MSO 202A : Complex Variables
Final Exam, 19th September 2022

Total Marks: 70

Time: 8 am - 10 am

- Answer all questions.
- Write each step clearly.

1. (a) Let $f, g : \overline{\mathbb{D}} \to \mathbb{C}^*$ be two analytic functions on the closed unit disc $\overline{\mathbb{D}} = \{z|\ |z| \leq 1\}$ such that $|f(z)| = |g(z)|$ for all $|z| = 1$. Then show that, there exists a $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} g(z)$ for all $z \in \overline{\mathbb{D}}$.

   [8]

   Solution

   f/g holomorphic on $\overline{\mathbb{D}}$ with $|f/g| = 1$ on the boundary circle. So by Maximum modulus principle $|f/g| \leq 1$ on $\overline{\mathbb{D}}$.

   [2]

   Similarly, $|g/f| \leq 1$ on $\overline{\mathbb{D}}$.

   [2]

   Hence $|f/g| = 1$ on $\overline{\mathbb{D}}$

   [1]

   Hence by Cauchy Riemann equation (or open mapping theorem) f/g is constant on $\overline{\mathbb{D}}$.

   [2]

   This constant is of modulus 1. So there exist θ such that $f = e^{i\theta} g$ on $\overline{\mathbb{D}}$

   [1]

(b) Let $f(z) = e^{\cos z} z^2$ and A be the closed disc $|z-5| \le 2$. Show that $\max_{z \in A} |f(z)|$ and $\min_{z\in A} |f(z)|$ are attained on $|z-5| = 2$.

[2+2]

Solution

Since f is holomorphic on the closed disc A, by Maximum modulus principle, $\max_{z\in A} |f(z)|$ is attained on $|z-5| = 2$.

[2]

Since $f \ne 0$ on the closed disc A, so $1/f$ is holomorphic on A. Hence applying Maximum modulus to $1/f$, the $\max_{z\in A} |1/f(z)| = \min_{z\in A} |f(z)|$ is attained on $|z-5| = 2$.

[2]

Remark: If someone does not write $f(z) \ne 0$ but apply Minimum modulus principle, then 1 mark only for the second part.

(c) Compute the integral

$$\int_0^{2\pi} \frac{dt}{\cos(t) - 2}.$$

[6]

Solution

Let C be the unit circle $z(t) = e^{it}$ $0 \le t \le 2\pi$. Replacing $\cos t = (z + \frac{1}{z})/2$ and $dt = -\frac{idz}{z}$

[1]

Then the given integral is $-2i \int_C \frac{dz}{z^2 - 4z + 1}$.

[1]

The poles are $2 \pm \sqrt{3}$.

[1]

The pole $z_0 = 2 - \sqrt{3}$ lies inside C.

[1]

Reside at z_0 = - $\sqrt{3}/6$.

[1]

So the given integral = $(-2i)(2\pi i)$ (Reside at z_0) = - $2\pi/\sqrt{3}$.

[1]

2. (a) Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a non-constant harmonic function. Show that u has at least one point (x_0, y_0) such that $u(x_0, y_0) = 0$.

[7]

Solution

Since $\mathbb{R}^2$ is convex/ simply connected the harmonic function u admits a harmonic conjugate v. Thus $f = u + iv$ is holomorphic on $\mathbb{C}$.

[1]

If u is never zero, then $u(x, y) > 0$ or $u(x, y) < 0$ for all (x, y).

[1]

Assume $u(x, y) > 0$ for all (x, y).

[1]

Consider the function $g = e^{-f} = e^{-u-iv}$.

[1]

$|g| = e^{-u} < 1$.

[1]

Thus g is entire bounded function, and so constant.

[1]

Differentiating g, we get $f' = 0$ and so f is constant which implies u constant.

[1]

(b) Let $f(z)$ be an entire function such that $|f(z)| \geq 1$ for all z. If $f(0) = 1$, then find the value of $f(1)$.

[5]

Solution

Note that $f(z) \neq 0$ for all z.

[1]

So $1/f$ is holomorphic on $\mathbb{C}$.

[1]

$|1/f(z)| < 1$ so $1/f$ is bounded and so constant by Liouville.

[1]

$f(1) = f(0) = 1$.

[2]

(c) Find the order of zero of the function $4\cos(z^4) + 2z^8 - 4$ at $z = 0$.

[3]

Solution

Expanding in power series about 0

$$4\cos(z^4) + 2z^8 - 4 = 4(1 - z^8/2 + z^{16}/6 - \cdots) + 2z^8 - 4 = 2z^{16}/3 - \cdots$$

[2]

So $z = 0$ is a zero of order 16.

[1]

(d) Can a power series of the form $\sum a_n(z-2)^n$ converge at $z = 6$ and diverge at $z = 2i$? Justify your answer. [3]

Solution

We know that if a power series $\sum a_n(z-a)^n$ converge at a point $z = z_0$, then it converge for all points in $|z - a| < |z_0 - a|$.

[2]

Here $|z_0 - a| = |6 - 2| = 4$. But $|2i - 2| = 2\sqrt{2} < 4$. So NOT possible.

[1]

3. (a) Evaluate

$$\int_0^\infty \frac{dx}{1+x^7}.$$

[10]

Solution

Let C_R be the path given as the sum of the paths $\gamma:\ Re^{it},\quad 0 \le t \le 2\pi/7$ and the intervals $[Re^{2\pi i/7}, 0]$ and $[0, R]$.

[2]

Let $f(z) = \frac{1}{1+z^7}$ and note that f has exactly one simple pole $z_0 = e^{\pi i/7}$ lying inside the region bounded by C_R.

[1]

$Res(f;\ z_0) = \lim_{z \to z_0}(z - z_0)f(z) = \lim_{z \to z_0} \frac{z-z_0}{z^7 - z_0^7} = \frac{1}{7z_0^6}$.

[1]

$|\int_\gamma f| \le \frac{2\pi R}{7(R^7-1)} \to 0$ as $R \to \infty$.

[1]

Integral over the interval $[0, R]$ is $\int\limits_0^R \frac{dx}{1+x^7}$

[1]

Integral over the interval $[Re^{2\pi i/7}, 0]$ is $-\int\limits_0^R \frac{z_0^2 dr}{1+(z_0^2 r)^7}$

[2]

Applying Cauchy's residue thereom

$$\int_{C_R} f(z)dz = \int_\gamma f(z)dz + (1 - z_0^2) \int\limits_0^R \frac{dx}{1+x^7} = 2\pi i Res(f;\ z_0) = 2\pi i \frac{1}{7z_0^6}$$

[1]

Letting $R \to \infty$, we get

$\int\limits_0^\infty \frac{dx}{1+x^7} = \frac{2\pi i}{7z_0^6(1-z_0^2)} = \frac{\pi}{7\sin(\pi/7)}$. [1]

Remark: Note that $0 = \int_{-\infty}^{\infty} \frac{dx}{1+x^7} \neq 2\int_0^{\infty} \frac{dx}{1+x^7}$. So trying to solve using the semicircle on the upper half plane is NOT going to work.

If you have tried with icircular ark of angle 0 to $\pi/4$, then the integral on the $\pi/4$ line is not going to give the required integral. For this approcah with righ calculation, partial marks has been awarded.

(b) Calculate the residue of the function $z^n e^{10/z}$ at ∞, $\quad n \in \mathbb{N}$.

[2]

Solution

Residue of the function $f(z) = z^n e^{10/z}$ at ∞ = - coefficients of z in Laurent series expansion of

$$f(1/z) = \frac{e^{10z}}{z^n} = \frac{1}{z^n}(1 + 10z + \cdots + \frac{(10z)^{n+1}}{(n+1)!} + \cdots).$$

[1]

Residue of the function $f(z)$ at ∞ is $z^n e^{10/z}$ is $-\frac{10^{n+1}}{(n+1)!}$.

[1]

(c) Let f be an entire function satisfying $|f(z)| < |z|^n$ for all $|z| > M$. Show that f is a polynomial.

[6]

Solution

Since f is an Entire function, it can written as power series on whole of complex plane $f(z) = \sum_{k \geq 0} a_k z^k$

[1]

$a_k = f^{(k)}(0)/k!$

[1]

Now by Cauchy's integral formula on the circle $C_R : |z| = R$ of radius R

$|f^{(k)}(0)| \leq \frac{k!}{R^k}||f||_{C_R}$ where $||f||_{C_R} = \sup_{z \in C_R} |f(z)|$.

[1]

Therefroe for $k \geq n+1$

$$|f^{(k)}(0)| \leq \frac{k!}{R^k}||f||_{C_R} \leq \frac{k!}{R^k}R^n = \frac{k!}{R^{k-n}} \to 0 \text{ as } R \to \infty$$

[2]

Hence $a_k = 0$ for all $k > n$. Hence f is a polynomial.

[1]

4. (a) Evaluate the integral $\int_C |z|^2 dz$ in the following two cases:

C: the line segment with initial point -1 and final point i.

C: the arc of the unit circle in $Im(z) \geq 0$ with initial point -1 and final point i.

[2+2]

Solution

C_1: the line segment with initial point -1 and final point i.

$\gamma(t) = t - 1 + ti$ where $0 \leq t \leq 1$.

$\int_{C_1} |z|^2 dz = \int_0^1 [t^2 + (t-1)^2](1+i)dt = (1+i) \int_0^1 2t^2 - 2t + 1 = 2/3(1+i)$.

C_2 : the arc of the unit circle in $Im(z) \geq 0$ with initial point -1 and final point i.

$\gamma(t) = e^{it}$ with t starting at π and ending at $\pi/2$.

$\int_{C_2} |z|^2 dz = \int_\pi^{\pi/2} 1.e^{it} i dt = e^{it}|_\pi^{\pi/2} = i + 1$.

(b) Suppose $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that $f(z) \neq 0$ for all z. Show that there exist a holomorphic function $g : \mathbb{D} \to \mathbb{C}$ such that $e^{g(z)} = f(z)$. Hence deduce that there exist a holomorphic function $h : \mathbb{D} \to \mathbb{C}$ such that $h^2(z) = f(z)$.

[6+2]

Solution

f'/f is holomorphic on $\mathbb{C}$

[2]

By Cauchy's Theorem, it has a primitive, say g. So $g' = f'/f$.

[1]

$(fe^{-g})' = f'e^{-g} - g'fe^{-g} = 0$.

[2]

So $fe^{-g} = .constant = e^{z_0}$. Thus $f = e^{g+z_0}$.

[1]

Take $h = e^{\frac{g}{2}}$. Then $h^2 = f$.

[2]

(c) Determine the domain of analyticity of the function $f(z) = \log_{\pi/2}(1+z)$. Expand it in power series about 0.

[2+2]

Solution

We know that $\log_{\pi/2}(z)$ is analytic on $\mathbb{C} - \{0 + iy \mid \; y \geq 0\}$ [1]

So the function $f(z) = \log_{\pi/2}(1+z)$ is analytic on the domain

$\mathbb{C} - \{-1 + iy \mid \; y \geq 0\}$

[1]

$f(0) = \log \log_{\pi/2}(1) = 2\pi i$

$f'(0) = \frac{1}{1+z}|_{z=0} = 1$

$f''(0) = \frac{-1}{(1+z)^2}|_{z=0} = -1$

$f'''(0) = 2!$

$f^{(n)}(0) = (-1)^n(n-1)!$ [1]

Thus $f(z) = \sum a_n z^n$ wheer $a_n = \frac{f^{(n)}(0)}{n!}$

$\log_{\pi/2}(1+z) = 2\pi i + z - z^2/2 + z^3/3 - z^4/4 + \cdots$ [1]