Complex integration

Recall

$$[a, b] \xrightarrow[\text{smooth}]{\gamma} \Omega \xrightarrow[\text{continuous}]{f} \mathbb{C}$$

$$\int_\gamma f(z)\,dz := \int_a f(\gamma(t))\,\gamma'(t)\,dt$$

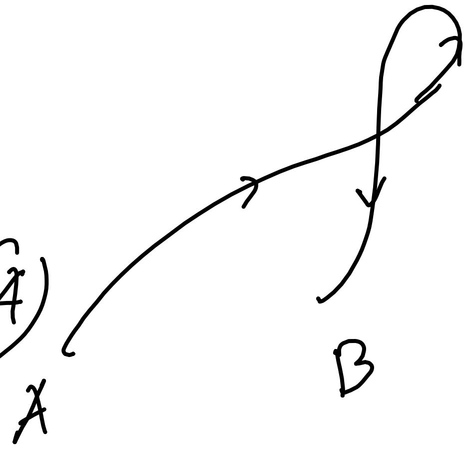
- $\left|\int_\gamma f(z)\,dz\right| \le ML$

  L = length of the curve

  $M = \sup_{z \in \gamma} \{|f(z)|\}$

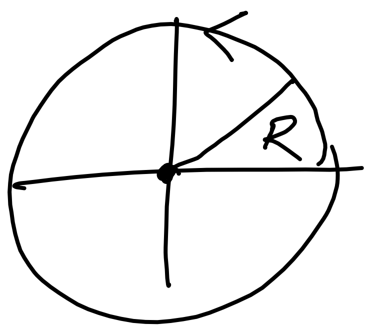
- If $f = F'$ where F is analytic, then

$$\int_\gamma f(z)\,dz = F(B) - F(A)$$



F - primitive of f.

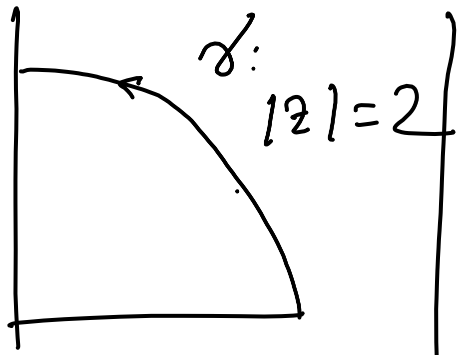
Example. $\int_{C_R} \frac{1}{z}\,dz = 2\pi i$



- $\int_{C_R} \frac{1}{z^2}\,dz = 0$

Example

(1) $f(z) = \dfrac{z+4}{z^3-1}$



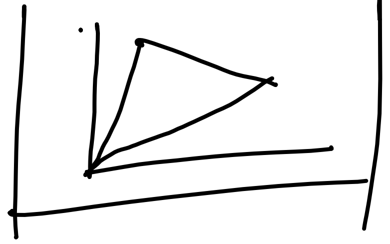
$$\left|\int_\gamma f(z)\,dz\right| \le ML.$$

$L = \pi$

$|z+4| \le |z| + |4| = 6$ on γ

$|z^3-1| \ge |z|^3 - 1$
$= 8 - 1 = 7$
on γ

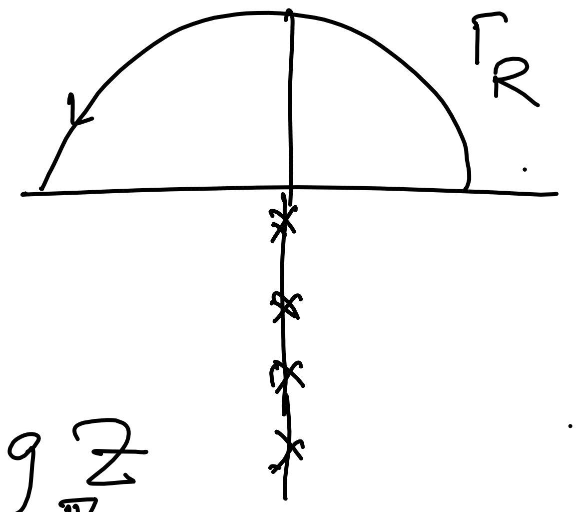
$|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$



$|f(z)| \le \dfrac{6}{7}$ on γ.

So $\left|\int_\gamma f(z)\,dz\right| \le \dfrac{6}{7}\cdot\pi$.

② $$\int_{\Gamma_R} \frac{\sqrt{z}}{z^2+1}\, dz$$



$$\sqrt{z} = e^{\frac{1}{2}\log_{-\pi/2} z}$$

$\log_{3\pi/2}$	$\log_{-\pi/2}$
$\frac{3\pi}{2} < \theta < \frac{3\pi}{2} + 2\pi$	$-\frac{\pi}{2} < \theta < -\frac{\pi}{2} + 2\pi$

$L = \pi R$

$|\sqrt{z}| = \sqrt{R}$ on Γ_R

$$|z^2+1| \geq \big||z|^2 - 1\big| = |R^2-1|$$

$$\left| \int_{\Gamma_R} \frac{\sqrt{z}}{z^2+1}\, dz \right| \leq \pi R \cdot \frac{\sqrt{R}}{|R^2-1|}$$

③ $$\int_\gamma (12z^2 - 4iz)\, dz$$

$\gamma: y = x^3 - 3x^2 + 4x - 1$ from $(1,1)$ to $(2,3)$

$$= \Big[4z^3 - 2iz^2\Big]_{z=1+i}^{2+3i}$$

$$= -156 + 38i$$

$$\int_{|z|=1} \frac{1}{z}\, dz = 2\pi i \quad (*)$$

$$u(x,y) = \frac{1}{2}\log(x^2+y^2).$$

This is harmonic function on $\mathbb{C}^*$

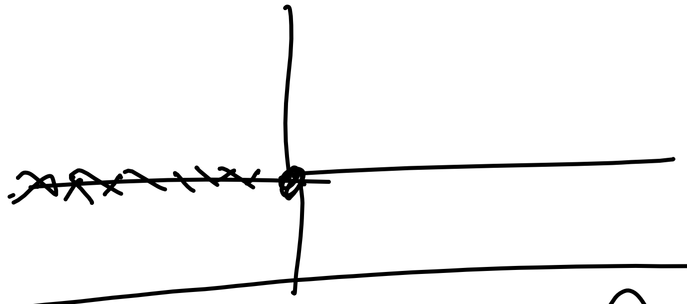
But it does not admit a harmonic conjugate on $\mathbb{C}^*$

If $\exists$ ~~a harmo~~ v s.t. $f = u + iv$ holomorphic on $\mathbb{C}^*$

$$f' = u_x - i u_y = \frac{1}{z}$$

This is a contradiction to (*).

- $\log z$



Theorem (Cauchy's Theorem).

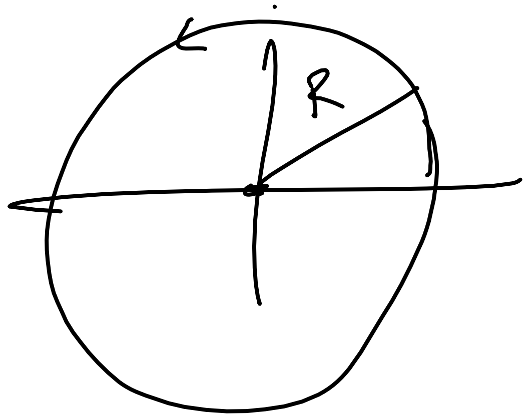
If f is holomorphic on a convex domain Ω, then f admits a primitive on Ω.

Example: $\Omega = \mathbb{C}$, $\mathbb{D}$,

Hence $\int_\gamma f(z)\, dz = 0$ for any closed curve γ in Ω.

Example $\displaystyle\int_{C_R} \frac{\sin(z^2+z^3+e^z)\cos(2z)}{dz}$

$= 0.$



Corollary A harmonic function on a convex domain admits a Harmonic conjugate.

Proof Ω - convex $u : \Omega \longrightarrow \mathbb{R}$

Harmonic. (u has second order continuous partial derivative)

$u_{xx} + u_{yy} = 0$

To find v s.t. $f = u + iv \in \mathcal{H}(\Omega)$

$f' = \boxed{u_x - iu_y}$

Let $g = u_x - iu_y = U + iV$

$U = u_x \qquad V = -u_y.$

$U_x = u_{xx} = V_y = -u_{yy}$

$U_y = u_{xy} \qquad V_x = -u_{yx}$

$U_y = -V_x.$

Also U_x, U_y, V_x, V_y are continuous.

$\Rightarrow g \in \mathcal{H}(\Omega).$

Since Ω is convex, $\exists$ $f \in \mathcal{H}(\Omega)$ s.t

$$\boxed{f' = g} = u_x - i u_y.$$

$f = A + iB$

$f' = A_x + iB_x = A_x - iA_y$

$\underbrace{u_x = A_x \quad u_y = A_y.}$

$\Downarrow$

$u = A + c.$

$f = u + c' + iB$

$= u + i\underline{B'} \in \mathcal{H}(\Omega)$

$h = f - c' = u + iB$
$\in \mathcal{H}(\Omega).$

$\blacksquare$

Corolly

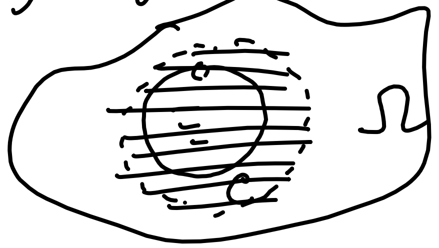
(1) $f \in \mathcal{H}(\mathbb{D})$ $\mathbb{D} = \{z / |z| < 1\}$

$\Rightarrow \int_\gamma f(z)\,dz = 0$

where γ is a closed curve in $\mathbb{D}$

(2) f is holomorphic on an open set containing a circle C & its inside

$\Rightarrow \int_C f(z)\,dz = 0$



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