MSO205A Quiz 2 Short answer-type Questions and Solutions

1. Qn 1 (2 marks)

Once the number A is determined, the p.d.f. is given by

$$f_X(x) = \begin{cases} \frac{1}{(105-A)-(-1)}, & \text{if } -1 < x < 105 - A, \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} \frac{1}{106-A}, & \text{if } -1 < x < 105 - A, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, $F_X(2) = \int_{-1}^{2} \frac{1}{106-A}\, dx = \frac{3}{106-A}$.

In the actual answer, exact value of A needs to be put to get credit.

2. Qn 2 (1.5 + 1.5 marks)

Set 1: (No, No)

$$F_1(x,y) := \begin{cases} xy, & \text{if } 0 < y < x < 1 \\ 0, & \text{otherwise.} \end{cases} \quad , \quad F_2(x,y) := \begin{cases} 1, & \text{if } x + y \geq 2 \\ 0, & \text{otherwise.} \end{cases}$$

Set 2: (Yes, No)

$$F_1(x,y) := \begin{cases} 1, & \text{if } x \geq -3 \text{ and } y \geq 2 \\ 0, & \text{otherwise.} \end{cases} \quad , \quad F_2(x,y) := \begin{cases} 1, & \text{if } 2x + y \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Set 3: (No, Yes)

$$F_1(x,y) := \begin{cases} 1, & \text{if } 3x + y \geq 1 \\ 0, & \text{otherwise.} \end{cases} \quad , \quad F_2(x,y) := \begin{cases} 1, & \text{if } x \geq 3 \text{ and } y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Set 4: (Yes, No)

$$F_1(x,y) := \begin{cases} 1, & \text{if } x \geq 0 \text{ and } y \geq 5 \\ 0, & \text{otherwise.} \end{cases} \quad , \quad F_2(x,y) := \begin{cases} 1, & \text{if } x + 2y \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Set 5: (No, Yes)

$$F_1(x,y) := \begin{cases} 1, & \text{if } x + 3y \geq 1 \\ 0, & \text{otherwise.} \end{cases} \quad , \quad F_2(x,y) := \begin{cases} 1, & \text{if } x \geq 4 \text{ and } y \geq -1 \\ 0, & \text{otherwise.} \end{cases}$$

Reasoning: The function

$$F(x,y) := \begin{cases} xy, & \text{if } 0 < y < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

is not a joint DF of a 2-dimensional random vector, since $\lim_{x\to\infty, y\to\infty} F(x,y) = 0$.

Functions of the form

$$F(x,y) := \begin{cases} 1, & \text{if } \alpha x + y \geq \beta \\ 0, & \text{otherwise.} \end{cases}$$

with fixed $\alpha, \beta > 0$ are not joint DFs of a 2-dimensional random vector. To see this, look at the intersection of the line $\alpha x + y = \beta$ with the co-ordinate axes. The points of intersection are $(0, \beta)$ and $(\frac{\beta}{\alpha}, 0)$. Consider the rectangle with corners given by $(0,0), (\frac{\beta}{\alpha}, 0), (\frac{\beta}{\alpha}, \beta), (0, \beta)$. Here,

$$F(\tfrac{\beta}{\alpha}, \beta) - F(\tfrac{\beta}{\alpha}, 0) - F(0, \beta) + F(0,0) = -1 < 0.$$

Therefore, the above F is not non-decreasing.

Functions of the form

$$F(x,y) := \begin{cases} 1, & \text{if } x \geq x_0 \text{ and } y \geq y_0 \\ 0, & \text{otherwise.} \end{cases}$$

are joint DFs of 2-dimensional degenerate random vectors taking value (x_0, y_0).

3. Qn 3 (2 + 2 marks)

Note that:

(i) If

$$f_X(x) = \exp(-x) 1_{(0,\infty)}(x), x \in \mathbb{R}$$

then $X \sim \mathit{Exponential}(1)$. Here

$$\mathbb{E}X = \int_0^\infty x \exp(-x)\, dx = 1, \quad \mathbb{E}X^2 = \int_0^\infty x^2 \exp(-x)\, dx = 2.$$

(ii) If

$$f_X(x) = 6x(1-x) 1_{(0,1)}(x), x \in \mathbb{R}$$

then $X \sim \mathit{Beta}(2,2)$. Here

$$\mathbb{E}X = \int_0^1 6x^2(1-x)\, dx = \tfrac{1}{2}, \quad \mathbb{E}X^2 = \int_0^1 6x^3(1-x)\, dx = \tfrac{3}{10}.$$

(iii) If $X \sim N(\mu, \sigma^2)$, then

$$\mathbb{E}X = \mu, \quad \mathbb{E}X^2 = Var(X) + (\mathbb{E}X)^2 = \sigma^2 + \mu^2.$$

Then, note the following computations:

Set 1: For

$$f_Y(x) := \frac{1}{4}\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-2)^2}{2}\right) + \frac{3}{4}\frac{1}{2\pi} \exp\left(-\frac{(x-3)^2}{4\pi}\right), \forall x \in \mathbb{R}$$

$$\mathbb{E}Y = \frac{1}{4}\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-2)^2}{2}\right) dx + \frac{3}{4}\int_{-\infty}^{\infty} x \frac{1}{2\pi} \exp\left(-\frac{(x-3)^2}{4\pi}\right) dx = \frac{2}{4} + \frac{9}{4} = \frac{11}{4}$$

and

$$\begin{aligned} \mathbb{E}Y^2 &= \frac{1}{4}\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-2)^2}{2}\right) dx + \frac{3}{4}\int_{-\infty}^{\infty} x^2 \frac{1}{2\pi} \exp\left(-\frac{(x-3)^2}{4\pi}\right) dx \\ &= \frac{1}{4}\left((2)^2 + 1\right) + \frac{3}{4}\left(3^2 + 2\pi\right) = 8 + \frac{3\pi}{2} \end{aligned}$$

Set 2: For

$$f_Y(x) := \frac{1}{5}\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x+2)^2}{2}\right) + \frac{4}{5}\exp(-x) 1_{(0,\infty)}(x), \forall x \in \mathbb{R}.$$

$$\mathbb{E}Y = \frac{1}{5}\int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x+2)^2}{2}\right)\,dx + \frac{4}{5}\int_0^{\infty} x\exp(-x)\,dx = -\frac{2}{5}+\frac{4}{5} = \frac{2}{5}$$

and

$$\begin{aligned}\mathbb{E}Y^2 &= \frac{1}{5}\int_{-\infty}^{\infty} x^2\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x+2)^2}{2}\right)\,dx + \frac{4}{5}\int_0^{\infty} x^2\exp(-x)\,dx \\ &= \frac{1}{5}\left((-2)^2+1\right) + \frac{8}{5} = \frac{13}{5}\end{aligned}$$

Set 3: For

$$f_Y(x) := \frac{1}{3}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x+2)^2}{2}\right) + 4x(1-x)1_{(0,1)}(x), \forall x\in\mathbb{R}.$$

$$\mathbb{E}Y = \frac{1}{3}\int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x+2)^2}{2}\right)\,dx + \frac{2}{3}\int_0^{1} x\,6x(1-x)\,dx = -\frac{2}{3}+\frac{1}{3} = -\frac{1}{3}$$

and

$$\begin{aligned}\mathbb{E}Y^2 &= \frac{1}{3}\int_{-\infty}^{\infty} x^2\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x+2)^2}{2}\right)\,dx + \frac{2}{3}\int_0^{1} x^2 6x(1-x)\,dx \\ &= \frac{1}{3}\left((-2)^2+1\right) + \frac{1}{5} = \frac{28}{15}\end{aligned}$$

Set 4: For

$$f_Y(x) := \frac{1}{5}\exp(-x)1_{(0,\infty)}(x) + \frac{4}{5}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x+2)^2}{2}\right), \forall x\in\mathbb{R}.$$

$$\mathbb{E}Y = \frac{1}{5}\int_0^{\infty} x\exp(-x)\,dx + \frac{4}{5}\int_{-\infty}^{\infty} x\,\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x+2)^2}{2}\right)\,dx = \frac{1}{5}-\frac{8}{5} = -\frac{7}{5}$$

and

$$\begin{aligned}\mathbb{E}Y^2 &= \frac{1}{5}\int_0^{\infty} x^2\exp(-x)\,dx + \frac{4}{5}\int_{-\infty}^{\infty} x^2\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x+2)^2}{2}\right)\,dx \\ &= \frac{2}{5} + \frac{4}{5}\left((-2)^2+1\right) = \frac{22}{5}\end{aligned}$$

Set 5: For

$$f_Y(x) := 2x(1-x)1_{(0,1)}(x) + \frac{2}{3}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-2)^2}{2}\right), \forall x\in\mathbb{R}.$$

$$\mathbb{E}Y = \frac{1}{3}\int_0^{1} x\,6x(1-x)\,dx + \frac{2}{3}\int_{-\infty}^{\infty} x\,\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-2)^2}{2}\right)\,dx = \frac{1}{6}+\frac{4}{3} = \frac{3}{2}$$

and

$$\begin{aligned}\mathbb{E}Y^2 &= \frac{1}{3}\int_0^{1} x^2\,6x(1-x)\,dx + \frac{2}{3}\int_{-\infty}^{\infty} x^2\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-2)^2}{2}\right)\,dx \\ &= \frac{1}{10} + \frac{2}{3}\left((-2)^2+1\right) = \frac{103}{30}\end{aligned}$$