MSO205A QUIZ 1 SOLUTIONS

Question 1. (3 marks) A secretary types 4 letters and prepares 4 corresponding envelopes. In a hurry, she places one letter in each envelope at random. What is the probability that at least one letter is in the correct envelope? You may write your answer as a fraction.

Answer: Accepted answer: $\frac{5}{8}$ ($\frac{15}{24}$ is also accepted)
For $i = 1, 2, 3, 4$, consider the events E_i that the i-th letter is in the correct envelope. We need to find $\mathbb{P}(E_1 \cup E_2 \cup E_3 \cup E_4)$.

Since one letter has been placed in each envelope, this is an experiment performed at random without replacement. Therefore for any $i = 1, 2, 3, 4$, in the event E_i, the other letters can be placed in any of the remaining $4 - 1 = 3$ envelopes in 3! ways. But the total number of ways in which 4 letters are distributed in 4 envelopes is 4! and hence $\mathbb{P}(E_i) = \frac{3!}{4!} = \frac{1}{4}$.

More generally, if $1 \leq k \leq 4$ letters are in the correct envelope, then remaining $4 - k$ letters can be placed in the remaining $4 - k$ envelopes with probability $\frac{(4-k)!}{4!}$. For example, for $1 \leq i < j \leq 4$, the probability that i and j-th letter are correctly placed is $\mathbb{P}(E_i \cap E_j) = \frac{2!}{4!}$.

By the Inclusion-Exclusion principle,

$$\mathbb{P}(E_1 \cup E_2 \cup E_3 \cup E_4) = \binom{4}{1} \times \frac{3!}{4!} - \binom{4}{2} \times \frac{2!}{4!} + \binom{4}{3} \times \frac{1!}{4!} - \binom{4}{4}\frac{0!}{4!} = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} = \frac{15}{24} = \frac{5}{8}.$$

Question 2. (1.5 + 1.5 marks) Suppose an integer is drawn at random from the set $\{2, 3, 5, 7, 8, 9\}$. If an even integer is drawn, we choose box A containing 2 white balls and 5 black balls. Otherwise, we choose box B containing 4 white balls and 3 red balls. All balls of the same colour are identical. Next, a ball is drawn at random from the chosen box. What is the probability that the ball drawn is white? If the ball drawn is white, what is the probability that box A was chosen?

Answer: Accepted answers: $\frac{10}{21}$ and $\frac{1}{5}$
We first find the probability that the ball drawn is white. Let W denote this event.

Suppose U_1 and U_2 denote the events that the box A and the box B is chosen respectively. Then the events $U_i, i = 1, 2$ are mutually exclusive and exhaustive.

Now,

$$\mathbb{P}(U_1) = \text{probability of obtaining an even integer, while drawing at random from the given set} = \frac{2}{6} = \frac{1}{3}$$

and hence $\mathbb{P}(U_2) = 1 - \frac{1}{3} = \frac{2}{3}$.

But, $\mathbb{P}(W \mid U_1) = \frac{2}{2+5} = \frac{2}{7}$ and $\mathbb{P}(W \mid U_2) = \frac{4}{4+3} = \frac{4}{7}$.

By the theorem of total probability, the probability that the ball drawn is white is $\mathbb{P}(W) = \mathbb{P}(U_1)\mathbb{P}(W \mid U_1) + \mathbb{P}(U_2)\mathbb{P}(W \mid U_2) = \frac{10}{21}$. (part marks: 1.5)

Now, the required probability that the box A was chosen, i.e. probability of event U_1 provided the ball drawn is white is given by $\mathbb{P}(U_1 \mid W)$. By Bayes Theorem, (part marks: 1.5)

$$\mathbb{P}(U_1 \mid W) = \frac{\mathbb{P}(U_1)\,\mathbb{P}(W \mid U_1)}{\mathbb{P}(W)} = \frac{1}{3}\frac{2}{7}\frac{21}{10} = \frac{1}{5}.$$

Question 3. (3 marks) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space such that there exist events A, B, C, D with $\mathbb{P}(A) = 0.99, \mathbb{P}(B) = 0.1, \mathbb{P}(C) = 0.6, \mathbb{P}(D) = 0.4$. Which of the following statement(s) is/are necessarily true? Choose all correct statement(s), by putting a tick (✓), to get credit.

(a) A, B, C are pairwise disjoint.
(b) C and D are pairwise disjoint.
(c) $\mathbb{P}(A \mid B) \geq 0.5$.
(d) $A \cup D = \Omega$.

Answer: Accepted answer: (c) only

If (a) were to be true, then $\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) > 1$, a contradiction. Hence, (a) is false.

Consider the equally likely probability model on $\Omega = \{1, 2, \cdots, 100\}$ (i.e. choosing an integer from Ω at random). If $A = \{1, 2, \cdots, 99\}, C = \{1, 2, \cdots, 60\}$ and $D = \{1, 2, \cdots, 40\}$ then $\mathbb{P}(A) = 0.99, \mathbb{P}(C) = 0.6$ and $\mathbb{P}(D) = 0.4$. This gives a counter-example to both of option (b) and (d).

To prove (c): note that $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1 = 0.09$ and hence $\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \geq 0.9 > 0.5$. Hence, (c) is true.

Question 4. (6 marks) Consider the random experiment of tossing a fair coin once. Let X be a random variable which takes the value 3 when head appears and 5 otherwise. Compute the law of X. Justify your answer.

Answer: The sample space Ω of the random experiment is $\{H, T\}$ and the event space $\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \Omega\}$ is the set of all subsets. Finally, the probability function/measure is given by $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = \frac{1}{2}, \mathbb{P}(\Omega) = 1$. (part marks: 1)

The given random variable $X : \Omega \to \mathbb{R}$ is defined by $X(H) = 3, X(T) = 5$. (part marks: 1)

Then for all subsets B of $\mathbb{R}$, we have (part marks: 2)

$$X^{-1}(B) = \begin{cases} \emptyset, \text{ if } 3, 5 \notin B \\ \{H\}, \text{ if } 3 \in B, 5 \notin B \\ \{T\}, \text{ if } 3 \notin B, 5 \in B \\ \Omega, \text{ if } 3, 5 \in B \end{cases}$$

Hence, the law of X is given by the function $\mathbb{P} \circ X^{-1}$ where (part marks: 2)

$$\mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) = \begin{cases} 0, \text{ if } 3, 5 \notin B \\ \frac{1}{2}, \text{ if } 3 \in B, 5 \notin B \\ \frac{1}{2}, \text{ if } 3 \notin B, 5 \in B \\ 1, \text{ if } 3, 5 \in B \end{cases}$$