Characteristic function

Definition 1

The characteristic function of a random variable Y is defined as

$$\phi_Y(\mathbf{t}) = \mathbb{E}[\exp(i\mathbf{t}'\mathbf{y})].$$

MVN

Definition 2

MVN Let $\mathbf{u}$ be a vector of order n whose components $u_1, \ldots, u_n$ are independent standard normal variables. Let X be an $r \times n$ matrix, and let $\boldsymbol{\mu}$ be a constant $r \times 1$ vector. The vector $\mathbf{y} = \mathrm{X}\mathbf{u} + \boldsymbol{\mu}$ is said to have (an r–dimensional) multivariate normal distribution.

Clearly, $\mathbb{E}(\mathbf{y}) = \boldsymbol{\mu}$ and $\mathrm{D}(\mathbf{y}) = \mathrm{X}\mathrm{X}' = \Sigma$.

Characteristic function:MVN

Definition 3

The characteristic function of a MVN RV $\mathbf{y}$ is given as

$$\phi_{\mathbf{y}}(t) = \exp\left(i\mathbf{t}'\boldsymbol{\mu} - \frac{1}{2}\mathbf{t}'\Sigma\mathbf{t}\right)$$

Characteristic function:MVN

Result 1: If $\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, then for any matrix B,

$$\mathrm{B}\mathbf{y} \sim \mathcal{N}(\mathrm{B}\boldsymbol{\mu}, \mathrm{B}\Sigma\mathrm{B}').$$

Proof:

$$\begin{aligned}
\phi_{\mathrm{B}\mathbf{y}}(\mathbf{t}) &= \mathbb{E}[\exp(i\mathbf{t}'\mathrm{B}\mathbf{y})] \\
&= \mathbb{E}[\exp(i\mathbf{t}^{*'}\mathbf{y})], \text{ where } \mathbf{t}^{*'} = (\mathrm{B}'\mathbf{t})' \\
&= \exp\left(i\mathbf{t}^{*'}\boldsymbol{\mu} - \frac{1}{2}\mathbf{t}^{*'}\Sigma\mathbf{t}^{*}\right) = \exp\left(i\mathbf{t}'\mathrm{B}\boldsymbol{\mu} - \frac{1}{2}\mathbf{t}'\mathrm{B}\Sigma\mathrm{B}'\mathbf{t}'\right) \\
&= \exp\left(i\mathbf{t}'\boldsymbol{\mu}^{*} - \frac{1}{2}\mathbf{t}'\Sigma^{*}\mathbf{t}'\right)
\end{aligned}$$

where $\boldsymbol{\mu}^* = \mathrm{B}\boldsymbol{\mu}$ and $\Sigma^* = \mathrm{B}\Sigma\mathrm{B}'$. Thus, $\mathrm{B}\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}^*, \Sigma^*)$.

MVN

Result 2: Let $\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$. Suppose that $\mathbf{y}$, $\boldsymbol{\mu}$, and Σ are conformally partitioned as

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}. \tag{1}$$

Then $\mathbf{y}_1 \sim \mathcal{N}(\boldsymbol{\mu}_1, \Sigma_{11})$ and $\mathbf{y}_2 \sim \mathcal{N}(\boldsymbol{\mu}_2, \Sigma_{22})$.

Proof: Note that for $\mathrm{B}_1 = \begin{pmatrix} \mathrm{I} & 0 \end{pmatrix}$, $\mathrm{B}_1\mathbf{y} = \mathbf{y}_1$ and for $\mathrm{B}_2 = \begin{pmatrix} 0 & \mathrm{I} \end{pmatrix}$, $\mathrm{B}_2\mathbf{y} = \mathbf{y}_2$. Now use **Result 1**.

MVN

Result 3: Let $\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, and suppose that $\mathbf{y}$, $\boldsymbol{\mu}$, and Σ are conformally partitioned as in (1). Then $\mathbf{y}_1$ and $\mathbf{y}_2$ are independent if and only if $\Sigma_{12} = 0$.

Proof: If $\mathbf{y}_1$ and $\mathbf{y}_2$ are independent then $\mathrm{Cov}(\mathbf{y}_1, \mathbf{y}_2) = \Sigma_{12} = 0$.
We now prove the converse part.
Suppose that $\Sigma_{12} = 0$. Then

$$\mathbf{t}'\Sigma\mathbf{t} = \mathbf{t}_1'\Sigma_{11}\mathbf{t}_2 + \mathbf{t}_2'\Sigma_{22}\mathbf{t}_2.$$

Therefore,

$$\phi_{\mathbf{y}}(\mathbf{t}) = \phi_{\mathbf{y}_1}(\mathbf{t}_1)\phi_{\mathbf{y}_2}(\mathbf{t}_2),$$

and hence $\mathbf{y}_1$ and $\mathbf{y}_2$ are independent.

MVN

Result 4: Let $\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$, and let A and B be matrices such that $\mathrm{A}\Sigma B' = 0$. Then, $\mathrm{A}\mathbf{y}$ and $\mathrm{B}\mathbf{y}$ are independent.

Proof: From **Result 1**

$$\begin{pmatrix} \mathrm{A} \\ \mathrm{B} \end{pmatrix} \mathbf{y} = \begin{pmatrix} \mathrm{A}\mathbf{y} \\ \mathrm{B}\mathbf{y} \end{pmatrix}$$

has multivariate normal distribution. Thus, by **Result 3** $\mathrm{A}\mathbf{y}$ and $\mathrm{B}\mathbf{y}$ are independent if $\mathrm{Cov}(\mathrm{A}\mathbf{y}, \mathrm{B}\mathbf{y}) = \mathrm{A}\Sigma\mathrm{B}' = 0$.

Cochran's Theorem

Theorem 4

Let $\mathbf{y} \sim \mathcal{N}(0, \mathrm{I}_n)$ *and let* A *be a symmetric* $n \times n$*. Let* $\mathbf{y}'\mathrm{A}\mathbf{y}$ *has the chi–square distribution with* r *degrees of freedom* (χ^2_r) *if and only if* A *is idempotent and Rank*$(\mathrm{A}) = r$*.*

Cochran's Theorem

To prove Theorem 4, following results are needed.

Result 5: If A is an idempotent matrix of rank r then there exists an orthogonal matrix P such that $\mathrm{A} = \mathrm{P}' \begin{pmatrix} \mathrm{I}_r & 0 \\ 0 & 0 \end{pmatrix} \mathrm{P}$.

Result 6: If $X_i \sim \mathcal{N}(0,1)$ and are independently for $i = 1\ldots,n$ then $\sum_{i=1}^{n} X_i^2 \sim \chi_n^2$.

Result 7: If $\mathrm{A}_{n\times n}$ is a real symmetric matrix then there exist a orthogonal matrix P such that $\mathrm{A} = \mathrm{P}\mathrm{Diag}(\lambda_1,\ldots,\lambda_n)\mathrm{P}'$, where λ_i's are eigen values of A.

Cochran's Theorem

Proof: Let A is an idempotent matrix with rank r then by **Result 5** there exists an orthogonal matrix P such that

$$\mathrm{A} = \mathrm{P}' \begin{pmatrix} \mathrm{I}_r & 0 \\ 0 & 0 \end{pmatrix} \mathrm{P}.$$

Let $\mathsf{z} = \mathrm{P}\mathbf{y}$. Then $\mathsf{z} \sim \mathcal{N}(0, \mathrm{I}_n)$. We have

$$\begin{aligned} \mathbf{y}'\mathrm{A}\mathbf{y} &= \mathbf{y}'\mathrm{P}' \begin{pmatrix} \mathrm{I}_r & 0 \\ 0 & 0 \end{pmatrix} \mathrm{P}\mathbf{y} = \mathsf{z}' \begin{pmatrix} \mathrm{I}_r & 0 \\ 0 & 0 \end{pmatrix} \mathsf{z} \\ &= z_1^2 + \cdots + z_r^2 \sim \chi_r^2. \text{ (by \textbf{Result 6})}. \end{aligned}$$

Cochran's Theorem

Conversely, suppose $\mathbf{y}'\mathrm{A}\mathbf{y} \sim \chi^2_r$. Since A is symmetric, there exists an orthogonal matrix P such that

$$\mathrm{A} = \mathrm{P}'\mathrm{Diag}(\lambda_1, \ldots, \lambda_n)\mathrm{P} = \mathrm{P}'\mathrm{D}\mathrm{P},$$

where $\mathrm{D} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$, $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Again, let $\mathrm{z} = \mathrm{P}\mathbf{y}$, so that $\mathrm{z} \sim \mathcal{N}(0, \mathrm{I}_n)$. The characteristic function of $\mathbf{y}'\mathrm{A}\mathbf{y} = \mathrm{z}'\mathrm{D}\mathrm{z}$ is given by

$$\begin{aligned} \phi(t) &= \mathbb{E}[\exp(it\mathbf{y}'\mathrm{A}\mathbf{y})] = \mathbb{E}[\exp(it\mathbf{y}'\mathrm{A}\mathbf{y})] \\ &= \mathbb{E}[\exp(it\sum_{j=1}^{n}\lambda_j z_j^2)] = \prod_{j=1}^{n}(1-2it\lambda_j)^{-\frac{1}{2}}, \qquad (2) \end{aligned}$$

since $z_j^2 \sim \chi^2_1$.

Cochran's Theorem

However, since $\mathbf{y}'\mathrm{A}\mathbf{y} \sim \chi^2_r$, its characteristic function is

$$\phi(t) = (1 - 2it)^{-r/2}. \tag{3}$$

Equating (2) an (3), we get

$$(1 - 2it)^{-r/2} = \prod_{j=1}^{n} (1 - 2it\lambda_j)^{-\frac{1}{2}} \tag{4}$$

for all t.

Cochran's Theorem

The left–hand side of (4) is a polynomial in t with r roots, all equal to $1/2i$. Therefore, the right-hand side also must have the same roots. This is possible precisely when r of the λ_i's are equal to 1, the rest being zero. Therefore, A is idempotent with rank r.