Taylor's Theorem

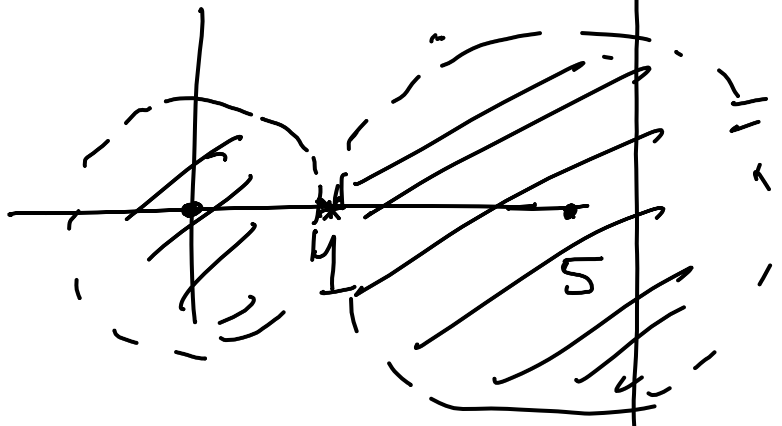
(Power series representation of a holomorphic function)

Example:

① $e^z = 1 + z + \frac{z^2}{\lfloor 2} + \cdots \quad \forall z \in \mathbb{C}$

② $\frac{1}{1-z} = 1 + z + z^2 + \cdots \quad |z| < 1$

Around the point $z_0 = 5$



$$\frac{1}{1-z}$$

$$= \frac{1}{-4 - (z-5)}$$

$$= \frac{-1}{4\left(1 + \frac{z-5}{4}\right)}$$

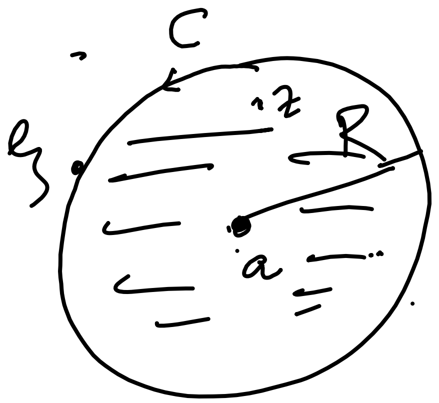
$$= \frac{-1}{4}\left(\sum_{n\geq 0}\left(\frac{z-5}{4}\right)^n\right)$$

$$= -\sum_{n\geq 0}\frac{(z-5)^n}{4^{n+1}}$$

$\left|\frac{z-5}{4}\right| < 1$

$|z-5| < 4$

Taylor's Th



The functn f is holomorphic on a the circle C & its inside. where C is a circle with centre at 'a'. Then.

$$f(z) = \sum a_n (z-a)^n \quad \forall\ |z-a| < R$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi-a)^{n+1}}\, d\xi$$

Proof.

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z}\, d\xi$$

By Cauchy's Integral formula

$$\frac{1}{\xi - z} = \frac{1}{(\xi-a) - (z-a)}$$

$$= \frac{1}{(\xi-a)\left(1 - \frac{z-a}{\xi-a}\right)}$$

$$= \frac{1}{(\xi-a)} \sum_{n \geq 0} \left(\frac{z-a}{\xi-a}\right)^n \quad \checkmark \left(\because \left|\frac{z-a}{\xi-a}\right| < 1\right)$$

$$f(z) = \frac{1}{2\pi i} \sum_{n \geq 0} \left(\int_C \frac{f(\xi)}{(\xi-a)^{n+1}}\right) (z-a)^n \quad \blacksquare$$

Corollary Thm

① $f \in \mathcal{H}(\Omega) \Rightarrow f' \in \mathcal{H}(\Omega)$

$\Rightarrow f'' \; \mathcal{H}(\Omega)$

$\Rightarrow \ldots$

$$f = u + iv$$
$$f' = u_x + i v_x$$
$$f'' = u_{xx} + i v_{xx}$$
$$f''' = u_{xxx} + i v_{xxx}$$

② $f(z) = \sum a_n (z - a)^n$

$$a_n = \frac{f^{(n)}(a)}{\lfloor n}$$

Comparing with the theorem

$$\frac{f^{(n)}(a)}{\lfloor n} = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - a)^{n+1}}\, d\xi.$$

$$\boxed{f^{(n)}(a) = \frac{\lfloor n}{2\pi i} \int_C \frac{f(\xi)}{(\xi - a)^{n+1}}\, d\xi}$$

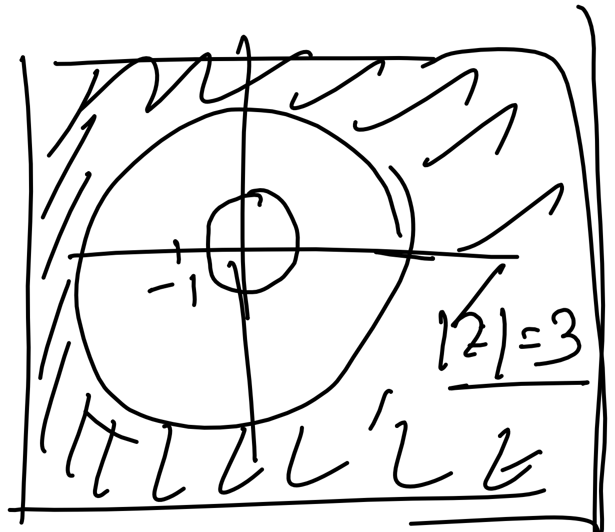
$$f^{(n)}(z) = \frac{\lfloor n}{2\pi i} \int_{C'} \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi$$

$$= \frac{\lfloor n}{2\pi i} \int_C \boxed{\frac{f(\xi)}{(\xi - z)^{n+1}}}\, d\xi \quad \checkmark$$

Example

$$\int_{|z|=3} \frac{e^{2z}}{(z+1)^4}\, dz$$

anticlockwise

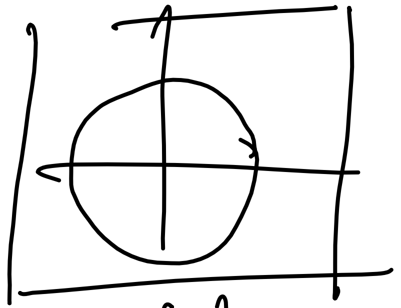


$$= \frac{2\pi i}{\lfloor 3} \frac{d^3}{dz^3}\left(e^{2z}\right)\Big|_{z=-1}$$

$$= \frac{8}{3}\pi i e^{-2}.$$

$$f^{(n)}(z) = \frac{\lfloor n}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi$$

(2) $$\int_{|z|=1} |z|^2\, dz$$



$z = e^{it}$

$$= \int_{t=0}^{2\pi} 1\, e^{it}\, i\, dt$$

$$= \left[i\frac{e^{it}}{i} \right]_{t=0}^{2\pi}$$

$$= \left[e^{2\pi i} - e^{0}\right]$$

$$= (1-1) = 0.$$

Liouville's Th

An Entire bounded function is constant. (entire $= \mathcal{H}(\mathbb{C})$)

Example

$\sin z : \mathbb{C} \longrightarrow \mathbb{C}$

$e^{z} : \mathbb{C} \longrightarrow \mathbb{C}^{*}$

Dense = "it is ~~ext~~ everywhere".

$A \subseteq \mathbb{R}^n$ is called dense in $\mathbb{R}^n$ if any open ball of $\mathbb{R}^n$ contains at ~~to~~ least one point of A.

Exmpls ① $\mathbb{Q} \subseteq \mathbb{R}$ dense

② $\mathbb{Q} \times \mathbb{Q} \subseteq \mathbb{C}$ dense

③ $\{\underset{\in \mathbb{R}}{\sin n} \mid n \in \mathbb{Z}\} \subseteq [-1\ \ 1]$ — dense in $[-1\ \ 1]$

Corollary Image of an non-constant entire function is a dense subset of $\mathbb{C}$.

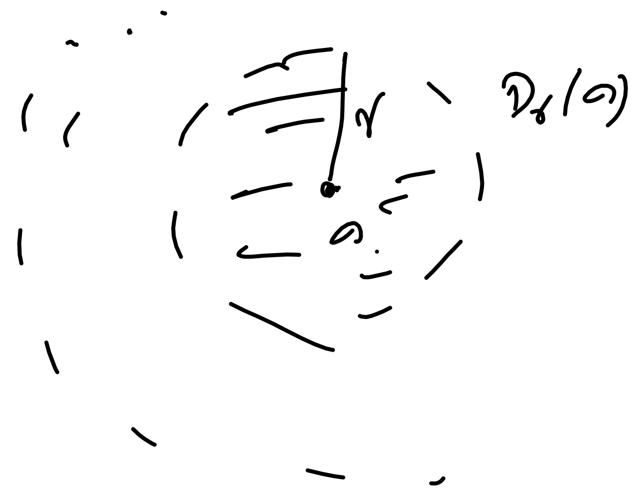
Prf $f : \mathbb{C} \longrightarrow \mathbb{C}$

If $Im(f)$ is not dense in $\mathbb{C}$,

then, $\exists$ disc $D_r(a) : |z-a| < r$ s.t

$Im(f) \cap D_r(a) = \varphi$.

$$|f(z) - a| \geq r \quad \forall z \in \mathbb{C}$$



$$g(z) = f(z) - a.$$

$$g \in H(\mathbb{C}) + g(z) \neq 0 \quad \forall z.$$

$$\frac{1}{g} \in H(\mathbb{C})$$

Note $\quad \dfrac{1}{|g|} = \dfrac{1}{|f(z)-a|} \leq \dfrac{1}{r} = M.$

$\Rightarrow \dfrac{1}{g}$ bonded + entire

$\Rightarrow \dfrac{1}{g}$ const $\Rightarrow f$ const. ∎

FACT (Picard Theorem)

Image of an entire function can at most miss one point.

Fundamental Th of algebra

$a_n \neq 0$

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

$$= z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right)$$

$\downarrow$

$a_n \neq 0$

$$\boxed{\lim_{z \to \infty} p(z) = \infty}$$

$$\left(z \cdot \left(\frac{1}{z^2}\right)\right)$$