Quiz: MSO202M (2023-2024 I)

Date: 06 September 2023 Time: 07:00 pm-08:00 pm Maximum marks: 30

Name: **Roll No.**

Instructions: (Read carefully)

- Please enter your NAME and ROLL NUMBER in the space provided on EACH page.
- Only those booklets with name and roll number on every page will be graded. All other booklets will NOT be graded.
- This answer booklet has 6 pages. Check to see if the print is either faulty or missing on any of the pages. In such a case, ask for a replacement immediately.
- Please answer each question ONLY in the space provided. Answers written outside the space provided for it WILL NOT be considered for grading. So remember to use space judiciously.
- For rough work, separate sheets will be provided to you. Write your name and roll number on rough sheets as well. However, they WILL NOT be collected back along with the answer booklet.
- No calculators, mobile phones, smart watches or other electronic gadgets are permitted in the exam hall.
- Notations: All notations used are as discussed in class.
- All questions are compulsory.
- Do NOT remove any of the sheets in this booket.

Q 1. Let F be a complex-valued analytic function defined on an open unit disc centered at origin of radius 1, i.e., $D = \{z \in \mathbb{C} : |z| < 1\}$ such that F satisfies $F\left(1 - \frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^2$ for all $n \in \mathbb{N}$. Does F necessarily be $F(z) = z^2$? Justify your answer. **6 marks**

Solution:

- No, not necessarily. **2 mark**
- Take
$$F(z) = z^2 + \sin\left(\frac{\pi}{1-z}\right) \qquad \text{or other suitable function} \quad \textbf{2 marks}$$
- For justification that $F\left(1 - \frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^2$ for all $n \in \mathbb{N}$ but $F(z) \neq z^2$. **2 marks**

Name:.. **Roll Number: ....**

Q 2. Let $P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$, $a_k \in \mathbb{C}$ be a polynomial of degree 3 in $z \in \mathbb{C}$. Evaluate

$$\int_{|z|=2} \frac{P(z)}{(z-1)^j}\, dz, \qquad \text{for all} \quad j \in \mathbb{Z}.$$

6 marks

Solution: Let

$$I = \int_{|z|=2} \frac{P(z)}{(z-1)^j}\, dz.$$

If $j \leq 0$, by Cauchy's theorem $I = 0$ as $P(z)(z-1)^{-j}$ is analytic. **1 mark**

If $j \geq 1$, applying Cauchy's integral formula, we have

$$I = \frac{2\pi i}{(j-1)!} P^{(j-1)}(1). \quad \textbf{2mark}$$

For $j = 1$,

$$I = 2\pi i\ (a_0 + a_1 + a_2 + a_3).$$

For $j = 2$,

$$I = 2\pi i\ (a_1 + 2a_2 + 3a_3).$$

For $j = 3$,

$$I = 2\pi i\ (a_2 + 3a_3).$$

For $j = 4$,

$$I = 2\pi i\ a_3.$$

2 mark for writing the value. For $j \geq 5$,

$$I = 0 \quad \textbf{1mark}.$$

Name:...   **Roll Number:** ....

Q 3 (a) If $f = u + iv$ is entire such that $au + bv \geq c$ for some real numbers $a \neq 0, b \neq 0$ and c, must f be a constant? Give details to support your answer.

(b) If two entire functions agree at infinitely many points, must they be equal ? Give details to support your answer.

4+2 marks

Solution 3(a):

Consider $g(z) = (-a + ib)\, f(z) = -au - bv + i(-av + bu)$ is an entire function. **2 mark**

Since $h(z) = e^{g(z)}$ is also an entire function with $\mathrm{Re}(g) = -au - bv \leq -c$. **1 mark**

$|h(z)| = e^{\mathrm{Re}(g)} \leq e^{-c}$. Hence by using Liouville's theorem $f(z)$ must be constant. **1 mark**

Solution 3(b):

Two such functions may not be equal. Consider $f(z) = \sin z$, and $g(z) = \sin 2z$ (or any other suitable functions). Both have the same value on the set $\{z = n\pi : n \in \mathbb{Z}\}$ but $f(z)$ and $g(z)$ are not the same functions. **2 mark**

Name:... **Roll Number:** ....

Q 4 Discuss all the finite singularities of the function

$$f(z) = \frac{z^3\,(z-1)^2\,(z-2)\,e^{1/z}}{\sin^2(\pi z)}, \qquad z \in \mathbb{C}.$$ **6 mark**

Also, explain in detail which of them are (i) Removable singularities. (ii) Poles and their order (iii) Essential singularities.

Solution: The set of all finite singularities of $f(z)$ will be the whole set of integers. **1 mark**

Removable singularity: Since the limit of the function $f(z)$ at $z = 1$ is non-zero and finite. Hence $z = 1$ is a Removable singularity. **1 mark**

Pole: Since, at every $z_0 \in \mathbb{Z} \setminus \{0, 1\}$, the limit of the function $f(z)$ at $z = z_0$ is infinite. Hence, all the elements of the set $\mathbb{Z} \setminus \{0, 1\}$ are poles. **2 mark**

Order of Poles: $z_0 = 2$ is a simple pole and all the other poles are of order 2. **1 mark**

Essential Singularity: Since the limit of the function $f(z)$ at $z = 0$ does not exist finitely or infinitely. Hence $z = 0$ is an essential singularity. **1 mark**

Name:... **Roll Number:** ....

Q 5. For $0 < |a| < |b|$, expand $f(z) = \frac{1}{(z-a)(z-b)}$ in Laurent series valid for (i) $|z| > |b|$ and (ii) $0 < |z-a| < |a-b|$. **3 + 3 marks**

Solution: (i) $|z| > |b|$,

$$\begin{aligned} f(z) &= \frac{1}{z^2\left(1-\frac{a}{z}\right)\left(1-\frac{b}{z}\right)} && \textbf{1 mark} \\ &= \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n \sum_{m=0}^{\infty}\left(\frac{b}{z}\right)^m, && \textbf{1 mark} \\ &= \frac{1}{a-b}\sum_{n=1}^{\infty}\frac{a^n-b^n}{z^{n+1}}. && \textbf{1 mark} \end{aligned}$$

(ii) $0 < |z-a| < |a-b|$

$$\begin{aligned} f(z) &= \frac{1}{(a-b)(z-a)\left(1+\frac{z-a}{a-b}\right)} && \textbf{2 marks} \\ &= \frac{1}{(a-b)(z-a)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-a}{a-b}\right)^n && \textbf{1 mark} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(z-a)^{n-1}}{(a-b)^{n+1}}. \end{aligned}$$