1. Section 2: Descriptive type

Question 3: Answers to all the sub-parts should be kept together.

(i) (1 + 2 marks) Let X be a discrete random variable with the property that $\mathbb{P}(-1 < X \leq 2) = 1$. Is X^2 also a discrete random variable? What can you say about the support of X^2?

(ii) (1 + 2 + 1 marks) A person plans to apply to 10 companies for a job. All the companies accept applications only through online mode, but 2 of these companies also require an additional supporting document to be sent by post. An application is considered to be complete once it reaches the intended company along with the supporting document, if any. Assume that online applications always reach the intended company, but the probability that a document sent by post reaches the destination is $\frac{1}{2}$. Further, assume that sending documents by post to different destinations are independent. Let Y denote the number of completed applications once the person sends the applications through online mode and documents by post. Identify the support of the discrete random variable Y and compute the p.m.f. of Y. Also compute $2\mathbb{E}Y^2 - \mathbb{E}Y$.

(iii) (2 marks) Suppose that all the cards numbered 2, 3, 4, 5 and 6 have been removed from a standard deck of cards and that three cards have been drawn at random from the remaining thirty-two cards. Given that at least two cards are from the diamond suite, what is the probability that all cards are from the diamond suite? You may write your answer as a fraction. (Note: binomial coefficients in the answer will not be acceptable.)

Answer:

(i) The RV X^2 can be written as $h(X)$, a function of the random variable X, for the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^2, \forall x \in \mathbb{R}$. As discussed in class, any function of a discrete RV is also a discrete RV and hence X^2 is discrete. (part marks: $\frac{1}{2}$ for reasoning, $\frac{1}{2}$ for saying X^2 is discrete)

Let S_X denote the support of X. Given that S_X is a finite or a countably infinite subset of $(-1, 2]$. Now,

$$S_{X^2} = S_{h(X)} = h(S_X) \subset h((-1, 2]) = [0, 4]. \text{ (part marks: 1)}$$

The support S_{X^2} is a finite or countably infinite subset of $[0, 4]$. (part marks: 1)

(Note: $(0, 4], [0, 4)$ or $(0, 4)$ are incorrect.)

(ii) Under the given information, $10 - 2 = 8$ applications have been sent online only mode and reach the intended company. Therefore, at least 8 applications are completed.

From the rest of the 2 applications with supporting documents, we have possibilities of $0, 1$ or 2 applications being completed. Hence, the possible values of Y are $8 + 0, 8 + 1$ and $8 + 2$. Hence, the support of Y is $S_Y = \{8, 9, 10\}$. (part marks: 1) (Note: If the support is not written as a set, then the answer is incorrect)

Now, using the independence of sending documents by post, we have (part marks: 1)

(a) $\mathbb{P}(Y = 8) = \mathbb{P}(\text{None of the supporting documents reach the intended companies}) = (1 - \frac{1}{2})^2 = \frac{1}{4}$.

(b) $\mathbb{P}(Y = 9) = \mathbb{P}(\text{exactly one supporting document reach the intended company}) = \binom{2}{1}\frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{2}$.

(c) $\mathbb{P}(Y = 10) = \mathbb{P}(\text{both the supporting documents reach the intended companies}) = (\frac{1}{2})^2 = \frac{1}{4}$.

Therefore, the p.m.f. of Y is

$$f_Y(y) := \begin{cases} \frac{1}{4}, & \text{if } x = 8 \text{ or } 10, \\ \frac{1}{2}, & \text{if } x = 9, \\ 0, & \text{otherwise. (part marks for mentioning `otherwise' case: 1)} \end{cases}$$

Since Y is supported on finitely many points, all the moments exist. Now,

$$\mathbb{E}Y = \sum_{y \in S_Y} y f_Y(y) = 8 \times \frac{1}{4} + 9 \times \frac{1}{2} + 10 \times \frac{1}{4} = 9$$

and

$$\mathbb{E}Y^2 = \sum_{y \in S_Y} y^2 f_Y(y) = 8^2 \times \frac{1}{4} + 9^2 \times \frac{1}{2} + 10^2 \times \frac{1}{4} = 81.5.$$

Then $2\mathbb{E}Y^2 - \mathbb{E}Y = 154$.

(iii) Let A represent the event that at least two cards are from the Diamond suit and let B denote the event that all cards are from the Diamond suit. Then $B \subset A$.

Now,

$$\mathbb{P}(A) = \frac{\binom{8}{3} + \binom{8}{2} \times \binom{24}{1}}{\binom{32}{3}}, \quad \mathbb{P}(B) = \frac{\binom{8}{3}}{\binom{32}{3}}.$$

Hence, the required probability is

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B)}{\mathbb{P}(A)} = \frac{\binom{8}{3}}{\binom{8}{3} + \binom{8}{2} \times \binom{24}{1}} = \frac{56}{56 + 28 \times 24} = \frac{1}{1 + 12} = \frac{1}{13}.$$

(Note: binomial coefficients in the answer will not be acceptable.)

Question 4: Answers to all the sub-parts should be kept together.

(i) (4 + 2 marks) Consider a continuous random variable X with the p.d.f.

$$f_X(x) = \begin{cases} 2\exp(-2x), & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Compute the distribution function of $Y = |X - 1|$. Is Y a continuous random variable with a p.d.f.? Justify your answer.

(ii) (3 marks) Let Z be a continuous random variable with p.d.f. f_Z such that $\mathbb{P}(Z \leq 0) = 0$ and $\mathbb{E}Z < \infty$. What can you say about $\lim_{n \to \infty} n\mathbb{P}(Z > n)$? Justify your answer. (Note: if you simply compute the limit for specific random variables Z, then no marks will be provided.)

Answer:

(i) X is a continuous RV with a p.d.f. given by

$$f_X(x) = \begin{cases} 2\exp(-2x), & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then for all $y \in \mathbb{R}$, (part marks: 1)

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(|X - 1| \leq y) = \begin{cases} 0, & \text{if } y < 0, \\ \mathbb{P}(1 - y \leq X \leq y + 1), & \text{if } y \geq 0 \end{cases}$$

For $y \geq 0$,

$$\mathbb{P}(1-y \leq X \leq y+1) = \int_{1-y}^{y+1} f_X(x)\,dx$$
$$= \begin{cases} \int_{1-y}^{y+1} 2\exp(-2x)\,dx, & \text{if } 0 \leq y < 1, \\ \int_{0}^{y+1} 2\exp(-2x)\,dx, & \text{if } y \geq 1 \end{cases}$$
$$= \begin{cases} e^{-2(1-y)} - e^{-2(y+1)}, & \text{if } 0 \leq y < 1, \\ 1 - e^{-2(y+1)}\,dx, & \text{if } y \geq 1 \end{cases}$$

Therefore, (part marks: 3)

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 0, \\ e^{2(y-1)} - e^{-2(y+1)}, & \text{if } 0 \leq y < 1, \\ 1 - e^{-2(y+1)}\,dx, & \text{if } y \geq 1 \end{cases}$$

Now, F_Y is continuous with

$$F'_Y(y) = \begin{cases} 0, & \text{if } y < 0, \\ 2e^{2(y-1)} + 2e^{-2(y+1)}, & \text{if } 0 < y < 1, \\ 2e^{-2(y+1)}, & \text{if } y > 1 \end{cases}$$

with possible discontinuities at the points 0 and 1. Also, (this verification – part marks: 1.5)

$$\int_{-\infty}^{\infty} F'_Y(y)\,dy = 2\int_0^1 [e^{2(y-1)} + e^{-2(y+1)}]\,dy + 2\int_1^{\infty} e^{-2(y+1)}\,dy = (1-e^{-2}) + (-e^{-4}+e^{-2}) + (-0+e^{-4}) = 1.$$

Therefore, Y is a continuous RV with the p.d.f. (part marks: 0.5)

$$f_Y(y) = \begin{cases} 0, & \text{if } y \leq 0, \\ 2e^{2(y-1)} + 2e^{-2(y+1)}, & \text{if } 0 < y < 1, \\ 0, & \text{if } y = 1, \\ 2e^{-2(y+1)}, & \text{if } y > 1 \end{cases}$$

(Note: values at 0 and 1 can be chosen arbitrarily)

(ii) We have $\mathbb{P}(Z < z) = 0, \forall z \leq 0$. Then, $\mathbb{E}Z = \int_0^{\infty} \mathbb{P}(Z > z)\,dz = \sum_{n=0}^{\infty} \mathbb{P}(Z > z)\,dz \geq \sum_{n=1}^{\infty} \mathbb{P}(Z > n)$. Hence, $\{\mathbb{P}(Z > n)\}_n$ is a non-increasing sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} \mathbb{P}(Z > n) < \infty$. Take, $S_m = \sum_{n=1}^{m} \mathbb{P}(Z > n), \forall m \geq 1$. This sequence of partial sums converge. (part marks: 1.5)

Now, observing that $S_{2m} - S_m \geq m\mathbb{P}(Z > 2m)$, we conclude $\lim_{m\to\infty} 2m\mathbb{P}(Z > 2m) = 0$. Observing $S_{2m+1} - S_m \geq (m+1)\mathbb{P}(Z > 2m+1)$, a similar argument gives $\lim_{m\to\infty}(2m+1)\mathbb{P}(Z > 2m+1) = 0$. (part marks: 1, The argument here is usually referred to Abel-Pringsheim Theorem. Stating this name also gets the part-marks)

We finally conclude $\lim_{n\to\infty} n\mathbb{P}(Z > n) = 0$. (part marks: 0.5) (Note: if you simply compute the limit for specific random variables Z, then no marks will be provided.)