Complex Integration

$f : [a\ b] \longrightarrow \mathbb{C}$

$f(t) = u(t) + i\,v(t)$

f continuous $\iff$ u, v are continuous.

$$\int_a^b f(t)\,dt := \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

- $f \longmapsto \int_a^b f(t)\,dt$ is complex linear.

$$\int \alpha f + \beta g = \alpha \int f + \beta \int g$$

$\alpha, \beta \in \mathbb{C}$.

- $$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt$$

Proof

$$\int_a^b f(t)\,dt = R e^{i\theta}$$

$$\left|\int_a^b f(t)\,dt\right| = R$$

$$= e^{-i\theta}\int_a^b f(t)\,dt$$

$$= \int_a^b \left[e^{-i\theta} f(t)\right] dt$$

$[x \le |z|$

$$= \int_a^b \operatorname{Re}\left[e^{-i\theta} f(t)\right] dt$$

$$\le \int_a^b |e^{-i\theta} f(t)|\,dt = \int_a^b |f(t)|\,dt$$

Now

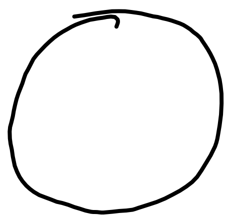
$$[a\ b] \xrightarrow{\gamma} \Omega \xrightarrow[\text{continuous}]{f} \mathbb{C} \quad (\Omega \subseteq \mathbb{C} \text{ open})$$

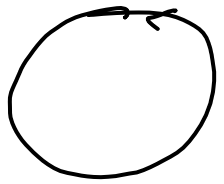
parametric

γ is a smooth curve: i.e. $\gamma'(t)$ is continuous with $|\gamma'(t)| \neq 0$

Then
$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

$|z|=1$ $\qquad$ $z = e^{it}$ $\quad t \in [0\ \ 2\pi]$

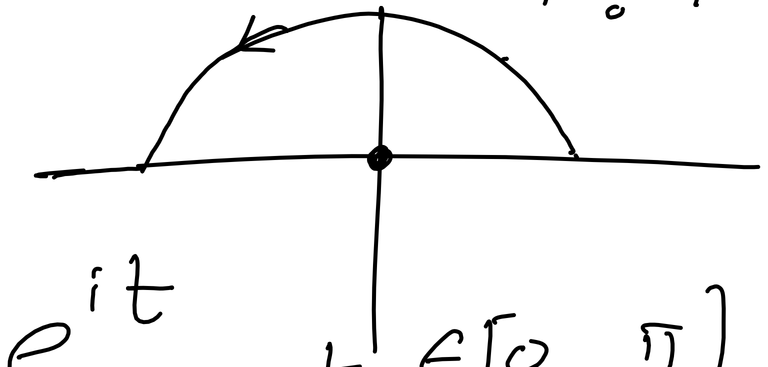




Example $\quad f(z) = \dfrac{1}{z}$

$$\Omega = \mathbb{C}^* \xrightarrow{f} \mathbb{C}$$

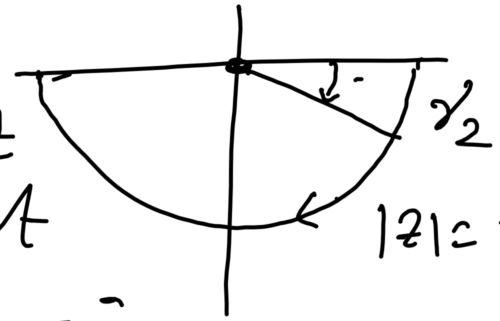
$\gamma_1 : |z| = 1$



$\gamma_1(t) = e^{it} \quad t \in [0\ \ \pi]$

$$\int_{\gamma_1} f(z)\,dz = \int_0^\pi \frac{1}{\cancel{e^{it}}}\, i\,\cancel{e^{it}}\,dt$$
$$= \pi i$$

$\gamma_2(t) = e^{-it} \quad t \in [0\ \ \pi]$



$$\int_{\gamma_2} f(z)\,dz = \int_0^\pi \frac{1}{e^{-it}} (-i) e^{-it}\,dt$$
$$= -\pi i$$

Properties

(i) $f \longmapsto \int_\gamma f(z)\,dz$ is complex linear.

$$\int_\gamma (\alpha f + \beta g) = \alpha \int_\gamma f + \beta \int_\gamma g$$

$\alpha, \beta \in \mathbb{C}$

(ii) $$\left|\int_\gamma f(z)\,dz\right| \leq M L$$

ML- inequality

L = length of γ.

$$M = \sup_{z \in \gamma} |f(z)| = \sup_{t \in [a\ b]} |f(\gamma(t))|$$

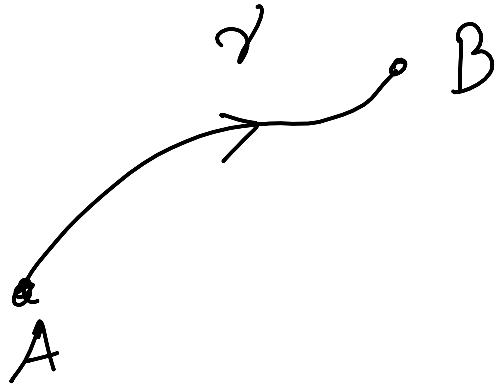
$$\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\,\gamma'(t)\,dt\right|$$

$$\leq \int_a^b \underbrace{|f(\gamma(t))|}\,|\gamma'(t)|\,dt$$

$$\leq M \underbrace{\int_a^b |\gamma'(t)|\,dt}_{L}$$

(iii) $\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz$

$\gamma : [a\ b] \longrightarrow \mathbb{C}$



$(-\gamma)(t) = \underline{\gamma(a+b-t)}$

$(-\gamma) : [a\ b] \longrightarrow \mathbb{C}$

$\Gamma = -\gamma$

$\int_{\Gamma} f(z)\,dz = \int_a^b f(\Gamma(t))\,\Gamma'(t)\,dt$

$[a\ b] \xrightarrow{a+b-t} [a\ b] \xrightarrow{\gamma} \mathbb{C}$

$= \int_a^b f(\gamma(a+b-t))\,(-\gamma'(a+b-t))\,dt$

$a+b-t = s$

$= \int_b^a f(\gamma(s))\,(+\gamma'(s))\,(+ds)$

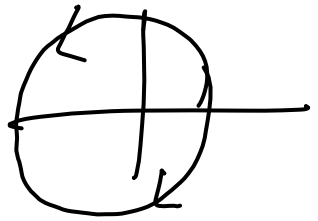
$= -\int_a^b f(\gamma(s))\,\gamma'(s)$

$= -\int_{\gamma} f(z)\,dz$

$\blacksquare$

Example

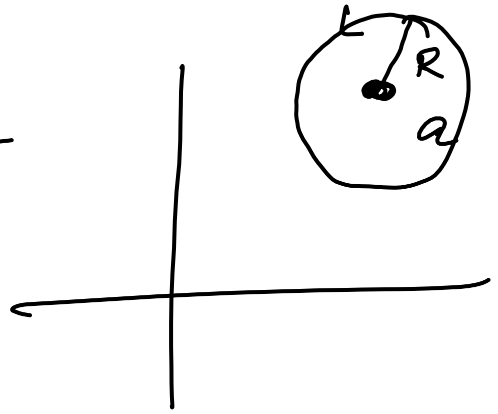
$$\int_{|z|=1} \frac{1}{z}\, dz$$



$$= \pi i + (-\pi i) = 2\pi i$$

Ingenl

$$\int_{|z-a|=R} \frac{1}{z-a}\, dz$$



$|z-a|=R$

anti clockwise

$z - a = Re^{it}$ $\quad t \in [0, 2\pi]$

$$= \int_0^{2\pi} \frac{1}{\cancel{Re^{it}}} \cancel{Re^{it}} \cdot i\, dt$$

$$= 2\pi i$$

$$\int_{|z|=1} \frac{1}{z^2}\, dz$$

$|z|=1$

anticlock

$z = e^{it}$

$$= \int_0^{2\pi} \frac{i e^{it}\, dt}{(e^{it})^2}$$

$$= i \int_0^{2\pi} e^{-it}\, dt$$

$$= i \left[\frac{e^{-it}}{-i} \right]_0^{2\pi}$$

$$= 0$$

Definition Suppose f is continuous on $\Omega \xrightarrow{f} \mathbb{C}$.

A primitive for f is an analytic function $F : \Omega \longrightarrow \mathbb{C}$ s.t.

$$F'(z) = f(z).$$

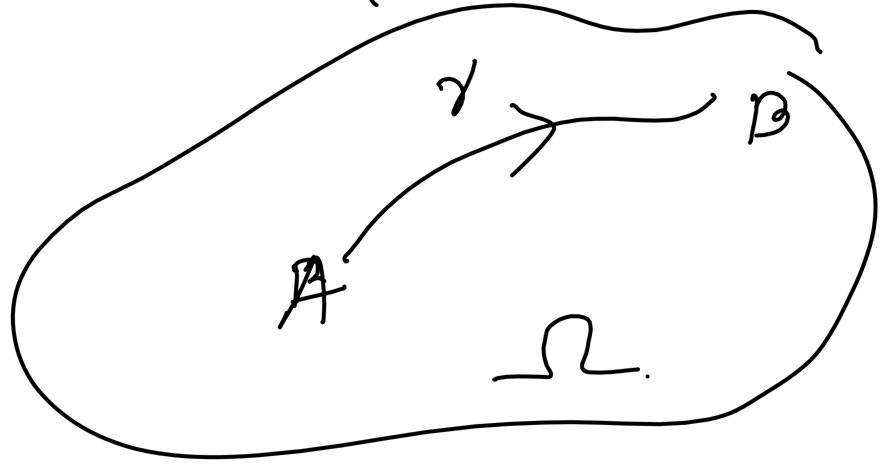
F is called a <u>primitive</u> for f.

- For Ω domain any two primitives differ by a constant.

Prop If f has a primitive F on Ω, then

$$\int_\gamma f(z)\,dz = F(B) - F(A)$$

γ is a curve in Ω.



Proof

$$\int_\gamma f(z)\,dz$$

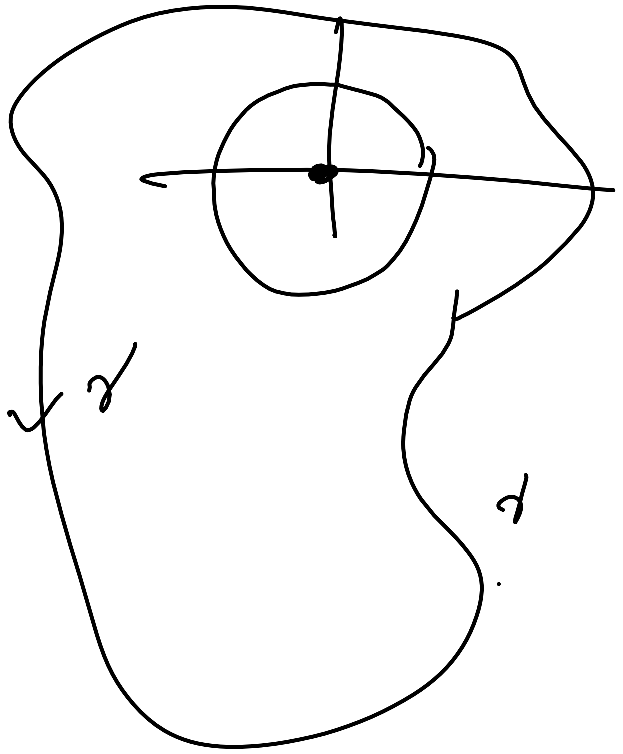
$$= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_a^b F'(\gamma(t))\,\gamma'(t)\,dt$$

$$= \int_a^b \frac{d}{dt}\big(F(\gamma(t))\big)\,dt = F(B) - F(A)$$

Example

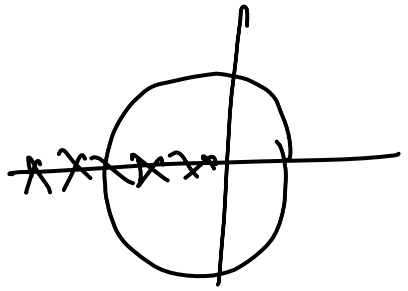
(1) $\int_\gamma \frac{1}{z^2}\, dz = 0$



In general

$\int_\gamma z^n\, dz$ $n \in \mathbb{Z}$
$n \neq -1$

(2) $\int_{|z|=1} \frac{1}{z} = 2\pi i$



$\frac{1}{z}$ does not admit a primitive on $\mathbb{C}^*$

$\int_{\gamma'} \frac{1}{z}\, dz = 0$