1. Week 10 Supplementary material

Example 1.1. Let $Z = (X, Y)$ be a 2-dimensional continuous random vector with the joint p.d.f. of the form

$$f_Z(x,y) = \begin{cases} \alpha xy, \text{ if } 0 < x < y < 1 \\ 0, \text{ otherwise} \end{cases}$$

for some constant $\alpha \in \mathbb{R}$. For f_Z to take non-negative values, we must have $\alpha > 0$. Now,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_Z(x,y)\,dxdy = \int_{y=0}^{1}\int_{x=0}^{y} \alpha xy\,dxdy = \int_{y=0}^{1} \alpha\frac{y^3}{2}\,dy = \frac{\alpha}{8}.$$

For f_Z to be a joint p.d.f., we need $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_Z(x,y)\,dxdy = 1$ and hence $\alpha = 8 > 0$. Also note that for this value of α, f_Z takes non-negative values. The marginal p.d.f. f_X of X can now be computed as follows.

$$f_X(x) = \int_{-\infty}^{\infty} f_Z(x,y)\,dy = \begin{cases} \int_{y=x}^{1} 8xy\,dy, \text{ if } x \in (0,1) \\ 0, \text{ otherwise} \end{cases} = \begin{cases} 4x[1-x^2], \text{ if } x \in (0,1) \\ 0, \text{ otherwise} \end{cases}.$$

The marginal p.d.f. f_Y of Y follows by a similar computation.

$$f_Y(y) = \int_{-\infty}^{\infty} f_Z(x,y)\,dx = \begin{cases} \int_{x=0}^{y} 8xy\,dx, \text{ if } y \in (0,1) \\ 0, \text{ otherwise} \end{cases} = \begin{cases} 4y^3, \text{ if } y \in (0,1) \\ 0, \text{ otherwise} \end{cases}.$$

Observe that $f_Z(\frac{1}{2}, \frac{1}{2}) = 0$ and $f_X(\frac{1}{2})f_Y(\frac{1}{2}) = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$. Hence X and Y are not independent.

In this case, we have, for fixed $x \in (0,1)$,

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \frac{2xy}{x(1-x^2)}, \text{ if } y \in (x,1) \\ 0, \text{ otherwise.} \end{cases}$$

Example 1.2. Let $U = (X, Y, Z)$ be a 3-dimensional continuous random vector with the joint p.d.f. of the form

$$f_U(x,y,z) = \begin{cases} \alpha xyz, \text{ if } x, y, z \in (0,1) \\ 0, \text{ otherwise} \end{cases}$$

for some constant $\alpha \in \mathbb{R}$. For f_Z to take non-negative values, we must have $\alpha > 0$. Now,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_U(x,y,z)\,dxdydz = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} \alpha xyz\,dxdydz = \frac{\alpha}{8}.$$

For f_U to be a joint p.d.f., we need $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_U(x,y,z)\,dxdydz = 1$ and hence $\alpha = 8 > 0$. Also note that for this value of α, f_U takes non-negative values. The marginal p.d.f. f_X of X can now be computed as follows.

$$f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_U(x,y,z)\,dydz = \begin{cases} \int_{z=0}^{1}\int_{y=0}^{1} 8xyz\,dydz, & \text{if } x \in (0,1) \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 2x, & \text{if } x \in (0,1) \\ 0, & \text{otherwise} \end{cases}.$$

By the symmetry of $f_U(x,y,z)$ in the variables x, y and z, we conclude that $X \overset{d}{=} Y \overset{d}{=} Z$. Observe that $f_{X,Y,Z}(x,y,z) = f_X(x)f_Y(y)f_Z(z), \forall x,y,z$ and hence the RVs X, Y, Z are independent.

Theorem 1.3. *Let $X = (X_1, \ldots, X_p)$ be a p-dimensional continuous random vector with joint p.d.f. f_X. Suppose that $\{x \in \mathbb{R}^p : f_X(x) > 0\}$ can be written as a disjoint union $\cup_{i=1}^{k} S_i$ of open sets in $\mathbb{R}^p$.*

Let $h^j : \mathbb{R}^p \to \mathbb{R}, j = 1, \cdots, p$ be functions such that $h = (h^1, \cdots, h^p) : S_i \to \mathbb{R}^p$ is one-to-one with inverse $h_i^{-1} = ((h_i^1)^{-1}, \cdots, (h_i^p)^{-1})$ for each $i = 1, \cdots, k$. Moreover, assume that $(h_i^j)^{-1}, i = 1, 2, \cdots, k; j = 1, \cdots, p$ have continuous partial derivatives and the Jacobian determinant of the transformation

$$J_i := \begin{vmatrix} \frac{\partial (h_i^1)^{-1}}{\partial y_1}(t) & \cdots & \frac{\partial (h_i^1)^{-1}}{\partial y_p}(y) \\ \vdots & \vdots & \vdots \\ \frac{\partial (h_i^p)^{-1}}{\partial y_1}(y) & \cdots & \frac{\partial (h_i^p)^{-1}}{\partial y_p}(y) \end{vmatrix} \neq 0, \forall i = 1, \cdots, k.$$

Then the p-dimensional random vector $Y = (Y_1, \cdots, Y_p) = h(X) = (h^1(X), \cdots, h^p(X))$ is a continuous with joint p.d.f.

$$f_Y(y) = \sum_{i=1}^{k} f_X((h_i^1)^{-1}(y), \cdots, (h_i^p)^{-1}(y))\,|J_i|\,1_{h(S_i)}(y).$$

Example 1.4. Fix $\lambda > 0$. Let $X_1 \sim \mathit{Exponential}(\lambda)$ and $X_2 \sim \mathit{Exponential}(\lambda)$ be independent RVs defined on the same probability space. The joint distribution of (X_1, X_2) is given by the joint p.d.f.

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2) = \begin{cases} \frac{1}{\lambda^2}\exp(-\frac{x_1+x_2}{\lambda}), & \text{if } x_1 > 0, x_2 > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Consider the function

$$h(x_1,x_2) = \begin{cases} (x_1+x_2, \frac{x_1}{x_1+x_2}), \forall x_1 > 0, x_2 > 0, \\ 0, \text{otherwise.} \end{cases}$$

Here, $\{(x_1,x_2) \in \mathbb{R}^2 : f_{X_1,X_2}(x_1,x_2) > 0\} = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ and $h : \{(x_1,x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\} \to \mathbb{R}^2$ is one-to-one with range $(0,\infty) \times (0,1)$. The inverse function is $h^{-1}(y_1,y_2) = (y_1y_2, y_1(1-y_2))$ for $(y_1,y_2) \in (0,\infty) \times (0,1)$ with Jacobian determinant given by

$$J(y_1,y_2) = \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{vmatrix} = -y_1 \neq 0.$$

Now, $Y = (Y_1,Y_2) = h(X_1,X_2) = (X_1+X_2, \frac{X_1}{X_1+X_2})$ has the joint p.d.f. given by

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} f_{X_1,X_2}(y_1y_2, y_1(1-y_2))|J(y_1,y_2)|, & \text{if } y_1 > 0, y_2 \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{1}{\lambda^2}y_1\exp\left(-\frac{y_1}{\lambda}\right), & \text{if } y_1 > 0, y_2 \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

Now, we compute the marginal distributions. The marginal p.d.f. f_{Y_1} is given by

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\,dy_2 = \begin{cases} \frac{1}{\lambda^2}y_1\exp\left(-\frac{y_1}{\lambda}\right), & \text{if } y_1 > 0 \\ 0, & \text{otherwise} \end{cases}$$

and the marginal p.d.f. f_{Y_2} is given by

$$f_{Y_2}(y_2) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\,dy_1 = \begin{cases} 1, & \text{if } y_2 \in (0,1) \\ 0, & \text{otherwise} \end{cases}.$$

Therefore $Y_1 = X_1 + X_2 \sim Gamma(2, \lambda)$ and $Y_2 = \frac{X_1}{X_1+X_2} \sim Uniform(0, 1)$. Moreover,

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2), \forall (y_1, y_2) \in \mathbb{R}^2$$

and hence Y_1 and Y_2 are independent.

Remark 1.5. Note that $Exponential(\lambda)$ distribution is the same as $Gamma(1, \lambda)$ distribution. Using the above computation, we can identify a $Gamma(2, \lambda)$ RV as a sum of two independent RVs each having distribution $Gamma(1, \lambda)$. A more general property in this regard is mentioned in the practice problem set.