1. Week 6 Supplementary material

Example 1.1. Consider a continuous RV X with DF $F_X : \mathbb{R} \to [0,1]$ and p.d.f. $f_X : \mathbb{R} \to [0,\infty)$ given by

$$F_X(x) := \begin{cases} 0, \text{ if } x < 0, \\ x, \text{ if } 0 \le x < 1, \\ 1, \text{ if } x \ge 1. \end{cases}, \qquad f_X(x) := \begin{cases} 1, \text{ if } 0 < x < 1, \\ 0, \text{ otherwise.} \end{cases}$$

To identify the support S, we consider the following cases.

(a) Let $x \in (-\infty, 0)$. Then for all h with $-x > h > 0$, we have $x - h < x + h < 0$ and consequently, $F_X(x+h) - F_X(x-h) = 0 - 0 = 0$. Therefore $x \notin S$.

(b) Let $x \in (1, \infty)$. Then for all $0 < h < x - 1$, we have $1 < x - h < x + h$ and consequently, $F_X(x+h) - F_X(x-h) = 1 - 1 = 0$. Therefore $x \notin S$.

(c) Let $x \in (0,1)$. For any $0 < h < \min\{x, 1-x\}$, we have $0 < x - h < x + h < 1$ and consequently, $F_X(x+h) - F_X(x-h) = (x+h) - (x-h) = 2h > 0$. For $h \ge \min\{x, 1-x\}$, at least one of $x-h, x+h$ is in $(0,1)^c$ and hence $F_X(x+h) - F_X(x-h) > 0$. Therefore $x \in S$.

(d) Let $x = 0$. Then for any $h > 0$, we have $F_X(0+h) - F_X(0-h) = F_X(0+h) > 0$. Then $0 \in S$. By a similar argument, $1 \in S$.

From the above discussion, we conclude that $S = [0,1]$.

Example 1.2. Let X be a discrete RV with p.m.f.

$$f_X(x) := \begin{cases} \frac{|x|}{110} & \text{if } x \in \{\pm 1, \pm 2, \dots, \pm 10\} \\ 0, & \text{otherwise} \end{cases}$$

and take $h : \mathbb{R} \to \mathbb{R}$ as $h(x) := |x|, \forall x \in \mathbb{R}$. Note that

$$h^{-1}((-\infty, y]) = \begin{cases} \emptyset, \text{ if } y < 0, \\ \{0\}, \text{ if } y = 0, \\ [-y, y], \text{ if } y > 0. \end{cases}$$

Then the DF of $Y = h(X) = |X|$ is given by

$$\begin{aligned} F_Y(y) &= \mathbb{P}(X \in h^{-1}((-\infty, y])) \\ &= \begin{cases} \mathbb{P}(X \in \emptyset), \text{ if } y < 0, \\ \mathbb{P}(X \in \{0\}), \text{ if } y = 0, \\ \mathbb{P}(X \in [-y, y]), \text{ if } y > 0. \end{cases} \\ &= \begin{cases} 0, \text{ if } y < 0, \\ \mathbb{P}(X = 0), \text{ if } y = 0, \\ \sum_{t \in [-y,y] \cap \{\pm 1, \pm 2, \ldots, \pm 10\}} f_X(t), \text{ if } y > 0. \end{cases} \\ &= \begin{cases} 0, \text{ if } y \leq 0, \\ \sum_{t \in [-y,y] \cap \{\pm 1, \pm 2, \ldots, \pm 10\}} \frac{|t|}{110}, \text{ if } y > 0. \end{cases} \end{aligned}$$

From the structure of the DF we conclude that the RV is discrete. The p.m.f. may be computed using the techniques discussed in the lectures.

Example 1.3. Let X be a continuous RV with p.d.f.

$$f_X(x) = \begin{cases} \frac{|x|}{2}, & \text{if } -1 < x < 1 \\ \frac{x}{3}, & \text{if } 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}$$

and take $h : \mathbb{R} \to \mathbb{R}$ as $h(x) := x^2, \forall x \in \mathbb{R}$. Note that

$$h^{-1}((-\infty, y]) = \begin{cases} \emptyset, \text{ if } y < 0, \\ \{0\}, \text{ if } y = 0, \\ [-\sqrt{y}, \sqrt{y}], \text{ if } y > 0. \end{cases}$$

Then the DF of $Y = h(X) = X^2$ is given by

$$F_Y(y) = \mathbb{P}(X \in h^{-1}((-\infty, y]))$$

$$= \begin{cases} \mathbb{P}(X \in \emptyset), \text{ if } y < 0, \\ \mathbb{P}(X \in \{0\}), \text{ if } y = 0, \\ \mathbb{P}(X \in [-\sqrt{y}, \sqrt{y}]), \text{ if } y > 0. \end{cases}$$

$$= \begin{cases} 0, \text{ if } y < 0, \\ \mathbb{P}(X = 0), \text{ if } y = 0, \\ \mathbb{P}(\{-\sqrt{y} \leq X \leq \sqrt{y}\}), \text{ if } y > 0. \end{cases}$$

$$= \begin{cases} 0, \text{ if } y < 0, \\ 0, \text{ if } y = 0, \\ \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\, dx, \text{ if } y > 0. \end{cases}$$

$$= \begin{cases} 0, \text{ if } y < 0, \\ 0, \text{ if } y = 0, \\ \int_{-\sqrt{y}}^{\sqrt{y}} \frac{|x|}{2} dx, \text{ if } 0 \leq y < 1 \\ \int_{-1}^{1} \frac{|x|}{2}\, dx + \int_{1}^{\sqrt{y}} \frac{x}{3} dx, \text{ if } 1 \leq y < 4 \\ 1, \text{ if } y \geq 4 \end{cases}$$

$$= \begin{cases} 0, \text{ if } y \leq 0, \\ \frac{y}{2}, \text{ if } 0 \leq y < 1 \\ \frac{y+2}{6}, \text{ if } 1 \leq y < 4 \\ 1, \text{ if } y \geq 4. \end{cases}$$

From the structure of the DF we conclude that the RV is continuous. The p.d.f. may be computed using the techniques discussed in the lectures.

Example 1.4. Let X be a discrete RV with p.m.f.

$$f_X(x) = \begin{cases} \frac{1}{7}, & \text{if } x \in \{-2,-1,0,1\} \\ \frac{3}{14}, & \text{if } x \in \{2,3\} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the RV $Y = X^2$. Here $S_X = \{-2,-1,0,1,2,3\}$ and $S_Y = \{0,1,4,9\}$. Observe that,

$$\begin{aligned}
\mathbb{P}(Y=0) &= \mathbb{P}\left(X^2=0\right) = \mathbb{P}(X=0) = \tfrac{1}{7}, \\
\mathbb{P}(Y=1) &= \mathbb{P}\left(X^2=1\right) = \mathbb{P}(X \in \{-1,1\}) = \tfrac{1}{7} + \tfrac{1}{7} = \tfrac{2}{7}, \\
\mathbb{P}(Y=4) &= \mathbb{P}\left(X^2=4\right) = \mathbb{P}(X \in \{-2,2\}) = \tfrac{1}{7} + \tfrac{3}{14} = \tfrac{5}{14} \\
\mathbb{P}(Y=9) &= \mathbb{P}\left(X^2=9\right) = \mathbb{P}(X \in \{-3,3\}) = 0 + \tfrac{3}{14} = \tfrac{3}{14}.
\end{aligned}$$

Therefore, the p.m.f. of Y is

$$f_Y(y) = \begin{cases} \frac{1}{7}, & \text{if } y = 0 \\ \frac{2}{7}, & \text{if } y = 1 \\ \frac{5}{14}, & \text{if } y = 4 \\ \frac{3}{14}, & \text{if } y = 9 \\ 0, & \text{otherwise,} \end{cases}$$

and the DF of Y is

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 0 \\ \frac{1}{7}, & \text{if } 0 \le y < 1 \\ \frac{3}{7}, & \text{if } 1 \le y < 4 \\ \frac{11}{14}, & \text{if } 4 \le y < 9 \\ 1, & \text{if } y \ge 9. \end{cases}$$

In fact, after identifying S_Y, we could have directly computed the DF F_Y as follows:

$$F_Y(y) = \mathbb{P}(Y \leq y) = \begin{cases} 0, \text{ if } y < 0, \\ \mathbb{P}(Y = 0), \text{ if } 0 \leq y < 1, \\ \mathbb{P}(Y = 0) + \mathbb{P}(Y = 1), \text{ if } 1 \leq y < 4, \\ \mathbb{P}(Y = 0) + \mathbb{P}(Y = 1) + \mathbb{P}(Y = 4), \text{ if } 4 \leq y < 9, \\ 1, \text{ if } y \geq 9. \end{cases}$$

and the p.m.f. f_Y from F_Y using standard techniques discussed in the lectures.

Example 1.5. Let X be a discrete RV with p.m.f.

$$f_X(x) = \begin{cases} \frac{x}{55} & \text{if } x \in \{1, 2, \ldots, 10\} \\ 0, & \text{otherwise.} \end{cases}$$

Now consider the RV $Y = X^2$. Note that the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) := x^2, \forall x \in \mathbb{R}$ is one-to-one on the support S_X. Here, Y is discrete with support $S_Y = \{1, 4, 9, \cdots, 100\}$. Hence, the p.m.f. f_Y is given by

$$f_Y(y) = \begin{cases} f_X(\sqrt{y}), \text{ if } y \in S_Y, \\ 0, \text{ otherwise} \end{cases} = \begin{cases} \frac{\sqrt{y}}{55}, \text{ if } y \in S_Y, \\ 0, \text{ otherwise} \end{cases}.$$

The DF F_Y can now be computed from the p.m.f. f_Y using standard techniques.

Example 1.6. Let X be a continuous RV with p.d.f.

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and consider $Y = X^2$. Here, $S_X = [0, \infty)$ and the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) := x^2, \forall x \in \mathbb{R}$ is continuous differentiable on $(0, \infty)$. Moreover, $h'(x) = 2x > 0, \forall x \in (0, \infty)$ and hence h is strictly monotone increasing on $(0, \infty)$. The inverse function is given by $h^{-1}(y) = \sqrt{y}, \forall y \in (0, \infty)$.

The p.d.f. f_Y is given by

$$f_Y(y) = \begin{cases} \frac{e^{-\sqrt{y}}}{2\sqrt{y}}, & \text{if } y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The DF F_Y can now be computed from the p.d.f. f_Y by standard techniques.

Example 1.7. Let X be a continuous RV with p.d.f.

$$f_X(x) = \begin{cases} \frac{|x|}{2}, & \text{if } -1 < x < 1 \\ \frac{x}{3}, & \text{if } 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}$$

and consider $Y = X^2$.

Observe that $\{x \in \mathbb{R} : f_X(x) > 0\} = (-1, 0) \cup (0, 2)$. Now, $h(x) = x^2$ is strictly decreasing on $(-1, 0)$ with inverse function $h_1^{-1}(t) = -\sqrt{t}$; and $h(x) = x^2$ is strictly increasing on $(0, 2)$ with inverse function $h_2^{-1}(t) = \sqrt{t}$. Note that $h\left((-1, 0)\right) = (0, 1)$ and $h\left((0, 2)\right) = (0, 4)$. Then, $Y = X^2$ has p.d.f. given by

$$\begin{aligned} f_Y(y) &= f_X(-\sqrt{y}) \left| \frac{d}{dy}(-\sqrt{y}) \right| 1_{(0,1)}(y) + f_X(\sqrt{y}) \left| \frac{d}{dy}(\sqrt{y}) \right| 1_{(0,4)}(y) \\ &= \begin{cases} \frac{1}{2}, & \text{if } 0 < y < 1 \\ \frac{1}{6}, & \text{if } 1 < y < 4 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We can now compute the DF of Y and verify that this matches with our earlier computation in Example 1.3.