1. Week 3 Supplementary material: Monty Hall problem

Statement: There are 3 doors with one door having an expensive car behind it and each of the other 2 doors having a goat behind them. Monty Hall, being a host of the game, knows what it behind each door. A contestant is asked to select one of the doors and he wins the item (car or goat) behind the selected door. The contestant selects one of the doors at random, and then Monty Hall opens one of the other two doors to reveal a goat behind it (Monty Hall knows the doors behind which there are goats). Monty Hall offers to trade the door the contestant has chosen for the other door that is closed. Should the contestant switch doors if the goal is to win the car? (The problem is based on the American Television game show 'Let's make a deal' hosted by Monty Hall).

Answer: Write the event of choosing doors 1, 2 and 3 as D_1, D_2 and D_3 respectively. Also write C for the event in which the contestant wins the car (i.e. selects the corresponding door).

Without loss of generality assume that the car is behind door 1.

Consider the following two cases.

Case when the contestant decides to switch doors. If the contestant's original choice was door 1, then after switching the new choice is one of door 2 or door 3. In either case, we have $\mathbb{P}(C \mid D_1) = 0$. If the original choice was door 2 or door 3, then Monty Hall must have opened door 3 or door 2 respectively. If the contestant now switches his choice, he chooses door 1 now. In this case, $\mathbb{P}(C \mid D_i) = 1, i = 2, 3$. The doors were originally chosen at random by the contestant. Hence $\mathbb{P}(D_i) = \frac{1}{3}, i = 1, 2, 3$. Further, these events are mutually exclusive and exhaustive. By the Theorem of Total Probability, $\mathbb{P}(C) = \sum_{i=1}^{3} \mathbb{P}(D_i)\ \mathbb{P}(C \mid D_i) = \frac{2}{3}$.

Case when the contestant does not switch: In this case, $\mathbb{P}(C \mid D_1) = 1$ and $\mathbb{P}(C \mid D_i) = 0, i = 2, 3$. Again by the Theorem of Total Probability, we have $\mathbb{P}(C) = \sum_{i=1}^{3} \mathbb{P}(D_i)\ \mathbb{P}(C \mid D_i) = \frac{1}{3}$.

At the time of the original choice, doors were chosen at random. However, after this choice, new information became available and we observe that by switching to the other option, the probability of winning the car increases. The contestant should switch.

Note that once new information becomes available, we should update the prior probabilities.