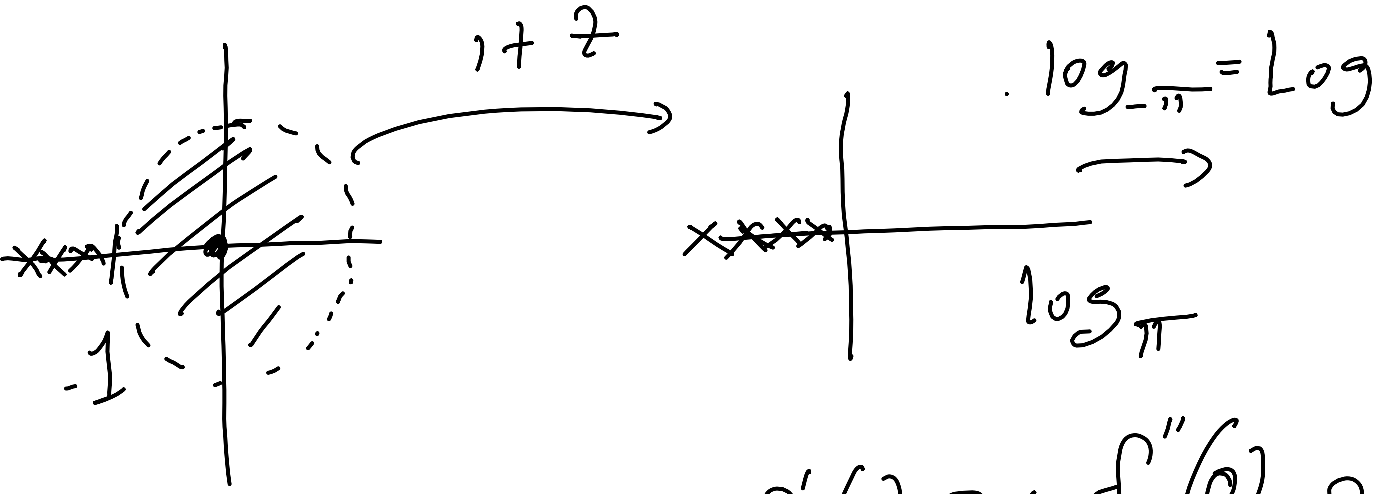
Example

① $\log_{-\pi}(1+z)$ expand as power series about $z=0$.



$$f(z) = f(0) + \frac{f'(0)}{} z + \frac{f''(0)}{\lfloor 2} z^2 + \frac{f'''(0)}{\lfloor 3} z^3$$

$f(0) = \log_{-\pi}(1) = 0$ $\qquad f(z) = \log_{-\pi}(1+z)$

$f'(z) = \dfrac{1}{1+z}$ $\qquad f'(0) = 1$

$f''(z) = -\dfrac{1}{(1+z)^2}$ $\qquad f''(0) = -1$

$f'''(z) = +\dfrac{\lfloor 2}{(1+z)^3} \qquad = \lfloor 2$

$|z|<1$

$\cancel{f(z}$

$$\log_{-\pi}(1+z) = 0 + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} - \cdots$$

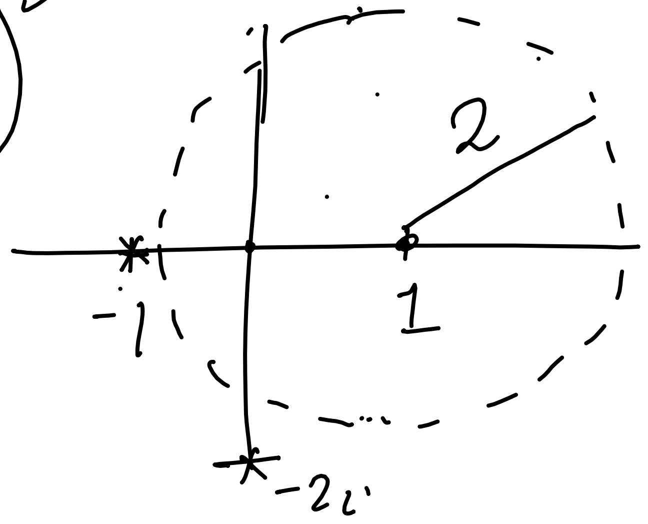
$$\log_{\pi}(1+z) = 2\pi + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} \cdots$$

(2) $\dfrac{1}{(z+1)(z+2i)} = f(z)$

$\Omega = \mathbb{C} \setminus \{-1, -2i\}$

$f \in H(\mathbb{C} \setminus \{-1, -2i\})$

Expand this as power series around. $z_0 = 1$



$f(z_0)$

$f'(z_0)$

$f''(z_0)$

$\cdots$

$|1+2i|$

$= \sqrt{1+4}$

$= \sqrt{5} > 2$

$$\frac{1}{(z+1)(z+2i)}$$

$$= \left[\frac{1}{z+1} - \frac{1}{z+2i}\right] \frac{1}{2i-1}$$

$$\frac{1}{z+1} = \frac{1}{(z-1)+2}$$

$$= \frac{1}{2\left[1 + \frac{z-1}{2}\right]}$$

$$= \frac{1}{2}\left(1 - \left(\frac{z-1}{2}\right) + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^3 \cdots\right)$$

$$\boxed{\begin{array}{l} \frac{1}{1+w} = 1 - w + w^2 - w^3 \cdots \\ |w| < 1 \end{array}}$$

Do similar expansion for the other term.

$\left|\frac{z-1}{2}\right| < 1$

$|z-1| < 2$

Maximum Modulus Principle

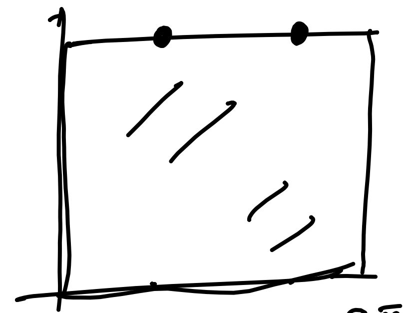
Suppose f is non-constant holomorphic on a domain Ω.
Then $|f|$ can not attain maximum on Ω. i.e.
$\nexists\, a \in \Omega$ s.t. $|f(a)| \geq |f(z)|$ $\forall z \in \Omega$

Corollary Ω bounded.
$f \in H(\Omega)$ + $f \in C(\overline{\Omega})$
$\overline{\Omega} = \Omega$ + boundary of Ω
Then $|f|$ attains max value on $(\overline{\Omega})$ boundary of Ω i.e. $(\overline{\Omega} \setminus \Omega)$

Example

(1) $\sin z$ $\qquad \Omega = [0\ 2\pi] \times [0\ 2\pi)$

Find $\max_{z\in\Omega} |\sin z|$



2π

Sol) $|\sin z|^2$
$= \sin^2 x + \sinh^2 y$ $\qquad z = x + iy$.

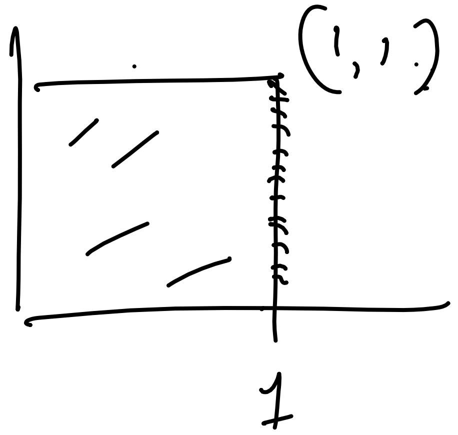
Maximum value of $\sin^2 x = 1$
$x = \pi/2,\ 3\pi/2$

Maximum value of $\sinh^2 y$ $\quad$ ($\sinh y$ increasing function)
$y = 2\pi$

$\text{Max} |\sin z| = \sqrt{1 + \sinh^2(2\pi)} = \cosh 2\pi$
attained at $(\pi/2,\ 2\pi)$ $\quad (3\pi/2,\ 2\pi)$

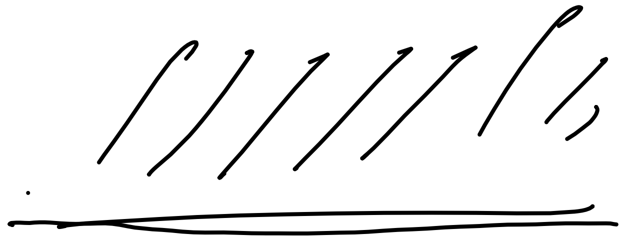
(2) $f(z) = e^{z}$ $\quad [0,1] \times [0,1] = \Omega$

$|f(z)| = e^{x}$



$\text{Max } |f(z)| = e^{1}$ at $x = 1$

(3) $f(z) = e^{-iz}$

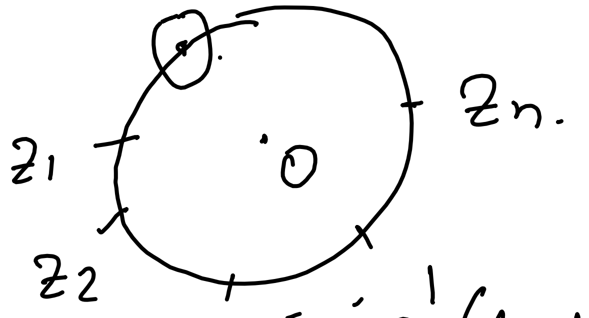


$\Omega = \{ z \mid y > 0 \}$.

$|f(z)| = e^{y}$

$\displaystyle\max_{z \in \Omega} |f(z)|$ – does not exist in Ω.

(4) Suppose $z_1, \cdots z_n$ are n points on circle $|z| = 1$

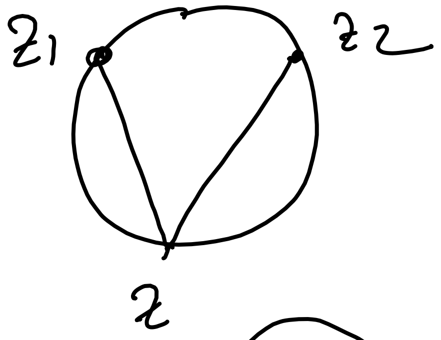


$S^1 : (|z| = 1)$

Then $\exists\, z \in S^1$ s.t.

$|z - z_1| \cdots |z - z_n| > 1$

Sol $f(z) = (z - z_1) \cdots (z - z_n)$



On the contrary.

$|f(z)| \leq 1 \quad \forall\, z \in S^1$ $\qquad |f(z)| \leq 1 \ \forall z$

$|f(0)| = 1$

This contradicts the Max Modulus Principle.

Proof of Max modulus

Assume $\exists\ a \in \Omega$

s.t $|f(z)| \leq |f(a)|$

$\forall\ z \in \Omega$.

$$f(a) = \frac{1}{2\pi i} \int_{C_r} \frac{f(\xi)}{\xi - a}\, d\xi.$$

where r small enough s.t

$C_r : |z-a| = r$ inside Ω.

$$\xi = re^{it} + a$$

$$f(a) = \frac{1}{2\pi \cancel{i}} \int_{t=0}^{2\pi} \frac{f(a+re^{it})}{\cancel{re^{it}}}\, \cancel{r}\cancel{e^{it}}\cancel{i}\, dt$$

$$|f(a)| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(a+re^{it})\, dt \right|$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(a+re^{it})|\, dt$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(a)|\, dt$$

$$= |f(a)|$$

$$|f(a)| = \frac{1}{2\pi} \int_0^{2\pi} |f(a+re^{it})|\, dt$$

$$\frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{it})|\, dt = |f(a)|$$

$$\Rightarrow \int_0^{2\pi} \underbrace{\left(|f(a)| - |f(a+re^{it})|\right)}_{\geq 0} dt = 0$$

$$\Rightarrow |f(a)| = |f(a+re^{it})|$$

$$\Rightarrow |f(a)| = |f(z)| \quad \forall\ |z-a| < r$$

$$\Rightarrow f = \text{const on } |z-a| \leq r$$

$$\Rightarrow f = \text{const on } \Omega.$$

(by Identity Principle) $\blacksquare$