1. Week 2 Supplementary material

Example 1.1. Suppose we perform a random experiment with the following steps.

(a) Suppose that there are two urns. The first urn contains 3 red balls and 5 green balls. The second urn contains 6 red balls and 3 green balls. All balls of the same colour are identical.

(b) Suppose a fair die is rolled and if the outcome is 1 or 6, then the first urn is chosen. Otherwise, the second urn is chosen.

(c) Finally, 2 balls are drawn at random from the chosen urn.

We want to find the probability that both the balls drawn are red. Let E denote this event. Suppose U_1 and U_2 denote the events that the first urn and the second urn is chosen respectively. Then the events $U_i, i = 1, 2$ are mutually exclusive and exhaustive. Moreover, $\mathbb{P}(U_1) = \frac{2}{6} = \frac{1}{3}$ and $\mathbb{P}(U_2) = \frac{4}{6} = \frac{2}{3}$. Further,

$$\mathbb{P}(E \mid U_1) = \frac{\binom{3}{2}}{\binom{8}{2}} = \frac{3}{28}, \mathbb{P}(E \mid U_2) = \frac{\binom{6}{2}}{\binom{9}{2}} = \frac{15}{36} = \frac{5}{12}.$$

Then the required probability can be computed as an application of Theorem of Total Probability as

$$\mathbb{P}(E) = \mathbb{P}(U_1)\ \mathbb{P}(E \mid U_1) + \mathbb{P}(U_2)\ \mathbb{P}(E \mid U_2) = \frac{1}{3}\frac{3}{28} + \frac{2}{3}\frac{5}{12} = \frac{1}{28} + \frac{5}{18} = \frac{79}{252}.$$

Example 1.2. Consider a rare disease X that affects one in a million people. A medical test is used to test for the presence of the disease. The test is 99% accurate in the sense that if a person does not have this disease, the chance that the test shows positive is 1% and if the person has this disease, the chance that the test shows negative is also 1%. Suppose a person is tested for the disease and the test result is positive. Let A be the event that the person has the disease X. Let B be the event that the test shows positive. As per the given information, the given data may be summarized as follows.

$$\mathbb{P}(A) = 10^{-6},\ \mathbb{P}(B^c \mid A) = 0.01,\ \mathbb{P}(B \mid A^c) = 0.01.$$

Then

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A) = 1 - 10^{-6},\ \mathbb{P}(B \mid A) = 1 - \mathbb{P}(B^c \mid A) = 0.99.$$

We are interested in the conditional probability that the person has the disease, given that the test result is positive. Here, A and A^c are mutually exclusive and exhaustive. By the Bayes' Theorem, we have

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B \mid A)\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\mathbb{P}(A^c)} = \frac{0.99 \times 10^{-6}}{0.99 \times 10^{-6} + 0.01 \times (1 - 10^{-6})} = 0.000099.$$

The posterior probability $\mathbb{P}(A \mid B) \approx 10^{-4}$ is updated from the prior probability $\mathbb{P}(A) = 10^{-6}$.