

## 1. A LIST OF STANDARD UNIVARIATE PROBABILITY DISTRIBUTIONS - PART 1

### 1.1. Discrete probability distributions and Discrete RVs.

**Example 1.1** (Degenerate RV). Fix  $c \in \mathbb{R}$ . Say that  $X$  is degenerate at  $c$  if its distribution is given by the p.m.f.

$$f_X(x) = \mathbb{P}(X = x) = \begin{cases} 1, & \text{if } x = c \\ 0, & \text{otherwise.} \end{cases}$$

This is a discrete RV with support  $S_X = \{c\}$ . As computed earlier,  $\mathbb{E}X = c$ . We also have  $\mathbb{E}X^n = c^n, \forall n \geq 1$  and  $M_X(t) = e^{tc}, \forall t \in \mathbb{R}$ . Note that  $\text{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 0$ .

*Remark 1.2* (Bernoulli Trial). Suppose that a random experiment has exactly two outcomes, identified as a ‘success’ and a ‘failure’. For example, while tossing a coin, we may think of obtaining a head as a success and a tail as a failure. Here, the sample space is  $\Omega = \{Success, Failure\}$ . A single trial of such an experiment is referred to as a Bernoulli trial. In this case,  $\text{Probability}(\{Success\}) = 1 - \text{Probability}(\{Failure\})$ . If we define an RV  $X : \Omega \rightarrow \mathbb{R}$  by  $X(Success) = 1$  and  $X(Failure) = 0$ , then  $X$  is a discrete RV with p.m.f.

$$f_X(x) = \mathbb{P}(X = x) = \begin{cases} 1 - \text{Probability}(\{Success\}), & \text{if } x = 0, \\ \text{Probability}(\{Success\}), & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

If  $\text{Probability}(\{Success\}) = 0$  or  $\text{Probability}(\{Failure\}) = 0$ , then  $X$  is degenerate at 0 or 1, respectively. The case when  $\text{Probability}(\{Success\}) \in (0, 1)$  is therefore of interest.

**Example 1.3** (Bernoulli( $p$ ) RV). Let  $p \in (0, 1)$ . An RV  $X$  is said to follow Bernoulli( $p$ ) distribution or equivalently,  $X$  is a Bernoulli( $p$ ) RV if its distribution is given by the p.m.f.

$$f_X(x) = \begin{cases} 1 - p, & \text{if } x = 0, \\ p, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

In relation with the Bernoulli trial described above,  $p$  may be treated as the probability of success. Here,  $\mathbb{E}X = p$ ,  $\mathbb{E}X^2 = p$ ,  $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = p - p^2 = p(1 - p)$ ,  $M_X(t) = 1 - p + pe^t$ ,  $t \in \mathbb{R}$ . By standard arguments, we can establish the existence of these moments.

**Notation 1.4.** We may write  $X \sim \text{Bernoulli}(p)$  to mean that  $X$  is a Bernoulli( $p$ ) RV. Similar notations shall be used for other RVs and their distributions.

**Example 1.5** (Binomial( $n, p$ ) RV). Fix a positive integer  $n$  and let  $p \in (0, 1)$ . By the Binomial theorem, we have

$$1 = [p + (1 - p)]^n = \sum_{k=0}^n \binom{n}{k} p^k (1 - p)^{n-k}$$

and hence the function  $f : \mathbb{R} \rightarrow [0, 1]$  given by

$$f(x) = \begin{cases} \binom{n}{x} p^x (1 - p)^{n-x}, & \text{if } x \in \{0, 1, \dots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

is a p.m.f.. An RV  $X$  is said to follow Binomial( $n, p$ ) distribution or equivalently,  $X$  is a Binomial( $n, p$ ) RV if its distribution is given by the above p.m.f.. Here,

$$\mathbb{E}X = \sum_{k=0}^n k \frac{n!}{k!(n-k)!} p^k (1 - p)^{n-k} = np \sum_{k=1}^n \binom{n-1}{k-1} p^{k-1} (1 - p)^{n-k} = np [p + (1 - p)]^{n-1} = np,$$

and

$$\mathbb{E}X(X-1) = \sum_{k=0}^n k(k-1) \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = n(n-1)p^2 \sum_{k=2}^n \binom{n-2}{k-2} p^{k-2} (1-p)^{n-k} = n(n-1)p^2.$$

Then  $\mathbb{E}X^2 = \mathbb{E}X(X-1) + \mathbb{E}X = n(n-1)p^2 + np$  and  $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = n(n-1)p^2 + np - n^2p^2 = np(1 - p)$ . Also

$$M_X(t) = \mathbb{E}e^{tX} = \sum_{k=0}^n e^{tk} \binom{n}{k} p^k (1 - p)^{n-k} = (1 - p + pe^t)^n, \forall t \in \mathbb{R}.$$

By standard arguments, we can establish the existence of these moments.

**Note 1.6.** Observe that Binomial( $1, p$ ) distribution is the same as Bernoulli( $p$ ) distribution. We shall explore the connection between Binomial and Bernoulli distributions later in the course.

*Remark 1.7* (Factorial moments). In the computation for  $\mathbb{E}X^2$  for  $X \sim \text{Binomial}(n, p)$ , we first computed  $\mathbb{E}X(X-1)$ , which is easy to compute. It turns out that expectations of the form  $\mathbb{E}X(X-1)$ ,  $\mathbb{E}X(X-1)(X-2)$  etc. are often easy to compute for integer valued RVs  $X$ . We refer to such expectations as factorial moments of  $X$ .

*Remark 1.8* (Symmetry of  $\text{Binomial}(n, \frac{1}{2})$  distribution). Let  $X \sim \text{Binomial}(n, p)$  and let  $Y := n - X$ . Since  $M_X(t) = (1 - p + pe^t)^n, \forall t \in \mathbb{R}$ , we have

$$M_Y(t) = \mathbb{E}e^{tY} = \mathbb{E}e^{t(n-X)} = e^{-nt}M_X(-t) = e^{nt}(1 - p + pe^{-t})^n = (p + (1 - p)e^t)^n.$$

Since MGFs determine the distribution, we conclude that  $Y \sim \text{Binomial}(n, 1 - p)$ . In particular, if  $p = \frac{1}{2}$ , then  $Y = n - X \stackrel{d}{=} X \sim \text{Binomial}(n, \frac{1}{2})$ . Rewriting the relation, we get  $\frac{n}{2} - X \stackrel{d}{=} X - \frac{n}{2}$ . Therefore,  $X \sim \text{Binomial}(n, \frac{1}{2})$  is symmetric about  $\frac{n}{2}$ .

We now look at more examples of discrete RVs. Later in the course, we shall discuss their motivation through various random experiments.

**Example 1.9** (Uniform RVs with support on a finite set). Consider a discrete RV  $X$  with support  $S_X = \{x_1, x_2, \dots, x_n\}$  and p.m.f.  $f_X : \mathbb{R} \rightarrow [0, 1]$  given by

$$f_X(x) = \begin{cases} \frac{1}{n}, & \text{if } x \in S_X, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\mathbb{E}X = \frac{1}{n} \sum_{x \in S_X} x, \quad \mathbb{E}X^2 = \frac{1}{n} \sum_{x \in S_X} x^2, \quad M_X(t) = \mathbb{E}e^{tX} = \frac{1}{n} \sum_{x \in S_X} e^{tx}, \forall t \in \mathbb{R}$$

and hence  $\text{Var}(X)$  can be computed by the formula  $\mathbb{E}X^2 - (\mathbb{E}X)^2$ . Observing that the support set is finite, we can establish the existence of these moments.

**Example 1.10** (Poisson ( $\lambda$ ) RV). Fix  $\lambda > 0$ . Note that  $e^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$  and hence the function  $f : \mathbb{R} \rightarrow [0, 1]$  given by

$$f(x) = \begin{cases} e^{-\lambda} \frac{\lambda^x}{x!}, & \text{if } x \in \{0, 1, 2, \dots\}, \\ 0, & \text{otherwise.} \end{cases}$$

is a p.m.f.. An RV  $X$  is said to follow Poisson( $\lambda$ ) distribution or equivalently,  $X$  is a Poisson( $\lambda$ ) RV if its distribution is given by the above p.m.f.. Recall that we have already computed the following  $\mathbb{E}X = \lambda$ ,  $Var(X) = \lambda$  and  $M_X(t) = e^{\lambda(e^t-1)}$ ,  $\forall t \in \mathbb{R}$ . As done for the case of Binomial( $n, p$ ) RVs, we can also compute the factorial moments. For example,

$$\mathbb{E}X(X-1) = \sum_{k=0}^{\infty} k(k-1)e^{-\lambda} \frac{\lambda^k}{k!} = \lambda^2 \sum_{k=2}^{\infty} e^{-\lambda} \frac{\lambda^{k-2}}{(k-2)!} = \lambda^2.$$

In fact,  $\mathbb{E}X(X-1)\cdots(X-(n-1)) = \lambda^n$  for all  $n \geq 1$ .

**Example 1.11** (Geometric ( $p$ ) RV). Fix  $p \in (0, 1)$ . Note that  $\sum_{k=0}^{\infty} p(1-p)^k = 1$  and hence the function  $f : \mathbb{R} \rightarrow [0, 1]$  given by

$$f(x) = \begin{cases} p(1-p)^x, & \text{if } x \in \{0, 1, 2, \dots\}, \\ 0, & \text{otherwise.} \end{cases}$$

is a p.m.f.. An RV  $X$  is said to follow Geometric( $p$ ) distribution or equivalently,  $X$  is a Geometric( $p$ ) RV if its distribution is given by the above p.m.f.. Let us compute the MGF. Here,

$$M_X(t) = \mathbb{E}e^{tX} = \sum_{k=0}^{\infty} e^{tk} p(1-p)^k = \frac{p}{1 - (1-p)e^t},$$

for all  $t$  such that  $0 < (1-p)e^t < 1$  or equivalently,  $t < \ln\left(\frac{1}{1-p}\right)$ . Looking at the derivatives of  $M_X$  and evaluating at  $t = 0$ , we have  $\mathbb{E}X = \frac{1-p}{p}$  and  $Var(X) = \frac{1-p}{p^2}$ .

## 1.2. Continuous probability distributions and Continuous RVs.

**Example 1.12** (Uniform( $a, b$ ) RV). Fix  $a, b \in \mathbb{R}$  with  $a < b$ . An RV  $X$  is said to follow Uniform( $a, b$ ) distribution or equivalently,  $X$  is a Uniform( $a, b$ ) RV if its distribution is given

by the p.d.f.

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{if } x \in (a, b), \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\mathbb{E}X = \int_a^b \frac{x}{b-a} dx = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}, \quad \mathbb{E}X^2 = \int_a^b \frac{x^2}{b-a} dx = \frac{b^3 - a^3}{3(b-a)} = \frac{a^2 + ab + b^2}{3}$$

and hence  $Var(X)$  can be computed by the formula  $\mathbb{E}X^2 - (\mathbb{E}X)^2$ . The MGF is given by

$$\mathbb{E}e^{tX} = \int_a^b \frac{e^{tx}}{b-a} dx = \begin{cases} \frac{e^{tb} - e^{ta}}{t(b-a)}, & \text{if } t \neq 0, \\ 1, & \text{if } t = 0. \end{cases}$$

By standard arguments, we can establish the existence of these moments. Further, observe that  $f_X(\frac{a+b}{2} - x) = f_X(\frac{a+b}{2} + x), \forall x \in \mathbb{R}$ . Here, we conclude that  $X$  is symmetric about its mean  $\frac{a+b}{2}$ .

**Example 1.13** (Cauchy( $\mu, \theta$ ) RV). Let  $\theta > 0$  and  $\mu \in \mathbb{R}$ . An RV  $X$  is said to follow Cauchy( $\mu, \theta$ ) distribution if its distribution is given by the p.d.f.

$$f_X(x) = \frac{\theta}{\pi} \frac{1}{\theta^2 + (x - \mu)^2}, \forall x \in \mathbb{R}.$$

The fact that  $f_X$  is a p.d.f. is easy to check. Set  $y = \frac{x-\mu}{\theta}$  and observe that

$$\int_{-\infty}^{\infty} f_X(x) dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+y^2} dy = \frac{2}{\pi} \int_0^{\infty} \frac{1}{1+y^2} dy = \frac{2}{\pi} \tan^{-1}(y) \Big|_0^{\infty} = 1.$$

We have seen earlier that  $\mathbb{E}X$  and the MGF do not exist for this distribution. In the general setting, note that  $\frac{X-\mu}{\theta} \sim \text{Cauchy}(0, 1)$  and by a similar argument, we can show that  $\mathbb{E}X$  and MGF do not exist. Moreover,  $f_X(\mu + x) = f_X(\mu - x), \forall x \in \mathbb{R}$ . Hence, we conclude that  $X$  is symmetric about  $\mu$ .

**Example 1.14** (Exponential( $\lambda$ ) RV). Let  $\lambda > 0$ . Note that  $\int_0^\infty \exp(-\frac{x}{\lambda}) dx = \lambda$  and hence the function  $f : \mathbb{R} \rightarrow [0, \infty)$  given by

$$f(x) = \begin{cases} \frac{1}{\lambda} \exp(-\frac{x}{\lambda}), & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

is a p.d.f.. An RV  $X$  is said to follow Exponential( $\lambda$ ) distribution or equivalently,  $X$  is an Exponential( $\lambda$ ) RV if its distribution is given by the above p.d.f.. We have

$$\mathbb{E}X^n = \lambda^n n!, \quad \text{Var}(X) = \lambda^2, \quad M_X(t) = (1 - \lambda t)^{-1}, \forall t < \frac{1}{\lambda}.$$

By standard arguments, we can establish the existence of these moments.

**Definition 1.15** (Gamma function). Recall that the integral  $\int_0^\infty x^{\alpha-1} e^{-x} dx$  exists if and only if  $\alpha > 0$ . On  $(0, \infty)$ , consider the function  $\alpha \mapsto \int_0^\infty x^{\alpha-1} e^{-x} dx$ . It is called the Gamma function and the value at any  $\alpha > 0$  is denoted by  $\Gamma(\alpha)$ .

*Remark 1.16.* We recall some important properties of the Gamma function.

- (a) For  $\alpha > 0$ , we have  $\Gamma(\alpha) > 0$ .
- (b)  $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ , if  $\alpha > 1$ .
- (c)  $\Gamma(1) = \int_0^\infty e^{-x} dx = 1$  and hence using (b),  $\Gamma(n) = (n - 1)!$  for all positive integers  $n$ .
- (d)  $\Gamma(\frac{1}{2}) = \int_0^\infty \frac{1}{\sqrt{x}} e^{-x} dx = \sqrt{\pi}$ . Putting  $x = \frac{y^2}{2}$ , this relation may be rewritten as

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{2} \int_0^\infty \exp\left(-\frac{y^2}{2}\right) dy = \sqrt{\pi}.$$

- (e) Fix  $\beta > 0$ . Putting  $x = \frac{y}{\beta}$ , in the integral for  $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ , we get  $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} \beta^{-\alpha} \exp(-\frac{y}{\beta}) dy$ .

**Example 1.17** (Gamma( $\alpha, \beta$ ) RV). Fix  $\alpha > 0, \beta > 0$ . By the properties of the Gamma function described above, the function  $f : \mathbb{R} \rightarrow [0, \infty)$  defined by

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} \beta^{-\alpha} \exp(-\frac{x}{\beta}), & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

is a p.d.f.. An RV  $X$  is said to follow Gamma( $\alpha, \beta$ ) distribution or equivalently,  $X$  is a Gamma( $\alpha, \beta$ ) RV if its distribution is given by the above p.d.f.. Note that for  $\alpha = 1$ , we get back the p.d.f. for an Exponential( $\beta$ ) RV (see Example 1.14), i.e. Gamma( $1, \beta$ ) distribution is the same as Exponential( $\beta$ ) distribution. For general  $\alpha > 0, \beta > 0$ , we have

$$\mathbb{E}X = \alpha\beta, \quad \text{Var}(X) = \alpha\beta^2, \quad M_X(t) = (1 - \beta t)^{-\alpha}, \forall t < \frac{1}{\beta}.$$

By standard arguments, we can establish the existence of these moments.

**Example 1.18** (Normal( $\mu, \sigma^2$ ) RV). Fix  $\mu \in \mathbb{R}, \sigma > 0$ . Note that  $\Gamma(\frac{1}{2}) = \int_0^\infty \frac{1}{\sqrt{t}} e^{-t} dt = \sqrt{\pi}$  (see Remark 1.16). Putting  $t = \frac{y^2}{2}$  and after suitable manipulation, we have  $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left(-\frac{y^2}{2}\right) dy = 1$ . Putting  $y = \frac{1}{\sigma}(x - \mu)$  (equivalently,  $x = \sigma y + \mu$ ), we have

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) dx = 1.$$

Therefore, the function  $f : \mathbb{R} \rightarrow [0, \infty)$  defined by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right), \forall x \in \mathbb{R}$$

is a p.d.f.. An RV  $X$  is said to follow Normal( $\mu, \sigma^2$ ) distribution or equivalently,  $X$  is a Normal( $\mu, \sigma^2$ ) RV, denoted by  $X \sim N(\mu, \sigma^2)$  if its distribution is given by the above p.d.f.. If  $X \sim N(\mu, \sigma^2)$ , from our above discussion we conclude that  $Y = \frac{X - \mu}{\sigma} \sim N(0, 1)$ . Now,

$$\begin{aligned} M_Y(t) &= \mathbb{E}e^{tY} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{ty} \exp\left(-\frac{y^2}{2}\right) dy \\ &= \exp\left(\frac{t^2}{2}\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \exp\left(-\frac{(y - t)^2}{2}\right) dy \end{aligned}$$

$$= \exp\left(\frac{t^2}{2}\right), \forall t \in \mathbb{R}.$$

In particular,  $\psi_Y(t) = \ln M_Y(t) = \frac{t^2}{2}, \forall t \in \mathbb{R}$  with  $\psi'(t) = t, \psi''(t) = 1, \forall t \in \mathbb{R}$ . Evaluating at  $t = 0$ , we conclude that  $\mathbb{E}Y = 0$  and  $\text{Var}(Y) = 1$ . But  $X = \sigma Y + \mu$  and hence  $\mathbb{E}X = \mu, \text{Var}(X) = \sigma^2$ . This yields the interpretation of the parameters  $\mu$  and  $\sigma$  in the distribution of  $X$ . Further,  $M_X(t) = \mathbb{E}e^{tX} = \mathbb{E}e^{t(\sigma Y + \mu)} = e^{\mu t} M_Y(\sigma t) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2), \forall t \in \mathbb{R}$ .

**Definition 1.19** (Standard Normal RV). We say  $X$  is a Standard Normal RV if  $X \sim N(0, 1)$ , i.e.  $\mathbb{E}X = 0$  and  $\text{Var}(X) = 1$ .

**Notation 1.20.** Normal RVs are also referred to as Gaussian RVs and Normal distribution as Gaussian distribution.

*Remark 1.21* (Symmetry of Gaussian Distribution). If  $X \sim N(\mu, \sigma^2)$ , note that  $f_X(\mu + x) = f_X(\mu - x), \forall x \in \mathbb{R}$ . Hence, we conclude that  $X$  is symmetric about its mean  $\mu$ .

*Remark 1.22* (Moments of a Standard Normal RV). Let  $X \sim N(0, 1)$ . Then  $X$  is symmetric about 0 and consequently,  $\mathbb{E}X^n = 0$  for all odd positive integers  $n$ . If  $n$  is an even positive integer, then  $n = 2m$  for some positive integer  $m$  and

$$\begin{aligned} \mathbb{E}X^n &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n \exp\left(-\frac{x^2}{2}\right) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2m} \exp\left(-\frac{x^2}{2}\right) dx \\ &= \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\infty} x^{2m} \exp\left(-\frac{x^2}{2}\right) dx \\ &= \frac{2^m}{\sqrt{\pi}} \int_0^{\infty} y^{m-\frac{1}{2}} \exp(-y) dy, \text{ (putting } y = \frac{x^2}{2}\text{)} \\ &= \frac{2^m}{\sqrt{\pi}} \Gamma\left(m + \frac{1}{2}\right) \\ &= 2^m \left(m - \frac{1}{2}\right) \times \cdots \times \frac{3}{2} \times \frac{1}{2} \\ &= (2m - 1) \times \cdots \times 3 \times 1 =: (2m - 1)!!, \end{aligned}$$



where we have used the properties of the Gamma function. In particular,  $\mathbb{E}X^4 = 3$ .

**Definition 1.23** (Beta function). Recall that the integral  $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx$  exists if and only if  $\alpha > 0$  and  $\beta > 0$ . On  $(0, \infty) \times (0, \infty)$ , consider the function  $(\alpha, \beta) \mapsto \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx$ . It is called the Beta function and the value at any  $(\alpha, \beta)$  is denoted by  $B(\alpha, \beta)$ .

*Remark 1.24.* Note that for  $\alpha > 0, \beta > 0$ , we have  $B(\alpha, \beta) > 0$  and  $B(\alpha, \beta) = B(\beta, \alpha)$ . Moreover,

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

**Example 1.25** (Beta( $\alpha, \beta$ ) RV). Fix  $\alpha > 0, \beta > 0$ . By the properties of the Beta function described above, the function  $f : \mathbb{R} \rightarrow [0, \infty)$  defined by

$$f(x) = \begin{cases} \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, & \text{if } x \in (0, 1) \\ 0, & \text{otherwise.} \end{cases}$$

is a p.d.f.. An RV  $X$  is said to follow Beta( $\alpha, \beta$ ) distribution or equivalently,  $X$  is a Beta( $\alpha, \beta$ ) RV if its distribution is given by the above p.d.f.. If  $\alpha = \beta$ , then  $f(1-x) = f(x), \forall x \in \mathbb{R}$  and hence  $X \stackrel{d}{=} 1 - X$ . Then,  $X - \frac{1}{2} \stackrel{d}{=} \frac{1}{2} - X$ , i.e.,  $X$  is symmetric about  $\frac{1}{2}$ . For all  $\alpha, \beta, r > 0$ , we have

$$\mathbb{E}X^r = \frac{1}{B(\alpha, \beta)} \int_0^1 x^{\alpha+r-1}(1-x)^{\beta-1} dx = \frac{B(\alpha+r, \beta)}{B(\alpha, \beta)}$$

and in particular,

$$\mathbb{E}X = \frac{B(\alpha+1, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{\alpha}{\alpha+\beta}$$

and

$$\mathbb{E}X^2 = \frac{B(\alpha+2, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}.$$

Then

$$\text{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}.$$

By standard arguments, we can establish the existence of these moments.