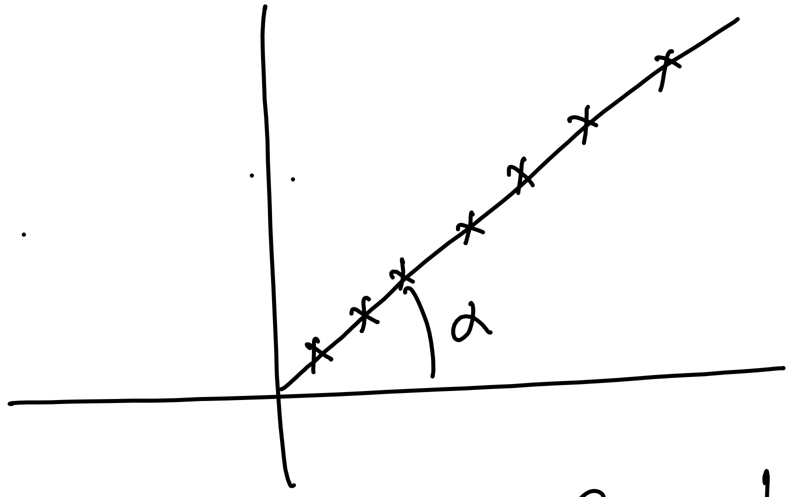
Complex logarithm

$z = re^{i\theta}$

$$\log_\alpha z := \log r + i\theta \qquad \theta \in (\alpha, \alpha + 2\pi)$$

$z \in U_\alpha$

$U_\alpha =$



Then $\log_\alpha$ is well-defined analytic function on U_α.

$$\mathrm{Log} = \log_{-\pi} \qquad \theta \in (-\pi, \pi)$$

– principal branch of log.

Example $z = -1 + \sqrt{3}\, i$

Calculate $\mathrm{Log}(z)$, $\log_{\pi/4}(z)$, $\log_{-\frac{3\pi}{2}}(z)$

Sol $z = -1 + \sqrt{3}\, i = 2e^{\frac{2\pi}{3} i}$

$$\mathrm{Log}(z) = \log 2 + i\,\frac{2\pi}{3}$$

$$\log_{\pi/4}(z) = \log 2 + \frac{2\pi}{3} i \qquad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{4} + 2\pi\right)$$

$$\log_{-\frac{3\pi}{2}}(z) = \log 2 + \left(\frac{2\pi}{3} - 2\pi\right) i$$

For any analytic branch of log

$$\frac{d}{dz}\left(\log_\alpha(z)\right)$$

$$= u_x + i v_x$$

$$= u_x - i u_y$$

$$= \frac{1}{z}$$

$$u = \frac{1}{2}\log(x^2+y^2)$$

$$u_x = \frac{1}{2}\cdot\frac{2x}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2}$$

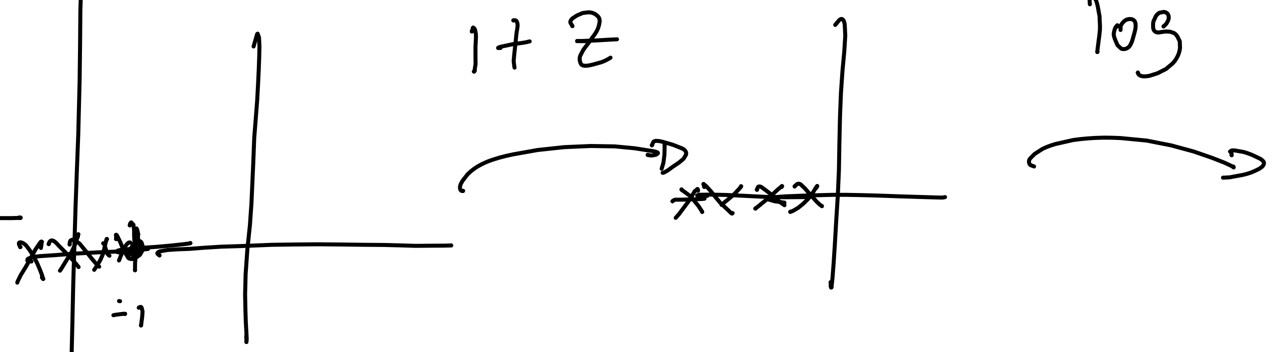
$$u_y = \frac{y}{x^2+y^2}$$

$$u_x - iu_y = \frac{x-iy}{x^2+y^2}$$

$$= \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}$$

Example

$f(z) = \log(1+z)$ where is this analytic?

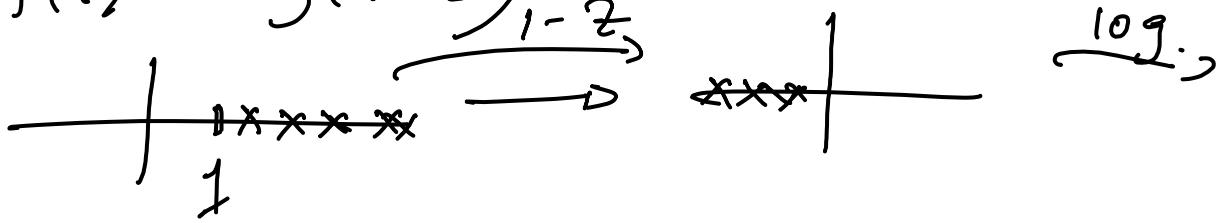


$1+z = x \qquad x<0$

$z = x-1 \qquad x<0$

$$f'(z) = \frac{1}{1+z}$$

$f(z) = \log(1-z)$ – where analytic.



Power Series

Recall • Series $\sum z_n \quad z_n \in \mathbb{C}$

• A series $\sum z_n$ is called convergent if $\{S_n\}$ $\quad S_n = z_1 + \cdots + z_n$ is convergent. $S_n \to w$

$$\sum z_n = w$$

• Absolute convergent

$$\sum |z_n| \text{ conv}$$

Fact. Absolute conv $\Rightarrow$ convergence.

• $\sum z_n$ conv $\Rightarrow$ $\lim z_n = 0$.

Example

$$1 + z + z^2 + \cdots + z^n + \cdots$$

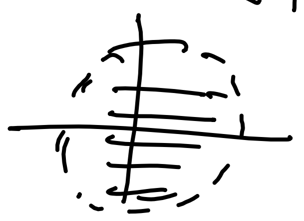
$$S_n = 1 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

If $|z| < 1$, then

$$S_n \to \frac{1}{1-z}$$

$$1 + z + z^2 + \cdots + z^2 + \cdots = \frac{1}{1-z} \quad \text{for } |z| < 1$$

– divergent for $|z| \geq 1$



$\mathbb{D}_1(0)$

Comparison test

If $|z_n| \le M_n$ + $\sum M_n$ convg $(M_n \geq 0)$

$\Rightarrow \sum |z_n|$ convg

$\Rightarrow \sum z_n$ convg

Prf $S_n = |z_1| + |z_2| + \cdots + |z_n|$

$S_n' = M_1 + \cdots + M_n$.

$S_n \le S_n' \le K$

$\Rightarrow \{S_n\}$ is bdd + monotone increasing

$\Rightarrow \{S_n\}$ convg.

- $\left.\lim \dfrac{|z_{n+1}|}{|z_n|} = L < 1 \right\} \Rightarrow \sum z_n$ convg.

- $\left.\lim \sqrt[n]{|z_n|} = L < 1 \right\} \Rightarrow \sum z_n$ convg

Power Series,

A series of the form

$$\sum a_n (z - z_0)^n \qquad a_n \in \mathbb{C}.$$

is called a power series about z_0.

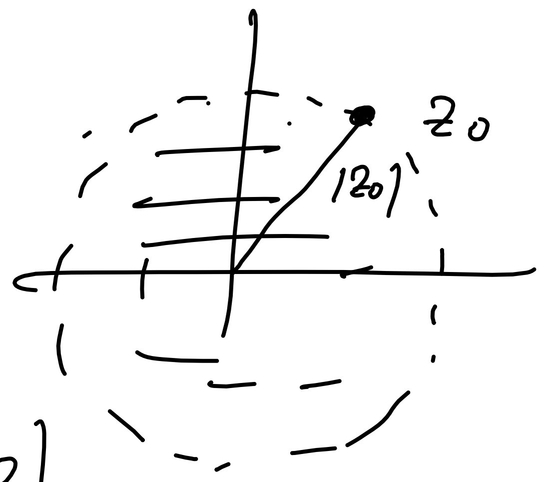
We will take $z_0 = 0$

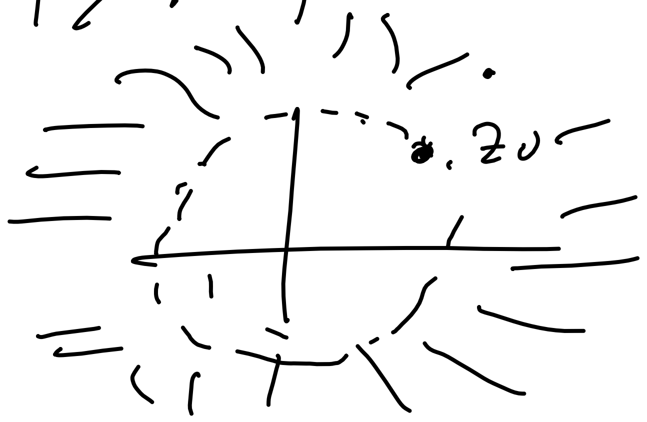
Example $1 + z + z^2 + \cdots = \frac{1}{1-z}$ $\quad |z| < 1$

Q) For what values of z, does the power series converge. $\sum a_n z^n$

Prop (1) $\sum a_n z^n$ conv for $z = z_0$.
$\Rightarrow \sum a_n z^n$ conv $\forall$ $|z| < |z_0|$

(2) $\sum a_n z^n$ divg for $z = z_0$.
$\Rightarrow \sum a_n z^n$ divg for $|z| > |z_0|$





Proof Assume $\sum a_n z^n$ conv for $z = z_0$.
Take z s.t. $|z| < |z_0|$.

$$|a_n z^n| = \left| a_n z_0^n \frac{z^n}{z_0^n} \right|$$

$$= \underbrace{|a_n z_0^n|} \cdot \underbrace{\left|\frac{z}{z_0}\right|^n}$$

$$\leq \underbrace{K \cdot \left|\frac{z}{z_0}\right|^n}_{M_n}.$$

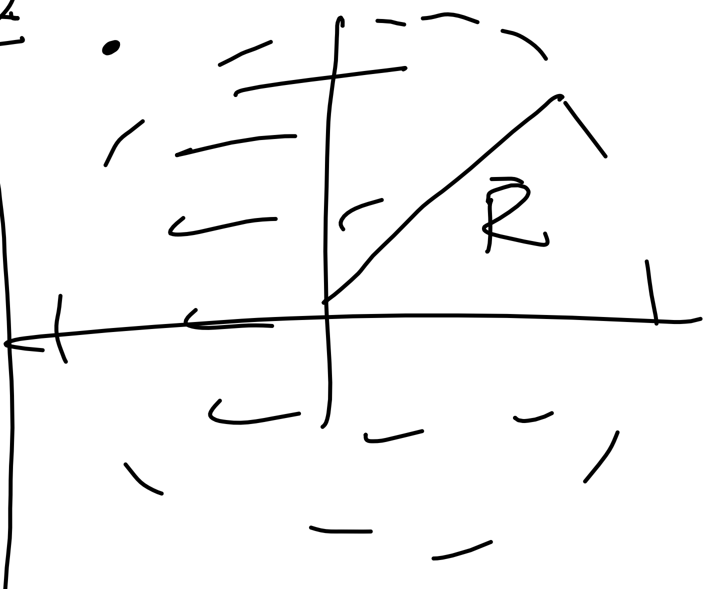
$\sum M_n$ conv since $\left|\frac{z}{z_0}\right| < 1$

Hence by comparison test $\sum a_n z^n$ is conv. ∎

$$R = \sup\left\{ |z| \,/\, \sum a_n z^n \text{ conv} \right\}.$$

- $|z| < R \Rightarrow \sum a_n z^n$ conv.
- $|z| > R \Rightarrow \sum a_n z^n$ div.

This R is called the radius of convergence of the $\sum a_n z^n$.



Th

$$\frac{1}{R} = \lim \frac{|a_n|}{|a_{n+1}|}.$$

$$\frac{1}{R} = \lim \sqrt[n]{|a_n|}$$

Example (1) $1+z+z^2+z^3+\cdots$

$R = 1$

On $|z|=1$. divg.

(2) $\sum \frac{z^n}{n^2}$ $\quad a_n = \frac{1}{n^2}$

$R = \lim \left|\frac{a_n}{a_{n+1}}\right| = \lim \frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}}$

$= 1$.

on $|z|=1$. $\left|\frac{z^n}{n^2}\right| = \frac{1}{n^2} = M_n$

$\therefore$ By comparison test $\sum \frac{z^n}{n^2}$ conv for $|z|=1$

Theorem $a_0 + a_1 z + a_2 z^2 \cdots = \sum a_n z^n = f(z)$ for $|z|<R$

R = radius of convergence.

- $f(z)$ is holomorphic on $|z|<R$.
- $f'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \cdots$
- The radius of convergence of $f'(z)$ is also R.

Example

$$f(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = e^z$$

$$a_n = \frac{1}{n!} \qquad (n+1)!$$

$$R = \lim \frac{|a_n|}{|a_{n+1}|} = \lim \frac{1}{n!} \cdots = \infty$$

$f(z)$ is conv $\forall\, z \in \mathbb{C}$.

- $\varphi(z) = f(z)\, f(a-z) \quad a \in \mathbb{C}$

$$\varphi'(z) = f'(z) f(z-a) - f(z) f'(a-z)$$

$$= 0$$

$$f'(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$= f(z).$$

$$\varphi(z) = \varphi(0) = f(a)$$

$$f(z)\, f(\underbrace{a-z}_{w}) = f(a)$$

$$\boxed{f(z) f(w) = f(z+w).}$$

$$f(z) = f(x+iy)$$
$$= f(x) f(iy)$$
$$= e^x (\cos y + i \sin y)$$
$$= e^z \quad \blacksquare$$