(8)

$$\frac{\partial \log \det (X'AX)}{\partial x_{ij}}$$

$$= tr\left((X'AX)^{-1} \frac{\partial (X'AX)}{\partial x_{ij}} \right)$$

$$= tr\left((X'AX)^{-1} \left[\left(\frac{\partial X}{\partial x_{ij}} \right)' AX + X'A \left(\frac{\partial X}{\partial x_{ij}} \right) \right] \right)$$

$$= tr\left[(X'AX)^{-1} \left(\frac{\partial X}{\partial x_{ij}} \right)' AX \right]$$

$$+ tr\left[(X'AX)^{-1} X'A \left(\frac{\partial X}{\partial x_{ij}} \right) \right]$$

$$= tr\left[(X'AX)^{-1} (u_i u_j')' AX \right]$$

$$+ tr\left[(X'AX)^{-1} X'A \, u_i u_j' \right]$$

$$= tr\left[(X'AX)^{-1} u_j u_i' AX \right]$$

$$+ tr\left[(X'AX)^{-1} X'A \, u_i u_j' \right]$$

$$= tr\left[u_i' AX (X'AX)^{-1} u_j \right] + tr\left[u_j' (X'AX)^{-1} X'A u_i \right]$$

$$= u_i' AX (X'AX)^{-1} u_j + u_j' (X'AX)^{-1} X'A u_i$$

$$= \underbrace{u_i' AX(X'AX)^{-1} u_j}_{\substack{ij\text{th element}\\ \text{of } AX(X'AX)^{-1}}} + \underbrace{u_j'(X'AX)^{-1}X'Au_i}_{\substack{ij\text{th element of}\\ (((X'AX)^{-1}X'A)'}}$$

$$\therefore \frac{\partial \log\det(X'AX)}{\partial X} = AX(X'AX)^{-1} + ((X'AX)^{-1}X'A)'$$

(7) $\dfrac{\partial\, tr(X^k)}{\partial x_{ij}}$ $\qquad X \cdots X$

$$= tr\left(\frac{\partial X^k}{\partial x_{ij}}\right)$$

$$= tr\left[X^{k-1}\cdot\frac{\partial X}{\partial x_{ij}} + X^{k-2}\frac{\partial X}{\partial x_{ij}}\cdot X + \cdots + \frac{\partial X}{\partial x_{ij}}X^{k-1}\right]$$

$$= tr\left[X^{k-1}\frac{\partial X}{\partial x_{ij}}\right] + tr\left[X^{k-2}\frac{\partial X}{\partial x_{ij}}\cdot X\right] + \cdots + tr\left[\frac{\partial X}{\partial x_{ij}}X^{k-1}\right]$$

$$= tr\left[X^{k-1}\frac{\partial X}{\partial x_{ij}}\right] + tr\left[X^{k-1}\frac{\partial X}{\partial x_{ij}}\right] + \cdots + tr\left[X^{k-1}\frac{\partial X}{\partial x_{ij}}\right]$$

$$= k \, tr\left[X^{k-1} \frac{\partial X}{\partial x_{ij}} \right]$$

$$= k \, tr\left[X^{k-1} \cdot u_i u_j' \right]$$

$$= k \, tr\left[u_j' X^{k-1} u_i \right]$$

$$= k \, \underbrace{u_j' X^{k-1} u_i}_{ij \text{ th element of } (X^{k-1})' = (X')^{k-1}}$$

$$\therefore \frac{\partial \, tr(X^k)}{\partial X} = k (X')^{k-1} .$$

(6) $\quad F^2 = F$

$$\Rightarrow F \cdot F = F$$

$$\Rightarrow F \frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial x_j} \cdot F = \frac{\partial F}{\partial x_j} .$$

$$\Rightarrow F^2 \frac{\partial F}{\partial x_j} + F \frac{\partial F}{\partial x_j} F = F \frac{\partial F}{\partial x_j} .$$

$$\Rightarrow F \frac{\partial F}{\partial x_j} + F \frac{\partial F}{\partial x_j} F = F \frac{\partial F}{\partial x_j} .$$

$$\Rightarrow F \frac{\partial F}{\partial x_j} F = 0 .$$

(5) $\qquad$ for $j \neq i$

$$\frac{\partial\, tr(AX)}{\partial x_{ij}}$$

$$= tr\, \frac{\partial (AX)}{\partial x_{ij}}$$

$$= tr\left[A \frac{\partial X}{\partial x_{ij}}\right]$$

$$= tr[A(u_i u_j' + u_j u_i')]$$

$$= tr[A u_i u_j'] + tr[A u_j u_i']$$

$$= tr[u_j' A u_i] + tr[u_i' A u_j]$$

$$= \underbrace{u_j' A u_i}_{ij\text{th element of } A'} + \underbrace{u_i' A u_j}_{ij\text{th element of } A}$$

for $j = i$

$$\frac{\partial}{\partial x_{ii}} tr(AX)$$

$$= tr\, \frac{\partial (AX)}{\partial x_{ii}} = tr\left[A \frac{\partial X}{\partial x_{ii}}\right]$$

$$= tr[A u_i u_i']$$

$$= tr[u_i' A u_i]$$

$= u_i' A u_i = a_{ii} \quad \forall\, i = 1(1)m.$

$$\therefore \frac{\partial \operatorname{tr}(AX)}{\partial X}$$

$$= A + A' - \operatorname{diag}(a_{11}, a_{22}, \ldots, a_{mm}).$$

(1) A matrix is projection matrix iff it is Symmetric and Idempotent.

Proof: Let X be a projection matrix, P_X

$$P_X = X(X'X)^- X'$$

only If :-

$$\therefore X = P_X$$

Now, $X^T = P_X^T = (X(X'X)^- X')'$

$$= X'(X'X)^- X$$

$$= P_X = X.$$

$\therefore$ X is Symmetric Matrix.

$$X^2 = X \cdot X$$
$$= P_X \cdot P_X$$
$$= X(X'X)^- X' X (X'X)^- X'$$
$$= P_X X (X'X)^- X'$$
$$= X(X'X)^- X'$$
$$= P_X.$$

$$\boxed{P_X X = X}$$

If X is symmetric and idempotent

$\therefore X^2 = X$ and $X' = X$.

Then, $P_X = X(X'X)^- X'$
$$= X(X \cdot X)^- X'$$
$$= X(X^2)^- X'$$
$$= X X^- X'$$
$$= X X^- X$$
$$= X.$$

$\therefore$ If X be a matrix which is symmetric and idempotent, then

$PX = X$. (Proved)

(Q)(2) Given, $\mathcal{C}(X) = \mathcal{U}$

Now, $w \in \mathcal{U} = \mathcal{C}(X)$

$$\therefore \ w = \underset{n\times p}{X} \underset{p\times 1}{b} \quad \text{(By Definition of Column space)}$$

Now

$$(y-w)'(y-w)$$

$$= (y - Xb)'(y - Xb)$$

$$= (y - Xb^* + Xb^* - Xb)'(y - Xb^* + Xb^* - Xb)$$

$$= (y - Xb^*)'(y - Xb^*) + (y - Xb^*)'(Xb^* - Xb) + (Xb^* - Xb)'(y - Xb^*) + (Xb^* - Xb)'(Xb^* - Xb)$$

$$= (y - Xb^*)'(y - Xb^*) + 2(y - Xb^*)'(Xb^* - Xb) + (Xb^* - Xb)'(Xb^* - Xb)$$

Now, $2(y - Xb^*)'(Xb^* - Xb)$

$$= 2(Xb^* - Xb)'(y - Xb^*)$$

$$= 2(b^* - b)'X'(y - Xb^*)$$

$$= 2(b^* - b)'(X'y - X'Xb^*)$$

$$= 0$$

By Normal Equation

$X^T X b^* = X^T \underset{\sim}{y}$

as, b^* is a

solution of

$X^T X b = X^T y$.

Now,

$$(y-w)'(y-w)$$

$$= (y-Xb^*)'(y-Xb^*) + (Xb^*-Xb)'(Xb^*-Xb)$$

$$\geq (y-Xb^*)'(y-Xb^*)$$

[As,

$2(y-Xb^*)'(Xb^*-Xb)$

$= 0$

and

$(b^*-b)'X'X(b^*-b)$

≥ 0]

So, '=' holds

iff $w = Xb^*$

So, $(y-w)'(y-w)$

minimized / attained

min. value at

$w = Xb^*$.

Now, If $w = P_X y = X(X^TX)^- X^T y$

If $X^TXb = X^T\underset{\sim}{y}$ is the normal eqn then

$$b^* = (X^TX)^- X^T\underset{\sim}{y}$$

when X^TX does not have full rank.

Now, Similarly

$$(y-w)'(y-w)$$

$$= (y - P_x y + P_x y - w)'(y - P_x y + P_x y - w)$$

$$= (y - P_x y)'(y - P_x y)$$
$$+ (y - P_x y)'(P_x y - w)$$
$$+ (P_x y - w)'(y - P_x y)$$
$$+ (P_x y - w)'(P_x y - w)$$

$$= (y - P_x y)'(y - P_x y)$$
$$+ 2(y - P_x y)'(P_x y - w)$$
$$+ (P_x y - w)'(P_x y - w)$$

Now, $(y - P_x y)'(P_x y - w)$

$$= y'(I - P_x)'(P_x y - w)$$

w can written as $w = Xb$ as $w \in \mathcal{C}(X)$

then,

$$(P_X y - w)$$

$$= (X(X'X)^- X'y - Xb)$$

(By Normal Equation $X'Xb = X'y$)

$$= (\underline{X(X'X)^- X'} X b - Xb)$$

$$= (P_X X b - Xb) \quad (\because P_X = X(X'X)^- X')$$

$$= (Xb - Xb)$$

$$= 0.$$

$$\therefore (y - P_X y)'(P_X y - w)$$

$$= 0.$$

$$\therefore (y - w)'(y - w) \geq (y - P_X y)'(y - P_X y)$$

$$\left[\because (P_X y - w)'(P_X y - w) \right.$$

$$= (X(X'X)^- X'y - Xb)'(X(X'X)^- X'y - Xb)$$

$$\left. = (P_X y - b)'X'X(P_X y - b) \geq 0 \right]$$

$\text{Min. } (y-w)'(y-w)$

$= (y - Xb^*)'(y - Xb^*)$

$= y'(y - Xb^*) - (Xb^*)'(y - Xb^*)$

$= y'(y - Xb^*) - (b^{*\prime}X'y - b^{*\prime}X'Xb^*)$

$= y'(y - Xb^*) - (b^{*\prime}X'Xb^* - b^{*\prime}X'Xb^*)$

$= y'(y - Xb^*) - 0$

$= y'(y - Xb^*)$ (<u>Proved</u>)

Again, $\text{Min}(y-w)'(y-w)$

$= (y - P_x y)'(y - P_x y)$

$= ((I - P_x)y)'(y - P_x y)$

$= y'(I - P_x)'(I - P_x)y$

$= y'(I - P_x - P_x + P_x^2)y$

$= y'(I - P_x - P_x + P_x)y$

[$\because P_x$ is idempotent $\therefore P_x^2 = P_x$]

$= y'(I - P_x)y.$ (Proved)

(P) (3) If $y \cdot x_i = 0 \quad \forall i = 1(1)s$

$\boxed{1st}$

Then, $\{x_1, \ldots, x_s\}$ be a set of matrices that spans u

So, $\text{Span}\{x_1 \ldots x_s\}$

$$= \left\{ \sum_{i=1}^{s} a_i x_i \; ; \; a_i \in \mathbb{R} \right\} \in u$$

Now, $y \cdot \left(\sum_{i=1}^{s} a_i x_i \right)$

$$= \sum_{i=1}^{s} a_i \cdot (y \cdot x_i) = 0$$

$\therefore y \perp u.$

Only If:-

$y \perp u$ $\quad \sum_{i=1}^{s} a_i x_i = Sp\{x_1, \ldots, x_s\}$

Given, $y \cdot \sum_{i=1}^{s} a_i x_i = 0$

$\Rightarrow \sum_{j=1}^{s} a_j \, y \cdot x_j = 0$

Now, WLG, $a_j = 0 \ \forall j \neq i$

i.e $a_i \neq 0$

Then, $a_i \, y \cdot x_i = 0$

$\Rightarrow y \cdot x_i = 0$

(as $a_i \neq 0$)

(Proved)

2nd

If $x_i \cdot z_j = 0$

Now, Let $x \in u$

$x = \sum_{i=1}^{s} a_i x_i$.

as $\{x_1, \ldots, x_s\}$ Span u.

let $z \in W$. $z = \sum_{j=1}^{t} b_j z_j$ as $\{z_1, \ldots, z_t\}$ Span W.

Now, $x \cdot z$

$$= \sum_{i=1}^{s} \sum_{j=1}^{t} a_i x_i \, b_j z_j$$

$$= \sum_{i=1}^{s} \sum_{j=1}^{t} a_i b_j \, x_i \cdot z_j$$

$$= 0. \quad [\because x_i \cdot z_j = 0 \;\; \forall\, i = 1(1)s,\; j = 1(1)t]$$

$\therefore$ $U \perp W$ (proved)

<u>Only If</u> :- If $U \perp W$

then $x \cdot z = 0$

for $x \in U$

and $z \in W$.

Now, $x \cdot z = 0$

$$\Rightarrow \sum_{r=1}^{s} \sum_{k=1}^{t} a_r x_r \, b_k z_k = 0$$

$$\Rightarrow \sum_{r=1}^{s} \sum_{k=1}^{t} a_r b_k \, x_r z_k = 0$$

so, WLG $a_r b_k = 0$

$\forall$ $r \neq i$

and $k \neq j$

$\therefore\ a_i b_j \neq 0$

$\therefore\ a_i b_j\, X_i \cdot Z'_j = 0$

$\Rightarrow X_i \cdot Z'_j = 0$ as, $a_i b_j \neq 0$.

$\forall\ i = 1(1)s \quad j = 1(1)t$

(proved)

P
(4) let $r = \dim(U)$ and $s = \dim(V)$

So, $r < s$

Let $\{P_1, P_2, \ldots, P_r\}$ be basis of

of U

and $\{Q_1, Q_2, \ldots, Q_s\}$ be

basis of V respectively.

Now, let $D = \sum_{i=1}^{s} x_i Q_i \in V$

Now, $P_i \cdot D = x_1 (P_i Q_1) + x_2 (P_i Q_2) + \cdots$

$\cdots + x_s (P_i Q_s)$

Let $\overset{r\times s}{N}$ be a matrix whose (i,j)th element

is $P_i \cdot Q_j = n_{ij}$

$r(N) \leq r < s$

So, $Nx = 0$

$\Rightarrow \exists$ some non-null vector $x = \{x_j\}$

Such that $Nx = 0$

as $r(N) < s$.

So, $D = \sum_{i=1}^{s} x_i Q_i \in V$

is non-null Matrix.

Because, $\{Q_1, \ldots, Q_s\}$

is a basis of V. So, they

Can not be null matrix.

$$\therefore P_i D = \sum_{j=1}^{s} x_j P_i Q_j$$

$$= \sum_{j=1}^{s} n_{ij} x_j .$$

$$\sum_{j=1}^{s} n_{ij} x_j = 0 \quad \text{as } Nx = 0$$

So, $\quad p_i \cdot D = 0$

$\therefore$ V contains a non-null matrix

or $\exists$ a non-null matrix in V for which that matrix is orthogonal to U.

(proved)