MTH207M (2023-24, ODD SEMESTER)

PROBLEM SET 3

1. If S and T are two subspaces of $\mathbb{R}^n$ having a common complement W, does it follow that $S = T$? Justify your answer.

2. If S is a subspace of $\mathbb{R}^n$ and x and y are fixed vectors such that $x \in S$ and $y \neq s$, show that there exists a complement T of S such that x is the projection of y into S along T.

3. If S and Y are subspaces such that $S \subseteq Y$ and Z is a complement of S, show that $Z \cap Y$ is a complement of S relative to Y (i.e., $S \bigoplus (Z \cap Y) = Y$).

4. If $\rho(A) = \rho(A^2)$ and $AB = BA = 0$, prove that $\rho(A + B) = \rho(A) + \rho(B)$. Show that none of $'AB = 0'$ and $'BA = 0'$ can be dropped here.

5. Let A be an idempotent matrix. Then show that

   (a) $\mathcal{C}(B) \subseteq \mathcal{C}(A)$ iff $AB = B$ and

   (b) $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ iff $BA = B$.

6. Let A and B be projectors of the same order. Then show that $A + B$ is a projector iff $\mathcal{C}(A) \subseteq \mathcal{N}(B)$ and $\mathcal{C}(B) \subseteq \mathcal{N}(A)$.

7. Let A and B be projectors of the same order. Show that the following statements are equivalent:

   (a) $A - B$ is a projector

   (b) $AB = BA = B$

   (c) $\rho(A - B) = \rho(A) - \rho(B)$

(d) $\mathcal{C}(B) \subseteq \mathcal{C}(A)$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A)$.

8. Let A and B be projectors of the same order.

   (a) If $A + B$ is a projector, show that it is the projector into $\mathcal{C}(A) \bigoplus \mathcal{C}(B)$ along $\mathcal{N}(A) \cap \mathcal{N}(B)$.

   (b) If $A - B$ is a projector, show that it is the projector into $\mathcal{C}(A) \cap \mathcal{N}(B)$ along $\mathcal{N}(A) \bigoplus \mathcal{C}(B)$.

   (c) Let $\mathcal{C}(AB) \subseteq \mathcal{C}(B)$. Then show that AB is a projector and that BA need not be a projector.