QUIZ 1 - MSO202M SOLUTIONS
IIT KANPUR - 2023–2024
DATE : 19 AUGUST, 2023, TIME : 60 MINS. (05:15–06:15 PM)
MAXIMUM MARKS: 30

(1) Let $a \in \mathbb{C}$ and $c > 0$ be a real number. Describe geometrically the set $S = \{z : |z-a| - |z+a| = 2c\}$ for following situations (i) $c > |a|$. (ii) $c = a > 0$. **[2 + 3 marks]**

Ans : (i) The triangle inequality gives that

$$|2a| = |z - a - (z+a)| \geq |z-a| - |z+a| = 2c, \text{ i.e., } c \leq |a| \quad \textbf{[1 marks]}.$$

Thus, there are no complex numbers in S if $c > |a|$. Hence $S = \emptyset$ **[1 mark]**.

(ii) $S = (-\infty, -a]$

$$\begin{aligned} & |z-a| - |z+a| = 2a \\ \iff & |z-a| = |z+a| + 2a \\ \iff & |z+a| = -Re(z+a) \quad \textbf{[2 marks]}\text{(After squaring both sides and simplifying)} \end{aligned}$$

This implies if $z \in S \iff Re(z+a) \leq 0 \iff Re(z) \leq -a$ **[1 mark]**.

(2) Let $f(z) = u(x,y) + iv(x,y)$ be defined by **[6 marks]**

$$f(z) = \begin{cases} \frac{(1+i/2)z^3}{|z|^2} & \textit{if } z \neq 0 \\ 0 & \textit{if } z = 0. \end{cases} \qquad \text{Then}$$

Ans :

(i) $u_x(0,0) = 1$ (ii) $u_y(0,0) = \frac{1}{2}$ (iii) $v_x(0,0) = \frac{1}{2}$ (iv) $v_y(0,0) = -1$

Solution:

First write $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, where

$$\begin{aligned} u(x,y) &= \frac{1}{x^2+y^2}\left(x^3 - 3xy^2 + \frac{1}{2}y^3 - \frac{3}{2}x^2y\right), \\ v(x,y) &= \frac{1}{x^2+y^2}\left(\frac{1}{2}x^3 - y^3 + 3x^2y - \frac{3}{2}xy^2\right). \qquad \textbf{[1+1 mark]} \end{aligned}$$

$$u_x(0,0) = \lim_{h\to 0} \frac{u(0+h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{u(h,0)}{h} = \lim_{h\to 0} \frac{h^3}{h^2.h} = 1. \quad \textbf{[1 mark]}$$

$$u_y(0,0) = \lim_{h\to 0} \frac{u(0,0+h) - u(0,0)}{h} = \lim_{h\to 0} \frac{u(0,h)}{h} = \lim_{h\to 0} \frac{h^3}{2h^2.h} = \frac{1}{2}. \quad \textbf{[1 mark]}$$

$$v_x(0,0) = \lim_{h\to 0} \frac{v(0+h,0) - v(0,0)}{h} = \lim_{h\to 0} \frac{v(h,0)}{h} = \lim_{h\to 0} \frac{h^3}{2h^2.h} = \frac{1}{2}. \quad \textbf{[1 mark]}$$

$$v_y(0,0) = \lim_{h\to 0} \frac{v(0,0+h)-v(0,0)}{h} = \lim_{h\to 0} \frac{v(0,h)}{h} = \lim_{h\to 0} \frac{-h^3}{h^2.h} = -1. \quad \textbf{[1 mark]}$$

(3) Let $u(x,y) = e^x \sin y$. Verify that u is harmonic. Find $f(z)$ and $f'(z)$, where $\Re f(z) = u(x,y)$. **[5 marks]**

Ans : (i) $f(z) = e^x \sin y - ie^x \cos y = -ie^z$ $\quad$ (ii) $f'(z) = f(z)$

Solution:

$$u_{xx}(x,y) = e^x \sin y, \qquad u_{yy}(x,y) = -e^x \sin y.$$

Verifying harmonic function condition : $u_{xx}(x,y) + u_{yy}(x,y) = 0.$ **[1 mark]**

Using C-R equations for $f(z) = u + iv$: $u_x(x,y) = e^x \sin y = v_y(x,y).$ **[1 mark]**

Integrating $v_y(x,y)$: $v(x,y) = -e^x \cos y + C(x)$ and getting $C(x) = 0$ using C-R equations and finally getting $f(z) = e^x \sin y - ie^x \cos y = -ie^z$ **[2 mark]**

Getting $f'(z) = u_x(x,y) + iv_x(x,y) = e^x \sin y - ie^x \cos y = -ie^z = f(z).$ **[1 mark]**

(4) Find the radius and domain of convergence of the following power series: **[6 marks]**

$$(i)\ \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} (z-i\pi)^n \quad (ii)\ \sum_{n=1}^{\infty} \left(\frac{2n+1}{3n+2}\right)(z-3)^n$$

Ans : (i) $R_1 = 1/e$

$$\begin{aligned} \frac{1}{R_1} &= \limsup |a_n|^{1/n} \quad \textbf{[1 mark]} \\ &= \limsup \left| \left(1+\frac{1}{n}\right) \right|^n = e \quad \textbf{[1 mark]}. \end{aligned}$$

Domain of convergence : $|z - i\pi| < \frac{1}{e}$ **[1 mark]**.

(ii) $R_2 = 1$

$$\begin{aligned} R_2 &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \quad \textbf{[1 mark]} \\ &= \lim_{n\to\infty} \frac{(2n+1)(3n+5)}{(3n+2)(2n+3)} = 1 \quad \textbf{[1 mark]} \end{aligned}$$

Domain of convergence : $|z-3| < 1$ **[1 mark]**.

(5) If $(-3)^{-i\sqrt{2}} = a + ib$, with $a, b \in \mathbb{R}$, then (where $z^s = \exp(s\,\mathrm{Log} z)$) **[5 marks]**

Ans : (i) $a = \exp(\pi\sqrt{2})\,\cos(\sqrt{2}\,\ln 3)$ (ii) $b = -\exp(\pi\sqrt{2})\,\sin(\sqrt{2}\,\ln 3)$

Solution :

$$(-3)^{-i\sqrt{2}} = \exp(-i\sqrt{2}\,\mathrm{Log}(-3)) = \exp(-i\sqrt{2}\,(\ln 3 + i\pi)) \qquad \textbf{[2 marks]}$$

$$= \exp(\pi\sqrt{2})\,\Big(\cos(\sqrt{2}\,\ln 3) - i\,\sin(\sqrt{2}\,\ln 3)\Big) = a + ib, \qquad \textbf{[2 mark]}$$

where

$$a = \exp(\pi\sqrt{2})\,\cos(\sqrt{2}\,\ln 3), \qquad b = -\exp(\pi\sqrt{2})\,\sin(\sqrt{2}\,\ln 3). \qquad \textbf{[1 mark]}$$

(6) Give example of complex numbers z_1 and z_2 such that $\sqrt{z_1 z_2} \neq \sqrt{z_1}\sqrt{z_2}$. **[3 marks]**

Ans : $(i) z_1 = -a$ $(ii) z_2 = -a$ where $a > 0$ is any real number. There are many other examples.