Finding a g-inverse of a matrix

Let A be a matrix with rank r. Recall the reduction of a matrix to its reduced echelon form (Section 4.4 in Rao-Bhima's book). Suppose A is reduced to its reduced echelon form F by a set of row operations, and we have $E_{m\times m}A_{m\times n} = F$ where E is the product of elementary matrices. Therefore, F has first r rows non-null and all other rows null, and assume that the independent columns are $F_{\cdot p_1}, F_{\cdot p_2}, \ldots, F_{\cdot p_r}$ where $p_1 < p_2 < \cdots < p_r$. By the definition of a reduced echelon form matrix, this means $F_{\cdot p_1} = e_1^m, F_{\cdot p_2} = e_2^m, \ldots,$ and $F_{\cdot p_r} = e_r^m$.[1] Construct a new matrix $G_{n\times m}$ where $G_{\cdot p_1} = E_{\cdot 1}, G_{\cdot p_2} = E_{\cdot 2} \ldots, G_{\cdot p_r} = E_{\cdot r}$, and $G_{\cdot j} = 0$ for all $j \notin \{p_1, p_2, \ldots, p_r\}$, i.e., the p_1-th row of G is the 1st row of E, the p_2-th row of G is the 2nd row of E, and so on till the p_r-th row which is the r-th row of E, and all other rows are null rows.

By construction GA has r non-null rows which are identical with the first r rows of F. Therefore, GA has rank r, hence, $\rho(A) = \rho(GA)$. We now show that $GAGA = GA$ by showing that $GA(GA)_{\cdot j} = (GA)_{\cdot j}$ for $1 \le j \le n$.[2] Verify that the columns of GA have the following form:

$$\begin{aligned}
\text{1st-}(p_1-1)\text{-th column} &: 0 \\
p_1\text{-thcolumn} &: e_{p_1}^n \\
(p_1+1)\text{-th-}(p_2-1)\text{-th column} &: \alpha_j e_{p_1}^n \text{ for all } j = p_1+1, \ldots, p_2-1, \\
p_2\text{-thcolumn} &: e_{p_2}^n \\
(p_2+1)\text{-th-}(p_3-1)\text{-th column} &: \beta_{1,j} e_{p_1}^n + \beta_{2,j} e_{p_2}^n \text{ for all } j = p_2+1, \ldots, p_3-1, \\
&\vdots \\
p_r\text{-thcolumn} &: e_{p_r}^n \\
(p_r+1)\text{-th-}n\text{-th column} &: \gamma_{1,j} e_{p_1}^n + \gamma_{2,j} e_{p_2}^n + \cdots + \gamma_{r,j} e_{p_r}^n \text{ for all } j = p_r+1, \ldots, n.
\end{aligned}$$

Note that here α_js can be obtained from the first row of F, ($\beta_{1,j}$, $\beta_{2,j}$)s are obtained from the first two rows of F, and so on. The following diagram may help to visualize why the columns of GA have those structures. Note the positions of the αs and βs in the illustration.

[1] Here e_j^m is the m-dimensional vector with j-th component 1 and all the other components are 0.

[2] $(GA)_{\cdot j}$ denotes the j-th column of GA.

$$F = \begin{array}{c} \begin{array}{cccccccccccc} & & & \text{p_1-th column} & & & & \text{p_2-th column} & & \end{array} \\ \begin{bmatrix} 0 & \cdots & 0 & 1 & \alpha_{p_1+1} & \alpha_{p_1+2} & \cdots & 0 & \beta_{1,p_2+1} & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & \beta_{2,p_2+1} & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \\ \vdots & & & & & & & & & \\ \cdots & & & & & & & & & \cdots \end{bmatrix}_{m\times n} \end{array}$$

$$GA = \begin{array}{cc} & \begin{array}{cccccccccc} & & & \text{p_1-th column} & & & & \text{p_2-th column} & & \end{array} \\ \begin{array}{r} \\ \\ \\ \text{p_1-th row} \\ \\ \text{(p_2-1)th row} \\ \text{p_2-th row} \\ \\ \\ \end{array} & \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & & & & & & & & & \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 1 & \alpha_{p_1+1} & \alpha_{p_1+2} & \cdots & 0 & \beta_{1,p_2+1} & \cdots \\ \vdots & & & & & & & & & \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & \beta_{2,p_2+1} & \cdots \\ \vdots & & & & & & & & & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \end{bmatrix}_{n\times n} \end{array}$$

Now, recall that for a matrix $X_{s\times t}$, $Xe^t_k = X_{\cdot k}$ for all $k \in \{1, \ldots, t\}$. This implies $GA(GA)_{\cdot p_j} = GAe^n_{p_j} = (GA)_{\cdot p_j}$, for all $j \in \{1, 2, \ldots, r\}$. Check that the same holds for other columns of GA. This concludes that $GA(GA)_{\cdot j} = (GA)_{\cdot j}$ for $1 \leq j \leq n$.