1. Week 7 Supplementary material

Example 1.1. Fix $c \in \mathbb{R}$. Let X be a discrete RV with p.m.f.

$$f_X(x) = \mathbb{P}(X = x) = \begin{cases} 1, & \text{if } x = c \\ 0, & \text{otherwise.} \end{cases}$$

Such RVs are called constant/degenerate RVs. Here, the support is a singleton set $S_X = \{c\}$ and $\sum_{x \in S_X} |x| f_X(x) = |c| < \infty$ and hence $\mathbb{E}X = \sum_{x \in S_X} x f_X(x) = c$.

Example 1.2. Let X be a discrete RV with p.m.f.

$$f_X(x) := \begin{cases} \frac{1}{6}, \forall x \in \{1, 2, 3, 4, 5, 6\} \\ 0, \text{ otherwise.} \end{cases}$$

Here, the support is $S_X = \{1, 2, 3, 4, 5, 6\}$, a finite set with all elements positive and hence $\sum_{x \in S_X} |x| f_X(x) = \sum_{x \in S_X} x f_X(x)$ is finite and

$$\mathbb{E}X = \sum_{x \in S_X} x f_X(x) = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{7}{2}.$$

Example 1.3. Let X be a discrete RV with p.m.f.

$$f_X(x) := \begin{cases} \frac{1}{2^x}, \forall x \in \{1, 2, 3, \cdots\} \\ 0, \text{ otherwise.} \end{cases}$$

Here, the support is $S_X = \{1, 2, 3, \cdots\}$, the set of natural numbers. To check the existence of $\mathbb{E}X$, we need to check the convergence of the series $\sum_{x \in S_X} |x| f_X(x) = \sum_{x=1}^{\infty} x \frac{1}{2^x}$. Now, the x-th term is $\frac{x}{2^x}$ and

$$\lim_{x \to \infty} \frac{\frac{x+1}{2^{x+1}}}{\frac{x}{2^x}} = \frac{1}{2} < 1.$$

By ratio test, we have the required convergence and the existence of $\mathbb{E}X$ follows.

Observe that

$$\mathbb{E}X = \sum_{x=1}^{\infty} x \frac{1}{2^x} = \frac{1}{2} + \sum_{x=2}^{\infty} x \frac{1}{2^x} = \frac{1}{2} + \sum_{x=1}^{\infty} (x+1) \frac{1}{2^{x+1}} = \frac{1}{2} + \frac{1}{2} \sum_{x=1}^{\infty} x \frac{1}{2^x} + \frac{1}{2} = 1 + \frac{1}{2}\mathbb{E}X,$$

which gives $\mathbb{E}X = 2$.

Note 1.4. It is fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Example 1.5. Let X be a discrete RV with p.m.f.

$$f_X(x) := \begin{cases} \frac{3}{\pi^2 x^2}, \forall x \in \{\pm 1, \pm 2, \pm 3, \cdots\} \\ 0, \text{ otherwise.} \end{cases}$$

Here, the support is $S_X = \{\pm 1, \pm 2, \pm 3, \cdots\}$. To check the existence of $\mathbb{E}X$, we need to check the convergence of the series $\sum_{x \in S_X} |x| f_X(x) = 2 \sum_{n=1}^{\infty} n \frac{3}{\pi^2 n^2} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n}$. However, this series diverges and hence $\mathbb{E}X$ does not exist.

Example 1.6. Let X be a continuous RV with the p.d.f.

$$f_X(x) = \begin{cases} 1, \text{ if } 0 < x < 1, \\ 0, \text{ otherwise.} \end{cases}$$

To check the existence of $\mathbb{E}X$, we need to check the existence of $\int_{-\infty}^{\infty} |x| f_X(x)\, dx$. Now,

$$\int_{-\infty}^{\infty} |x| f_X(x)\, dx = \int_0^1 x\, dx = \frac{1}{2}$$

and hence $\mathbb{E}X = \frac{1}{2}$.

Example 1.7. Let X be a continuous RV with the p.d.f.

$$f_X(x) = \frac{1}{2} e^{-|x|}, \forall x \in \mathbb{R}.$$

To check the existence of $\mathbb{E}X$, we need to check the existence of $\int_{-\infty}^{\infty} |x| f_X(x)\, dx$. Now,

$$\int_{-\infty}^{\infty} |x| f_X(x)\, dx = \int_{-\infty}^{\infty} |x| \frac{1}{2} e^{-|x|}\, dx = \int_0^{\infty} x e^{-x}\, dx = 1 < \infty$$

and hence $\mathbb{E}X$ exists and

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-\infty}^{\infty} x \frac{1}{2} e^{-|x|}\, dx = 0.$$

Example 1.8. Let X be a continuous RV with the p.d.f.

$$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \forall x \in \mathbb{R}.$$

To check the existence of $\mathbb{E}X$, we need to check the existence of $\int_{-\infty}^{\infty} |x| f_X(x)\, dx$. Now,

$$\int_{-\infty}^{\infty} |x| f_X(x)\, dx = \int_{-\infty}^{\infty} |x| \frac{1}{\pi}\frac{1}{1+x^2}\, dx = \frac{2}{\pi}\int_0^{\infty} \frac{x}{1+x^2}\, dx = \infty$$

and hence $\mathbb{E}X$ does not exist.

Example 1.9. Let X be a discrete RV with p.m.f.

$$f_X(x) := \begin{cases} \frac{1}{6}, \forall x \in \{1,2,3,4,5,6\} \\ 0, \text{ otherwise.} \end{cases}$$

Here, existence of $\mu_1' = \mathbb{E}X$ and $\mu_2' = \mathbb{E}X^2$ can be established by standard calculations. Moreover,

$$\mathbb{E}X = \sum_{x \in S_X} x f_X(x) = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$$

and

$$\mathbb{E}X^2 = \sum_{x \in S_X} x^2 f_X(x) = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{91}{6}.$$

Variance can now be computed using the relation $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$.

Example 1.10. In Example 1.6, we had shown $\mathbb{E}X = \frac{1}{2}$, where X is a continuous RV with the p.d.f.

$$f_X(x) = \begin{cases} 1, \text{ if } 0 < x < 1, \\ 0, \text{ otherwise.} \end{cases}$$

Now, $\mathbb{E}X^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^1 x^2\, dx = \frac{1}{3}$. Then $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

Example 1.11. Let X be a discrete RV with p.m.f.

$$f_X(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!}, & \text{if } x \in \{0,1,2,\ldots\} \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda > 0$. We have

$$M_X(t) = \mathbb{E}\left[e^{tX}\right] = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda\left(e^t - 1\right)} \ \forall\ t \in \mathbb{R}$$

since $A = \left\{t \in \mathbb{R} : \mathbb{E}\left(e^{tX}\right) < \infty\right\} = \mathbb{R}$. Now,

$$M_X^{(1)}(t) = \lambda e^t e^{\lambda\left(e^t-1\right)} \quad \text{and} \quad M_X^{(2)}(t) = \lambda e^t e^{\lambda\left(e^t-1\right)}\left(1 + \lambda e^t\right)\ \forall\ t \in \mathbb{R}.$$

Then,

$$\mu_1' = \mathbb{E}(X) = M_X^{(1)}(0) = \lambda,\ \mu_2' = \mathbb{E}(X^2) = M_X^{(2)}(0) = \lambda(1+\lambda),\ Var(X) = \mu_2 = \mu_2' - (\mu_1')^2 = \lambda.$$

Again, for $t \in \mathbb{R}$, $\psi_X(t) = \ln\left(M_X(t)\right) = \lambda\left(e^t - 1\right)$, which yields $\psi_X^{(1)}(t) = \psi_X^{(2)}(t) = \lambda e^t, \forall t \in \mathbb{R}$. Then, $\mu_1' = \mathbb{E}(X) = \lambda, \mu_2 = Var(X) = \lambda$. Higher order moments can be calculated by looking at higher order derivatives of M_X.

Example 1.12. Let X be a continuous RV with p.d.f.

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$M_X(t) = \mathbb{E}\left(e^{tX}\right) = \int_0^\infty e^{tx}e^{-x}\, \mathrm{d}x = \int_0^\infty e^{-(1-t)x}\, \mathrm{d}x = (1-t)^{-1} < \infty,\ \text{if } t < 1.$$

In particular, M_X exists on $(-1,1)$ and $A = \left\{t \in \mathbb{R} : \mathbb{E}\left(e^{tX}\right) < \infty\right\} = (-\infty, 1) \supset (-1,1)$. Now,

$$M_X^{(1)}(t) = (1-t)^{-2} \quad \text{and} \quad M_X^{(2)}(t) = 2(1-t)^{-3}, t < 1.$$

Then,

$$\mu_1' = \mathbb{E}(X) = M_X^{(1)}(0) = 1,\ \mu_2' = \mathbb{E}(X^2) = M_X^{(2)}(0) = 2,\ Var(X) = \mu_2 = \mu_2' - (\mu_1')^2 = 1.$$

Again, for $t < 1$, $\psi_X(t) = \ln\left(M_X(t)\right) = -\ln(1-t)$, which yields $\psi_X^{(1)}(t) = \frac{1}{1-t}, \psi_X^{(2)}(t) = \frac{1}{(1-t)^2}, \forall t < 1$. Then, $\mu_1' = \mathbb{E}(X) = 1, \mu_2 = Var(X) = 1$.

Now, consider the Maclaurin's series expansion for M_X around $t = 0$. We have

$$M_X(t) = (1-t)^{-1} = \sum_{r=0}^{\infty} t^r, \forall t \in (-1, 1)$$

and hence $\mu'_r = r!$, which is the coefficient of $\frac{t^r}{r!}$ in the above power series.

Example 1.13. Let X be a continuous RV with p.d.f.

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \forall x \in \mathbb{R}.$$

As observed earlier in Example 1.8, $\mathbb{E}X$ does not exist. Since the existence of moments is a necessary condition for the existence of MGF, we conclude that the MGF does not exist for this RV X.