Review of set relations and functions

Thesis: set theory (ZFC) is the foundation of mathematics. All "mathematical objects" can be coded as sets & all math statements can be expressed & all math statements can be expressed in set theory & all math proofs can be formalised in ZFC.

(Everything is a set & $\in$) membership equality

Axiom: $\exists x \, \forall y \quad (y \notin x)$
(There is a set with no numbers)

MTH 301

Ex: $f : X \to Y$, (X, d_1) & (Y, d_2) metric spaces

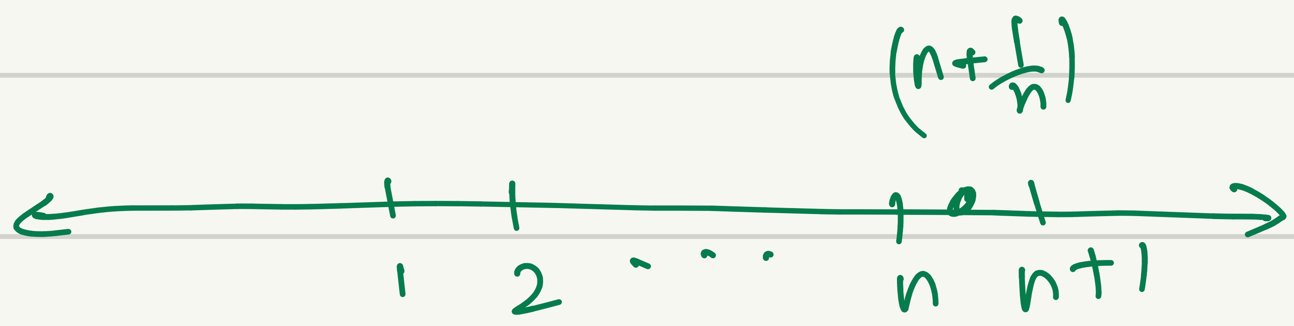
Show that the following are equivalent.

(1) f is uniformly continuous on X

(2) For all sequences $\langle (x_n, y_n) : n \geq 1 \rangle$ of pairs of points in X

$$d_1(x_n, y_n) \to 0 \Rightarrow d_2(f(x_n), f(y_n)) \to 0$$

Application: $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$ is not uniformly cont. on $\mathbb{R}$



$\langle (n, n + \frac{1}{n}) : n \geq 1 \rangle$ $\quad |n - (n + \frac{1}{n})| = \frac{1}{n} \to 0$

$\langle (n^2, (n + \frac{1}{n})^2) : n \geq 1 \rangle$ $\quad |2 + \frac{1}{n^2}| \to 2 \neq 0$

Proof of: (X, d_1) compact metric sp., (Y, d_2) metric space & $f : X \to Y$ continuous

$\Downarrow$

f is uniformly cont. on X.

fix $\varepsilon > 0$. Will find $\delta > 0$ s.t.

$(\forall x, y \in X)\ (d(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon)$ $\quad (*)_\delta$

For each $x \in X$, fix $\delta_x > 0$ s.t.

$(\forall y \in B_X(x, \delta_x))\ (d_2(f(y), f(x)) < \varepsilon)$

Define $\mathcal{F} = \{ B_X(x, \frac{\delta_x}{2}) : x \in X \}$

$\mathcal{F}$ is an open cover of X. X compact $\Rightarrow$
$(\exists\, x_1, \dots x_n \in X$ s.t.$)$ $\left(B_X(x_1, \frac{\delta_{x_1}}{2}) \cup B_X(x_2, \frac{\delta_{x_2}}{2}) \dots \cup B_X(x_n, \frac{\delta_{x_n}}{2}) \right)$

$(\ast\ast)$

Put $\delta = \min\left(\frac{\delta x_1}{2}, \dots \frac{\delta x_n}{2} \right) > 0$

Claim: $(\ )_{\delta/4}$ holds fo this δ.

Proof: Let $x, y \in X$ & suppose $d_1(x,y) < \delta/4$

We claim that, for some $k \in \{1, 2, \dots n\}$
both $x, y \in B_X(x_k, \delta x_k)$

By $(\ast\ast)$ fix $1 \le k \le n$ s.t. $x \in B_X(x_k, \delta x_k)$

Claim: $y \in B_X(x_k, \delta x_k)$

$\hookrightarrow$ $d_1(x, x_k) < \frac{\delta x_k}{2} \Rightarrow \left. d_1(x,y) + d_1(x, x_k) \right] \frac{\delta x_k}{2}$
$\ge d_1(y, x_k)$

$\Downarrow$

$d_1(y, x_k) \le \frac{\delta x_k}{2} + \frac{\delta x_k}{8} < \delta x_k$

HW

Ex:- $E \subseteq \mathbb{R}$ compact & non-empty $\Rightarrow$ Both $\sup(E)$, $\inf(E) \in E$

Connectedness : Let (X,d) be a metric space

(1) $E \subseteq X$ is clopen in E iff E is
closed & open
both closed & open in E.

(2) X is connected iff the only clopen subsets of X are $\emptyset$ & X.

(3) X is disconnected iff it's not connected.

example:- (1) $X = (0,1) \cup (2,3)$
$(0,1)$ is open in X
$(0,1)$ is closed in X

clopen sets in X are : $\emptyset, X, (0,1), (2,3)$

2 $X = [0,1) \cup (1,2]$, clopen sets, $\emptyset, X$, $[0,1)$, $(1,2]$

Theorem : (1) (X, d_1) connected, (Y, d_2) metric space $f: X \to Y$ cont.
$\Rightarrow f[X]$ range(f) is connected

(2) $E \subseteq \mathbb{R}$ is connected iff E is an interval

$((\forall x,y \in E)(x<y \Rightarrow [x,y] \subseteq E))$

$(-\infty, a)$, $(-\infty, a]$ $(a, \infty]$, $[a, \infty)$
$(-\infty, \infty)$ (a,b), $[a,b]$,
$[a,b)$, $(a,b]$

Proof: ① Exercise

② First suppose E is connected. If E is not an interval we can fix $x<y$ both in E & $x<z<y$ s.t. $z \notin E$

Check $A = \{w \in E : w < z\}$ is clopen in E & $A \notin \{\emptyset, E\} \Rightarrow E$ is disconnected

$E \subseteq \mathbb{R}$ is connected $\Leftrightarrow$ E is an interval

($\Rightarrow$) Last time

($\Leftarrow$) Let E be a non-empty interval
Suppose (towards a contradiction), E is disconnected & fix $A \subseteq E$ s.t. A is clopen in E & $A \notin \{\emptyset, E\}$

Fix $a \in A$ & $b \in E \setminus A$
Note that $[a,b] \subseteq E$

Put $W = A \cap [a,b]$. Then $a \in W$ & $b \notin W$

Let $\sup(W) = x$ Then $a \le x \le b$ so $x \in E$

$x = \sup(W) \Rightarrow x \in cl(W)$ & $W = A \cap [a,b]$,
$W \cup W'$

W is closed in $E \Rightarrow x \in W$ & $x \neq b$

Also $x \in ([a,b] \setminus W)'$ $\quad ([a,b] \setminus W \supseteq (x, b])$
Hence $x \in (E \setminus A)'$

As $E \setminus A$ is closed in E, we get $x \in E \setminus A$

But now $x \in A$ & $x \in E \setminus A$: A contradiction.

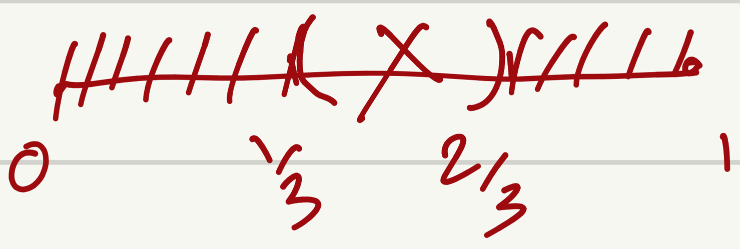
Defn: $E \subseteq \mathbb{R}$, nonempty

(1) $\sup(E) = +\infty$ if E is unbounded from above in $\mathbb{R}$.

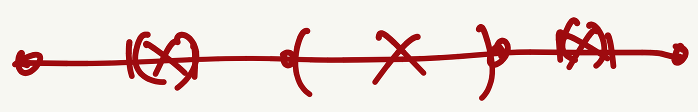
(2) $\inf(E) = -\infty$ if E is unbounded from below in $\mathbb{R}$.

Cantor Set :-

$C_0 = [0,1]$



$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$



$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [8/9, 9/9]$

$C \overset{dfn}{=} \bigcap_{k \geq 1} C_k$

$C_3 = 8$ intervals of length $\frac{1}{27}$

(1) C is compact in $\mathbb{R}$

(2) $|C| = |2^{\mathbb{N}}| = |\mathbb{R}|$

C_k = disj union of 2^k interval each of length $\frac{1}{3^k}$.

(3) $x \in C \iff \exists \langle a_n : n \geq 1 \rangle$ each $a_n \in \{0, 2\}$ &

$x = \sum_{n \geq 1} \frac{a_n}{3^n}$

(4) C is nowhere dense in $\mathbb{R}$.

Tree representation of C

$[0,1] = C_0$

$[0, \frac{1}{3}] \cup [\frac{2}{3}, 1] = C_1$

$[0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1] = C_2$

$$F : \{0,2\}^{\mathbb{N}} \xrightarrow{\text{bij}} C$$

$$F(\langle a_n : n \geq 1 \rangle)$$

$$= \sum_{n \geq 1} \frac{a_n}{3^n}$$

Subsequence :- Let $\langle x_n : n \geq 1 \rangle$ & $\langle y_n : n \geq 1 \rangle$ be sequence $\langle y_n : n \geq 1 \rangle$ is a subsequence of $\langle x_n : n \geq 1 \rangle$ iff for some $n_1 < n_2 \dots$

$$\langle y_k : k \geq 1 \rangle = \langle x_{n_k} : k \geq 1 \rangle$$

Theorem: (X, d) be compact metric space. Every sequence in (X, d) has a convergent subsequence

Corollary: Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence

Proof of corollary: Let $\langle x_n : n \geq 1 \rangle$ be a bounded sequence in $\mathbb{R}^n$ Choose $k > 0$ s.t. each

$$x_n \in \underbrace{[-k, k] \times [-k, k] \dots [-k, k]}_{n \text{ times}} \text{ — compact}$$

Proof of theorem: let $\langle x_n : n \geq 1 \rangle$ be a sequence in X. Let $A = \{x_n : n \geq 1\}$

Case 1: A is finite Easy (why?)

Case 2: A is infinite
By a prev. thm, $A' \neq \emptyset$, Fix $x \in A'$.
We'll construct a subsequence of $\langle x_n : n \geq 1 \rangle$

that converges to x.

To each $k \geq 1$, choose n_k s.t.

(a) $x_{n_k} \in B(x, \frac{1}{k})$

(b) $n_k < n_{k+1}$

Begin by choosing n_1 s.t. $x_{n_1} \in B(x, 1)$

Having chosen $n_1 < n_2 < \dots < n_k$ observe that $A \cap B(x, \frac{1}{k})$ is infinite. So we can chose

$n_{k+1} > n_k$ s.t. $x_{n_{k+1}} \in B(x, \frac{1}{k})$

$\lim_{k \to \infty} d(x_{n_k}, x) = 0$ (as $d(x_{n_k}, x) < \frac{1}{k}$)

So $\lim_{k \to \infty} x_{n_k} = x$.

<u>Real sequences</u>

Let $\langle a_n : n \geq 1 \rangle$ be a sequence in $\mathbb{R}$

$a_1 \; a_2 \; a_3 \dots a_n \; a_{n+1}$

n^{th} tail of $\langle a_k : k \geq 1 \rangle$

Define $s_n = \sup(\{a_k : k \geq n\})$

$t_n = \inf(\{a_k : k \geq n\})$

Observe that $s_1 \geq s_2 \geq s_3 \geq \dots$

$t_1 \leq t_2 \leq t_3 \leq \dots$

<u>Define:</u> $\limsup_{n \to \infty} a_n = \lim_{n \to \infty} s_n = \inf(\{s_n : n \geq 1\})$

$\vee |$

$\liminf_{n \to \infty} a_n = \lim_{n \to \infty} t_n = \sup(\{t_n : n \geq 1\})$

Examples: (1) $a_n = (-1)^n$

$\underset{a_1}{-1}, \underset{a_2}{1}, \underset{a_3}{-1}, 1, -1, 1, \ldots\ldots,$

$s_n = 1$, $t_n = -1$

So, $\limsup_{n\to\infty} (-1)^n = 1$

$\liminf_{n\to\infty} (-1)^n = -1$

So, $\limsup_{n\to\infty} a_n \geq \liminf_{n\to\infty} a_n$

(2) $a_n = \frac{1}{n}$, $s_n = \frac{1}{n}$ $t_n = 0$

$\limsup_{n\to\infty} \frac{1}{n} = 0$

$\liminf_{n\to\infty} \frac{1}{n} = 0$

(3) Let $\langle a_n : n \geq 1 \rangle$ be a 1-1 listing of $(0,1) \cap \mathbb{Q}$

$s_n = \sup \{ a_k : k \geq n \}) = 1$

$t_n = 0$

(4) $a_n = (-1)^n n$, $s_n = \infty$

$t_n = -\infty$

$-1, 2, -3, 4, -5, \ldots\ldots$

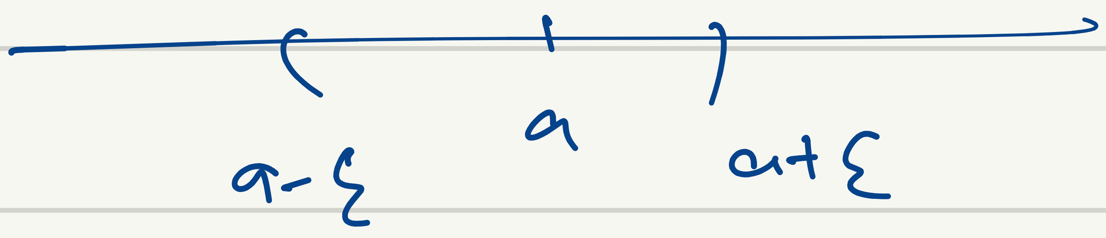
$\limsup_{n\to\infty} a_n = \infty$, $\liminf_{n\to\infty} a_n = -\infty$

Homework $\langle a_n : n \geq 1 \rangle$ bounded seq. in $\mathbb{R}$

$\langle a_n : n \geq 1 \rangle$ converges iff $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$

Proof: $(\Rightarrow)$ fix $a \in \mathbb{R}$ s.t. $\lim_{n\to\infty} a_n = a$

Let $\varepsilon > 0$ be any positive no. There exists $N \geq 1$ s.t. $(\forall n \geq N)\ (|a_n - a| < \varepsilon)$



$a-\varepsilon < a_n < a+\varepsilon \quad \forall n \geq N$

$$a-\varepsilon \leq \inf(\{a_k : k \geq n\}) = t_n \leq S_n = \sup(\{a_k : k \geq n\}) \leq a+\varepsilon$$

$$\Rightarrow a-\varepsilon \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq a+\varepsilon \quad \text{--- } (\cancel{*})_\varepsilon$$

$(*)_\varepsilon$ holds $\forall \varepsilon > 0$

$$\Rightarrow \limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a$$

Series (1) $\langle a_n : n \geq 1 \rangle$ seq. in $\mathbb{R}$

"$\sum_{n=1}^{\infty} a_n$" denotes $\langle S_n : n \geq 1 \rangle$ where

series associated with $\langle a_n : n \geq 1 \rangle$

$S_n = \sum_{k=1}^{n} a_k$ (n^{th} partial sum of $\langle a_n : n \geq 1 \rangle$)

$\sum_{n=1}^{\infty} a_n$ converges to S iff $\lim_{n\to\infty} S_n = S$

(2) $\sum_{n=1}^{\infty} a_n$ converges absolutely iff $\sum_{n=1}^{\infty} |a_n|$ converges

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges but not absolutely

Facts from calculus

(1) $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$

(2) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $1 < p < \infty$

(3) (Leibnitz test for alternating series)

Suppose a sequence $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq a_{n+1} \dots$ is where each $a_n \geq 0$

Then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges iff $\lim_{n\to\infty} a_n = 0$

(4) $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$

Fact: e is irrational

Proof: Suppose not fix $m, n \geq 1$ & $n \geq 2$

$e = \frac{m}{n}$ & $n \geq 10$ then $\frac{m}{n} = 2 + \frac{1}{2!} + \frac{1}{3!} + \dots$

Multi. by $(n!)$

$$\underbrace{\frac{m}{n} \cdot (n!)}_{\text{interger}} = \overbrace{\left[2 + \frac{1}{2!} + \frac{1}{3!} + \dots \frac{1}{n!}\right]}^{\text{Integer}} + \underbrace{\frac{1}{(n+1)!} + \dots}_{\text{not an integer}}$$

$$\left(\frac{m}{n}(n!)\right) - [[nt]] = \underbrace{\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots}_{\text{between } 0 \,\&\, 1}$$

$$= A$$

$$A \le \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots$$

$$A \le \frac{1}{\cancel{n+1}} \cdot \frac{\cancel{n+1}}{n} \Rightarrow A \le \frac{1}{n} < 1$$

Thm: Let $a > 0$ & $n \ge 1$ (n integer). Then there exists a unique $x > 0$ s.t. $x^n = a$ (we write $x = a^{1/n} = \sqrt[n]{a}$)

Proof: (Uniqueness): If $x \ne x'$, say $0 < x < x'$, then $x^n < (x')^n$ (Use induction)

(Existence) Put $E = \{y > 0 : y^n < a\}$

① $E \ne \emptyset$ ($\frac{a}{1+a} \in E$. why? Put $b = \frac{a}{1+a}$

Then $0 < b < 1$, so $b^n < b < a$

② E is bounded from above by $1+a$

$(1+a)^n > 1 + na > a$

So $\sup(E) = x > 0$ exists

Claim: $x^n = a$

Suppose not.

case I: $x^n < a$ $\quad (a - x^n > 0)$

We claim that $\exists\ h>0$ s.t.

$x+h \in E$ (impossible as $x = \sup(E)$

↳ proved similarly as $p^2<2$ doesn't have a largest no. in its set.

$(x+h)^n - x^n \leq hn(x+h)^{n-1} < a - x^n$ for all sufficiently small h.

$\Rightarrow (x+h)^n < a$ for all small $h>0$

$\Rightarrow \underline{x+h \in E}$

for rudin. $h < \frac{(x+1)^{n-1}}{n(a-x^n)}$

(note: $u^2 - v^2 = (u-v)(u+v)$

$u^3 - v^3 = (u-v)(u^2+uv+v^2)$

$u^n - v^n = (u-v)(u^{n-1}+u^{n-2}v + \dots + uv^{n-2}+v^{n-1})$

If $0<v<u$ then

$u^n - v^n = (u-v)(\overbrace{u^{n-1}+\dots+u^{n-1}}^{n \text{ times}})$

$= nu^{n-1}(u-v)^n$

~~HW~~

Case 2:

Lemma: ① $a>0 \Rightarrow \lim_{n\to\infty} a^{1/n} = 1$

② $\lim_{n\to\infty} n^{1/n} = 1$

Proof: ① first assume $a>1$. Then $a^{1/n} > 1\ \forall n\geq 1$

(other case $a=1$ (trivial)
$0<a<1 \Rightarrow 1<b=\frac{1}{a}$)

(otherwise $(a^{1/n})^n \leq (1)^n$
$\Rightarrow a \leq 1$)

Put $x_n = a^{1/n}-1>0$ (Finish the proof).

$a = (1+x_n)^n \geq 1+nx$

$$\Rightarrow \quad x_n \leq \frac{a-1}{n} \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \quad \lim_{n\to\infty} x_n = 0 \Rightarrow \lim_{n\to\infty} a^{1/n} = 1$$

(2) Proof: $x_n = n^{1/n} - 1 > 0 \quad \forall\, n \geq 2$

$$n = (1+x_n)^n > \binom{n}{2} x_n^2 \qquad \begin{matrix}\text{For all} \\ (n \geq 2)\end{matrix}$$

$$n > \frac{n(n-1)}{2} x_n^2$$

$$x_n^2 < \frac{2}{n-1}$$

$$\Rightarrow \quad 0 \leq x_n < \sqrt{\frac{2}{n-1}} \to 0 \quad \text{as } n \to \infty$$

So, $\lim_{n\to\infty} x_n = 0$

$$\Rightarrow \lim_{n\to\infty} n^{1/n} = 1$$

Ratio test: $\sum_{n=1}^{\infty} a_n$, $\alpha = \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

each $a_n \neq 0$

(1) $\alpha < 1 \Rightarrow \sum_{n=1}^{\infty} |a_n| < \infty$ (So $\sum_{n=1}^{\infty} a_n$ is absolutely convergent)

(2) $\alpha \geq 1 \Rightarrow$ Nothing can be said about convergence of $\sum_{n=1}^{\infty} a_n$

$\alpha > 1$ Give counterexample.

Root test: $\sum_{n=1}^{\infty} a_n$, $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$

(1) $\alpha < 1 \Rightarrow \sum_{n=1}^{\infty} |a_n| < \infty$

(2) $\alpha > 1 \Rightarrow \lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges

(3) $\alpha = 1 \Rightarrow$ Nothing can be said about convergence $\sum_{n=0}^{\infty} a_n$

Power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

Let $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$

Define $R = \begin{cases} \infty & \alpha = 0 \\ 0 & \alpha = \infty \\ 1/\alpha & \text{otherwise} \end{cases}$

Then (1) $|x| < R \Rightarrow \sum_{n=0}^{\infty} |a_n x^n| < \infty$

(2) $|x| > R \Rightarrow \sum_{n=0}^{\infty} a_n x^n$ diverges

(3) $|x| = R \Rightarrow$ Inconclusive

Proof of ratio test:

(1) $\alpha < 1$, Let $\beta = \alpha + \frac{1-\alpha}{2}$

Then $\alpha < \beta < 1$

As $\alpha = \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < \beta$

$\exists N \geq 1$ s.t. $(\forall n \geq N)\left(\left|\frac{a_{n+1}}{a_n}\right| < \beta \right)$

Ex: $\limsup_{n\to\infty} a_n < \beta \Rightarrow (\exists N \geq 1)(\forall n \geq N)(a_n < \beta)$

$|a_1| \; |a_2|, \ldots, |a_N|. \; |a_{N+1}| \ldots |a_{N+2}| \ldots$

$|a_1|, |a_2| \ldots \; |a_N|. \; \beta|a_N|, \beta^2|a_N| \ldots \beta^k|a_N|$

$$\Rightarrow \sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{N-1} |a_n| + (|a_N| + |a_N|\beta + \ldots)$$

$$|a_N|+|a_N|\beta+\ldots = |a_N|\left(\frac{1}{1-\beta}\right)$$

(2) Ex/ H.W.

$$\frac{1}{2}, \frac{1}{2^2}, \boxed{\frac{1}{2^3}, \frac{1}{2^4}} \ldots \boxed{\frac{1}{2^n}, \frac{1}{2^{n+1}} \ldots} \ldots$$

$$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{10^3}, \frac{1}{2^4} \ldots \frac{1}{10^n}, \ldots$$

$$\sum_{n=1}^{\infty} |a_n| < \infty$$

Proof of ratio test:

(1) $\alpha < 1$ Put $\beta = 1 + \frac{(1-\alpha)}{2}$

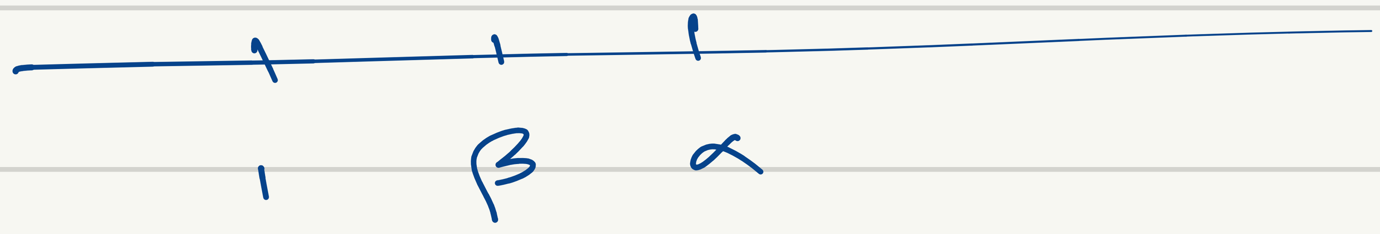
$\Rightarrow \alpha < \beta < 1$

$\limsup_{n\to\infty} |a_n|^{1/n} = \alpha < \beta$

$\Rightarrow (\exists N \geq 1)(\forall n \geq N)(|a_n|^{1/n} < \beta)$

Now, $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n| \leq \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} \beta^n$

$$= \frac{\beta^N}{1-\beta} < \infty$$



$\alpha > 1$: Put $\beta = \alpha - \left(\frac{\alpha - 1}{2}\right)$

$$\limsup_{n\to\infty} |a_n|^{1/n} = \alpha > \beta > 1$$

$\Rightarrow (\forall N)(\exists n > N)\left(|a_n|^{1/n} > \beta\right)$ → There exist infinitely many terms for which $|a_n| > \beta^n > 1$ $\Rightarrow \boxed{\lim_{n\to\infty} a_n \neq 0}$ (So, $\sum_{n=1}^{\infty} a_n$ is divergent)

Exercise $\limsup_{n\to\infty} x_n > \beta$

$\Rightarrow (\forall N)(\exists n > N)(x_n > \beta)$

$$\underbrace{x_1, x_2, \ldots x_N, x_{N+1}}_{\text{sup of them} > \beta}$$

H.W

Lemma $a_n \neq 0 \quad \forall n \geq 1$

$$\limsup_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

Riemann rearrangement theorem

Thm: $\sum_{n=1}^{\infty} |a_n| < \infty \Rightarrow \forall$ bijection $f : \mathbb{N} \to \mathbb{N}$

$$\sum_{n=1}^{\infty} a_n = \underbrace{\sum_{n=1}^{\infty} a_{f(n)}}_{\text{rearrangement of } \sum_{n=1}^{\infty} a_n \text{ via } f}$$

(2) $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n| = \infty$

(so $\sum_{n=1}^{\infty} a_n$ is conditionally convergent (not absolutely convergent))

Then for every $-\infty \le \alpha \le \infty$, there exists a bijection $f: \mathbb{N}^+ \to \mathbb{N}^+$ s.t. $\sum_{n=1}^{\infty} a_{f(n)} = \alpha$

Proof of (1): for each $N \geq 1$ $\sum_{n=1}^{N} |a_{f(n)}| \le \sum_{n=1}^{\infty} |a_n|$

So $\sum_{n=1}^{\infty} |a_{f(n)}| = \lim_{n\to\infty} \sum_{n=1}^{N} |a_{f(n)}| \le A < \infty$ so,

Put $\alpha = \sum_{n=1}^{\infty} a_n$ Will show $(\forall \varepsilon > 0)(\exists N \geq 1)$ $(\forall n \geq N)\left(\left|\sum_{k=1}^{n} a_{f(k)} - \alpha\right| < \varepsilon\right)$

Choose $N_1 \geq 1$ s.t. $\forall n \geq N_1$, $\left|\sum_{k=1}^{n} a_k - \alpha\right| < \varepsilon/2$

Choose $N_2 \geq N_1$ s.t. $\{1, 2, 3, \ldots, N_1\} \subseteq \{f(1), f(2), \ldots f(N_2)\}$

Now if $n \geq N_2$, $\left|\sum_{k=1}^{n} a_{f(k)} - \alpha\right| = \underbrace{\left|\sum_{k=1}^{N_1} a_k - \alpha\right|}_{<\varepsilon/2} + \underbrace{\left|\sum_{k \in F} a_k\right|}_{\le \sum_{k=N_1}^{\infty} |a_k| \to 0 \text{ for } n\to\infty}$

when $F \underset{\text{(finite)}}{\subseteq} \{N_1 + 1, N_1 + 2, \ldots\}$

(2) (a) $\sum_{n=1}^{\infty} a_n$ converge & $\sum_{n=1}^{\infty} |a_n| = \infty \Rightarrow \{n \in \mathbb{N} : a_n \geq 0\} = P_+$

$\{n \in \mathbb{N} : a_n < 0\} = P_-$

are both infinite sets

and $\underbrace{\sum_{n \in P_+} a_n}_{= \sum_{n\ge1} b_n} = \sum_{n \in P_-} |a_n| = \infty$

$ \sum_{n\ge1} -c_n = \sum_{n \in P_-} |a_n|$

Let $\langle b_1, b_2, b_3, \ldots \rangle$ list $\langle a_n : n \in P^+ \rangle$

Let $\langle c_1, c_2, c_3, \dots \rangle$ list $\langle a_n : n \in P^- \rangle$

+ve term $b_1 \quad b_2 \quad b_3 \; \dots \; b_{n_1} \;]\; b_{n_1+1} \dots$

-ve term $c_1 \quad c_2 \quad c_3 \; \dots$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

$a_1 \quad a_2 \quad a_3 \quad a_4$

$1 \quad \frac{1}{3} \quad \frac{1}{5} \quad \frac{1}{7}$

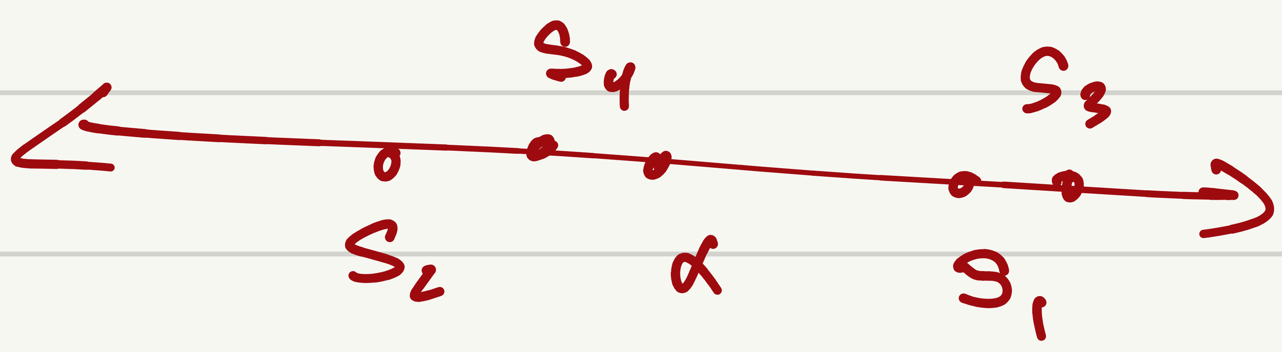
$b_1 \quad b_2 \quad b_3 \quad b_4$

$-\frac{1}{2} \quad -\frac{1}{4} \quad -\frac{1}{6}$

$c_1 \quad c_2 \quad c_3$

Let $-\infty \le \alpha \le \infty$

Case 1: $-\infty < \alpha < \infty$] choose $n_1 \ge 1$ least s.t.
$s_1 = b_1 + b_2 + \dots + b_{n_1} > \alpha$



Choose $n_2 \ge 1$ least s.t.

$s_2 = s_1 + c_1 + c_2 + \dots + c_n < \alpha$

$\lim_{n \to 0} a_n = \lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n = 0$

$\boxed{b_1 + b_2 + \dots + b_{n_1}} + \boxed{c_1 + c_2 + \dots + c_{n_2}} + \boxed{b_{n_1+1} + \dots + b_n}$

it crosses α

pulls it below α

makes it bigger than α

Cauchy criterion for summability

Recall that for a sequence $\langle a_n : n \geq 1 \rangle$ of real nos $\lim_{n\to\infty} a_n$ exists iff $(\forall \varepsilon > 0)(\exists N \geq 1)$
$(\forall m, n > N)$
$(|a_m - a_n| < \varepsilon)$

$\Leftrightarrow$ $\langle a_n : n \geq 1 \rangle$ is cauchy

$\lim_{n\to\infty} \text{diam}(\{a_k : k \geq 1\}) = 0$

Prove that following are equivalent for any series $\sum_{n=1}^{\infty} a_n$

(1) $\sum_{n=1}^{\infty} a_n$ converges

(2) $(\forall \varepsilon > 0)(\exists N \geq 1)(\forall N \leq m \leq n)\left(|a_m + a_{m+1} + \dots + a_n| < \varepsilon\right)$

Proof. Let's start by.
$(1 \Rightarrow 2)$

If $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow$ $\lim_{n\to\infty} a_n \to 0$.

$\forall \varepsilon > 0$ $\exists N \geq 1$ s.t. $|a_n - 0| < \frac{\varepsilon}{n}$ $\Rightarrow$ $|a_n| < \frac{\varepsilon}{n-m+1}$
$\forall n \geq N$.

So, $|a_m + a_{m+1} + \dots a_n| \leq |a_m| + |a_{m+1}| + \dots |a_n|$

$\leq \frac{\varepsilon}{n-m+1} + \frac{\varepsilon}{n-m+1} + \dots$

$\leq \frac{\varepsilon}{n-m+1}(n-m+1)$

Differentiation review

Let $f:[a,b]\to\mathbb{R}$ & $x\in[a,b]$

$$f'(x) = \lim_{y\to x} \frac{f(y)-f(x)}{y-x}$$

$$f'(a) = \lim_{\substack{y\neq x \\ y\to a^+}} \frac{f(y)-f(a)}{y-a}$$

$$f'(b) = \lim_{\substack{y\neq x \\ y\to b^-}} \frac{f(y)-f(b)}{y-b}$$

Facts: (1) $(f\pm g)'(x) = f'(x) \pm g'(x)$

(2) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

(3) $\left(\frac{f}{g}\right)' =$

(4) chain rule

(5) Derivative of f^{-1}

Lemma: $f:[a,b]\to\mathbb{R}$ $x\in[a,b]$

i.e.

(a) $f'(x)>0 \Rightarrow (\exists h>0)\left(\begin{array}{l} (x<y<x+h \Rightarrow f(y)>f(x) \\ (x-h<y<x \Rightarrow f(y)<f(x) \end{array}\right)$

(b) $f'(x)<0 \Rightarrow (\exists h>0)\left(\begin{array}{l} (x<y<x+h \Rightarrow f(y)<f(x) \\ (x-h<y<x \Rightarrow f(x)<f(y) \end{array}\right)$

Cor: $f:[a,b]\to\mathbb{R}$

f has a local $\overbrace{\text{max./min}}^{\text{extremum}}$ at $x\in[a,b)$ and
f is differentiable at x

$$f'(x)=0$$

Roll's thm $f:[a,b]\to\mathbb{R}$

f is continuous on $[a,b]$
f is differentiable on (a,b)
$f(a)=f(b)=L$

$$\Rightarrow (\exists\, x\in(a,b))\ (f'(x)=0)$$

Proof: Put $M_1 = \max(\{f(x) : x\in[a,b]\})$

$M_2 = \min(\{f(x) : x\in[a,b]\})$

$M_1 \geq M_2$

case1: $M_1=M_2 \Rightarrow$ f is constantly $L=M_1=M_2$

$\Rightarrow f'(x)=0 \quad \forall x\in(a,b)$

Case 2: $M_1 > M_2$

$M_1 \geq L \geq M_2 \Rightarrow M_1 > L$ or $L > M_2$

case 2a: $M_1 > L \Rightarrow$ fix $x\in(a,b)$ s.t. $f(x)=M_1$

$\Rightarrow f'(x)=0$

Case 2b: $M_2 < L$

Mean value Theorem:

$f:[a,b]\to\mathbb{R}$, f continuous on $[a,b]$ & f is differentiable on (a,b)

$$\Rightarrow \left(\exists x\in(a,b)\right)\left(\frac{f(b)-f(a)}{b-a}=f'(x)\right)$$

proof:

$$g(x)=\big(f(b)-f(a)\big)x-(b-a)f(x)$$

$g:[a,b]\to\mathbb{R}$

Apply Roll's to g

HW

Exercise: $f:\mathbb{R}\to\mathbb{R}$ satisfies

$$|f(x)-f(y)|\le (x-y)^2$$

Show that f is constant

Cor: suppose $f:(a,b)\to\mathbb{R}$

(1) $f'(x)\ge 0$ $\forall x\in(a,b)$

$\Rightarrow$ f is monotonous increasing on (a,b)

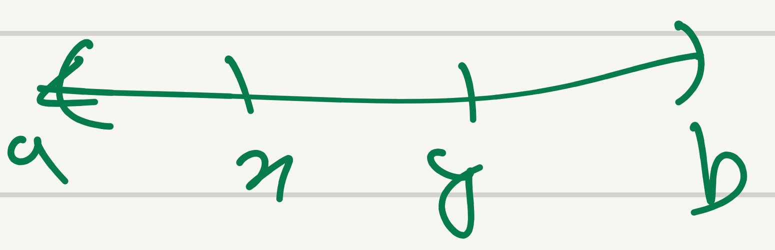
(2) $f'(x)\le 0$ $\forall x\in(a,b)$

$\Rightarrow$ f is monotonically decreasing on (a,b)

(3) $f'(x)=0$ $\forall x\in(a,b)$

$\Rightarrow$ f is const. on (a,b)

Proof:



$z\in(x,y)$

$$\frac{f(y)-f(x)}{y-x}=f'(z)$$

Darboux property of f'

$f:[a,b]\to\mathbb{R}$ is differentiable on $[a,b]$
Then for all α between $f'(a)$ & $f'(b)$

$$(\exists x\in[a,b])\ (f'(x)=\alpha)$$

Proof: WLOG $f'(a)<f'(b)$ & $f'(a)<\alpha<f'(b)$

Define $g:[a,b]\to\mathbb{R}$ by $g(x)=f(x)-\alpha x$
g is differentiable on $[a,b]$

$$\left.\begin{aligned} g'(a) &= f'(a)-\alpha<0 \\ g'(b) &= f'(b)-\alpha>0 \end{aligned}\right]$$

$$\Downarrow$$

$$(\exists h>0)\left(\begin{aligned} &a<x<a+h \Rightarrow g(x)<g(a) \\ &b-h<x<b \Rightarrow g(x)<g(b) \end{aligned}\right)$$

$\Rightarrow$ f attains its minimum value on $[a,b]$ at some $x\in(a,b)$

$$\Downarrow$$

$$g'(x)=0$$

$$\Downarrow$$

$$\boxed{f'(x)=\alpha}$$

$f:\mathbb{R}\to\mathbb{R}$

$$f(x)=\begin{cases} 0 & \text{if } x=0 \\ x^2\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0 \end{cases}$$

$$f'(x)=\begin{cases} 2x\sin\frac{1}{x}-x\cos\left(\frac{1}{x}\right) & \text{if } x\neq 0 \\ 0 & x=0 \end{cases}$$

$$\lim_{x\to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x} = 0$$

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$$

$$= -\lim_{x\to 0}\left(\cos\frac{1}{x}\right) \text{ doesn't exist}$$

HW

Exercise: $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $(x-h, x+h) \setminus \{x\}$

& $\lim_{y\to x} f'(y) = L$ exists

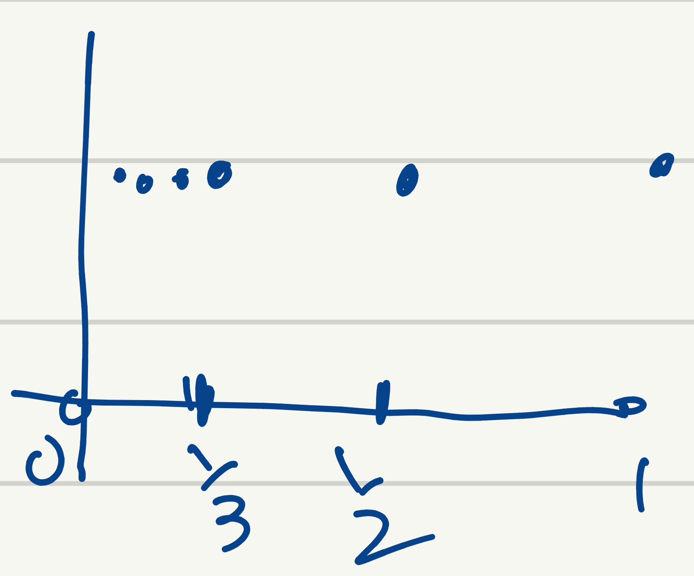
& f is continuous at x $\Rightarrow$ $\boxed{f'(x) = L}$

& so f' is continuous at x.

After Mid-Sem

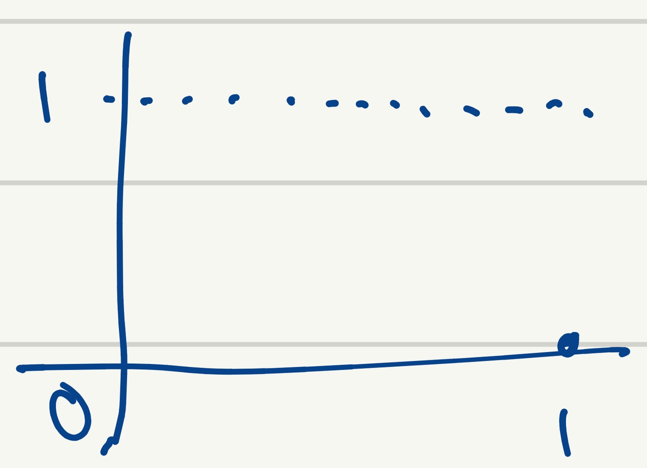
Riemann Integral

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \geq 2 \\ 0 & \text{if } x \in [0,1] \setminus \{\frac{1}{n} : n \geq 2\} \end{cases}$$



$$\int_a^b f(x)\,dx = 0$$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

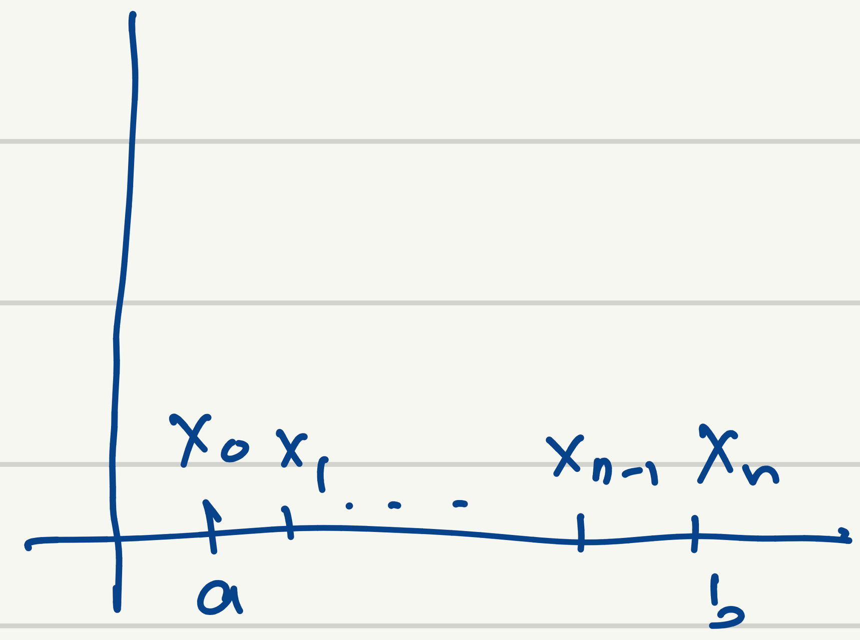


$$\int_0^1 f(x)\,dx = \text{Not riemann Integrable}$$

Def^n: Suppose $f:[a,b]\to\mathbb{R}$ is bounded

An interval partition of $[a,b]$ is a finite $P \subseteq [a,b]$ s.t. $a, b \in P$

Let $P = \{\underset{a}{\underset{\shortparallel}{x_0}} < x_1 < \dots < \underset{b}{\underset{\shortparallel}{x_n}}\}$ be an interval partition of $[a,b]$



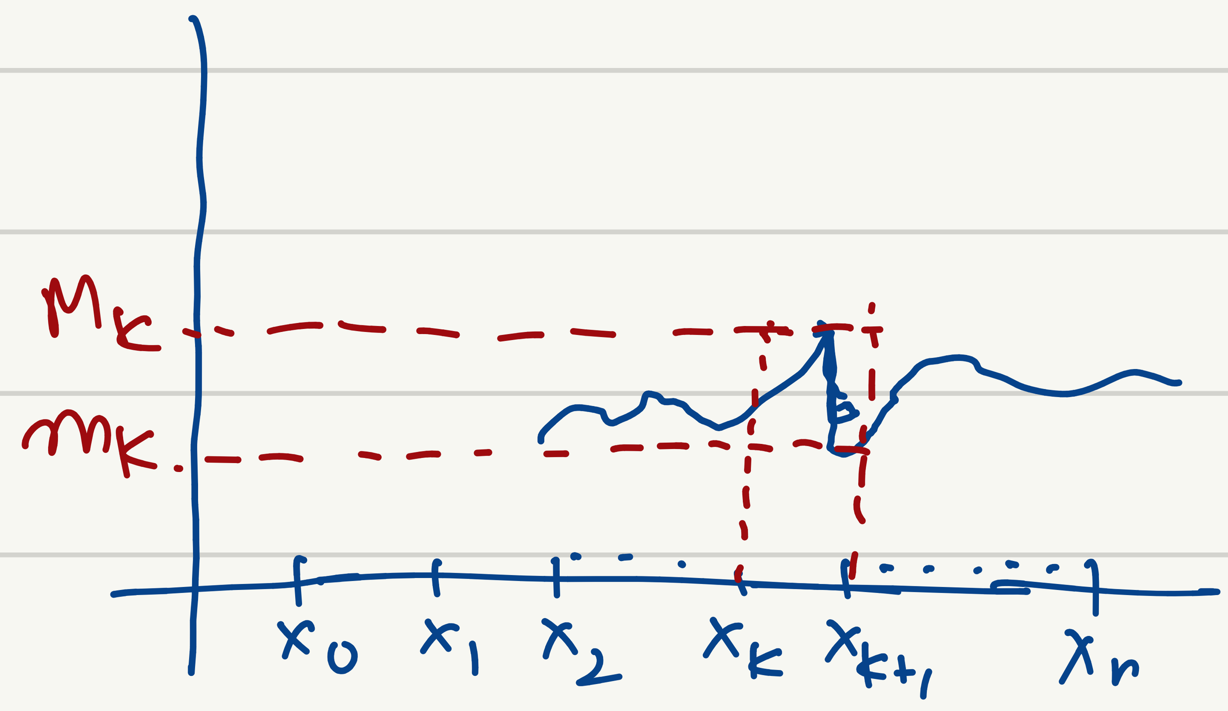
Define for each

$0 \le k \le n-1$, $\sup(f \restriction [x_k, x_{k+1}]) = M_k$

, $\inf(f \restriction [x_k, x_{k+1}]) = m_k$

$P = \{\underset{a}{\underset{\shortparallel}{x_0}} < x_1 \dots < \underset{b}{\underset{\shortparallel}{x_n}}\}$

Define the upper riemann sum of f with respect to P by



$$\mathcal{U}(f,P) = \sum_{k=0}^{n-1} M_k (x_{k+1} - x_k)$$

$$L(f,P) = \sum_{k=0}^{n-1} m_k (x_{k+1} - x_k)$$

(2) The upper/lower riemann integral of f

$$\overline{\int_{[a,b]}} f(x)\,dx = \inf\{U(f,P) : P \text{ is an interval position of } [a,b]\}$$

$$\underline{\int^{[a,b]}} f(x)\,dx = \sup\{L(f,P) : P \text{ is an interval position of } [a,b]\}$$

(3) f is Riemann integrable if

$\overline{\int_{[a,b]}} f(x)\,dx = \underline{\int^{[a,b]}} f(x)\,dx$ & the common value is denoted by $\int_{[a,b]} f(x)\,dx$

P, Q interval partitions of $[a,b]$
Q is a refinement of P iff $P \subseteq Q$

Obvious.

Lemma:- (1) Q refines $P \Rightarrow$ $L(f,P) \leq \boxed{\begin{matrix} L(f,Q) \\ \wedge| \\ U(f,Q) \end{matrix}}$
$$\begin{matrix} \wedge| \\ U(f,P) \end{matrix}$$

(2) f is Riemann integrable iff

$(\forall \varepsilon > 0)$ ($\exists$ interval partition P) $(U(f,P) - L(f,P) < \varepsilon)$

Proof: (1) Exercise
(2)

Lemma: $f : [a,b] \to \mathbb{R}$ bounded

(a) $P = \{x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots x_n\}$ is an interval partition of $[a,b]$, $y \in [a,b] \setminus P$, with $x_0 = a$, $x_n = b$.

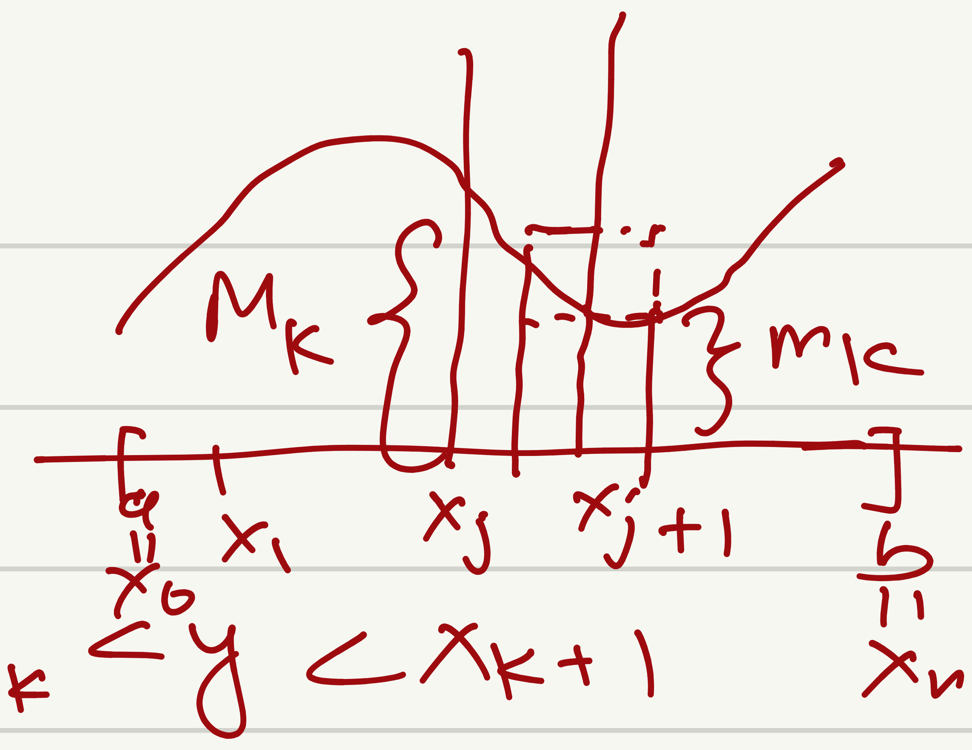
$P' = P \cup \{y\}$. Then $L(f,P) \leq \overbrace{L(f,P') \leq U(f,P')}^{\text{clear}}$
$$\begin{matrix} \wedge| \\ U(f,P) \end{matrix}$$

(b) For any interval partitions P, Q of $[a,b]$

(i) $P \subseteq Q \Rightarrow L(f,P) \leq L(f,Q) \leq U(f,Q)$
$$\begin{matrix} \wedge| \\ U(f,P) \end{matrix}$$

(ii) $L(f,P) \leq U(f,Q)$

(iii) $\overline{\int_{[a,b]}} f(x)dx \geq \underline{\int_{[a,b]}} f(x)dx$



<u>Proof:</u> (a) fix $0 \leq k < n-1$ s.t. $x_k < y < x_{k+1}$

$$L(f,P) = \sum_{k=0}^{n-1} m_k (a_{k+1} - a_k)$$

where $m_k = \inf(\{f(x) : x \in [a_k, a_{k+1}]\})$

$$= \inf(f[[a_k, a_{k+1}]])$$

$$L(f,P') = \sum_{k=0}^{j-1} m_k' (a_{k+1} - a_k) + m_j'(y - x_j) + m_{j+1}'(x_{j+1} - y) + \sum_{k=j+1}^{n-1} m_k' (x_{k+1} - x_k)$$

where $m_k' = m_k$ for all $k \notin \{j, j+1\}$

$$\left.\begin{aligned} m_j' &= \inf(f[[x_j, y]]) \\ m_{j+1}' &= \inf(f[[y, x_{j+1}]]) \end{aligned}\right\} \geq \underbrace{\inf(f[[x_j, x_{j+1}]])}_{= m_j}$$

$$L(f,P) - L(f,P') = \underbrace{m_j(a_{j+1} - a_j) - m_j'(y - a_j) - m_{j+1}'(a_{j+1} - y)}_{\leq 0}$$

$$= \underbrace{(m_j - m_j')}_{\leq 0} \underbrace{(y - a_j)}_{\geq 0} + (m_j - m_{j+1}') \underbrace{(a_{j+1} - y)}_{\geq 0}$$

≤ 0

<u>Proof (b):</u>

(i) Follows from (a) by induction on $|Q \setminus P|$

$|Q \setminus P| = 1$ ✓ by (a)

b (ii) Let $S = P \cup Q$. Then $P \subseteq S$ & $Q \subseteq S$

by b(i) $\boxed{L(f,P) \leq L(f,S)} \leq U(f,S) \leq U(f,P)$

$L(f,Q) \leq L(f,S) \leq U(f,S) \leq U(f,Q)$

$\Rightarrow L(f,P) \leq U(f,Q)$

b (iii) $W_L = \{L(f,P) : P$ is an interval partition of $[a,b]\}$

$W_R = \{U(f,P) : P$ is an interval partition of $[a,b]\}$

b(ii) $\Rightarrow$ $(\forall s_1 \in W_L)(\forall s_2 \in W_R)(s_1 \leq s_2)$

$$\underline{\int_{[a,b]}} f(x)dx = \sup(W_L) \leq \inf(W_R) = \overline{\int_{[a,b]}} f(x)dx$$

Lemma:- $f : [a,b] \to \mathbb{R}$ is bounded

f is Riemann Integrable (this by def^n means $\overline{\int_{[a,b]}} f(x)dx = \underline{\int_{[a,b]}} f(x)dx$)

iff

$(\forall \varepsilon > 0)$ ($\exists$ interval partition P of $[a,b]$)

$(0 \leq U(f,P) - L(f,P) < \varepsilon)$

Proof:- $(\Rightarrow)$ Fix $\varepsilon > 0$. Assume $\underline{\int} f = \overline{\int} f$

Then $(\exists P)(\exists P')\,(\,U(f,P) - L(f,P') < \varepsilon\,)$

interval s
partition of $[a,b]$

Put $Q = P \cup P'$

Claim: $U(f,Q) - L(f,Q) < \varepsilon$

$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$

Examples:- (1) $f : [0,2] \to \mathbb{R}$, $f(x) = 5 \quad \forall x \in [0,2]$

P is any interval partition of $[0,2]$, then

$U(f,P) = L(f,P) = 5 \cdot 2 = 10$

$\Rightarrow \overline{\int}_{[a,b]} f(x)\,dx = \underline{\int}_{[a,b]} f(x)\,dx = 10$

(2) $f(x) = \begin{cases} 1 & 0 \leq x \leq 3/2 \\ 2 & 3/2 < x \leq 2 \end{cases}$

$3/2 \in P$

$U(f,P) = 2.5 \quad \Rightarrow \overline{\int}_{[0,2]} f(x)\,dx = 2.5$

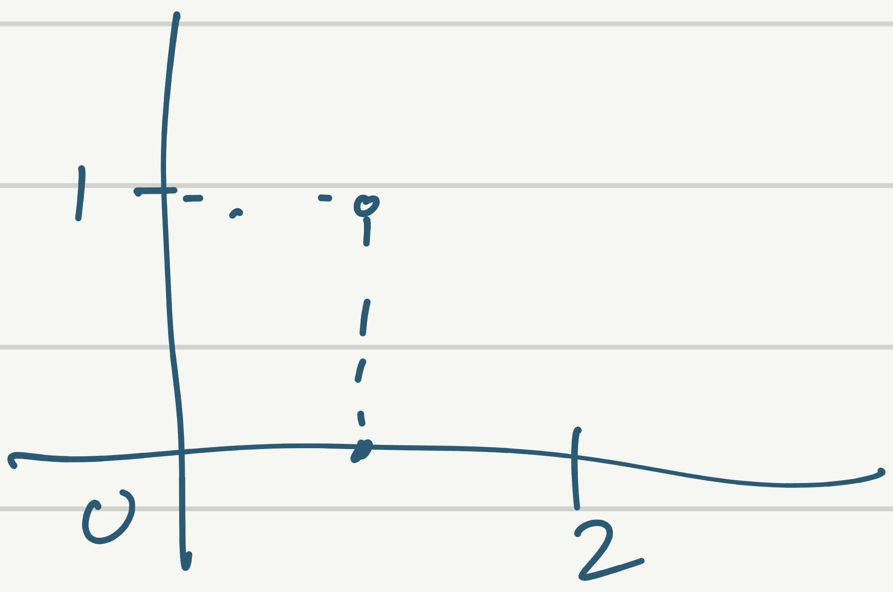
$L(f,P) = 1.5 + l + 1 - 2l$
$= 2.5 - l$

$\underline{\int}_{[0,2]} f(x)\,dx = 2.5$

(3) $$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,2] \\ 0 & x \in [0,2] \setminus \mathbb{Q} \end{cases}$$

$L(f, P) = 0 \cdot 2 = 0$

$U(f, P) = 1 \cdot 2 = 2$



$$\underline{\int_{[0,2]}} f(x)\,dx = 0 < \overline{\int_{[0,2]}} f(x)\,dx = 2$$

Definition: $E \subseteq \mathbb{R}$ has measure zero / is Lebesgue null iff

$(\forall \varepsilon > 0)$ $(\exists \langle J_n : n \geq 1 \rangle$ a sequence of open intervals$)$

$\left(\bigcup_{n \geq 1} J_n \supseteq E \ \& \ \sum_{n \geq 1} \text{length}(J_n) < \varepsilon \right)$

Lemma:

(1) $F \subseteq E \subseteq \mathbb{R}$ & E has measure zero $\Rightarrow$ F has measure zero

(2) $(\forall n \geq 1)$ $(E_n \subseteq \mathbb{R}$ has measure zero$)$ $\Rightarrow \bigcup_{n \geq 1} E_n$ has measure zero

(3) $[a, b]$ doesn't have measure zero

$(\forall a < b)$

$\star$ $\emptyset$ & singleton sets have measure zero in $\mathbb{R}$.

Proof: In Exercise

(2) Put $E = \bigcup_{n\geq 1} E_n$ fix $\varepsilon > 0$. We'll find a sequence $\langle J_n : n \geq 1 \rangle$ of open intervals

$E \subseteq \bigcup_{n\geq 1} J_n$ & $\sum_{n\geq 1} \text{length}(J_n) < \varepsilon$

E_1 has measure zero $\Rightarrow$ choose $\langle J_{1,n} : n \geq 1 \rangle$

s.t. $\bigcup_{n\geq 1} J_{1,n} \supseteq E_1$ & $\sum_{n\geq 1} \text{length}(J_{1,n}) < \varepsilon/2$

E_2 has measure zero $\Rightarrow$ choose $\langle J_{2,n} : n \geq 1 \rangle$

s.t. $\bigcup_{n\geq 1} J_{2,n} \supseteq E_2$ & $\sum_{n\geq 1} \text{length}(J_{2,n}) < \varepsilon/4$

$\vdots$

E_k has measure $\Rightarrow$ choose $\langle J_{k,n} : n \geq 1 \rangle$

s.t. $\bigcup_{n\geq 1} J_{k,n} \supseteq E_k$ & $\sum_{n\geq 1} \text{length}(J_{k,n}) < \frac{\varepsilon}{2^k}$

Let $\langle J_m : m \geq 1 \rangle$ list $\{J_{k,n} : k, n \geq 1\}$ is a one-one function.

$\left(h : \mathbb{N}^2 \xrightarrow{\text{bij.}} \mathbb{N}^1 \times \mathbb{N}^1 \quad J_m = J_{h(n)} \right)$

Then $\sum_{m\geq 1} J_m = \sum_{k\geq 1} \left(\sum_{n\geq 1} J_{k,n} \right) < \sum_{k\geq 1} \frac{\varepsilon}{2^k} = \varepsilon$

& $\bigcup_{n\geq 1} J_n \supseteq E$

Exercise:- Cantor set has measure zero.

(3) $[0,1]$ doesn't have measure zero.

Suppose not let $\varepsilon = \frac{1}{2}$ Choose $\langle J_n : n \geq 1 \rangle$ a sequence of open subsets s.t. $\bigcup_{n\geq 1} J_n \supseteq [0,1]$

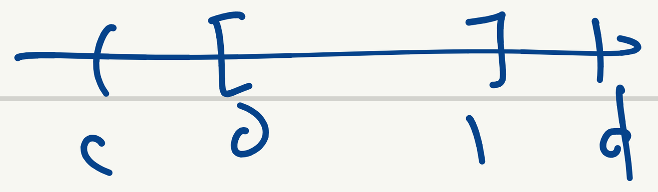
& $\sum_{n\geq 1} \text{length}(J_n) < \frac{1}{2}$

As $[0,1]$ is compact, there are finitely many $\underset{I_1}{\underset{\shortparallel}{J_{n_1}}}, \dots, \underset{I_k}{\underset{\shortparallel}{J_{n_k}}}$ s.t. $[0,1] \subseteq I_1 \cup I_2 \dots \cup I_k$

$$\sum \text{length}(I_j) < \frac{1}{2}$$

By induction on k we get contradiction

$k = 1$



$[0,1] \subseteq I_1$

$$\text{length}(I_1) = d - c > 1$$

Inductive set $k+1$: Homework

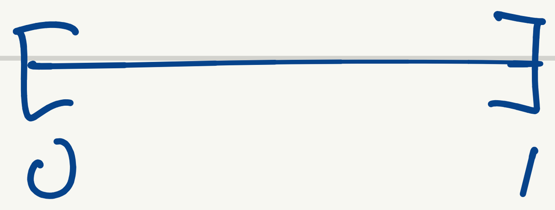
(0) Cantor set C has measure zero

Let $\varepsilon > 0$ be given.

fix k s.t.

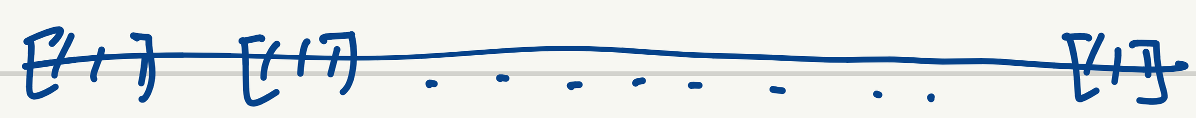
$$\left(\frac{2}{3}\right)^k < \frac{\varepsilon}{2}$$

$C_1 = [0,1]$



$C_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

C_k is disjoint union of 2^k sets s.t. each is of length $\frac{1}{3^k}$.



Let $C_k = I_1 \cup I_2 \cup \dots I_{2^k}$

where $\text{length}(I_n) = \frac{1}{3^k}$

(Lebesgue, 1903)

Thm: $f : [a,b] \to \mathbb{R}$ is bounded.

f is Riemann integrable iff $\{x \in [a,b] : f$ is discontinuous at $x\}$ has measure zero.

Proof:

$(\Rightarrow)$

Def^n Oscillation

$f : [a,b] \to \mathbb{R}$ bounded

$A \subseteq [a,b]$

$$\operatorname{osc}(f, A) = \sup\{|f(y) - f(x)| : x, y \in A\}$$

$$= \sup(f[A]) - \inf(f[A])$$

$$= \operatorname{diam}(f[A])$$

$$\operatorname{osc}(f, x) = \operatorname{osc}(f, [x-h, x+h])$$

$$f[A] = \{f(x) : x \in A\}$$

$$A \subseteq B \Rightarrow f[A] \subseteq f[B]$$

for $0 < h_2 < h_1$, $\quad f[[x-h_2, x+h_2]] \subseteq f[[x-h_1, x+h_1]]$

Theorem: $f : [a,b] \to \mathbb{R}$ is bounded, f is Riemann integrable iff $D_f = \{x \in [a,b] : f$ is discontinuous at $x\}$ has measure zero.

Note:- $f : [a,b] \to \mathbb{R}$ bounded $\phi \neq A \subseteq [a,b]$

define $osc(f,A) = \sup(\{|f(x)| - |f(y)| : x,y \in A\})$

for $x \in [a,b]$, define

$osc(f,x) = \lim_{h \downarrow 0} osc(f, [x-h, x+h])$

HW.

(1) f is continuous at x iff $osc(f,x) = 0$

(2) $\{x \in (a,b) : osc(f,x) > a\}$ is open for every $a > 0$

Proof: Note that $D_f = \bigcup_{N \geq 1} \{x \in (a,b) : osc(f,x) \geq \frac{1}{N}\}$

let $A_N = \{x \in (a,b) : osc(f,x) \geq \frac{1}{N}\}$ is compact

($\Rightarrow$) Assume f is Riemann integrable

will show A_N has measure zero $\forall N \geq 1$

It'll show that $D_f = \bigcup_{N \geq 1} A_N$ has measure zero.

Fix $\varepsilon > 0$. we will find a sequence of open intervals $\langle J_n : n \geq 1 \rangle$ s.t. $\bigcup_{n \geq 1} J_n \supseteq A_N$

& $\sum_{n \geq 1} \text{length}(J_n) < \varepsilon$

As f is Riemann integrable, we can find an interval partition

$$P = \{ \underset{\substack{\| \\ a}}{x_0} < x_1 < \dots < \underset{\substack{\| \\ b}}{x_n} \} \text{ s.t.}$$

$[U(f,P) - L(f,P) < \varepsilon]$ — ⊛

$S = \{ k : 1 \le k < n \ \& \ (x_k, x_{k+1}) \cap A_N \ne \emptyset \}$

$k \in S \implies \exists x \in (x_k, x_{k+1}) \cap A_N$

$\implies \exists x \in (x_k, x_{k+1}) \ (\operatorname{osc}(f, x) \ge \frac{1}{N})$

$\implies \boxed{(M_k - m_k) \ge \frac{1}{N} \quad \forall k \in S}$

$\bigcup \{ (x_k, x_{k+1}) : k \in S \} \supseteq A_N \setminus \{x_0, \dots x_n\}$

$$\sum_{k=0}^{n-1} (M_k - m_k)(x_{k+1} - x_k) < \varepsilon$$

$$\vee|$$

$$\sum_{k \in S} (M_k - m_k)(x_{k+1} - x_k)$$

$$\vee|$$

$$\sum_{k \in S} \frac{1}{N} (x_{k+1} - x_k)$$

$$\sum_{k \in S} (x_{k+1} - x_k) < N\varepsilon$$

$$< \varepsilon' \quad (\varepsilon' = \tfrac{\varepsilon}{N})$$

for $\{x_0, \dots x_n\}$

$\bigcup_{i=0}^{n-1} \{ x_i - \delta, x_i + \delta \}$, length $= 2\delta n$

Take $\delta < \frac{1}{2n}$ $\left(\varepsilon' - \sum_{k \in S} (x_{k+1} - x_k) \right)$

$\frac{?}{}$ or $3n$

($\Leftarrow$) Note that $D_f = \bigcup_{N\geq 1} \{x \in (a,b) : osc(f,x) \geq \frac{1}{N}\}$

let $A_N = \{x \in (a,b) : osc(f,x) \geq \frac{1}{N}\}$ is compact

Now assume D_f has measure zero.

$\Rightarrow$ A_N has measure zero for every $N \geq 1$

<u>lemma</u>: $f : [c,d] \to \mathbb{R}$ bounded & $\forall x \in (c,d)$ $osc(f,x) < \varepsilon$. Then there is an interval partition $Q = \{y_0 < y_1 < \dots y_m\}$ of $[c,d]$ ($y_0 = c$, $y_m = d$)

s.t. $\sum_{k=c}^{m-1} (M_k - m_k)(y_{k+1} - y_k) < 2\varepsilon(d-c)$

<u>continued proof</u>

Want to show $(\forall \varepsilon > 0)$ ($\exists$ partition $P = \{x_0 < x_1 \dots < x_n\}$ s.t. $(U(f,P) - L(f,P) < \varepsilon)$

fix $\varepsilon > 0$. choose N s.t. $\frac{1}{N} < \frac{\varepsilon}{8(b-a)}$ A_N has measure zero & A_N is a compact set

So we can choose a finite list of open intervals $\{J_1, \dots J_p\}$ s.t. $\bigcup_{i=1}^{p} J_i \supseteq A_N$

& $\sum_{i=1}^{p} \text{length}(J_i) < \cancel{\frac{\varepsilon}{(b-a)}}$

Put $[a,b] \setminus \bigcup_{i=1}^{p} J_i = [c_1,d_1] \cup \dots \cup [c_s,d_s]$

for each $1 \le j \le s$ and $x \in (c_j,d_j)$

$\text{osc}(f,x) < \frac{1}{N}$. Using lemma,

choose a partition $Q_j = \{ y_0^j < y_1^j \dots < y_{m_j}^j \}$

s.t. $\sum_{k=0}^{m_j-1} (M_k - m_k)(y_{k+1}^* - y_k) < 2\left(\frac{1}{N}\right)(d_j - c_j)$

Let P be the partition of $[a,b]$, obtained by taking the end pt. of all intervals in Q_j's for $j=1$ to $j=s$

$$U(P,f) - L(P,f) = \underbrace{\sum \text{over } J_1,\dots,J_n}_{< \varepsilon/4} + \sum \text{over } Q_j\text{'s}$$

$$\sum \left(\sup(f \text{ on } J_i) - \inf(f \text{ on } J_i)\right) \text{length}(J_i)$$

$$\le \sum M \text{ length}(J_i) < M \cdot \frac{\varepsilon}{4M} < \frac{\varepsilon}{4}$$

$$\le \sum_{j=1}^{s} 2\left(\frac{1}{N}\right)(d_j - c_j)$$

So, $U - L < \frac{\varepsilon}{2}$

$$\le \frac{2}{N}(b-a) < \frac{\varepsilon}{4}$$

Defn Let $R([a,b])$ be the set of all bounded riemann integrable function from $[a,b]$ to $\mathbb{R}$

Thm:

(1) $C([a,b]) \subseteq R([a,b])$

(2) $f:[a,b] \to \mathbb{R}$ increasing/decreasing $\Rightarrow f \in R([a,b])$

(3) $f \in R([a,b])$ & $a<c<d<b$ $\Rightarrow f \restriction [c,d] \in R([c,d])$

Proof: (1) Let D_f be the set of pts of discontinuity of $f:[a,b] \to \mathbb{R}$, $f \in C([a,b])$

$\Rightarrow f$ is bounded & $\underset{\substack{\uparrow \\ \text{has measure zero}}}{D_f = \emptyset}$. So $f \in R([a,b])$

(2) $f:[a,b] \to \mathbb{R}$ increasing/decreasing $\Rightarrow f$ is bound D_f is countable (by HW) $\Rightarrow D_f$ has measure zero.

(3) Put $g := f \restriction [c,d]$, $D_g \subseteq D_f$ & $f \in R([a,b])$

$\Rightarrow \underset{\substack{\cap \\ D_g}}{}D_f$ has measure zero $\Rightarrow$ D_g has measure zero.

g is bounded as f is bounded

$\Rightarrow g \in R([c,d])$

Thm: Let $f, g \in R([a,b])$ & $\alpha \in \mathbb{R}$

① $f+g \in R([a,b])$ & $\displaystyle\int_{[a,b]} (f+g)(x)\,dx = \int_{[a,b]} f(x)\,dx + \int_{[a,b]} g(x)\,dx$

② $\alpha f \in R([a,b])$ & $\int_{[a,b]} (\alpha f)(x)\,dx = \alpha \int_{[a,b]} f(x)\,dx$

③ $f, g \in R([a,b])$ & if $g \neq 0$ on $[a,b]$

& $\frac{f}{g}$ is bounded on $[a,b]$, then $f, g \in R([a,b])$

④ $f \leq g \Rightarrow \int_{[a,b]} f(x)\,dx \leq \int_{[a,b]} g(x)\,dx$

⑤ $a \leq c \leq b$, $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

Exercise

Proof : (1) f, g bounded $\Rightarrow$ $f+g$ is bounded

$D_{f+g} \subseteq D_f \cup D_g$ & D_f, D_g have measure zero

$\Downarrow$

$D_{f+g} \subseteq D_f \cup D_g$ has measure zero.

$\Downarrow$

D_{f+g} has measure zero.

(a) Hence $f+g \in R([a,b])$

(3) $\frac{f}{g}$ is bounded & $D_{f/g} \subseteq D_f \cup D_g$

$\Rightarrow$ $D_{f/g}$ has measure zero $\Rightarrow$ $D_{f/g} \in R([a,b])$

fg is bounded & $D_{fg} \subseteq D_f \cup D_g$

$\Rightarrow$ D_{fg} has measure zero $\Rightarrow$ $fg \in R([a,b])$

(4) Enough to show $h \in R([a,b])$ & $h(x) \geq 0$ $\forall x \in [a,b]$

$$\Rightarrow \int_{[a,b]} h(x)dx \geq 0$$

If P is any interval partition of $[a,b]$, then $U(f,P) \geq 0$. Hence $\overline{\int}_{[a,b]} f(x)dx \geq 0$

Proof ① (b) We'll show that for every $\varepsilon > 0$,

$$\left| \int f + \int g - \int (f+g) \right| < \varepsilon$$

choose interval partitions P_1 & P_2 of $[a,b]$ s.t.

$$L(f,P) \leq \int_{[a,b]} f(x)\,dx \leq U(f,P)$$

$$\begin{cases} U(f,P) - L(f,P) < \varepsilon/2 \\ U(g,P) - L(g,P) < \varepsilon/2 \end{cases}$$

$$(*) \begin{cases} U(f,P) - \varepsilon/2 \leq \int_{[a,b]} f(x)\,dx \leq L(f,P) + \frac{\varepsilon}{2} \\ U(g,P) - \frac{\varepsilon}{2} \leq \int_{[a,b]} g(x)\,dx \leq L(g,P) + \frac{\varepsilon}{2} \end{cases}$$

$$\int_{[a,b]} (f+g)(x)\,dx \leq \boxed{\overline{\int}_{[a,b]} (f+g)(x)\,dx \leq U(f,P) + U(g,P)}$$

Let $P = \{ x_0 < x_1 < \cdots < x_n \}$, $x_0 = a$, $x_n = b$

$$U(f+g,P) = \sum_{k=0}^{n-1} \left(\sup(f+g \restriction [x_k, x_{k+1}]) \right)(x_{k+1} - x_k)$$

$$\leq \sum_{k=0}^{n-1} \left(\sup(f \restriction [x_k, x_{k+1}]) + \sup(g \restriction_{[x_k, x_{k+1}]}) \right.$$

$h_1, h_2 : [c,d] \to \mathbb{R}$ bounded

$\sup(\text{range}(h_1+h_2)) \le \sup(\text{range}(h_1)) + \sup(\text{range}(h_2))$

from (#)

(#)
$$\left\{\begin{array}{l}
\overline{\int_{[a,b]}}(f+g)(x)\,dx \le U(f,P)+U(g,P) \le \int_{[a,b]} f(x)\,dx + \int_{[a,b]} g(x)\,dx + \varepsilon \\
\quad \vee| \\
\int_{[a,b]}(f+g)(x)\,dx \\
\quad \vee| \\
\underline{\int_{[a,b]}}(f+g)(x)\,dx \ge L(g,P)+L(f,P) \\
\qquad\qquad\qquad\qquad\qquad\qquad \mid\vee \\
\qquad\qquad\qquad\qquad \int_{[a,b]} f(x)\,dx + \int_{[a,b]} g(x)\,dx - \varepsilon
\end{array}\right.$$

&

from (#)

$$\int f + \int g - \varepsilon \le \int (f+g) \le \int f + \int g + \varepsilon$$

$$\Downarrow \quad \left|\int f + \int g - \int (f+g)\right| < 2\varepsilon$$

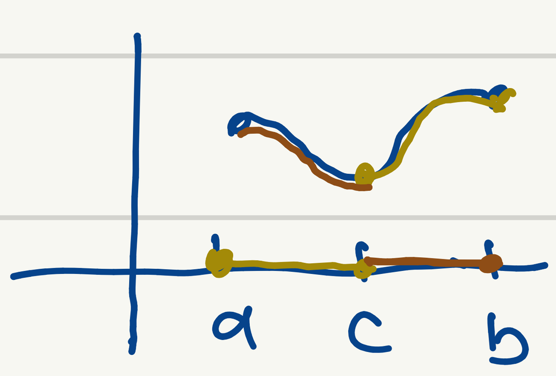
Thm: (3) $f \in R([a,b])$, $\alpha \in \mathbb{R} \Rightarrow \alpha f \in R([a,b])$

and $\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$

(5) $f \in R([a,b])$, $a \le c < d \le b \Rightarrow f \restriction [c,d] \in R([c,d])$

and $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Proof (5) Define $g, h : [a,b] \to \mathbb{R}$



$$h(x) = \begin{cases} 0 & \text{if } a \le x < c \\ f(x) & \text{if } c \le x \le b \end{cases}$$

$$g(x) = \begin{cases} h(x) & \text{if } a \le x \le c \\ 0 & c < x \le b \end{cases}$$

D_f has measure zero

$D_g \subseteq D_f \cup \{c\}$

$D_h \subseteq D_f \cup \{c\}$

$\Rightarrow$ D_g, D_h both have measure zero & as f is bounded, both g, h are bounded.

So, $g, h \in R([a,b])$

By a previous part, $\int_a^b (g+h)(x)\,dx = \int_a^b g(x)\,dx + \int_a^b h(x)\,dx$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Check.

(3) f is bounded $\Rightarrow \alpha f$ is bounded

$D_{\alpha f} \subseteq D_f$ & D_f has measure zero $\Rightarrow$ $D_{\alpha f}$ has measure zero

Hence $\alpha f \in R([a,b])$

Case 1: $\alpha = 0$

Case 2: $\alpha > 0$

for any interval partition $P = \{x_0 < x_1 < \cdots < x_n\}$, $x_0 = a$, $x_n = b$, of $[a,b]$

$$U(\alpha f, P) = \sum_{k=0}^{n-1} \sup(\{\alpha f(x) ; x_k \le x \le x_{k+1}\})(x_{k+1} - x_k) \quad - (1)$$

$$\left.\begin{array}{l} \sup(-A) = -\inf(A) \\ \inf(-A) = -\sup(A) \\ \alpha>0 \Rightarrow \begin{cases} \sup(\alpha A) = \alpha \sup(A) \\ \inf(\alpha A) = \alpha \inf(A) \end{cases} \end{array}\right\} \underline{HW}.$$

① <u>continued</u>.

$$= \alpha \sum_{k=0}^{n-1} \sup\left(f\left[[x_k, x_{k+1}]\right]\right)(x_{k+1}-x_k)$$

$$= \alpha\, U(f, P)$$

$$\operatorname{Inf} \{ U(\alpha f, P) : P \text{ interval partition of } [a,b] \}$$

$$= \inf(\{ \alpha U(f,P) : P \text{ int. part. of } [a,b] \}$$

$$\overset{HW}{=} \alpha \inf(\{ U(f,P) : P \text{ in part of } [a,b] \})$$

$$= \alpha \overline{\int_{[a,b]}} f(x)\,dx = \alpha \overline{\int_a^b} f(x)\,dx$$

<u>Case 3: $\alpha < 0$</u>

$$\alpha < 0 \Leftrightarrow -\alpha > 0$$

$$\alpha A = -((-\alpha)(A))$$

$$U(\alpha f, P) = \alpha L(f, P)$$

$$L(\alpha f, P) = \alpha U(f, P)$$

[Check]

$$\sup(\alpha A) = \sup(-(-\alpha)(A)) = -\inf(-\alpha A) \overset{HW}{=} (-1)(-\alpha)\inf(A) = \alpha \inf(A)$$

$$\int_a^b (\alpha f)(x)\,dx = \overline{\int_{[a,b]}} (\alpha f)(x)\,dx = \inf(\{ U(\alpha f, P) : P \text{ int. part. of } [a,b] \})$$

$$= \inf(\{ \alpha L(f,P) : P \text{ int. part. of } [a,b] \})$$

$$= \alpha \sup(\{ L(f,P) : P \text{ int. part. of } [a,b] \})$$

$$= \alpha \int_{[\bar{a},b]} f(x)\,dx = \alpha \int_a^b f(x)\,dx$$

(HW)(52) $f : [a,b] \to \mathbb{R}$ is Riemann integrable
Show that $|f|$ is Rieman integrable

and $$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx$$

$f : [a,b] \xrightarrow{RI} \mathbb{R}$

$\Rightarrow \forall [c,d] \subseteq [a,b]$, f is also RI on $[c,d]$

Indefinite Integral $\left\{ F(x) = \int_a^x f(t)\,dt \right.$, $x \in [a,b]$

$$|F(x) - F(y)| = \left| \int_y^x f(t)\,dt \right| \le \int_y^x |f(t)|\,dt \le M(x-y)$$

$$\le \int_x^y |f(t)|\,dt \le M(y-x)$$

So $|F(x) - F(y)| \le M|x-y|$

Then F is Lip, in particular cts.

$c \in [a,b]$, Assume f is cts at c.

$$\frac{F(c+h) - F(c)}{h} = \frac{1}{h} \int_c^{c+h} f(t)\,dt \qquad \text{Assume } h > 0.$$

$$\Rightarrow \left| \frac{F(c+h) - F(c)}{h} - f(c) \right| = \frac{1}{h} \left| \int_c^{c+h} (f(t) - f(c))\,dt \right|$$

$$\le \frac{1}{h} \int_c^{c+h} |f(t) - f(c)|\,dt$$

$$\le \frac{1}{h} \cdot \varepsilon \cdot h = \varepsilon$$

Let $\varepsilon > 0$. $\because f$ is cts at c i.e. $\exists\ \delta > 0$ s.t.
$x, y \in [a, b]$ & $|x - y| < \delta$ $|f(x) - f(y)| < \varepsilon$

Thm 6.20 (Rudin)

Let $0 < h < \delta$ $|f(t) - f(c)| < \varepsilon$ whenever $t \in [c, c+h]$

1st F.T.C (Fundamental Theorem of Calculus)

Conclusion -

1) If f is cts at c then F is diff. at c
& $F'(c) = f(c)$

2) If f is cts then F is diff & $F' = f$

COROLLARY :

$f : \mathbb{R} \xrightarrow{\text{cts}} \mathbb{R}$ $F(x) = \int_0^x f(t)\, dt$

$\Rightarrow$ F is diff. & $F' = f$

f cts $\Rightarrow$ F diff
$F' = f$

Thm 6.21 Rudin

2nd Fundamental Theorem

G is diff, $G' \text{ is } RI \Rightarrow \int_a^b G' = G(b) - G(a)$

$f : [a, b] \xrightarrow{\text{diff}} R$ & $f' \in RI$. (Riemann integrable)

$\Rightarrow \int_a^b f' = f(b) - f(a)$

Proof: let P : $a = x_0 < x_1 < x_2 \ldots < x_n = b$ be a partition of $[a, b]$, Then we prove $L(f', P) \leq f(b) - f(a) \leq U(f', P)$

$f(b) - f(a) = (f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + \ldots (f(x_n) - f(x_{n-1}))$

$= f'(c_0)(x_1 - x_0) + f'(c_1)(x_2 - x_1) + \ldots + (f(x_n) - f(x_{n-1}))$

(From LMVT)

$a = x_0 < c_0 < x_1$

$x_1 < c_1 < x_2$

$\vdots$

$x_{n-1} < c_{n-1} < x_n$

$$\leq \underset{[x_0, x_1]}{\text{lub}} f' (x_1 - x_0) + \underset{[x_1, x_2]}{\text{lub}} f' (x_2 - x_1) + \dots + \underset{[x_{n-1}, x_n]}{\text{lub}} f' (x_n - x_{n-1})$$

$$= U(f', P)$$

$\text{lub} \{ L(f', P) : P \text{ is a partition of } [a,b] \}$

$\text{glb} \{ U(f', P) : \quad " \quad \} = \int_a^b f'$

$$\Rightarrow \underline{\int_a^b} f' \leq f(b) - f(a) \leq \overline{\int_a^b} f'$$

since f' is RI, $f(b) - f(a) = \int_a^b f'$

change of variable

$$[c,d] \xrightarrow[\text{diff}]{\phi} [a,b] \xrightarrow[\text{cts.}]{f} \mathbb{R}$$

Assume $\phi' \in RI$ $\qquad (F \circ \phi)'(x)$

$F(x) = \int_a^x f(t)\, dt$. $a \leq x \leq b$ is diff.

$$(F \circ \phi)'(x) = F'(\phi(x)) \cdot \phi'(x) = \overbrace{f(\phi(x))\phi'(x)}^{RI}$$

$\forall x \in [a, b]$

From F.T.

$$\int_c^d (F \circ \phi)' = \int_c^d f(\phi(x))\, \phi'(x)\, dx$$

$$\| $$

$$F(\phi(d)) - F(\phi(c)) = \int_{\phi(c)}^{\phi(d)} f$$

$y = \phi(x)$

Integration by parts :-

$f, g : [a,b] \xrightarrow{\text{diff.}} \mathbb{R}$

$f', g' \in RI.$

$(fg)' = f'g + fg'$

$$\int_a^b (fg)' = \int_a^b f'g + \int_a^b fg'$$

$$\Rightarrow f(b)g(b) - f(a)g(a) = \int_a^b f'g + \int_a^b g'f$$

$$\Rightarrow \int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'$$

X-set (non-empty set)

$f_n : X \to \mathbb{R}$

$\{f_n\}_{n=1}^{\infty}$ & $f : X \to \mathbb{R}$

$x \in X$, $f_1(x)$, $f_2(x)$, ...

* We say $f_n \xrightarrow[n\to\infty]{} f$ "point-wise" converges if $\forall x \in X$
$f_n(x) \xrightarrow[n\to\infty]{} f(x)$

* We say $\sum_{n=1}^{\infty} f_n$ converges to f pointwise if

$\forall x \in X$, $\sum_{n=1}^{\infty} f_n(x) = f(x)$

Then we say that $f = \sum_{n=1}^{\infty} f_n$

$a \in X$

$\lim_{n\to\infty} f_n(x) = f(x)$

$$\lim_{n\to\infty} \lim_{x\to a} f_n(x) = \lim_{x\to a} f(x) = \lim_{x\to a} \lim_{n\to\infty} f_n(x)$$

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx$$

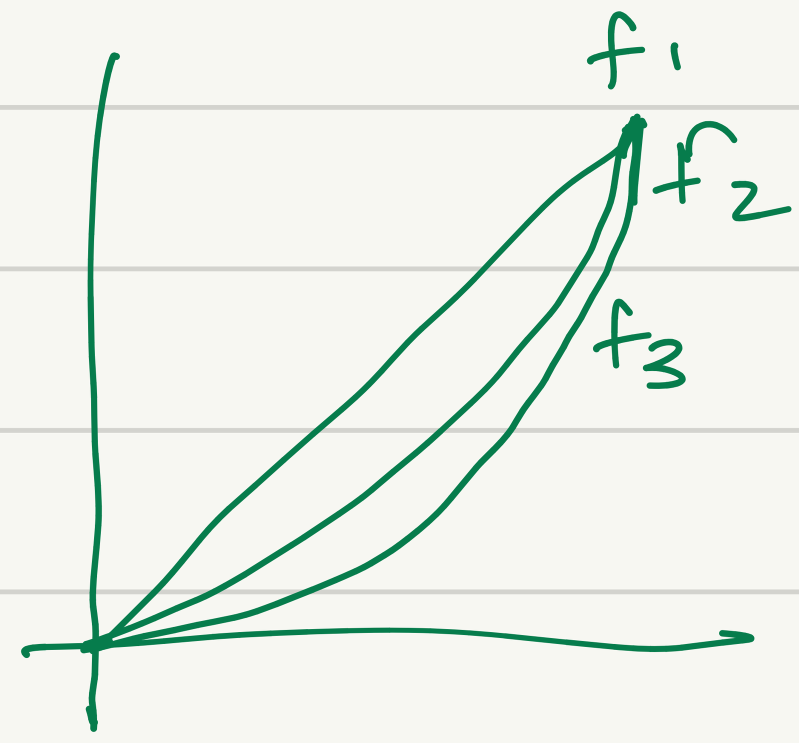
Examples:-

1. $f_n(X) = a_n$

$\{f_n\}_{n=1}^{\infty}$ is convergent iff $\{a_n\}_{n=1}^{\infty}$ is convergent

2) $f_n(x) = x^n, \quad x \in [0,1].$

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$



3) $f_n(X) = \dfrac{x^{2n}}{1+x^{2n}}, \quad x \in [-1,1]$ i.e. $|x| = 1$

$$f(x) = \begin{cases} 0 & |x| < 1 \\ \frac{1}{2} & x = \pm 1 \end{cases}$$

$\dfrac{(n+1)^2 \cancel{x} + x}{n \quad x^n}$

r

4) $f_n(x) = n^2 x (1-x)^n, \quad x \in [0,1]$

So $n^2 r^n, \quad 0 \le r < 1$

$(1+h)^n > nh$

$0 < r < 1 \qquad r^n \to 0$

$n^k r^n = n^k (1-h)^n = n^k \frac{1}{p^n}$

$= n^k \cdot \frac{1}{(1+h)^n} \leq \frac{k!\, n^k}{n(n-1)\cdots} \cdot \frac{1}{h^{k+1}}$

$f(x) = 0 \quad \forall x \in [0,1]$

$\lim_{n\to\infty} \frac{k!}{n(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k}{n})} \cdot \frac{1}{h^{k+1}}$ (k is fixed)

$= 0$

$\int_0^1 f_n(x)\, dx = n^2 \int_0^1 (1-t)t^n\, dt$

$= n^2 \left(\int_0^1 t^n\, dt - \int_0^1 t^{n+1}\, dt \right) = n^2 \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$

$= \frac{n^2}{n+1} - \frac{n^2}{n+2}$

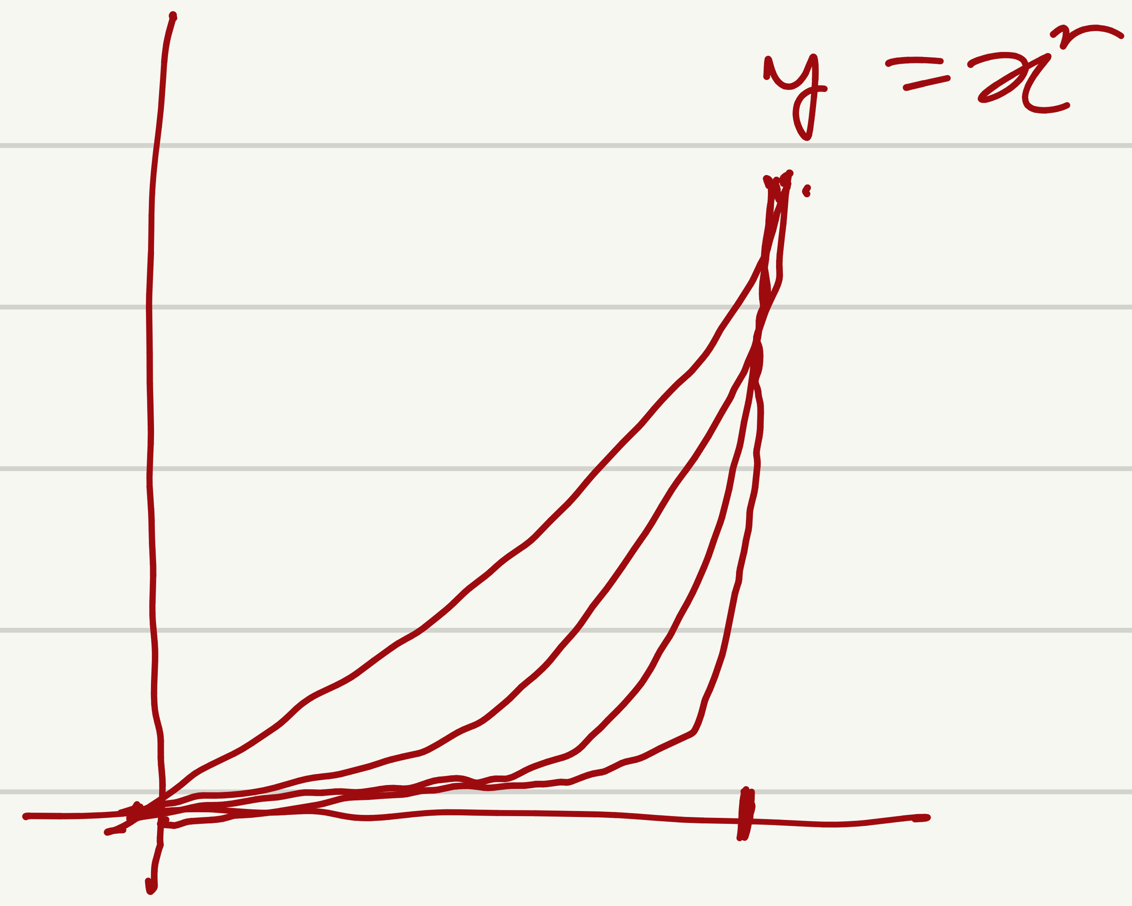
$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\, dx$ (Integrability & limit can't be interchanged freely)

5. $f_n(x) = \frac{\sin nx}{\sqrt{n}} \qquad x \in [0,1]$

$f_n'(x) = \sqrt{n} \cos nx$

Pointwise convergence isn't enough to preserve the qualities of element function to limit function.

$f_n \xrightarrow[n\to\infty]{} f$ point-wise



$\forall x \in X, \quad \forall \varepsilon > 0, \quad \exists N \in \mathbb{N}$ s.t.

$\forall n \geq N, \quad |f_n(x) - f(x)| < \varepsilon$

We want choice of x, that shouldn't affect N.

$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$

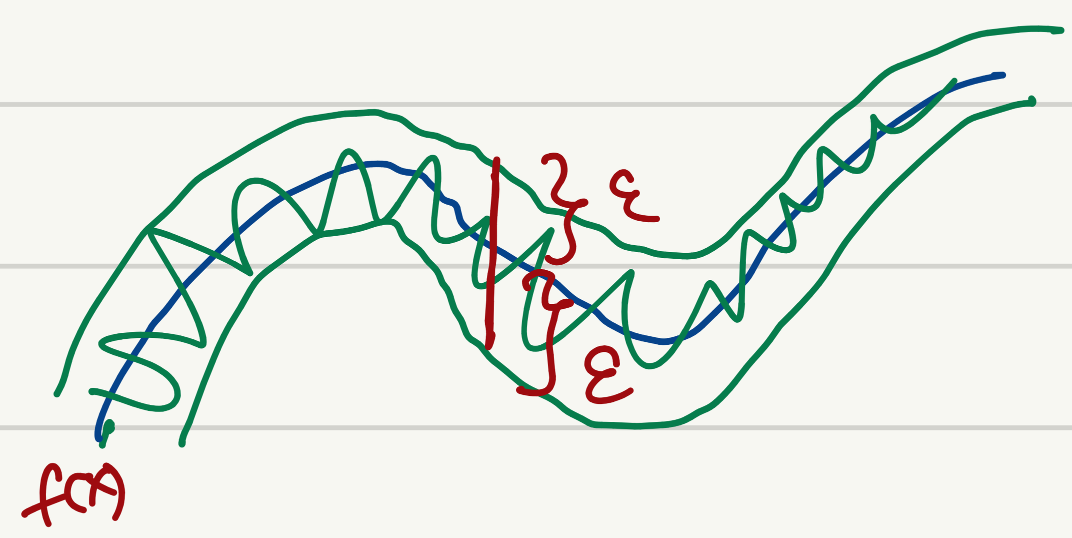
$$|f_n(x) - f(x)| < \varepsilon \quad \forall x \in X$$

If this happens we say $f_n \xrightarrow[n\to\infty]{} f$ "uniformly".

$$f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon$$

Given $\varepsilon > 0$

$\exists N \in \mathbb{N}$ s.t.



X - metric sp. $\quad a \in X$

$f_n : X \to \mathbb{R}$

$f_n \xrightarrow[n\to\infty]{} f$ uniformly

Assume each f_n is cts at a

$$|f(x) - f(a)| \leq |f(x) - f_n(x)| + |f_n(x) - f(a)|$$

$$\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|$$

let $\varepsilon > 0$, since $f_n \xrightarrow[n\to\infty]{} f$ uniformly. so $\exists N \in \mathbb{N}$
s.t. $\forall n \geq N \quad |f_n(x) - f(x)| < \varepsilon \quad \forall x \in X$

From (*) we obtain

$$|f(x)-f(a)| \le 2\varepsilon + |f_N(x) - f_N(a)|$$

As f_N is cts at a $\exists\, \delta>0$ s.t. $\forall x \in B(a,\delta)$

$|f_N(x) - f_N(a)| < \varepsilon \Rightarrow \forall x \in B(a,\delta)$, $|f(x)-f(a)|$
$< 2\varepsilon+\varepsilon = 3\varepsilon$

X - metric sp. $a \in X$

$f_n : [a,b] \to \mathbb{R}$

$f_n \xrightarrow[n\to\infty]{} f$ uniformly

Assume each f_n is RI

$\Rightarrow$ f is RI & $\int_a^b f_n \xrightarrow[n\to\infty]{} \int_a^b f$

If f is RI then $\rightarrow \left| \int_a^b f_n - \int_a^b f \right| = \left| \int_a^b f_n - f \right| \le \int_a^b |f_n(x) - f(x)|\,dx$
$< \varepsilon(b-a)\ \forall n \ge N$

$\varepsilon = 1$, $\forall x \in [a,b]$, $|f_n(x) - f(x)| < 1 \quad \forall n \ge N$

boundedness - $|f(x)| \le \underbrace{|f(x) - f_N(x)|}_{<1} + |f_N(x)|$
$< 1 + M \quad \forall x \in [a,b]$

In order to show RI of f, show that the set of discontinuities has measure zero.

$f_n : X \to \mathbb{R}$

$f_n \xrightarrow[n\to\infty]{} f$ uniform

X - metric space.

$B_c(X) = \{ f : X \xrightarrow{cts} \mathbb{R} : f \text{ is bounded} \}$

$f, g \in B_c(X)$, $d(f,g) = \sup_{x \in X} |f(x) - g(x)|$

$|f(x) - g(x)| \le |f(x)| + |g(x)| \le M_1 + M_2 \quad \forall x$

suppose that $\{f_n\}_{n=1}^{\infty}$ is a seq^n is $B(X)$

$f_n \xrightarrow[n \to \infty]{d} f$

$\Rightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \ge N$

$d(f_n, f) < \varepsilon$

$\Rightarrow$ ( ) $\forall x \in X$, $|f_n(x) - f(x)| < \varepsilon$

conversely, $f_n \xrightarrow[n \to \infty]{} f$ uniformly then $f_n \xrightarrow[n \to \infty]{d} f$

X - set

$f_n : X \to \mathbb{R}$, $\forall n \ge 1$

we say $\{f_n\}_{n=1}^{\infty}$ is uniformly cauchy if.

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall m, n \ge N$ $|f_m(x) - f_n(x)| < \varepsilon$ $\underline{\forall x \in X}$

__Observe__ : $\{f_n\}_{n=1}^{\infty}$ is a seq^n in $B(X)$

$\{f_n\}_{n=1}^{\infty}$ is cauchy $\underbrace{\text{in } B_c(X)}_{d(f_m, f_n) < \varepsilon}$ $\Leftrightarrow$ $\{f_n\}_{n=1}^{\infty}$ is uniformly cauchy

$f_n : X \to \mathbb{R}$; $n \geq 1$

Assume $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent

$$f_n \xrightarrow[n\to\infty]{\text{unif.}} f$$

$$|f_n(x) - f_m(x)| \leq \underbrace{|f_n(x) - f(x)| + |f_m(x) - f(x)|}_{< \varepsilon + \varepsilon}$$

$n \geq N'$
$m \geq N'$

$f_n : X \longrightarrow \mathbb{R}$; $n \geq 1$

Assume $\{f_n\}_{n=1}^{\infty}$ is uniformly cauchy

Let $\varepsilon > 0$; $\exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N$

$$|f_m(x) - f_n(x)| < \varepsilon \quad \forall x \in X$$

$\forall x \in X$, $\{f_n(x)\}_{n=1}^{\infty}$ is cauchy

Go through pt. wise convergence & we get a limit.

We define $f(x) = \lim_{n\to\infty} f_n(x)$ $\forall x \in X$

(We show that this conv. is uniform)

As $\{f_n\}_{n=1}^{\infty}$ is uniformly cauchy

$\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N$ $|f_m(x) - f_n(x)| < \varepsilon$ $\forall x \in X$

$n \geq N$, $x \in X$

$$|f_n(x) - f(x)| \leq |f_n(x) - f_{m_x}(x)| + |f_{m_x}(x) - f(x)| < \varepsilon + \varepsilon = 2\varepsilon$$

As, $f_n(x) \xrightarrow[n\to\infty]{} f(x)$ $\exists N_x \in \mathbb{N}$ $\forall m \geq N_x$

$|f_m(x) - f(x)| < \varepsilon$ $\quad m_x : \max\{N, N_x\}$

Recap

unit interval $= [0,1]$

Is there a seq $\langle f_n : n \geq 1 \rangle$ of continuous f's on $[0,1]$ s.t. $f_n \xrightarrow{\text{pointwise}} f$

where $f(x) = \begin{cases} 1 & x \in [0,1] \cap \mathbb{Q} \\ 0 & x \in [0,1] \setminus \mathbb{Q} \end{cases}$

No

Rene Baire

fact: If $f_n : [a,b] \to [-M,M]$ is Riemann Integrable for every $n \geq 1$ & $f_n \xrightarrow{\text{pointwise}} f : [a,b] \to [-M,M]$

If f is also Riemann intergrable, then
| why assume this? see example 3 |

$$\lim_{n \to \infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

$\langle f_n(x) = n^2 x(1-x)^n \quad n \geq 1 \rangle$ not uniformly bounded on $[0,1]$

Example 3 $\quad \mathbb{Q} \cap [0,1] = \{a_1, a_2 \ldots\}$ lists each member of $\mathbb{Q} \cap [0,1]$ one

Define $f_n : [0,1] \to [0,1]$ fn $n \geq 1$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{a_1, a_2, \ldots a_n\} \\ 0 & \text{if } x \in [0,1] \setminus \{a_1, a_2 \ldots a_n\} \end{cases}$$

$f_n \xrightarrow{\text{pointwise}} f$ where $f : [0,1] \to [0,1]$ is

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise} \end{cases}$$

f is not riemann integrable ($D_f = [a, 1]$ doesn't have measure zero)

Thm (Baire) : Suppose $f_n : [a,b] \to \mathbb{R}$ is conti. for each $n \geq 1$. Let $f : [a,b] \to \mathbb{R}$ be s.t.

$$f_n \xrightarrow{\text{pointwise}} f$$

Then $\underbrace{\{x \in [a,b] : f \text{ is continuous at } x\}}_{[a,b] \setminus D_f}$ is dense in $[a,b]$

Definition: $f : [a,b] \to \mathbb{R}$ is said to be a Baire class one function iff there is a sequence $\langle f_n : n \geq 1 \rangle$ of continuous function on $[a,b]$ s.t.

$$f_n \xrightarrow{\text{pointwise}} f$$

Examples: (1) every continuous $f : [a,b] \to \mathbb{R}$ is Baire class one

(2) If $f : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable then f' is Baire class one why?

then $g_n \xrightarrow{\text{pointwise}} f'$

$$g_n(x) = \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}}$$

Each g_n is continuous on $\mathbb{R}$

Corrollary: (1) $f : [a,b] \to \mathbb{R}$ differentiable on $[a,b]$
$\Rightarrow$ Then f' is continuous on a dense subset of $[a,b]$

(2) $f : [0,1] \to [0,1]$ $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise} \end{cases}$

is not Baire class one

Proof of Theorem

$\forall \varepsilon > 0$, let $W_\varepsilon = \{ x \in \mathbb{R} : osc(f, x) < \varepsilon \}$

$\bigcap_{N \geq 1} W_{\frac{1}{N}} = \mathbb{R} \setminus D_f$

each W_ε is open & dense in $\mathbb{R}$

So by Baire category thm

$\cap W_{\frac{1}{N}}$ is dense in $\mathbb{R}$

To prove W_ε is dense

$\forall m, n \geq 1$, define

$A_{m,n} = \{ x \in \mathbb{R} : | f_m(x) - f_n(x) | \leq \frac{\varepsilon}{3} \}$

pre-image of $[-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}]$ under

$f_m - f_n$, continuous.

$\therefore A_{m,n}$ is closed (?)

$B_k = \bigcap \{ A_{m,n} : m, n \geq k \}$ is closed in $\mathbb{R}$

Observe that $\bigcup_{k\geq 1} B_k = \mathbb{R}$

(why?)

choose $\lim_{n\to\infty} f_n(x) = f(x)$

we can choose $k \geq 1$ s.t. for $\forall\, n \geq k$

$$|f_n(x) - f(x)| < \frac{\varepsilon}{6}$$

$\therefore$ $\forall\, m, n \geq k$

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2} \quad \forall x \in \mathbb{R}$$

Let $[a,b]$ be any interval in $\mathbb{R}$

$$\bigcup_{k\geq 1} (\underbrace{B_k \cap [a,b]}_{\text{closed}}) = [a,b]$$

By baire category thm

$\exists\, k$ s.t. $B_k \cap [a,b]$ is not nowhere dense in $[a,b]$

So, $\exists\, [c,d] \subseteq [a,b]$ s.t.

$$[c,d] \subseteq B_k \cap [a,b]$$

As f_k is cts. at x, $\exists\, \delta > 0$ s.t.

(a) $(x-\delta, x+\delta) \subseteq [c,d]$

(b) $\left(\forall y \in (x-\delta, x+\delta) \right) \left(|f_k(x) - f_k(y)| < \frac{\varepsilon}{3} \right)$

Now for any $y \in (x-\delta, x+\delta)$ and $N \geq k$

$$|f_N(x) - f_N(y)| < \underbrace{|f_N(x) - f_k(x)|}_{<\varepsilon/3} + \underbrace{|f_k(x) - f_k(y)|}_{<\varepsilon/3}$$

$+ |f_k(y) - f_N(y)|$
$< \varepsilon/3$

$< \varepsilon$

Take $N \to \infty$

$|f(x) - f(y)| < \varepsilon$

$\forall y \in (x-\delta, x+\delta)$

$|f(x) - f(y)| < \varepsilon$

$\therefore \operatorname{osc}(f, (x-\delta, x+\delta)) \leq 2\varepsilon$

$\therefore \operatorname{osc}(f, x) \leq 2\varepsilon$

$\therefore x \in W_{3\varepsilon}$

Exercise: Let (X,d) be a metric space. Let $C_0(X) = C_0(X,\mathbb{R})$ = set of all bounded continuous function from X to $\mathbb{R}$

for $f \in C(X)$, define $\|f\|_\infty = \sup(\{|f(x)| : x \in X\})$

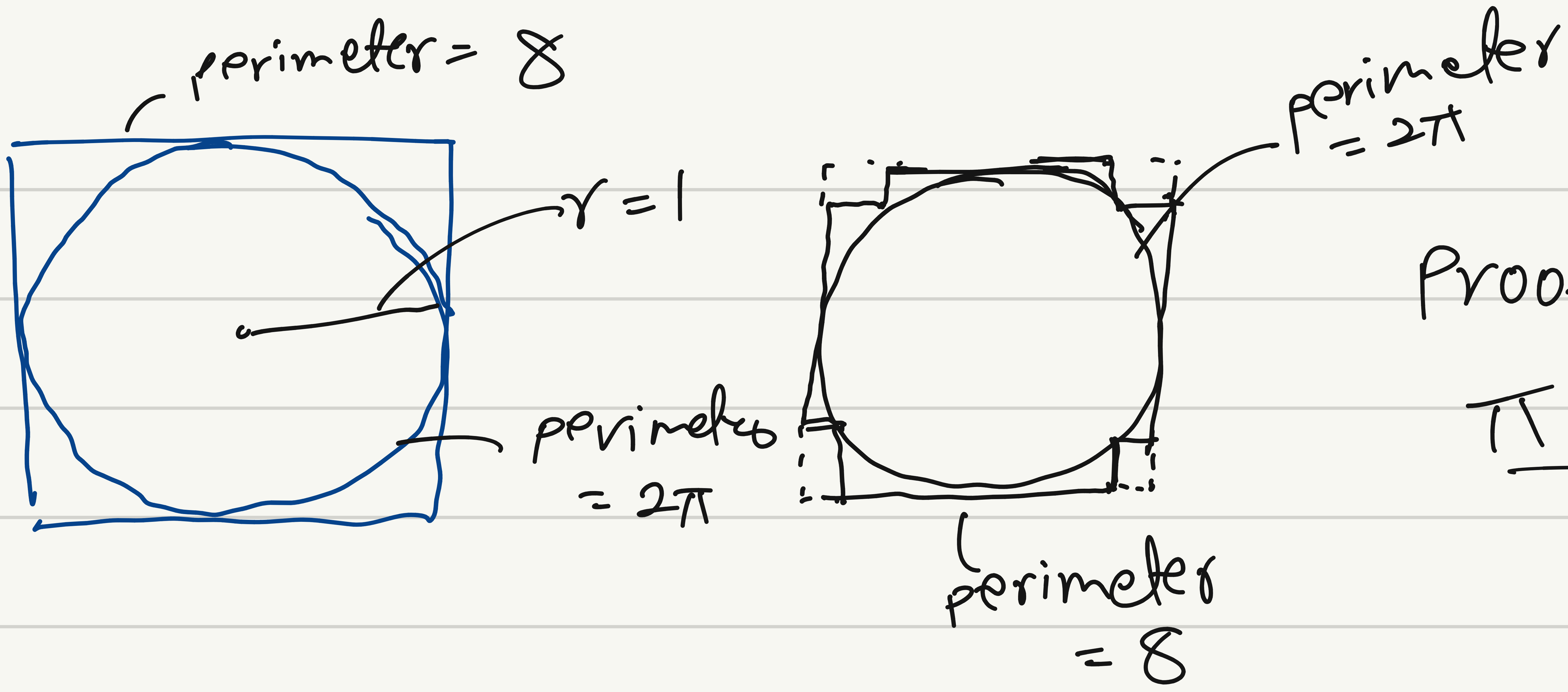
sup norm of f

Define $d_\infty(f,g) = \|f-g\|_\infty$. Show that

$(C_0(X), d_\infty)$ is a complete metric space

* If X is compact, $C_0(X) := C(X)$ = set of all cont. functions from X to $\mathbb{R}$

Continuity is preserved under uniform convergence But NOT DIFFERENTIABILITY.

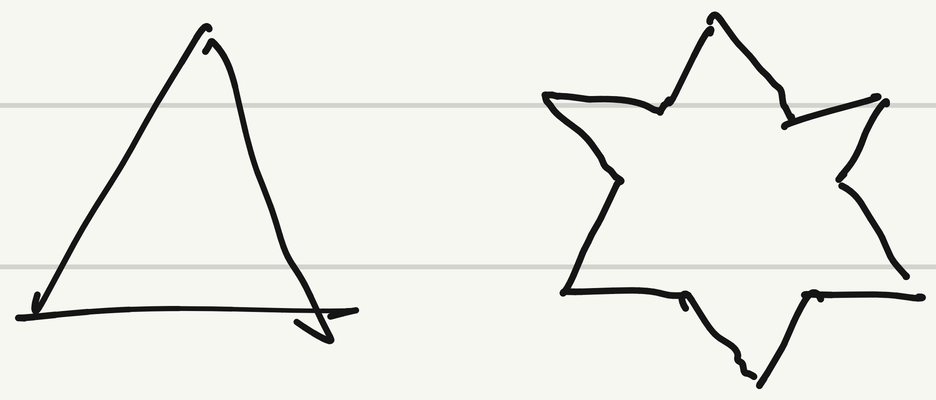


Proof of $\pi = 4$

$f : S^1 \to \mathbb{R}^2 \rightsquigarrow$ a curve

S^1: circle.

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx = \text{length of curve}$$



Theorem $f_n : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ for each $n \geq 1$

1. $\langle f_n : n \geq 1 \rangle$ uniformly converges on $[a,b]$ to $f : [a,b] \to \mathbb{R}$ (so f is continuous)

e.g. $f_n : \mathbb{R} \to \mathbb{R}$

$$f_n(x) = \frac{\sin(n^2 x)}{n}$$

$f_n \xrightarrow{\text{unif}} 0$ on $\mathbb{R}$

$f_n'(0) = n \longrightarrow$ doesn't converge.

(for understanding differentiability isn't transferred by only uniform convergence)

2. $\langle f_n' : n \geq 1 \rangle$ uniformly converges on $[a,b]$ to $g : [a,b] \to \mathbb{R}$. Then f is differentiable on $[a,b]$ & $f'(x) = g(x) \quad \forall x \in [a,b]$

$$= \lim_{n \to \infty} f_n'(x)$$

Proof: 1 can be replaced by

(1)' $(\exists x_0 \in [a,b])(\lim_{n\to\infty} f_n(x_0)$ exists$)$

(1)' + (2) $\Rightarrow$ (1)

Proof: fix $x_0 \in [a,b]$ s.t. $f_n(x_0)$ converges as $n \to \infty$

Want to show $\langle f_n : n \geq 1 \rangle$ is uniformly convergent

Enough to show: (*) $(\forall \varepsilon > 0)(\exists N \geq 1)(\forall m, n \geq N)(\forall x \in [a,b])(|f_n(x) - f_m(x)| < \varepsilon)$

$\langle f_n : n \geq 1 \rangle$ is cauchy in $(C[a,b], d_\infty)$

$\Downarrow$

$(\forall x \in [a,b])(\langle f_n(x) : n \geq 1 \rangle$ is Cauchy in $\mathbb{R})$

$\downarrow$

It converges to say $f(x)$

fix $\varepsilon > 0$. We'll show (*)

$\langle f_n' : n \geq 1 \rangle$ is uniformly convergent to g on $[a,b]$

fix $N_1 \geq 1$ s.t. $(\forall n \geq N_1)(\forall x \in [a,b])\left(|f_n'(x) - g(x)| < \frac{\varepsilon}{4(b-a)}\right)$

By Δ-ineq., $(\forall n, m \geq N_1)\left(|f_n'(x) - f_m'(x)| < \frac{\varepsilon}{2(b-a)}\right)$

By mean value theorem $(\forall x, y \in [a,b])(\exists t$ bet. x & $y)$

$$|(f_n - f_m)(y) - (f_n - f_m)(x)| \overset{MVT}{=} |f_n'(t) - f_m'(t)|\,|y - x|$$

$\forall m,n \geq N_1 \quad \forall x,y \in [a,b] \qquad \leq \frac{\varepsilon|y-x|}{2(b-a)} \quad -(A)$

Let $x \in [a,b] \quad (\forall n,m \geq N_1)$

$$|f_n(x) - f_m(x)| = |(f_n-f_m)(x) - (f_n-f_m)(x_0) + (f_n-f_m)(x_0)|$$

$$\leq |(f_n-f_m)(x) - (f_n-f_m)(x_0)| + |(f_n-f_m)(x_0)|$$

$$\leq \frac{\varepsilon|x_0-x|}{2(b-a)} + \frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

($f_k(x_0) \xrightarrow{k\to\infty} z_0$
N_1 is large enough show that
$\forall m,n \geq N_1$)
$|(f_n-f_m)(x_0)| < \frac{\varepsilon}{2}$

Now we'll show that f in diff. on $[a,b]$
& $f' = g$

Let $x \in [a,b]$ be fixed

$$f'(x) = \lim_{y\to x} \frac{f(y)-f(x)}{y-x} \overset{?}{=} g(x) = \lim_{n\to\infty} f_n'(x)$$

$$\phi_n(y) = \frac{f_n(y) - f_n(x)}{y-x}$$

This is the claim

$\boxed{\text{Claim: } \phi_n(y) \xrightarrow[\text{on } [a,b]\setminus\{x\}]{\text{uniformly}} \frac{f(y)-f(x)}{y-x}}$

proof arijit ne diya hai?

$$(\forall \varepsilon>0)\ (\exists N \geq 1)\ (\forall m,n \geq N)\ \left(|\phi_n(y) - \phi_m(y)| < \varepsilon\right)$$

$$= \frac{(f_n-f_m)(y) - (f_n-f_m)(x)}{y-x} \leq \frac{\varepsilon}{2(b-a)} \text{ (by A)}$$

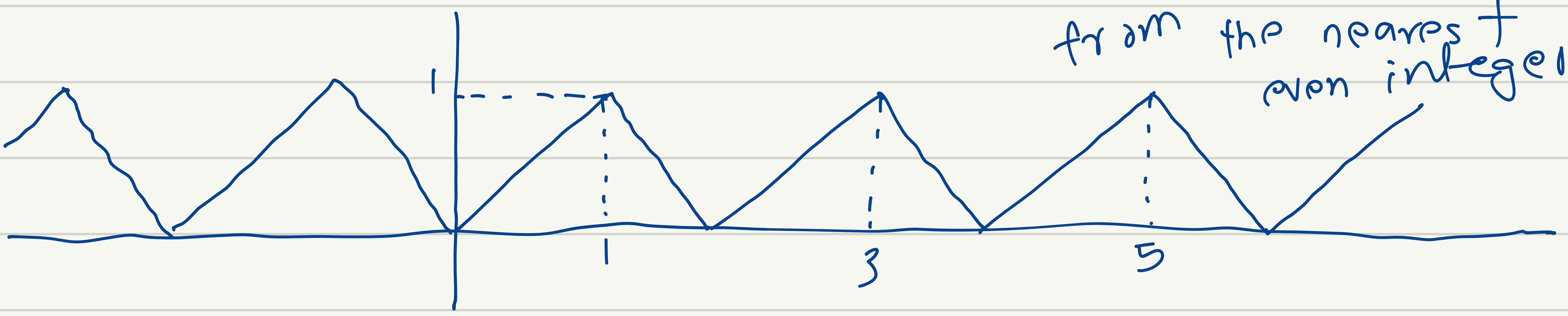
$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

$$= \lim_{y \to x} \Big(\underbrace{\lim_{n \to \infty}}_{\text{unif. by claim}} \frac{f_n(y) - f_n(x)}{y - x} \Big)$$

$$= \lim_{n \to \infty} \lim_{y \to x} \frac{f_n(y) - f_n(x)}{y - x}$$

$$= \lim_{n \to \infty} f_n'(x) = g(x)$$

$y = t(x)$ = distance of x from the nearest even integer



Ex:-

(1) $t(x)$ is continuous & differentiable everywhere except on $\mathbb{Z}$

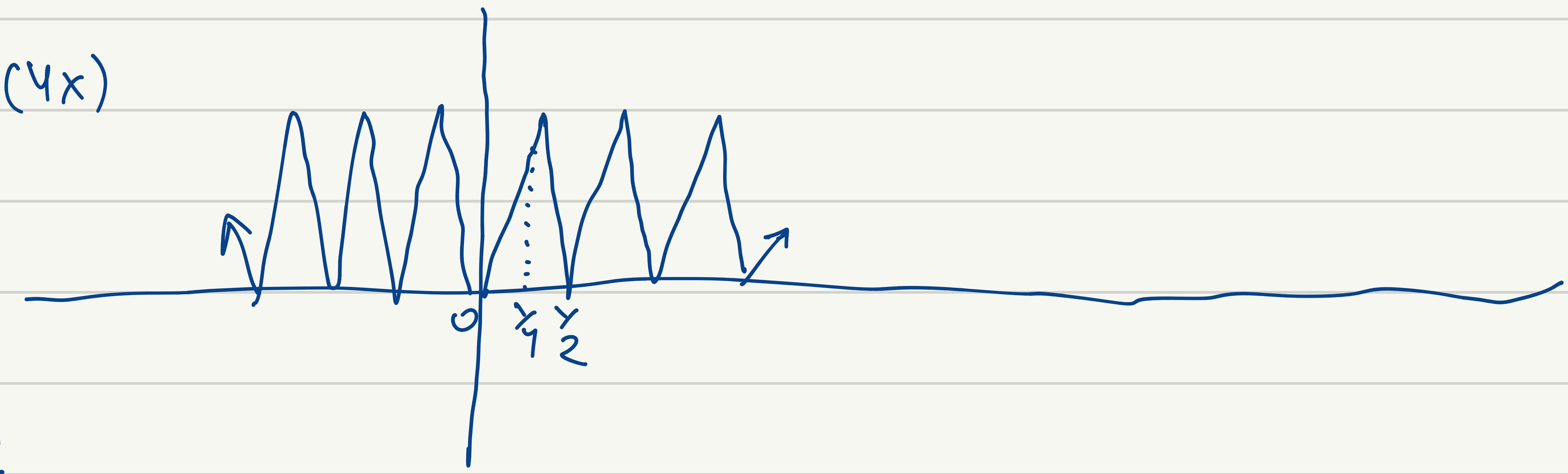
(2) $0 \le t(x) \le 1$ for all $x \in \mathbb{R}$

(3) $|t(x) - t(y)| \le |x - y|$

(4) $t(x+2) = t(x) \ \forall x \in \mathbb{R}$

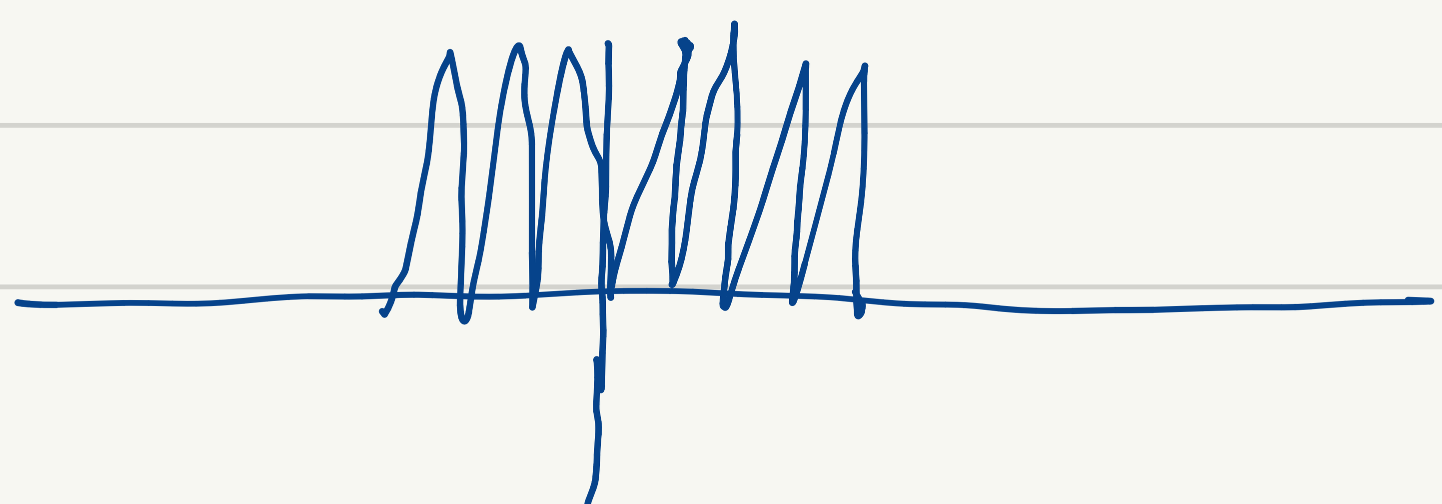
(5) If $x \ne y$ & there is no integer in between x & y then $|t(x) - t(y)| = |x - y|$

$t(4x)$



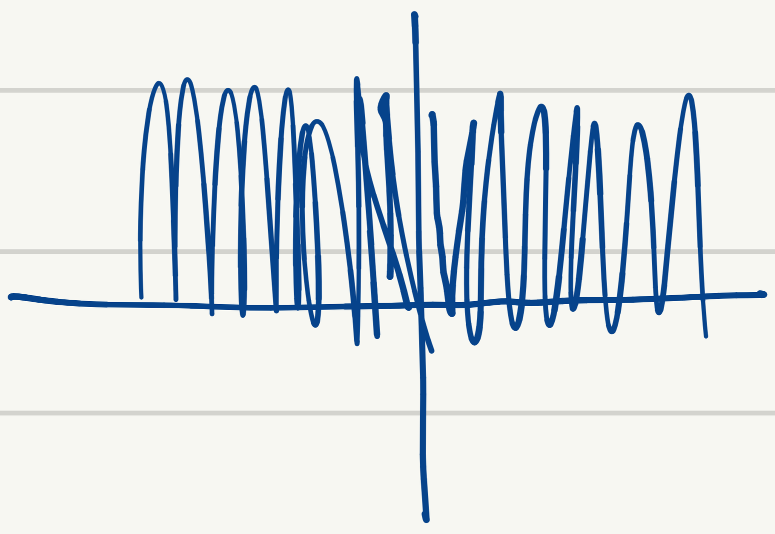
⋮

$t(8x)$



$\vdots$

$t(4^n x)$



$x = \frac{1}{3}$?

$\sum_{n=0}^{\infty} t(4^n x) = ??$

Does it conv. at x?

Depends of x.

Theorem: Let $t(x)$ be as before

Then $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n t(4^n x)$ uniformly converges on $\mathbb{R}$

to $f : \mathbb{R} \to \mathbb{R}$ (so f is continuous on $\mathbb{R}$)

AND f is nowhere differentiable. In fact, for all $x \in \mathbb{R}$

$$\limsup_{h \downarrow 0} \left| \frac{f(x+h) - f(x)}{h} \right| = \infty$$

$h \downarrow 0 \equiv h \to 0^+$

$$\lim_{h \downarrow 0} \left(\sup \left\{ \left| \frac{f(y) - f(x)}{y - x} \right| : 0 < |y - x| < h \right\} \right)$$

Proof: Put $S_n(x) = \sum_{k=0}^{n} \left(\frac{3}{4}\right)^k t(4^k x)$ (Note that each $S_n(x)$ is conti. on $\mathbb{R}$)

Claim: $(\forall \varepsilon > 0)(\exists N \geq 1)(\forall m, n \geq N)(\forall x \in \mathbb{R})$
$(|S_m(x) - S_n(x)| < \varepsilon)$

Proof: trivial as $|t| \leq 1$ & $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n < \infty$

So $S_n \xrightarrow[\text{on } \mathbb{R}]{\text{uniformly}} f$ & f is continuous

fix $\varepsilon > 0$. Choose $N \geq 1$ s.t. $\sum_{k=N}^{\infty} \left(\frac{3}{4}\right)^k < \varepsilon$

If $x \in \mathbb{R}$ & $m, n \geq N$, then $|S_m(x) - S_n(x)|$

$$\le \sum_{k=N}^{\infty} \left| \left(\frac{3}{4}\right)^k t(4^k x) \right|$$

$$\le 1$$

Claim: f is nowhere differentiable

Proof: fix $x \in \mathbb{R}$. Let $m \ge 1$

choose $\delta_m = \pm \frac{1}{2} (4^{-m})$ with a sign

s.t. there is no integer strictly b/w $4^m x$ & $4^m x + (\delta_m) 4^m$
$\qquad$ $\left(x \pm \frac{1}{2}\right)$

As $m \to \infty$, $\delta_m \to 0$

we will show that $\dfrac{f(x+\delta_m) - f(x)}{\delta_m}$ blows up as $m \to \infty$

$$\frac{f(x+\delta_m) - f(x)}{\delta_m} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \frac{\left[t(4^n(x+\delta_m)) - t(4^n x) \right]}{\delta_m}$$

(*)

case I: $n = m$

$$w_m = \frac{t(4^m(x+\delta_m)) - t(4^m x)}{\delta_m}$$

$$= \left| \frac{t(4^m x + (4^m \delta_m)) - t(4^m x)}{\delta_m} \right|$$

$$= \left| \frac{4^m \cancel{\delta_m}}{\cancel{\delta_m}} \right|$$

$$= 4^m$$

($|t(a) - t(b)| = |a-b|$ if there is no integer is in b/w a & b)

case II: $n > m$

$$w_n = \frac{t(4^n x + 4^n \delta_m) - t(4^n x)}{\delta_m}$$

$$= 0$$

because $\quad 4^n \delta_m = \left(\pm \frac{1}{2} 4^{-m} \right) \cdot 4^n = \pm \frac{1}{2} \left(4^{(n-m)} \right.$

$=$ even integer

& $\quad f(y+2) = f(y) \quad (\forall y \in \mathbb{R})$.

case 3: $0 \le n < m$

$$|W_n| = \left| \frac{f(4^n x + 4^n \delta_m) - f(4^n x)}{\delta_m} \right| \le \left| \frac{4^n \cancel{\delta_m}}{\cancel{\delta_m}} \right|$$

$$\le 4^n$$

$|f(a) - f(b)| \le |a - b|$

So, $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \left[\frac{f(4^n(x+\delta_m)) - f(4^n x)}{\delta_m} \right]$

$$= \underbrace{\sum_{n=0}^{n-1} \left(\frac{3}{4}\right)^n \left[\frac{f(4^n(x+\delta_m)) - f(4^n x)}{\delta_m} \right]}_{\substack{\text{Abs value} \\ \le 4^n}} + \left(\frac{3}{4}\right)^m \left(\mp 4^m \right)$$

$$\qquad\qquad\qquad\qquad = \pm 3^m$$

$$\left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| \ge 3^m - (1 + 3 + 3^2 + \ldots + 3^{m-1})$$

$$= 3^m - \left(\frac{3^m - 1}{2} \right)$$

$$\ge \frac{3^m}{2} \to \infty \text{ as } m \to \infty$$

weirstrass approximation theorem:

conv: Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are both continuous and g has compact support (this means some $M > 0$, g vanishes outside $[-M, M]$)

Define $(f * g)(x) = \int_{-\infty}^{\infty} f(x-t) g(x)\, dt$

Ex: Show that $(f*g)(x) = \int_{-\infty}^{\infty} f(t) g(x-t)\, dt$

polynomials are dense in $\left(C[a,b], d_\infty \right)$

$$d_\infty(f,g) = \sup_{x \in [a,b]} |(f-g)(x)|$$

Theorem: Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous. Then for every $\varepsilon > 0$, there is a polynomial $p(x)$ such that $(\forall x \in [a,b]) \left(|f(x) - p(x)| < \varepsilon \right)$

Ex:-

Suppose $f(x) = e^x$, $f : [a,b] \longrightarrow \mathbb{R}$

$e^x = \sum_{n=0}^{\infty} \left(\frac{1}{n!} \right) x^n$ & the series unimormly converges to e^x on $[a,b]$

$$\Rightarrow (\forall \varepsilon > 0)(\exists N)(\forall x \in [a,b]) \left(\left| e^x - \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^N}{N!} \right) \right| < \varepsilon \right)$$

This shows that for real analytic functions the theorem holds.

Proof: Define $h_n : [-1,1] \longrightarrow \mathbb{R}$ by
$h_n(x) = (1 - x^2)^n$ for $n \geq 1$

Note $h_n(x)$ is a polynomial in x of degree $2n$

and $h_n(x) \xrightarrow{n\to\infty} \begin{cases} 0 & \text{if } x \in [-1,1]\setminus\{0\} \\ 1 & \text{if } x = 0 \end{cases}$

Claim 1:

$$\int_{-1}^{1} h_n(x)\,dx > \frac{1}{\sqrt{n}} \quad \forall n \geq 1$$

Proof of claim 1:

$$\int_{-1}^{1} h_n(x)\,dx = 2\int_0^1 (1-x^2)^n\,dx$$

Ex: $0 \leq x \leq 1 \Rightarrow (1-x)^n \geq 1 - nx \quad \forall n \geq 1$

HINT: Let $\theta(x) = (1-x)^n - 1 + nx$. Show that $\theta(0) = 0$ & $\theta'(x) \geq 0$ for $0 \leq x < 1$

So,
$$2\int_0^1 (1-x^2)^n\,dx \geq 2\int_0^{1/\sqrt{n}} (1-x^2)^n\,dx \geq 2\int_0^{1/\sqrt{n}} (1-nx^2)\,dx$$

$$= 2\left([x]_0^{1/\sqrt{n}} + \left[\frac{-nx^3}{3}\right]_0^{1/\sqrt{n}} \right)$$

$$= 2\left(\frac{1}{\sqrt{n}} - \frac{\cancel{n}}{3\cancel{n}\sqrt{n}}\right) = \frac{1}{\sqrt{n}}\left(4/3\right) > \frac{1}{\sqrt{n}}$$

Define $g_n : [-1,1] \to \mathbb{R}$ by $g_n(x) = c_n(1-x^2)^n$ where $c_n = \dfrac{1}{\int_{-1}^{1}(1-x^2)^n\,dx}$

Note $\int_{-1}^{1} g_n(x)\,dx = 1 \quad \left(\int_{-1}^{1} c_n(1-x^2)^n\,dx = c_n \int_{-1}^{1}(1-x^2)^n\,dx = 1 \right)$

Note that by claim 1, $c_n < \sqrt{n} \quad \forall n \geq 1$

WLOG $a = 0$, $b = 1$, $f(0) = f(1) = 0$

$K(x) = f((b-a)x + a)$, $K_1 : [0,1] \longrightarrow \mathbb{R}$

$K_2(x) = (K_1(x) - K_1(0)) - x(K_1(1) - K_1(0))$

$K_2 : [0,1] \to \mathbb{R}$ $\qquad ((b-a)x + a) \longleftarrow\!\shortmid\, x$

$K_2(0) = K_2(1) = 0$

Define $p_n(x) = \int_{-1}^{1} f(x+t)\, g_n(t)\, dt$ for $0 \le x \le 1$

Claim 2: $p_n(x)$ is a polynomial in x.

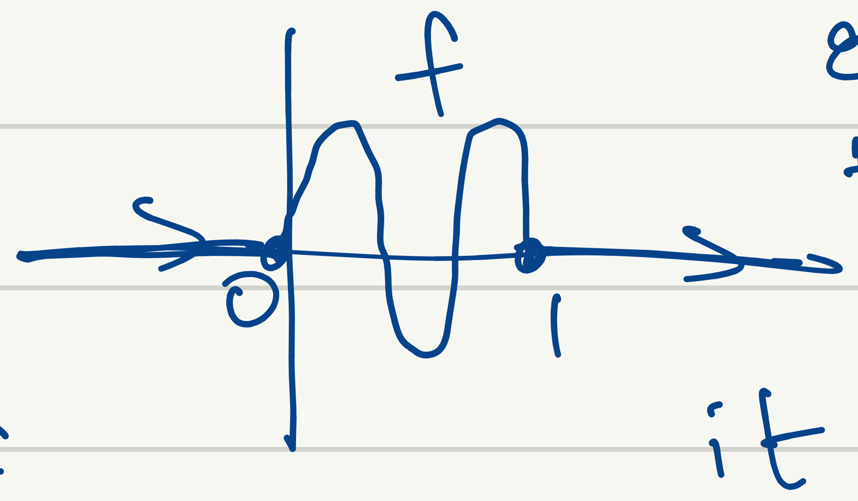
Proof of claim 2: $p_n(x) = \int_{-1}^{1} f(x+t)\, g_n(t)\, dt$

$p_n(x) = \int_{-x}^{1-x} f(x+t)\, g_n(t)\, dt$

[Extend f to all of $\mathbb{R}$ by $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ if $x \notin [0,1]$

$u = x + t$

$= \int_0^1 f(u)\, g_n(u-x)\, du$



extend g_n to $\mathbb{R}$ by declaring it 0 outside $[-1,1]$

poly. in x with coeff as function of u

= poly. in x.

$-1 \le x \le 1$

$g_n(x) = c_n(1-x^2)^n \quad x \in [-1,1]$

$= \int_0^1 f(u)\, g_n(u-x)\, du$

$= \int_0^1 f(u)\, (c_n)\underbrace{(1-(u-x)^2)^n}_{a_{2n}(u)x^{2n} + a_{2n-1}(u)x^{2n-1} + \dots + a_0(u)} du$

$= \int_0^1 [(a_{2n}(u) f(u) c_n) x^{2n} + f(u)\, c_n\, a_{2n-1}(u)\, x^{2n-1}$

$+ \dots + a_0(u)\, c_n f(u)]\, du$

$= \left(\int_0^1 (a_{2n}(u) f(u) c_n)\, du\right) x^{2n} + \dots$

= poly. in x of degree $2n$.

- Arzela Ascoli theorem

$(C([a,b]), d_\infty)$ is a $\underbrace{\text{complete separable metric space}}_{\text{Polish spaces}}$

Is it compact?

(No it's not bounded)

$f_n : [a,b] \to \mathbb{R}$

$f_n(x) = n \quad \forall x$

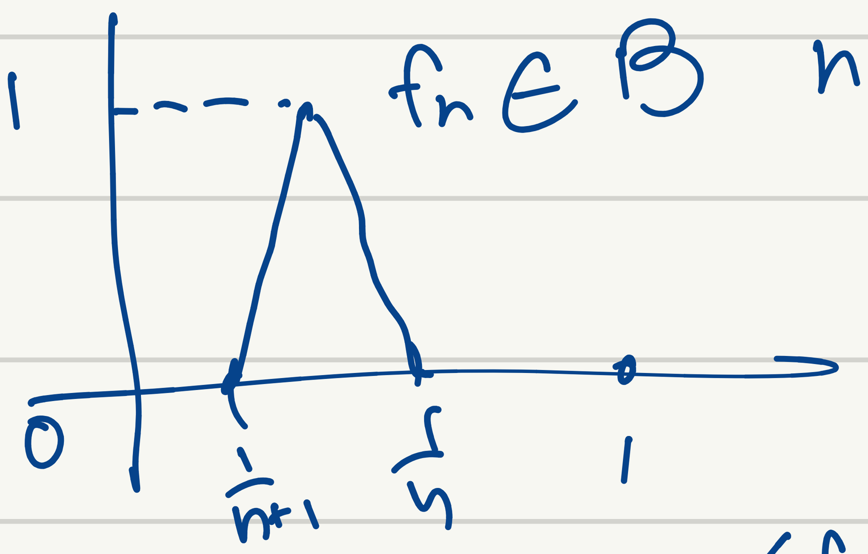
$d_\infty(f_n, f_0) = \|f_n - 0\|_\infty = n$

B is closed in $C([0,1])$

$\Rightarrow$ (B, d_∞) is compact & bounded

Is it compact?

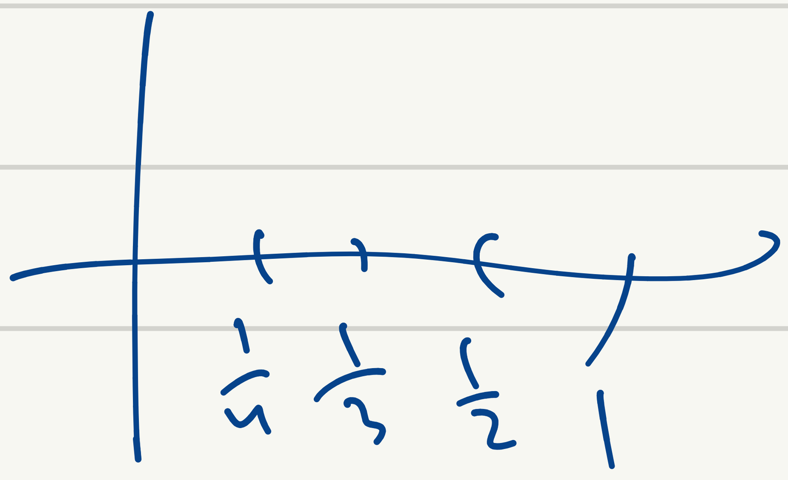
(No) $\langle f_n : n \geq 1 \rangle$

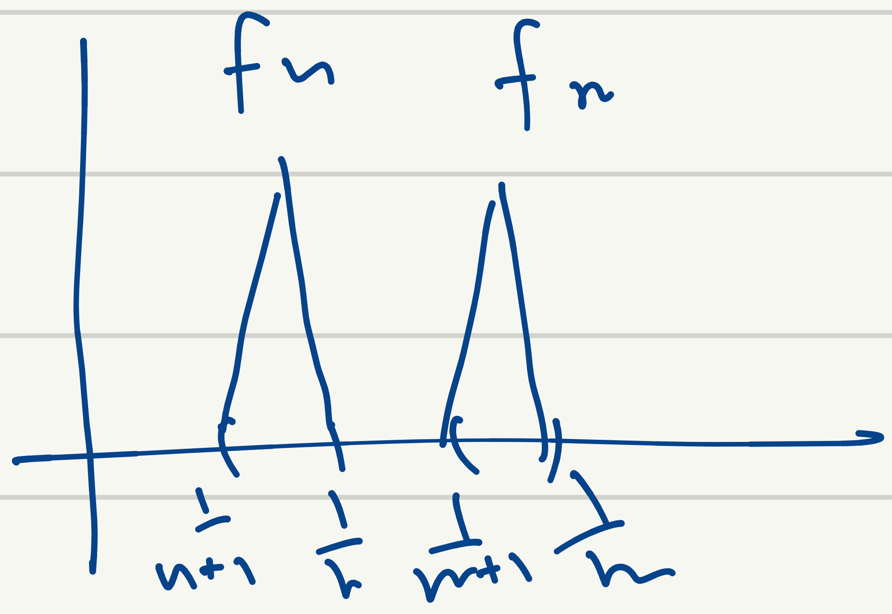


$f_n \in B$, $n \neq m \Rightarrow \|f_n - f_m\|_\infty = 1$

$\Downarrow$

$\langle f_n : n \geq 1 \rangle$ doesn't have a convergent subsequence





<u>Defn:</u> $\mathcal{F}$ is a family of functions from $[a,b]$ to $\mathbb{R}$

(1) Let $x \in [a,b]$. We say that $\mathcal{F}$ is equiconti at x iff "each $f \in \mathcal{F}$ is continuous at x" <u>AND</u>

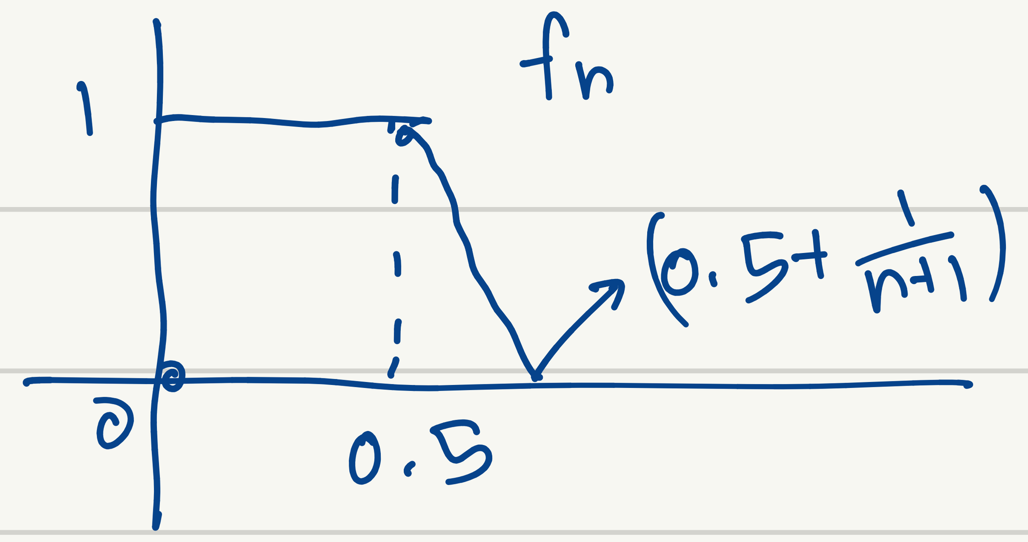
$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall f \in \mathcal{F})(\forall y \in [a,b])(|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$$

(2)

$\mathcal{F}$ is uniformly equicontinuous on $[a,b]$ iff "each $f \in \mathcal{F}$ is unif. cont. on $[a,b]$"

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall f \in \mathcal{F})(\forall x, y \in [a,b])(|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$$

Exercise :-



$\{f_n : n \geq 1\}$ is not equicontinuous at $x = 0.5$ }

Thm: (Arzela - Ascoli thm for $C([a,b])$) $C(X)$ ↑ compact metric sp.

Suppose $\langle f_n : n \geq 1 \rangle$ is bounded in $(C([a,b]), d_\infty)$

Assume $\mathcal{F} = \{f_n : n \geq 1\}$ is uniformly equicontinuous on $[a,b]$.

Then $\exists$ subseq. $\langle f_{n_k} : k \geq 1 \rangle$ of $\langle f_n : n \geq 1 \rangle$ that is convergent in $(C([a,b]), d_\infty)$

So, $f_n \xrightarrow{\text{uniformly}}$ on $[a,b]$

Ex: Suppose $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \supseteq A_k \supseteq A_{k+1} \supseteq \cdots$

is a sequence of infinite subsets of $\mathbb{N}$ (eg. $A_k = \mathbb{N} \setminus \{1, 4, \dots 4^k\}$)

Then there is an infinite $X \subseteq \mathbb{N}$

s.t. for every $k \geq 1$, $\underbrace{\text{" } X \setminus A_k \text{ is finite"}}_{\text{all but finitely many members of } X \text{ belongs to } A_k.}$

Lemma: Let D be countable. Let $f_n : D \to \mathbb{R}$ for $n \geq 1$. Assume for each $x \in D$, $\{f_n(x) : n \geq 1\}$ is bounded in $\mathbb{R}$

Then, there is a subsequence $\langle f_{n_k} : k \geq 1 \rangle$ of $\langle f_n : n \geq 1 \rangle$ and there is an $f : D \to \mathbb{R}$ such that

$f_{n_k} \xrightarrow[\text{on } D]{\text{pointwise}} f \quad \left((\forall x \in D)\cdot\left(\lim_{k\to\infty} f_{n_k}(x) = f(x)\right)\right)$

Proof of lemma:- WLOG, $|D| = |\mathbb{N}|$

Let $D = \{d_1, d_2 \ldots\}$ list all points in D

(1) $\langle f_n(x_1) : n \geq 1 \rangle$ is bounded in $\mathbb{R}$ $\Rightarrow \exists \langle n_k : k \geq 1 \rangle$ s.t. $n_1 < n_2 < n_3 < \ldots$ & $\langle f_{n_k}(x_1) : k \geq 1 \rangle$ is convergent to say $y_1 \in \mathbb{R}$

(2) $\langle f_{n_k}(x_2) : k \geq 1 \rangle$ is bounded in $\mathbb{R}$ $\Rightarrow \exists \langle n_{k_\ell} : \ell \geq 1 \rangle$ s.t. $k_1 < k_2 \ldots$ &

$\langle f_{n_{k_\ell}}(x_2) : \ell > 1 \rangle$ converges to say $y_2 \in \mathbb{R}$

$A_1 = \{n_1, n_2 \ldots\} = \{n_k : k \geq 1\}$

$A_2 = \{n_{k_1}, n_{k_2}, \ldots\} = \{n_{k_\ell} : \ell \geq 1\}$

Let $A_{n+1} \subseteq A_n$ be infinite s.t.

if $A_{n+1} = \{a_1 < a_2 < a_3 < \ldots\}$ then $f_{a_k}(x_{n+1}) \xrightarrow{k\to\infty}$ some $y_{n+1} \in \mathbb{R}$

$= \{\min(A_{n+1} \setminus \{1, 2, \ldots n\}) : n \geq 1\}$

Let $X \subseteq \mathbb{N}$ s.t. X is finite & $X \setminus A_n$ is finite $\forall n \geq 1$

Then $\langle f_\ell : \ell \in X \rangle$ is as required

Arzelia-Ascoli theorem (X,d) compact metric space (e.g. $X=[a,b]$) $\langle f_n : n\geq 1\rangle$ is a sequence in $(C(X), \|\ \|_\infty)$

Assume for some $M>0$, $\|f_n\|_\infty \leq M$ for every $n\geq 1$

Also assume that $\{f_n : n\geq 1\}$ is unif. equicontinuous on X

$(\forall \varepsilon>0)\ (\exists \delta>0)\ (\forall n\geq 1)(\forall x,y\in X)$
$(d(x,y)<\delta \Rightarrow |f_n(x)-f_n(y)|<\varepsilon)$

Then $\exists$ a subsequence $\langle f_{n_k} : k\geq 1\rangle$ of $\langle f_n : n\geq 1\rangle$ that converges in $(C(X), \|\ \|_\infty)$

$\Rightarrow f_{n_k}$ uniformly converges on X.

Proof: Fix $D\subseteq X$ s.t. D is countable dense

For each $x\in D$, $\langle f_n(x) : n\geq 1\rangle \subseteq [-M,M]$

(*) By previous lemma, $\exists \langle n_k : k\geq 1\rangle$ s.t. $n_1<n_2<\ldots$ and $(\forall x\in D)(f_{n_k}(x)$ converges as $k\to\infty)$

Claim: $\langle f_{n_k} : k\geq 1\rangle$ is Cauchy in $(C(X), \|\ \|_\infty)$

Proof: fix $\varepsilon>0$, we'll find $N\geq 1$ s.t.
$(\forall j,k\geq N)(\|f_{n_j}-f_{n_k}\|_\infty<\varepsilon)$

As $\langle f_n : n \geq 1 \rangle$ is unif. equi. on X we can fix $\delta > 0$ s.t.

$(\forall n \geq 1)(\forall x, y \in X)\ (d(x,y) < \delta$

$\Downarrow$

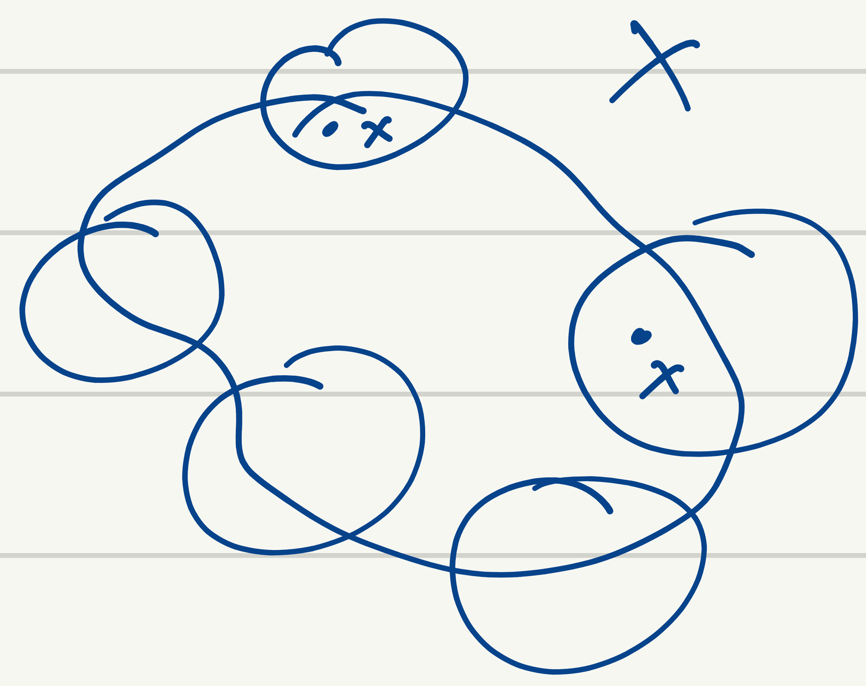
$|f_n(x) - f_n(y)| < \varepsilon/4\ )$

$(**)$

Let $\mathcal{F} = \{ B(x, \delta) : x \in D \}$

As D is dense in X, $\bigcup_{x \in D} B(x, \delta) = X$

(If $y \in X$, then $B(y, \delta) \cap D \neq \phi$
choose $x \in B(y, \delta) \cap D$. Then
$d(x,y) < \delta \implies y \in B(x, \delta)$)



$y \in B(x, \delta) \Leftrightarrow x \in B(y, \delta)$)

$\mathcal{F}$ has a finite subcover,

By $(*)$ $\{ B(x_1, \delta), \ldots B(x_k, \delta) \}$ $x_1, \ldots x_k \in D$

(As $\langle f_{n_k}(x_i) : k \geq 1 \rangle$ converges in $\mathbb{R}$ for all $1 \leq i \leq k$

Say $f_{n_k}(x_i) \xrightarrow{k \to \infty} b_i \in \mathbb{R}$. $\boxed{\text{Choose } N \geq 1}$ s.t.

$(***)$ $(\forall j \geq N)(\forall 1 \leq i \leq k)\ (|f_{n_j}(x_i) - b_i| < \varepsilon/4\ )$

Let $y \in X$ Let $j, k \geq N$. Fix $1 \leq i \leq k$ s.t. $y \in B(x_i, \delta)$ (so $d(x_i, y) < \delta$)

$$\begin{aligned} |f_{n_j}(y) - f_{n_k}(y)| &\leq |f_{n_j}(y) - f_{n_j}(x_i)| \\ &+ |f_{n_j}(x_i) - f_{n_k}(x_i)| \\ &+ |f_{n_k}(x_i) - f_{n_k}(y)| \end{aligned}$$

(Δ-ineq)

$$\boxed{|f_{n_j}(y) - f_{n_j}(x_i)| < \frac{\varepsilon}{4}}$$ By $(\cancel{*}*)$

IIIy $\quad |f_{n_k}(x_i) - f_{n_k}(y)| \leq \frac{\varepsilon}{4}$ By $(**)$

& $\quad |f_{n_j}(x_i) - f_{n_k}(x_i)| \leq \underbrace{|f_{n_j}(x_i) - b_i|}_{< \varepsilon/4 \text{ as } j \geq N}$

$+ \underbrace{|b_i - f_{n_k}(x_i)|}_{< \varepsilon/4 \text{ as } k \geq N}$

from $(***)$

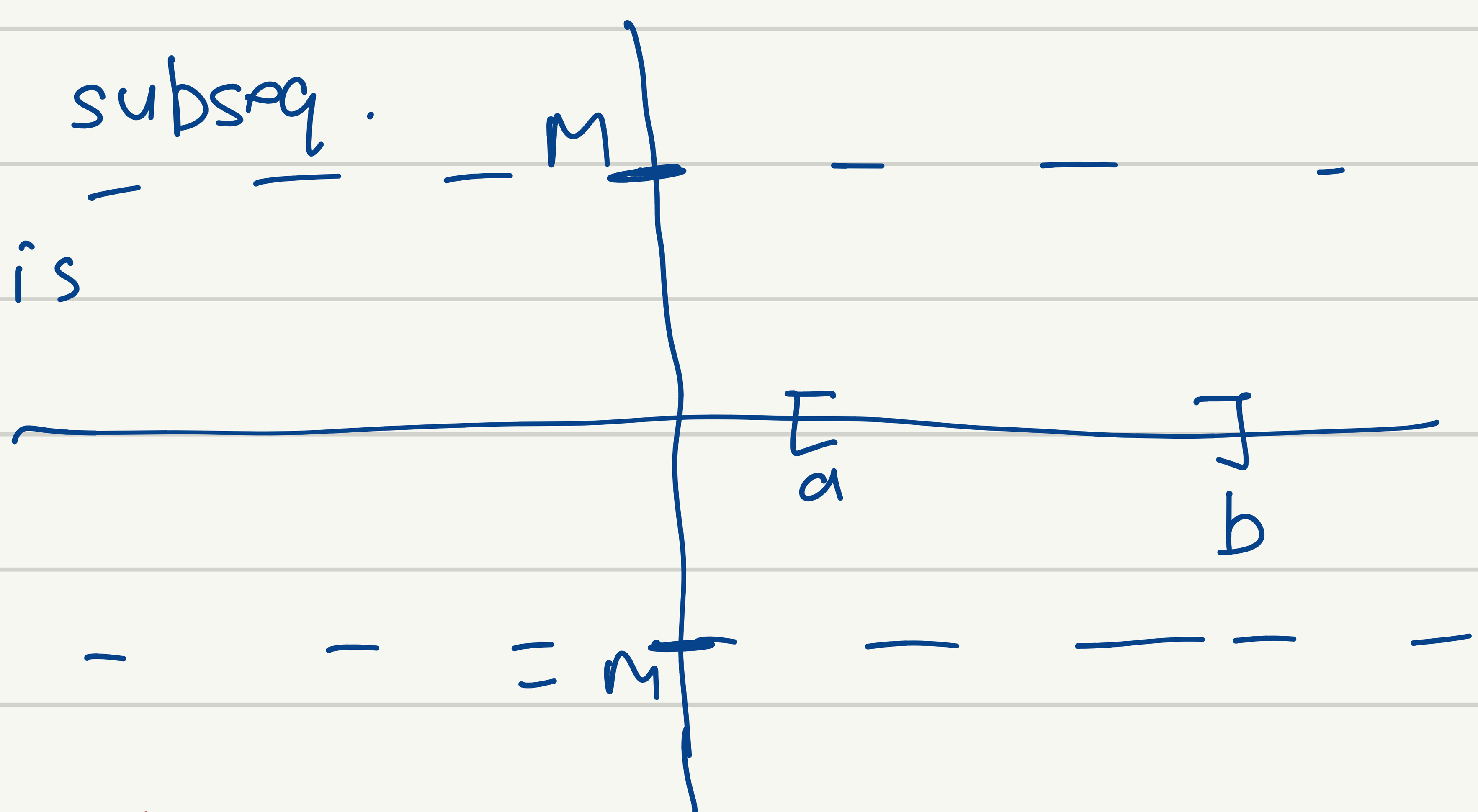
$\leq \varepsilon/2$

Hence $\boxed{\|f_{n_j} - f_{n_k}\|_\infty < \varepsilon}$

Cor: $f_n : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ for every $n \geq 1$.

Assume $\exists M > 0$ s.t. $(\forall x \in [a,b])\big(|f_n(x)| \leq M$ & $|f_n'(x)| \leq M\big)$

Then $\exists \langle f_{n_k} : k \geq 1 \rangle$ subseq. of $\langle f_n : n \geq 1 \rangle$ that is uniformly convergent.



Proof: (Just check uniform equicontinuity on $[a,b]$)

$(\forall x < y \text{ in } [a,b])\big(|f_n(x) - f_n(y)| = |f_n'(z)(y-x)|$

Fix $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{M}$ $\quad (\forall x,y \in [a,b])(|x-y| < \delta$

$$\Rightarrow \quad |f_n(x) - f_n(y)| = |f_n'(z)(x-y)|$$
$$\leq M|x-y| \leq M\delta < \varepsilon$$

Ex: $\quad f_n : [0,1] \longrightarrow \mathbb{R} \quad f_n(x) = x^n$ for $n \geq 1$

$$f_n(x) \longrightarrow \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

It follows that $\langle f_n : n \geq 1 \rangle$ doesn't have any uniformly convergent subsequence

$\|f_n\|_\infty \leq 1$

$f_n'(x) = n x^{n-1}$

$\|f'_n\|_\infty = n$ not bounded.