MSO202- INTRODUCTION TO COMPLEX ANALYSIS

1. Assignment 4

Notation: Let C_r denotes the circle with radius r and centre at origin and oriented anticlockwise, with $C := C_1$.

(1) Let a be a positive real number and Γ be the rectangle with vertices $0, a, a + 2\pi i$, and $2\pi i$. Explicitly compute the integral
$$\int_\Gamma e^z dz$$
and verify the Cauchy's Theorem.

(2) Let L be a path which consists of the half circle $z = Re^{it}$, $0 \leqslant t \leqslant \pi$ and the straight line segment: $-R \leqslant \Re z \leqslant R, Imz = 0$. Find the integral
$$\int_L |z|^2 \overline{z} dz.$$

(3) Evaluate the contour integral $\int_L f(z)dz$ using the parametric representation of L, where
$$f(z) = \frac{z^2 - 1}{z} \quad \textit{and } L = (i) \textit{ the semicircle } z = 2e^{i\theta},\ 0 \leqslant \theta \leqslant \pi.$$
$$L = (ii) \textit{ the semicircle } z = 2e^{i\theta},\ \pi \leqslant \theta \leqslant 2\pi. \quad L = (iii) z = 2e^{i\theta},\ 0 \leqslant \theta \leqslant 2\pi.$$

Also, calculate the integral using an anti-derivative of $f(z)$.

(4) Show that
$$f(R) := \left| \int_{C_R} \frac{Log(z^2)}{z^2} dz \right| \leqslant 2\pi \left(\frac{\pi + 2\log R}{R} \right).$$
Conclude that $\lim_{R\to\infty} f(R) = 0$.

(5) Let L be a path and $\overline{L}$ the path which is the image of L by the function $z \to \overline{z}$. Let f be a continuous function on L. Prove that the function $z \to \overline{f(\overline{z})}$ is continuous on L and
$$\overline{\int_L f(z)dz} = \int_{\overline{L}} \overline{f(\overline{z})} dz.$$

(6) Evaluate
$$\int_L \left(e^z + \frac{1}{z} \right) dz,$$
where L is the lower half of the circle with radius 1, centre 0, negatively oriented. Also evaluate by finding an antiderivative.

(7) Let $|a| < r < |b|$, prove that
$$\int_{C_r} \frac{1}{z-a} dz = 2\pi i \quad \textit{and} \quad \int_{C_r} \frac{dz}{(z-a)(z-b)} = 2\pi i/(a-b).$$

(8) Evaluate

$$(i) \int_{C_5} \frac{\sin z}{(z+1)^7} dz \quad (ii) \int_{C_5} \frac{\cos(\pi z^2)}{(z^2-1)(z-2)(z+3)} dz \quad (iii) \int_{C_5} \frac{e^{2z}}{z(z+1)^4} dz.$$

(9) Evaluate the integral

$$\int_L \frac{dz}{(z^2-1)(z+3)} dz$$

for all possible contour which does not passes through $z = \pm 1, \pm i, 2, 3$.

(10) Suppose $f(z)$ is anlytic and satisfies the relation $|f(z) - 2| < 1$ in a region Ω. Show that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0$$

for every closed curve γ in Ω.

(11) Show that $\int_\gamma \overline{f(z)} f'(z) dz$ is purely imaginary where γ is any curve in a region Ω and f is holomorphic in Ω.

(12) Let f be analytic on a region Ω and let C be a circle with interior contained in Ω. For any $a \in \Omega$ not on C show that

$$\int_C \frac{f'(\xi)}{(\xi - a)} d\xi = \int_C \frac{f(\xi)}{(\xi - a)^2} d\xi.$$

(13) Show that successive derivatives of an analytic function f at a point z_0 can never satisfy the inequality $|f^{(n)}(z_0)| > n^n n!$ for all $n \in \mathbb{N}$.

(14) Let τ be a complex number which is not real. Suppose that f is an entire function such that $f(z+1) = f(z)$ and $f(z+\tau) = f(z)$. Then show that f is a constant. (This exercise says that a doubly periodic entire function is constant.)

(15) Let f be an entire function satisfying $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. Show that f is constant.

(16) The Bernoulli numbers B_n are defined by the series power series

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

Show that $\frac{z}{e^z-1} + \frac{z}{2} = \frac{z}{2} \coth \frac{z}{2}$. Conclude that $B_1 = -\frac{1}{2}$ and $B_{2n+1} = 0, \ n \geq 1$. Deduce that

$$z \cot z = \sum_{n=0}^{\infty} (-1)^n \frac{B_{2n}}{(2n)!} z^{2n}.$$