Home Work 2 (2023)
MTH 212
Elementary Stochastic Process

1. Prove or give a counter example that any relation operation is reflexive, i.e. a is related to a, for all $a \in S$.

2. Prove or give a counter example that any relation operation is symmetric.

3. Prove or give a counter example that any relation operation is transitive.

4. Prove that if S is countable, then an equivalence relation divides S into disjoint equivalent classes.

5. Find the transition probability matrix $\mathbf{P}$ of the following Markov Chain. Consider a sequence of tosses of a coin with the probability of head p. At time n, after n tosses of the coin, the state of the process is the number of heads in the n tosses minus the number of tails. Find the number of equivalent classes of the Markov Chain.

6. Find the transition probability matrix $\mathbf{P}$ of the following Markov Chain. N black balls and N white balls are placed in two urns so that each urn contains N balls. At each step one ball is selected at random from each urn, and the two balls are interchanged. The state of the system is the number of white balls in the first urn. Find the number of equivalent classes of the Markov Chain. Is it an irreducible Markov Chain? Find the periods of each state.

7. Determine the classes and the periodicity of the various states for a Markov Chain with the following transition probability matrix.

$$(a) \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \qquad (b) \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \end{bmatrix}$$

8. Give a counter example that a stochastic matrix need not be a two-step transition matrix of a Markov Chain.

9. Let a Markov Chain has r states. Prove that if the state k can be reached from j, then it can be reached in $r-1$ steps or less.

10. Let

$$\mathbf{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \quad \text{then} \quad \mathbf{P}^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}$$