MSO205A PRACTICE PROBLEMS SET 8

Question 1. Let $X \sim Binomial(n, p)$ for some integer $n \geq 3$ and $p \in (0, 1)$. Compute $\mathbb{E}X(X - 1)(X - 2)$, if it exists.

Question 2. Verify that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Question 3. Let $X \sim N(\mu, \sigma^2)$ for $\mu \in \mathbb{R}, \sigma > 0$. Compute $\mathbb{E}X^k$ for $k = 2, 3, 4$. [Hint: When $X \sim N(0, 1)$, these moments has been computed in the lecture notes.]

Question 4. Fix $\alpha > 0, \beta > 0$ and let $X \sim Beta(\alpha, \beta)$. Compute the MGF of X, if it exists.

Question 5. Let $X \sim Beta(1, 1)$. Does the distribution of X match with any other distribution discussed in the lecture notes?

Question 6. An RV X has the MGF given by the following expressions. Identify the distribution of X.

(1) $M_X(t) = (1 - \frac{t}{2})^{-3}, \forall t < 2$.
(2) $M_X(t) = \frac{1}{3}e^{-t} + \frac{2}{3}, \forall t \in \mathbb{R}$.

Question 7. Let X be a continuous RV with $\mathbb{P}(X > 0) = 1$ and such that $\mu_1' = \mathbb{E}X$ exists. Prove that $\mathbb{P}(X > 2\mu_1') \leq \frac{1}{2}$.

Question 8. Let $x_1, x_2, \cdots, x_k > 0$ be distinct real numbers and let n be a positive integer. Using Jensen's inequality discussed in the lecture notes, show that

$$\left(\frac{x_1 + x_2 + \cdots + x_k}{k}\right)^n \leq \frac{x_1^n + x_2^n + \cdots + x_k^n}{k}$$

Question 9. Let $x_1, x_2, \cdots, x_k, p_1, p_2, \cdots, p_k > 0$ be such that $\sum_{i=1}^k p_i = 1$. Prove the classical AM-GM-HM inequality using the AM-GM-HM inequality for RVs discussed in the lecture notes,

$$\sum_{i=1}^k x_i p_i \geq \prod_{i=1}^k x_i^{p_i} \geq \frac{1}{\sum_{i=1}^k \frac{p_i}{x_i}}$$

Question 10. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X = (X_1, X_2, X_3) : \Omega \to \mathbb{R}^3$ be a 3-dimensional random vector. State and prove the non-decreasing property of the joint DF of X.