1. Week 5 Supplementary material

Remark 1.1. Let X be an RV defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with law $\mathbb{P}_X$ and DF F_X. Consider the set $D := \{x \in \mathbb{R} : F_X \text{ is discontinuous at } x\}$. We claim that it is either finite or countably infinite.

Solution 1: For each $n = 1, 2, \cdots$, consider the set $D_n := \{x \in \mathbb{R} : F_X(x+) - F_X(x-) > \frac{1}{n}\} = \{x \in \mathbb{R} : F_X(x) - F_X(x-) > \frac{1}{n}\} = \{x \in \mathbb{R} : \mathbb{P}(X = x) > \frac{1}{n}\}$. Then $D = \cup_n D_n$.

Note that each D_n is a finite set. If not, then for some positive integer m, D_m is infinite and we can choose a sequence $\{x_k\}_k$ from D_m with distinct points. Write $A = \{x_k : k \geq 1\}$. Then $\mathbb{P}(X = x_k) > \frac{1}{m}, \forall k$ and

$$1 \geq \mathbb{P}(X \in A) = \sum_{k=1}^{\infty} \mathbb{P}(X = x_k) > \sum_{k=1}^{\infty} \frac{1}{m} = \infty,$$

which is a contradiction. Therefore, each D_n is a finite set. In fact, by the above argument, we have $\#D_n < n$.

Now, $D = \cup_{n=1}^{\infty} D_n$ and hence D is either finite or countably infinite.

Solution 2: For each $x \in D$, we have an open interval $(F_X(x-), F_X(x))$ contained in $[0, 1]$.

Let $x, y \in D$ with $x < y$. Then, we have $F_X(x-) < F_X(x) \leq F_X(y-) < F_X(y)$ and hence the open intervals $(F_X(x-), F_X(x))$ and $(F_X(y-), F_X(y))$ are pairwise disjoint. For each $x \in D$, choose a rational number $r_x \in (F_X(x-), F_X(x))$. All the chosen rationals r_x are therefore distinct.

By the above construction, the points x in D are in one-to-one correspondence with $r_x \in [0, 1]$. Since the set of rational numbers in $[0, 1]$ are countable, D must be finite or countably infinite.

Example 1.2. Consider the function $F : \mathbb{R} \to [0, 1]$ defined by

$$F(x) := \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{4} + \frac{x}{2}, \text{ if } 0 \leq x \leq 1, \\ \frac{1}{2} + \frac{x}{4}, \text{ if } 1 < x < 2, \\ 1, \text{ if } x \geq 2. \end{cases}$$

We have F is non-decreasing on $(-\infty, 0), [0, 1], (1, 2)$ and on $[2, \infty)$. Moreover, for $x < 0, y \in (0, 1), z \in (1, 2), w > 2$, we have

$$F(x) = 0 < \frac{1}{4} = F(0) < F(y) < \frac{1}{4} + \frac{1}{2} = F(1) < F(z) < 1 = F(2) = F(w).$$

Therefore F is non-decreasing on $\mathbb{R}$.

By definition, F is continuous on $(-\infty, 0), (0, 1), (1, 2)$ and on $(2, \infty)$. So possible discontinuities may arise at the points $0, 1, 2$. We check the right continuity at these points. We have

$$\begin{aligned}
F(0+) &= \lim_{h \downarrow 0} F(0 + h) = \lim_{h \downarrow 0} \left[\frac{1}{4} + \frac{h}{2}\right] = \frac{1}{4} = F(0), \\
F(1+) &= \lim_{h \downarrow 0} F(1 + h) = \lim_{h \downarrow 0} \left[\frac{1}{2} + \frac{1 + h}{4}\right] = \frac{1}{2} + \frac{1}{4} = F(1), \\
F(2+) &= \lim_{h \downarrow 0} F(2 + h) = 1 = F(2).
\end{aligned}$$

Therefore F is right continuous on $\mathbb{R}$.

Finally $\lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} 0 = 0$ and $\lim_{x \to \infty} F(x) = \lim_{x \to \infty} 1 = 1$. Hence, F is a distribution function.

Remark 1.3. Let X be an RV with DF F_X. Then, for all $x < y$, we have

$$\mathbb{P}(x < X \leq y) = F_X(y) - F_X(x),$$

$$\mathbb{P}(x < X < y) = F_X(y-) - F_X(x),$$

$$\mathbb{P}(x \leq X < y) = F_X(y-) - F_X(x-),$$

$$\mathbb{P}(x \leq X \leq y) = F_X(y) - F_X(x-).$$