1. Week 9 Supplementary material

Notation 1.1. Let $\prod_{j=1}^p (a_j, b_j]$ be a rectangle in $\mathbb{R}^p$. Observe that the co-ordinates of the vertices are made up of either a_j or b_j for each $j = 1, 2, \cdots, p$. Let Δ_k^p denote the set of vertices where exactly k many a_j's appear. Then the complete set of vertices is $\cup_{k=0}^p \Delta_k^p$. For example,

$$\Delta_0^2 = \{(b_1, b_2)\}, \quad \Delta_1^2 = \{(a_1, b_2), (b_1, a_2)\}, \quad \Delta_2^2 = \{(a_1, a_2)\}.$$

Proposition 1.2. *Let $X = (X_1, X_2, \cdots, X_p) : \Omega \to \mathbb{R}^p$ be a p-dimensional random vector. Let $a_1 < b_1, a_2 < b_2, \cdots, a_p < b_p$. Then, the non-decreasing property for F_X is stated as follows:*

$$\mathbb{P}(X \in \prod_{j=1}^p (a_j, b_j]) = \sum_{k=0}^p (-1)^k \sum_{x \in \Delta_k^p} F_X(x) = \mathbb{P}(a_1 < X_1 \leq b_1, a_2 < X_2 \leq b_2, \ldots, a_p < X_p \leq b_p) \geq 0.$$

Example 1.3. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) := \begin{cases} 0, & \text{if } x < 0 \text{ or } y < 0 \text{ or } x + y < 1, \\ 1, & \text{otherwise.} \end{cases}$$

Note that $F(1, 1) = 1, F(\frac{1}{3}, \frac{1}{3}) = 0, F(1, \frac{1}{3}) = 1, F(\frac{1}{3}, 1) = 1$. Then,

$$F(1, 1) - F\left(1, \frac{1}{3}\right) - F\left(\frac{1}{3}, 1\right) + F\left(\frac{1}{3}, \frac{1}{3}\right) < 0.$$

The non-decreasing property does not hold. Hence, F is not a joint DF of a 2-dimensional random vector.

Example 1.4. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) := \begin{cases} 0, & \text{if } x \leq 0, y \in \mathbb{R}, \\ x, & \text{if } x \in (0, 1), y \in \mathbb{R}, \\ 1, & \text{if } x \geq 1, y \in \mathbb{R}. \end{cases}$$

Here, for all $x \in (0, 1)$, $\lim_{y \to -\infty} F(x, y) = x \neq 0$. Hence, F is not a joint DF of a 2-dimensional random vector.

Example 1.5. Let $Z = (X, Y)$ be a 2-dimensional discrete random vector with the joint p.m.f. of the form

$$f_Z(x, y) = \begin{cases} \alpha(x + y), \text{ if } x, y \in \{1, 2, 3, 4\} \\ 0, \text{ otherwise} \end{cases}$$

for some constant $\alpha \in \mathbb{R}$. For f_Z to take non-negative values, we must have $\alpha > 0$. Now, $\sum_{x,y \in \{1,2,3,4\}} \alpha(x + y) = 1$ simplifies to $80\alpha = 1$ and hence $\alpha = \frac{1}{80}$. Also note that for this value of α, f_Z takes non-negative values. The support of Z is $\{(x, y) : x, y \in \{1, 2, 3, 4\}\} = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$. The support of X is $\{1, 2, 3, 4\}$ and the marginal p.m.f. f_X can now be computed as

$$f_X(x) = \begin{cases} \sum_{y \in \{1,2,3,4\}} \frac{1}{80}(x + y), \text{ if } x \in \{1, 2, 3, 4\} \\ 0, \text{ otherwise} \end{cases} = \begin{cases} \frac{1}{40}(2x + 5), \text{ if } x \in \{1, 2, 3, 4\} \\ 0, \text{ otherwise} \end{cases}$$

By the symmetry of $f_Z(x, y)$ in the variables x and y, we conclude that $X \stackrel{d}{=} Y$. Note that $f_Z(1, 1) = \frac{1}{40}$ and $f_X(1)f_Y(1) = \frac{49}{1600}$. Hence X and Y are not independent.
We have, for fixed $x \in \{1, 2, 3, 4\}$,

$$f_{Y|X}(y \mid x) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_X(x)}, \text{ if } y \in \{1, 2, 3, 4\} \\ 0, \text{ otherwise.} \end{cases} = \begin{cases} \frac{x+y}{2(2x+5)}, \text{ if } y \in \{1, 2, 3, 4\} \\ 0, \text{ otherwise.} \end{cases}$$

Example 1.6. Let $U = (X, Y, Z)$ be a 3-dimensional discrete random vector with the joint p.m.f. of the form

$$f_U(x, y, z) = \begin{cases} \alpha xyz, \text{ if } x = 1, y \in \{1, 2\}, z \in \{1, 2, 3\} \\ 0, \text{ otherwise} \end{cases}$$

for some constant $\alpha \in \mathbb{R}$. For f_U to take non-negative values, we must have $\alpha > 0$. Now, $\sum_{x=1, y \in \{1,2\}, z \in \{1,2,3\}} \alpha xyz = 1$ simplifies to $18\alpha = 1$ and hence $\alpha = \frac{1}{18}$. Also note that for this value of α, f_U takes non-negative values. The support of U is $\{(x, y, z) : x = 1, y \in \{1, 2\}, z \in \{1, 2, 3\}\} = \{1\} \times \{1, 2\} \times \{1, 2, 3\}$. The support of X is $\{1\}$ and the marginal p.m.f. f_X can now be computed as

$$f_X(x) = \begin{cases} \sum_{y\in\{1,2\},z\in\{1,2,3\}} \frac{1}{18}yz, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}$$

as expected. Similar computation yields

$$f_Y(y) = \begin{cases} \frac{1}{3}, & \text{if } y = 1 \\ \frac{2}{3}, & \text{if } y = 2 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{y}{3}, & \text{if } y \in \{1,2\} \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_Z(z) = \begin{cases} \frac{1}{6}, & \text{if } z = 1 \\ \frac{1}{3}, & \text{if } z = 2 \\ \frac{1}{2}, & \text{if } z = 3 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{z}{6}, & \text{if } z \in \{1,2,3\} \\ 0, & \text{otherwise} \end{cases}$$

Observe that $f_{X,Y,Z}(x, y, z) = f_X(x)f_Y(y)f_Z(z), \forall x, y, z$ and hence the RVs X, Y, Z are independent.