Recall

f holomorphic on $|z-a|\le R$

$$f(z)=\sum a_n (z-a)^n \qquad |z-a|<R$$

holomorphic on Ω		Analytic
$f'(z)$ exist on Ω	$\Longleftarrow$ $\Longrightarrow$	can be represented by power series $\sum a_n z^n$

$$\left(\sum_{m,n} a_{m,n} z^n w^m\right)$$

Zeros of an analytic function

$f:\Omega\to\mathbb{C}$ holomorphic.

$$Z_f=\{z\in\Omega \,/\, f(z)=0\}.$$

Example

1) $f(z)=e^z$

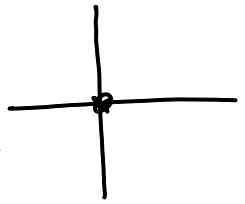
$$Z_f=\emptyset.$$

2) $f(z)=\sin z$

$$Z_f=\{n\pi \,/\, n\in\mathbb{Z}\}.$$

3) $f=\sin\frac{1}{z}$ $\qquad \Omega=\mathbb{C}^*$

$$Z_f=\left\{\frac{1}{n\pi} \,/\, n\in\mathbb{Z}\right\}.$$



4) $f(z)=\cos\frac{1+z}{1-z}$ $\quad (\Omega=|z|<1)$

$$Z_f=\left\{\frac{n\pi-2}{n\pi+2} \,/\, n \text{ odd}\right\}$$

$(a\ \ b) \subseteq Z_f$

$(a\ \ b) \subseteq \mathbb{R}$

$A \subseteq \mathbb{R}^n$. A point $a \in \mathbb{R}^n$ is called a limit point of A if $\exists \{x_n\} \subseteq A$ s.t $x_n \to a$.

limit points of $(a \cdot b) = [a\ ,\ b]$

$x_n = a + \frac{b-a}{n}$

Theorem

Ω - domain. $f \in \mathcal{H}(\Omega)$.

Then the following are equivalent.

(a) $f \equiv 0$ on Ω.

(b) $\exists$ a point $z_0 \in \Omega$ s.t $f^{(n)}(z_0) = 0 \ \forall\ n$.

(c) Z_f has a limit point in Ω.

$f \in \mathcal{H}(\Omega)$

$f \equiv 0$

$f \not\equiv 0 \Rightarrow Z_f$ has no limit point in Ω.

Corollary $f, g \in \mathcal{H}(\Omega)$

Ω - domain.

If $\{ z \in \Omega \mid f(z) = g(z) \}$

has a limit point in Ω.

$\Rightarrow f \equiv g$ on Ω.

(Apply the previous theorem to

$F = f - g$)

(Identity Principle)

Application

① $\sin^2 z + \cos^2 z = 1$

$f(z) = \sin^2 z + \cos^2 z - 1$

$\in \mathcal{H}(\mathbb{C})$

$\mathbb{R} \subseteq Z_f$

$\Rightarrow$ each point of x-axis is a limit point of Z_f

$\Rightarrow f \equiv 0$

② $\sin(z+w) = \sin z \cos w + \cos w \sin z$

(3) Find all holomorphic function on $\mathbb{D}$ s.t $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ $n=2,3,4$

$\mathbb{D} : \{|z|<1\}$ $\qquad \mathbb{D} \xrightarrow{f} \mathbb{C}$

$$f(z) = z^2$$

If g is another such function.

then $\{f(z) = g(z)\} = \{\frac{1}{n} / n=2,3,4 \dots\}$

— has a limit point $0 \in \mathbb{D}$.

$\Rightarrow f \equiv g$.

(4) Does $\exists$ a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ s.t

$$f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n^2}.$$

Sol NO!

$f\left(\frac{1}{2n}\right) = \frac{1}{(2n)^2}$ $\Rightarrow f(z) = z^2$

$f\left(\frac{1}{2n+1}\right) = -\frac{1}{(2n+1)^2}$

Suppose $f \not\equiv 0$ on Ω.

& $f(z_0) = 0$. $z_0 \in \Omega$.

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

$$a_n = \frac{f^{(n)}(z_0)}{\lfloor n}.$$

Not all a_n's are zero.

$a_0 = 0$

$$\frac{1}{1-z} = 1 + z + z^2 - -$$

$|z| < 1$

$$\frac{1}{1-z} =$$

$\exists !\ k$ s.t

$$f(z) = \underline{(z - z_0)^k} \underbrace{\left[a_k + a_{k+1}(z - z_0) + a_{k+2}(z - z_0)^2 \cdots \right]}_{g(z)}$$

k = least integer s.t $a_k \neq 0$.

$$\boxed{f(z) = (z - z_0)^k g(z).}$$

$g(z_0) = a_k \neq 0$.

$\Rightarrow g(z) \neq 0$ in a nbd of z_0.

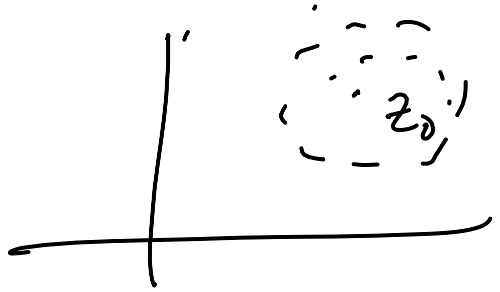
$\Rightarrow f(z) \neq 0$ in a nbd of z_0.

This k is called the order of the zero z_0.

Zeros of an holomorphic function are isolated points

If $f(z_0) = 0$, then $\exists$ nbd $|z - z_0| < r$ s.t. $f(z) \neq 0$

$\forall\ 0 < |z - z_0| < r$



Corollary $f, g \in \mathcal{H}(\Omega)$.

If $f \cdot g \equiv 0$ then

either $f \equiv 0$ on Ω

or $g \equiv 0$ on Ω.

$$(f \cdot g)(z) = f(z)\, g(z)$$

Proof Suppose $f \not\equiv 0$

$z_0 \in \Omega$

- $\to f(z_0) = 0$ $\Rightarrow$ Since zeros of f are isolated $\Rightarrow g(z_0) = 0$
- $\to f(z_0) \neq 0 \Rightarrow g(z_0) = 0$