MSO205A Quiz 3 Solutions

1. Question 1 (1.5 + 1 + 1.5 + 1 marks)

Consider a continuous random vector (X, Y) with the joint p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} \frac{c}{\pi}, & \text{if } x^2 + y^2 < r^2 \ (r^2 = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{6}) \\ 0, & \text{otherwise,} \end{cases}$$

where c is a positive constant. Then $\boxed{c =\quad}$,

$\boxed{\mathbb{E}X =\quad}$ and $\boxed{Cov(X,Y) =\quad}$

Are the RVs X and Y independent? $\boxed{\text{Yes/No}}$ (underline the correct answer)

Solution: From $\int_{\mathbb{R}}\int_{\mathbb{R}} f_{X,Y}(x,y)\,dxdy = 1$ we have $\frac{c}{\pi}\pi r^2 = 1$. Therefore, $c = 2$ (Set 1), 3 (Set 2), 5 (Set 3), 6 (Set 4).

Now,

$$\mathbb{E}X = \int_{\mathbb{R}}\int_{\mathbb{R}} x f_{X,Y}(x,y)\,dxdy = 0$$

using the p.d.f. is symmetric and for the same reason, $\mathbb{E}Y = \int_{\mathbb{R}}\int_{\mathbb{R}} y f_{X,Y}(x,y)\,dxdy = 0$ and $\mathbb{E}XY = \int_{\mathbb{R}}\int_{\mathbb{R}} xy f_{X,Y}(x,y)\,dxdy = 0$.

Therefore, $Cov(X,Y) = \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y = 0$.

The marginal p.d.f. of X is given by

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dy = \begin{cases} \frac{2c}{\pi}\sqrt{r^2 - x^2}, & \text{if } |x| < r, \\ 0, & \text{otherwise.} \end{cases}$$

Y also has the same p.d.f.. Checking that $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$, we conclude X and Y are not independent.

2. Question 2 (3 marks)

Given that $\Phi(0.84) = 0.8, \Phi(0.525) = 0.7, \Phi(0.675) = 0.75$, where Φ denotes the DF of $X \sim N(0,1)$. By the symmetry of $N(0,1)$ distribution, we have $\Phi(-0.84) = 1 - 0.8 = 0.2, \Phi(-0.525) = 1 - 0.7 = 0.3, \Phi(-0.675) = 1 - 0.75 = 0.25$.

For fixed $a \neq 0, b > 0$ (the scalars in the question sets satisfy this condition), $Y = a + bX \sim N(a, b^2)$. Note that the quantiles for both X and Y are unique, since the corresponding DFs are one-to-one.

Let x_p and y_p denote the quantile of order p for X and Y, respectively. Then, $\mathbb{P}(X \leq x_p) = p$ and

$$p = \mathbb{P}(Y \leq y_p) = \mathbb{P}(a + bX \leq y_p) = \mathbb{P}(X \leq \frac{y_p - a}{b}).$$

Using uniqueness of the quantiles, we have $\frac{y_p - a}{b} = x_p$ or $y_p = a + bx_p$.

Solution:

(Set 1) $p = 0.25, x_p = -0.675, \alpha = y_p = 1 + 2(-0.675) = -0.35$ and therefore,

$$2\alpha = -0.7$$

(Set 2) $p = 0.2, x_p = -0.84, \alpha = y_p = 1 + 3(-0.84) = -1.52$ and therefore,

$$2\alpha = -3.04$$

(Set 3) $p = 0.25, x_p = -0.675, \alpha = y_p = -1 + 2(-0.675) = -2.35$ and therefore,

$$2\alpha = -4.7$$

(Set 4) $p = 0.3, x_p = -0.525, \alpha = y_p = -1 + 2(-0.525) = -2.05$ and therefore,

$$2\alpha = -4.1$$

3. Question 3 (4 marks)

Let X and Y be two non-degenerate discrete RVs defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the MGF M_X exists on $\mathbb{R}$, $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{E}Y^2 < \infty$. Which of the following statement(s) is/are necessarily true? Put a tick (✓) beside all correct statement(s) to get credit. No partial marking is applicable.

(a) $\exp(\mathbb{E}X) \leq M_X(1)$.
(b) $\left(\mathbb{E}X^3\right)^2 > \mathbb{E}X^6$
(c) $\mathbb{P}(X \geq \alpha) \leq e^{-\lambda\alpha} M_X(\lambda)$ for all $\alpha > 0, \lambda > 0$.
(d) $F_X(0) = 0$
(e) $\mathbb{E}(X+Y)^2$ may not exist.

Solution: only (a) and (c)

Since, the MGF M_X exists on $\mathbb{R}$, all moments exist for X. Now, (a) follows from Jensen's inequality and (c) from Chernoff's inequality.

Taking $Z = X^3$, $(\mathbb{E}X^3)^2 = (\mathbb{E}Z)^2 \leq \mathbb{E}Z^2 = \mathbb{E}X^6$. Hence, (b) is false.

If X is a Binomial or Poisson RV, then $F_X(0) \neq 0$ and this provides a counter-example to (d).

We have $\mathbb{E}X^2 < \infty, \mathbb{E}Y^2 < \infty$. Since $(X+Y)^2 \leq 2(X^2+Y^2)$, we have the existence of $\mathbb{E}(X+Y)^2$. Therefore, (e) is false.

4. Question 4 - (1.5 + 1.5 marks)

(Set 1) Let $X \sim N(1,4), Y \sim N(3,9), Z \sim N(-2,16)$ be independent RVs. Then

$\boxed{\left(\frac{Y-3}{3}\right)^2 + \left(\frac{Z+2}{4}\right)^2 \sim \qquad\qquad}$ and $\boxed{2\frac{X-1}{Z+2} \sim \qquad\qquad}$

Solution: Refer to sampling distributions and Problem 9 of Practice set 11.

$$\left(\frac{Y-3}{3}\right)^2+\left(\frac{Z+2}{4}\right)^2 \sim \chi^2_2 = Gamma(1,2), \quad 2\frac{X-1}{Z+2}=\frac{\frac{X-1}{2}}{\frac{Z+2}{4}} \sim t_1 = Cauchy(0,1)$$

(Set 2) Let $X \sim N(1,4), Y \sim N(3,9), Z \sim N(-2,16)$ be independent RVs and set $W = X + 2Y$. Then $\boxed{\frac{(\frac{Y-3}{3})^2+(\frac{Z+2}{4})^2}{2(\frac{X-1}{2})^2} \sim \quad}$ and $\boxed{F_W(W) \sim \quad}$ where F_W denotes the DF of W.

Solution: Refer to sampling distributions, Problem 6 of Practice set 10 and probability integral transform.

Here, $W \sim N(2\times 3+1, 2^2\times 9+4) = N(7,40)$ and

$$\frac{\left(\frac{Y-3}{3}\right)^2+\left(\frac{Z+2}{4}\right)^2}{2\left(\frac{X-1}{2}\right)^2} = \frac{\frac{(\frac{Y-3}{3})^2+(\frac{Z+2}{4})^2}{2}}{\left(\frac{X-1}{2}\right)^2} \sim F_{2,1}, \quad F_W(W) \sim Uniform(0,1).$$

(Set 3) Let $X \sim N(1,4), Y \sim N(3,9), Z \sim N(-2,16)$ be independent RVs and set $W = 2Y - X$. Then $\boxed{W \sim \quad}$. Moreover, $\beta\frac{Z+2}{W-5} \sim t_1$ for $\boxed{\beta = \quad}$

Solution: Refer to Problem 6 of Practice set 10 and Problem 9 of Practice set 11..

$$W \sim N(2\times 3-1, 2^2\times 9+4) = N(5,40)$$

and

$$\frac{\frac{z+2}{4}}{\frac{W-5}{\sqrt{40}}} \sim t_1 = Cauchy(0,1).$$

Therefore,

$$\beta = \frac{\sqrt{40}}{4} = \sqrt{\frac{5}{2}}$$

(Set 4) Let $X \sim N(1,4), Y \sim N(-2,16)$ be independent RVs. Then,

$\boxed{(\frac{X-1}{2})^2+(\frac{Y+2}{4})^2 \sim \quad}$ and $\boxed{\frac{1}{2}\frac{Y+2}{X-1} \sim \quad}$

Solution: Refer to sampling distributions and Problem 9 of Practice set 11.

$$\left(\frac{X-1}{2}\right)^2+\left(\frac{Y+2}{4}\right)^2 \sim \chi^2_2 = Gamma(1,2), \quad \frac{1}{2}\frac{Y+2}{X-1}=\frac{\frac{Y+2}{4}}{\frac{X-1}{2}} \sim t_1 = Cauchy(0,1)$$