MSO202: Introduction To Complex Analysis

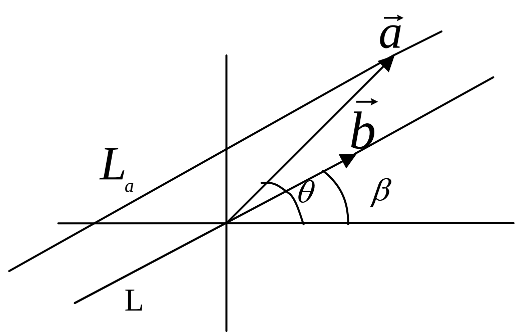
Lecture 2

Geometrical Interpretation of $H_a = \{z : \text{Im}(\frac{z-a}{b}) > 0\}$, $b \neq 0$

We first give the geometrical of $H_0 = \{z : \text{Im}(\frac{z}{b}) > 0, |b| = 1\}$, i.e. when a = 0 and $|b| = 1$.

In this case, write $z = r(\cos\theta + i\sin\theta)$ and $b = \cos\beta + i\sin\beta$, then

$$\frac{z}{b} = r[\cos(\theta - \beta) + i\sin(\theta - \beta)].$$



Now, $z \in H_0 \quad \Leftrightarrow \sin(\theta - \beta) > 0 \quad (\because \text{Im}(\frac{z}{b}) > 0)$

$$\Leftrightarrow 0 < \theta - \beta < \pi$$

$$\Leftrightarrow \beta < \theta < \beta + \pi$$

$\Leftrightarrow H_0$ is the half plane lying to the left of line L passing through origin, if one walks along L in the direction of $\vec{b}$.

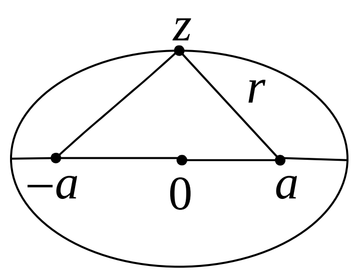
Now,

$$w \in H_0 \Leftrightarrow a + w \in H_a \quad (\because \operatorname{Im}(\frac{a+w-a}{b}) = \operatorname{Im}(\frac{w}{b}) > 0).$$

Therefore, H_a is the half plane lying to the left of line L_a passing through a and in the direction of $\vec{b}$ (as we walk in the direction of $\vec{b}$).

Ellipses in terms of Complex Numbers

$|z-a|+|z+a|=2r$ represents an ellipse for r > 0 and $|a|<r$

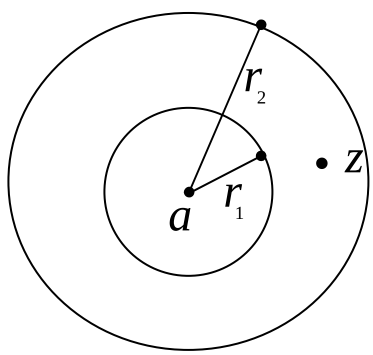


$|z-a|+|z+a|<2r$ represents the interior of the above ellipse

$|z-a|+|z+a|>2r$ represents the exterior of the above ellipse.

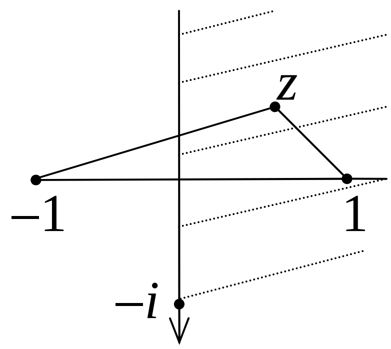
Annulus

$r_1 < |z-a| < r_2$



Half -Plane in terms of Complex Numbers

$H_0 = \{z : |z-1| < |z+1|\}$ represents the right half plane.



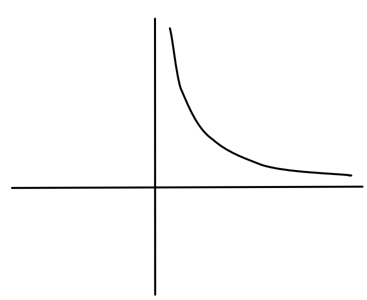
Another way to represent H_0 is

$$H_0 = \left\{ z : \operatorname{Im}\left(\frac{z}{-i} \right) > 0 \right\} = \{ z : \operatorname{Re} z > 0 \}$$

Hyperbola in terms of Complex Numbers

The hyperbola is represented by the parametric equation

$$z = t + i\frac{1}{t} \ (x = t,\ y = \frac{1}{t} \Rightarrow xy = 1)$$



Interior Point of a Set, Open Set, Connected Set and Domain of Complex Numbers:

Interior Point, Exterior Point of a Set: A point is called ***interior point*** of a set $A \subset \mathbf{C}$ if an open disk centered at this point and contained in A can be found.
A point is called ***exterior point*** of a set A if it is an interior point of A^c (complement of A).

Example: Every point of $A = \{z : |z-a| + |z+a| < 2r\}$ is an *interior point* of set A. Every point of $A^* = \{z : |z-a| + |z+a| > 2r\}$ is an *exterior point* of set A.

Open Set: A set $G \subset \mathbf{C}$ is called an ***open set*** if every point of G is an interior point of G.

Example: The set $A = \{z : |z-a| + |z+a| < 2r\}$ is an open set.

Connected Set: A set $A \subset \mathbf{C}$ is called connected, if for every pair of points in A, a continuous curve contained in A can be found that joins these points.

Example: (i) The set $A = \{z : |z-a| + |z+a| < 2r\}$ is connected
(ii) union of two disjoint disks is not a connected set.

Domain: A set $A \subset \mathbf{C}$ which is both open and connected is called **domain**.

Example: (i) The set $A = \{z : \left|z-a\right| + \left|z+a\right| > 2r\}$ is a domain (ii) union of two disjoint disks is not a domain (iii) The set $A = \{z : \left|z-a\right| + \left|z+a\right| = 2r\}$ is not a domain.

Convergent Sequences of Complex Numbers. A sequence $\{z_n\}$ of complex numbers is said to be convergent if, for some $z_0 \in \mathcal{C}$ and every $\varepsilon > 0$, there exists a non-negative integer n_0 such that is

$$\left|z_n - z_0\right| < \varepsilon \ \ for\ all\, n > n_0 \qquad (*)$$

z_0 is called limit of the sequence and we use the notation $z_0 = \lim_{n\to\infty} z_n$.

It is easily seen that $\{z_n = x_n + i\,y_n\}$ converges to $z_0 = x_0 + i\,y_0$ iff and only if x_n converges to x_0 and y_n converges to y_0 (use $|\,x_n - x_0\,| \le |\,z_n - z_0\,|, |\,y_n - y_0\,| \le |\,z_n - z_0\,|$)

All the results as well as their proofs about convergence of sequences of complex numbers are analogous to corresponding results and their proofs for convergence of sequences of real numbers.

For example, it can be easily shown by arguments similar to those for real sequences that the sequence $\{z^n\}$converges to complex number 0 as $n \to \infty$ if $\left|z\right| < 1$ (using (*)), while, for $|z| = 1$, the sequence $\{z^n\}$ does not converge, since $\left\{\cos n\theta + i\sin n\theta\right\}, \theta\ real,$does not converge (since, the sequence $\left\{\cos n\theta\right\}$ of its real part does not converge as $n \to \infty$).

Continuous Functions. A function $f:\mathcal{C}\to\mathcal{C}$ is called continuous at $z_0\in\mathbb{C}$, if for any $\in>0$, there exists a $\delta>0$ such that

$$\left|f(z)-f(z_0)\right|<\varepsilon \;\; for \;\left|z-z_0\right|<\delta.$$

The function f is said to be continuous in a set A if f is continuous at all the points of A.

The definition of continuity is meaningful only if f is defined in some neighbourhood of z_0 (i.e. a disk centered at z_0).

Example 1: Let

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}\equiv \operatorname{Im}\left(\dfrac{z^2}{2|z|^2}\right) & if \;\; (x,y)\neq(0,0)\\ 0 & if \;\; (x,y)=(0,0)\end{cases}$$

Then, $f(x,y)\to\dfrac{m}{1+m^2}$ *along the line* $y=mx$ as $(x,y)\to(0,0)$.
Consequently, f is not continuous at (0, 0).

Example 2: Let

$$f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2} & \textit{if } (x,y)\neq(0,0)\\ 0 & \textit{if } (x,y)=(0,0)\end{cases}$$

Then, $f(x,y)\to 0$ *along the line* $y=mx$ *as* $(x,y)\to(0,0)$

but $f(x,y)\to\dfrac{1}{2}$ *along* $y=x^2$ *as* $(x,y)\to(0,0)$.

Consequently, *f* is not continuous at (0, 0).

Proposition 1. TFAE (The following are equivalent)

(i) f is continuous at z_0

(ii) $z_n \to z_0 \Rightarrow f(z_n) \to f(z_0)$

Proof.

Equivalence of (i) and (ii)

$(i) \Rightarrow (ii)$: Let for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left| f(z) - f(z_0) \right| < \varepsilon \ \ for \ \left| z - z_0 \right| < \delta. \ldots\ldots\ldots\ldots\ldots (1)$$

Now, $z_n \to z_0 \Rightarrow z_n \in \left| z - z_0 \right| < \delta$ for all $n > n_0$

$$\Rightarrow \left| f(z_n) - f(z_0) \right| < \varepsilon \ \ for \ all \ \ n > n_0$$

$$\Rightarrow f(z_n) \to f(z_0)$$

$(ii) \to (i)$: Let $z_n \to z_0 \Rightarrow f(z_n) \to f(z_0)$. Let f be not continuous at z_0. Then, $\exists \, \varepsilon_0$ such that, for every natural number n,

$0 < \left| z - z_0 \right| < 1/n$ contains a point z_n^* satisfying $\left| f(z_n^*) - f(z_0) \right| \geq \varepsilon_0$.

$\Rightarrow z_n^* \to z_0 \ but \ \ f(z_n^*) \to f(z_0)$.

Proposition 2. The functions $f \pm g, \alpha f$, fg, f/g $(g \neq 0)$ are continuous whenever f and g are continuous. Converse need not be true.

Proposition 3. If f is continuous in A and g is continuous in the range of f, then $g \circ f$ is continuous in A.

The proofs of above propositions are analogous to corresponding proofs for real valued functions of real variables.

Examples.

(i) Any polynomial in z is continuous in $\mathcal{C}$
(*use Proposition 2*).

(ii) f is continuous if and only if $\mathrm{Re}(f)$ *and* $\mathrm{Im}(f)$ are continuous
(*use Proposition 2 and* $|\mathrm{Re}(f)| < |f|, |\mathrm{Im}(f)| < |f|$).

(iii) f is continuous if and only if $\overline{f}$ is continuous
(*use* $|f| = |\overline{f}|$).

(iv) f is continuous, then $|f|$ is continuous. Converse need not be true, e.g. consider, $f(z) = u(z) + i\,v(z)$, where

$$u(z) = \begin{cases} \alpha, & if\ \ z\ has\ rational\ coordinates \\ -\alpha, & otherwise \end{cases}, \quad v(z) = \beta.$$

Differentiable Functions.

A function $f:\mathcal{C}\to\mathcal{C}$ is called differentiable at z_0 if

$$\lim_{\varsigma\to 0}\frac{f(z_0+\varsigma)-f(z_0)}{\varsigma} \qquad (1)$$

exists finitely. In that case, the limit in (1) is called the derivative of f(z) at the point z_0 and is denoted by $f'(z_0)$. Note that $f'(z_0)$ can also be written as

$$f'(z_0)=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

$$(take\ z=z_0+\Delta z)$$

Remark. All the results on differentiability of functions $f:\mathcal{R}\to\mathcal{R}$ are true for differentiability of functions $f:\mathcal{C}\to\mathcal{C}$ and can be proved analogously.

Note: If f is diff. it is cont. but the converse need not be true. (Ex. $f(z) = |z|^2$ is cont. everywhere but is diff. only at 0)

$$\lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z_0|^2 + |\Delta z|^2 + \overline{z}_0 \Delta z + z_0 \overline{\Delta z} - |z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (\overline{z}_0 + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z})$$

= 0 if $z_0 = 0$ (limit does not exist if $z_0 \neq 0$)

Analytic Function. A function f is said to be analytic at z_0 if f (i) f is differentiable at z_0 and (ii) f is differentiable in some neighbourhood of z_0 (i.e. in a disk centered at z_0).
The function f is said to be analytic in a set A, if f is analytic at all points of A.

It is clear that if f is differentiable in any open set $G \subseteq \mathcal{C}$, then f is analytic in G. Converse holds obviously.

Examples:
(i) $|z|^2$ is not analytic anywhere.
(ii) Any polynomial is analytic at all points of $\mathcal{C}$.