MSO 202A: Complex Variables
August-September 2022
Assignment-0

Exercises marked (**T**) are to be discussed in the tutorials.

1. (**T**)Let $P(z)$ be a polynomial with real coefficients. Show that if z_0 is a root of P then so is $\overline{z}_0$.

2. Solve the following equations in polar form and locate the roots in the complex plane:
   (a) $z^4 = -1$
   (b) (**T**)$z^4 = -1 + \sqrt{3}\iota$

3. Simplify $(1+\iota)^{17}$ into the form $a + b\iota$.

4. Show that if two integers can be expressed as the sum of two squares, then so can their product.

5. (**T**)Show that the n-th roots of 1 (aside from 1) satisfy the cyclotomic equation $z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$

6. (**T**)Consider the $n-1$ diagonals of a regular n-gon inscribed in a unit circle obtained by connecting one vertex with all the others. Show that the product of their lengths is n.

7. Let ω be a p-th root of unity. Define
$$\chi(p) = \sum_{n=0}^{p-1} \omega^{n^2}.$$
   Verify that $\chi(3)^2 = -3,\ \chi(5)^2 = 5,\ \chi(7)^2 = -7.$

   (Remark: The expression $\chi(p)$ is known as Gauss Sum. For odd prime p it can be shown that $\chi(p)^2 = (-1)^{\frac{p-1}{2}} p$.)

8. For each of the following equations, give a geometric description of the set of complex numbers. (a) (**T**)$|z - z_1| = |z - z_2|$ (b) $|z - z_1| + |z - z_2| = c$ (c) $|z - 2 + 3\iota| < 1$ (d) (**T**)$0 \le z < \pi/4$ (e) $|z - 4| \ge |z|$ (f) $|\operatorname{Re} z| \ge a > 0$

9. In each following functions $f(z)$, compute the limit $\lim_{z\to 0} f(z)$. Hence conclude whether the functions can be defined at $z = 0$ to become continuous.

   (**T**)(a) $2z\frac{\operatorname{Re} z}{|z|}$ (**T**)(b) $\frac{\iota z}{|z|}$ (c) $3\frac{\operatorname{Re} z}{z}$

10. (**T**)Let
$$f(z) = \frac{\{(1-\iota)z + (1+\iota)\overline{z}\}^2}{z\overline{z}}.$$
    Show that $\lim_{x\to 0}\lim_{y\to 0} f(z) = \lim_{y\to 0}\lim_{x\to 0} f(z)$ but $\lim_{z\to 0} f(z)$ does not exist.