MTH207M (2023-24, ODD SEMESTER)

PROBLEM SET 1

1. Show that $\rho(A) = \text{tr}(GA)$ if G is a g-inverse of A.

2. Let A be an $m \times n$ matrix. Show that an $n \times m$ matrix G is a g-inverse of A iff $\rho(I - GA) = n - \rho(A)$.

3. For a g-inverse G of A, $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ iff $CGA = C$.

4. Show that $B^- A^-$ need not, in general, be a g-inverse of AB. However, if $\rho(AB) = \rho(A)$, show that $B(AB)^-$ is a g-inverse of A and that $B^- B(AB)^-$ is a g-inverse of AB.

5. Let G be a g-inverse of A. Then prove that

$$\{G + (I - GA)U + V(I - AG) \mid U, V \text{ arbitrary }\}$$

   is the class of all g-inverses of A. (Hint: If H is a g-inverse of A, take $U = H - G$.)

6. Let B be an $m \times n$ matrix and G an $n \times m$ matrix. Then, for any $r \times m$ full column rank matrix A and $n \times p$ full row rank matrix C, G is a generalized inverse of ABC if and only if $CGA = H$ for some generalized inverse H of B.

7. Show that if G is a g-inverse of $X^T X$, then

   (a) G^T is a g-inverse of $X^T X$, and

   (b) GX^T is a g-inverse of X.

8. Let $\rho(A + B) = \rho(A) + \rho(B)$. Then, show that

(a) $\mathcal{C}(A) \subseteq \mathcal{C}(A+B)$ and $\mathcal{C}(B) \subseteq \mathcal{C}(A+B)$, and

(b) for every g-inverse G of $A+B$, show that $AGA = A$ and $AGB = 0$.

9. Let A and B be two matrices such that AB exists. Then, show that

   (a) $B^- A^-$ is a g-inverse of AB iff $A^- ABB^-$ is idempotent.

   (b) If A has full column rank and B has full row rank, then $B^- A^-$ is a g-inverse of AB.