__Recall__

__Cauchy's Theorem__

f holomorphic on a convex domain Ω

$\Rightarrow$ f admits a primitive on Ω.

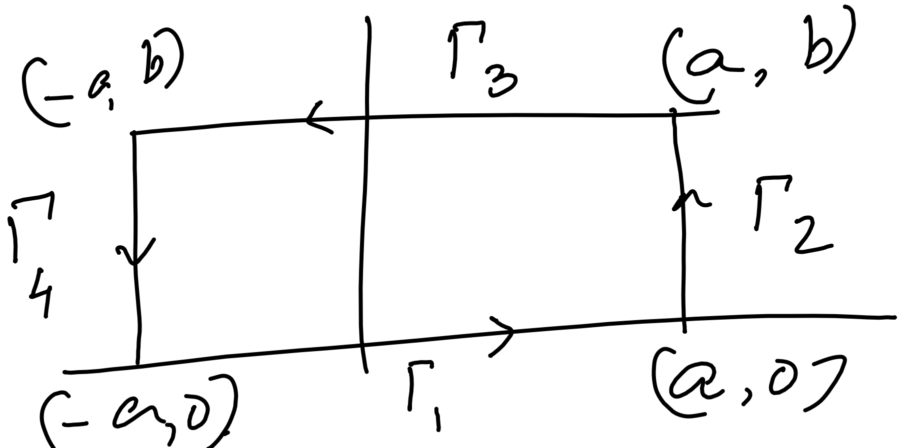
$\Rightarrow$ $\int_\gamma f(z)\,dz = 0$ for any closed curve γ in Ω.

__Application__

$$\int_{-\infty}^{\infty} e^{-x^2}\cos 2bx = e^{-b^2}\sqrt{\pi}$$

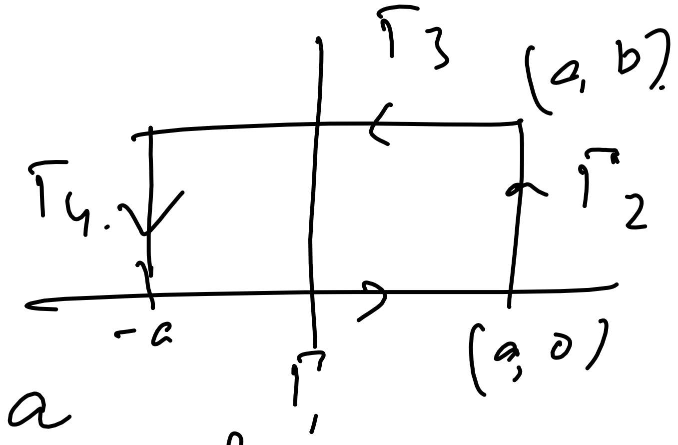
$(b>0)$.

__sol__ $f(z) = e^{-z^2} \in \mathcal{H}(\mathbb{C})$



By Cauchy's Theorem

$$\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4} = 0$$



On Γ_1

$$\int_{\Gamma_1} f(z)\,dz = \int_{-a}^{a} e^{-x^2}\,dx$$

On Γ_2

$$\int_{\Gamma_2} f(z)\,dz = \int_0^b e^{-(a+iy)^2}\, i\,dy \qquad z = a+iy$$

on Γ_3

$$\int_{\Gamma_3} f(z)\,dz = \int_a^{-a} e^{-(x+ib)^2}\,dx \qquad z = x+ib$$

$$= -e^{b^2}\int_{-a}^{a} e^{-x^2-2ibx}\,dx = -e^{b^2}\int_{-a}^{a} e^{-x^2}(\cos 2bx - i\sin 2bx)\,dx$$

$$\int_{\Gamma_4} f(z)\,dz = \int_b^0 e^{-(-a+iy)^2}\, i\,dy$$

- Letting $a \to \infty$, first integral.

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

- Letting $a \to \infty$ this integral

$$-e^{+b^2}\int_{-\infty}^{\infty} e^{-x^2}\cos 2bx\,dx.$$

$$I = \int_0^\infty e^{-x^2}\,dx$$
$$I = \int_0^\infty e^{-y^2}\,dy$$
$$I^2 = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

- **On Γ_2**

$$\left|\int_{\Gamma_2} f(z)\,dz\right| = \left|\int_0^b e^{-(a^2-y^2+2aiy)}\, i\,dy\right|$$

$$\leq \int_0^b e^{-a^2+y^2}\,dy$$

$$\leq \int_0^b e^{-a^2+b^2}\,dy = \frac{b\,e^{b^2}}{e^{a^2}} \to 0 \text{ as } a \to \infty$$

Similarly on Γ_4

$$\left|\int_{\Gamma_4} f(z)\,dz\right| \longrightarrow 0 \text{ as } a \longrightarrow \infty.$$

Thus we get

$$\sqrt{\pi} - e^{-b^2}\int_{-\infty}^{\infty} e^{-x^2}\cos 2bx\,dx = 0.$$

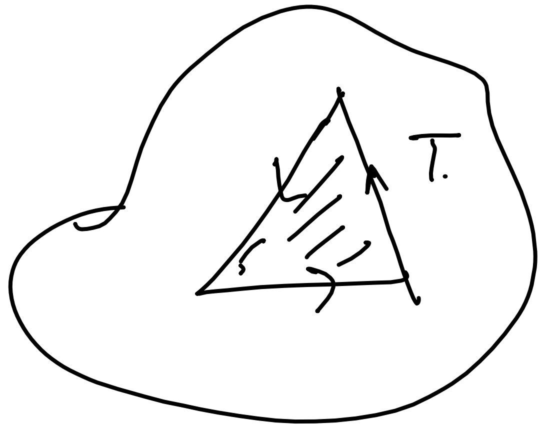
$$\int_{-\infty}^{\infty} e^{-x^2}\cos 2bx = e^{b^2}\sqrt{\pi}$$

Proof of Cauchy's Theorem.

The proof depends on a result by Goursat

Goursat's Theorem

If $f \in H(\Omega)$ & Ω contains a triangle T & its interior.



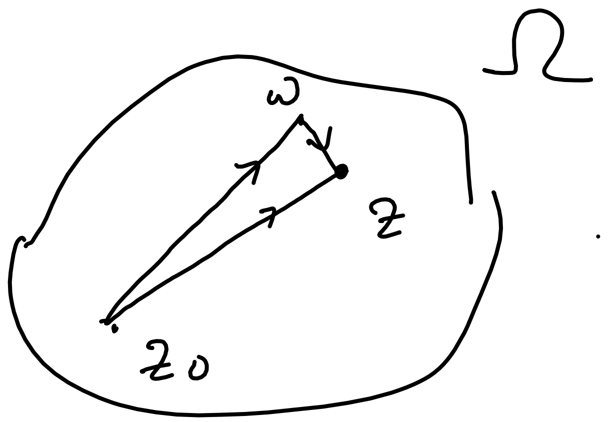
Then $\displaystyle\int_T f(z)\,dz = 0$

Assuming Goursat's Th, let us now complete the proof of Cauchy's Th.

$f \in \mathcal{H}(\Omega)$

$\boxed{\text{Fix } z_0}$

$$F(z) = \int_{\overrightarrow{z_0 z}} f(\xi)\, d\xi$$



To show

$$\lim_{w \to z} \frac{F(w) - F(z)}{w - z} = f(z).$$

$$F(w) - F(z) = \int_{\overrightarrow{zw}} f(\xi)\, d\xi$$

By Goursat's Th

$$\int_{\overrightarrow{z_0 w}} + \int_{\overrightarrow{wz}} + \int_{\overrightarrow{z z_0}} = 0$$

$$F(w) + \int_{\overrightarrow{wz}} - F(z) = 0$$

$$F(w) - F(z) = \int_{\overrightarrow{zw}} f(\xi)\, d\xi$$

$$\frac{F(w) - F(z)}{w - z} = \frac{1}{(w - z)} \int_{\overrightarrow{zw}} f(\xi)\, d\xi$$

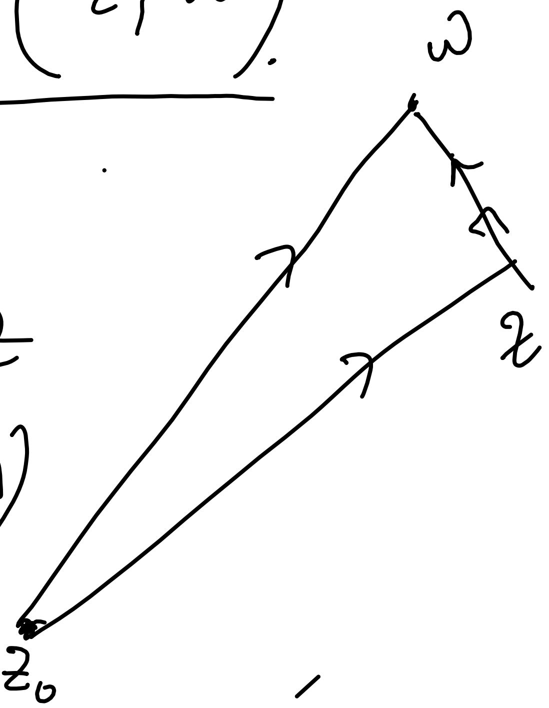
$$\left| \frac{F(w) - F(z)}{w - z} - f(z) \right| = \left| \frac{1}{w - z} \left[\int_{\overrightarrow{zw}} f(\xi)\, d\xi - \int_{\overrightarrow{zw}} f(z)\, d\xi \right] \right|$$

$$\left| \frac{1}{w-z} \int_{[z,w]} [f(\xi) - f(z)]\, d\xi \right|$$

$$\le \frac{1}{\cancel{|w-z|}} \, \cancel{|w-z|} \cdot M(z,w)$$

$$M(z,w) \to 0 \text{ as } w \to z$$

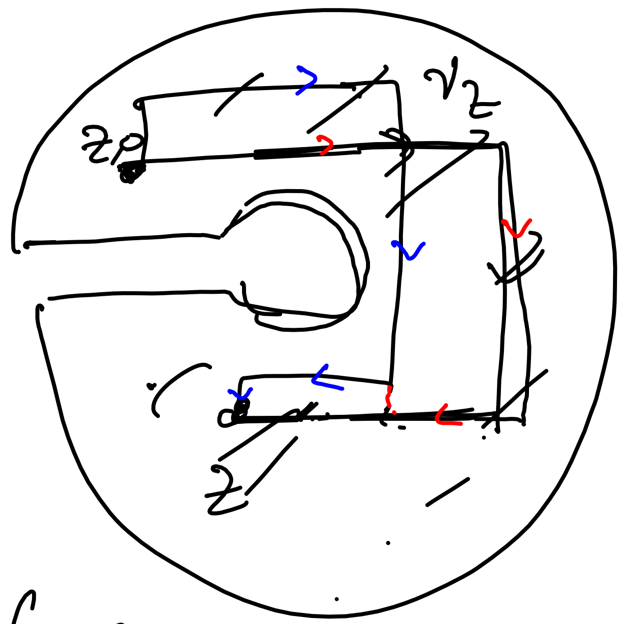
($\because$ f is continuous)



Hence $\displaystyle\lim_{w \to z} \frac{F(w) - F(z)}{w-z} = f(z)$

so $F'(z) = f(z)$

Keyhole contour



$$F(z) = \int_{\gamma_z} f(\xi)\, d\xi.$$