MTH301A: *Analysis I*

Homework solutions: These will be periodically updated. Odd Semester, 2023/24

(1) Read the proofs of Propositions 1.14, 1.15, 1.16 and 1.18 in Rudin's Chapter 1.

(2) Let $(F, \oplus, \cdot, \prec)$ be an ordered field. Show that F contains an isomorphic copy of the ordered field of rationals. More precisely, show that there is an injection $h : \mathbb{Q} \to F$ such that the following hold for every $x, y \in \mathbb{Q}$.

(a) $h(x + y) = h(x) \oplus h(y)$

(b) $h(xy) = h(x) \cdot h(y)$

(c) $h(0) = 0_F$ and $h(1) = 1_F$

(d) $x < y \iff h(x) \prec h(y)$

Solution.

Recall that for $x \in F$, we write $-x$ to denote the additive inverse of x in F and if $x \neq 0_F$, we write x^{-1} to denote the multiplicative inverse of x in F. We will define h in stages.

Stage 1: Define $h_1 : \mathbb{N} \to F$ by induction as follows. $h_1(0) = 0_F$ and $h_1(n + 1) = h_1(n) \oplus 1_F$. Using induction you can check that h_1 satisfies (a)-(d) above for all $x, y \in \mathbb{N}$.

Some details:

(a) By induction on n, we will prove the following statement

$$(\forall m \in \mathbb{N})(h_1(m + n) = h_1(m) \oplus h_1(n))$$

When $n = 0$, this is clear. So assume that the statement holds at n and we will show it for $n + 1$. For any $m \in \mathbb{N}$,

$$h_1(m + (n + 1)) = h_1((m + n) + 1) = h_1(m + n) \oplus 1_F = (h_1(m) \oplus h_1(n)) \oplus 1_F$$

$$= h_1(m) \oplus (h_1(n) \oplus 1_F) = h_1(m) \oplus h_1(n + 1)$$

where we used the inductive hypothesis for the 3rd equality, associativity of $\oplus$ for the 4th equality and definition of h_1 for the final equality.

(b) By induction on n, we will prove the following statement

$$(\forall m \in \mathbb{N})(h_1(mn) = h_1(m) \cdot h_1(n))$$

When $n = 0$, this follows from the fact that $0_F \cdot x = 0_F$ for any $x \in F$ (Proposition 1.16(a) in Rudin). So assume that the statement holds at n and we will show it for $n + 1$. For any $m \in \mathbb{N}$,

$$h_1(m(n + 1)) = h_1(mn + n) = h_1(mn) \oplus h_1(n) = (h_1(m) \cdot h_1(n)) \oplus h_1(n)$$

$$= h_1(m) \cdot (h_1(n) \oplus 1_F) = h_1(m) \cdot h_1(n+1)$$

where we used clause (a) for the 2nd equality, inductive hypothesis for the 3rd equality and the distributive property for the 4th equality.

(c) Clear from the definition of h_1.

(d) It is enough to show that $h_1(n) \prec h_1(n+1)$ for every $n \in \mathbb{N}$ since the result then follows from the fact that $\prec$ is transitive. But this follows by adding $h_1(n)$ to both sides of $0_F \prec 1_F$.

(e) Observe that for every $0 \leq n \leq m$ we have

$$h_1(m-n) \oplus h_1(n) = h_1((m-n)+n) = h_1(m)$$

Adding $-h_1(n)$ on both sides gives

$$h_1(m-n) = h_1(m) \oplus (-h_1(n))$$

This will be used below.

Stage 2: Extend h_1 to $h_2 : \mathbb{Z} \to F$ by defining $h_2(-n) = -h_1(n)$ (the additive inverse of $h_1(n)$ in F) for each $n \in \mathbb{N}$. Once again check that h_2 satisfies (a)-(d) above for all $x, y \in \mathbb{Z}$.

Some details: First check that $-a = (-1_F) \cdot a$ for every $a \in F$. We will repeatedly use this below.

(a) We consider three cases.

The case when $x, y \geq 0$ follows from clause (a) for h_1.

Next suppose x, y are both negative. Say $x = -m$ and $y = -n$ where $m, n \geq 0$. Then

$$h_2(x+y) = h_2(-(m+n)) = -h_1(m+n) = -(h_1(m) \oplus h_1(n))$$

$$= -1_F \cdot (h_1(m) \oplus h_1(n)) = -h_1(m) \oplus (-h_1(n)) = h_2(x) + h_2(y)$$

where we used $-a = -1_F \cdot a$ in the 4th and 5th equality and the distributive property for the 5th equality.

Now suppose exactly one of x, y is negative. Say $x = m > 0$ and $y = -n \leq 0$. We have to show

$$h_2(m-n) = h_2(m) \oplus h_2(-n)$$

If $m \geq n$, this follows from clause (e) above.

So assume $m < n$. Then $m - n = -(n-m)$ and $n - m > 0$. Using clause (e), we get

$$h_2(m-n) = h_2(-(n-m)) = -h_1(n-m) = -1_F \cdot h_1(n-m)$$

$$= -1_F \cdot (h_1(n) \oplus (-h_1(m))) = (-h_1(n)) \oplus h_1(m) = h_2(m) \oplus h_2(-n)$$

(b) If $x, y \geq 0$ this is clear.

If $x, y < 0$, then put $x = -m$ and $y = -n$. Now consider

$$h_2(xy) = h_2(mn) = h_1(mn) = h_1(m) \cdot h_1(n) = (-1_F \cdot -1_F) \cdot (h_1(m) \cdot h_1(n))$$
$$= (-h_1(m)) \cdot (-h_1(n)) = h_2(x) \cdot h_2(y)$$

The case when exactly one of x, y is negative is left to you.

(c) Clear.

(d) Using clause (d) for h_1, we get $0_F \prec h_1(n)$ for every $n \geq 1$. Adding $-h_1(n)$ on both sides, we get $-h_1(n) \prec 0_F$. So for all $z \in \mathbb{Z}$, $0 < z \iff 0_F \prec h_2(z)$.

Note that for any $x, y \in \mathbb{Z}$, $x < y \iff y - x > 0$. Since we have already shown $h_2(y - x) = h_2(y + (-x)) = h_2(y) \oplus (-h_2(x))$, it follows that $x < y$ iff $0_F \prec h_2(y) \oplus (-h_2(x))$. The latter is equivalent to $h_2(x) \prec h_2(y)$.

Stage 3: Finally define $h : \mathbb{Q} \to F$ by setting $h(m/n) = h_2(m) \cdot (h_2(n))^{-1}$ for each $m \in \mathbb{Z}$ and $n \geq 1$. You should first check that h is well-defined in the sense that if $m/n = m'/n'$, then $h_2(m) \cdot (h_2(n))^{-1} = h_2(m') \cdot (h_2(n'))^{-1}$. Now check that for every $x, y \in \mathbb{Q}$ clauses (a)-(d) hold.

This should not be too hard now. I will post details if someone asks.

Note that the fact that h is injective follows from Clause (d). $\square$

(3) Let $\prec$ be any linear order on $\mathbb{C}$ (the field of complex numbers). Show that $(\mathbb{C}, +, \cdot, \prec)$ is not an ordered field.

Solution.

Towards a contradiction, suppose there is a linear order $\prec$ on $\mathbb{C}$ such that $(\mathbb{C}, +, \cdot, \prec)$ is an ordered field. By Proposition 1.18 (d) in Rudin's Chapter 1, it follows that for every $x \in \mathbb{C}$, $0 \prec x^2$. In particular, $0 \prec 1^2 = 1$ and $0 \prec i^2 = -1$. Adding 1 to both sides of $0 \prec -1$, we get $1 \prec 0$. As $\prec$ is a transitive relation, $0 \prec 1$ and $1 \prec 0$, we get $0 \prec 0$ which contradicts the fact that $\prec$ is an irreflexive relation. $\square$

(4) Read the Appendix at the end of Chapter 1 in Rudin's book.

(5) Show that any two complete ordered fields are isomorphic. This means that if $(F_1, +_1, \cdot_1, <_1)$ and $(F_2, +_2, \cdot_2, <_2)$ are both complete ordered fields, then there is a **bijection** $h : F_1 \to F_2$ such that the following hold for any $x, y \in F_1$.

(a) $h(x +_1 y) = h(x) +_2 h(y)$

(b) $h(x \cdot_1 y) = h(x) \cdot_2 h(y)$

(c) $h(0_{F_1}) = 0_{F_2}$ and $h(1_{F_1}) = 1_{F_2}$

(d) $x <_1 y \iff h(x) <_2 h(y)$

Solution

It is enough to show that any complete ordered field is isomorphic to $(\mathbb{R}, +, \cdot, <)$. Let $(F, \oplus, \cdot, \prec)$ be a complete order field. Let $h : \mathbb{Q} \to F$ be the function satisfying problem (2)(a)-(d) for every $x, y \in \mathbb{Q}$. Extend h to $H : \mathbb{R} \to F$ by defining

$$H(x) = \sup(\{h(r) : r \in \mathbb{Q} \text{ and } r < x\})$$

where the supremum is taken in $(F, \prec)$. Now use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ and $h[\mathbb{Q}]$ is dense in F (by archimedean property) to conclude that H is an order preserving bijection from $\mathbb{R}$ to F.

Checking that H preserves addition and multiplication is routine and left to the reader. $\square$

(6) Show that a set A is countable iff there is an injective function $f : A \to \mathbb{N}$.

Solution

We first prove the left to right implication. Assume that A is countable. Then either A is finite or there is a bijection $f : A \to \mathbb{N}$. If the latter holds, then f is also an injective function so we are done. If A is finite then we can fix some $n \in \mathbb{N}$ and a bijection $h : \{0, 1, \cdots, n-1\} \to A$. Define $f : A \to \mathbb{N}$ by $f(x) = h^{-1}(x)$. Then f is an injection.

Now assume that there is an injective function $f : A \to \mathbb{N}$. Assume that A iis nonempty otherwise it is countable. and we are done. Fix some $x \in A$. Define $g : \mathbb{N} \to A$ as follows. If $n \in \text{range}(f)$, define $g(n) = f^{-1}(n)$. Otherwise, define $g(n) = x$. It is clear that g is a surjection from $\mathbb{N}$ to A. So A is countable (by a theorem proved in class). $\square$

(7) A real number $x \in [0, 1]$ is said to be computable if there is a computer program that on all inputs $n \geq 1$ halts and outputs the nth decimal digit of x. Show that the set of computable real numbers is countable.

Solution

Let P be the set of all C-programs. Each C-program is determined by its code which is a finite sequence of keyboard characters. For each $n \geq 1$, let P_n be the set of all programs with code length n. Then each P_n is finite. So $P = \bigcup_{n \geq 1} P_n$ is countable (proved in class). Let S be the set of all computable $x \in [0, 1]$. Define $F : S \to P$ such that for each $x \in S$, $F(x) \in P$ is a program that computes x. Then $F : S \to P$ is injective since one program cannot compute two different numbers. Hence $|S| \leq |P| \leq |\mathbb{N}|$. It follows that S is countable. $\square$

(8) An algebraic real number is a real number which is the root of a polynomial equation

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with integer coefficients ($\{a_0, a_1, \cdots, a_n\} \subseteq \mathbb{Z}$ and $a_n \neq 0$). Show that the set of all algebraic real numbers is countable.

Solution

For each $n \geq 1$, let P_n be the set of all nonzero polynomials with integer coefficients of degree $\leq n$. Then $|P_n| \leq |\mathbb{Z}^{n+1}| = |\mathbb{N}|$. Hence $P = \bigcup_{n\geq 1} P_n$ is countable. Now each polynomial $p(x) \in P_n$ has at most n roots. Let $R_{p(x)}$ be the set of all real roots of $p(x)$. Let A be the set of all algebraic real numbers. By definition

$$A = \bigcup_{p(x)\in P} R_{p(x)}$$

is a countable union of finite sets. Hence A is countable. □

(9) Let $f : \mathbb{R} \to \mathbb{R}$ be a increasing function. Show that $\{a \in \mathbb{R} : f \text{ is discontinuous at } a\}$ is countable.

Solution

First show that f is discontinuous at x iff

$$\lim_{y\to x^-} f(y) = \sup(\{f(y) : y < x\}) < \inf(\{f(y) : y > x\}) = \lim_{y\to x^+} f(y)$$

Let D be the set of all points of discontinuity of f. Define $h : D \to \mathbb{Q}$ as follows. For each $x \in D$, $f(x)$ is a rational strictly between $\lim_{y\to x^-} f(y)$ (left hand limit of f at x) and $\lim_{y\to x^+} f(y)$ (right hand limit of f at x). Now check that h is injective. So $|D| \leq |\mathbb{Q}| = |\mathbb{N}|$. It follows that D is countable. □

(10) Let $C(\mathbb{R})$ be the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $|C(\mathbb{R})| = |\mathbb{R}|$. Hint: First show that if $f, g \in C(\mathbb{R})$ and $f \restriction \mathbb{Q} = g \restriction \mathbb{Q}$, then $f = g$. Then use the fact that the set of all functions $h : \mathbb{Q} \to \mathbb{R}$ has cardinality equal to $|\mathbb{R}^{\mathbb{N}}| = |\mathbb{R}|$.

Solution

It is clear that $|C(\mathbb{R})| \geq |\mathbb{R}|$ since every constant function is continuous. To show that $|C(\mathbb{R})| \leq |\mathbb{R}|$, we'll construct an injective function from $C(\mathbb{R})$ to $\mathbb{R}^{\mathbb{N}}$. This suffices since $|\mathbb{R}^{\mathbb{N}}| = |\mathbb{R}|$. Since $|\mathbb{Q}| = |\mathbb{N}|$, it is enough to construct an injective function $H : C(\mathbb{R}) \to \mathbb{R}^{\mathbb{Q}}$ where $\mathbb{R}^{\mathbb{Q}}$ is the set of all functions from $\mathbb{Q}$ to $\mathbb{R}$. Given $f \in C(\mathbb{R})$, define $H(f) = f \restriction \mathbb{Q}$. We claim that H is injective. To see this assume that $H(f) = H(g)$ and we'll show that $f = g$. Let $x \in \mathbb{R}$. Let $\langle a_n : n \geq 1\rangle$ be a sequence of rationals converging to x. Since f, g are continuous, $f(a_n)$ converges to $f(x)$ and $g(a_n)$ converges to $g(x)$. As $f \restriction \mathbb{Q} = g \restriction \mathbb{Q}$, we must have $f(a_n) = g(a_n)$ for every $n \geq 1$. Hence $f(x) = g(x)$. So $f = g$ and H is injective. So we have shown $|C(\mathbb{R})| \leq |\mathbb{R}|$ and $|\mathbb{R}| \leq |C(\mathbb{R})|$. By Cantor-Schroder-Bernstein theorem, we get $|C(\mathbb{R})| = |\mathbb{R}|$. □

(11) For every set X, show that $|X| < |\mathcal{P}(X)|$. This means that there is an injection from X to $\mathcal{P}(X)$ but there is no injection from $\mathcal{P}(X)$ to X.

Solution

The function $f : X \to \mathcal{P}(X)$ defined by $f(y) = \{y\}$ is injective so $|X| \leq |\mathcal{P}(X)|$. Next, towards a contradiction suppose there is an injective function $g : \mathcal{P}(X) \to X$. Then there is a surjective function from $H : X \to \mathcal{P}(X)$ (Why?). Define $D = \{y \in X : y \notin H(y)\}$. Now **check** that $D \notin \text{range}(H)$. So H is not surjective. A contradiction. □

(12) Prove the Cauchy-Schwarz inequality in $\mathbb{R}^n$. For all $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$|\vec{x} \cdot \vec{y}| \leq ||\vec{x}||||\vec{y}||.$$

Recall that $\vec{x} \cdot \vec{y} = \sum_{1 \leq i \leq n} x_i y_i$ and $||\vec{x}|| = (\vec{x} \cdot \vec{x})^{1/2}$ where $\vec{x} = (x_1, x_2, \cdots, x_n)$ and $\vec{y} = (y_1, y_2, \cdots, y_n)$.

Solution

Put $||\vec{x}|| = A$ and $||\vec{y}|| = B$. If $A = 0$, then $\vec{x} = \vec{0}$ so the inequality holds. Similarly if $B = 0$, the inequality also holds. So we can assume that both $A, B > 0$. Now

$$0 \leq \sum_{i=1}^{n} (x_i/A \pm y_i/B)^2 = \sum_{i=1}^{n} x_i^2/A^2 + \sum_{i=1}^{n} y_i^2/B^2 \pm 2 \sum_{i=1}^{n} x_i y_i/AB$$

which gives

$$0 \leq ||\vec{x}||^2/A^2 + ||\vec{y}||^2/B^2 \pm 2(\vec{x} \cdot \vec{y})/AB = 1 + 1 \pm 2(\vec{x} \cdot \vec{y})/AB$$

It follows that $\pm(\vec{x} \cdot \vec{y}) \leq AB$ or $|\vec{x} \cdot \vec{y}| \leq ||\vec{x}||||\vec{y}||$. □

(13) Show that $d(\vec{x}, \vec{y}) = ||\vec{x} - \vec{y}||$ defines a metric on $\mathbb{R}^n$.

Solution

Checking the first two conditions for being a metric is easy and left to the reader. We only check the Δ-inequality. First show that $||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$ by squaring both sides and using the Cauchy-Schwarz inequality. Use this to conclude that for every $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$,

$$||\vec{x} - \vec{z}|| = ||(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})|| \leq ||\vec{x} - \vec{y}|| + ||\vec{y} - \vec{z}||.$$

□

(14) Consider $\mathbb{R}$ under the usual metric ($d(x, y) = |x - y|$). Let $E \subseteq \mathbb{R}$ be as in (a)-(e) below. Compute E' (the set of limit points of E), $\mathsf{cl}(E)$ (the closure of E) and $\mathsf{Int}(E)$ (the interior of E) in each case.

(a) $E = (0, 1)$.

(b) $E = [0, 1)$.

(c) $E = \mathbb{Z}$.

(d) $E = \mathbb{Q}$.

(e) $E = \{1/n : n \geq 1\}$.

Solution

(a) $E' = \mathsf{cl}(E) = [0, 1]$ and $\mathsf{Int}(E) = (0, 1)$.

(b) $E' = \mathsf{cl}(E) = [0, 1]$ and $\mathsf{Int}(E) = (0, 1)$.

(c) $E' = \mathsf{Int}(E) = \emptyset$ and $\mathsf{cl}(E) = \mathbb{Z}$.

(d) $E' = \mathsf{cl}(E) = \mathbb{R}$ and $\mathsf{Int}(E) = \emptyset$.

(e) $E' = \{0\}$, $\mathsf{cl}(E) = \{1/n : n \geq 1\} \cup \{0\}$ and $\mathsf{Int}(E) = \emptyset$.

(15) Construct a bounded subset $E \subseteq \mathbb{R}$ such that $|E'| = 5$ (i.e., E has exactly 5 limits points in $\mathbb{R}$).

Solution

Let $E = \{k + 1/n : n \geq 1 \text{ and } k \in \{0, 1, 2, 3, 4\}\}$. Then $E' = \{0, 1, 2, 3, 4\}$. □

(16) Let (X, d) be a metric space and $E \subseteq X$. Show that $E'' \subseteq E'$. Conclude that E' is closed in X.

Solution

Let $y \in E''$. We must show that $y \in E'$. For this it is enough to show that every open ball centered at y intersects E at at least one point other than y.

So let $B(y, r)$ be an open ball centered at y of radius $r > 0$. It is enough to show that $B(y, r) \cap E$ is an infinite set. Since $y \in E''$, we can fix some $x \neq y$ such that $x \in B(y, r) \cap E'$. Put $d = r - d(x, y) > 0$ and note that $B(x, d) \subseteq B(y, r)$ (Why?). As $x \in E'$, we must have $B(x, d) \cap E$ is infinite (by a theorem proved in class). As $B(x, d) \subseteq B(y, r)$, it follows that $B(y, r) \cap E$ is also infinite and we are done. □

(17) Try problems 1-11 (as many as you can) in Rudin Chapter 2.

(18) Let (X, d) be a metric space and $E \subseteq X$. Show that the following are equivalent.

(a) E is closed in X.

(b) For every sequence $\langle x_n : n \geq 1 \rangle$ of points in E, if $\langle x_n : n \geq 1 \rangle$ converges to $y \in X$, then $y \in E$.

Solution

Proof of (a) implies (b): Assume E is closed in X and let $\langle x_n : n \geq 1 \rangle$ be a sequence of points in E that converges to $y \in X$. We have to show that $y \in E$.

Towards a contradiction, assume that $y \notin E$. We are going to show that $y \in E'$. As E is closed in X, we must have $E' \subseteq E$ which implies that $y \in E$: A contradiction. Now to show that $y \in E'$, let us consider any open ball $B(y, r)$ centered at y. As $d(x_n, y) \to 0$, we can fix some $n \geq 1$ such that $d(x_n, y) < r$. Then $x_n \in B(y, r)$. As

$y \notin E$ and $x_n \in E$, it follows that E contains a point of $B(y, r)$ (namely x_n) other than y. So $y \in E'$ and we are done.

Proof of (b) implies (a): Assume (b). We have to show $E' \subseteq E$. Let $y \in E'$. Since $y \in E'$, for each $n \geq 1$, the set $B(y, 1/n) \cap E$ is infinite. So we can choose $x_n \in B(y, 1/n) \cap E$ for each $n \geq 1$. Now $d(x_n, y) < 1/n$ so $\langle x_n : n \geq 1 \rangle$ is a sequence of points in E that converges to y. By (b), we get $y \in E$ as required. □

(19) Let (X, d) be a metric space and $E \subseteq X$. Show that the following are equivalent.

(a) (X, d) is a complete metric space.

(b) For every sequence $\langle C_n : n \geq 1 \rangle$ of closed balls in X, if $C_{n+1} \subseteq C_n$ for all $n \geq 1$ and if radius of C_n goes to zero as $n \to \infty$, then $\bigcap_{n \geq 1} C_n \neq \emptyset$.

Solution

Recall that $C(x, r) = \{y \in X : d(x, y) \leq r\}$ denotes the closed ball with center x and radius r and $B(x, r) = \{y \in X : d(x, y) < r\}$ denotes the open ball with center x and radius r.

Proof of (a) implies (b): Let (X, d) be a complete metric space and let $C_n = C(x_n, r_n) = \{y \in X : d(x_n, y) \leq r_n\}$ (so C_n is the closed ball with center x_n and radius r_n). Assume $r_n \to 0$ as $n \to \infty$ and $C_{n+1} \subseteq C_n$ for all $n \geq 1$. We have to show that $\bigcap_{n \geq 1} C_n \neq \emptyset$.

Consider the sequence $\langle x_n : n \geq 1 \rangle$. We will show that this is a Cauchy sequence. For this, we have to show that $\mathsf{diam}(\{x_k : k \geq n\}) \to 0$ as $n \to \infty$. Using Δ-inequality, first check that for each $n \geq 1$, $\mathsf{diam}(C_n) \leq 2r_n$. As $\{x_k : k \geq n\} \subseteq C_n$, it follows that $\mathsf{diam}(\{x_k : k \geq n\}) \leq \mathsf{diam}(C_n) \leq 2r_n$ and hence $\mathsf{diam}(\{x_k : k \geq n\}) \to 0$ as $n \to \infty$. So $\langle x_n : n \geq 1 \rangle$ is Cauchy.

As (X, d) is complete, there exists $x \in X$ such that $\langle x_n : n \geq 1 \rangle$ converges to x. We claim that $x \in C_n$ for all $n \geq 1$. To see this, observe that $\langle x_k : k \geq n \rangle$ is a sequence of points all of which lie in C_n. As C_n is a closed set (Why?), the limit x must be also in C_n (by Problem 18). So $x \in C_n$ for every $n \geq 1$ and it follows that $\bigcap_{n \geq 1} C_n \neq \emptyset$.

Proof of (b) implies (a): Assume (b). Let $\langle x_n : n \geq 1 \rangle$ be a Cauchy sequence in X. We have to show that this sequence converges to some $x \in X$. Put $t_n = \mathsf{diam}(\{x_k : k \geq n\})$. Then $t_n \to 0$ as $n \to \infty$ because the sequence is Cauchy. We can assume that each $t_n > 0$. Otherwise, the sequence must be eventually constant and therefore will have a limit.

Key Claim: There are $1 \leq n_1 < n_2 < \cdots$ such that for each $k \geq 1$,

$$C(x_{n_{k+1}}, 2t_{n_{k+1}}) \subseteq C(x_{n_k}, 2t_{n_k}).$$

How do we choose such n_k's? Start by defining $n_1 = 1$. Suppose n_j has been chosen for all $j \leq k$. Choose n_{k+1} as follows. Since $t_n \to 0$, we can find an $n_\star > n_k$ such that $t_{n_\star} \leq t_{n_k}/2$. Define $n_{k+1} = n_\star$.

We claim that $C(x_{n_{k+1}}, 2t_{n_{k+1}}) \subseteq C(x_{n_k}, 2t_{n_k})$. Let $y \in C(x_{n_{k+1}}, 2t_{n_{k+1}})$. Then

$$d(x_{n_{k+1}}, y) \leq 2t_{n_{k+1}} \leq t_{n_k}.$$

Also observe that

$$d(x_{n_k}, x_{n_{k+1}}) \leq \mathsf{diam}(\{x_j : j \geq n_k\}) = t_{n_k}.$$

By Δ-inequality, we get

$$d(y, x_{n_k}) \leq d(y, x_{n_{k+1}}) + d(x_{n_k}, x_{n_{k+1}}) \leq t_{n_k} + t_{n_k} = 2t_{n_k}.$$

Hence $y \in C(x_{n_k}, 2t_{n_k})$. This proves the key claim.

Now (b) implies that there exists some $x \in \bigcap_{k \geq 1} C(x_{n_k}, 2t_{n_k})$. Fix such an x. We will show that $d(x_n, x) \to 0$ (so x_n's converge to x and the proof will be complete). Let $\varepsilon > 0$ be given. Choose k large enough so that $t_{n_k} \leq \varepsilon/4$. We will show that for every $n \geq n_k$, $d(x, x_n) < \varepsilon$.

Let $n \geq n_k$. By Δ-inequality, $d(x_n, x) \leq d(x, x_{n_k}) + d(x_{n_k}, x_n)$. Since $x \in C(x_{n_k}, 2t_{n_k})$, we get $d(x, x_{n_k}) \leq 2t_{n_k} \leq \varepsilon/2$. Also since $n \geq n_k$, $d(x_{n_k}, x_n) \leq t_{n_k} \leq \varepsilon/4$. Hence

$$d(x_n, x) \leq d(x, x_{n_k}) + d(x_{n_k}, x_n) \leq \varepsilon/2 + \varepsilon/4 < \varepsilon.$$

$\square$

Remark: In (b), the requirement that radius of C_n goes to zero cannot be dropped. See Exercise 13.8.14 here:
`https://classicalrealanalysis.info/documents/TBB-AllChapters-Landscape.pdf`

(20) Show that $\mathbb{R}$ is a complete metric space under the usual metric.

Solution

By problem (19), it suffices to show that whenever $\langle [a_n, b_n] : n \geq 1 \rangle$ is a nested sequence of closed intervals in $\mathbb{R}$ satisfying $\lim_{n \to \infty} b_n - a_n = 0$, we must have

$$\bigcap_{n \geq 1} [a_n, b_n] \neq \emptyset$$

But this is clear since $a = \sup(\{a_n : n \geq 1\}) \in [a_n, b_n]$ for every $n \geq 1$.]qed

(21) Show that $\mathbb{R}^n$ is complete metric space under the usual metric.

Solution

Let $\langle x_k : k \geq 1 \rangle$ be a Cauchy sequence in $\mathbb{R}^n$. Let $x_k = (x_{k,1}, x_{k,2}, \cdots, x_{k,n})$ where each $x_{k,i} \in \mathbb{R}$. First check that for each $k \in \{1, \ldots, n\}$, $\langle x_{k,j} : j \geq 1 \rangle$ is a Cauchy sequence in $\mathbb{R}$. By problem (20), we can fix $a_k \in \mathbb{R}$ such that $\lim_{j \to \infty} x_{k,j} = a_k$. Put $a = (a_1, \cdots, a_n)$. Now show that $\lim_{k \to \infty} x_k = a$. $\square$

(22) Show that the conclusion of the Baire category theorem doesn't hold if we do not assume that (X, d) is complete.

Solution

Consider $\mathbb{Q}$ under the usual metric. For each $x \in \mathbb{Q}$, the set $\{x\}$ is nowhere dense in $\mathbb{Q}$. Hence $\mathbb{Q} = \bigcup_{x \in \mathbb{Q}} \{x\}$ is a countable union of nowhere dense subsets of $\mathbb{Q}$ whose complement (being empty) in $\mathbb{Q}$ is not dense in $\mathbb{Q}$. $\square$

(23) Let $\{a_n : n \geq 1\}$ be a one-to-one list of the set all rationals $\mathbb{Q}$. For each $n \geq 1$, define

$$U_n = \bigcup_{k \geq 1} \left(a_k - 2^{-(n+k)}, a_k + 2^{-(n+k)}\right).$$

Is $\bigcap\limits_{n \geq 1} U_n = \mathbb{Q}$? Justify your answer.

Solution

The answer is no. In fact, we will show that $W = \bigcap\limits_{n \geq 1} U_n$ is uncountable. Suppose not and we will produce a contradiction.

First note that for each $n \geq 1$, $E_n = \mathbb{R} \setminus U_n$ is nowhere dense in $\mathbb{R}$. Consider the family $\mathcal{F} = \{E_n : n \geq 1\} \bigcup \{\{x\} : x \in E\}$. $\mathcal{F}$ is a countable family of nowhere dense subsets of $\mathbb{R}$. By the Baire category theorem applied to the complete metric space $\mathbb{R}$, we get that $\mathbb{R} \setminus \bigcup \mathcal{F}$ must be dense in $\mathbb{R}$. But $\mathbb{R} \setminus \bigcup \mathcal{F} = \emptyset$ (Why?). A contradiction. $\square$

(24) Show that the intersection of any family of compact subsets of a metric space (X, d) is compact.

Solution

Let (X, d) be a metric space and $\mathcal{F}$ be a family of compact subsets of X. We can assume that both $\mathcal{F}$ and $\bigcap \mathcal{F}$ are nonempty otherwise the result is clear. Fix $E \in \mathcal{F}$ and define $\mathcal{E} = \{E \cap A : A \in \mathcal{F}\}$. Note that each member of $\mathcal{E}$ is a closed subset of the compact space E (Why?). Hence $\bigcap \mathcal{F} = \bigcap \mathcal{E}$ is also closed in E (Why?). Since every closed subset of a compact space is compact, $\bigcap \mathcal{F}$ is compact. $\square$

(25) Suppose (X, d) is a metric space and $\mathcal{F}$ is a family of compact subsets of X. Assume that for every finite $\mathcal{E} \subseteq \mathcal{F}$, $\bigcap \mathcal{E} \neq \emptyset$. Show that $\bigcap \mathcal{F} \neq \emptyset$.

Solution

See Theorem 2.36 in Rudin Chapter II. $\square$

(26) Show that every compact metric space is both complete and bounded.

Solution

Let (X,d) be compact. Fix $x \in X$ and define $\mathcal{F} = \{B(x,n) : n \geq 1\}$. Then $\mathcal{F}$ is an open cover of X. Fix a finite subcover $\{B(x,n_1), \cdots, B(x,n_k)\}$ where $n_1 < \cdots < n_k$. Then $X = B(x,n_k)$ and so X is bounded.

Next, towards a contradiction, suppose X is not complete and fix a Cauchy sequence $\langle x_n : n \geq 1\rangle$ which does not converge to any point in X.

First check that $\{x_n : n \geq 1\}$ must be an infinite set.

Next show that for each $y \in X$, we can choose $r_y > 0$ such that

$$B(y, r_y) \cap \{x_n : n \geq 1\} \subseteq \{y\}.$$

Now $\{B(y, r_y) : y \in X\}$ is an open cover of X so by compactness it has a finite subcover $\mathcal{E} = \{B(y_1, r_1), \cdots, B(y_n, r_n)\}$. But this is impossible since $\mathcal{E}$ does not even cover $\{x_n : n \geq 1\}$.

Another proof of compact implies complete: Let $\langle x_n : n \geq 1\rangle$ be a Cauchy sequence in X. Note that $\langle x_n : n \geq 1\rangle$ is a bounded sequence in a compact space X so by a theorem proved in class (Theorem 3.6(a) in Chapter 3 of Rudin's book), there is a subsequence $\langle x_{n_k} : k \geq 1\rangle$ that converges to some $y \in X$. We claim that $\langle x_n : n \geq 1\rangle$ also converges to y.

Suppose $\varepsilon > 0$ is given. As $\langle x_n : n \geq 1\rangle$ is Cauchy, we can choose some $N \geq 1$ such that for all $m, n \geq N$, $d(x_m, x_n) < \varepsilon/2$. We claim that for all $n \geq N$, $d(x_n, y) < \varepsilon$. Fix $n \geq N$. Since $\lim_{k\to\infty} x_{n_k} = y$, we can choose some $n_k > N$ such that $d(x_{n_k}, y) < \varepsilon/2$. As both $n_k, n \geq N$, we get $d(x_n, y) \leq d(x_n, x_{n_k}) + d(x_{n_k}, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

(27) Give an example of a complete and bounded metric space which is not compact.

Solution

Any infinite discrete space. □

(28) Let $a_k < b_k$ for $k \in \{1, 2, \cdots, n\}$. Show that $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is compact under the Euclidean metric.

Solution

See Theorem 2.40 in Rudin Chapter II. □

(29) Suppose (X,d) is a compact metric space and $f : X \to \mathbb{R}$ is continuous. Show that there are $x_0, x_1 \in X$ such that f attains its maximum value at x_0 and f attains its minimum value at x_1.

Solution

Put $E = \text{range}(f) = f[X]$. Then E is a continuous image of a compact space and therefore it is also compact. So E is closed and bounded in $\mathbb{R}$. Put $\sup(E) = y_0$ and $\inf(E) = y_1$.

We claim that $y_0 \in \mathsf{cl}(E) = E$. To see this assume $y_0 \notin E$ and we'll show that $y_0 \in E'$. As y_0 is the supremum of E, for every $r > 0$, $y_0 - r$ is not an upper bound of E and so $E \cap (y_0 - r, y_0) \neq \emptyset$. It follows that $y_0 \in E'$. A similar argument shows that $y_1 \in \mathsf{cl}(E) = E$. Hence both $y_0, y_1 \in E = f[X]$. Choose $x_0, x_1 \in X$ such that $f(x_0) = y_0$ and $f(x_1) = y_1$. Then f attains its maximum value at x_0 and minimum value at x_1. $\square$

(30) Suppose (X, d) is a connected metric space and $f : X \to \mathbb{R}$ is continuous. Assume that $x, y \in X$ and $f(x) < f(y)$. Show that for every $a \in [f(x), f(y)]$, there exists $z \in X$ such that $f(z) = a$.

Solution

Put $E = \text{range}(f) = f[X]$. Then E is a continuous image of a connected space and therefore it is also connected. It follows that E is an interval in $\mathbb{R}$. The claim follows. $\square$

(31) Suppose (X, d_1) and (Y, d_2) are metric spaces and $f : X \to Y$. Show that the following are equivalent.

(a) f is uniformly continuous on X.

(b) For every sequence $\langle (x_n, y_n) : n \geq 1 \rangle$ of pairs of points in X,

$$\lim_{n\to\infty} d_1(x_n, y_n) = 0 \implies \lim_{n\to\infty} d_2(f(x_n), f(y_n)) = 0.$$

Solution

(a) $\implies$ (b): Assume f is uniformly continuous on X. Let $\langle (x_n, y_n) : n \geq 1 \rangle$ be a sequence of pairs of points in X such that $\lim_{n\to\infty} d_1(x_n, y_n) = 0$. We will show that $\lim_{n\to\infty} d_2(f(x_n), f(y_n)) = 0$.

Let $\varepsilon > 0$ be arbitrary. As f is uniformly continuous on X, there exists $\delta > 0$ such that

$$(\forall x, y \in X)(d_1(x, y) < \delta \implies d_2(f(x), f(y)) < \varepsilon) \tag{1}$$

As $d_1(x_n, y_n) \to 0$, we can choose $N \geq 1$ such that

$$(\forall n \geq N)(d_1(x_n, y_n) < \delta) \tag{2}$$

From (1) and (2), we get

$$(\forall n \geq N)(d_2(f(x_n), f(y_n)) < \varepsilon) \tag{3}$$

It follows that $\lim_{n\to\infty} d_2(f(x_n), f(y_n)) = 0$.

(b) $\implies$ (a): Assume (b). Towards a contradiction, suppose f is not uniformly continuous on X. Then there exists an $\varepsilon > 0$, such that for every $\delta > 0$, there are $x, y \in X$ such that $d_1(x, y) < \delta$ and $d_2(f(x), f(y)) \geq \varepsilon$. In particular for each $n \geq 1$, letting $\delta = 1/n$, we can find $x_n, y_n \in X$ such that $d_1(x_n, y_n) < 1/n$ and $d_2(f(x_n), f(y_n)) \geq \varepsilon$. But now the sequence $\langle (x_n, y_n) : n \geq 1 \rangle$ satisfies $\lim_{n\to\infty} d_1(x_n, y_n) = 0$ and $d_2(f(x_n), f(y_n))$ does not converge to 0. This contradicts (b). $\square$

(32) Let $f : \mathbb{R} \to \mathbb{R}$. Define

$$\mathsf{osc}(f, x) = \lim_{h\to 0} \sup(\{|f(a) - f(b)| : a, b \in (x - h, x + h)\})$$

(a) Show that f is continuous at x iff $\mathsf{osc}(f, x) = 0$.

(b) Show that for each $a > 0$, the set $\{x \in \mathbb{R} : \mathsf{osc}(f, x) < a\}$ is open in $\mathbb{R}$.

(c) A subset $E \subseteq \mathbb{R}$ is a G_δ (G-delta) set if it is the intersection of some countable family of open sets. Show that $\{x \in \mathbb{R} : f \text{ is continuous at } x\}$ is a G_δ set.

(d) Show that $\mathbb{Q}$ is not a G_δ set.

(e) Show that there is no function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at every rational and discontinuous at every irrational.

Solution

For each $h > 0$, define the oscillation of f over $(x - h, x + h)$ by

$$\mathsf{osc}(f, x, h) = \sup(\{|f(a) - f(b)| : a, b \in (x - h, x + h)\})$$

and observe that $0 < h_1 < h_2 \implies 0 \leq \mathsf{osc}(f, x) \leq \mathsf{osc}(f, x, h_1) \leq \mathsf{osc}(f, x, h_2)$.

(a) First assume that f is continuous at x. We will show that $\mathsf{osc}(f, x) \leq \varepsilon$ for every $\varepsilon > 0$. It will follows that $\mathsf{osc}(f, x) = 0$. Let $\varepsilon > 0$. Since f is continuous at x, we can choose $h > 0$ such that for every $y \in (x - h, x + h)$, $|f(x) - f(y)| < \varepsilon/2$. Now for every $a, b \in (x - h, x + h)$,

$$|f(a) - f(b)| \leq |f(a) - f(x)| + |f(b) - f(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

It follows that $\mathsf{osc}(f, x, h) \leq \varepsilon$. Hence also $\mathsf{osc}(f, x) \leq \varepsilon$.

Next assume that $\mathsf{osc}(f, x) = \lim_{h\to 0} \mathsf{osc}(f, x, h) = 0$. We will show that f is continuous at x.

Let $\varepsilon > 0$. Choose $\delta > 0$ such that for every $h \leq \delta$, $\mathsf{osc}(f, x, h) < \varepsilon$. It follows that for every $a, b \in (x - \delta, x + \delta)$, $|f(a) - f(b)| < \varepsilon$. in particular, for every $y \in (x - \delta, x + \delta)$, $|f(x) - f(y)| < \varepsilon$.

It follows that f is continuous at x. $\square$

(b) Put $U = \{x \in \mathbb{R} : \mathsf{osc}(f, x) < a\}$. Let $x \in U$. We will show that x is an interior point of U. Since $\mathsf{osc}(f, x) < a$, we can fix $h_\star > 0$ such that $\mathsf{osc}(f, x, h_\star) < a$. We claim that $(x - h_\star, x + h_\star) \subseteq U$. To see this, suppose $y \in (x - h_\star, x + h_\star)$. Choose $h > 0$ such that $(y - h, y + h) \subseteq (x - h_\star, x + h_\star)$. It follows that

$$\mathsf{osc}(f, y) \leq \mathsf{osc}(f, y, h) \leq \mathsf{osc}(f, x, h_\star) < a.$$

Hence $y \in U$ and therefore $(x - h_\star, x + h_\star) \subseteq U$. □

(c) Put $D_f = \{x \in \mathbb{R} : f \text{ is continuous at } x\}$. Then by part (a),

$$D_f = \bigcap_{n \geq 1} \{x \in \mathbb{R} : \mathsf{osc}(f, x) < 1/n\}$$

Since $\{x \in \mathbb{R} : \mathsf{osc}(f, x) < 1/n\}$ is open in $\mathbb{R}$ for each $n \geq 1$, it follows that D_f is the intersection of a countable family of open sets. Hence D_f is a G_δ set. □

(d) Suppose not and let $\{U_n : n \geq 1\}$ be a countable family of open sets such that $\mathbb{Q} = \bigcap_{n \geq 1} U_n$. Then each U_n is open and dense in $\mathbb{R}$ (as $\mathbb{Q} \subseteq U_n$). But this implies that $\bigcap_{n \geq 1} U_n$ must be uncountable (see the solution to (23)). So $\mathbb{Q}$ is not a G_δ set. □

(e) Follows from (c)+(d). □

(33) Let C denote the Cantor set. Show that $C' = C$. Also show that C is nowhere dense in $\mathbb{R}$.

Solution

Recall that $C = \bigcap_{n \geq 1} C_n$ where C_n is a union of 2^n pairwise disjoint closed intervals each of length 3^{-n}. Since each C_n is closed in $\mathbb{R}$, C is also closed in $\mathbb{R}$. Therefore $C' \subseteq C$.

To see that $C \subseteq C'$, assume $x \in C$ and $r > 0$. We will show that $C \cap (x - r, x + r)$ contains some $y \neq x$. Choose $n \geq 1$ such that $3^{-n} < r$. Since $x \in C \subseteq C_n$, there is a unique interval $[a, b] \subseteq C_n$ such that $x \in C_n$. Note that both $a, b \in C$. Also, either $x = a$ or $x = b$ or $a < x < b$. If $x = a$, then $y = b \in C$ and $|x - y| \leq b - a = 3^{-n} < r$. If $x \neq a$, then $y = a \in C$ and $|x - y| \leq b - a = 3^{-n} < r$. So in all cases there is a point $y \in C \cap (x - r, x + r)$ such that $y \neq x$. Hence $x \in C'$.

Next we will show that C is nowhere dense in $\mathbb{R}$. Let (a, b) be any open interval. We will find a subinterval $(c, d) \subseteq (a, b)$ such that $C \cap (c, d) = \emptyset$. We can assume that $0 \leq a < b \leq 1$ otherwise this is clear (if $a < 0$, then $(c, d) = (a, 0)$ works and if $b > 1$, then $(c, d) = (1, b)$ works). Choose $n \geq 1$ large enough so that $3^{-n} < (b - a)/5$. Now observe that if we divide $[0, 1]$ into 3^n consecutive closed subintervals each of length 3^{-n}, then there must be 3 of these consecutive subintervals say J_1, J_2, J_3 which are all subsets of (a, b). Now observe that at least one of the open intervals $\mathsf{Int}(J_k)$ for

$k \in \{1, 2, 3\}$ must be disjoint from C_n (Why?). So we can take (c, d) to be this interval. □

(34) We say that $E \subseteq \mathbb{R}$ has measure zero iff for every $\varepsilon > 0$, there exists a sequence of open intervals $\langle J_n : n \geq 1 \rangle$ such that $\sum_{n \geq 1} \mathsf{length}(J_n) < \varepsilon$ and $E \subseteq \bigcup_{n \geq 1} J_n$. Show that the Cantor set has measure zero.

Solution

Let $\varepsilon > 0$. Choose $n \geq 1$ such that $(2/3)^n < \varepsilon/2$. Recall that C_n is a union of 2^k closed intervals each of length 3^{-n}. Let these intervals be $\{I_k : 1 \leq k \leq 2^n\}$. Let $\langle J_k : k \geq 1 \rangle$ be a sequence of open intervals defined as follows. For each $k \leq 2^n$, let $J_k \supseteq I_k$ be an open interval such that $\mathsf{length}(J_k) < 2^{-n} + \varepsilon/4^k$. If $k > 2^n$, define $J_k = \emptyset$. Now check that $\sum_{n \geq 1} \mathsf{length}(J_n) < \varepsilon$ and $C \subseteq C_n \subseteq \bigcup_{n \geq 1} J_n$. □

(35) Let $\langle a_n : n \geq 1 \rangle$ be a bounded sequence of reals. Show that $\langle a_n : n \geq 1 \rangle$ is convergent iff

$$\limsup_{n \to \infty} a_n = \liminf_{n \to \infty} a_n$$

Solution

First assume that $\langle a_n : n \geq 1 \rangle$ is a convergent sequence and fix $a \in \mathbb{R}$ such that $\lim_{n \to \infty} a_n = a$. Put $\liminf_{n \to \infty} a_n = \alpha$ and $\limsup_{n \to \infty} a_n = \beta$. We will show that $a - \varepsilon \leq \alpha \leq \beta \leq a + \varepsilon$ for every $\varepsilon > 0$. It will follow that $\alpha = \beta = a$. So let $\varepsilon > 0$. Since $\lim_{n \to \infty} a_n = a$, we can find $N \geq 1$ such that for every $n \geq N$, $|a_n - a| < \varepsilon$. It follows that for all $n \geq N$, $a - \varepsilon < a_n < a + \varepsilon$. So $a + \varepsilon$ is an upper bound for $\{a_n : n \geq N\}$ and $a - \varepsilon$ is a lower bound for $\{a_n : n \geq N\}$. Therefore $a - \varepsilon \leq \alpha \leq \beta \leq a + \varepsilon$ (Why?) and we are done.

Next assume that $\liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n = a$. We will show that $\langle a_n : n \geq 1 \rangle$ converges to a. Let $\varepsilon > 0$. We will find $N \geq 1$ such that for all $n \geq N$, $|a_n - a| < \varepsilon$. Since $\liminf_{n \to \infty} a_n = a$, we can find $N_1 \geq 1$ such that

$$n \geq N_1 \implies a - \varepsilon < a_n \text{ (Why?)}$$

Also, since $\limsup_{n \to \infty} a_n = a$, we can find $N_2 \geq 1$ such that for all

$$n \geq N_2 \implies a_n < a + \varepsilon \text{ (Why?)}$$

It follows that for every $n \geq N = \max(N_1, N_2)$, we have $|a_n - a| < \varepsilon$. □

(36) Assume $\limsup_{n \to \infty} x_n < \beta$. Show that there exists $N \geq 1$ such that for all $n \geq N$, $x_n < \beta$.

Solution

Put $\limsup_{n\to\infty} x_n = s$. Let $s_n = \sup(\{x_k : k \geq n\})$. Then s_n's are decreasing with n and $\lim_{n\to\infty} s_n = s$. Put $\beta - s = \varepsilon > 0$. Choose $N \geq 1$ such that for all $n \geq N$, $|s_n - s| < \varepsilon = \beta - s$. It follows that for every $n \geq N$, we have $x_n \leq s_n < s + \varepsilon = \beta$. $\square$

(37) Assume $\limsup_{n\to\infty} x_n > \beta$. Show that there exist infinitely many n such that $x_n > \beta$.

Solution

Put $\limsup_{n\to\infty} x_n = s$. Let $s_n = \sup(\{x_k : k \geq n\})$. Then s_n's are decreasing with n and $\lim_{n\to\infty} s_n = s$. As $s > \beta$, we get $s_n > \beta$ for every $n \geq 1$. It follows that for every $n \geq 1$, β is not an upper bound of $\{x_k : k \geq n\}$. So for every n there exists $k \geq n$ such that $x_k > \beta$. Therefore $\{n : x_n > \beta\}$ is infinite. $\square$

(38) Show that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Solution

Put $\sum_{k=1}^{n} a_k = s_n$ and $\lim_{n\to\infty} s_n = s$. To show that $\lim_{n\to\infty} a_n = 0$, we will show that for every $\varepsilon > 0$, there exists $N \geq 1$ such that for all $n \geq N$, $|a_n| < \varepsilon$.
Since $\lim_{n\to\infty} s_n = s$, we can choose $N \geq 2$ such that for all $n \geq N-1$, $|s_n - s| < \varepsilon/2$. It follows now that for every $n \geq N$,

$$|a_n| = |s_{n+1} - s_n| \leq |s_{n+1} - s| + |s_n - s| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$\square$

(39) Construct a sequence $a_n > 0$ such that $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges.

Solution

$$a_n = \begin{cases} 10^{-n} & \text{if } n \text{ is odd.} \\ 2^{-n} & \text{if } n \text{ is even.} \end{cases}$$

$\square$

(40) Construct a sequence $a_n > 0$ such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} a_n^{1/n} = 1$ and $\sum_{n=1}^{\infty} a_n$ converges/diverges.

Solution

$a_n = n^{-2}$ and $a_n = n^{-1}$. $\square$

(41) Let $a_n > 0$ for every $n \geq 1$. Show that

$$\limsup_{n\to\infty} a_n^{1/n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$$

Solution

See Theorem 3.37 in Rudin's Chapter III. $\square$

(42) Describe $\{x : \sum_{n=1}^{\infty} a_n x^n \text{ converges}\}$ in each one of the following cases.

(a) $a_n = n^n$.

(b) $a_n = 1/n!$.

(c) $a_n = 2^n/n^2$.

(d) $a_n = n^3/3^n$.

Solution

(a) $\limsup_{n\to\infty}(a_n)^{1/n} = \infty$. So the radius of convergence is 0. Therefore, this power series converges only at $x = 0$.

(b) For any $x \neq 0$, we have $\limsup_{n\to\infty} \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \limsup_{n\to\infty} \frac{|x|}{n} = 0 < 1$. By the ratio test, this power series converges for all x.

(c) For any x, we have $\limsup_{n\to\infty} |a_n x^n|^{1/n} = \limsup_{n\to\infty} \frac{2|x|}{n^{2/n}} = 2|x|$. By the root test, this power series converges if $|x| < 1/2$ and diverges if $|x| > 1/2$. Furthermore, at $x = \pm 1/2$, the series converges absolutely as $\sum n^{-2}$ converges.

(d) For any x, we have $\limsup_{n\to\infty} |a_n x^n|^{1/n} = \limsup_{n\to\infty} \frac{|x| n^{3/n}}{3} = |x|/3$. By the root test, this power series converges if $|x| < 3$ and diverges if $|x| > 3$. Furthermore, at $x = \pm 3$, the series diverges as $(\pm n)^3$ does not converge to 0.

(43) Suppose $a_n > 0$ for all $n \geq 1$. Show that $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges.

Solution

Since $1 + a_n > 1$, we get $a_n > \frac{a_n}{1+a_n}$ for all $n \geq 1$. So $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n} \leq \sum_{n=1}^{\infty} a_n$. Hence if $\sum_{n=1}^{\infty} a_n < \infty$, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n} < \infty$.

Now assume that $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n} < \infty$. Then $\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0$. So we can choose $N \geq 1$ such that for every $n \geq N$, we have

$$\frac{a_n}{1+a_n} < \frac{1}{2} \implies a_n < 1.$$

It follows that

$$\sum_{n=N}^{\infty} \frac{a_n}{2} \leq \sum_{n=N}^{\infty} \frac{a_n}{1+a_n} < \infty$$

Therefore, $\sum_{n=1}^{\infty} a_n < \infty$. □

(44) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $(x-h, x+h) \setminus \{x\}$ for some $h > 0$ and continuous at x. Assume that $\lim_{y\to x} f'(y) = L$. Show that f is differentiable at x and $f'(x) = L$.

Solution

We will show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every z,

$$0 < |x-z| < \delta \implies \left| \frac{f(z) - f(x)}{z - x} - L \right| < \varepsilon.$$

It will follow that $f'(x)$ exists and is equal to L.

Since $\lim_{y\to x} f'(y) = L$, we can choose $0 < \delta < h$ such that for every y,

$$0 < |x-y| < \delta \implies |f'(y) - L| < \varepsilon.$$

First suppose that $0 < x - y < \delta$. By the mean value theorem applied to $f \restriction [x, y]$, there must be some $z \in (x, y)$ such that $\frac{f(y)-f(x)}{y-x} = f'(z)$. As $0 < |z - x| < \delta$, it follows that $|f'(z) - L| < \varepsilon$. Hence $\left| \frac{f(y)-f(x)}{y-x} - L \right| < \varepsilon$.

The case when $0 < y - x < \delta$ is similar. □

(45) Suppose $f, g : [a, b] \to \mathbb{R}$ are both continuous on $[a, b]$ and differentiable on (a, b). Show that there exists $x \in (a, b)$ such that

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x).$$

Solution

Apply Rolle's theorem to $h : [a, b] \to \mathbb{R}$ defined by

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

□

(46) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leq |x - y|^2$ for every $x, y \in \mathbb{R}$. Show that f is constant.

Solution

For any $x \neq y$, we have

$$0 \leq \left| \frac{f(y) - f(x)}{y - x} \right| \leq |y - x|.$$

Taking limits as $y \to x$, we get $f'(x)$ exists and is equal to 0. Hence f is everywhere differentiable and f' is identically 0. So f must be constant.

Another solution: Given $x < y$, we will show $f(x) = f(y)$. For each $n \geq 1$, divide the interval $[x, y]$ into n equal subintervals with end-points

$$x_0 = x < x_1 = x + \frac{y-x}{n} < \cdots < x_k = x + \frac{k(y-x)}{n} < \cdots < x_n = x + \frac{n(y-x)}{n} = y.$$

Since $f(x) - f(y) = \sum_{k=0}^{n-1} f(x_k) - f(x_{k+1})$, by Δ-inequality, we get

$$|f(x) - f(y)| \leq \sum_{k=0}^{n-1} |f(x_k) - f(x_{k+1})| \leq \sum_{k=0}^{n-1} |x_k - x_{k+1}|^2 = n\left(\frac{1}{n^2}\right) = \frac{1}{n}.$$

It follows that $|f(x) - f(y)| \leq 1/n$ for all $n \geq 1$. Hence $f(x) = f(y)$. □

(47) Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b). Assume that $\sup(\{|f'(x)| : x \in (a, b)\}) < \infty$. Show that f is uniformly continuous on (a, b).

Solution

Put $\sup(\{|f'(x)| : x \in (a, b)\}) = M < \infty$. Let $\langle (x_n, y_n) : n \geq 1 \rangle$ be a sequence of pairs of points in the interval (a, b) such that $|x_n - y_n| \to 0$. We will show that $|f(x_n) - f(y_n)| \to 0$. This suffices by problem (31). Now for any $n \geq 1$, either $x_n = y_n$ in which case $|f(x_n) - f(y_n)| = 0$ or $x_n \neq y_n$ in which case by the mean value theorem, there exists some z_n between x_n and y_n such that

$$|f(x_n) - f(y_n)| = |f'(z_n)(x_n - y_n)| \leq M|x_n - y_n|.$$

So $|f(x_n) - f(y_n)| \to 0$ as $n \to \infty$. Therefore f is uniformly continuous on (a, b).

Urjasween Chakraborty showed that the converse is false. Consider $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$. Check out his HelloIITK forum post for more details. □