Name: ______________________

Roll Number: ________________

Practice Midsem

MTH301A - Analysis I

(Odd Semester 2023/24, IIT Kanpur)

INSTRUCTIONS

1. Write your **Name** and **Roll number** above.
2. This exam contains **4 + 1** questions and is worth **40%** of your grade.
3. Answer **ALL** questions.

Question 1. [**5** × **2 Points**]

For each of the following statements, determine whether it is **true or false**. No justification required.

(i) There is a linear order $\prec$ on $\mathbb{C}$ such that $(\mathbb{C}, +, \cdot, \prec)$ is an ordered field.

(ii) A set X is countably infinite iff there is a surjection from X to $\mathbb{N}$.

(iii) $\{2^{-n} : n \geq 1\} \cup \{0\}$ is a compact metric space under the usual metric.

(iv) If $E \subseteq (0, 1)$ is infinite, then $\sup(E)$ is a limit point of E.

(v) If $E \subseteq \mathbb{R}$ is infinite and bounded, then $E \cap E' \neq \emptyset$.

Solution

(i) False. See Homework 3.

(ii) False. Take $X = \mathbb{R}$.

(iii) True. It is both closed and bounded in $\mathbb{R}$.

(iv) False. Take $E = (0, 1/3) \cup \{1/2\}$.

(v) False. Take $E = \{1/n : n \geq 1\}$.

Question 2. [10 Points]

Let (X, d) be a metric space and $E \subseteq X$. A point $y \in X$ is said to be a **boundary point of** E **in** X iff for every $r > 0$, both $B(y, r) \cap E \neq \emptyset$ and $B(y, r) \cap (X \setminus E) \neq \emptyset$. Let $\partial(E)$ denote the set of all boundary points of E in X.

(a) [**4 Points**] Show that $\mathsf{cl}(E) = E \cup \partial(E)$. Conclude that E is closed in X iff $\partial(E) \subseteq E$.

(b) [**3 Points**] Show that $\{\mathsf{Int}(E), \partial(E), \mathsf{Int}(X \setminus E)\}$ is a partition of X.

(c) [**3 Points**] Let $E \subseteq \mathbb{R}$ be uncountable. Show that there exists $x \in E$ such that x is a limit point of E.

Solution

(a) Since $\mathsf{cl}(E) = E \cup E'$, it suffices to show that $E \cup E' \subseteq E \cup \partial(E)$ and $E \cup \partial(E) \subseteq E \cup E'$.

First suppose $x \in E \cup E'$. If $x \in E$, then $x \in E \cup \partial(E)$. If $x \in E'$ and $x \notin E$, then for every $r > 0$, $B(x, r) \cap (X \setminus E) \neq \emptyset$ (as $x \in X \setminus E$) and $B(x, r) \cap E \neq \emptyset$ (as x is a limit point of E). So $x \in \partial(E)$.

Next suppose $x \in E \cup \partial(E)$. If $x \in E$, then $x \in E \cup E'$. If $x \in \partial(E)$ and $x \notin E$, then for every $r > 0$, $B(x, r) \cap E \setminus \{x\} \neq \emptyset$ (as $x \in \partial(E)$ and $x \notin E$). So $x \in E'$.

Finally, E is closed in X iff $E' \subseteq E$ iff $E \cup E' \subseteq E$ iff $E \cup \partial(E) \subseteq E$ iff $\partial(E) \subseteq E$. $\square$

(b) Let $x \in X$. We have to show that exactly one of the following holds: $x \in \mathsf{Int}(E)$, $x \in \mathsf{Int}(X \setminus E)$, $x \in \partial(E)$. We consider two cases.

Case 1: $x \in \partial(E)$. This means that for every $r > 0$, $B(x, r)$ intersects both E and $X \setminus E$. So there is no $r > 0$ for which $B(x, r)$ is a subset of E or a subset of $X \setminus E$. Hence $x \notin \mathsf{Int}(E)$ and $x \notin \mathsf{Int}(X \setminus E)$.

Case 2: $x \notin \partial(E)$. This means that for some $r > 0$, either $B(x, r) \cap E = \emptyset$ or $B(x, r) \cap (X \setminus E) = \emptyset$. If $B(x, r) \cap E = \emptyset$, then $B(x, r) \subseteq X \setminus E$ which means that $x \in \mathsf{Int}(X \setminus E)$. If $B(x, r) \cap (X \setminus E) = \emptyset$, then $B(x, r) \subseteq E$ which means that $x \in \mathsf{Int}(E)$. So either $x \in \mathsf{Int}(E)$ or $x \in \mathsf{Int}(X \setminus E)$. Also we cannot have both since $\mathsf{Int}(E) \cap \mathsf{Int}(X \setminus E) \subseteq E \cap (X \setminus E) = \emptyset$. $\square$

(c) Let $\mathcal{F}$ be the family of all open intervals J with rational end points such that $J \cap E$ is countable. Note that $\mathcal{F}$ is countable since $|\mathbb{Q} \times \mathbb{Q}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. Put $A = \bigcup\{J \cap E : J \in \mathcal{F}\}$. Then A is also countable because it is the union of a countable family of countable sets. Since E is uncountable and A is a countable subset of E, $E \setminus A$ must be uncountable and therefore also nonempty. Let $x \in E \setminus A$. We claim that $x \in E'$. To see this, fix $r > 0$ and we will show $(x - r, x + r) \cap E$ is infinite. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we can find rationals $a, b \in \mathbb{Q}$ such that $x - r < a < x$ and $x < b < x + r$. Then $J = (a, b)$ an open interval with rational end points such that $x \in J$ and $J \subseteq (x - r, x + r)$. As $x \notin A$, we must have $J \notin \mathcal{F}$. So $J \cap E$ is uncountable. Hence $(x - r, x + r) \cap E$ is also uncountable. $\square$

Question 3. [10 Points]

(a) [**5 Points**] Let $\langle a_n : n \geq 1 \rangle$ be a sequence of nonzero reals numbers. Assume that $\liminf_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} > 1$. Show that $\sum_{n=1}^{\infty} a_n$ diverges.

(b) [**5 Points**] Let $a_n \geq 0$ for all $n \geq 1$. Assume that $\sum_{n=1}^{\infty} a_n^2$ converges. Show that $\sum_{n=1}^{\infty} \dfrac{a_n}{n}$ also converges. Is the converse true?

Solution
Will be discussed on Friday Sept. 15.

Question 4. [10 Points]

(a) [**2 Points**] Give the definition of a complete metric space.

(b) [**2 Points**] Give the definition of a compact metric space.

(c) [**4 Points**] Show that every compact metric space is complete.

(d) [**2 Points**] Give an example of a complete metric space that is not compact.

Solution

(a) A metric space (X, d) is complete iff every Cauchy sequence in X converges to some point in X.

(b) A metric space (X, d) is compact iff every open cover of X has a finite subcover.

(c) See Homework 26.

(d) $\mathbb{R}$. Also see Homework 27.

Bonus Question [5 Points]

Let $f : [0,1] \to [0,1]$ be a continuous function. Show that there exists $x \in [0,1]$ such that $f(x) = x$.

Solution

Define $g : [0,1] \to \mathbb{R}$ by $g(x) = f(x) - x$. Then g is continuous (being the difference of two continuous functions). Note that $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$. If either $g(0) = 0$ or $g(1) = 0$, we are done so assume $g(0) > 0$ and $g(1) < 0$. By the intermediate value theorem (Homework 30 applied to $X = [0,1]$), there exists some $x \in [0,1]$ such that $g(x) = 0$ and so $f(x) = x$. $\square$

End of Exam