Home Work 5
MTH 212
Elementary Stochastic Process

1. Consider an urn with a total of D tokens; some numbered +1, and some -1. The composition of the urn changes over time as follows. At each turn a token is picked at random, and the sign is changed and put back in the urn. Let X_n denote the number of +1 sign at time n. Show that X_n is a Markov chain, and find the transition probability matrix. Is this an irreducible Markov chain?

2. Prove that for a finite Markov chain all states cannot be null recurrent.

3. Is it possible that in an infinite Markov Chain all states are null recurrent?

4. Consider the following transition probability matrix with the state space $S = \{0, 1, 2, 3, 4\}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

   Find $\lim_{n\to\infty} p_{ij}^{(n)}$, for all $i, j \in S$.

5. Consider the following transition probability matrix with the state space $S = \{0, 1, 2, 3, 4, \ldots, N\}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & & \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & \ldots & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

   Find $\lim_{n\to\infty} p_{ij}^{(n)}$, for all $i, j \in S$.

6. Consider the following transition probability matrix with the state space $S = \{1, 2, 3, 4\}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

   Find $\lim_{n\to\infty} p_{ij}^{(n)}$, for all $i, j \in S$.

7. Consider the following transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & \dots \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & \dots \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Prove that the Markov Chain is an irreducible aperiodic recurrent Markov Chain. Find $\lim p_{00}^{(n)}$.

8. Consider the following transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 2/3 & 1/3 & 0 & 0 & 0 & 0 & \dots \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 & \dots \\ 2/3 & 0 & 0 & 1/3 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Prove that the Markov Chain is an irreducible aperiodic recurrent Markov Chain. Find $\lim p_{00}^{(n)}$.

9. Consider the following transition probability matrix

$$\mathbf{P} = \begin{bmatrix} p & 1-p & 0 & 0 & 0 & 0 & \dots \\ p & 0 & 1-p & 0 & 0 & 0 & \dots \\ p & 0 & 0 & 1-p & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Here $0 < p < 1$. Find the values of p such that the Markov Chain is an irreducible aperiodic recurrent Markov Chain. Find $\lim p_{00}^{(n)}$.

10. Consider the following transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 1-p & p & 0 & 0 & 0 & 0 & \dots \\ 1-p^2 & 0 & p^2 & 0 & 0 & 0 & \dots \\ 1-p^3 & 0 & 0 & p^3 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Here $0 < p < 1$. Find the values of p such that the Markov Chain is an irreducible aperiodic recurrent Markov Chain. Find $\lim p_{00}^{(n)}$.