MSO 202A: Complex Variables
August-September 2022
Assignment-3

Throughout C will denote the unit circle around origin, oriented counterclockwise.

1. **(T)**Find all the zeros of the function $f(z) = 2 + \cos z$.

2. **(T)**The Bernoulli numbers B_n are defined by the series power series
$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$
Show that $\frac{z}{e^z-1} + \frac{z}{2} = \frac{z}{2}\coth\frac{z}{2}$. Conclude that $B_1 = -\frac{1}{2}$ and $B_{2n+1} = 0,\ n \geq 1$. Deduce that
$$z \cot z = \sum_{n=0}^{\infty} (-1)^n \frac{B_{2n}}{(2n)!} z^{2n}.$$

3. **(T)**Let a be a positive real number and Γ be the rectangle with vertices $0, a, a+2\pi\iota, 2\pi\iota$. Explicitly compute the integral $\int\limits_{\Gamma} e^z dz$ and verify that the integral is 0.

4. Calculate by hand $\int\limits_{C} \frac{1}{z} dz$ and $\int\limits_{-C} \frac{1}{z} dz$, where $-C$ is C with opposite orientation.

5. Show that $1/z$ is holomorphic on $\mathbb{C}^*$ but it does not admit a primitive/antiderivative on $\mathbb{C}^*$. Use it to show that $u = 1/2 \log(x^2 + y^2)$ does not admit a harmonic conjugate on $\mathbb{C}^*$.

6. Let γ be the upper half of the unit circle described anticlockwise. Show that
$$\left| \int_{\gamma} \frac{\exp(z)}{z} dz \right| \leq \pi e.$$

7. **(T)**Show that
$$\left| \int_{C_3} \frac{1}{z^2 + \iota} dz \right| \leq \frac{3\pi}{4}.$$

8. **(T)**Show that
$$\left| \int_{\gamma} \log(z) dz \right| \leq \frac{\pi^2}{4},$$
where γ is the first quadrant portion of the circle C. Here principal branch of log is used.

9. Suppose $f(z)$ is anlytic and satisfies the relation $|f(z) - 1| < 1$ in a region Ω. Show that
$$\int_{\gamma} \frac{f'(z)}{f(z)} dz = 0$$
for every closed curve γ in Ω.

10. (**T**)Show that $\int_\gamma \overline{f(z)} f'(z) dz$ is purely imaginary where γ is any smooth closed curve in a region Ω and f is holomorphic in Ω.

11. Compute the following integrals:

   (a) (**T**) $\int_{|z|=1} e^z z^{-n} dz; \quad n \in \mathbb{Z};$

   (b) $\int_{|z|=2} z^n (1-z)^m dz; \quad m, n \in \mathbb{Z}$

   (c) (**T**) $\int_{|z|=1} \frac{\cos z}{\sin z} dz$

   (d)

$$\int_{|z|=1} (z - \frac{1}{z})^n \frac{dz}{z} = \begin{cases} 2\pi i \binom{n}{n/2} (-1)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

   Use it to show that

$$\int_0^{2\pi} \sin^n t \, dt = \begin{cases} \frac{\pi}{2^{n-1}} \binom{n}{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$