MSO 202A : Complex Variables
Final Exam, 19th September 2022

Total Marks: 70

Time: 8 am - 10 am

- Answer all questions.
- Write each step clearly.

1. (a) Let $f, g : \overline{\mathbb{D}} \to \mathbb{C}^*$ be two analytic functions on the closed unit disc $\overline{\mathbb{D}} = \{z | \; |z| \leq 1\}$ such that $|f(z)| = |g(z)|$ for all $|z| = 1$. Then show that, there exists a $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} g(z)$ for all $z \in \overline{\mathbb{D}}$.

[8]

(b) Let $f(z) = e^{\cos z} z^2$ and A be the closed disc $|z - 5| \leq 2$. Show that $\max_{z \in A} |f(z)|$ and $\min_{z \in A} |f(z)|$ are attained on $|z - 5| = 2$.

[2+2]

(c) Compute the integral

$$\int_0^{2\pi} \frac{dt}{\cos(t) - 2}.$$

[6]

2. (a) Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a non-constant harmonic function. Show that u has at least one point (x_0, y_0) such that $u(x_0, y_0) = 0$.

[7]

(b) Let $f(z)$ be an entire function such that $|f(z)| > 1$ for all z. If $f(0) = 1$, then find the value of $f(1)$.

[5]

(c) Find the order of zero of the function $4\cos(z^4) + 2z^8 - 4$ at $z = 0$.

[3]

(d) Can a power series of the form $\sum a_n (z - 2)^n$ converge at $z = 6$ and diverge at $z = 2i$? Justify your answer. [3]

3. (a) Evaluate

$$\int_0^\infty \frac{dx}{1+x^7}.$$

[10]

(b) Calculate the residue of the function $z^n e^{10/z}$ at ∞, $\quad n \in \mathbb{N}$.

[2]

(c) Let f be an entire function satisfying $|f(z)| < |z|^n$ for all $|z| > M$. Show that f is a polynomial.

[6]

4. (a) Evaluate the integral $\int_C |z|^2 dz$ in the following two cases:

C: the line segment with initial point -1 and final point i.

C: the arc of the unit circle in $Im(z) \geq 0$ with initial point -1 and final point i.

[2+2]

(b) Suppose $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that $f(z) \neq 0$ for all z. Show that there exist a holomorphic function $g : \mathbb{D} \to \mathbb{C}$ such that $e^{g(z)} = f(z)$. Hence deduce that there exist a holomorphic function $h : \mathbb{D} \to \mathbb{C}$ such that $h^2(z) = f(z)$.

[6+2]

(c) Determine the domain of analyticity of the function $f(z) = \log_{\pi/2}(1+z)$. Expand it in power series about 0.

[2+2]