MSO 202A : Complex Variables
Quiz, 23rd August 2022

Total Marks: 30 **Time: 6:10 pm - 7 pm**

- Answer all questions.
- Write each step clearly.

1. Prove or disprove the following statements. Explain your answer with complete details. Here $\mathbb{D} = \{z|\ |z| < 1\}$ is the unit disc.

   (a) If $f(z) = u + \iota v : \mathbb{D} \to \mathbb{C}$ is a function such that $f^2(z)$ is analytic, then $f(z)$ itself is analytic.

   [5]

   (b) If $f(z) = u + \iota v : \mathbb{D} \to \mathbb{C}$ is a function such that u, v has continuous partial derivatives on $\mathbb{D}$ and $f^2(z)$ is analytic, then $f(z)$ itself is analytic.

   [5]

   Solution:

   (a)

   This statement is false. Let $f(z) = \sqrt{z} = e^{\frac{1}{2}\log z}$, then $f^2(z) = z$ which is holomorphic on $\mathbb{D}$, but $f(z)$ is not holomorphic on $\mathbb{D}$.

   [3]

   Justifying why f is not (and can not be) analytic on $\mathbb{D}$

   If $f(z)$ were holomorphic on $\mathbb{D}$ that means that $\log z$ is holomorphic on $\mathbb{D}$ which is not possible since $\frac{d}{dz}\log z = \frac{1}{z}$ and $\int_\gamma \frac{1}{z} dz = 2\pi\iota$ where γ is a circle with center at the origin.

   [2]

   Remark: If no counter example is provided then 0 marks. Because to disprove a statement, you need to provide a counter example.

   (b)

   Given $f = u + \iota v$, so $f^2 = u^2 - v^2 + 2\iota uv = U + \iota V$ where $U = u^2 - v^2$ and $V = 2uv$.

   [1]

   $U_x = 2uu_x - 2vv_x = V_y = 2uv_y + 2vu_y$
   $\implies u(u_x - v_y) - v(v_x + u_y) = 0$

   [1]

   $U_y = 2uu_y - 2vv_y = -V_x = -2uv_x - 2vu_x$
   $\implies v(u_x - v_y) + u(v_x + u_y) = 0$

   [1]

If $u^2 + v^2 \neq 0$ then $u_x - v_y = 0$ and $v_x + u_y = 0$. So the CR equations are satisfied for f and it has continuous partial derivatives. So f is holomorphic on $\mathbb{D}$.

[1]

If If $u^2 + v^2 = 0$ then $f = 0$ and so f is of course holomorphic.

[1]

2. We know that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} x\sin(1/x) = 0$, where $x \in \mathbb{R}$. Using the definitions of limits only, determine whether the following limits exist or not. If it exist, find its value.

$$(a) \quad \lim_{z\to 0} \frac{\sin z}{z}; \quad (b) \quad \lim_{z\to 0} z\sin(1/z), \quad z \in \mathbb{C}.$$

[5+5]

Solution:

(a)

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots$

[1]

$= zg(z)$ where $g(z) = 1 - \frac{z^2}{3!} +$ is holomorphic on $\mathbb{C}$ with $g(0) = 1$.

[2]

$\lim_{z\to 0} \frac{\sin z}{z} = \lim_{z\to 0} \frac{zg(z)}{z} = \lim_{z\to 0} g(z) = g(0) = 1.$

[2]

Remark1: If someone use"L'Hispital" rule, without proving the rule, then 2 marks only. $\lim_{z\to 0} \frac{\sin z}{z} = \lim_{z\to 0} \frac{\cos z}{1} = \cos 0 = 1.$

Assume $f(a) = g(a) = 0$ and $g'(a) \neq 0$. Then

$$\lim_{z\to a} \frac{f(z)}{g(z)} = \lim_{z\to a} \frac{f(z) - f(a)}{g(z) - g(a)} = \lim_{z\to a} \frac{(f(z) - f(a))/(z-a)}{(g(z) - g(a))/(z-a)} = \frac{f'(a)}{g'(a)}.$$

Remark2: If different directional limits are calculated only, then 0 marks, since it does not prove that the limit exist.

(b)

If $z \to 0$ along x-axis then the limit is $\lim_{z\to 0} z\sin\frac{1}{z} = \lim_{x\to 0} x\sin\frac{1}{x} = 0$, since $\sin(y)$ is bounded for $y \in \mathbb{R}$.

[1]

Let us now calculate the limit for $z \to 0$ along y-axis. In this case,

$$\begin{aligned}\lim_{z\to 0} z\sin\frac{1}{z} &= \lim_{a\to 0} a\iota\sin\frac{1}{a\iota} \\ &= \lim_{a\to 0} a\iota\frac{e^{1/a} - e^{-1/a}}{2\iota} \\ &= \lim_{a\to 0} a\frac{e^{1/a} - e^{-1/a}}{2}.\end{aligned}$$

[2]

Now the last expression is a limit of real variables. We can see that $\lim\limits_{a\to 0+} a\frac{e^{1/a}-e^{-1/a}}{2} = \lim\limits_{a\to 0-} a\frac{e^{1/a}-e^{-1/a}}{2} = \infty$.

[1]

Thus $\lim\limits_{z\to 0} z\sin\frac{1}{z}$ does not exists.

[1]

3. (a) Use ML-ineuality to show that

$$|\int_\gamma \frac{e^z dz}{z^2+1}| \leq e^2\frac{8\pi}{3},$$

where γ is the circle $|z| = 2$ travelled twice anticlockwise.

[6]

Solution:

$\gamma(t) = 2e^{it}$ where $t \in [0, 4\pi]$

[1]

L = length of the curve = 8π.

[1]

$|e^z| = e^x \leq e^2$ on γ.

[1]

$|z^2+1| \geq |z|^2 - 1 = 3$ on γ.

[1]

Thus $|\frac{e^z}{z^2+1}| \leq e^2/3 = M$ on γ.

[1]

Hence by ML-inequality $|\int_\gamma \frac{e^z dz}{z^2+1}| \leq ML = e^2\frac{8\pi}{3}$.

[1]

Remark: If someone takes $\gamma(t) = 2e^{it}$ where $t \in [0, 2\pi]$ and does the calculations correctly to reach bound as $e^2\frac{4\pi}{3}$, then 4 marks.

(b) Evaluate $\int_\gamma e^{z^2}dz$, where $\gamma(t) = t(1-t)e^t + i\cos(2\pi t^3), \quad t \in [0, 1]$. Explain your answer clearly.

[4]

Solution:

Note that γ is a closed curve i.e., $\gamma(0) = \gamma(1)$.

[2]

e^{z^2} is analytic on $\mathbb{C}$.

[1]

Hence by Cauchy's Theorem, the integral is zero.

[1]