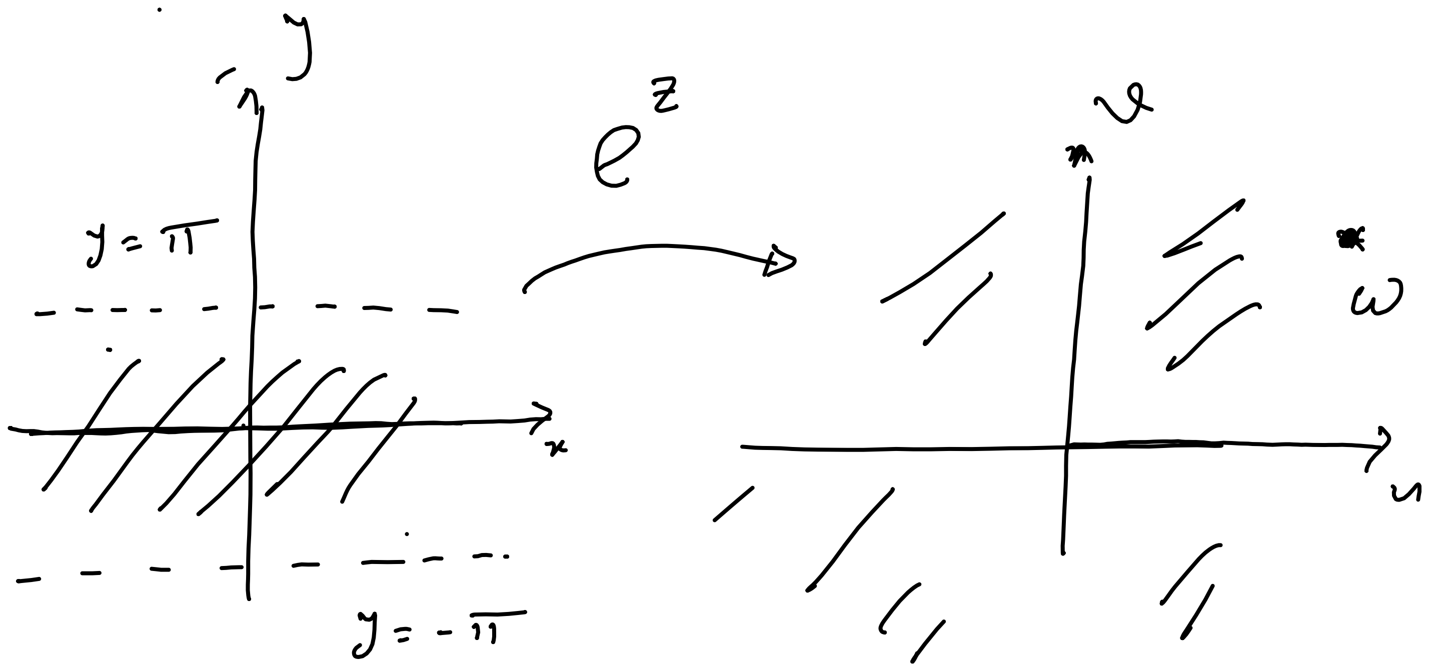
Complex logarithm

Recall

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$



$$e^z = e^{z+2\pi i}$$

Image of $\{z \mid -\pi < y \leq \pi\}$ under e^z-map will be $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

Defintn

$$\log z := \log|z| + i\,\arg(z)$$

$$\operatorname{Log} z = \log|z| + i\,\operatorname{Arg}(z)$$

$$-\pi < \operatorname{Arg}(z) \leq \pi$$

principal value of log.

Exmple

① $-4 = z = 4e^{i\pi}$

$$\log(-4) = \log 4 + i(\pi + 2k\pi)$$

$k \in \mathbb{Z}$.

$$\text{Log}(-4) = \log(4) + i\pi$$

② $z = \sqrt{3} - i = 2e^{-i\pi/6}$.

$$\text{Log}\, z = \log 2 + \left(-\frac{\pi}{6} i\right)$$

$$\log(z) = \text{Log} + 2k\pi i$$

$k \in \mathbb{Z}$.

• $e^{\log z} = e^{\log|z| + i\arg(z)}$

$= |z| e^{i\arg(z)}$

$= z$

• $\log(e^z) = z + 2k\pi i$

k

$$e^z = e^x \cdot e^{iy}$$

$$\log(e^z) = \log e^x + i(y + 2k\pi)$$

$= x + iy + 2k\pi i \quad k \in \mathbb{Z}$

$= z + 2k\pi i$

$$\boxed{\text{Log}(e^z) \overset{?}{\neq} z}$$

$$\mathrm{Log}(e^{i}) =$$

$$e^{i} = e^{0 + i\cdot 1}$$

$$= e^{0} \cdot e^{i(1)}$$

$$\mathrm{Log}(e^{i}) = \log e^{0} + i\,1 = i$$

$$-\pi < 1 < \pi$$

$$\mathrm{Log}\left(e^{\frac{3\pi}{2} i}\right) = \log(1) + i\left(\frac{3\pi}{2} - 2\pi\right)$$

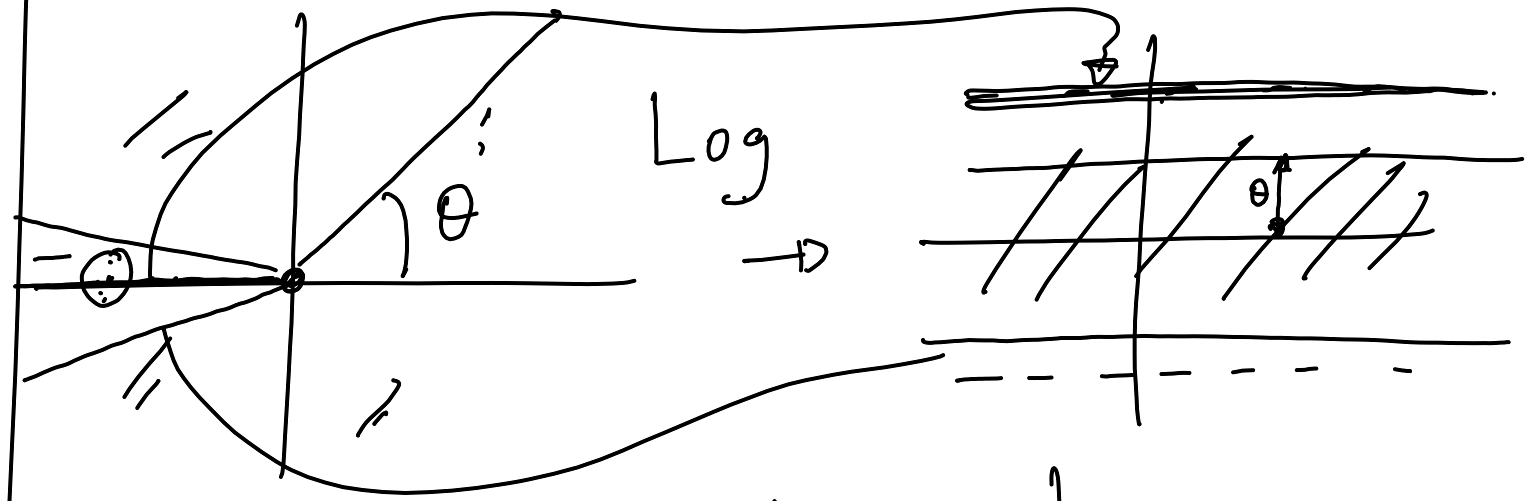
<u>Branch of log</u>

$\log(z)$ not a function

$\mathrm{Log}(z)$ – function.

$$\mathbb{C}^{*} \xrightarrow{\mathrm{Log}(z)} \mathbb{C}$$

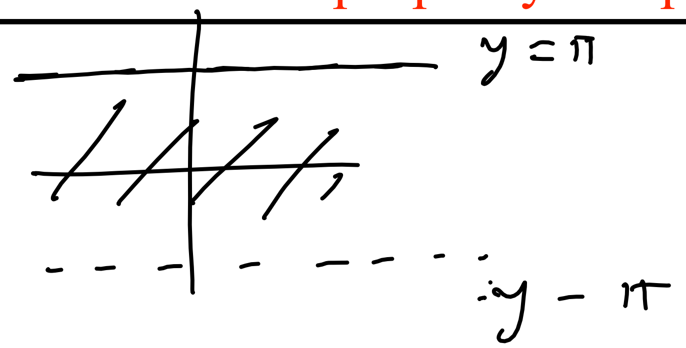
$$\mathrm{Log}(z) = \log|z| + i\theta \quad -\pi < \theta \le \pi$$



- $\mathrm{Log}(z)$ is discontinuous along negative x-axis.

note

On the rhs, the domain is the strip -\pi < y <= \pi

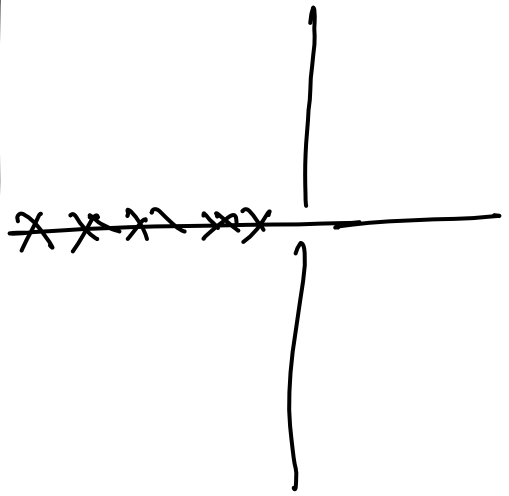
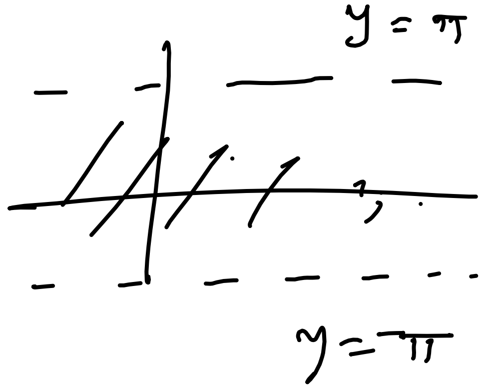
① $\mathbb{C}^* \underset{e^z}{\overset{\text{Log}}{\rightleftarrows}}$ 

are inverse of each other

② Log is continuous on $\Omega = \mathbb{C}^* \setminus \{\text{negative } x\text{-axis}\}$

Prop If f is analytic + f bijective + $f'(z) \neq 0$ + f^{-1} is continuous

$\Rightarrow g = f^{-1}$ is analytic

 $\underset{\substack{e^z \\ \shortparallel \\ f}}{\overset{\text{Log}}{\rightleftarrows}}$ 

Thm $\mathrm{Log}(z)$ is ~~any~~ analytic on $\mathbb{C}^* \setminus$ negative x-axis.

This called an analytic branch of logarithm.

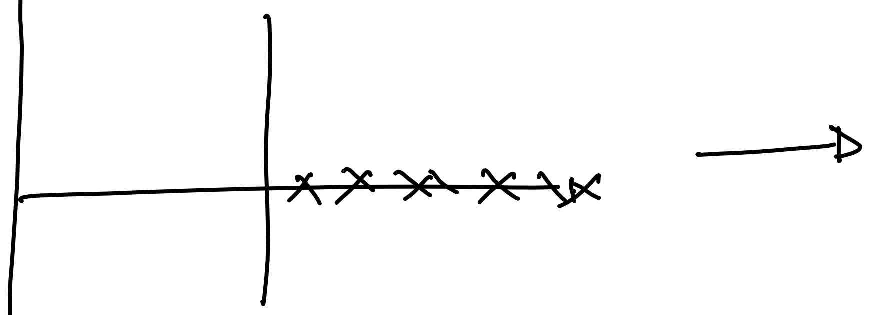
This ~~~~ is called principal branch of log.

$$\mathrm{Log}(z) = \log|z| + i\,(\mathrm{Arg}(z))$$

$$-\pi < (\mathrm{Arg}\, z) < \pi$$

— analytic

Other branch



$$\log(z) = \log|z| + i\,\theta$$

$$0 < \theta < 2\pi$$

This will be analytic on $\mathbb{C}^* \setminus \{\text{positive } x\text{-axis}\}$.

$$\boxed{\mathrm{Log}(zw) \neq \mathrm{Log}(z) + \mathrm{Log}(w)}$$

$$z = -\sqrt{3} + i = 2e^{i\frac{5\pi}{6}}$$

$$w = \sqrt{2}(1+i) = 2e^{i\pi/4}.$$

$$\mathrm{Log}(z) = \log 2 + i\frac{5\pi}{6}$$

$$\mathrm{Log}(w) = \log 2 + i\,\pi/4.$$

$$zw = 4e^{i\frac{13\pi}{12}}$$

$$\mathrm{Log}(zw) = \log 4 + i\left(\frac{13}{12}\pi - 2\pi\right)$$

$$\mathrm{Log}(z^n) \neq n\,\mathrm{Log}\, z$$

~~Example~~

$\sqrt{z}$ $\qquad z = re^{i\theta}$

$= \sqrt{r}\cdot e^{i\left(\frac{\theta}{2}\right) + \frac{2k\pi i}{2}} \qquad k \in 0, 1$

$= \sqrt{r}\, e^{i\frac{\theta}{2}}, \quad -\sqrt{2}\, e^{i\theta/2}.$

$e^{\frac{1}{2}\mathrm{Log}(z)}$

$e^{\frac{1}{2}\left(\log|z| + i\theta\right)}$

$= e^{\frac{1}{2}\log|z|} \cdot e^{i\frac{\theta}{2}}$

$= \sqrt{|z|}\, e^{i\theta/2}.$

$$\sqrt{z} = e^{\frac{1}{2}(\log z)}$$

Then we can say that

$z^a := e^{a\log(z)} \qquad a \in \mathbb{R}$

This is multivalued function.

~~Di~~ An analytic branch of this correspond to an analytic branch of log.

$$\boxed{z_1^a z_2^a \neq (z_1 z_2)^a}$$

$$\boxed{\sqrt{z_1 z_2} \neq \sqrt{z_1}\sqrt{z_2}}$$

$z = w = e^{\frac{3\pi}{4}i}$

$$a^z := e^{z \log a} \qquad a \in \mathbb{C} \text{ is fixed}$$

$a^z = f(z)$ analytic on $\mathbb{C}$.

Expl $a = i$

$$i^z = e^{z \operatorname{Log} i}$$

$$= e^{-\frac{\pi}{2} y} \left(\cos \frac{\pi}{2} x + i \sin \frac{\pi}{2} x \right)$$

$\blacksquare$