Quadratic forms

Theorem 1

Let $\mathbf{y} \sim \mathcal{N}(0, \mathrm{I}_n)$ and let A_1 and A_2 be symmetric idempotent matrices. Then $\mathbf{y}'\mathrm{A}_1\mathbf{y}$ and $\mathbf{y}'\mathrm{A}_2\mathbf{y}$ are independent if and only if $\mathrm{A}_1\mathrm{A}_2 = 0$.

Proof: Suppose that $\mathrm{A}_1\mathrm{A}_2 = 0$. Then by **Result 4** of Linear Model III, $\mathrm{A}_1\mathbf{y}$ and $\mathrm{A}_2\mathbf{y}$ are independent. Hence

$$\mathbf{y}'\mathrm{A}_1\mathbf{y} = (\mathrm{A}_1\mathbf{y})'(\mathrm{A}_1\mathbf{y}),\ \mathbf{y}'\mathrm{A}_2\mathbf{y} = (\mathrm{A}_2\mathbf{y})'(\mathrm{A}_2\mathbf{y})$$

are independent, since they are (measurable) functions of independent random variables.

Quadratic forms

Conversely, let $\mathbf{y}'A_1\mathbf{y}$ and $\mathbf{y}'A_2\mathbf{y}$ are independent. By Cochran Theorem $\mathbf{y}'A_1\mathbf{y}$ and $\mathbf{y}'A_2\mathbf{y}$ are distributed as chi–squared RVs. SInce the sum of independent chi–squared RVs is also a chi-squared RV i.e.

$$\mathbf{y}'A_1\mathbf{y} + \mathbf{y}'A_2\mathbf{y} = \mathbf{y}'(A_1 + A_2)\mathbf{y}$$

must be chi-square.
Again by Cochran Theorem, $A_1 + A_2$ is independent. Therefore,

$$\begin{aligned} A_1 + A_2 &= (A_1 + A_2)^2 \\ &= A_1^2 + A_2^2 + A_1A_2 + A_2A_1 \\ &= A_1 + A_2 + A_1A_2 + A_2A_1. \end{aligned}$$

Quadratic forms

Hence

$$A_1A_2 + A_2A_1 = 0.$$

This gives, upon postmultiplying by A_2,

$$A_1A_2 + A_2A_1A_2 = 0. \tag{1}$$

Premultiplying (1) by A_2 to get

$$A_2A_1A_2 + A_2^2A_1A_2 = 2A_2A_1A_2 = 0.$$

Hence $A_2A_1A_2 = 0$. Substituting in (1), we get $A_1A_2 = 0$.

Quadratic and linear form

Theorem 2

Let $\mathbf{y} \sim \mathcal{N}(0, \mathrm{I}_n)$. *Let* A *be a symmetric idempotent matrix and let* $\mathbf{l} \in \mathbb{R}^n$ *be a nonzero vector. Then* $\mathbf{y}'\mathrm{A}\mathbf{y}$ *and* $\mathbf{l}'\mathbf{y}$ *are independent if and only if* $\mathrm{A}\mathbf{l} = 0$.

Quadratic and linear form

Proof: We assume, without loss of generality, that $||\mathbf{l}|| = 1$. Then $\mathrm{B} = \mathbf{l}\mathbf{l}'$ is a symmetric and idempotent matrix.
First suppose that $\mathbf{y}'\mathrm{A}\mathbf{y}$ and $\mathbf{l}'\mathbf{y}$ are independent. Then using, as before, the fact that (measurable) functions of independent random variables are independent, we conclude that $\mathbf{y}'\mathrm{A}\mathbf{y}$ and $\mathbf{y}'\mathrm{B}\mathbf{y}$ are independent. Thus it follows from Theorem 1 that $\mathrm{AB} = 0$, and then it is easy to prove that $\mathrm{A}\mathbf{l} = 0$.
Conversely, if $\mathrm{A}\mathbf{l} = 0$, then by **Result 4** of Linear Model III, $\mathrm{A}\mathbf{y}$ and $\mathbf{l}'\mathbf{y}$ are independent. Hence $\mathbf{y}'\mathrm{A}\mathbf{y} = (\mathrm{A}\mathbf{y})'(\mathrm{A}\mathbf{y})$ and $'\mathbf{y}$ are independent. This completes the proof.

Cochran's theorem:MTF

Theorem 3

Let $\mathrm{A}_1, \ldots, \mathrm{A}_k$ *be* $n \times n$ *matrices with* $\sum_{i=1}^{k} \mathrm{A}_i = \mathrm{I}$. *Then the following conditions are equivalent:*

(i) $\sum_{i=1}^{k} Rank(\mathrm{A}_i) = n$.

(ii) $\mathrm{A}_i^2 = \mathrm{A}_i$, $i = 1, \ldots, k$.

(iii) $\mathrm{A}_i \mathrm{A}_j = 0$, $i \neq j$.

Cochran's theorem:MTF

Proof: (i) $\Longrightarrow$ (ii): Let $\mathrm{A}_i = \mathrm{B}_i\mathrm{C}_i$ be a rank factorization, $i = 1, \ldots, k$. Then

$$\mathrm{B}_1\mathrm{C}_1 + \cdots + \mathrm{B}_k\mathrm{C}_k = \mathrm{I},$$

and hence

$$\begin{bmatrix} \mathrm{B}_1 & \cdots & \mathrm{B}_k \end{bmatrix} \begin{bmatrix} \mathrm{C}_1 \\ \vdots \\ \mathrm{C}_k \end{bmatrix} = \mathrm{I}.$$

Since $\sum_{i=1}^{k} \mathrm{Rank}(\mathrm{A}_i) = n$, $\begin{bmatrix} \mathrm{B}_1 & \cdots & \mathrm{B}_k \end{bmatrix}$ is a square matrix.

Cochran's theorem:MTF

Therefore,

$$\begin{bmatrix} \mathrm{C}_1 \\ \vdots \\ \mathrm{C}_k \end{bmatrix} \begin{bmatrix} \mathrm{B}_1 & \cdots & \mathrm{B}_k \end{bmatrix} = \mathrm{I}.$$

Thus $\mathrm{C}_i \mathrm{B}_j = 0$, $i \neq j$. It follows that for $i \neq j$,

$$\mathrm{A}_i \mathrm{A}_j = \mathrm{B}_i \mathrm{C}_i \mathrm{B}_j \mathrm{C}_j = 0.$$

Cochran's theorem:MTF

(iii) $\implies$ (ii): Since $\sum_{i=1}^{k} \mathrm{A}_i = \mathrm{I}$,

$$\mathrm{A}_j(\sum_{i=1}^{k} \mathrm{A}_i) = \mathrm{A}_j,\ j = 1, \ldots, k.$$

It follows that $\mathrm{A}_j^2 = \mathrm{A}_j$.

Cochran's theorem:MTF

(ii) $\implies$ (i): Since A_i is idempotent, $\text{Rank}(\mathrm{A}_i) = \text{trace}(\mathrm{A}_i)$. Now

$$\sum_{i=1}^{k} \text{Rank}(\mathrm{A}_i) = \sum_{i=1}^{k} \text{trace}(\mathrm{A}_i) = \text{trace}(\sum_{i=1}^{k} \mathrm{A}_i) = n.$$

This completes the proof.

GLH

Assume that in our usual linear regression model we have $\mathbf{y} \sim \mathcal{N}(\mathrm{X}\boldsymbol{\beta}, \sigma^{\mathrm{I}})$, where $\mathbf{y}$ is $n \times 1$, X is $n \times p$, and $\boldsymbol{\beta}$ is $p \times 1$. Let $\mathsf{R}(\mathrm{X}) = r$.

We have seen that

$$RSS = \min_{\boldsymbol{\beta}} (\mathbf{y} - \mathrm{X}\boldsymbol{\beta})'(\mathbf{y} - \mathrm{X}\boldsymbol{\beta}),$$

and the minimum is attained at

$$\widehat{\boldsymbol{\beta}} = (\mathrm{X}'\mathrm{X})^{-}\mathrm{X}'\mathbf{y}.$$

GLH

Theorem 4

$$\frac{RSS}{\sigma^2} \sim \chi^2_{n-r}.$$

Proof: We know that $RSS = \mathbf{y}'\mathrm{P}\mathbf{y}$ where $\mathrm{P} = \mathrm{I} - \mathrm{X}(\mathrm{X}'\mathrm{X})^{-}\mathrm{X}'$. Since $\mathrm{PX} = 0$, it can be shown that $RSS = (\mathbf{y} - \mathrm{X}\boldsymbol{\beta})'\mathrm{P}(\mathbf{y} - \mathrm{X}\boldsymbol{\beta})$. Next, let us define $\mathbf{y}^* = (\mathbf{y} - \mathrm{X}\boldsymbol{\beta})/\sigma$. Then $\mathbf{y}^* \sim \mathcal{N}(0, \mathrm{I})$. Also, P is symmetric and idempotent with rank $n - r$. Therefore by Cochran Theorem, $\mathbf{y}^{*'}\mathbf{y}^* = RSS/\sigma^2 \sim \chi^2_{n-r}$.

GLH

Consider the following hypothesis testing problem

$$H_0 : \mathrm{L}\boldsymbol{\beta} = \mathrm{z} \text{ versus } H_1 : \mathrm{L}\boldsymbol{\beta} \neq \mathrm{z}$$

where $\mathcal{R}(\mathrm{L}) \subset \mathcal{R}(\mathrm{X})$ and that the system of equations $\mathrm{L}\boldsymbol{\beta} = \mathrm{z}$ is consistent.

As before, let $\mathrm{L} = \mathrm{W}\mathrm{X}'\mathrm{X}$, $\mathrm{W}\mathrm{X}' = \mathrm{T}$. Then

$$RSS_H = \min_{\boldsymbol{\beta}:\mathrm{L}\boldsymbol{\beta}=\mathrm{z}} (\mathbf{y} - \mathrm{X}\boldsymbol{\beta})'(\mathbf{y} - \mathrm{X}\boldsymbol{\beta})$$

is attained at $\widetilde{\boldsymbol{\beta}}$, where

$$\widetilde{\boldsymbol{\beta}} = \widehat{\boldsymbol{\beta}} - (\mathrm{X}'\mathrm{X})^{-}\mathrm{L}'(\mathrm{T}\mathrm{T}')^{-}(\mathrm{L}\widehat{\boldsymbol{\beta}} - \mathrm{z}).$$

GLH

Therefore,

$$
\begin{aligned}
\mathrm{X}\widetilde{\boldsymbol{\beta}} &= (\mathrm{I}-\mathrm{P})\mathbf{y} - \mathrm{T}'(\mathrm{T}\mathrm{T}')^{-}(\mathrm{T}\mathbf{y}-\mathsf{z}) \\
&= (\mathrm{I}-\mathrm{P})\mathbf{y} - \mathrm{T}'(\mathrm{T}\mathrm{T}')^{-}(\mathrm{T}\mathbf{y}-\mathrm{T}\mathrm{X}\boldsymbol{\beta}+\mathrm{T}\mathrm{X}\boldsymbol{\beta}-\mathsf{z}) \qquad (2)
\end{aligned}
$$

Under the null hypothesis (when H_0 is true), $\mathrm{TX}\boldsymbol{\beta} = \mathrm{L}\boldsymbol{\beta} = \mathsf{z}$, therefore by (2),

$$\mathbf{y} - \mathrm{X}\widetilde{\boldsymbol{\beta}} = \mathrm{P}\mathbf{y} + \mathrm{U}(\mathbf{y} - \mathrm{X}\boldsymbol{\beta}),$$

where $\mathrm{U} = \mathrm{T}'(\mathrm{T}\mathrm{T}')^{-}\mathrm{T}$.

GLH

Thus

$$RSS_H = (\mathbf{y} - \mathrm{X}\widetilde{\boldsymbol{\beta}})'(\mathbf{y} - \mathrm{X}\widetilde{\boldsymbol{\beta}}) = \mathbf{y}'\mathrm{P}\mathbf{y} + (\mathbf{y} - \mathrm{X}\boldsymbol{\beta})'\mathrm{U}(\mathbf{y} - \mathrm{X}\boldsymbol{\beta}),$$

as $\mathrm{PU} = 0$. Since U is symmetric and idempotent, we conclude that

$$\frac{RSS_H - RSS}{\sigma^2} \sim \chi^2_{\mathrm{Rank}(\mathrm{U})}. \tag{3}$$

By Theorem 16 of Projection and Theorem 4 of Linear Model II, $\mathrm{Rank}(\mathrm{U}) = \mathrm{Rank}(\mathrm{T}) = \mathrm{Rank}(\mathrm{L})$.

GLH

Moreover, $\mathrm{PU} = 0$. Thus, two quadratic forms RSS and $RSS_H - RSS$ are independent. We conclude that

$$\frac{(RSS_H - RSS)/\mathrm{Rank}(\mathrm{L})}{RSS/(n-r)} \sim F(\mathrm{Rank}(\mathrm{L}), n-r) \tag{4}$$

which can be used to test H_0.

Remark: Conclusion in (4) follows from the following result:

Theorem 5

Let X and Y be two independent chi–sqrue RVs with dfs n_1 and n_2 respectively. Then

$$\frac{X/n_1}{Y/n_2} \sim F(n_1, n_2).$$