Lecture 9

Line Integrals Independent of Path

Definition (Simply Connected Domain): *A domain G is called* ***simply connected*** *if every simple closed curve in G encloses only points of G (i.e. the domain G has no holes).*

Let G be a simply connected domain and the points $a, z \in G$. Let the function f be continuous on G. The indefinite integral $\int_a^z f(w)\,dw$ is called independent of path if the value of the integral is the same for all simple piecewise differentiable curves C lying in G and joining the points a and z.

It is easily seen that an indefinite integral is independent of path, if

(i) f is analytic in G

or

(ii) $\int_C f(w)dw = 0$ for every closed piece-wise differentiable curve C lying in G.

Note that (i)$\Rightarrow$(ii) so it is sufficient to prove that indefinite integrals are independent of path by using (ii). This can be done as *follows by using the definition of integration:*

Let C_1, C_2 *be any two piecewise differentiable curves joining a and* z *.*

Consider the curve $C = C_1 U(-C_2)$ *. Since* C *is a closed curve*

$$\int_C f(w)dw = 0 \Rightarrow \int_{C_1} f(w)dw = -\int_{-C_2} f(w)dw = \int_{C_2} f(w)dw$$

thus the integral $\int_a^z f(w)\,dw$ *is independent of path*).

Note: An indefinite integral $\int_a^z f(w)\,dw$ defines a function F(z) by F(z) = $\int_a^z f(w)\,dw$ only if $\int_a^z f(w)\,dw$ is independent of path.

Proposition. *Let* $\int_a^z f(w)\,dw$ *be independent of path,* f *is continuous in a simply connected domain* G *containing the points* a *and* z *and* $F(z)=\int_a^z f(w)\,dw.$ *Then,* $F(z)$ *is differentiable and* $F'(z)=f(z)$ *for all* $z\in G$ *.*

Proof. We have

$$F(z+\Delta z)-F(z)=\int_z^{z+\Delta z} f(w)\,dw$$

Choose the path of integration from z to Δz to be a straight line segment (*this is possible, since, by assumption, the value of integral is same along every path joining* z *and* Δz)

$$\Rightarrow \frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)=\frac{1}{\Delta z}\int_z^{z+\Delta z} f(w)\,dw-\frac{1}{\Delta z}\int_z^{z+\Delta z} f(z)\,dw$$

$$=\frac{1}{\Delta z}\int_z^{z+\Delta z}\big(f(w)-f(z)\big)\,dw \quad . \qquad (*)$$

Now, f is continuous at z

$\Rightarrow$ *for* $\varepsilon>0,\ \exists\delta>0,\ s.t.\ |f(w)-f(z)|<\varepsilon$ *whenever* $|w-z|<\delta$.

Therefore, (*) gives,

$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)\right|<\frac{1}{|\Delta z|}.\varepsilon.|\Delta z|=\varepsilon,\ \text{whenever } |\Delta z|<\delta.$$

$\Rightarrow F'(z)=f(z).$

Proposition (Morera's Theorem, *Converse of Cauchy Theorem*):

If f is continuous in a simply connected domain G and $\int_C f(w)\,dw = 0$, for every closed curve C in G, then f is analytic in G.

Proof. By the hypothesis of Morera's Theorem, $F(z) = \int_a^z f(w)\,dw$, $a, z \in G$, is independent of path.

The previous proposition $\Rightarrow F'(z) = f(z)$ exists for every $z \in G$.

$\Rightarrow F$ is analytic, so has derivatives of all orders in G (*by a Proposition based on Taylor's Theorem*); in particular, the second derivative of F in G exists.

$\Rightarrow$ the derivative of f exists in G .

$\Rightarrow$ f is analytic in G

Zeros of Analytic Functions

The point 'a ' is called a zero of order m of a function *f(z), analytic at the point* a, if

$$f(a) = f'(a) = \ldots = f^{(m-1)}(a) = 0 \; but \;\; f^{m}(a) \neq 0.$$

If the function *f(z)* has a zero of order m at the point a, then

$$f(z) = \sum_{n=m}^{\infty} b_n (z-a)^n = (z-a)^m g(z), \; where \; g(z) = \sum_{n=m}^{\infty} b_n (z-a)^{n-m}$$

.

Since, $g(a) = b_m = \dfrac{f^{(m)}(a)}{m!}$, it follows that $g(a) \neq 0$.

Isolated Zeros Theorem. *The zeros of functions analytic in a domain D are isolated unless the function is identically zero.*

(A zero 'a' of function f is called isolated if a disk centered at 'a' can be found which does not contain any other zero of f)

Proof: Let $f(z)$ be analytic in a domain D and $a \in D$ be such that $f(a)=0$. Consider the Taylor series expansion $\sum_{n=0}^{\infty} b_n (z-a)^n$ of $f(z)$ convergent in a disk $\{z : |z-a| < R\} \subset D$.

Let $b_j = 0$ for $1 \le j \le k-1$ and $b_k \neq 0$. Then,

$$f(z) = (z-a)^k \sum_{n=0}^{\infty} b_{n+k} (z-a)^n \equiv (z-a)^k g(z) \qquad (say)$$

Since, $\sum_{n=0}^{\infty} b_{n+k} (z-a)^n$ has same radius of convergence as $\sum_{n=0}^{\infty} b_n (z-a)^n$, the function $g(z)$ represented by it is analytic, hence is continuous, in $|z-a| < R$.

The continuity of g(z) at a and $g(a) = b_k \neq 0 \Rightarrow$ there exists a $\delta > 0$ such that $|g(z) - b_k| < \dfrac{|b_k|}{2}$ for all z in $|z-a| < \delta$.

$\Rightarrow$ $g(z) \neq 0$ for all z in $|z-a| < \delta$.

Let $\delta^* = \min(\delta, R)$. Then $g(z) \neq 0$ in the disk $|z-a| < \delta^*$ contained in D .

Consequently, $f(z) \neq 0$ in the disk $|z-a| < \delta^*$, except at $a = 0$. Thus, the zero a of $f(z)$ is isolated.

Corollary 1: *If* f *and* g *are analytic in a domain* D *and* $\exists$ *a sequence* $\{z_n\}$ *with a limit point in D, such that* $f(z_n) = g(z_n)$ *for all n, then* $f(z) \equiv g(z)$ *in* D.

Proof: *Apply the above theorem for the function* $\varphi(z) = f(z) - g(z)$.

Corollary 2: *If* f *and* g *are analytic in a domain* D *and* $f(\varsigma) = g(\varsigma)$ *for all the points lying on some curve in D, then* $f(z) \equiv g(z)$ *in* D.