MTH 207A: Assignment 2

P 1. Prove that a matrix is a projection matrix if and only if it is symmetric and idempotent.

P 2. Let $\mathcal{U}$ represent a subspace of the linear space $\mathbb{R}^n$ of all n–dimensional column vectors, take X to be an $n \times p$ matrix such that $\mathcal{C}(\mathrm{X}) = \mathcal{U}$, and let $\mathbf{y}$ represent a vector in $\mathbb{R}^n$. Then, for $\mathbf{w} \in \mathcal{U}$, the sum of squares $(\mathbf{y} - \mathbf{w})'(\mathbf{y} - \mathbf{w})$ of the elements of the difference $\mathbf{y} - \mathbf{w}$ between $\mathbf{y}$ and $\mathbf{w}$ is minimized uniquely by taking $\mathbf{w} = \mathrm{X}\mathbf{b}^*$, where $\mathbf{b}^*$ is any solution to the normal equations $\mathrm{X}'\mathrm{X}\mathbf{b} = \mathrm{X}'\mathbf{y}$, or, equivalently, by taking $\mathbf{w} = \mathrm{P_X}\mathbf{y}$. Further, the minimum value of the sum of squares is expressible as

$$(\mathbf{y} - \mathrm{X}\mathbf{b}^*)'(\mathbf{y} - \mathrm{X}\mathbf{b}^*) = \mathbf{y}'(\mathbf{y} - \mathrm{X}\mathbf{b}^*) = \mathbf{y}'(\mathrm{I} - \mathrm{P_X})\mathbf{y}.$$

P 3. Let Y represent a matrix in a linear space $\mathcal{V}$, let $\mathcal{U}$ and $\mathcal{W}$ represent subspaces of $\mathcal{V}$, and take $\{\mathrm{X}_1, \ldots, \mathrm{X}_s\}$ to be a set of matrices that spans $\mathcal{U}$ and $\{\mathrm{Z}_1, \ldots, \mathrm{Z}_t\}$ to be a set that spans $\mathcal{W}$.Then, $\mathrm{Y} \perp \mathcal{U}$ if and only if $\mathrm{Y} \cdot \mathrm{X}_i = 0$ for $i = 1, \ldots, s$; that is, Y is orthogonal to $\mathcal{U}$ if and only if Y is orthogonal to each of the matrices $\mathrm{X}_1, \ldots, \mathrm{X}_s$, and similarly, $\mathcal{U} \perp \mathcal{W}$ if and only if $\mathrm{X}_i \cdot \mathrm{Z}_j = 0$ for $i = 1, \ldots, s$ and $j = 1, \ldots, t$; that is, $\mathcal{U}$ is orthogonal to $\mathcal{W}$ if and only if each of the matrices $\mathrm{X}_1, \ldots, \mathrm{X}_s$ is orthogonal to each of the matrices $\mathrm{Z}_1, \ldots, \mathrm{Z}_t$.

P 4. Let $\mathcal{U}$ and $\mathcal{V}$ represent subspaces of $\mathbb{R}^{m \times n}$. Show that if $\dim(\mathcal{V}) > \dim(\mathcal{U})$, then $\mathcal{V}$ contains a nonnull matrix that is orthogonal to $\mathcal{U}$.

P 5. Suppose that X is a symmetric matrix of dimensions $m \times m$ and $\mathrm{A} = \{a_{ts}\}$ to be an $m \times m$ matrix of constants. Then find $\partial tr(\mathrm{AX})/\partial \mathrm{X}$.

P 6. Let F represent a $p \times p$ matrix of functions, defined on a set $\mathcal{S}$, of a vector $\mathbf{x} = (x_1, \ldots, x_m)'$ of m variables. Let $\mathbf{c}$ represent any interior point of $\mathcal{S}$ at which F is continuously differentiable. Show that if F is idempotent at all points in some neighborhood of $\mathbf{c}$, then (at $\mathbf{x} = \mathbf{c}$)

$$\mathrm{F}\frac{\partial \mathrm{F}}{\partial x_j}\mathrm{F} = 0.$$

P 7. Let $\mathrm{X} = \{x_{ij}\}$ represent an $m \times n$ matrix of "independent" variables, and suppose that X is free to range over all of $\mathbb{R}^{m \times n}$. Show that, for $k = 2, 3, \ldots,$

$$\frac{\partial \mathrm{tr}(\mathrm{X}^k)}{\partial \mathrm{X}} = k(\mathrm{X}')^{k-1}.$$

P 8. Let $\mathrm{X} = \{x_{st}\}$ represent an $m \times n$ matrix of "independent" variables, let A represent an $m \times n$ matrix of constants, and suppose that the range of X is a set $\mathcal{S}$ comprising some or all X-values for which $\det(\mathrm{X}'\mathrm{AX}) > O$. Show that $\log \det(\mathrm{X}'\mathrm{AX})$ is continuously differentiable at any interior point C of $\mathcal{S}$ and that (at X = C)

$$\frac{\partial \log \det(\mathrm{X}'\mathrm{AX})}{\partial \mathrm{X}} = \mathrm{AX}(\mathrm{X}'\mathrm{AX})^{-1} + [(\mathrm{X}'\mathrm{AX})^{-1}\mathrm{X}'\mathrm{A}]'.$$