MSO 202A: Complex Variables
August-September 2022
Assignment-2

1. Show that absolute convergence implies convergence of a series.

2. Assume $a_n \neq 0$ except finitely many terms. Show that $1/R = \lim \frac{|a_{n+1}|}{|a_n|}$, provided the limit exists.

3. Let $z_n = x_n + \iota y_n$, where $x_n, y_n \in \mathbb{R}$. Show that $\sum z_n$ is convergent if and only if $\sum x_n$ and $\sum y_n$ are convergent. Moreover $\sum z_n = \sum x_n + \iota \sum y_n$. Use this to conclude that $e^{iy} = \cos y + \iota \sin y, \quad y \in \mathbb{R}$.

4. Given an example to show that $\operatorname{Log}(z^n) \neq n \operatorname{Log}(z)$.

5. Assume that we choose the branch $\sqrt{z} = e^{1/2 \operatorname{Log} z}$. Given an example to show that $\sqrt{z_1 z_2} \neq \sqrt{z_1}\sqrt{z_2}$.

6. Draw the domain and range of the complex log branches $\log_0, \log_{2\pi}, \log_{-2\pi}, \log_{-\pi}, \log_{\pi}, \log_{\pi/4}$. Calculate complex logrithm of -1 using the first 3 branches.

   Calculate complex logrithm of 1 using the last 3 branches. Can you do it using the first three branch?

7. Where is the function $f(z) = \log_{3\pi/2}(3 - 5z)$ analytic? What is $f(1)$ and $f(0)$.

8. Let $\Omega, U \subseteq \mathbb{C}$ be open sets such that $f : \Omega \to U$ f is bijective analytic with $f'(z) \neq 0$ and $f^{-1} = g$ is continuous. Then show that g is analytic.

   (Remark: Think of the situation $U = \mathbb{C}^*$ and $\Omega = \{z| \ -\pi < y < \pi\}, \quad f(z) = e^z, \ g =$ Log. We used this result to prove that Log or any other branch $\log_\alpha$ is analytic.)

9. Write the following in the form $a + \iota b$.

   (a) $\log(\operatorname{Log}\ \iota)$ (b) $(\iota)^{-\iota}$

10. Prove or disprove:
$$\lim_{z\to 0} z \sin\frac{1}{z} = 0$$

11. Prove that the each of the three series the radius of convergence is 1. Further show the following:

    (a) Show that $\sum_{n=1}^{\infty} n z^n$ does not converge at any point on the unit circle.

    (b) Show that $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges at all points on the unit circle except.

    (c) Show that $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges at all points on the unit circle except at $z = 1$.

12. Consider $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = e^{-\frac{1}{x^2}}$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. Then:

(a) Calculate $f', \quad f'', \quad f'''$.

(b) Prove derivative of $\frac{c}{x^p}e^{-1/x^2}$ consists of sum of terms of similar form. Hence deduce that $f^{(n)}(x)$ consists of sum terms of the form $\frac{c}{x^p}e^{-1/x^2}$ for different $c, p \in \mathbb{N}$.

(c) Prove that

$$\lim_{x\to 0} \frac{c}{x^p} e^{-1/x^2} = 0, \quad c, p \in \mathbb{N}.$$

(d) Deduce that $f^{(n)}(0) = 0$ for all n.

(e) Thus conclude that f is infinitely differentiable but f can not be represented by a power series..

[Recall: A real function $f : \mathbb{R} \to \mathbb{R}$ is said to be *real analytic* at x_0 if $f(x)$ can be written as a convergent power series $\sum a_n(x - x_0)^n$. We know that any (complex) analytic function is infinitely differentiable BUT there exists infinitely differentiable real valued function which is NOT real analytic.]

13. Prove that if p is a polynomial then

$$\lim_{z\to\infty} |p(z)| = \infty$$

. However,

$$\lim_{z\to\infty} |e^z| \neq \infty.$$

14. Identify the following series as a holomorphic function f(z):

$$(a) \sum_{n=1}^{\infty} nz^n, \quad (b) \sum_{n=1}^{\infty} n^2 z^n, \quad (c) \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{(2n+1)!}, \quad (d) \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{2^n n!}$$

15. Let $f(z) = \frac{1}{z(z-1)}$. Where is the function analytic? Can you write f as a power series around $z = 2\iota$? What is the radius of convergence of this power series?