1. Week 4 Supplementary material

Example 1.1. If we toss a fair coin twice independently, then the sample space is $\Omega = \{HH, HT, TH, TT\}$ with $\mathbb{P}(\{HH\}) = \mathbb{P}(\{HT\}) = \mathbb{P}(\{TH\}) = \mathbb{P}(\{TT\}) = \frac{1}{4}$. Consider the RV $X : \Omega \to \mathbb{R}$ which denotes the number of heads. Here,

$$X(HH) = 2, \quad X(HT) = X(TH) = 1, \quad X(TT) = 0.$$

Consider the induced probability measure $\mathbb{P} \circ X^{-1}$ on $\mathbb{B}$. We have

$$\mathbb{P} \circ X^{-1}(\{0\}) = \mathbb{P}(X^{-1}(\{0\})) = \mathbb{P}(\{TT\}) = \frac{1}{4},$$
$$\mathbb{P} \circ X^{-1}(\{1\}) = \mathbb{P}(X^{-1}(\{1\})) = \mathbb{P}(\{HT, TH\}) = \frac{1}{2},$$
$$\mathbb{P} \circ X^{-1}(\{2\}) = \mathbb{P}(X^{-1}(\{2\})) = \mathbb{P}(\{HH\}) = \frac{1}{4}.$$

More generally, for any $A \in \mathbb{B}$, we have

$$\mathbb{P} \circ X^{-1}(A) = \mathbb{P}(\{\omega : X(\omega) \in A\}) = \sum_{i \in \{0,1,2\} \cap A} \mathbb{P} \circ X^{-1}(\{i\}).$$

Then for all $x \in \mathbb{R}$, we have

$$F_X(x) = \mathbb{P}_X((-\infty, x])) = \sum_{i \in \{0,1,2\} \cap (-\infty, x]} \mathbb{P}_X(\{i\}) = \begin{cases} 0, \text{ if } x < 0, \\ \mathbb{P}_X(\{0\}), \text{ if } 0 \leq x < 1, \\ \mathbb{P}_X(\{0\}) + \mathbb{P}_X(\{1\}), \text{ if } 1 \leq x < 2, \\ \mathbb{P}_X(\{0\}) + \mathbb{P}_X(\{1\}) + \mathbb{P}_X(\{2\}), \text{ if } x \geq 2. \end{cases}$$

Therefore,

$$F_X(x) = \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{4}, \text{ if } 0 \leq x < 1, \\ \frac{3}{4}, \text{ if } 1 \leq x < 2, \\ 1, \text{ if } x \geq 2. \end{cases}$$

Example 1.2. Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} 0, \text{ if } x < 0, \\ x, \text{ if } 0 \le x < 1, \\ 1, \text{ if } x \ge 1. \end{cases}$$

The function is a constant on $(-\infty, 0)$ and on $[1, \infty)$. Moreover, it is non-decreasing in the interval $[0, 1)$. Further for $x < 0, y \in (0, 1), z > 1$, we have

$$F(x) = F(0) < F(y) < F(1) = F(z).$$

Hence, F in non-decreasing over $\mathbb{R}$. Again, by definition F is continuous on the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$. We check for right continuity at the points 0 and 1. We have

$$\lim_{h \downarrow 0} F(0 + h) = \lim_{h \downarrow 0} h = 0 = F(0), \quad \lim_{h \downarrow 0} F(1 + h) = \lim_{h \downarrow 0} 1 = 1 = F(1).$$

Hence, F is right continuous on $\mathbb{R}$. Finally, $\lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} 0 = 0$ and $\lim_{x \to \infty} F(x) = \lim_{x \to \infty} 1 = 1$. Hence, F is the DF of some RV. Later on, we shall identify the corresponding RV.

Example 1.3. Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{4} + \frac{x}{2}, \text{ if } 0 \le x \le 1, \\ \frac{1}{2} + \frac{x}{4}, \text{ if } 1 < x < 2, \\ 1, \text{ if } x \ge 2. \end{cases}$$

Assume that F is the DF of some RV X (the verification is left as an exercise in practice problem set 4). Since F is continuous on the intervals $(-\infty, 0), (0, 1), (1, 2)$ and $(2, \infty)$, discontinuities may arise only at the points $0, 1, 2$.

We have $F(0-) = \lim_{h \downarrow 0} F(0 - h) = 0$ and $F(0) = \frac{1}{4}$. Therefore F is discontinuous at 0 with jump $F(0) - F(0-) = \frac{1}{4}$.

We have $F(1-) = \lim_{h \downarrow 0} F(1 - h) = \lim_{h \downarrow 0}[\frac{1}{4} + \frac{1-h}{2}] = \frac{3}{4}$ and $F(1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$. Therefore F is continuous at 1.

We have $F(2-) = \lim_{h\downarrow 0} F(2-h) = \lim_{h\downarrow 0}[\frac{1}{2} + \frac{2-h}{4}] = 1$ and $F(2) = 1$. Therefore F is continuous at 2.

Only discontinuity of F is at the point 0. In particular, $\mathbb{P}(X = 0) = F(0) - F(0-) = \frac{1}{4}$. At all other points F is continuous and hence $\mathbb{P}(X = x) = 0, \forall x \neq 0$.

Observe that $\mathbb{P}(0 \leq X < 1) = F(1-) - F(0-) = \frac{3}{4}$. Again, $\mathbb{P}(\frac{3}{2} < X \leq 2) = F(2) - F(\frac{3}{2}) = 1 - [\frac{1}{2} + \frac{3}{8}] = \frac{1}{8}$.