MSO202 - INTRODUCTION TO CPMPLEX ANALYSIS
IIT KANPUR - 2023–2024

Instructions: Course materials and calculators are not allowed.

Assignment 1

The exercises labeled as **(T)** will be covered during the tutorials.

(1) **(T)** Let z, w be two complex numbers. Show that

(a)

$$|z+w|^2 = |z|^2 + |w|^2 + 2\Re(z\overline{w}) \qquad (b).$$

(b) **Law of cosine** Let ΔABC be a triangle with $\angle ACB = \theta$. Prove that $|a|^2 + |b|^2 - 2ab\cos\theta = c^2$, where a, b, c are sides opposite to angles A, B, C respectively.

(c) $|z+w| = |z| + |w|$ if and only if $zw = 0$ of $z = kw$ for some real number k.

(2) Suppose that z_1 and z_2 are complex numbers, with $z_1 z_2$ real and non-zero. Show that there exists a real number r such that $z_1 = r z_2$.

(3) Express following in form of $x + iy$, with $x, y \in \mathbb{R}$.

(a)

$$(a)\left(\frac{1+i}{\sqrt{2}}\right)^{2023}. \quad (b)\left(1+\sqrt{3}i\right)^{1001}. \quad (c)(1-i)^{31}.$$

(4) **(T)** Consider the $n-1$ diagonals of a regular n-gon inscribed in a unit circle obtained by connecting one vertex with all the others. Show that the product of their lengths is n.

(5) Sketch the following sets and determine which ones of these are domains:

$$(a)|z-4| \geqslant |z|. \quad (b)|arg(z+i)| \leqslant \pi/4 \quad (c)|\Im z| < |\Re z|. \quad (d)|z+ia| < |z-a|.$$

(6) **(T)** Let z, w be two complex numbers such that $\overline{z}w \neq 1$. Prove that

$$|\frac{w-z}{1-\overline{w}z}| < 1 \text{ if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$|\frac{w-z}{1-\overline{w}z}| = 1 \text{ if } |z| = 1 \text{ or } |w| = 1,$$

(7) Show that for

$$f(z) = \frac{[(1-i)z + (1+i)\overline{z}]^2}{|z|^2}$$

$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0} f(x,y)$ but limit $\lim_{z\to 0} f(z)$ does not exists.

(8) **T** Show that

$$\frac{R^3 - 4R}{R^2 + R + 1} \leqslant \left| \frac{z^3 + 4z}{z^2 + z + 1} \right| \leqslant \frac{R^3 + 4R}{(R-1)^2},$$

for $|Z| = R > 1$.

MSO202 - INTRODUCTION TO CPMPLEX ANALYSIS
IIT KANPUR - 2023–2024

Instructions: Course materials and calculators are not allowed.

ASSIGNMENT 2

The exercises labeled as **(T)** will be covered during the tutorials.

(1) **(T)** Which of the following functions $f(z)$ can be defined at $z = 0$ so that they become continuous at $z = 0$:

$$(a)\frac{\Re z}{|z|} \quad (b)\frac{\Re z^2}{|z|^2} \quad (c)\frac{a\Re z}{z} \quad (d)\frac{iz^2}{|z|}.$$

(2) **(T)**Show that

$$f(z) = \begin{cases} \frac{z^2}{|z|} & if \quad z \neq 0 \\ 0 & \quad z = 0 \end{cases}$$

is continuous at $z = 0$, first order partial derivatives of its real and imaginary part exist at $z = 0$, but $f(z)$ is not differentiable at $z = 0$.

(3) Prove that for a fixed $w \in D = \{z \in \mathbb{C} : |z| < 1\}$, the mapping

$$F : z \mapsto \frac{w - z}{1 - \overline{w}z}$$

satisfy the following conditions:

- F maps D to itself and $F : D \to D$ is holomorphic.
- F interchanges 0 and w, i.e., $F(0) = w$ and $F(w) = 0$.
- $|F(z)| = 1$ if $|z| = 1$.
- F is bijective.

(4) Write the following functions $f(z)$ in the forms $f(z) = u(x, y) + iv(x, y)$.

$$(a)f(z) = \overline{\exp(z^2)} \quad (b)f(z) = \frac{1}{i - z}.$$

(5) Which of the following maps are holomorphic?
  (a) $P(x + \iota y) = x^3 - 3xy^2 - x + \iota(3x^2y - y^3 - y)$
  (b) $P(x + \iota y) = x^2 + \iota y^2$
  (c) **(T)** $P(x + \iota y) = 2xy + \iota(y^2 - x^2)$

(6) Determine if there exist an analytic function with u as real part. Also, find the harmonic conjugate.

$(a)\ x^2 - y^2 \quad (b)\ \sinh x \cos y \quad (c) 2x(1 - y) \quad (d)\ x^2 - y^2 - x + y \quad (e)\ e^x \sin y$

(f) **(T)**$e^{(x^2 - y^2)} \cos(2xy)$.

(7) **(T)** Show that the set of natural numbers N can not be partitioned into finite number of subsets that are in arithmetic progression with distinct common difference.

(8) **(T)** Prove that

(a) If $f(z)$ and $\overline{f(z)}$ both are analytic in a domain, then it is a constant function.

(b) If $f(z)$ is analytic in a domain D and $f'(z) \equiv 0$ in D, then it is a constant function.

(c) If $f(z)$ is analytic in a domain D and $u_x + v_y \equiv 0$ in D , then $f'(z)$ is a constant function.

(d) Let $f(z) = u + iv = Re^{i\theta}$ be an analytic function in a domain D. If either of the functions u, v, R, θ is a constant function in D, then f is a constant function.

(9) Write down C-R equation in polar co-ordinates. Express $f'(z)$ in terms of polar co-ordinates.

MSO202- INTRODUCTION TO COMPLEX ANALYSIS

1. Assignment 3

(1) **(T)** Show that $f(z) = e^z$ is the only analytic solutions of the functional equation $f(z_1 + z_2) = f(z_1)f(z_2)$ which satisfies the condition $f(x) = e^x$ for all real number x.

(2) **(T)** Show that $|\sin(z)|^2 = \sin^2 x + \sinh^2 y$ and $|\cos(z)|^2 = \cos^2 x + \sinh^2 y$.

(3) **(T)** find all complex roots of equations:

$$(i)\ \cos z = 4 \quad (ii)\ \log z = 3i \quad (iii)\ z^i = i.$$

(4) Prove that

$$(i)\ \sin^{-1}(z) = -i\log\left(i(z+\sqrt{z^2-1})\right) \quad (ii)\ \cos^{-1}(z) = -i\log\left((z+\sqrt{z^2-1})\right)$$

$$(iii)\ \tan^{-1}(z) = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right) = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) \quad (iv)\ \cot^{-1}(z) = \frac{i}{2}\log\left(\frac{z-i}{z+i}\right)$$

$$(v)\ \sinh^{-1}(z) = \log\left((z+\sqrt{z^2+1})\right) \quad (vi)\ \cosh^{-1}(z) = \log\left((z+\sqrt{z^2-1})\right)$$

$$(vii)\ \tanh^{-1}(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \quad (viii)\ \coth^{-1}(z) = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right).$$

(5) **(T)** Give examples to show that (i) $\mathrm{Log}(z^n) \neq n\mathrm{Log} z$ $(ii)\sqrt{z_1 z_2} \neq \sqrt{z_1}\sqrt{z_2}$.

(6) **(T)** Find $F(0), F(1), F(-1)$, where $F(z)$ is the branch of the function $\sqrt[4]{z-i}$ which for $z = 1+i$ takes the value 1.

(7) **(T)** Let $z_1, z_2 \in \mathbb{C}$. Prove that $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$.

(8) Do limits $\lim_{z\to 0} z\sin(1/z)$ and $\lim_{z\to\infty} e^z$ exist?

(9) Write Laurent series expansion of

$$f(z) = \frac{1}{z(z^2+1)}$$

in (i) $0 < |z| < 1$ and (ii) $1 < |z| < \infty$.

(10) **(T)** If series $f(z) = \sum a_n z^n$ has radius of convergence $0 < R < \infty$. Let k be a natural number. Find radius of convergence of

$$(i)\sum_{n=1}^{\infty} a_n z^{kn} \quad (ii)\sum_{n=1}^{\infty} n^k a_n z^n \quad (iii)\sum_{n=1}^{\infty} \frac{a_n}{n!} z^n.$$

(11) **(T)** Find radius of convergence of following series.

$$(i)\sum_{n=1}^{\infty} \frac{1}{n!} z^{2n+3} \quad (ii)\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3}(z+2)^n \quad (iii)\sum_{n=1}^{\infty} \frac{1}{n!} z^{2n^2} \quad (iv)\sum_{n=1}^{\infty} (3z-2)^{2n}$$

$$(v)\sum_{n=1}^{\infty} \frac{3n+8}{7n+9}(z+2)^n \quad (v)\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^4}(z+2)^n.$$

Preliminary version – August 20, 2023

MSO202- INTRODUCTION TO COMPLEX ANALYSIS

1. Assignment 4

Notation: Let C_r denotes the circle with radius r and centre at origin and oriented anticlockwise, with $C := C_1$.

(1) Let a be a positive real number and Γ be the rectangle with vertices $0, a, a + 2\pi i$, and $2\pi i$. Explicitly compute the integral

$$\int_\Gamma e^z dz$$

and verify the Cauchy's Theorem.

(2) Let L be a path which consists of the half circle $z = Re^{it}$, $0 \leqslant t \leqslant \pi$ and the straight line segment: $-R \leqslant \Re z \leqslant R, Imz = 0$. Find the integral

$$\int_L |z|^2 \overline{z} dz.$$

(3) Evaluate the contour integral $\int_L f(z)dz$ using the parametric representation of L, where

$$f(z) = \frac{z^2-1}{z} \quad \textit{and } L = (i) \textit{ the semicircle } z = 2e^{i\theta},\ 0 \leqslant \theta \leqslant \pi.$$

$$L = (ii) \textit{ the semicircle } z = 2e^{i\theta},\ \pi \leqslant \theta \leqslant 2\pi. \quad L = (iii) z = 2e^{i\theta},\ 0 \leqslant \theta \leqslant 2\pi.$$

Also, calculate the integral using an anti-derivative of $f(z)$.

(4) Show that

$$f(R) := \left| \int_{C_R} \frac{Log(z^2)}{z^2} dz \right| \leqslant 2\pi \left(\frac{\pi + 2\log R}{R} \right).$$

Conclude that $\lim_{R\to\infty} f(R) = 0$.

(5) Let L be a path and $\overline{L}$ the path which is the image of L by the function $z \to \overline{z}$. Let f be a continuous function on L. Prove that the function $z \to \overline{f(\overline{z})}$ is continuous on L and

$$\overline{\int_L f(z)dz} = \int_{\overline{L}} \overline{f(\overline{z})} dz.$$

(6) Evaluate

$$\int_L \left(e^z + \frac{1}{z} \right) dz,$$

where L is the lower half of the circle with radius 1, centre 0, negatively oriented. Also evaluate by finding an antiderivative.

(7) Let $|a| < r < |b|$, prove that

$$\int_{C_r} \frac{1}{z-a} dz = 2\pi i \quad \textit{and} \quad \int_{C_r} \frac{dz}{(z-a)(z-b)} = 2\pi i/(a-b).$$

(8) Evaluate

$$(i) \int_{C_5} \frac{\sin z}{(z+1)^7} dz \quad (ii) \int_{C_5} \frac{\cos(\pi z^2)}{(z^2-1)(z-2)(z+3)} dz \quad (iii) \int_{C_5} \frac{e^{2z}}{z(z+1)^4} dz.$$

(9) Evaluate the integral

$$\int_L \frac{dz}{(z^2-1)(z+3)} dz$$

for all possible contour which does not passes through $z = \pm 1, \pm i, 2, 3$.

(10) Suppose $f(z)$ is anlytic and satisfies the relation $|f(z)-2| < 1$ in a region Ω. Show that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0$$

for every closed curve γ in Ω.

(11) Show that $\int_\gamma \overline{f(z)} f'(z) dz$ is purely imaginary where γ is any curve in a region Ω and f is holomorphic in Ω.

(12) Let f be analytic on a region Ω and let C be a circle with interior contained in Ω. For any $a \in \Omega$ not on C show that

$$\int_C \frac{f'(\xi)}{(\xi - a)} d\xi = \int_C \frac{f(\xi)}{(\xi - a)^2} d\xi.$$

(13) Show that successive derivatives of an analytic function f at a point z_0 can never satisfy the inequality $|f^{(n)}(z_0)| > n^n n!$ for all $n \in \mathbb{N}$.

(14) Let τ be a complex number which is not real. Suppose that f is an entire function such that $f(z+1) = f(z)$ and $f(z+\tau) = f(z)$. Then show that f is a constant. (This exercise says that a doubly periodic entire function is constant.)

(15) Let f be an entire function satisfying $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. Show that f is constant.

(16) The Bernoulli numbers B_n are defined by the series power series

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

Show that $\frac{z}{e^z-1} + \frac{z}{2} = \frac{z}{2} \coth \frac{z}{2}$. Conclude that $B_1 = -\frac{1}{2}$ and $B_{2n+1} = 0,\ n \geq 1$. Deduce that

$$z \cot z = \sum_{n=0}^{\infty} (-1)^n \frac{B_{2n}}{(2n)!} z^{2n}.$$

MSO202- INTRODUCTION TO COMPLEX ANALYSIS

1. Assignment 5

(1) Let $f(z)$ be an entire function such that $f(\mathbb{R}) \subset \mathbb{R}$. Show that $g(z) = \overline{f(\overline{z})}$ is also entire and $\overline{f(z)} = f(\overline{z})$.

(2) Suppose that Ω is simply connected domain with $1 \in \Omega$, and $0 \notin \Omega$. Then there is a branch of the logarithm $F(z) = \log_\Omega z$ so that
  (i) F is holomorphic in Ω.
  (ii) $e^{F(z)} = z$ for all in Ω.
  (iii) $F(r) = \log r$ whenever r is a real number and near 1.
  (**Hint:** Page 98, Theorem 6.1 of Stein and Shakarchi)

(3) Calculate

$$(i) \int_{C_2} \frac{z^{2009}}{z^{2010} + z^2 + 1} dz \quad (i) \int_{C_2} \frac{z^{2023}}{z^{2023} + z^{2022} + z^{2000} + 1} dz.$$

(4) Let f be a complex valued function in the unit disk $D := \{z : |z| < 1\}$ such that f^2 and f^3 are both analytic. Prove that f is analytic.

(5) Let $f, g : \Omega \to \mathbb{C}$ be analytic functions. If $fg \equiv 0$, then either $f \equiv 0$ or $g \equiv 0$. Also, if $\overline{f}g$ is analytic. Show that f is constant function or g is identically zero.

(6) Fibonacci number are given by the recurrence relation $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Let F_n be the n-th Fibonacci number. Show that (informally)

$$F(z) := \sum_{n=0}^{\infty} F_n z^n = \frac{1}{1 - z - z^2}.$$

Find radius of convergence of $F(z)$.

(7) Let $f(z)$ be an entire function such that $|f(z)| \leqslant C|z|^{1/2}$ for all z, then show that f is a constant function.

(8) Expand each of the following functions in Laurent series in the neighbourhood of the indicated points z_0 and, in each case, determine the largest domain where the resulting Laurent series converges:

$$(i)$$

(9) Find the Laurent series of the function

$$f(z) = \frac{z+4}{z^2(z^2 + 3z + 2)} \qquad in$$

$$(i)\ 0 < |z| < 1. \quad (ii)\ \ 1 < |z| < 2\ (iii)\ \ 0 < |z+1| < 1\ (iv)\ \ |z| > 2.$$

(10) **T** Find the Laurent series of the function

$$f(z) = \frac{z^2}{z^2 - 3z + 2} \qquad in$$

$$(i)\ \ 1 < |z| < 2\ (ii)\ \ 1 < |z-3| < 2$$

(11) Find the order of the zero at $z = 0$ for the following functions:

$(i) z^3 \left(\exp(\exp z^3) - 1\right)$ (ii) $6\sin(z^3) + z^3(z^6 - 6)$ (iii) $\exp(\sin z) - \exp(\tan z)$.

(12) Find all the holomorphic functions $f : B_1(0) \to \mathbb{C}$ satisfying:

$$(i)\ \ f\left(\frac{1}{n}\right) = \frac{1}{n^2}\ (ii)\ \ f\left(\frac{1}{n}\right) = \frac{1}{n+1}\ \ (iii)\ \ f\left(\frac{1}{n^2}\right) = \frac{1}{n}.$$

(13) Find the order of all the zeros of the following functions

$$(i)\ z\sin z\ \ (ii)\ \ (1 - \exp z))(z^2 - 9)^4\ \ (iii)\frac{\sin^3 z}{z}$$

MSO202- INTRODUCTION TO COMPLEX ANALYSIS

1. Assignment 6

(1) Find residues of the following functions at all its poles:

$$(i)\cot(\pi z) \quad (ii)\frac{z}{z^n-1} \quad (iii)\frac{z^2(z-1)^3(z-2)}{\sin^2(\pi z)}$$

(2) Evaluate $(a,b>0)$

$$(i)\int_{\mathbb{R}}\frac{x\sin(ax)}{x^2+b^2}dx \quad (ii)\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}(|a|>|b|) \quad (iii)\int_0^{2\pi}\frac{d\theta}{1+a\sin\theta}(|a|<1)$$

$$(iv)\int_0^{2\pi}\frac{\cos(n\theta)d\theta}{a+\cos\theta}\ (a>1, n\in\mathbb{N}) \quad (v)\int_{\mathbb{R}}\frac{1}{(x-1)(x^2+4)}dx \quad (vi)\int_{\mathbb{R}}\frac{\sin^2 x}{x^2+4}dx$$

(3) Let $\phi\in(0,\pi)$ and $n\in\mathbb{N}$. Prove that

$$\int_{|z|=2}\frac{z^n dz}{1-2z\cos\phi+z^2}=\frac{\sin(n\phi)}{\sin\phi}.$$

(4) Use Argument principle to evaluate

$$(i)\int_{|z+1+i|=2}\frac{z+i}{z^2+2iz-4}dz \quad (ii)\int_{|z|=2}\frac{z+2}{z(z+1)}dz$$

(5) Use Rouche's Theorem to determine the roots of polynomial

$$(i)p(z)=z^{10}-6z^9-3z+1 \quad in \quad |z|<1 \quad (ii)z^5+6z^3+2z+10 \quad in \quad 1<|z|<3.$$

(6) Use Rouche's Theorem to prove fundamental theorem of algebra.

(7) Let g be analytic for $|z|\leqslant 1$ and $|g(z)|<1$ for $|z|=1$. Then prove that g has a unique fixed point in $|z|<1$. What happens if we replace $|g(z)|<1$ with $|g(z)|\leqslant 1$ in above condition.

(8) If $f(z)$ is analytic at a with $f'(a)\neq 0$, then f(z) is one-to-one in some neighborhood of a. Conversely, if $f(z)$ is analytic and one-to-one in a domain D, then $f'(z)\neq 0$, in D.

(9) Find the linear fractional transformation that maps the points $z_1=-i$, $z_2=0$ $z_3=i$ to $w_1=-1$, $w_2=i$, $w_3=1$.

(10) Let f be analytic on an open set D, and $f'(a)\neq 0$ for some $a\in D$. Evaluate

$$\frac{1}{2\pi i}\int_C\frac{dz}{f(z)-f(a)},$$

where C is a sufficiently small circle centered at a.

Preliminary version – September 9, 2023