## 1. Week 8 Supplementary material

**Example 1.1.** If $Y$ is an RV with the MGF $M_Y(t) = (1-t)^{-1}, \forall t \in (-1, 1)$, then recalling that the MGF (if it exists) determines the distribution uniquely, we conclude that $Y$ is a continuous RV with p.d.f.

$$f_Y(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

**Example 1.2.** If $X$ is a discrete RV with support $S_X$ and p.m.f. $f_X$, then the MGF $M_X$ is of the form

$$M_X(t) = \sum_{x \in S_X} e^{tx} f_X(x).$$

We can also make a converse statement. Since the MGF uniquely identifies a distribution, if an MGF is given by a sum of the above form, we can immediately identify the corresponding discrete RV with its support and p.m.f.. For example, if $M_X(t) = \frac{1}{2} + \frac{1}{3}e^t + \frac{1}{6}e^{-t}$, then $X$ is discrete with the p.m.f.

$$f_X(x) := \begin{cases} \frac{1}{2}, & \text{if } x = 0, \\ \frac{1}{3}, & \text{if } x = 1, \\ \frac{1}{6}, & \text{if } x = -1, \\ 0, & \text{otherwise.} \end{cases}$$

*Remark* 1.3. In general, the upper bound in Markov's inequality is very conservative. However, they can not be improved further. To see this, consider a discrete RV $X$ with p.m.f. given by

$$f_X(x) := \begin{cases} \frac{3}{4}, & \text{if } x = 0, \\ \frac{1}{4}, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathbb{P}(X \geq 1) = \frac{1}{4} = \mathbb{E}X$, which is sharp. If we consider

$$f_X(x) := \begin{cases} \frac{3}{4}, & \text{if } x = 0, \\ \frac{1}{4}, & \text{if } x = 2, \\ 0, & \text{otherwise,} \end{cases}$$

then, $\mathbb{P}(X \geq 1) = \frac{1}{4} < \frac{1}{2} = \mathbb{E}X$.

*Remark* 1.4. Some special cases of Jensen's inequality are of interest.

(a) Consider $h(x) = x^2, \forall x \in \mathbb{R}$. Here, $h''(x) = 2 > 0, \forall x$ and hence $h$ is convex on $\mathbb{R}$. Then $(\mathbb{E}X)^2 \leq \mathbb{E}X^2$, provided the expectations exist. We had seen this inequality earlier as a consequence of $Var(X)$ being non-negative.

(b) For any integer $n \geq 2$, consider the function $h(x) = x^n$ on $[0, \infty)$. Here, $h''(x) = n(n-1)x^{n-2} \geq 0, \forall x \in (0, \infty)$ and hence $h$ is convex. Then $(\mathbb{E}|X|)^n \leq \mathbb{E}|X|^n$, provided the expectations exist.

(c) Consider $h(x) = e^x, \forall x \in \mathbb{R}$. Here, $h''(x) = e^x > 0, \forall x$ and hence $h$ is convex on $\mathbb{R}$. Then $e^{\mathbb{E}X} \leq \mathbb{E}e^X$, provided the expectations exist.

(d) Consider any RV $X$ with $\mathbb{P}(X > 0) = 1$ and look at $h(x) := -\ln x, \forall x \in (0, \infty)$. Then $h''(x) = \frac{1}{x^2} > 0, \forall x \in (0, \infty)$ and hence $h$ is convex. Then $-\ln(\mathbb{E}X) \leq \mathbb{E}(-\ln X)$, i.e. $\ln(\mathbb{E}X) \geq \mathbb{E}(\ln X)$, provided the expectations exist.

(e) Consider any RV $X$ with $\mathbb{P}(X > 0) = 1$. Then $\mathbb{P}(\frac{1}{X} > 0) = 1$ and hence by (d), $-\ln(\mathbb{E}\frac{1}{X}) \leq \mathbb{E}(-\ln \frac{1}{X}) = \mathbb{E}(\ln X)$. Then $(\mathbb{E}\frac{1}{X})^{-1} = e^{-\ln(\mathbb{E}\frac{1}{X})} \leq e^{\mathbb{E}(\ln X)} \leq \mathbb{E}X$, by (c). This inequality holds, provided all the expectations exist. We may think of $\mathbb{E}X$ as the arithmetic mean (A.M.) of $X$, $e^{\mathbb{E}(\ln X)}$ as the geometric mean (G.M.) of $X$, and $\frac{1}{\mathbb{E}[\frac{1}{X}]}$ as the harmonic mean (H.M.) of $X$. The inequality obtained here is related to the classical A.M.-G.M.-H.M. inequality.