$$y_{ij} = \mu + \alpha_i + \epsilon_{ij} \quad \forall\ i = 1(1)k,\ j = 1(1)n.$$

$$\Rightarrow \epsilon_{ij}^2 = (y_{ij} - \mu - \alpha_i)^2$$

$$L = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij} - \mu - \alpha_i)^2$$ is the Sum of Square of error (SSE)

Now, we have to find, Residual sum of Square that is Minimized error sum of Square where $\hat{\mu}$ and $\hat{\alpha_i}$ are to be used as the estimator of $\mu$ and $\alpha_i$ respectively.

$$\frac{\partial L}{\partial \mu} = 0 \Rightarrow \sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij} - \mu - \alpha_i) = 0$$

$$\Rightarrow \sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij} = \sum_{i=1}^{k} n_i \mu + \sum_{i=1}^{k} \alpha_i n_i$$

$$\Rightarrow \sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij} = n\mu + \sum_{i=1}^{k} \alpha_i n_i.$$

$$\frac{\partial L}{\partial \alpha_i} = 0 \Rightarrow 2 \cdot \sum_{j=1}^{n_i}(y_{ij} - \mu - \alpha_i)(-1) = 0$$

$$\Rightarrow \sum_{j=1}^{n_i}(y_{ij} - \mu - \alpha_i) = 0.$$

$$\Rightarrow \sum_{j=1}^{n_i} y_{ij} = n_i\mu + n_i\alpha_i$$

$$\Rightarrow \bar{y}_{i0} = \mu + \alpha_i \quad \forall\, i = 1(1)k.$$

$$\bar{y}_{00} = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij}}{\sum_{i=1}^{k} n_i} = \frac{\sum_{i=1}^{n} n_i \bar{y}_{i0}}{\sum_{i=1}^{k} n_i}.$$

Q> (1) ( Matrix Approach )

Our Model Set up,

$$y_{ij} = \mu + \alpha_i + e_{ij} \qquad \forall\, i = 1(1)k, \quad j = 1(1)n_i$$

$$\left.\begin{aligned} y_{1j} &= \mu + \alpha_1 + e_{1j} \\ y_{2j} &= \mu + \alpha_2 + e_{2j} \\ &\vdots \\ y_{kj} &= \mu + \alpha_k + e_{kj} \end{aligned}\right\} \quad \forall\, j = 1(1)n_i$$

Model can be written as,

$$\begin{bmatrix} y_{11} \\ \vdots \\ y_{1n_1} \\ \vdots \\ y_{k1} \\ \vdots \\ y_{kn_k} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & & 1 \\ 0 & 0 & \cdots & & 1 \\ \vdots & & & & \\ 0 & 0 & \cdots & & 1 \end{bmatrix} \underset{k \times 1}{\begin{bmatrix} \mu + \alpha_1 \\ \mu + \alpha_2 \\ \vdots \\ \mu + \alpha_k \end{bmatrix}} + \epsilon$$

$$\epsilon = \begin{bmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{1n_1} \\ \vdots \\ \epsilon_{k1} \\ \vdots \\ \epsilon_{kn_k} \end{bmatrix}$$

$X'X$

$= \operatorname{diag}(n_1, n_2, \ldots, n_k)$.

$$\theta = \begin{pmatrix} \mu+\alpha_1 \\ \mu+\alpha_2 \\ \vdots \\ \mu+\alpha_k \end{pmatrix}$$

$X'X$

$$= \begin{bmatrix} 0 & \cdots & 0 & \cdots & 1 & \cdots & 1 \\ 0 & \cdots & 1 & \cdots 1 & 0 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 \cdots 0 & 1 & 1 \cdots 1 & \cdots & 0 \cdots & 0 \\ \underbrace{1 \; 1 \cdots 1} & 0 & \cdots & & & & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ \vdots & \vdots & & & \vdots \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 1 & \cdots & & 0 \\ 0 & 0 & \cdots & 0 \cdots & 1 \\ \vdots & & & \vdots & \vdots \\ 0 & & & 0 & 1 \end{bmatrix}$$

$= \operatorname{diag}(n_1, n_2, \ldots, n_k)$

$$X'y = \begin{pmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{k0} \end{pmatrix} \qquad y_{i0} = \sum_{j=1}^{n_i} y_{ij} \quad \forall i = 1, \ldots, k$$

The model of full rank and BLUE's of $\alpha_i$ are given by the components of $\hat{\theta}$,

where $\hat{\theta} = (X'X)^{-1} X'\underset{\sim}{y} = \operatorname{diag}(1/n_1, \ldots, 1/n_k) \begin{pmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{k0} \end{pmatrix}$

$$= \begin{pmatrix} \bar{y}_{10} \\ \bar{y}_{20} \\ \vdots \\ \bar{y}_{k0} \end{pmatrix}$$

$$\widehat{\mu+\alpha_1} = \bar{y}_{10} \qquad \hat{\mu} + \hat{\alpha}_1 = \bar{y}_{10}$$

$$\widehat{\mu+\alpha_2} = \bar{y}_{20} \qquad \hat{\mu} + \hat{\alpha}_2 = \bar{y}_{20}$$

$$\vdots \qquad \vdots$$

$$\widehat{\mu+\alpha_k} = \bar{y}_{k0} \qquad \hat{\mu} + \hat{\alpha}_k = \bar{y}_{k0}$$

$$\therefore \boxed{\hat{\alpha}_i = \bar{y}_{i0} - \hat{\mu}}$$

As the model is of full rank,

So,

RSS

$$= y'y - y'x(x'x)^{-1}x'y$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij}^2 - (y_{10} \cdots y_{k0}) \begin{pmatrix} \bar{y}_{10} \\ \bar{y}_{20} \\ \vdots \\ \bar{y}_{k0} \end{pmatrix}$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij}^2 - \sum_{i=1}^{k} \frac{y_{i0}^2}{n_i}.$$

Now, $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2) \quad \left.\begin{array}{l} \forall\, i = 1(1)k \\ j = 1(1)n_i \end{array}\right\}$

$\therefore\ y_{ij} \sim N(\mu + \alpha_i, \sigma^2). \quad \forall\, i = 1(1)k, \; j = 1(1)n_i$

$$RSS = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij} - \hat{\mu} - \hat{\alpha}_i)^2$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_{io})^2 = \sum_{i=1}^{k}(n_i - 1)s_i^2$$

$y_{ij} \sim N(\mu + \alpha_i, \sigma^2) \quad \forall\, i = 1(1)k, \; j = 1()n_i$

ind.

$$\bar{y}_{io} \sim N\left(\mu + \alpha_i, \frac{\sigma^2}{n_i}\right)$$

$$\frac{\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_{io})^2}{\sigma^2} \sim \chi^2_{n_i - 1}$$

(By Cochran's Theorem)

So, $$\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{(y_{ij} - \bar{y}_{io})^2}{\sigma^2} \sim \chi^2_{n-k}$$

$$\therefore\ \boxed{\frac{RSS}{\sigma^2} \sim \chi^2_{n-k}}$$

Q 1 (1)

( Estimation Approach )

Model is,

$$y_{ij} = \mu + \alpha_i + e_{ij}$$

$$\Rightarrow e_{ij} = y_{ij} - \mu - \alpha_i$$

Now, $$L = \sum_{i=1}^{k}\sum_{j=1}^{n_i} (y_{ij} - \mu - \alpha_i)^2$$ ✓

$$\frac{\partial L}{\partial \mu} = 0$$

$$\Rightarrow \sum_{i=1}^{k}\sum_{j=1}^{n_i} (y_{ij} - \mu - \alpha_i)(-1) = 0$$

$$\Rightarrow \sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij} = n\mu + \sum_{i=1}^{k} n_i\alpha_i \quad \text{-(1)}$$

$$\frac{\partial L}{\partial \alpha_i} = 0 \Rightarrow \sum_{j=1}^{n_i} (y_{ij} - \mu - \alpha_i)(-1) = 0.$$

$$\Rightarrow \sum_{j=1}^{n_i} y_{ij} = n_i\mu + n_i\alpha_i$$

$$\boxed{\bar{y}_{i0} = \mu + \alpha_i}$$

$$\Rightarrow \hat{\alpha}_i = \bar{y}_{i0} - \hat{\mu}.$$

and $\sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij} = n\mu + \sum_{i=1}^{k} n_i\alpha_i$

$$\Rightarrow \boxed{\bar{y}_{00} = \mu + \frac{\sum_{i=1}^{k} n_i\alpha_i}{n}.}$$

If the restriction (Identification Constraints) $\sum_{i=1}^{k} n_i\alpha_i = 0$

then,

$$\underline{\bar{y}_{00} = \hat{\mu}} \quad \text{and} \quad \hat{\alpha}_i = \bar{y}_{i0} - \hat{\mu}$$
$$= \bar{y}_{i0} - \bar{y}_{00}$$
$$\forall i = 1(1)k$$

But If the restriction $\sum_{i=1}^{k} n_i\alpha_i \neq 0$

then, $\hat{\alpha}_i = \bar{y}_{i0} - \hat{\mu}$ $\quad \forall i = 1(1)k$

and $\hat{\mu} = \bar{y}_{00} - \dfrac{\sum_{i=1}^{k} n_i\hat{\alpha}_i}{n}$.

(2) $SS_{Reg} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} (\hat{y}_{ij} - \bar{y})^2$

$= \sum_{i=1}^{k}\sum_{j=1}^{n_i} (\hat{\mu} + \hat{\alpha}_i - \bar{y})^2$

$= \sum_{i=1}^{k}\sum_{j=1}^{n_i} (\bar{y} - (\hat{\mu} + \hat{\alpha}_i))^2 = \sum_{i=1}^{k}\sum_{j=1}^{n_i} (\bar{y}_{00} - \bar{y}_{i0})^2$

$= \sum_{i=1}^{k}\sum_{j=1}^{n_i} (\bar{y}_{i0} - \bar{y})^2$

$y_{ij} \sim N(\mu + \alpha_i, \sigma^2)$ $\quad = \underbrace{\sum_{i=1}^{k} n_i (\bar{y}_{i.} - \bar{y})^2}$

$\bar{y}_{i0} \sim N(\mu + \alpha_i, \frac{\sigma^2}{n_i})$

$\bar{y}_{00} \sim N(\mu + \alpha_i, \frac{\sigma^2}{n})$

$$\underbrace{\sum_{i=1}^{k}\sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{00})^2}_{SS\ (Total)} = \underbrace{\sum_{i=1}^{k}\sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i0})^2}_{SSE} + \underbrace{\sum_{i=1}^{k} n_i (\bar{y}_{i0} - \bar{y}_{00})^2}_{SSB}$$

Now, $y_{ij} \sim N(\mu + \alpha_i, \sigma^2)$

$\bar{y}_{00} \sim N(\mu + \alpha_i, \frac{\sigma^2}{n})$

By Cochran's Theorem ? ind.

$$\frac{\sum_{i=1}^{k}\sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{00})^2}{\sigma^2} \sim \chi^2_{n-1}$$

Also, SSE and SSB are orthogonal to each other.

So, SSE and SSB are <u>uncorrelated</u> also.

So,

$$\frac{SS(\text{total})}{\sigma^2} = \frac{SSE}{\sigma^2} + \frac{SSB}{\sigma^2}$$

$$\Rightarrow \frac{SSB}{\sigma^2} = \frac{SS(\text{total})}{\sigma^2} - \frac{SSE}{\sigma^2}$$

$$\sim \chi^2_{(n-1)-(n-k)}$$

(By <u>Additive property of $\chi^2$-distribution</u>)

$$\overset{D}{=} \chi^2_{k-1}$$

$$\therefore \frac{SSA}{\sigma^2} = \frac{SS_{Res}}{\sigma^2} = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(\bar{y}_{i\cdot} - \bar{y}_{00})^2}{\sigma^2}$$

$$\sim \chi^2_{k-1}$$

P.(3)

$$y_1 = \theta_1 + \theta_2 + \epsilon_1$$

$$y_2 = 2\theta_1 + \epsilon_2$$

$$y_3 = \theta_1 - \theta_2 + \epsilon_3$$

$$\epsilon_i \overset{iid}{\sim} N(0, \sigma^2) \quad \forall\, i = 1(1)3.$$

$$\underset{\sim}{Y} = X \underset{\sim}{\theta} + \underset{\sim}{\epsilon}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}^{3\times1} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix}^{3\times2} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}^{2\times1} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix}^{3\times1}$$

$$X = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix}^{3\times2}$$

$$X^T X = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$$

$X^T X$ is non-singular. So, $(X^T X)^{-1}$ exists.

$$\hat{\theta} = (X^T X)^{-1} X^T \underset{\sim}{Y}$$

$$= \frac{\begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}^T}{12} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}}{12} \begin{pmatrix} y_1 + 2y_2 + y_3 \\ y_1 - y_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1/6 & 0 \\ 0 & 1/2 \end{pmatrix}^{2\times 2} \begin{pmatrix} y_1 + 2y_2 + y_3 \\ y_1 - y_3 \end{pmatrix}^{2\times 1}$$

$$= \begin{pmatrix} \dfrac{y_1 + 2y_2 + y_3}{6} \\ \dfrac{y_1 - y_2}{2} \end{pmatrix}^{2\times 1}$$

$$\therefore \hat{\theta} = \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{6}(y_1 + 2y_2 + y_3) \\ \frac{1}{2}(y_1 - y_2) \end{pmatrix}$$

As, It is a full rank Model,

$$RSS = \underset{\sim}{y}'\underset{\sim}{y} - \underset{\sim}{y}' X (X^T X)^{-1} X^T \underset{\sim}{y}$$

$$= \underset{\sim}{y}'\underset{\sim}{y} - \hat{\theta}' X^T X \hat{\theta}$$

From Normal Equation,

$X^T X \hat{\theta} = X^T \underset{\sim}{y}$

$$= \sum_{i=1}^{3} y_i^2 - \left( \frac{1}{6}(y_1 + 2y_2 + y_3) \quad \frac{1}{2}(y_1 - y_2) \right) \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{6}(y_1 + 2y_2 + y_3) \\ \frac{1}{2}(y_1 - y_2) \end{pmatrix}$$

$$= \sum_{i=1}^{3} y_i^2 - (y_1 + 2y_2 + y_3 \quad y_1 - y_2) \begin{pmatrix} \frac{1}{6}(y_1 + 2y_2 + y_3) \\ \frac{1}{2}(y_1 - y_2) \end{pmatrix}$$

$$= \sum_{i=1}^{3} y_i^2 - \left[ \frac{1}{6}(y_1 + 2y_2 + y_3)^2 + \frac{1}{2}(y_1 - y_2)^2 \right]$$

$$= \sum_{i=1}^{3} y_i^2 - \hat{\theta}_1^2 - \hat{\theta}_2^2$$

Now, for $H_0 : \theta_1 = \theta_2 = \theta$ (say)

$$\underset{\sim}{\epsilon}' \underset{\sim}{\epsilon} = \sum_{i=1}^{3} \epsilon_i^2$$

$$= (y_1 - 2\theta)^2 + (y_2 - 2\theta)^2 + y_3^2$$

Now, $\dfrac{\partial(\epsilon'\epsilon)}{\partial\theta} = 0$

$$\Rightarrow 2(y_1 - 2\theta)(-2) + 2(y_2 - 2\theta)(-2) = 0$$

$$\Rightarrow y_1 - 2\theta + y_2 - 2\theta = 0$$

$$\Rightarrow 2(y_1 - y_2) = 4\theta$$

$$\Rightarrow y_1 - y_2 = 2\theta$$

$$\Rightarrow \hat{\theta}_{H_0} = \frac{y_1 - y_2}{2}.$$

Now, $RSS_{(H_0)} = \underset{H}{\text{Min}}\ \underset{\sim}{\epsilon}' \underset{\sim}{\epsilon}$

$$= (y_1 - 2\hat{\theta}_{H_0})^2 + (y_2 - 2\hat{\theta}_{H_0})^2 + y_3^2$$

<u>Effect of $H_0$</u>, $SSH_0 = RSS_{(H_0)} - RSS$

$$= (y_1 - 2\hat{\theta}_{H_0})^2 + (y_2 - 2\hat{\theta}_{H_0})^2 + y_3^2 - \sum_{i=1}^{3} y_i^2 + \hat{\theta}_1^2 + \hat{\theta}_2^2$$

$$= \left[y_1 - 2\left(\frac{y_1 - y_2}{2}\right)\right]^2 + \left(y_2 - 2\left(\frac{y_1 - y_2}{2}\right)\right)^2 + y_3^2 - \sum_{i=1}^{3} y_i^2 + \hat{\theta}_1^2 + \hat{\theta}_2^2$$

$$= y_2^2 + (2y_2 - y_1)^2 + y_3^2 - y_1^2 - y_2^2 - y_3^2 + \hat{\theta}_1^2 + \hat{\theta}_2^2$$

$$= (2y_2 - y_1)^2 - y_1^2 + \hat{\theta}_1^2 + \hat{\theta}_2^2$$

$$= 4y_2^2 + y_1^2 - 4y_1y_2 - y_1^2 + \hat{\theta}_1^2 + \hat{\theta}_2^2$$

$$= 4y_2^2 - 4y_1y_2 + \hat{\theta}_1^2 + \hat{\theta}_2^2$$

$\therefore$ Now, $\dfrac{RSS}{\sigma^2} \sim \chi^2_{3-r(X)}$

$\stackrel{D}{\equiv} \chi^2_{df(SSE)}$

$r(X) = r(X^TX) = 2$

$df(SSE)$

$=$ total no. of variable − no. of parameter estimated

$= 3 - 2 = 1$

$$\boxed{\frac{RSS}{\sigma^2} \sim \chi^2_{3-2} \stackrel{D}{\equiv} \chi^2_1}$$

Also, $\dfrac{RSS(H_0)}{\sigma^2} \sim \chi^2_{df(RSS(H_0))}$

$df(RSS(H_0))$

$=$ No. of variables − no. of parameter estimated

$= 3 - 1 = 2$

So,

$$\boxed{\frac{SSH_0}{\sigma^2} = \frac{RSS(H_0)}{\sigma^2} - \frac{RSS}{\sigma^2} \sim \chi^2_1}$$

[ $\because$ we know that,

$RSS(H_0) = \underbrace{SSH_0 + RSS}_{\text{orthogonal}}$ ]

and $\left.\begin{array}{l} \dfrac{SSH_0}{\sigma^2} \sim \chi^2_1 \\ \dfrac{RSS}{\sigma^2} \sim \chi^2_1 \end{array}\right\}$ ind.

$$F = \frac{(SSH_0/\sigma^2)}{(RSS/\sigma^2)} = \frac{\chi^2_1/1}{\chi^2_1/1} \sim F_{1,1}$$

$$y_1 = \theta_1 + \theta_2 + \epsilon_1$$

$$y_2 = 2\theta_1 + \epsilon_2$$

$$y_3 = \theta_1 - \theta_2 + \epsilon_3$$

$$L = \sum_{i=1}^{3} \epsilon_i^{\checkmark} = \left\{ y_1 - (\theta_1 + \theta_2) \right\}^{\checkmark} + \left\{ y_2 - 2\theta_1 \right\}^{\checkmark} + \left\{ y_3 - (\theta_1 - \theta_2) \right\}^{\checkmark}$$

$$\frac{\partial L}{\partial \theta_1} = 0$$

$$\Rightarrow 2(y_1 - \theta_1 - \theta_2)(-1) + 2(y_2 - 2\theta_1)(-2) + 2(y_3 - \theta_1 + \theta_2)(-1) = 0.$$

$$\underset{\sim}{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

$$\hat{\underset{\sim}{\theta}} =$$