MSO205A PRACTICE PROBLEMS SET 13

Question 1. Refer to Question 6 of problem set 12. Show by an example that the continuous mapping theorem does not hold for converge in r-th mean/moment.

Question 2. Construct an example of a sequence of RVs $\{X_n\}_n$ converging in law/distribution, but not in probability.

Question 3. Let $\{X_n\}_n$ be a sequence of i.i.d. RVs with finite second moment. Show that:

(1) $\frac{2}{n(n+1)} \sum_{j=1}^{n} jX_j \xrightarrow[n\to\infty]{P} \mathbb{E}X_1$.

(2) $\frac{6}{n(n+1)(2n+1)} \sum_{j=1}^{n} j^2 X_j \xrightarrow[n\to\infty]{P} \mathbb{E}X_1$.

Question 4. Let $a, b \in \mathbb{R}$ and let $\{X_n\}_n$ be a sequence of RVs such that $X_n \xrightarrow[n\to\infty]{P} a$ as well as $X_n \xrightarrow[n\to\infty]{P} b$. Show that $a = b$.

Question 5. Consider a sequence $\{X_n\}_n$ of RVs with $X_n \sim N(\frac{1}{n}, 1 - \frac{1}{n}), \forall n$. Does this sequence converge in law/distribution?

Question 6. Suppose that a continuous RV X has a quantile of order $\frac{1}{3}$ at 5. Consider a random sample of size 100 from the distribution of X. What is the probability (approximately) that more than 40 sample values are more than 5? Express the approximate value in terms of Φ, the DF of $N(0,1)$ distribution.

Question 7. Fix $\lambda > 0$. Let $X_1, X_2, \cdots$ be a sequence of i.i.d. RVs with *Exponential*(λ) distribution. Consider the sample mean $\bar{X}_n := \frac{1}{n} \sum_{j=1}^{n} X_j, \forall n$. Show that

$$\sqrt{n}\left(\frac{1}{\bar{X}_n} - \frac{1}{\lambda}\right) \xrightarrow[n\to\infty]{d} N\left(0, \frac{1}{\lambda^2}\right).$$

Question 8. Compute the mode of *Binomial*(n, p) distribution.

Question 9. Let $X \sim Poisson(\lambda)$ for some $\lambda > 0$. Compute the coefficient of skewness and excess kurtosis.

Question 10. Let X be a p-dimensional random vector, $a \in \mathbb{R}^m$ and A be an $m \times p$ real matrix. Then the Characteristic function of the m-dimensional random vector $Y = a + AX$ given by

$$\Phi_Y(u) = \exp(iu^t a)\, \Phi_X(A^t u), u \in \mathbb{R}^m.$$

Question 11. Show that $\mathbb{E}|X|^\alpha < \infty, \forall \alpha \in (0,1)$ when $X \sim Cauchy(0,1)$.

Question 12. Compute the Characteristic functions for standard probability distributions.