MSO 202A: Complex Variables
August-September 2022
Assignment-4

Throughout C_R will denote the circle of radius R around origin, oriented counterclockwise. and $C_1 = C$.

1. **(T)**Show that
$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}dx = \pi/2$$

   Solution: Let $R > 1$. Let $f(z) = \frac{1}{(z^2+1)^2}$ and Γ be the closed curve consisting of semicircular arc of radius R (C_R) on the upper half place union the segment $[-R, R]$ (C_1) on the x-axis.

   Then
$$\int_{C_R+C_1} f = \int_{C_R+C_1} \frac{1/(z+i)^2}{(z-i)^2}dz = 2\pi i[\frac{d}{dz}1/(z+i)^2]_{z=i} = \pi/2.$$

   $\int_{C_1} f = \int_{-R}^{R} \frac{1}{(1+x^2)^2}dx \to \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}dx$ as $R \to \infty$.
   $|\int_{C_R} f| \le \pi R \frac{1}{(R^2-1)^2} \to 0$ as $R \to \infty$.

2. Suppose $f(z)$ is defined by the integral
$$f(z) = \int_{C_3} \frac{2\xi^2 + 7\xi + 1}{\xi - z}d\xi.$$

   Find $f'(1+i)$

   Solution:

   Clearly, comparing with cauchy's integral formula, we conclude that

   $f(z) = 2z^2 + 7z + 1$ for $|z| < 3$. So $f'(z) = 4z + 7$. So $f'(1+i) = -12\pi + 26\pi i$.

3. Compute $\int_{C_4} \frac{z}{z^2+4}dz$ where C_4 is the circe $|z| = 4$ oriented anticlockwise.

   Solution:

   $\int_{C_4} \frac{z}{z^2+4}dz = 1/2[\int_{C_4} \frac{1}{z+2i}dz + \int_{C_4} \frac{1}{z-2i}dz] = 1/2.[2\pi i + 2\pi i] = 2\pi i.$

4. **(T)**Suppose that $f = u + iv$ is an entire function and u is bounded (or v is bounded). Show that f is constant.

   Solution:

   The function $g = e^{f(z)} = e^{u+\iota v}$ is entire and $|g| = e^u$ which is bounded. Thus g is constant. Differentiating $g' = f'e^f = 0$ implies $f' = 0$ on $\mathbb{C}$ implies f is constant.

   (Remark: taking log to g is not correct way to show f is constant on $\mathbb{C}$ since log does not have an analytic branch on $\mathbb{C}$)

5. Using Liouville's theorem, conclude that $\sin z, \cos z$ are not bounded functions.

   Solution:

   If they were bounded they should be constant by Liouville theorem.

6. (**T**)Suppose that $f = u + iv$ is an entire function and $|f(z)| < |z|^n$ for some $n \geq 0$ and for all sufficiently large $|z|$. Show that f is a polynomial.

   Solution: Any holomorphic function f can be written as a power series $f = \sum a_n z^n$ where $a_n = \frac{f^{(n)}(0)}{n!}$.

   By the given condition, there exists integer $n \geq 0$ and a real number $R > 0$ such that $|f(z)| < |z|^n$ for all $|z| \geq R$. Using Cauchy's inequallities for any circle $|z| = R_1 > R$,

   $$|f^{n+1}(0))| \leq \frac{n!||f||_{C_{R_1}}}{R_1^{n+1}} < n!/R_1 \to 0, \quad \text{as } R_1 \to \infty.$$

   Similarly, $f^k(0) = 0$ for all $k > n$. Hence f is a polynomial by the power series expansion about origin.

7. Suppose that $f = u + iv$ is an entire function and u(or v) is a polynomial. Then show that f is a polynomial.

   Solution: Suppose u is a polynomial in x, y. Then for large n, $\frac{\partial}{\partial^n x}u = \frac{\partial}{\partial^n y}u = 0$. Then $f^{(n)} = \frac{\partial}{\partial^n x}u + \iota\frac{\partial}{\partial^n x}u$ is analytic and takes only imaginary values. So $f^{(n)}$ must be constant and so $f^{(n+1)} = 0$. Hence in the ppwoer series expandion of f we must have $a_k = 0$ for $k \geq n+1$. Hence f is a polynomial.

8. Show that if u is a bounded harmonic function on $\mathbb{C}$ then u is constant.

   Solution:

   Since u is harmonic on a simply connected domain $\mathbb{C}$, it admits a harmonic conjugate. Thus $f = u + \iota v$ is analytic on $\mathbb{C}$. By a previous exercise f is constant.

9. (**T**)Let τ be a complex number which is not real. Suppose that f is an entire function such that $f(z+1) = f(z)$ and $f(z+\tau) = f(z)$. Then show that f is a constant. (This exercise says that a doubly periodic entire function is constant.)

   Solution:

   By the periodicity condition the image of f is determined by the image of f on a parallelogram with sides $z = 1$ and $z = \tau$. Paralleogram is compact. So the image of the parallegogram is bounded. Hence f is bounded and so it is constant.

10. Let f be an entire function satisfying $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. Show that f is constant.

    Solution:

    By the given condition f never vanishes. Thus $g = 1/f$ is entire and $|g| \leq 1$. Hence g is constant and so is f.

11. $(\mathbf{T})$Suppose that $f\colon \mathbb{D} \to \mathbb{C}$ is anlytic on unit disc $\mathbb{D} = \{z : \ |z| < 1\}$. Show that $|f'(0)| \le d/2$, where $d = \sup\limits_{z,w\in\mathbb{D}} |f(z) - f(w)|$ is the diameter of the image of f.

Solution:

Let C_r be the circle of radius $r < 1$ with centre at the origin. By Cauchy's integral formula

$$2\pi\iota f'(0) = \int_{C_r} \frac{f(\xi)}{\xi^2} d\xi.$$

Replacing ξ by $-\xi$ this can also be written as

$$2\pi\iota f'(0) = \int_{C_r} \frac{f(-\xi)}{\xi} - d\xi.$$

Adding this two, we get

$$4\pi\iota f'(0) = \int_{C_r} \frac{f(\xi) - f(-\xi)}{\xi^2} d\xi$$

Taking modulus and applying ML estimates, we get $4\pi|f'(0| \le 2\pi r.d.(1/r^2)$. Thus $|f'(0)| \le \frac{d}{2r}|$ for all $0 < r < 1$. Letting $r \to 1-$ we get $|f'(0)| \le d/2$.

12. $(\mathbf{T})$Let Ω be a bounded open subset of $\mathbb{C}$ and $f\colon \Omega \to \Omega$ is a holomorphic function. Prove that if there exists a point $a \in \Omega$ such that $f(a) = a$ and $f'(a) = 1$ then f is linear.

Solution:

The power series of f around $z = a$ is given by $f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$

By the given condition $f(z) = z + a_2(z-a)^2 + O((z-a)^3)$

Since $f : \Omega \to \Omega$, we have $f^2 = f \circ f : \Omega \to \Omega$. Simple calculation gives

$f^2 = z + 2a_2(z-a)^2 + O((z-a)^3)$. Proceeding similarly,

$f^k = z + ka_2(z-a)^2 + O((z-a)^3)$

Therefore, taking C_r a circle of radius r around a inside Ω,

$$ka_2 = (1/2)(f^k)''(a) = \frac{1}{2\pi\iota} \int_{C_r} \frac{f^k(\xi)}{(\xi - a)^2}$$

$k|a_2| \le \frac{1}{2\pi} 2\pi r M(1/r^2) = M/r$ where $M=$ diameter of $\Omega < \infty$.. This is true for all k. Hence $a_2 = 0$.

Thus $f(z) = z + a_3(z-a)^3 + O((z-a)^4)$.Proceeding similarly we have $a_3 = 0$. Continuing the process, we have $f(z) = z$.

13. Show that successive derivatives of an analytic function f at a point z_0 can never satisfy the inequality $|f^{(n)}(z_0)| > n^n n!$ for all $n \in \mathbb{N}$.

Solution:

We know that f admits power series expansion $\sum a_n(z-z_0)^n$ where $a_n = f^{(n)}(z_0)/n!$. The radius of convergence is given by $1/R = \limsup \sum \sqrt[n]{|a_n|} < \infty$. If $|f^{(n)}(z_0)| > n^n n!$ for all n imples that the $\sqrt[n]{|a_n|} > n$ for all n so $\sqrt[n]{|a_n|}$ diverge to infinity. This is a contradiction.

14. Let f be anlytic on a region Ω and let C be a circle with interior containd in Ω. For any $a \in \Omega$ not on C show that

$$\int_C \frac{f'(\xi)}{(\xi - a)} d\xi = \int_C \frac{f(\xi)}{(\xi - a)^2} d\xi$$

Solution:

By Cauchy's integral formula for f' we get $\int_C \frac{f'(\xi)}{(\xi-a)} d\xi = 2\pi\iota f'(a)$.

By Cauchy's integral formula for f we get $\int_C \frac{f(\xi)}{(\xi-a)^2} = 2\pi\iota f'(a)$.

15. (a) If $f(z)$ is a holomorphic inside and on a circle C containng a prove that

$$f(a)^n = \frac{1}{2\pi i} \int_C \frac{f(z)^n}{(z-a)} dz.$$

Solution:

Apply Cauchy's integral formula to the anytic function $(f(z))^n = f(z).f(z) \cdots f(z)$.

(b) Use (a) to show that $|f(a)|^n \leq LM^n/(2\pi D)$ where D is the distance of a from C, L is the length of C and M is the maximum value of $|f(z)|$ on C.

Solution:

Apply ML-estimates to get this ineuqality.

(c) Use (b) to show that $|f(a)| \leq M$. In other words, the maximum value of $|f(z)|$ is obtained on the boundary. This result is known as Maximum Modulus Principle.

Solution:

$|f(z)| \leq M(k)^{1/n}$. Taking limit as $n \to \infty$, we get $|f(z)| \leq M$.

(d) The maximum modulus value of $f(z) = 1/z$ on unit circle is 1, yet $|f(1/2)| = 2$. Expalin why this does not contradict (c).

Solution:

Since here f is not holomorphic inside the unit circle.

16. This exercise gives a generalization of Goursat's and Cauchy's theorem.

Let T be a triangle whose interior is contained in an open set Ω of $\mathbb{C}$. Suppose that $f : \Omega \to \mathbb{C}$ is a continuous function which is holomorphic on Ω in except possibly at a point z_0. Prove that

$$\int_T f(z)dz = 0.$$

17. Let $\mathbb{D}$ be an open disc and $f\colon \mathbb{D} \to \mathbb{C}$ be a continuous function which is holomorphic on $\mathbb{D} \setminus \{z_0\}$ for some fixed $z_0 \in \mathbb{D}$. Then prove that f has a primitive on $\mathbb{D}$.

    (Remark: Hence we conclude that: Let $f : \Omega \to \mathbb{C}$ is a continuous function on an open set Ω and anlytic on $\Omega \setminus \{z_0\}$ where $z_0 \in \Omega$. Then show that f is analytic on Ω.)