MSO202- INTRODUCTION TO COMPLEX ANALYSIS

1. Assignment 5

(1) Let $f(z)$ be an entire function such that $f(\mathbb{R}) \subset \mathbb{R}$. Show that $g(z) = \overline{f(\overline{z})}$ is also entire and $\overline{f(z)} = f(\overline{z})$.

(2) Suppose that Ω is simply connected domain with $1 \in \Omega$, and $0 \notin \Omega$. Then there is a branch of the logarithm $F(z) = \log_\Omega z$ so that
  (i) F is holomorphic in Ω.
  (ii) $e^{F(z)} = z$ for all in Ω.
  (iii) $F(r) = \log r$ whenever r is a real number and near 1.
  (**Hint:** Page 98, Theorem 6.1 of Stein and Shakarchi)

(3) Calculate

$$(i) \int_{C_2} \frac{z^{2009}}{z^{2010} + z^2 + 1} dz \quad (i) \int_{C_2} \frac{z^{2023}}{z^{2023} + z^{2022} + z^{2000} + 1} dz.$$

(4) Let f be a complex valued function in the unit disk $D := \{z : |z| < 1\}$ such that f^2 and f^3 are both analytic. Prove that f is analytic.

(5) Let $f, g : \Omega \to \mathbb{C}$ be analytic functions. If $fg \equiv 0$, then either $f \equiv 0$ or $g \equiv 0$. Also, if $\overline{f}g$ is analytic. Show that f is constant function or g is identically zero.

(6) Fibonacci number are given by the recurrence relation $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Let F_n be the n-th Fibonacci number. Show that (informally)

$$F(z) := \sum_{n=0}^{\infty} F_n z^n = \frac{1}{1 - z - z^2}.$$

Find radius of convergence of $F(z)$.

(7) Let $f(z)$ be an entire function such that $|f(z)| \leqslant C|z|^{1/2}$ for all z, then show that f is a constant function.

(8) Expand each of the following functions in Laurent series in the neighbourhood of the indicated points z_0 and, in each case, determine the largest domain where the resulting Laurent series converges:

$$(i)$$

(9) Find the Laurent series of the function

$$f(z) = \frac{z+4}{z^2(z^2 + 3z + 2)} \qquad in$$

$$(i)\ 0 < |z| < 1. \ (ii)\ \ 1 < |z| < 2\ (iii)\ \ 0 < |z+1| < 1\ (iv)\ \ |z| > 2.$$

(10) **T** Find the Laurent series of the function

$$f(z) = \frac{z^2}{z^2 - 3z + 2} \qquad in$$

$$(i)\ \ 1 < |z| < 2\ (ii)\ \ 1 < |z-3| < 2$$

(11) Find the order of the zero at $z = 0$ for the following functions:
$(i) z^3 \left(\exp(\exp z^3) - 1\right)$ (ii) $6\sin(z^3) + z^3(z^6 - 6)$ (iii) $\exp(\sin z) - \exp(\tan z)$.

(12) Find all the holomorphic functions $f : B_1(0) \to \mathbb{C}$ satisfying:

$$(i)\ \ f\left(\frac{1}{n}\right) = \frac{1}{n^2}\ (ii)\ \ f\left(\frac{1}{n}\right) = \frac{1}{n+1}\ \ (iii)\ \ f\left(\frac{1}{n^2}\right) = \frac{1}{n}.$$

(13) Find the order of all the zeros of the following functions

$$(i)\ z\sin z\ \ (ii)\ \ (1 - \exp z))(z^2 - 9)^4\ \ (iii)\frac{\sin^3 z}{z}$$