MSO205A PRACTICE PROBLEMS SET 11

Question 1. Let $X_i \sim Poisson(\lambda_i), i = 1, 2, \cdots, n$ be independent RVs, with $\lambda_i > 0, \forall i$. Show that $X_1 + X_2 + \cdots + X_n \sim Poisson(\sum_{i=1}^n \lambda_i)$.
(Note: A special case of this result is the following: If $X_1, X_2, \cdots, X_n$ be a random sample from $Poisson(\lambda)$ distribution, then $X_1 + X_2 + \cdots + X_n \sim Poisson(n\lambda)$.)

Question 2. Let $X \sim Poisson(\lambda), Y \sim Poisson(\mu)$ be independent RVs. Find the conditional distribution of X given $X + Y = k$ for $k = 0, 1, \cdots$.

Question 3. Let X, Y be RVs defined on the same probability space. Fix $a, b, c, d \in \mathbb{R}$ and set $U = a + bX, V = c + dY$. Express $\rho(U, V)$ in terms of $\rho(X, Y)$.

Question 4. Compute the factorial moments for Negative Binomial and Hypergeometric distribution.

Question 5. Suppose a pair of fair die are rolled seven times independently. Find the probability that the sum of the dots obtained is 12 once and 8 twice.

Question 6. If $X_1, X_2, \cdots, X_n$ are i.i.d. $Geometric(p)$ RVs, for some $p > 0$, then find the distribution of $X_1 + X_2 + \cdots + X_n$.

Question 7. Let X be a continuous RV with p.d.f. f_X. If X is symmetric about $\mu \in \mathbb{R}$ and if $\mathbb{E}X$ exists, show that

$$\mathbb{E}X = \mu = m = \frac{\mathfrak{z}_{0.25} + \mathfrak{z}_{0.75}}{2},$$

where $m, \mathfrak{z}_{0.25}, \mathfrak{z}_{0.75}$ denotes the median, the lower and upper quartiles respectively. Assume that these are unique.

Question 8. Let X be an RV with $\mathbb{E}|X| < \infty$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) := \mathbb{E}|X - x|, x \in \mathbb{R}$. Show that $g(m) \leq g(x), \forall x \in \mathbb{R}$, where m is the median of X. (Note: This shows that the mean deviation is minimized at the median).

Question 9. Let X and Y be i.i.d. $N(0, 1)$ RVs. Identify the distribution of $\frac{X}{Y}$ and $\frac{X}{|Y|}$.

Question 10. Let $X \sim F_{m,n}$. Identify the distribution of $\frac{n}{n+mX}$.

Question 11. Let X and Y be i.i.d. *Exponential*(λ) RVs, for some $\lambda > 0$. Identify the distribution of $\frac{X}{Y}$.

Question 12. Verify that for a discrete RV X with the DF

$$F_X(x) := \begin{cases} 0, \text{ if } x < 0, \\ \frac{1}{2}, \text{ if } 0 \leq x < 1, \\ 1, \text{ if } x \geq 1, \end{cases}$$

the median is not unique. Given $p \in (0,1)$, construct an example of discrete RV X (by specifying the DF F_X or the p.m.f. f_X) such that the quantile of order p is not unique.

Question 13. Verify that for a continuous RV X with the DF

$$F_X(x) := \begin{cases} 0, \text{ if } x < 0, \\ \frac{x}{2}, \text{ if } 0 \leq x < 1, \\ \frac{1}{2}, \text{ if } 1 \leq x < 2, \\ \frac{x-1}{2}, \text{ if } 2 \leq x < 3, \\ 1, \text{ if } x \geq 3, \end{cases}$$

the median is not unique. Given $p \in (0,1)$, construct an example of continuous RV X (by specifying the DF F_X or the p.d.f. f_X) such that the quantile of order p is not unique.

Question 14. Consider the set

$$A := \left\{ t = (t_1, t_2, \ldots, t_p) \in \mathbb{R}^p : \mathbb{E}\left(e^{\sum_{i=1}^p t_i X_i} \right) < \infty \right\}$$

for a given random vector $X = (X_1, X_2, \cdots, X_p)$ and look at $\Psi_X(t) := \ln M_X(t), t \in A$. Verify that

$$\left[\frac{\partial^2}{\partial t_i \partial t_j} \Psi_X(t) \right]_{(t_1, t_2 \ldots t_p) = (0, \ldots, 0)} = Cov(X_i, X_j).$$