MSO202- INTRODUCTION TO COMPLEX ANALYSIS

1. Assignment 3

(1) **(T)** Show that $f(z) = e^z$ is the only analytic solutions of the functional equation $f(z_1 + z_2) = f(z_1)f(z_2)$ which satisfies the condition $f(x) = e^x$ for all real number x.

(2) **(T)** Show that $|\sin(z)|^2 = \sin^2 x + \sinh^2 y$ and $|\cos(z)|^2 = \cos^2 x + \sinh^2 y$.

(3) **(T)** find all complex roots of equations:

$$(i)\ \cos z = 4 \quad (ii)\ \log z = 3i \quad (iii)\ z^i = i.$$

(4) Prove that

$$(i)\ \sin^{-1}(z) = -i\log\left(i(z+\sqrt{z^2-1})\right) \quad (ii)\ \cos^{-1}(z) = -i\log\left((z+\sqrt{z^2-1})\right)$$

$$(iii)\ \tan^{-1}(z) = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right) = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) \quad (iv)\ \cot^{-1}(z) = \frac{i}{2}\log\left(\frac{z-i}{z+i}\right)$$

$$(v)\ \sinh^{-1}(z) = \log\left((z+\sqrt{z^2+1})\right) \quad (vi)\ \cosh^{-1}(z) = \log\left((z+\sqrt{z^2-1})\right)$$

$$(vii)\ \tanh^{-1}(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \quad (viii)\ \coth^{-1}(z) = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right).$$

(5) **(T)** Give examples to show that (i) $\mathrm{Log}(z^n) \neq n\mathrm{Log}z$ $(ii)\sqrt{z_1 z_2} \neq \sqrt{z_1}\sqrt{z_2}$.

(6) **(T)** Find $F(0), F(1), F(-1)$, where $F(z)$ is the branch of the function $\sqrt[4]{z-i}$ which for $z = 1+i$ takes the value 1.

(7) **(T)** Let $z_1, z_2 \in \mathbb{C}$. Prove that $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$.

(8) Do limits $\lim_{z\to 0} z\sin(1/z)$ and $\lim_{z\to\infty} e^z$ exist?

(9) Write Laurent series expansion of

$$f(z) = \frac{1}{z(z^2+1)}$$

in (i) $0 < |z| < 1$ and (ii) $1 < |z| < \infty$.

(10) **(T)** If series $f(z) = \sum a_n z^n$ has radius of convergence $0 < R < \infty$. Let k be a natural number. Find radius of convergence of

$$(i)\sum_{n=1}^{\infty} a_n z^{kn} \quad (ii)\sum_{n=1}^{\infty} n^k a_n z^n \quad (iii)\sum_{n=1}^{\infty} \frac{a_n}{n!} z^n.$$

(11) **(T)** Find radius of convergence of following series.

$$(i)\sum_{n=1}^{\infty} \frac{1}{n!} z^{2n+3} \quad (ii)\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3}(z+2)^n \quad (iii)\sum_{n=1}^{\infty} \frac{1}{n!} z^{2n^2} \quad (iv)\sum_{n=1}^{\infty} (3z-2)^{2n}$$

$$(v)\sum_{n=1}^{\infty} \frac{3n+8}{7n+9}(z+2)^n \quad (v)\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^4}(z+2)^n.$$

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