P 1. $E(Y_1) = \theta_1 - \theta_3$

$E(Y_2) = \theta_1 + \theta_2 - \theta_3$

$E(Y_3) = \theta_1 - \theta_3$

$E(Y_4) = \theta_1 - \theta_2 - \theta_3$

So, our Model is,

$$\underset{\sim}{Y} = X\underset{\sim}{\theta} + \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}_{4\times 1} \quad X = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{pmatrix}_{4\times 3} \quad \underset{\sim}{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}_{3\times 1}$$

X is the Design Matrix ; θ_i's are unknown $\forall\, i = 1(1)3$

$$X = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X^T X = \overset{3\times 4}{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}} \overset{4\times 3}{\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{pmatrix}}$$

$$= \begin{pmatrix} 4 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & -4 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(1)

$\theta_1 = \underset{\sim}{c}_1^T \underset{\sim}{\theta}$ will be estimable if $\underset{\sim}{c}_1^T \in R(X)$
$= R(X^T X)$

$$\theta_1 = (1\ 0\ 0) \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$$

$\underset{\sim}{c_1}^T = (1\ 0\ 0) \notin R(X^T X) = R(X)$

$\therefore \theta_1$ is not estimable.

$$\theta_3 = (0\ 0\ 1)\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = \underset{\sim}{c_3}^T \underset{\sim}{\theta}$$

$\underset{\sim}{c_3}^T = (0\ 0\ 1) \notin R(X^T X) = R(X)$

$\therefore \theta_3$ is not estimable.

(2) $$\theta_1 + \theta_2 = (1\ 1\ 0)\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix}$$

$$= \underset{\sim}{d}'\underset{\sim}{\theta}$$

$\underset{\sim}{d}' \notin R(X^T X) = R(X)$.

$\therefore \theta_1 + \theta_2$ is not estimable.

(3) $\theta_1 - \theta_3 = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$

$= \underset{\sim}{e}^T \underline{\theta}$.

$\underset{\sim}{e}^T = (1\ 0\ -1) \in R(X^T X) = R(X)$

$\therefore\ \theta_1 - \theta_3$ is estimable.

(4) $\theta_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$

$= \underset{\sim}{d}' \underline{\theta}$

$\underset{\sim}{d}' = (0\ 1\ 0) \in R(X^T X) = R(X)$

$\therefore\ \theta_2$ is estimable.

P(2) In a linear Regression Model,

$$\underset{n\times 1}{E(y)} = \underset{n\times p}{X}\underset{p\times 1}{\beta}.$$

So, $\underset{n\times 1}{\underset{\sim}{Y}} = \underset{n\times p}{X}\underset{p\times 1}{\underset{\sim}{\beta}} + \underset{n\times 1}{\underset{\sim}{e}}$ is our Model.

Now, If $r(\underset{n\times p}{X}) = p$

then $r(\underset{p\times p}{X^T X}) = p$

$\therefore X^T X$ possesed

full rank.

Now, $(X^T X)^{-1}$ exist.

and $\hat{\beta} = (X^T X)^{-1} X^T \underset{\sim}{Y}$ is the

unique sol^n.

Now $E(l^T \hat{\beta}) = E(l^T (X^T X)^{-1} X^T \underset{\sim}{Y})$

$= l^T (X^T X)^{-1} X^T X \underline{\beta}$

$= l^T \beta \quad \forall \underset{\sim}{l}'$

$\therefore$ All linear parametric function estimable.

1(3) $E(y_1) = 2\beta_1 - \beta_2 - \beta_3$

$E(y_2) = \beta_2 - \beta_4$

$E(y_3) = \beta_2 + \beta_3 - 2\beta_4$

So, our model is $\underset{\sim}{Y} = X\underline{\beta} + \underline{e}$

$$X = \begin{pmatrix} 2 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & -2 \end{pmatrix} \quad \underline{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

$$X = \begin{pmatrix} 2 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}_{3\times 4}$$

$r(X) = 4$

X has full coloumn rank 4.

So, $(X^T X)$ has full rank.

Now, $\hat{\beta}$ is unique.

So, all linear parametric function

$\underset{\sim}{l}^T \underset{\sim}{\beta}$ estimable.

So, $E(\underset{\sim}{c}^T \underset{\sim}{y}) = \underset{\sim}{l}^T \underset{\sim}{\beta}$

$$\Rightarrow c_1(2\beta_1 - \beta_2 - \beta_3) + c_2(\beta_2 - \beta_4) + c_3(\beta_2 + \beta_3 - 2\beta_4)$$

$$= l_1\beta_1 + l_2\beta_2 + l_3\beta_3 + l_4\beta_4$$

$$\Rightarrow \beta_1(2c_1) + \beta_2(-c_1 + c_2 + c_3) + \beta_3(-c_1 + c_3) + \beta_4(-c_2 - 2c_3)$$
$$= l_1\beta_1 + l_2\beta_2 + l_3\beta_3 + l_4\beta_4$$

$\therefore \lambda_1 = 2c_1$

$\lambda_2 = c_2 + c_3 - c_1$

$\lambda_3 = -c_1 + c_3$

$\lambda_4 = -c_2 - 2c_3.$

$\therefore$ all linear pararametric function

$\underset{\sim}{l}^T \underset{\sim}{\beta}$ estimable where

$(\lambda_1, \lambda_2 \lambda_3 \lambda_4) = \underset{\sim}{l}^T$ is such that

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$

$= 2c_1 + c_2 + c_3 - c_1 - c_1 + c_3$
$- c_2 - 2c_3$

$= 2c_1 - 2c_1 + c_2 - c_2 + 2c_3 - 2c_3$

$= 0.$

P 4.

Our Model is, $\underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{e}$ — (1)

$$X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix}^{3\times 2} \quad \underset{\sim}{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$X^T X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}^{2\times 2}$$

$X^T X$ is non-singular and $(X^T X)^{-1}$ exists. So, $X^T X$ has full rank 2.

Now, all parametric functions $\underset{\sim}{c}^T \underset{\sim}{\beta}$ estimable.

So the Solution of (1) $\hat{\beta}$ is unique.

So, $\hat{\beta} = (X^T X)^{-1} X^T \underline{y}$ is unique.

So, LSE $\hat{\beta}$ exist.

Now, By <u>Gauss-Markov</u> Theorem $\underline{c}^T \underline{\beta}$

1. $\underset{\sim}{c}^T \hat{\beta}$ is <u>unique</u> <u>and</u> <u>unbiased</u>.
2. $\underset{\sim}{c}^T \hat{\beta}$ is the <u>BLUE</u> of $\underline{c}^T \underline{\beta}$.

So, $c^T \underset{\sim}{\hat{\beta}}$ is BLUE of $c^T \underset{\sim}{\beta}$

$$2\beta_1 + \beta_2 = \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$= \underline{c}^T \underline{\beta}$$

$$\therefore \underline{c}^T = \begin{pmatrix} 2 & 1 \end{pmatrix}$$

Now,

$$\hat{\beta} = \overset{2\times 2}{\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}^{-1}} \overset{2\times 3}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}} \overset{3\times 1}{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}$$

$$= \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}^T \Big/ 14 \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix} \Big/ 14 \quad \begin{pmatrix} y_1 + y_2 + y_3 \\ y_1 - y_2 + 2y_3 \end{pmatrix}$$

$$= \begin{pmatrix} 3/7 & -1/7 \\ -1/7 & 3/14 \end{pmatrix}^{2\times 2} \begin{pmatrix} y_1 + y_2 + y_3 \\ y_1 - y_2 + 2y_3 \end{pmatrix}^{2\times 1}$$

$$= \begin{pmatrix} 3/7 \, (y_1 + y_2 + y_3) - 1/7 \, (y_1 - y_2 + 2y_3) \\ -1/7 \, (y_1 + y_2 + y_3) + 3/14 \, (y_1 - y_2 + 2y_3) \end{pmatrix}$$

$$= \begin{pmatrix} 2/7 \, y_1 + 4/7 \, y_2 + 1/7 \, y_3 \\ \frac{1}{14} y_1 - \frac{5}{14} y_2 + 2/7 \, y_3 \end{pmatrix}$$

$\therefore$ BLUE of $2\beta_1 + \beta_2$ is

$$c^T \hat{\beta} = \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 2/7 \, y_1 + 4/7 \, y_2 + 1/7 \, y_3 \\ 1/14 \, y_1 - 5/14 \, y_2 + 2/7 \, y_3 \end{pmatrix}$$

$$= \frac{4}{7} y_1 + \frac{8}{7} y_2 + \frac{2}{7} y_3$$

$$+ \frac{1}{14} y_1 - \frac{5}{14} y_2 + \frac{2}{7} y_3$$

$$= \frac{9}{14} y_1 + \frac{11}{14} y_2 + \frac{4}{7} y_3$$

$$= \frac{9y_1 + 11y_2 + 8y_3}{14} .$$

$\therefore$ BLUE of $2\beta_1 + \beta_2$ is

$$\frac{1}{14}(9y_1 + 11y_2 + 8y_3).$$

$$V\left(\frac{1}{14}(9y_1 + 11y_2 + 8y_3)\right)$$

$$= \left(\frac{1}{14}\right)^2 \left[81\, V(y_1) + 121\, V(y_2) + 64\, V(y_3)\right]$$

$$= \left(\frac{1}{14}\right)^2 \cdot \left[202 + 64\right] \sigma^2$$

$$= \frac{1}{196} \cdot 266\, \sigma^2 = \frac{266}{196} \sigma^2$$

$$= \frac{133}{98} \sigma^2 .$$

(b) $\underset{\sim}{Y} = X\underset{\sim}{\beta} + \underset{\sim}{e}$ is our Model.

$\underset{\sim}{e}$ is such that $E(\underset{\sim}{e}) = 0$

$D(\underset{\sim}{e}) = \sigma^2 I_n$.

Now, X is of full rank.

By Method of LS, we will estimate

$\underset{\sim}{\beta}$ by minimizing $\sum_{i=1}^{n} e_i^2 = \underset{\sim}{e}^T \underset{\sim}{e}$

$= (\underset{\sim}{y} - X\underset{\sim}{\beta})'(\underset{\sim}{y} - X\underset{\sim}{\beta})$

$= L(\underset{\sim}{\beta})$

$L(\underset{\sim}{\beta}) = \underset{\sim}{y}^T \underset{\sim}{y} - 2\underset{\sim}{y}^T X\underset{\sim}{\beta} + \underset{\sim}{\beta}' X' X\underset{\sim}{\beta}$.

$\therefore \dfrac{\partial L(\underset{\sim}{\beta})}{\partial \underset{\sim}{\beta}} = 0$

$\Rightarrow \dfrac{\partial}{\partial \underset{\sim}{\beta}} \left\{ \underset{\sim}{y}^T \underset{\sim}{y} - 2\underset{\sim}{y}^T X\underset{\sim}{\beta} + \underset{\sim}{\beta}' X' X\underset{\sim}{\beta} \right\} = 0$

$\Rightarrow \dfrac{\partial}{\partial \underset{\sim}{\beta}} \left\{ \underset{\sim}{y}^T \underset{\sim}{y} - \right.$ $\left. + \underset{\sim}{\beta}' X' X\underset{\sim}{\beta} \right] = 0$

$$\Rightarrow \quad -2X'\underset{\sim}{y} + 2X'X\underset{\sim}{\beta} = 0$$

$$\Rightarrow \quad \underbrace{X'X\underset{\sim}{\beta} = X'\underset{\sim}{y}}_{\text{Normal Equation}} \qquad -(1)$$

As, X is of full rank.

$\therefore X^T X$ is of full rank.

$\therefore (X^T X)^{-1}$ exists.

$\therefore$ From (1)

$$(X'X)^{-1}(X^T X)\underset{\sim}{\beta} = (X^T X)^{-1} X^T \underset{\sim}{y}$$

$\Rightarrow \underset{\sim}{\hat{\beta}} = (X^T X)^{-1} X^T \underset{\sim}{y}$ is the unique solution of β.

$\therefore \hat{\beta}$ is the unique LSE of β.

Now, As, $E(\underset{\sim}{y}) = X\underset{\sim}{\beta}$ and

$D(\underset{\sim}{y}) = \sigma^2 I_n$ and

$\hat{\beta}$ is the LSE of β so,

$c^T \hat{\beta}$ is the BLUE of $c^T \beta$

$$\underset{\sim}{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{pmatrix}$$

$$\hat{\underset{\sim}{\beta}} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_p \end{pmatrix}$$

By Gauss Markov Theorem,

$\underset{\sim}{c}^T \hat{\underset{\sim}{\beta}}$ is BLUE of $\underset{\sim}{c}^T \underset{\sim}{\beta}$.

Now, $\begin{array}{c} (0\ 0 - 1\ 0 - 0) \\ \uparrow_{ith} \end{array} \begin{pmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_p \end{pmatrix}$

$= \underset{\sim}{c}^T \hat{\underset{\sim}{\beta}}$ is

BLUE of

$\begin{array}{c} (0\ 0 - 1\ 0 - 0) \\ \uparrow_{ith} \end{array} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}$

$= c^T \underset{\sim}{\beta}$

$\Rightarrow \hat{\beta}_i$ is BLUE of $\beta_i \quad \forall\, i = 1(1)p$

So, $\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_p \end{pmatrix}$ is BLUE of

$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{pmatrix}$.

By Gauss-Markov Theorem.

Now, $E(\hat{\beta}) = E\left((X^T X)^{-1} X^T \underset{\sim}{y}\right)$

$= (X^T X)^{-1} X^T X \underset{\sim}{\beta}$

$= \underset{\sim}{\beta}$.

$V(\hat{\beta}) = V\left((X^T X)^{-1} X^T \underset{\sim}{y}\right)$

So, $V(\hat{\beta}) = D(\hat{\beta})$

$= D\left((X^T X)^{-1} X^T \underset{\sim}{y}\right)$

$$D\left((X^T X)^{-1} X^T \underset{\sim}{Y}\right)$$

$$= D(A\underset{\sim}{Y}) \qquad \left(\text{where, } A = (X^T X)^{-1} X^T\right)$$

$$= A\, D(\underset{\sim}{Y})\, A^T \qquad D(\underset{\sim}{Y}) = \sigma^2 I_n$$

$$= A(\sigma^2 I_n) A^T \qquad \text{, from the Model.}$$

$$= (X^T X)^{-1} X^T (\sigma^2 I_n) \left(X (X^T X)^{-1}\right)$$

$$= \sigma^2 (X^T X)^{-1} X^T X (X^T X)^{-1}$$

$$= \sigma^2 (X^T X)^{-1}$$

$$D(\hat{\beta})$$

$$= D((X^T X)^{-1} X^T Y)$$

$$= D(AY)$$

$$= A D(Y) A^T$$

$$= A \sigma^2 I_n A^T$$

$$(X^T X)^{-1} X^T \sigma^2 I_n \quad X (X^T X)^{-1}$$

$$= (X^T X)^{-1} X^T X (X^T X)^{-1} \sigma^2$$

$$= (X^T X)^{-1} \sigma^2$$

$$= \sigma^2 (X^T X)^{-1}$$

$$E(1'\hat{\beta})$$

$$= E(1'(X^T X)^- X^T \underset{\sim}{y})$$

$$= 1'(X^T X)^- X^T X \underset{\sim}{\beta}$$