(1) for any $r \times m$ matrix C of full Coloumn rank i.e of rank m

$\exists$ a left inv. of C say X

So, $XC = I$

Also, for any $n \times p$ matrix D of full row rank i.e rank n $\exists$ a right inv. of D say Y

So, $DY = I$

So, $CAD = CBD$

$\Rightarrow XCAD = XCBD$

$\Rightarrow IAD = IBD$

$\Rightarrow AD = BD$

$\Rightarrow ADY = BDY$

$\Rightarrow AI = BI$

$\Rightarrow A = B.$ (proved)

(2) <u>If</u> $CGA = H$ (Given)

$(ABC)G(ABC)$

$= A(BHB)C$

$= ABC$

$\therefore$ G is generalised inverse of ABC.

$\because$ H is g-inverse of B

$\therefore BHB = B$

<u>Only If</u> :- G is g-inverse of ABC

$\therefore (ABC)G(ABC) = ABC$ (Given)

Given, A has full column rank.

$\therefore \exists$ a left inverse of A say N

$\therefore NA = I$

Again, C has full row rank.

$\therefore$ $\exists$ a right inverse of C, say L.

$$\therefore CL = I$$

$$\therefore ABCGABL = ABL$$

$$\Rightarrow NABCGABLL = NABLL$$

$$\Rightarrow BCGAB = B$$

CGA is G-inverse of B

$$\therefore CGA = H.$$

(proved)

(P) (3) $\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ is g-inv. of $\begin{pmatrix} A & B \end{pmatrix}$

$$(A|B)\left(\frac{G_1}{G_2}\right)(A|B) = (A|B)$$

$$\Rightarrow (AG_1 + BG_2)(A \mid B) = (A \mid B)$$

$$\Rightarrow (AG_1A + BG_2A \mid AG_1B + BG_2B) = (A \mid B)$$

$$\therefore \quad AG_1A + BG_2A = A$$

$$AG_1B + BG_2B = B$$

Then, $\begin{pmatrix} A & KB \end{pmatrix} \begin{pmatrix} G_1 \\ K^{-1}G_2 \end{pmatrix} \begin{pmatrix} A & KB \end{pmatrix}$

$$= (AG_1 + BG_2)(A \mid KB)$$

$$= (AG_1A + BG_2A \mid K(AG_1B + BG_2B))$$

$$= (A \mid KB)$$

$$\therefore \begin{pmatrix} G_1 \\ K^{-1} G_2 \end{pmatrix} \text{ is g-inverse of } (A \; KB)$$

Again,

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \text{ is g-inv. of } (A | C)$$

$$(A|C) \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} (A|C) = (A|C)$$

$$\Rightarrow (AH_1 + CH_2)(A | C) = (A | C)$$

$$\Rightarrow (AH_1A + CH_2A \mid AH_1C + CH_2C) = (A | C)$$

$$\therefore AH_1A + CH_2A = A$$

$$AH_1C + CH_2C = C$$

Now,

$$(A \mid KC) \begin{pmatrix} H_1 \\ \hline K^{-1}H_2 \end{pmatrix} (A \mid KC)$$

$$= (AH_1 + CH_2)(A \mid KC)$$

$$= (AH_1A + CH_2A \mid K(AH_1C + CH_2C))$$

$$= (A \mid KC)$$

$$\therefore \begin{pmatrix} H_1 \\ \hline K^{-1}H_2 \end{pmatrix} \text{ is g-inv of } (A \mid KC).$$

P(4) Let A be of rank r. Choose any $r \times r$ non-singular submatrix A such that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$ where A_{11} is $r \times r$

and nonsingular. So, then we know

$$G = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$ is g-inv. of A.

Then $AGA = A$

$$\Rightarrow \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} I & 0 \\ A_{21}A_{11}^{-1} & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{21}A_{11}^{-1}A_{12} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$\therefore$ By comparing the element of two matrices we get $A_{22} = A_{21}A_{11}^{-1}A_{12}$

(Proved)

(1)(5) If $A'A$ is idempotent

MP inv. of A is A^+

Now,

$$A'A = A'A\,A'A \quad -(1)$$

pre-multiplying (1) by $A^+(A^+)'$ and post-multiplying (1) by A^+ on both side we get,

$$A^+(A^+)'A'AA^+ = A^+(AA^+)'AA^+$$
$$= A^+\underline{AA^+A}A^+$$
$$= A^+AA^+$$
$$= A^+$$

($(AA^+)' = AA^+$
as A^+ is Least Square
g-inv. of A.)

[$AA^+A = A$
as, A^+ is g-inv. of A
and A^+ is Reflexive g-inverse of A
$\therefore A^+AA^+ = A^+$]

Now,

$$A^+(A^+)'A'AA'AA^+$$
$$= A^+(AA^+)'AA'AA^+$$
$$= A^+\underline{AA^+A}A'AA^+$$
$$= A^+AA'AA^+$$
$$= (A^+A)'A'AA^+$$
$$= A'(A^+)'A'AA^+$$
$$= (AA^+A)'AA^+$$
$$= A'AA^+ = A'(AA^+)'$$
$$= A'(A^+)'A' = (AA^+A)'$$
$$= A'$$

[A^+ is Minimum-norm g-inv. of A
$\therefore (A^+A)' = A^+A$]

<u>only If</u> :- If $A^+ = A'$

then

As, A^+ is g-inv. of A

So, $AA^+A = A$.

$\therefore$ $A'AA'A$

$= A^+\underbrace{AA^+A}$

$= A^+A$ [A^+ is minimum norm g-inv. of A, so $(A^+A)' = A^+A$]

$= (A^+A)'$

$= A'(A^+)'$

$= A'(A')'$

$= A'A$ (proved)

1 P) (b)

$$C = tr(A'A)$$
$$= tr(vu'uv')$$
$$= tr(\underbrace{v'v\,u'u}_{\text{Scaler}})$$
$$= v'v\,u'u$$

$; u, v \in \mathbb{R}^n$

$$ABA$$
$$= A\,C^{-1}A'A$$
$$= uv'\,\frac{1}{v'v\,u'u}\,vu'uv'$$
$$= \frac{u(v'v)(u'u)v'}{(v'v)(u'u)} = uv' = A.$$

BAB

$= C^{-1}A'AC^{-1}A'$

$= C^{-1}vu'uv'C^{-1}vu'$

$$= \frac{v(u'u)(v'v)u'}{(v'v)(u'u)(v'v)(u'u)}$$

$$= \frac{vu'}{v'vu'u} = C^{-1}A' = B.$$

$(BA)' = A'B'$

$$= uv' \cdot \frac{1}{v'v\,u'u} \cdot uv'$$

$$= \frac{v(u'u)v'}{v'v(u'u)}$$

$$= \frac{vv'}{v'v}$$

$BA = C^{-1}A'A$

$$= \frac{1}{(v'v)(u'u)} v(u'u)v'$$

$$= \frac{vv'}{v'v}.$$

$\therefore (BA)' = BA.$

$$(AB)' = B'A'$$

$$= \frac{1}{v'v\,u'u} \cdot uv'vu'$$

$$= \frac{uu'}{u'u}$$

$$AB = AC^{-1}A'$$

$$= uv' \frac{1}{v'v\,u'u} \cdot vu'$$

$$= \frac{uu'}{u'u}$$

$$\therefore (AB)' = AB.$$

$\therefore B = C^{-1}A'$ is MP inverse of A.