Department of Mathematics and Statistics
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MSO202 Assignment 1 Solutions
Introduction To Complex Analysis

The problems marked (T) need an explicit discussion in the tutorial class. Other problems are for enhanced practice.

1. Sketch the following sets and determine which ones of these are domains:
(a) $|z-2+i|<1$ (b) $S=\{z:|z-1|<1$ *or* $|z+1|<1$ (T)(c) $0\le \arg z\le \pi/4$
(T)(d) $|z-4|\ge |z|$ (e) $|\operatorname{Re} z|>a>0$ (f) $|\operatorname{Im} z|\le |\operatorname{Re} z|$ (T)(g) $|z+ia|<|z-a|$ *for* $a>0$.

(a) open and connected, being an open disc, so a domain (b) open and disconnected, being union of two disjoint open discs, so not a domain (c) closed and connected, being the closed region between two rays, so not a domain (d) closed half plane, given by $x\le 2$, so not a domain (e) open and disconnected, being the union of two half planes x > a and x < - a, so not a domain (f) closed region below the line y = x, so not a domain (g) open and connected region defined by y < -x, since a > 0, so a domain.

2. Which of the following functions f(z) can be defined at z = 0 so that they become continuous at z = 0:
(T)(a) $2z\dfrac{\operatorname{Re} z}{|z|}$ (b) $\dfrac{\operatorname{Re}(z^2)}{|2z|^2}$ (T)(c) $\dfrac{3\operatorname{Re} z}{z}$ (d) $\dfrac{iz}{|z|}$ (e) $\dfrac{(\operatorname{Re} z)^2 \operatorname{Im} z}{(\operatorname{Re} z)^4+(\operatorname{Im} z)^2}$

(a) Since $\left|2z\dfrac{\operatorname{Re} z}{|z|}-0\right| = 2|\operatorname{Re} z|\to 0$ *as* $z\to 0$, limit of f(z) as $z\to 0$ exists, so that f(0) can be defined to be 0 to make it continuous at z = 0. (b)As $z\to 0$, $f(z)=\dfrac{x^2-y^2}{4(x^2+y^2)}\to 0$ along y =x and f(z)$\to 1/4$ along y = 0, so it cannot be made continuous at z = 0 howsoever its defined at z = 0 (c) As $z\to 0$, $f(z)=\dfrac{3x}{x+iy}\to 0$ along x = 0 and f(z)$\to 3$ along y = 0 and x > 0, so it cannot be made continuous at z = 0 howsoever its defined at z = 0 (d) As $z\to 0$, $f(z)=\dfrac{ix-y}{\sqrt{x^2+y^2}}\to -1$ along x = 0, y > 0 and f(z)$\to i$ along y = 0, x > 0, so it cannot be made continuous at z = 0 howsoever its defined at z = 0 (e) As $z\to 0$, f(z) = $\dfrac{(\operatorname{Re} z)^2 \operatorname{Im} z}{(\operatorname{Re} z)^4+(\operatorname{Im} z)^2} = \dfrac{x^2 y}{x^4+y^2}\to 0$ along any line y = mx and f(z) $\to 1/2$ along y = x^2, so it cannot be made continuous at z = 0 howsoever its defined at z = 0.

3. Show that, for $f(z)=\dfrac{[(1-i)z+(1+i)\bar{z}]^2}{z\,\bar{z}}$,
$\lim_{x\to 0}[\lim_{y\to 0} f(z)] = \lim_{y\to 0}[\lim_{x\to 0} f(z)]$ but $\lim_{z\to 0} f(z)$ does not exist.

It can be easily verified that $\lim_{x\to 0}[\lim_{y\to 0} f(z)] = \lim_{y\to 0}[\lim_{x\to 0} f(z)] =$ 4, since $\lim_{y\to 0} f(z) = \dfrac{\{(1-i)x+(1+i)x\}^2}{x^2} = 4.$
However, as z $\to 0$ along the line y = - x, f(z)$\to 0$ as $z\to 0$ so that the limit of f(z) as z $\to 0$ does not exist.

4. Show that (a) *f(z) = Re z* is not differentiable for any z (b) $f(z) = |z|^2$ is differentiable only at z = 0.

(a) By Problem 2(c), the quotient $\dfrac{f(z+\Delta z) - f(z)}{\Delta z} = \dfrac{\operatorname{Re} \Delta z}{\Delta z}$ does not have a limit as $\Delta z \to 0$. Consequently, the given function is not differentiable for any z. (b) As $\Delta z \to 0$, the quotient $\dfrac{f(z+\Delta z) - f(z)}{\Delta z} = \dfrac{\bar{z}\Delta z + z\,\overline{\Delta z} + \Delta z\,\overline{\Delta z}}{\Delta z}$ has limit only if z = 0, hence the result.

5. (T)Show that the function

$$f(z) = \begin{cases} \dfrac{z^5}{|z|^4} & if \;\; z \neq 0 \\ \\ 0 & if \;\; z = 0 \end{cases}$$

is continuous at z = 0, fiirst order partial derivatives of its real and imaginary part exist at z = 0, but f(z) is not differentiable at z = 0.

Continuity: $|f(z)| = |z| \to 0 \;\; as \;\; z \to 0 \Rightarrow f(z) \to 0 \;\; as \;\; z \to 0$, implying continuity at z = 0.
First Order Partial Derivatives: Let $f(z) = u(x,y) + i\,v(x,y)$, then $u(x,0) = x$ = x, $v(x,0) = 0$, $u(0,y) = 0$, $v(0,y) = y$. Since, $\lim\limits_{x\to 0} \dfrac{u(x,0) - u(0,0)}{x} = \lim\limits_{x \to 0} \dfrac{x-0}{x}$ exists, $u_x(0,0)$ exists and equals 1 . Similarly, it can be shown $u_y(0,0)$, $v_x(0,0)$ and $v_y(0,0)$ exist.
Differentiability: As z $\to 0$, the quotient $\dfrac{f(z) - f(0)}{z} = \dfrac{z^4}{|z|^4} \to 1$ along z = x (real axis), while this quotient $\to -1$ along the line z = x + i x, x real, showing that f(z) is not differentiable at z = 0.

6. Prove that if a function f(z) is differentiable at z = 0, it is continuous at z = 0.

Follows by standard arguments.

7. Show that for each of the following functions Cauchy-Riemann equations are satisfied at the origin. Also determine whether these functions are differentiable at z = 0. Are these functions analytic at z = 0?

(T) (i) $f(z) = \sqrt{|\operatorname{Re}(z) \operatorname{Im}(z)|}$ (ii) $f(z) = xy^2 + i\,yx^2$, $z = x + i\,y$ (iii) $f(z) = \begin{cases} \dfrac{\operatorname{Im}(z)^2}{|z|^2} & if \;\; z \neq 0 \\ \\ 0 & if \;\; z = 0 \end{cases}$

(i) $f(z) = \sqrt{|xy|} \Rightarrow u(x,y) = \sqrt{|xy|} \;\; and \;\; v(x,y) = 0 \Rightarrow u_x(0,0) = \lim\limits_{h \to 0} \dfrac{u(h,0) - u(0,0)}{h} = 0$. Similarly, $u_y(0, 0) = 0$.
Further, since v $\equiv 0$, $v_x(0, 0) = v_y(0, 0) = 0$. $\Rightarrow$ CR equations are satisfied at z = 0.

As $z \to 0$, the quotient $\dfrac{f(0+z)-f(0)}{z} = \dfrac{\sqrt{|xy|}}{x+iy} \to 0$ along the line x = 0, while this quotient $\to$ 1/(1 + i) along the line x = y. Consequently, f(z) is not differentiable at z = 0.

(ii) Observe that u_x, u_y, v_x, v_y exist in a neighbourhood of z = 0, are continuous and satisfy CR equations at z = 0, consequently f(z) is differentiable at z = 0.Since CR equations are satisfied only at z = 0, f(z) is not analytic at z = 0.

(iii) $u(x,y) = \dfrac{2xy}{x^2+y^2}$ *and* $v \equiv 0 \Rightarrow u_x = u_y = v_x = v_y = 0$ at z = 0. Therefore, CR equations are satisfied at z = 0.

However, $u_x(x,y) = \dfrac{2y(y^2-x^2)}{(x^2+y^2)^2}$ is not continuous at z = 0. Consequently, the derivative of f(z) does not exist at z = 0.

8. Find the domain in which the function

$$f(z) = |\operatorname{Re} z^2| + i|\operatorname{Im} z^2|$$

is analytic.

Observe that $f(z) = |x^2-y^2| + 2i\,|xy|$ can be written as

$$f(z) = z^2 \quad for \quad 0<\theta<\pi/4 \quad and \quad \pi<\theta<5\pi/4,$$
$$f(z) = -\bar{z}^2 \quad for \quad \pi/4<\theta<\pi/2 \quad and \quad 5\pi/4<\theta<3\pi/2,$$
$$f(z) = -z^2 \quad for \quad \pi/2<\theta<3\pi/4 \quad and \quad 3\pi/2<\theta<7\pi/4,$$
$$f(z) = \bar{z}^2 \quad for \quad 3\pi/4<\theta<\pi \quad and \quad 7\pi/4<\theta<2\pi.$$

Consequently, the function is analytic in the regions
$0<\theta<\pi/4,\ \pi<\theta<5\pi/4,\ \pi/2<\theta<3\pi/4,\ 3\pi/2<\theta<7\pi/4.$
Further, along the rays $\theta = 0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4,$ either the real part or the imaginary part of f(z) is zero, so it is not analytic on these rays.

9. (T)Show that the derivative of a real valued function *f(z)* of a complex variable z, at any point, is either zero or it does not exist.

Consider the quotient $F(\Delta z) = \dfrac{f(z+\Delta z)-f(z)}{\Delta z}$. Since f is real valued, the numerator of the quotient is real. Consequently, $F(\Delta z) \to a\ real\ value$, if $\Delta z \to 0$ along real axis (i.e., $\Delta y = 0$) and $F(\Delta z) \to a\ purely\ imaginary\ value$, if $\Delta z \to 0$ along imaginary axis (i.e., $\Delta x = 0$). Therefore, $\lim_{\Delta z \to 0} F(\Delta z)$ either does not exist, or if it exists, it must be zero.

10. (T)Prove that

(a) If $f(z)$ *and* $\overline{f(z)}$ both are analytic in a domain D, then *f(z)* is a constant function in *D*.
(b) If *f(z)* is analytic and $f'(z) \equiv 0$ in a domaind D, then *f(z)* is a constant function in *D*.
(c) If *f(z)* is analytic in a domain *D* and $u_x + v_y = 0$ in *D*, then $f'(z)$ is constant in *D*.

(a) $f(z) = u + i\,v$, $\overline{f(z)} = u - i\,v$ analytic in a domain D implies $u_x = v_y$ and $u_x = -v_y \Rightarrow u_x = 0 = v_y$ and
$u_y = -v_x$ and $u_y = -(-v_x) \Rightarrow u_y = 0 = v_x$
$\Rightarrow$ u and v are constants in D.
(b) $f'(z) = u_x + i\,v_x \equiv 0$ in a domain D $\Rightarrow$ $u_x = 0 = v_y$ and $v_x = 0 = -u_y$ in D
$\Rightarrow$ u and v are constants in D.

(c) $u_x + v_y = 0 \Rightarrow u_x = -v_y$
But, by CR equations, $u_x = v_y$
$\Rightarrow u_x = 0 = v_y \Rightarrow$ u is a function of y alone and v is a function of x alone.
Again, by CR equations, $u_y = -v_x$.
$\Rightarrow$ A function of y alone (i.e. u_y) = A function of x alone (i.e. v_x)
$\Rightarrow u_y = v_x$ = constant (say, K) in D.
$\Rightarrow f'(z) = u_x + i\,v_x = iK \text{ in } D.$

11. (T)Let $f(z) = u + i\,v = R\,e^{i\varphi}$ be an analytic function in a domain D. Prove that if any of the functions u, v, R, φ is identically constant in D, then f(z) is a constant function in D.
(i) $u \equiv constant \Rightarrow$ (by CR equations) $v_x = v_y = 0 \Rightarrow v \equiv constant$
(ii) $v \equiv constant \Rightarrow$ (by CR equations) $v_x = v_y = 0 \Rightarrow v \equiv constant$
(iii) $R \equiv constant \Rightarrow R^2 = u^2 + v^2$ is constant $\Rightarrow u\,u_x + v\,v_x = 0$ and $u\,u_y + v\,v_y = 0$

$\Rightarrow$ (By CR equations) $u\,u_x - v\,u_y = 0$ and $u\,u_y + v\,u_x = 0 \Rightarrow u_x = 0,\ u_y = 0$

$\Rightarrow u \equiv constant \Rightarrow$ (by (i)) f(z) is constant.

(iv) $Arg\ f \equiv constant \Rightarrow \tan^{-1}\dfrac{v}{u} = a\ real\ constant = c\ (say) \Rightarrow v = u \tan c.$

$\Rightarrow$ $f(z) = (1 + i \tan c)\,u$ is analytic
$\Rightarrow$ $g(z) = (1 - i \tan c)\,f(z)$ is analytic
$\Rightarrow g(z) = (1 + \tan^2 c)\,u$ is analytic
$\Rightarrow$ (since $Im(g) = 0$) $g \equiv constant$. $\Rightarrow$ $f \equiv constant$.

12. If f(z) is an analytic function in a domain D, prove that

$$\nabla^2 |f(z)|^2 = 4|f'(z)|^2 .$$

f(z) = u + i v is analytic in D $\Rightarrow u \,\&\, v$ satisfy CR equations. Let $\varphi = |f(z)|^2 = u^2 + v^2$. Then,

$\varphi_{xx} = 2\{u\ u_{xx} + v\ v_{xx} + u_x^2 + v_x^2\}$ and $\varphi_{yy} = 2\{u\ u_{yy} + v\ v_{yy} + u_y^2 + v_y^2\}$

$\Rightarrow \Delta^2\varphi = \Delta^2 |f(z)|^2 = 2u\,\Delta^2 u + 2v\,\Delta^2 v + 4|f'(z)|^2 = 4|f'(z)|^2$.

13. Using CR equations in cartesian coordinates, obtain the following CR equations in the polar coordinates:
$r\,u_r = v_\theta$, $r\,v_r = -u_\theta$. Express $f'(z)$ in terms of the partial derivatives with respect to r and θ.

(a) Put x = r cosθ and y = r cosθ and express the first partial derivatives with respect to x and y in terms of the first partial derivatives with respect to r and θ. The CR equations in Cartesian Coordinates then transform in to the given CR equations in the Polar Coordinates.

(b) Use $f'(z) = u_x + i\,v_x$ and the transformation of the first partial derivatives with respect to cartesian coordinates to the first partial derivatives with respect to polar coordinates found in (a) above.