MSO205A PRACTICE PROBLEMS SET 10

Question 1. Let $X = (X_1, X_2, X_3)$ be a continuous random vector with joint p.d.f.

$$f_X(x_1, x_2, x_3) = \begin{cases} \frac{\alpha}{x_1 x_2}, & \text{if } 0 < x_3 < x_2 < x_1 < 1, \\ 0, & \text{otherwise} \end{cases}$$

for some constant $\alpha \in \mathbb{R}$. Find the value of α and identify the marginal p.d.f.s of X_1, X_2 and X_3. Are X_1, X_2, X_3 independent? If not independent, find the conditional DF and conditional p.d.f. of X_2 given $(X_1, X_3) = (x_1, x_3)$ with $0 < x_3 < x_1 < 1$.

Question 2. Let $X = (X_1, X_2)$ be a bivariate continuous random vector with joint p.d.f. given by

$$f_X(x_1, x_2) = \begin{cases} 1, & \text{if } 0 < |x_2| \leq x_1 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the marginal p.d.f.s of X_1 and X_2 and show that X_1, X_2 are not independent.

Question 3. Let $X \sim Exponential(\lambda)$ for some $\lambda > 0$. For $r, s > 0$, show that

$$\mathbb{P}(X > r + s \mid X > r) = \mathbb{P}(X > s).$$

Note: This property is usually referred to as the 'no memory' property of the Exponential distribution.

Question 4. Let $X_i \sim Gamma(\alpha_i, \beta), i = 1, 2, \cdots, n$ be independent RVs, with $\alpha_i > 0, \forall i$ and $\beta > 0$. Show that $X_1 + X_2 + \cdots + X_n \sim Gamma(\sum_{i=1}^n \alpha_i, \beta)$.

Question 5. Let $X \sim Gamma(\alpha_1, \beta), Y \sim Gamma(\alpha_2, \beta)$ be independent RVs, for some $\alpha_1, \alpha_2, \beta > 0$. Identify the distribution of $\frac{X}{X+Y}$.

Question 6. If $X_1, X_2, \cdots, X_n$ are independent RVs with $X_i \sim N(\mu_i, \sigma_i^2)$, then find the distribution of $X_1 + X_2 + \cdots + X_n$.

Question 7. Let X and Y be i.i.d. $N(0,1)$ RVs. Fix $a \neq 0, b \neq 0$ and set $U := aX + bY, V := bX - aY$. Find the joint p.d.f. of U, V. Are U and V independent?