Hint Solutions for EndSem

April 28, 2023

- The linear regression model is written as follows:

  $$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

  where $\boldsymbol{Y} = (y_1, \ldots, y_{200})^\top$, $\boldsymbol{X} = \begin{bmatrix} e^1 & e^{-1} \\ \vdots & \vdots \\ e^{200} & e^{-200} \end{bmatrix}$, $\boldsymbol{\beta} = (\beta_1, \beta_2)^\top$ and $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_{200})^\top$.

  Now, the least squares estimator of $\boldsymbol{\beta}$ is $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{X}^\top \boldsymbol{Y}$. Note that $\mathbb{E}(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$ and $\mathrm{Var}(\hat{\boldsymbol{\beta}}) = \sigma^2(\boldsymbol{X}^\top \boldsymbol{X})^{-1} = \frac{\sigma^2}{det(\boldsymbol{X}^\top \boldsymbol{X})} \begin{bmatrix} \sum_{i=1}^{200} e^{-2i} & -N \\ -N & \sum_{i=1}^{200} e^{2i} \end{bmatrix}$, and $det(\boldsymbol{X}^\top \boldsymbol{X}) > 0$.

- The optimum value is 7. Also, see the .R code.

- Using the conditional distribution, first compute the joint mgf $M(t_1, t_2)$. It has the following expression:

  $$M(t_1, t_2) = 1/3e^{t_1^2/2} + 1/2e^{t_1^2/2+t_1+t_2} + 1/6e^{t_1^2/2+2t_1+2t_2}.$$

  Now, evaluate the partial derivatives of M at $(0, 0)$ to obtain $Cov(X, Y) = 17/36$ and $Var(X) = 53/36$. Final answer is 87.

- Using standard calculations, one can show that $\widehat{\theta}_{MLE} = \sqrt{\frac{\sum_{i=1}^n x_i^k}{kn}}$.

  $E(X) = \theta^{2/k} a_k$, where $a_k = k^{1/k}\Gamma\Big(1+\frac{1}{k}\Big) \implies \widehat{\theta}_{MOM} = \left(\frac{\bar{X}}{a_k}\right)^{k/2}$, where $\bar{X}$ is the sample mean.

- Using the Neyman-Pearson lemma, the critical region for this test is $\prod_{i=1}^n X_i > c/2$. Setting $\alpha = 0.10$ and the fact that $-2\log X \sim \chi^2_2$ under H_0, we get $c = 2e^{-\chi^2_{20}(0.10)/2} = 0.003973303$.

- The two correct choices are:

  – Given that $P[C] = 0.6, P[B \cap C^c] = 0.2, P[A^c|B^c \cap C^c] = 0.05$, then $P[A \cup B \cup C] = 0.99$.
  – There exists a triplet (X, Y, Z) such that Y is positively correlated with both X and Z, but X and Z are negatively correlated.

- We have

  $$P(Y = k) = \int_{ak}^{a(k+1)} \lambda e^{-\lambda x} dx = e^{-\lambda ak}[1 - e^{-\lambda a}]$$

  for $k = 0, 1, \ldots$
  This implies $Y \sim Geometric$ distribution with $p = 1 - e^{-\lambda a}$. Thus, $\mathbb{E}[Y] < \mathrm{Var}(Y)$.

- For $r > 0$, we have the following:
  $\mathbb{E}[X_1^{2r}] = (2r-1)(2r-3)\cdots 3 \cdot 1$.
  $\mathbb{E}[X_2^{2r}] = \mathbb{E}[X_3^{2r}] = e^{\frac{r^2}{2}}$.
  $\mathbb{E}[X_4^{2r}] = \frac{3^{r/2}}{6}[1 + (-1)^r]$.

- In our case, both A_n and B_n are positive definite matrices. Hence, the following hold:

  – $det((A_n + B_n)/2) \geq det(A_n B_n)^{1/2}$, and
  – $\{det(A_n) + det(B_n)\}^{1/n} \geq det(A_n)^{1/n} + det(B_n)^{1/n}$ for any $n \in \mathbb{N}$.