MSO 202A: Complex Variables
August-September 2022
Assignment-5

Throughout C_R will denote the circle of radius R around origin, oriented counterclockwise. and $C_1 = C$.

1. **(T)**Find the max value of $|f|$ on $|z| \leq 1$ where $f(z) = z^2 + 3z - 1$. Where is it attained?

   Solution:

   By the Maximum modulus principle the max value of $|f|$ is attained on the boundary $|z| = 1$. Putting $z = e^{it}$ we have $|f|^2 = 11 - 2\cos 2t$, which is maximum for $t = \pi/2, 3\pi/2$. Hence max value of $|f|$ on $|z| \leq 1$ is $\sqrt{13}$ attained at e^{it} for $t = \pi/2, 3\pi/2$.

2. **Analytic functions satisfies Mean Value Property.** Let f be holomorphic function on a an open set containing the closed disc $|z - z_0| \leq r$. Show that u satisfies :

   $$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})dt.$$

   (Remark: $\frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})dt$ called the **Mean Value of** f over the circle $|z - z_0| = r$.)

   Solution:

   Use Cauchy's integral formula. Done in class in the proof of Maximum Modulus principle.

3. **(T)Harmonic functions satisfies Mean Value Property.** Let u be Harmonic function on a an open set Ω containing the closed disc $|z - z_0| \leq r$. Show that u satisfies :

   $$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})dt.$$

   Solution:

   We can find a $r' > r$ such that $\mathbb{D}_{r'}(z_0) : |z - z_0| < r' \subset \Omega$. Since $\mathbb{D}_{r'}(z_0)$ is simply connected, u admits a harmoninc conjugate on $\mathbb{D}_{r'}(z_0)$. So there exist an holomorpfic function $f = u + iv$ on $\mathbb{D}_{r'}(z_0)$. Apply the previous exercise and take the real part.

4. Find the mean value of the function $u(x, y) = x^2 - y^2 + x$ on the circle $|z - i| = 2$.

   Solution:

   Note that u is harmoninc on $\mathbb{C}$. So by the mean value property, we have the Mean value $= u(i) = u(0, 1) = -1$.

5. **(T)**Let f be a nowhere zero, entire function. Prove that there exists an entire function g such that $\exp(g) = f$.

Solution:

The function f'/f is entire. By Cauchy's Theorem, it has a primitive g that is $g' = f/f'$. Now $\frac{d}{dz}(fe^{-g}) = f'e^{-g} - fe^{-g}g' = 0$. Thus $fe^{-g} = constant = e^c$. So $f = e^{g+c}$.

6. Find out all entire functions f such that $f(x) = e^x$ for $x \in \mathbb{R}$.

Solution:

$f(z) = e^z$ is one such function. If g is any such function then $\mathbb{R} \subseteq \{z : \ f(z) = g(z)\}$. Each point of $\mathbb{R}$ is a limit point of the set $\{z : \ f(z) = g(z)\}$. Hence by Identity Princple $f = g$.

7. Find out the order of the all zeros for

(T)(a) $f(z) = z^2(e^{z^2} - 1)$ (b) $z \sin z$. (c) $\sin^3(z)/z$

Solution:

(a) Order of the zero $z = 0$ is 4 since the Taylor series about 0 start with z^4. Other zeros of $f(z)$ are given by $e^{z^2} = 1$ i .e $z^2 = 2k\pi i$. These are of order 1 since f' does not vanish at these points.

(b) Zeros are $z = k\pi$ with $k \in \mathbb{Z}$. The zero $z = 0$ is of order 2 are other zeros are simple zero, i.e. have order 1.

(c) The function is not defined at $z = 0$. The zeros of the function are given by $\sin z = 0$ that is $z = k\pi$ where $k = \pm 1, \pm 2, \cdots$. All these are zeros of order 3.

8. **(T)**Does there exist a holomorphic function f on $\mathbb{D}$ such that $Z_f = \{z \in \mathbb{D} : \ f(z) = 0\}$ is equal to the following set? Justify your answer.

(a) $S_1 = \{1/n : \ n \in \mathbb{N}\}$. (b) $S_2 = \{1 - 1/n : \ n \in \mathbb{N}\}$. (c) $S_1 = \{z \in \mathbb{D} : \mathrm{Re}(z) = 0\}$ (d) $S_4 = \{z \in \mathbb{D} : -1/2 < y < 1/2\}$.

Solution:

(a) If such a function f exist, then the zero set $Z_f = \{z : \ f(z) = 0\}$ has a limit point $z = 0$ which is in $\mathbb{D}$. So $f = 0$.

(b) $f(z) = \sin \frac{\pi}{z-1}$

(c) In this case $Z_f = \{z : \ f(z) = 0\}$ has a limit point in $\mathbb{D}$ and so $f = 0$.

(d) In this case $Z_f = \{z : \ f(z) = 0\}$ has a limit point in $\mathbb{D}$ and so $f = 0$.

9. **(T)**Evaluate $\int_\Gamma \frac{1}{z^4} dz$ where Γ is the part of clockwise oriented ellipse $\frac{(x-3)^2}{1} + \frac{y^2}{4} = 1$ lying on the upper half plane $y > 0$.

Solution: Let Γ^* be the clockwise oriented closed curve consisting of the part of the given ellipse in the upper half plane and the line segment L with initial point $(4, 0)$

and end point $(2,0)$. Since the function $1/z^4$ is analytic inside and on $\Gamma*$, we have by Cauchy's theorem

$\int\limits_{\Gamma*} \frac{1}{z^4}dz = \int\limits_{\Gamma} \frac{1}{z^4}dz + \int\limits_{L} \frac{1}{z^4}dz = 0$

So $\int\limits_{\Gamma} \frac{1}{z^4}dz = \int\limits_{-L} \frac{1}{z^4}dz = \int\limits_{2}^{4} \frac{1}{x^4}dx = 7/192.$

10. (**T**)Write down the power series expansion $\sum a_n(z-z_0)^n$ of the function $Log(z) = \log_{-\pi}(z)$ around $z_0 = -1+\iota$.

    What is the radius of convergence of the power series?

    For which values of z, we have $Log(z) = \sum a_n(z-z_0)^n$.?

    Solution:

    $f(z) = Logz$. So $f'(z) = 1/z, \quad f''(z) = -1/z^2, \cdots, f^{(n)}(z) = (-1)^{n-1}(n-1)!/z^{n+1}$

    $a_n = f^{(n)}(z_0)/n! = (-1)^{n-1}\frac{1}{nz_0^{n+1}}$

    Therefore by Ratio test the radius of convergence of the Taylor series is $R = \lim |a_n|/|a_{n+1}| = |z_0| = \sqrt{2}$. So the power series is convergent for $|z-z_0| \leq \sqrt{2}$.

    The function $Logz$ is analytic on the domain $\Omega = \mathbb{C}$-nonpositive x axis. The largest disc in Ω with center at z_0 is $|z-z_0| < 1$. Thus

    $$Logz = \sum(-1)^{n-1}\frac{1}{nz_0^{n+1}}(z-z_0)^n, \quad \text{for } |z-z_0| < 1.$$

11. (**T**)Evaluate the following integrals on the square γ, oriented on the counterclockwise direction with sides $x = \pm 2, y = \pm 2$.

    (a) $\int_\gamma \frac{\cos z}{z(z^2+8)}dz$ (b) $\int_\gamma \frac{\cosh z}{z^4}dz$.

    Solution:

    (a) Given integral $= \int\limits_{C} \frac{\cos z/(z^2+8)}{z}dz = 2\pi i \cos z/(z^2+8)|_{z=0} = \pi i/4.$

    (b) Given integral $= \frac{2\pi i}{3!}\frac{d^3}{dz^3}(\cosh z)|_{z=0} = 0$

12. Find out

    $$\int_C \frac{2z^3+z^2+4}{z^4+4z^2}; \quad C: |z-2| = 4$$

    Solution:

    The given integral $= \int\limits_{C}(\frac{1}{z^2} + \frac{1}{z+2i} + \frac{1}{z-2i})dz = 4\pi i.$