MSO205A SOLUTIONS: MID SEMESTER EXAM SHORT ANSWER TYPE

Question 1: (1 + (2 + 2) + 2 + 1 + 2 marks) It is known that the function $F : \mathbb{R} \to \mathbb{R}$, defined by

$$F(x) = \begin{cases} 0, \text{ if } x < 0, \\ \beta x + \frac{1}{8}, \text{ if } 0 \leq x \leq 1, \\ \frac{x}{6} + \frac{1}{12}, \text{ if } 1 < x < 2, \\ \alpha x - \frac{1}{2}, \text{ if } 2 \leq x \leq 3, \\ 1, \text{ otherwise} \end{cases}$$

is a distribution function (DF) for some $\alpha, \beta \in \mathbb{R}$. Are α, β unique? $\boxed{\text{Yes/No}}$ (underline the correct answer). A possible value of (α, β) for which F is a DF is $\boxed{(\alpha,\beta) = \quad}$. For this (α, β), let γ denote the number of discontinuity points of F. Then, $\boxed{8\alpha + 4\beta + 2\gamma = \quad}$ and the corresponding random variable X is $\boxed{\text{discrete/ continuous / neither discrete nor continuous}}$ (underline the correct answer). Here, $\boxed{\mathbb{P}(X > \frac{3}{2} \mid X \leq \frac{5}{2}) = \quad}$

Answer: Using right-continuity of F at 1, we have $\frac{1}{6} + \frac{1}{12} = F(1+) = F(1) = \beta + \frac{1}{8}$, which gives $\beta = \frac{1}{8}$.

Using right-continuity of F at 3, we have $1 = F(3+) = F(3) = 3\alpha - \frac{1}{2}$, which gives $\alpha = \frac{1}{2}$.

In particular, α and β are uniquely determined. (Note: In the other set of questions, the roles of α and β have been exchanged. They are still uniquely determined, but $(\alpha, \beta) = (\frac{1}{8}, \frac{1}{2})$)

The DF is continuous on the intervals $(-\infty, 0), (0, 1), (1, 2), (2, 3)$ and $(3, \infty)$. We check for discontinuities at the points $0, 1, 2, 3$.

At 0, $F(0) - F(0-) = \frac{1}{8} - 0 = \frac{1}{8}$. Hence, F is discontinuous at 0.

Observe that $F(x) = \frac{x+1}{8}, \forall x \in [0, 1]$. Then, F is left-continuous and hence, continuous at 1.

At 2, $F(2) - F(2-) = (1 - \frac{1}{2}) - (\frac{2}{6} + \frac{1}{12}) = \frac{1}{4}$. Hence, F is discontinuous at 2.

Observe that $F(x) = \frac{x-1}{2}, \forall x \in [2, 3]$. Then, F is left-continuous and hence, continuous at 3.

We have $\gamma = 2$ and hence $8\alpha + 4\beta + 2\gamma = 8.5$. (Note: in the other set of questions, we get $8\alpha + 4\beta + 2\gamma = 7$.)

Now, $\mathbb{P}(X \leq \frac{5}{2}) = F(\frac{5}{2}) = \frac{5}{4} - \frac{1}{2} = \frac{3}{4}$. Also, $F(\frac{3}{2}) = \frac{3}{12} + \frac{1}{12} = \frac{1}{3}$. Then, $\mathbb{P}(\frac{3}{2} < X \leq \frac{5}{2}) = F(\frac{5}{2}) - F(\frac{3}{2}) = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$ and hence

$$\mathbb{P}(X > \tfrac{3}{2} \mid X \leq \tfrac{5}{2}) = \frac{\mathbb{P}(\frac{3}{2} < X \leq \frac{5}{2})}{\mathbb{P}(X \leq \frac{5}{2})} = \frac{5}{9}.$$

Question 2: (2 marks) Consider the following two functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} \frac{2}{3^x}, & \text{if } x \in \{1, 2, 3, \cdots\}, \\ 0, & \text{otherwise} \end{cases} \qquad g(x) := \frac{4}{\pi} \frac{1}{4^2 + x^2}, \quad \forall x \in \mathbb{R}.$$

Choose the correct option below, by putting a tick (✓) to get credit.

(i) f is a p.m.f. and g is a p.d.f.

(ii) g is a p.d.f., but f is not a p.m.f.

(iii) f is a p.m.f., but g is not a p.d.f.

(iv) f is not a p.m.f. and g is not a p.d.f.

Answer: The function f takes non-negative values and

$$\sum_{x=1}^{\infty} \frac{2}{3^x} = \frac{2}{3} \frac{1}{1 - \frac{1}{3}} = 1.$$

Hence, f is a p.m.f.

The function g takes non-negative values. Now, changing variables by $y = 4x$, we have

$$\int_{-\infty}^{\infty} \frac{4}{\pi} \frac{1}{4^2 + x^2} \, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1 + y^2} \, dy = 1.$$

Hence, g is p.d.f. Therefore, (i) is correct.

(Note: In the other set of questions, we get $\int_{-\infty}^{\infty} \frac{4^2}{\pi} \frac{1}{4^2+x^2} \, dx = 4$. Here, g is not a p.d.f. and (iii) is correct.)