MTH 301 — Assignment 1

9/8/22

1) Construct a bounded set of real numbers with exactly five limit points.

2) Let (X, d) be a metric space and $E \subseteq X$. If E' denotes the set of all limit points of E, then show that E' is closed.

3) Take $\overline{E} = E \cup E'$ and prove that E and $\overline{E}$ have the same limit points. Do E and E' always have the same limit points?

4) Let $A_1, A_2, \ldots$ be subsets of a metric space. If $B = \bigcup_{j=1}^{\infty} A_j$, prove that $\overline{B} \supseteq \bigcup_{j=1}^{\infty} \overline{A_j}$. Show, by an example, that this inclusion can be proper.

5) Is every point of every open set $E \subseteq \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

6) Let E° denote the set of all interior points of a set E.

(i) Prove that E° is always open.

(ii) E is open iff $E = E^\circ$.

(iii) If $G \subseteq E$ and G is open, then $G \subseteq E^\circ$.

(iv) Do E and $\overline{E}$ always have the same interiors?

(v) Do E and E° always have the same closures?

7) Let X be an infinite set. For $p, q \in X$ define

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}.$$

Show that d is a metric. Which subsets of the resulting metric space are open and which are closed? Which are compact?

8) For $x, y \in \mathbb{R}$, define

$d_1(x,y) = (x-y)^2$, $d_2(x,y) = \sqrt{|x-y|}$,

$d_3(x,y) = |x^2 - y^2|$, $d_4(x,y) = |x - 2y|$

$$d_5(x,y) = \frac{|x-y|}{1+|x-y|}.$$

Determine, for each of these, whether it is a metric or not.

9) Construct a compact set of real numbers whose limit points form a countable set.

10) Construct an open cover of the subset $(0,1)$ of $\mathbb{R}$ which has no finite subcover.

11) A metric space is called separable if it contains a countable ~~dens~~ dense set. Show that $\mathbb{R}^k$ is separable.

12) Prove that every separable metric space has a countable base. [A collection of open subsets of X is called a base if every open set in X is union of a subcollection of that collection]

13) Consider $(\mathbb{Q}, d)$ a sub-metric space of $(\mathbb{R}, d)$ with $d(x,y) = |x-y|$. Let $E = \{ p \in \mathbb{Q} : 2 < p^2 < 3 \}$. Show that E is closed, bounded in $\mathbb{Q}$ but not compact. Is E open in $\mathbb{Q}$?

14) Is there a non-empty perfect set in $\mathbb{R}$ which contains no rational number?

15) Are closures and interiors of connected sets always connected?

16) Show that every convex subset of $\mathbb{R}^K$ is connected.

[A subset $E \subseteq \mathbb{R}^K$ is called convex if for all $x, y \in E$, the line segement $(1-t)x + ty$ belongs to E where $t \in [0,1]$.]