Name Dootika Vats MTH210 (2024): Quiz 2 Roll. No. Solutions.

Show all mathematical details and steps. Marks will not be given unless all work is shown.

RoU for a partially unknown target: Let $g(x)$ be a target density with support $\mathcal{X}$ and distribution function G. Suppose $g(x) = m\tilde{g}(x)$ such that $m > 0$ is unknown, and $\tilde{g}(x)$ is known. Define the set

$$D = \left\{ (u, v) : 0 \le u \le \sqrt{\tilde{g}\left(\frac{v}{u}\right)} \right\}.$$

If D is bounded, let (U, V) be uniformly distributed over the set D.

8 1. **Prove** that $V/U \sim G$. You are not allowed to assume the existence of any result taught in class.

2 2. Assume, we are able to find a box $[0, a] \times [b, c]$ that contains D. Is there any disadvantage of using RoU for such a partially unknown target?

① Let $Z = \frac{V}{U}$. We will show the density of Z is $g(z)$. By definition, the joint density of (U, V) is

$$g_{(U,V)}(u, v) = \frac{1}{\iint_D ds\,dt} I((u,v) \in D).$$

−5 if RoU proof assumed.

Consider transformation $(U, V) \to (U, Z)$ with $Z = \frac{V}{U}$. Then $U = U$ and $V = UZ$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \frac{\partial U}{\partial U} & \frac{\partial V}{\partial U} \\ \frac{\partial U}{\partial Z} & \frac{\partial V}{\partial Z} \end{vmatrix} = \begin{vmatrix} 1 & z \\ 0 & u \end{vmatrix} = u.$$

−1 if Jacobian calculation not shown.

So,

$$g_{(U,Z)}(u, z) = \frac{u}{\iint_D ds\,dt} I\left(0 \le u \le \sqrt{\tilde{g}(z)}\right)$$

To find the marginal distribution of Z, $g_Z(z)$, we integrate w.r.t u:

$$g_Z(z) = \int_0^{\sqrt{\tilde{g}(z)}} \frac{u}{\iint_D ds\,dt} du = \frac{\tilde{g}(z)}{2\iint_D ds\,dt}. \quad \text{—} (\dagger)$$

−4 if stuck here.

Since $g_Z(z)$ must be a density, $\int_{\mathcal{X}} g_Z(z) = 1 \Rightarrow \frac{\int_{\mathcal{X}} \tilde{g}(z)}{2\iint_D ds\,dt} = 1$

$$\Rightarrow \frac{1}{m} \underbrace{\int_{\mathcal{X}} m\tilde{g}(z)}_{=1} = 2\iint_D ds\,dt \Rightarrow \boxed{\iint_D ds\,dt = \frac{1}{2m}} \text{—} (*)$$

−2 if area incorrectly identified.

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Using $(*)$ in $(\dagger)$, we get $\boxed{g_Z(z) = g(z)}$ and we are done

(2) If we are able to enclose D in a $[0,a]\times[b,c]$ box, then we can implement A-R.

The efficiency of the A-R algorithm will be determined by the bound "c".

$$\text{"c"} = \frac{\text{Area of Box}}{\text{Area of D}} = \frac{a(b-c)}{\frac{1}{2m}} = \frac{2ma(b-c)}{\text{unknown}}.$$

This efficiency is unknown! So will not know the expected number of loops of the algorithm.

(−2) if reasoning not correct.