_{\text{analytic part}}$$

- b_1 is called the Residue of f at $z = z_0$

- If $b_n = 0 \ \forall n$, then $z = z_0$ is called a removable singularity.

- If $b_n = 0 \ \forall n > k$ & $b_k \neq 0$

$$f(z) = \frac{b_k}{(z - z_0)^k} + \frac{b_{k-1}}{(z - z_0)^{k-1}} + \cdots + \frac{b_1}{(z - z_0)} + \sum_{n \geq 0} a_n (z - z_0)^n$$

we call $z = z_0$ is pole of order k.

- If $b_n \neq 0$ for infinitely many n, then $z = z_0$ is called Essential singularity.

Example

(1) $\dfrac{\cos z - 1}{z^2} = f(z)$.

Singular points $= \{0\}$.

$$\frac{\cos z - 1}{z^2} = \frac{1 - \frac{z^2}{\lfloor 2} + \frac{z^4}{\lfloor 4} - \cdots - 1}{z^2}$$

$$= -\frac{1}{\lfloor 2} + \frac{z^2}{\lfloor 4} - \frac{z^4}{\lfloor 6} \cdots$$

$z = 0$ is removeable singular

$f(0) = -\dfrac{1}{\lfloor 2}$

- $f(z) = e^{\frac{1}{z^2}}$

Singular point $z = 0$.

Laurent Series around $z = 0$

$$f(z) = \underbrace{1}_{\text{analytic}} + \underbrace{\frac{1}{z^2} + \frac{1}{z^4 \lfloor 2} + \frac{1}{z^6 \lfloor 3} \cdots}_{\text{principal part}}$$

$z = 0$ essential singularity.

$\operatorname{Res}(e^{\frac{1}{z^2}}; 0) = 0$.

$f(z) = e^{\frac{1}{z}}$

$\operatorname{Res}(e^{\frac{1}{z}}; 0) = 1$.

- $z = z_0$ is a pole of order k of f $\iff$ $\frac{1}{f}$ has zero of order k at z_0.

$\iff f(z) = \frac{g(z)}{(z-z_0)^k}$. $g(z_0) \neq 0$

g is holomorphic near z_0.

$$\frac{1}{f} = (z-z_0)^k \frac{1}{g(z)}$$

($\because \frac{1}{g(z)}$ is holomorphic near z_0 & $\frac{1}{g(z_0)} \neq 0$)

$$f(z) = \frac{b_k}{(z-z_0)^k} + \cdots + \frac{b_1}{(z-z_0)} + \sum_{n \geq 0} a_n (z-z_0)^n$$

$$= \frac{b_k + b_{k-1}(z-z_0) + \cdots + b_1(z-z_0)^{k-1} + a_0(z-z_0)^k + \cdots}{(z-z_0)^k}$$

$$= \frac{g(z)}{(z-z_0)^k} \qquad g(z_0) = b_k \neq 0$$

$z = z_0$ is a pole of order k iff

$$f(z) = \frac{g(z)}{(z - z_0)^k} \qquad g(z_0) \neq 0.$$

$\mathrm{Res}(f; z_0) =$ Coeff of $(z - z_0)^{k-1}$ in Taylor Series of $g(z)$.

$$= \frac{1}{\lfloor k-1} \frac{d^{k-1}}{dz^{k-1}} g(z) \Big|_{z = z_0}$$

$$= \lim_{z \to z_0} \frac{1}{\lfloor k-1} \frac{d^{k-1}}{dz^{k-1}} \left[(z - z_0)^k f(z) \right]$$

Example

$$f(z) = \frac{1}{z(z^2+1)(z-2)^2}$$

Singular point.

$= 0, \pm i, 2.$ poles.

0: order 1; $\pm i$: order 1; 2: order 2.

$$\mathrm{Res}(f; 0) = \lim_{z \to 0} z \cdot f(z) = \frac{1}{4}.$$

$$\mathrm{Res}(f; 2) = \lim_{z \to 2} \frac{d}{dz} (z-2)^2 f(z).$$

$$= \lim_{z \to 2} \qquad = -\frac{13}{100}$$

③ $f(z) = \dfrac{\sinh(z)\, e^{z}}{z^{5}}$

singular point : $z = 0$

$$f = \left(z + \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} \cdots\right) \quad \sinh z = \frac{e^{z} - e^{-z}}{2}$$

$$\frac{\left(1 + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \frac{z^4}{\lfloor 4}\right)}{z^5}$$

$$= \frac{z + z^2 + z^3\left(\frac{1}{\lfloor 3} + \frac{1}{\lfloor 2}\right) + z^4\left(\quad\right)}{z^5}$$

. $z = 0$ is a pole of order 4.

$$\operatorname{Res}(f; 0) = \frac{2^4}{\lfloor 4} \cdot \frac{1}{2} =$$

$$f(z) = \frac{e^{z} - e^{-z}}{2} \cdot e^{z} \frac{1}{z^5}$$

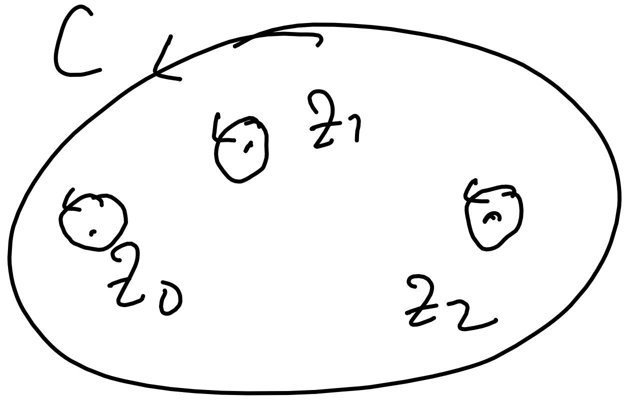
$$= \frac{e^{2z} - 1}{2} \cdot \frac{1}{z^5}$$

$$= \frac{1}{2}\left(2z + \frac{(2z)^2}{\lfloor 2} + \frac{(2z)^3}{\lfloor 3} \cdots\right)$$

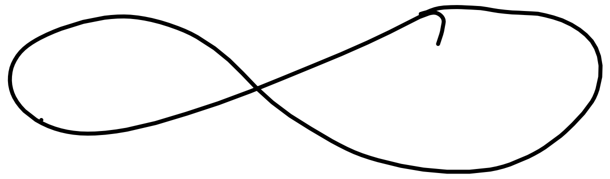
Residue Theorem

Suppose $f(z)$ is analytic on & inside C except for finitely many isolated singularity inside C. (C does not pass through singular points). C oriented anticlockwise, C simple closed curve

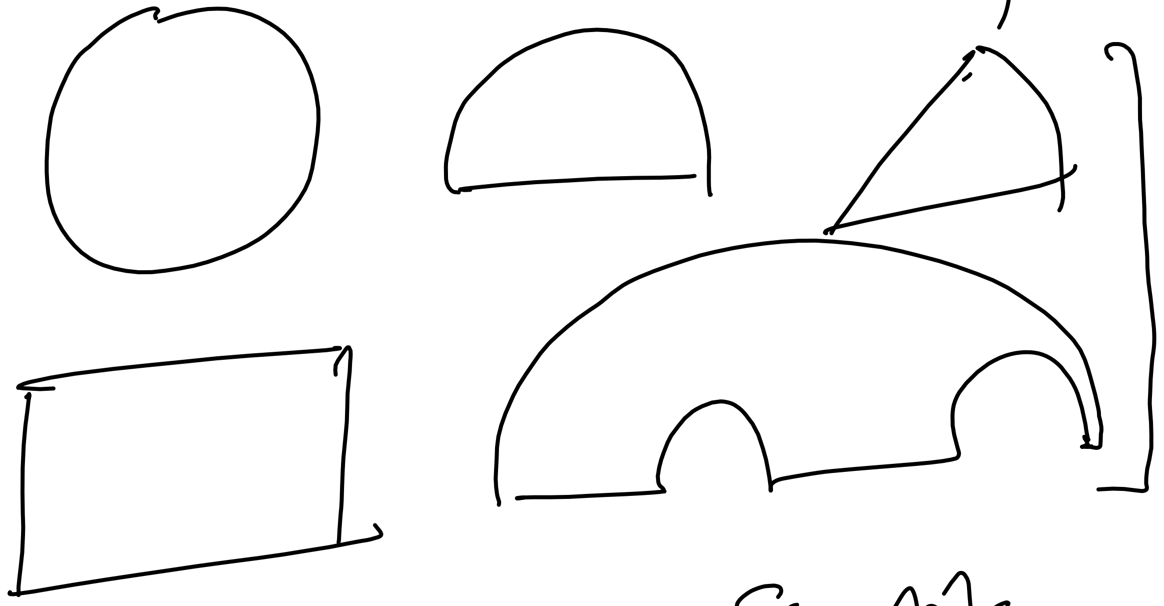
$$\int_C f(z)\,dz = 2\pi i \left(\text{sum of the residues of } f \text{ inside } C\right)$$



C is simple if it has no self intersects



— closed but not simple.

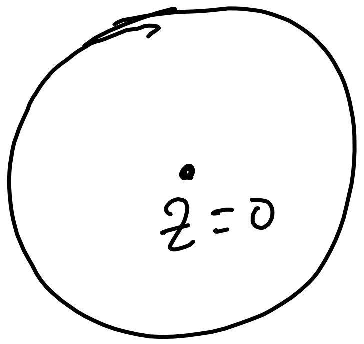


Simple closed curve

Example

① $\int \underbrace{z^2 \sin\left(\frac{1}{z}\right)}_{f(z)} dz = 2\pi i\left(-\frac{1}{6}\right)$.

$|z| = 1$.



$z = 0$ Isolated singularity of f

It is an essential singularity

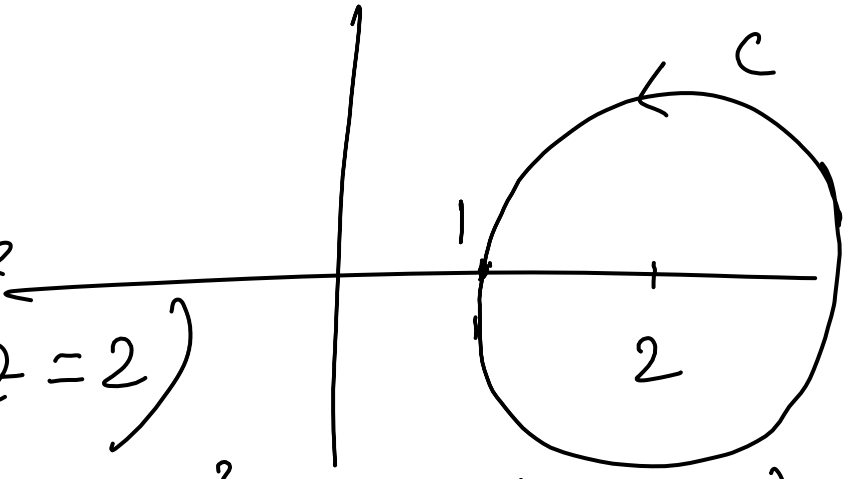
$$z^2 \sin\left(\frac{1}{z}\right) = z^2\left(\frac{1}{z} - \frac{1}{z^3}\frac{1}{3!} + \frac{1}{z^5}\frac{1}{5!} \right.$$

$$\left. = z - \frac{1}{z}\cdot\frac{1}{3!} + \cdots\cdots\right)$$

$$\mathrm{Res}(f; 0) = -\frac{1}{3!} = -\frac{1}{6}.$$

② $\int \frac{dz}{z(z-2)^4}\, dz$

$|z-2| = 1$



$= 2\pi i\,(\text{Residue at } z=2)$

$$= 2\pi i\left(\lim_{z\to 2} \frac{1}{3!}\frac{d^3}{dz^3}(z-2)^4 f(z)\right)$$

$$= 2\pi i\left(\lim_{z\to 2} \frac{1}{6}\frac{d^3}{dz^3}\left(\frac{1}{z}\right)\right).$$

$$= \frac{\pi i}{3}.$$

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