MSO 202A: Complex Variables
August-September 2022
Assignment-3

Throughout C will denote the unit circle around origin, oriented counterclockwise.

1. **(T)**Find all the zeros of the function $f(z) = 2 + \cos z$.

   Solution: The zeros of $f(z)$ are solutions of the equation

$$\cos(x + \iota y) = \cos x \cosh y - \iota \sin x \sinh y = -2.$$

   So we must have $\cos x \cosh y = -2$ and $\sin x \sinh y = 0$.

   Now $\sin x \sinh y = 0$ if and only if either $\sinh y = 0$ or $\sin x = 0$. The first case is excluded because it requires $y = 0$ and $f(z)$ can not have real zero. (no real z can satisfy $\cos z = -2$)

   So we must have $\sin x = 0$ i.e. $x = k\pi$ for $k \in \mathbb{Z}$. Hence from the other equation $\cosh y = (-1)^k 2$. But $\cosh y = (e^y + e^{-y})/2 \geq 1$ for all real y with equality if and only if $y = 0$; otherwise, $\cosh y = C$ has two distinct real roots for every $C > 1$. We conclude that $-2 = \cos x \cosh y = \cos k\pi \cosh y = (-1)^k \cosh y$ has a solution if and only if $x = k\pi$ for some odd integer k and y is one of the two real roots of $\cosh y = 2$.

2. **(T)**The Bernoulli numbers B_n are defined by the series power series

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

   Show that $\frac{z}{e^z-1} + \frac{z}{2} = \frac{z}{2}\coth\frac{z}{2}$. Conclude that $B_1 = -\frac{1}{2}$ and $B_{2n+1} = 0,\ n \geq 1$. Deduce that

$$z \cot z = \sum_{n=0}^{\infty} (-1)^n \frac{B_{2n}}{(2n)!} z^{2n}.$$

   Solution:

$$\frac{z}{e^z - 1} + \frac{z}{2} = \frac{z}{2}\frac{e^z - 1}{e^z + 1} = \frac{z}{2}\frac{e^{z/2} - e^{-z/2}}{e^{z/2} + e^{-z/2}} = \frac{z}{2}\coth\frac{z}{2}$$

   Since $\frac{z}{2}\coth\frac{z}{2}$ is an even function, the power series expansion will have only even powers of z. Thus

$$\frac{z}{2}\coth\frac{z}{2} = \frac{z}{e^z - 1} + \frac{z}{2} = \frac{z}{2} + \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \sum_{n=0}^{\infty} \frac{B_{2n}}{2n!} z^{2n}$$

   Hence $B_1 = 1/2$ and $B_{2n+1} = 0$ for $n > 0$

$$\cot z = \frac{\cos z}{\sin z} = \frac{(e^{\iota z} + e^{-\iota z})/2}{(e^{\iota z} - e^{-\iota z})/2\iota} = \iota \coth(\iota z)$$

Replacing z by $2\iota z$ in the formula for $\frac{z}{2}\coth\frac{z}{2}$, we get

$$z\cot z = \iota z\coth(\iota z) = \sum_{n=0}^{\infty}\frac{B_{2n}}{2n!}(2\iota z)^{2n} = \sum_{n=0}^{\infty}(-1)^n\frac{B_{2n}}{(2n)!}z^{2n}.$$

3. **(T)** Let a be a positive real number and Γ be the rectangle with vertices $0, a, a+2\pi\iota, 2\pi\iota$. Explicitly compute the integral $\int\limits_{\Gamma} e^z dz$ and verify that the integral is 0.

   Solution:

   Orient the rectangle anticlockwise. Then the integral is sum of four integrals.

   On the x-axis the integral is $\int_0^a e^x dx = e^a - 1$.

   On the horizontal line $y = 2\pi$ the integral is $-\int_0^a e^{x+2\pi\iota}dx = -\int_0^a e^x dx = -(e^a-1)$.

   On the $x = a$ line, the integral is $\int_0^{2\pi} e^{a+\iota y}dy = e^a(e^{2\pi i}-1) = 0$

   On the $x = 0$ line, the integral is $-\int_0^{2\pi} e^{\iota y}dy = (e^{2\pi i}-1) = 0$

   Thus adding all these integrals we get 0.

4. Calculate by hand $\int\limits_C \frac{1}{z}dz$ and $\int\limits_{-C}\frac{1}{z}dz$, where $-C$ is C with opposite orientation.

   Solution:

   $C : z = \cos t + \iota\sin t = e^{\iota t}$ so $\int_C \frac{1}{z}dz = \int_{t=0}^{2\pi}\frac{1}{e^{\iota t}}\iota e^{\iota t}dt = 2\pi\iota$

   $-C : z = \cos t - \iota\sin t = e^{-\iota t}$ so $\int_{-C}\frac{1}{z}dz = \int_{t=0}^{2\pi}\frac{1}{e^{-\iota t}}(-\iota e^{-\iota t})dt = -2\pi\iota$

5. Show that $1/z$ is holomorphic on $\mathbb{C}^*$ but it does not admit a primitive/antiderivative on $\mathbb{C}^*$. Use it to show that $u = 1/2\log(x^2+y^2)$ does not admit a harmonic conjugate on $\mathbb{C}^*$.

   Solution: $\int_C \frac{1}{z}dz = 2\pi\iota$. So it does not admit a primitive on $\mathbb{C}^*$.

   If the harmonic function $u = 1/2\log(x^2+y^2)$ on $\mathbb{C}^*$ admits a harmonic conjugate $i.e f = u+\iota v$ holomorphic, then $f' = u_x - \iota u_y = 1/z$. So $1/z$ has a primitive on $\mathbb{C}^*$ which is a contradiction.

6. Let γ be the upper half of the unit circle described anticlockwise. Show that

   $$\left|\int_\gamma \frac{\exp(z)}{z}dz\right| \leq \pi e.$$

   Solution:

   $L = \pi$.

   $M = \sup_{z\in\gamma}|\frac{e^z}{z}| = \sup_{z\in\gamma}|e^z| = \sup_{t\in[0,\pi]} e^{\cos t} = e$

   Hence by ML-estimate we have the required inequality.

7. (**T**)Show that

$$|\int_{C_3} \frac{1}{z^2+\iota} dz| \leq \frac{3\pi}{4}.$$

Solution:

Here $L = 6\pi$

$$M = \sup_{z \in C_3} |\frac{1}{z^2+\iota}| = \frac{1}{\inf\limits_{z \in C_3} |z^2+\iota|} = \frac{1}{\inf\limits_{w \in C_9} |w+\iota|} = 1/8.$$

Thus by ML-estimates $|\int_{C_3} \frac{1}{z^2+\iota} dz| \leq 6\pi/8 = \frac{3\pi}{4}$.

8. (**T**)Show that

$$|\int_\gamma \log(z) dz| \leq \frac{\pi^2}{4},$$

where γ is the first quadrant portion of the circle C. Here principal branch of log is used.

Solution:

Here $L =$ length of $\gamma = \pi/2$.

For $z \in \gamma$, we have $\log z = \iota\theta$ where $0 \leq \theta \leq \pi/2$. Thus $M = \sup_{z \in \gamma} |\log z| = \pi/2$. So by ML-estimate we have $|\int_\gamma \log(z) dz| \leq \frac{\pi}{2}.\frac{\pi}{2}$.

9. Suppose $f(z)$ is anlytic and satisfies the relation $|f(z)-1| < 1$ in a region Ω. Show that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0$$

for every closed curve γ in Ω.

Solution: The function $Logf(z)$ is holomorphic on Ω with derivative f'/f.

10. (**T**)Show that $\int\limits_\gamma \overline{f(z)} f'(z) dz$ is purely imaginary where γ is any smooth closed curve in a region Ω and f is holomorphic in Ω.

Solution:

$f = u + \iota v$. Then $\bar{f} f' = (u - \iota v)(u_x + \iota v_x) = uu_x + vv_x + \iota(uv_x - vu_x)$ Write $\gamma(t) = x(t) + \iota y(t)$. Then

$$\bar{f} f' \gamma' = [uu_x + vv_x + \iota(uv_x - vu_x)](x' + \iota y') = [(uu_x + vv_x)x' - (uv_x - vu_x)y'] + \iota(\cdots)$$

Using CR-equations, the real part is

$(uu_x + vv_x)x' - (uv_x - vu_x)y' = uu_x x' + uu_y y' + vv_x x' + vv_y y'$
$= \frac{d}{dt}[u^2(x(t), y(t)) + v^2(x(t), y(t))]$

Therefore integral of the real part over closed curve is zero. Hence the given integral is purely imaginary.

11. Compute the following integrals:

(a) **(T)** $\int\limits_{|z|=1} e^z z^{-n} dz; \quad n \in \mathbb{Z};$

Solution:

If $n \geq 0$, then the function is analytic on and inside $|z| = 1$and so the integral is zero.

When $n < 0$, take $f(z) = e^z$. Then by Cauchy's integral formula, the integral is $2\pi i f^{(n)}(0)/(n!) = 2\pi i/(n!)$

(b) $\int_{|z|=2} z^n (1-z)^m dz; \quad m, n \in \mathbb{Z}$

Solution:

Done in class using Cauchy's theorme.

(c) **(T)** $\int\limits_{|z|=1} \frac{\cos z}{\sin z} dz$

Solution:

The value of the integral is 0 since the function $\cos z/\sin z$ is analytic inside $|z| = 1$.

(d)

$$\int_{|z|=1} (z - \frac{1}{z})^n \frac{dz}{z} = \begin{cases} 2\pi i \binom{n}{n/2} (-1)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Use it to show that

$$\int_0^{2\pi} \sin^n t \, dt = \begin{cases} \frac{\pi}{2^{n-1}} \binom{n}{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Solution: The integral is $\int \frac{(z^2-1)^n}{z^{n+1}} dz = \frac{2\pi\iota}{n!} \frac{d^n}{d^n z} (z^2-1)^n|_{z=0}$. Now $\frac{d^n}{d^n z} (z^2-1)^n|_{z=0}$ $=$ (coefficient of z^n in $(z^2-1)^n$)(n!). Hence the result.

For the second part, put $z = e^{\iota t}$ and simplify.