1. Week 1 Supplementary material

Example 1.1 (Placing r balls in m bins)**.** Fix two positive integers r and m. Suppose that there are r labelled balls and m labelled bins/boxes/urns. Assume that each bin can hold all the balls, if required. One by one, we put the balls into the bins 'at random'. Then, by letting ω_i be the bin-number into which the i-th ball is placed, we can capture the full configuration by the vector $\underline{\omega} = (\omega_1, \ldots, \omega_r)$. Let Ω be the list of all configurations. Therefore, Ω is the sample space of this random experiment. We have

$$\Omega = \{\underline{\omega} : \underline{\omega} = (\omega_1, \ldots, \omega_r) \text{ with } 1 \leq \omega_i \leq m \text{ for each } 1 \leq i \leq r\}.$$

The cardinality of Ω is m^r (since each ball may be placed in one of the m bins). Since the experiment has been performed at random, we have $\mathbb{P}(\{\underline{\omega}\}) = p_{\underline{\omega}} = m^{-r}, \forall \underline{\omega} \in \Omega$. We now consider the probabilities of the following events.

(a) Let A be the event that the r-th ball is placed in the first bin. Then $A = \{\underline{\omega} \in \Omega : \omega_r = 1\}$. Here, balls numbered 1 to $r-1$ can be placed in any of the m bins. Therefore, the number of outcomes $\underline{\omega}$ favourable to A is m^{r-1}. Hence, $\mathbb{P}(A) = \frac{m^{r-1}}{m^r} = \frac{1}{m}$.

(b) Let B be the event that the first bin is empty. Then $B = \{\underline{\omega} \in \Omega : \omega_i \neq 1, \forall i = 1, 2, \cdots, r\}$. Here, each ball can be placed in any of the remaining bins numbered 2 to m. Since there are $m-1$ choices for each ball, the number of outcomes $\underline{\omega}$ favourable to B is $(m-1)^r$. Hence $\mathbb{P}(B) = \frac{(m-1)^r}{m^r}$.

(c) Consider $r \leq m$ and let C be the event that all the balls are placed in distinct bins, i.e. no bins contain more than one ball (a bin may remain empty). Then, $C = \{\underline{\omega} \in \Omega : \omega_i \neq \omega_j, \forall i \neq j\}$. Here, we are choosing/sampling bins for each ball and the sampling is being done without replacement. Hence, the number of outcomes $\underline{\omega}$ favourable to C is ${}^mP_r$. Hence $\mathbb{P}(C) = {}^mP_r \, m^{-r} = \frac{m(m-1)\cdots(m-r+1)}{m^r} = \frac{(m-1)\cdots(m-r+1)}{m^{r-1}}$.

Example 1.2 (Birthday Paradox)**.** There are n people at a party. What is the chance that two of them have the same birthday? Assume that none of them was born on a leap year and that days are equally likely to be a birthday of a person. The problem structure remains the same as in the previous balls in bin problem, where the bins are labelled as $1, 2, \ldots, 365$ (days of the year), and the balls are labelled as $1, 2, \ldots, n$ (people). In the notations of the previous example, $r = n$ and $m = 365$. Here, we wish to find the probability of the following event

$$D = \{\underline{\omega} \in \Omega : \omega_i = \omega_j, \text{for some } i \neq j\}.$$

Note that $D = C^c$, where C is as in the previous example. Therefore, $\mathbb{P}(D) = 1 - \mathbb{P}(C) = 1 - \frac{(365-1)\cdots(365-n+1)}{365^{n-1}}$. The reason this is called a 'paradox' is that even for n much smaller than 365, the probability becomes significantly large. For example, $n = 25$ gives $\mathbb{P}(D) > 0.5$.