A^*

Meow  i'm Cat Man Meow

Axiom 0 → $\exists X$ st $\forall y$, $y \notin X$

[There exists a set with no members)

Axiom 1 → $\forall x \forall y$,

$$(x \subseteq y \ \& \ y \subseteq x) \Leftrightarrow y = x$$

$x \subseteq y$, x is subset of y

$\forall x \in X$, $x \in y$

Axiom 2 → $\forall x \forall y$, $\exists Z$, $Z = (x, y)$

$$\forall v, \ (v \in Z) \Leftrightarrow (v = x \text{ or } v = y)$$

[This can create new sets]

Axiom 3 → Every set has an union.

$$\forall X, \exists Y, \text{ st } (Y = \cup X)$$

If $X = \{\{1,2\}, \{a, b\}\}$,

$Y = \{1, 2, a b\}$

Union of all sets in X.

$Y = \cup X$ abbreviates $(\forall v)(v \in Y \Leftrightarrow (\exists W)(W \in X \ \& \ v \in W))$

$X = \{\{\emptyset\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}\}$

$\cup X = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$

$(\forall X\ \forall Y\ \exists Z)\quad (Z = \bigcup\{X, Y\})$

$\|$ defn

$Z = X \cup Y$

__Ordered Set__

$(x,y) \overset{\text{defn}}{=} \{\{x\}, \{x,y\}\}$

Ex:- $(x,y) = (a,b) \Leftrightarrow x = a \ \& \ y = b$

(Axiom of Comprehension)

__Axiom of Subset__ → $\forall Y\ \forall P(x)\quad \exists Z$

$$Z = \{x \in Y : P(x)\}$$

$$(\forall x)\ (x \in Z \Leftrightarrow x \in Y \ \& \ P(x))$$

($P(x)$ built using $\in, =, \&, \text{or}, \Rightarrow, \Leftrightarrow, \neg, \forall, \exists$)

$\neg$ not

__Intersection__

$A \neq \emptyset \Rightarrow$

$\bigcap A = \{y : (\forall X \in A)\ (y \in X)\}$ exists

__Proof__ → Fix $X \in A$,

Put $Y = \{x \in X : (\forall W \in A)\ (x \in W)\}$

Then, $Y = \bigcap A$

Exists by Axiom of Comprehension

Axiom of Power Set → $(\forall X \, \exists Y)\,(Y = P(X))$

$Y = P(X)$ means $Y = \{A : A \subseteq X\}$

$$\Updownarrow$$

$$(\forall A)\,(A \in Y \Leftrightarrow A \subseteq X)$$

Natural Numbers (Meow)

$$\begin{aligned} 0 &= \emptyset \\ 1 &= \{0\} = \{\emptyset\} \\ 2 &= \{0,1\} \\ 3 &= \{0,1,2\} \\ &\vdots \end{aligned}$$

$$n+1 = n \cup \{n\} = S(n)$$

Successor of n

$$\boxed{2 < 4 \Leftrightarrow 2 \in 4}$$

$$S(X) \overset{\text{defn}}{=\!=} X \cup \{X\}$$

Axiom of Infinity → A set A is called Inductive iff

$$0 \in A \;\&\; (\forall x \in A)\,(S(x) \in A)$$

(Ax of ∞) says that $(\exists A)$ (A is inductive)

Natural Number Set

Let A be any inductive Set.

$$\mathbb{N} = \bigcap \{B \subseteq A : B \text{ is inductive}\}$$

Power Set, Comprehension

Smallest inductive set

Cartesian Product

$$A \times B = \{ (x,y) : x \in A \ \& \ y \in B \} \subseteq P(P(A \cup B))$$

Relations

R is a relation iff every member of R is an ordered pair

$$\mathrm{dom}(R) = \{ x : (\exists y)((x,y) \in R) \}$$
$$\mathrm{range}(R) = \{ y : (\exists x)((x,y) \in R) \}$$

R is a relation from A to B iff

$$\mathrm{dom}(R) \subseteq A \ \& \ \mathrm{range}(R) \subseteq B \iff R \subseteq A \times B$$

If $(x,y) \in R$, we write xRy

Function

f is a function iff f is a relation.

$$(\forall (x,y))(\forall (x,z))((x,y) \in f \ \& \ (x,z) \in f) \Rightarrow y = z$$

If $(x,y) \in f$, $f(x) = y$

$$f : A \to B$$

↳ co domain

$\mathrm{dom}(f) = A$ $\quad$ $\mathrm{Range}(f) \subseteq B$

$X \subseteq A$, $\quad f|X : X \to B$

$$(\forall y \in X)((f|X)(y) = f(y))$$

$f[X] = \text{range}(f|X)$

Pre Image

$f: A \to B$

$Y \subseteq B$

$$f^{-1}[Y] \overset{\text{defn}}{=} \{x \in A : f(x) \in Y\}$$

Pre Image of Y under f

$$f^{-1}[Y_1 \cup Y_2] = f^{-1}[Y_1] \cup f^{-1}[Y_2]$$
$$f^{-1}[Y_2 \cap Y_2] = f^{-1}[Y_1] \cap f^{-1}[Y_2]$$

Linear Ordering

$(X, \prec)$ is a linear ordering iff

(1) $\prec$ is a relation from X to X $(\prec \subseteq X \times X)$

(2) $(\forall a \in X)(\neg(a \prec a))$

Transitive

(3) $(\forall a, b, c \in X)((a \prec b) \,\&\, (b \prec c) \Rightarrow (a \prec c))$

Comparitive

(4) $(\forall a, b \in X)$ ($a = b$ or $a < b$ or $a > b$)

$\mathbb{N} = \{0, 1, 2, \dots\}$ $\qquad$ $\mathbb{N}^+ = \{1, 2, \dots\}$

$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$

$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z} \text{ \& } n \in \mathbb{N}^+\}$

Thm There is no $x \in \mathbb{Q}$ st $x^2 = 2$

Proof → Trivial

If p is prime, there is no $x \in \mathbb{Q}$, st $x^n = p$

Motivation for $\mathbb{R}$ → There are holes in $\mathbb{Q}$ $[x^2 = 2, x^n = p \dots]$

So, to "complete" $\mathbb{Q}$, we need $\mathbb{R}$

Ordered Field

~~Def~~ $(F, +, \cdot, <)$ is an Ordered Field if

(1) $F \neq \emptyset$, F has at least 2 members

(2) $+$ is a binary operation on F $\left(\begin{array}{l} + : F \times F \to F \\ +(x, y) = x + y \end{array} \right)$

(a) $(a+b)+c = a + ~~(b+c)~~ (b+c)$ $\forall a, b, c \in F$

(b) $a + b = b + a$

(c) $\exists y \in F$ st $(x + y = x ; \forall x \in F)$

Such a y is unique, we denote it by 0_F or 0.

(d) $(\forall x \in F)$ $(\exists y \in F)$ st $(x + y = 0)$

Such a y is unique. We denote it by $-x$.

(3) $\cdot$ is a binary operation on F

(a) $a.(b.c) = (a.b).c$

(b) $a.b = b.a$

(c) Mult. Identity 1_F or 1. $X.1 = X$ $(\forall X \in F)$

(d) Inverse: $(\forall x \in F)(X \neq 0 \Rightarrow (\exists y \in F)(X.y = 1))$

y is Unique $(\frac{1}{X}, X^{-1})$

(4) $X.(y+Z) = (X.y) + (X.Z)$

(5) $(F, <)$ is a linear ordering.

(a) $X < y \Rightarrow X+a < y+a$

(b) $X > 0,\ y > 0 \Rightarrow X.y > 0$

Defn → Let $(A, <)$ be a linear ordering.

(i) Let $X \subseteq A$ and $y \in A$,
we say that y is an upperbound for X in $(A, <)$ if

$\forall (x \in X)\ (x \leq y)$

(ii) $\sup(X) = y$ means y is the smallest upper bound of X in $(A, <)$

$\inf(X) = y$. - - .

(6) Complete Ordered Field (Dedekind)

$(\forall X \subseteq F)\ [(X \neq \emptyset\ \&\ X$ is bounded from above in $(F, <)$

$\Downarrow$

$(\exists a \in F)(\sup(X) = a)\ ]$

Theorem Let $(F, +, \cdot, <)$ be a complete ordered field.
Let $A \subseteq F$, $A \neq \emptyset$ & assume A has a lower bound in F.
Then, A has a greatest lower bound ($\inf(A)$)

If $A \subseteq F$, $-A \subseteq F$,
$-A = \{-x, x \in A\}$

$-A$ is bounded from above by $-y$.

$\inf(A) = -\sup(-A)$

What do Ordered Field look like?

$1_F + 1_F = 2_F > 1_F$

$1_F + 1_F = 3_F > 2_F$

$n_F = \underbrace{1_F + 1_F + \ldots}_{n \text{ times}}$

$(-n)_F = -(n)_F$

$\frac{m}{n} \longmapsto \frac{m_F}{n_F}$

Thm Let $(F, +, \cdot, <)$ be a complete ordered field.

(a) If $x>0$, then $\{nx : n \geq 1\}$ is not bounded from above in F.

(b) If $0<x<y$, then $(\exists\, m,n \geq 1)$

$$\left(x < \frac{m_F}{n_F} < y \right)$$

Pf (a) If bounded, $y = \sup(A)$

So, $y-x$ not upper bound,

So, $nx > y-x$, for some n

$(n+1)x > y$

(b) Choose $n \cdot 1 > \frac{1}{y-x}$

$y-x > \frac{1}{n}$

$\frac{1}{n} < y-x$

$\exists\, m$, st

$x < \frac{m}{n} < y$ [As $\frac{1}{n} < y-x$, $\frac{1}{n} \times$ factor can't cross x & y together]

Ex:- $\mathbb{Q}$ is not complete ordered field

Pf ~~Sup~~ $\alpha = \{x \in \mathbb{Q} : x^2 < 2\}$

Sup (α) does not exist.

Say $x = \sup(\alpha)$

~~$x^2 \leq 2$~~

Consider $y = x + \dfrac{2-x^2}{2+x}$

$y^2 < 2$

$\therefore y > x, \ y \in \alpha,$

$\therefore x \neq \sup(\alpha)$

$\therefore$ No $\sup(\alpha)$ exist.

~~Def~~

Defn → A (Dedekind) cut in $(\mathbb{Q}, <)$ is a subset $\alpha \in \mathbb{Q}$ satisfying

1. $\alpha \neq \emptyset$, $\alpha \neq \mathbb{Q}$
2. If $x \in \alpha$ & $y < x$, then $y \in x$
3. α doesn't have a largest rational.

$\{r \in \mathbb{Q} : r \leq \frac{1}{2}\}$ not a cut [$\frac{1}{2}$ belongs to set]

$0_* = \{r \in \mathbb{Q} : r < 0\}$

$r_* = \{x \in \mathbb{Q} : x < r\}$

$\{x \in \mathbb{Q} : x \leq 0 \text{ or } x^2 < 2\}$

Defn → $\mathbb{R} = \{\alpha \in P(\mathbb{Q}) : \alpha$ is a cut$\}$

Defn → For $\alpha, \beta \in \mathbb{R}$, define

$$\alpha <_* \beta \text{ iff } \alpha \subset \beta$$

$r < s$ are rationals,

$r_* <_* s_*$

Thm $(\mathbb{R}, <_*)$ is a Dedekind - complete Linear Ordering

(a) $\alpha \not<_* \alpha$

(b) $\alpha <_* \beta$ & $\beta <_* \gamma \Rightarrow \alpha <_* \gamma$

(c) $\alpha, \beta \in \mathbb{R}$. Assume $\alpha \neq \beta$ ~~& $\alpha \not<_* \beta$~~

WLOG, say $\beta \not\subseteq \alpha$. Fix $r \in \beta \setminus \alpha$

$\therefore$ r is an upper bound of α

As $r \in \beta$, $\alpha \subseteq \beta$ & $\alpha \neq \beta$

$\therefore$ $\alpha \subset \beta$

(d) Let $A \subseteq \mathbb{R}$, $A \neq \emptyset$ and $<_*$ - bounded from above by say β

Let $r \in \mathbb{Q} \setminus \beta$

$$(\forall \alpha \in A)\ (\forall s \in \alpha)\ (s < r)$$

Put $\gamma = \cup A$ (Union of all cuts in A)

Claim $\gamma \in \mathbb{R}$

Proof → Exercise

Claim $(\forall \alpha \in A)(\alpha \leq_* \gamma)$

$(\alpha \subseteq \gamma)$

Claim If $\beta < \gamma$,

then $(\exists \alpha \in A)(\beta <_* \alpha)$

$(\exists r \in \gamma \setminus \beta)$

As $t \in \gamma = \cup A$,

Fix $\alpha \in A$ st $t \in \alpha$

$\Rightarrow \beta <_* \alpha$

$\Downarrow$

$\beta \subset \alpha$

Addition:

$$\alpha +_* \beta = \{ r+s : r \in \alpha, s \in \beta \}$$

Check:
(1) $\alpha +_* \beta \in \mathbb{R}$
(2) $(\alpha +_* \beta) +_* \gamma = \alpha +_* (\beta +_* \gamma)$
(3) $\alpha +_* \beta = \beta +_* \alpha$
(4) $0_* +_* \alpha = \alpha$
(5) For each $\alpha \in \mathbb{R}$, define

$$-\alpha = \{ r \in \mathbb{Q} : (\exists p \in \mathbb{Q}^+)(-r-p \notin \alpha) \}$$

Show that $\alpha +_* (-\alpha) = 0_*$

$\alpha <_* \beta$

$\Rightarrow \alpha + \gamma <_* \beta + \gamma$

Multiplication

$\mathbb{R}^+ = \{\alpha \in \mathbb{R} : 0_* <_* \alpha\}$

$0 \in \alpha$

$\alpha, \beta \in \mathbb{R}^+,$

$\alpha \cdot_* \beta = \{p \in \mathbb{Q} : (\exists r \in \alpha)(\exists s \in \beta)$
$(r > 0, \& s > 0 \text{ \& } p < rs)\}$

Check if cut

$\forall \alpha, \beta \in \mathbb{R}^+,$

(1) $\alpha \cdot_* \beta \in \mathbb{R}^+$

(2) $(\alpha \cdot_* \beta) \cdot_* \gamma = \alpha \cdot_* (\beta \cdot_* \gamma)$

(3) $\alpha \cdot_* \beta = \beta \cdot_* \alpha$

(4) $1_* \cdot \alpha = \alpha$

(5) α^{-1} exists $(\alpha \cdot_* \alpha^{-1} = 1_*)$

(6) $\alpha \cdot_* (\beta +_* \gamma) = \alpha \cdot_* \beta +_* \alpha \cdot_* \gamma$

(7) Extend to $\mathbb{R}$

Assignment

A2. Define $h_1 : \mathbb{N} \rightarrow F$

as $h_1(0_F) = 0_F$

$h_1(n+1) = h_1(n) + 1_F$

(a) By induction,

$h_1(n+0) = h_1(n) + 0_F$ ✓

Say, works till K,

$h_1(n+k) = h_1(n) +_F h_1(k)$

$h_1(n+k) +_F 1_F = h_1(n) +_F h_1(k) +_F 1_F$

$h_1(n+(k+1)) = h_1(n) +_F h_1(k+1)$

∴ ✓

(d) $1_F >_F 1_F$

$h_1(1_F) > h_2(0)$

Use induction

Finite Set

A set X is finite iff for some $n \in \mathbb{N}$, $\exists f : \{0, 1, \dots n\} \to X$

f is a bijection.

Else it is infinite.

We say that X, Y be sets such that X & Y have same cardinality $(X \sim Y)$ iff

$$\exists f : X \to Y, \quad st \ f \text{ is a bijection.}$$

$$f : X \xrightarrow{bij} Y$$
$$f : X \leftrightarrow Y$$

X is countably infinite iff $X \sim \mathbb{N}$

Thm (Cantor- Schroder - Bornoteux)

X, Y are sets. Suppose there are

$$f : X \xrightarrow{inj} Y$$
$$g : Y \xrightarrow{inj} X$$

$$X \sim Y$$

Thm →

X is countable $\iff$ Either $X = \emptyset$ or

$\exists f : \mathbb{N} \to X$ st range $(f) = X$

Proof Assume X is countable & non empty.
If X is countably infinite, then

$\exists f : \mathbb{N} \xleftrightarrow{bij} X$, so range $(f) = X$

If X is finite, fix $n \geq 1$ & a bijection

$h : \{0, \dots n-1\} \xleftrightarrow{bij} X$

define $f : \mathbb{N} \to X$ by $f(x) \begin{cases} h(x) & x \in \{0, \dots n-1\} \\ h(0) & \text{else} \end{cases}$

Then, range $(f) = X$

$\therefore$ ① $\Rightarrow$ ②

~~X is finite $\Rightarrow$ X is countable~~

(a) $X = \emptyset \Rightarrow$ X is finite $\Rightarrow$ X is countable

(b) X is finite $\Rightarrow$ X is countable

(c) X is infinite
Fix $f : \mathbb{N} \to X$ st
range $(f) = X$
Define $h : \mathbb{N} \to X$ as

① $h(0) = f(0)$

② $h(n+1) = \{ f(k) : k \in \mathbb{N}$ is least number for which $f(k) \notin \{h(0), \dots, h(n)\}$

Exists as f is infinite $[\text{range} = X]$

$\therefore$ h is a bijection, $\therefore$ X is countably infinite.

$\therefore$ ② $\Rightarrow$ ①

Sequences

A function in X

$$\langle S_n : n \in \mathbb{N} \rangle$$

$$S_n = S(n)$$

A finite sequence in X is a function $S : \{0 \ldots n-1\} \to X$

length $(S) = n$

$$\langle S_k : 0 \leq k \leq n-1 \}$$

Facts about countable sets

(1) A, B countable $\Rightarrow$ $A \times B$, $A \cup B$ are countable

(2) If $\langle A_n : n \geq 1 \rangle$ is a sequence of countable sets,

$\bigcup_{n \geq 1} A_n$ is countable.

(3) If A is countable, then the set of all finite sequences in A are countable

(4) $\mathbb{Z}$, $\mathbb{Q}$ are countable.

(5) $P(\mathbb{N})$, $2^{\mathbb{N}}$, $\mathbb{R}$ are uncountable.

If X is countable, $Y \subseteq X \Rightarrow Y$ is countable.

Proof: (1) Let $A = \{a_1, a_2, \ldots\}$
$B = \{b_1, b_2, \ldots\}$

$h : \mathbb{N} \to A \times B$

$$h(x) = \begin{cases} (a_m, b_n) & x = 2^m 3^n \\ (a_1, b_1) & \text{otherwise} \end{cases}$$

[Meow]

$A \cup B$ follows from (2)

(2) A_n is countable for each $n \geq 1$

$A_1 = \{a_{11}, a_{12}, a_{13}, \ldots\}$
$A_2 = \{a_{21}, a_{22}, a_{23}, \ldots\}$
$\vdots$
$A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}$

$$h(x) = \begin{cases} a_{m,n} & x = 2^m 3^n \\ a_{1,1} & \text{else} \end{cases}$$

(3) X is countable, $A \neq \emptyset$

S_n = Set of all sequences in X of length n.
(a) S_1 is countable
(b) S_2 is countable $(A \times A)$
(c) If S_n is countable, S_{n+1} is countable,
$S_{n+1} \sim S_n \times A$
$\therefore \bigcup S_n$ is countable

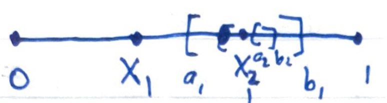
(4) Trivial.

(5) $\mathbb{R}$ is uncountable [Big Whoop.]

↳ $[0,1]$

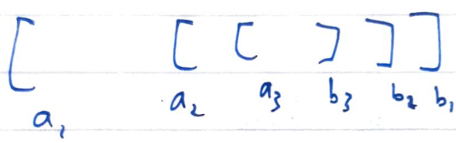
Suppose not.

Let $[0,1] = \{ x_1, x_2, \dots \}$



↳ first number in sequence in (a,b)

↓



$\sup(a_n) = y$

$y \in [0,1]$

$y \in [a_n, b_n] \ \forall \ n \in \mathbb{N}$

$y \neq x_1 \quad [y > x_1]$

$y \neq x_n \ \forall \ n \in \mathbb{N},$

or $y > a_n \quad [a_n > x_n]$

$\therefore \ y \notin [0,1]$

Boom Bitch !!!

Proposition

Prop ① $\mathbb{R}$ is uncountable

? ② $|\mathbb{R}| = |2^{\mathbb{N}}| = |\mathbb{R}^n|$ $\quad n \in \mathbb{N}^+$

③ $|\mathbb{R}| = |\mathbb{R}^{\mathbb{N}}|$

Notation A B are sets. Define

① $|A| \leq |B| \Rightarrow \exists f : A \xrightarrow{inj} B$

② $|A| = |B| \Rightarrow \exists f : A \xrightarrow{bij} B$

Facts ① $|A| \leq |A|$ $\quad (|A| \leq |B|$ & $|B| \leq |C| \Rightarrow |A| \leq |C|)$

② $(|A| \leq |B|$ & $|B| \leq |A|) \Rightarrow |B| = |A|$

③ For any sets A, B,

either $|A| \leq |B|$ or $|A| \geq |B|$

Proof Prop① Take $[a,b]$

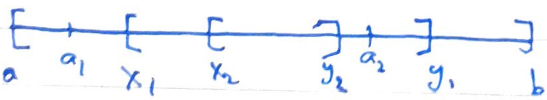
$[a,b] = \{a_n : n \geq 1\}$

Choose $\langle [x_n, y_n] : n \geq 1 \rangle$ st

(a) $(\forall n \geq 1)$ $(x_n < x_{n+1} < y_{n+1} < y_n)$

(b) $a < x_1 < y_1 < b$

(c) $(\forall n \geq 1)$ $(a_n \notin [x_n, y_n])$



Define : $x = \sup \{x_n : n \geq 1\}$

$a < x_n < x_{n+1} < y_{n+1} < y_n < b$

$\downarrow$

Each y_n is an upper bound

$\{x_n\}$ increasing

$x_n \leq x \leq y_n \qquad \forall n \geq 1$

① $x \in [a,b]$

② $(\forall n \geq 1)\ (x \in [x_n, y_n])$

③ By ① and ②, a contradiction, $[a,b]$ uncountable.

Proof Prop ② $\quad 2^{\mathbb{N}} = \{f : f: \mathbb{N} \to \{0,1\}\}$

$=$ Set of all sequences in $\{0,1\}$

$P(\mathbb{N}) = \{X : X \subseteq \mathbb{N}\}$

$\boxed{X^Y = \{f : f: Y \to X\}}$ Define $H(X) = 1_X$

$1_X : \mathbb{N} \to \{0,1\}$

$$1_X(n) = \begin{cases} 1, & n \in X \\ 0, & n \notin X \end{cases}$$

Show that H is a bijection.

$\therefore\ |2^{\mathbb{N}}| = |P(\mathbb{N})|$

Show that $|P(\mathbb{N})| = |P(\mathbb{Q})|$

Claim: $\exists f : \mathbb{R} \xrightarrow{inj} P(\mathbb{N})$

$\Downarrow$

$f : \mathbb{R} \xrightarrow{inj} P(\mathbb{Q})$

$f(x) = \{r \in \mathbb{Q} : r < x\} \qquad x \in \mathbb{R}$

$\therefore\ f$ is 1-1

$\therefore\ |\mathbb{R}| \le |2^{\mathbb{N}}|$

Claim: $\exists g : 2^{\mathbb{N}} \xrightarrow{inj} \mathbb{R}$

Define $\langle a_n : n \in \mathbb{N} \rangle \in 2^{\mathbb{N}}$,

$$g(\langle a_n : n \in \mathbb{N} \rangle) = \frac{a_0}{10} + \frac{a_1}{10^2} + \cdots$$

$$= \sum_{n \ge 0} \frac{a_n}{10^{n+1}}$$

Show that g is 1-1.

So, $|2^{\mathbb{N}}| \le |\mathbb{R}|$

$\therefore\ |\mathbb{R}| = |2^{\mathbb{N}}|$ [CSB]

Proof Prop ③

$|\mathbb{R}^n| \le |\mathbb{R}^{\mathbb{N}}|$

Prove $|\mathbb{R}^{\mathbb{N}}| \le |\mathbb{R}|$

$\downarrow$

$|(2^{\mathbb{N}})^{\mathbb{N}}| \le |2^{\mathbb{N}}|$

$$2^{\mathbb{N}\times\mathbb{N}} = \{f : f : \mathbb{N}\times\mathbb{N} \to \{0,1\}\}$$

$$|(2^{\mathbb{N}})^{\mathbb{N}}| = |2^{(\mathbb{N}\times\mathbb{N})}| = |2^{\mathbb{N}}|$$

Define $G : (2^{\mathbb{N}})^{\mathbb{N}} \to 2^{\mathbb{N}\times\mathbb{N}}$

$G(\langle f_n : n \in \mathbb{N} \rangle)(j,k)$

$= f_j(k)$

Meowtric Spaces ()

$A \cong B$, iff

$\exists f : A \xrightarrow{bij} B$ st

$(\forall x, y \in A) \left(\|x-y\| = \|f(x) - f(y)\| \right)$

[Proper defn of ~~Convergent~~ Congruency]

A metric space is a pair (X, d) st $X \neq \emptyset$ &

$d : X \times X \to [0, \infty)$ &

(1) $d(x,y) = 0$ iff $x = y$

(2) $d(x,y) = d(y,x)$

(3) $d(x,y) + d(y,z) \geq d(x,z)$, $\forall x, y, z \in X$

$(\mathbb{R}, d)$ where $d(x,y) = |x-y|$

$(\mathbb{R}^n, d)$ where $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$

$\|\vec{x}\| = (x_1^2 + x_2^2 + \dots x_n^2)^{\frac{1}{2}}$

st d follows (3).

(X, d) , $X \neq \emptyset$

$$d \to \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

d is Discrete Matrix on X

$$l_\infty(\mathbb{N}) = \{ \langle x_n : n \geq 1 \rangle : x_n \in \mathbb{R} \ \& \ \sup\{|x_n| : n \geq 1\} \text{ finite} \}$$

$$\| \langle x_n : n \geq 1 \rangle \|_\infty = \sup(\{|x_n| : n \geq 1\})$$

$$d(\vec{x}; \vec{y}) = \| \vec{x} - \vec{y} \|_\infty$$

$$\vec{x} = \langle x_n : n \geq 1 \rangle$$
$$\vec{y} = \langle y_n : n \geq 1 \rangle$$

$$C[a,b] = \{ f : f : [a,b] \to \mathbb{R} \quad \text{is cts} \}$$

If $f, g \in C[a,b]$, $f \pm g \in C[a,b]$

$$\| \ \|_\infty : C[a,b] \to [0, \infty)$$

$$\|f\|_\infty = \sup(\text{range}(|f|))$$
$$= \sup(\{|f(x)| : a \leq x \leq b\})$$

$$d(f,g) = \|f-g\|_\infty$$

Defn → Let (X, d) be a metric space, $E \subseteq X$ and $y \in X$

(1) E is bounded in (X,d) if $\sup(\{d(x,y) : x, y \in E\})$ is finite

(2) For $r > 0$, define the open ball in (X,d) with center y & radius r by $B_{(X,d)}(y,r) = \{x \in X : d(x,y) < r\}$

(3) p is an interior point of E iff there exists $r > 0$ st $B(p,r) \subseteq E$

(4) E is open in (X,d) iff $\forall p \in E$, p is an interior point.

(5) $Int(E) = \{z \in E : z$ is an interior point$\}$
E is open iff $Int(E) = E$

(6) $E \subseteq X$ & $y \in X$
y is a limit point of E

iff $(\forall r > 0)\left((B(y,r) \cap E) \setminus \{y\} \neq \emptyset\right)$

(7) $E' = \{y \in X : y$ is a limit point of $E\}$

(8) $Closure(E) = cl(E) = E \cup E'$

(9) E is closed in (X,d) iff

$cl(E) = E$

$[E' \subseteq E]$

(10) E is dense in X iff $cl(E) = X$

(11) E is open iff $Int(E) = E$

Facts: Let (X, d) be a metric space

(0) $\emptyset, X$ are both closed & open in X

(1) $B(y, r)$ is open in X for all $y \in X$ & $r > 0$.

(2) $[y \in X$ & $E \subseteq X$ & $y \in E'] \Rightarrow (\forall r > 0)$ $(B(y,r) \cap E)$ is infinite

(3) E is open iff $X \setminus E$ is closed

(4) (a) Union of any family of open sets is open.

(b) Intersection of closed is closed

(c) ''

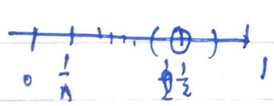
(d) ''

$E = \{\frac{1}{n} : n \geq 1\}$

$\{0\}$ is a limit point.

You can always take a ball around 0, and get a point $X \in E$ contained in the ball.

Say, you take $\frac{1}{2}$, you can't take any ball



Interior points →

$(0,1) \to$ open in $\mathbb{R}$

$[0,1) \to$ not open in $\mathbb{R}$

$\{0\}$ not interior point

Not in $[0,1)$

classmate

Date

Page

Proof - ① Let $X \in B(y,r)$ $[d(x,y) < r]$

To show $(\exists r' > 0)(B(X,r') \subseteq B(y,r))$

Take $r' = r - d(X,y) > 0$

Let $Z \in B(X,r')$, then $d(X,z) < r'$

$d(X,z) < r - d(X,y)$

$\therefore d(y,z) < r$ [Δ inequality]

$\therefore Z \in B(y,r)$

$\therefore B(X,r') \subseteq B(y,r)$

② Suppose not. List all such points.

Q → For any limit point, there are points in Ball around it.

$B(y,r) \cap (E \setminus \{y\}) = \{y_1, y_2, \dots y_n\}$

$r' = \dfrac{\min \{d(y, y_k)\}}{2}$

$B(y,r') \cap (E \setminus \{y\}) = \emptyset$

Boo hoo

(3) Say, E is open.
Let $y \in X$ be a limit point of $X \setminus E$.
We will show $y \in X \setminus E$

Say $y \notin X/E$, then $y \in E$
As E is open, $\exists r > 0$ st $B(y,r) \subseteq E$,
which says $B(y,r) \cap (X/E) = \emptyset$
which means y is not a limit point of X/E.

Bang Bang

$\therefore y \in X \setminus E$

$\therefore$ E is open $\Rightarrow$ X/E is closed

Say, E is closed.
Let $y \in X \setminus E$. We will show that y is an interior point of $X \setminus E$.

As $y \notin E$, y can't be limit point of E (E is closed)

$\therefore \exists r > 0$ st
$$B(y,r) \cap (E \setminus \{y\}) = \emptyset$$
As $y \notin E$, $E \setminus \{y\} = E$

$\therefore B(y,r) \cap E = \emptyset$

$\therefore B(y,r) \subseteq X \setminus E$

$\therefore$ y is interior point of $X \setminus E$

(4) (a) Let $\mathcal{F}$ be a family of open sets in X.

$\bigcup \mathcal{F} = E$

Show that E is open.

Let $y \in E$, need to show that y is an interior point of E.

$\exists\, U \in \mathcal{F}$ st $y \in U$.

As $U \in \mathcal{F}$, U is open in X. As $y \in U$, y is an interior point.

$\therefore$ $\exists\, r > 0$, st

$B(y,r) \subseteq U \subseteq E$ $[\bigcup \mathcal{F} = E]$

$\therefore$ $B(y,r) \subseteq E$

(a) $\xRightarrow[\text{de Morgan}]{\textcircled{3}}$ (b)

(c) Let U_1, U_2 be open in X.

Put $U_1 \cap U_2 = U$.

Take $y \in U$.

As $y \in U_1$ & $y \in U_2$,

$\exists\, r_1, r_2 > 0$

st $B(y,r_1) \subseteq U_1$

$B(y,r_2) \subseteq U_2$

$r = \min(\{r_1, r_2\})$

$\therefore$ $B(y,r) \subseteq U_1$

$B(y,r) \subseteq U_2$

$\therefore$ $B(y,r) \subseteq U$

(c) $\Rightarrow$ (d)

Defn → Let (X,d) be a metric space. $x \in X$ & $\langle x_n : n \geq 1 \rangle$ be a sequence of points in X. We say that $\langle x_n \rangle$ converges to x if $\lim_{n\to\infty} d(x, x_n) = 0$

→ (X,d) metric space, $E \subseteq X$.
$\text{diam}(E) = \sup(\{d(x,y) : x,y \in E\})$

→ $\langle x_n : n \geq 1 \rangle$ is Cauchy iff

$$\lim_{n\to\infty} \text{diam}(\langle x_n : n \geq 1 \rangle) = 0$$ (Luv u bro)

$$A \subseteq B \Rightarrow \text{diam}(A) \leq \text{diam}(B)$$

$$d_k = \text{diam}(\langle x_n : n \geq k \rangle)$$

$$\therefore \; d_{k+1} \leq d_k$$

~~dk~~ is

If $\langle x_n : n \geq 1 \rangle$ converges to x, it should be Cauchy.

[Δ inequality]

Defn → A metric space (X,d) is complete iff every Cauchy sequence in (X,d) converges to some point in X.

$\mathbb{Z}$ under usual metric is complete

$\mathbb{R}$ is complete

$(0,1]$ not complete

$\mathbb{Q}$ not complete

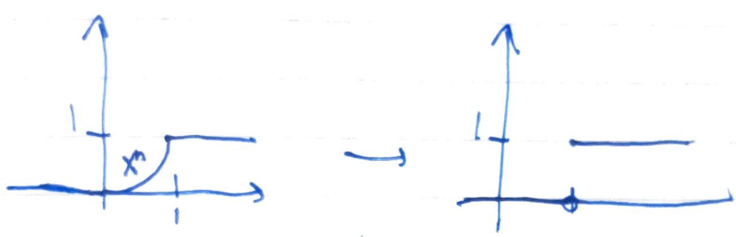
$\mathbb{R}^n$ complete

Baire Category Theorem

Let $f_n : \mathbb{R} \xrightarrow{\text{continuous}} \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$

$f_n \xrightarrow{\text{pointwise}} f$ if $(\forall x \in \mathbb{R})(\lim_{n\to\infty} f_n(x) = f(x))$

Could f be everywhere discontinuous?



$f_n(x)$ $\qquad$ $f(x)$

Defn → (X,d) metric space.

(1) Closed ball with centre y & radius r in (X,d) is
$C(y,r) = \{x \in X : d(x,y) \leq r\}$

$C(y,r)$ can be shown closed by $C(y,r)^c$ is open

(2) $E \subseteq X$ is called nowhere dense in X iff for every open ball $B \subseteq X$, there is an open ball $B' \subseteq B$ st
$B' \cap E = \emptyset$

$\mathbb{Z}$ nowhere dense, $\mathbb{Q}$ not

Thm → Let (X,d) be a complete metric space. Let E_n be nowhere dense in (X,d) for each $n \geq 1$.

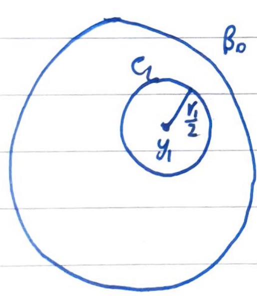
Then, $X \setminus \bigcup_{n\geq 1} E_n \neq \emptyset$

Moreover, $(X \setminus \bigcup_{n \geq 1} E_n)$ is dense in X.

Proof → Let B_0 be an open ball in (X,d).
We will show

$$(X \setminus \bigcup_{n\geq 1} E_n) \cap B_0 \neq \emptyset$$

(a) E_1 is nowhere dense $\Rightarrow \exists$ Open ball $B_1(y_1, r_1) = B_1$
st $B_1 \cap E_1 = \emptyset$. Let $C_1 = C(y_1, \frac{r_1}{2})$



Like that suppose

$$C_n \subseteq C_{n-1} \subseteq \dots \subseteq C_2 \subseteq C_1 \text{ have}$$

been chosen st these are closed balls and

$$C_1 \cap E_1 = \emptyset \;\&\; C_2 \cap E_2 = \emptyset \;\&\; \dots$$
$$C_n \cap E_n = \emptyset$$

Choose C_{n+1} as

$$C_n = C(y_n, r_n)$$

Put $B_n = B(y_n, r_n)$

E_{n+1} is nowhere dense $\Rightarrow \exists$ open ball $B_{n+1} = B(y_{n+1}, r_{n+1})$

st $B_{n+1} \cap E_{n+1} = \emptyset$

Define $C_{n+1} = C(y_{n+1}, \frac{r_{n+1}}{2})$

So we get $C_1 \supseteq C_2 \supseteq \dots C_{n+1} \supseteq \dots$

st $C_n \cap E_n = \emptyset \quad \forall n \geq 1$

Let $C_n = C(y_n, a_n)$ then $a_n \to 0$

Hence, $\langle y_n : n \geq 1 \rangle$ is a Cauchy sequence in (X,d)

$(\operatorname{diam}(\{y_k : k \geq n\}) \leq \operatorname{diam}(C_n) \leq 2a_n)$

As (X,d) is complete, $\exists\, y \in X$ st

$\langle y_n : n \geq 1 \rangle$ converges to y.

Show that $(\forall n \geq 1)(y \in C_n)$

$\therefore y \notin E_n$

$\therefore y \in B_0 \cap (X \setminus \bigcup E_n$

Suppose (X,d) is a metric space & $Y \subseteq X$ is non-empty. Then $(Y, d \restriction (Y \times Y))$ is also a metric space. We sometimes just say (Y,d) is a subspace of (X,d).

Restriction

Lemma: (X,d) metric space $Y \subseteq X$ non empty

(1) For each $y \in Y$, $B_Y(y,r) = B_X(y,r) \cap Y$

(2) $E \subseteq Y$ is open in $Y \Rightarrow \exists$ open $U \subseteq X$ st $E = U \cap Y$

(3) $U \subseteq X$ is open in $X \Rightarrow U \cap Y$ is open in Y.

Proof : ① $B_Y(y,r) = \{x \in Y : d(x,y) < r\}$

$B_X(y,r) = \{x \in X : d(x,y) < r\}$

So, $B_Y(y,r) = B_X(y,r) \cap Y$

② $E \subseteq Y$ open in Y.

$\Rightarrow (\forall y \in E)(\exists r_y > 0)(B_Y(y, r_y) \subseteq E)$

Define $U = \bigcup_{y \in E} B_X(y, r_y)$

U is open in X & $U \cap Y = E$
[Union of open] $[0, 0.5) \underset{\text{open}}{\subseteq} [0,1)$

③ $U \subseteq X$ open in X
$E = U \cap Y$
$y \in E$. Show that y is int point of E in Y

As $y \in E \subseteq U$ & U open in X.

$\exists r > 0$, $B_X(y,r) \subseteq U$

$\Rightarrow B_Y(y,r) = B_X(y,r) \cap Y \subseteq U \cap Y = E$

Compactness

Defn → (X,d) is a metric space & $E \subseteq X$. An open ~~set~~ cover of E is a family $\mathcal{F}$ of open subsets of X such that $E \subseteq \cup\mathcal{F}$

Suppose $\mathcal{F}$ is an open cover of E and $\mathcal{G} \subseteq \mathcal{F}$ is a subcover of E if $\cup\mathcal{G} \supseteq E$

Defn → (X,d) metric space, $E \subseteq X$. We say that E is compact in X iff for every open cover $\mathcal{F}$ of E, there is a finite $\mathcal{G} \subseteq \mathcal{F}$ st $\cup\mathcal{G} \supseteq E$

Ex: (X,d) metric space & $E \subseteq X$ if for every open cover $\mathcal{F}$ of E st each member of $\mathcal{F}$ is an open ball in X, there is a finite $\mathcal{G} \subseteq \mathcal{F}$ st $\cup\mathcal{G} \supseteq E$, then E is compact.

$E \subseteq X$ is finite $\Rightarrow$ E is compact.

Say $E = \{y_1, y_2, \dots, y_n\}$

Let $\mathcal{F}$ be an open cover of E, then

(a) Each $U \in \mathcal{F}$ is an open set in X

(b) $\cup\mathcal{F} \supseteq E$

Choose $U_1 \in \mathcal{F}$ st $y_1 \in U_1$

Choose $U_2 \in \mathcal{F}$ st $y_2 \in U_2$

$\vdots$

$U_k \in \mathcal{F}$ st $y_k \in U_k$

$\mathcal{G} = \{U_1, \dots U_n\} \underset{\text{finite}}{\subseteq} \mathcal{F}$

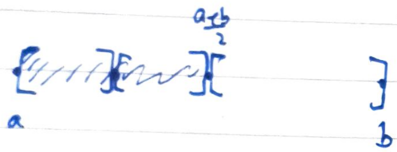
$\cup\mathcal{G} = \bigcup_{k=1}^{n} U_k \supseteq E$

Thm → All closed intervals in $\mathbb{R}$ are compact.
$[a,b]$

Proof → Let $\mathcal{F}$ be an open cover of $[a,b]$.

Observation: If A is a cover of $E \cup F$ & no finite $\mathcal{B} \subseteq A$ is a cover of $E \cup F$, either no finite $\mathcal{B} \subseteq A$ is cover of E or no finite $\mathcal{B} \subseteq A$ is cover of F

Suppose no finite $\mathcal{E} \subseteq \mathcal{F}$ st $\bigcup \mathcal{E} \supseteq [a,b]$



$$[a,b] = \left[a, \tfrac{a+b}{2}\right] \cup \left[\tfrac{a+b}{2}, b\right]$$

Either no finite $\mathcal{E} \subseteq \mathcal{F}$ for $\left[a, \frac{a+b}{2}\right]$
or for $\left[\frac{a+b}{2}, b\right]$

Inductively choose $\langle [a_n, b_n] : n \geq 1 \rangle$ st

(1) $[a_1, b_1] = [a,b]$

(2) $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ &
$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$

(3) $(\forall n \geq 1)$ (there is no finite $\mathcal{E} \subseteq \mathcal{F}$ st $\bigcup \mathcal{E} \supseteq [a_n, b_n]$

Put $x = \sup(\{a_n : n \geq 1\})$

$= \inf(\{b_n : n \geq 1\})$

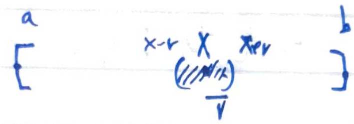
$$\bigcap [a_n, b_n] = \{x\}$$

As $x \in [a,b] \subseteq \bigcup f$,

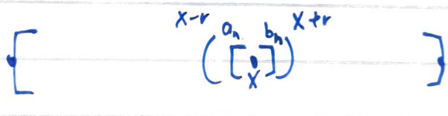
$\exists\, V \in f$ st $x \in V$

As V is open in $\mathbb{R}$, $\exists\, r > 0$ st

$$(x-r, x+r) \subseteq V$$



Choose n large enough so that $x-r < a_n \leq x \leq b_n < x+r$



So, $[a_n, b_n] \subseteq V$

Finite Sub cover

$\mathcal{E}_1 = \{V\}$

$\bigcup \mathcal{E}_1 \supseteq [a_n, b_n]$

Contradicts (3)

$\therefore$ Compact

$[a,b] \times [c,d]$ is compact in $\mathbb{R}^2$

Thm → Let (X,d) be a metric space $E \subseteq X$. Then E is compact in X iff E is compact in E.

Proof → Suppose E is compact in X.
Let $\mathcal{F}$ be open cover of E where each $V \in \mathcal{F}$ is open in E.

$\forall V \in \mathcal{F}$, fix an open $U_V \underset{open}{\subseteq} X$

st $V = U_V \cap E$

$\boxed{\begin{matrix} V_1 = U_{V_1} \cap E \\ V_\cdot \ldots U_{V_\cdot} \cap E \\ \vdots \end{matrix}}$

Put $\mathcal{F}' = \{ U_V : V \in \mathcal{F} \}$

$\therefore$ $\mathcal{F}'$ is an open cover of E, via open sets in X. As E is compact in X,

$\exists \mathcal{G}' \subseteq \mathcal{F}'$ st $\bigcup \mathcal{G}' \supseteq E$

$\mathcal{G}' = \{ U_{V_1}, U_{V_2}, \ldots U_{V_n} \}$

Then $\mathcal{G} = \{ V_1, V_2, \ldots V_n \}$ is cover of E.

Next, suppose E is compact in E.
Let $\mathcal{F}$ be a cover of E st each $U \in \mathcal{F}$ is open in X.
Consider $\mathcal{F}' = \{ U \cap E : U \in \mathcal{F} \}$
$\mathcal{F}'$ is still a cover of E & each set in $\mathcal{F}'$ is open in E.
As E is compact in E, we can find $\mathcal{G}' \underset{finite}{\subseteq} \mathcal{F}'$ st $\bigcup \mathcal{G}' \supseteq E$, say
$\mathcal{G}' = \{ U_1 \cap E, U_2 \cap E, \ldots, U_n \cap E \}$

Put $\mathcal{E}_1 = \{U_1, U_2, \dots U_n\}$

then $\mathcal{E}_1 \underset{\text{finite}}{\supseteq} f$

& $\bigcup \mathcal{E}_1 \supseteq E$

$\therefore$ E is compact in X [MEOW ]

<u>Thm</u> → (X, d) metric space, $E \subseteq X$

(1) Assume E is compact, then

(a) E is closed in X.

(b) E is bounded in X.

(c) ~~$\forall A \subseteq E$, $\exists x \in E$~~

$\forall$ infinite $A \subseteq E$,

$\exists x \in E$ st x is a limit point of A in E

(2) If X is compact & E is closed in X, then E is compact

<u>Thm</u> → Let $E \subseteq \mathbb{R}^n$, then

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)

(i) E is closed in $\mathbb{R}^n$ & bounded.

(ii) E is compact

(iii) $\forall$ infinite $A \subseteq E$, A has a limit point in E

<u>Proof</u> → (1) (b) Easy proof using basic balls [He He]

(a) Say E is not closed.

$\therefore \exists\, y \in E'$ st $y \notin E$

& $(\forall r > 0)\ (B(y, r) \cap E$ is infinite$)$

$\bigcup \{X \setminus C(y, \frac{1}{n}) : n \geq 1\} \supseteq E$ [As $y \notin E$]

C closed ball

E compact $\Rightarrow$ $(\exists m \geq 0\ (X \setminus C(y, \frac{1}{m}) \supseteq E) \Rightarrow B(y, \frac{1}{m}) \cap E = \emptyset$

[Ta da dhum]

(⇐) E is compact. Let $A \subseteq E$ be infinite. Suppose no point in E is in A'.

For each $x \in E$, fix $r_x > 0$ st

$$B(x, r_x) \cap A = \begin{cases} \{x\}, & \text{if } x \in A \\ \emptyset, & \text{otherwise} \end{cases}$$

Put

$f = \{B(x, r_x) : x \in E\}$

is an open cover of E.

Let $\{B(x_1, r_{x_1}), B(x_2, r_{x_2}), \ldots, B(x_n, r_{x_n})\}$ is finite subcover of E.

As $A \subseteq E$,

$$\underbrace{A}_{\text{Infinite}} \subseteq \underbrace{B(x_1, r_{x_1}) \cup \ldots B(x_n, r_{x_n})}_{\text{Finite}}$$

(⇐) Let f be an open cover of E (in X)

$\underbrace{f'}_{\text{Open}} = f \cup \underbrace{\{X \setminus E\}}_{\text{open in } X}$ is open cover of E

$\cup \xi' \supseteq X$

$\xi = \xi' \setminus \{X \setminus E\}$ covers E

Proof → (i) ⇒ (ii)

$E \subseteq \mathbb{R}^n$ closed and bounded

Bounded ⇒ $(\exists K>0)\ (E \subseteq \underbrace{[-K, K] \times \dots [-K, K]}_{\text{Compact}})$

$\therefore$ by (2), E is compact.

(ii) ⇒ (iii)

By (1) (c)

(iii) ⇒ (i)

E is bounded is easy

If not, choose $\{x_k : k \geq 1\} \subseteq E$, $\|x_k\| \geq k$
↓
No limit point.

Closed easy.

If not limit point not in E.

Fix $y \in E' \setminus E$, Choose $\langle x_n : n \geq 1 \rangle$ st
$d(y, x_n) < \frac{1}{n}$

Put $A = \{x_n : n \geq 1\}$

y limit point of A, $y \notin E$

Say, A has another lp, $z \in E$

$d(y, z) > 0$

Fix $n \geq 1$, $d(y, z) > \frac{1}{n}$

How do you define limits with $d = \begin{cases} 0 \\ 1 \end{cases}$ metric?

$\therefore$ y & z is disjoint.

And $X_{n \geq n}$ is y ball, not z ball.

$\therefore$ z can't be limit point.

(X, d) is a metric space.

Let $f : X \to \mathbb{R}$

$a \in X$

$\lim_{x \to a} f(x) = c$

means than $\forall \varepsilon > 0$, $\exists \delta_a > 0$ st whenever,

$d(x, a) < \delta_a$,

$|f(x) - c| < \varepsilon$

$\forall \varepsilon > 0 \quad \exists \delta > 0$ st

$\forall x \in B_X(a, \delta)$ we have

$|f(x) - c| < \varepsilon$

f is cts at a iff

$\lim_{x \to a} f(x) = f(a)$

f is uniformly cts if $\forall \varepsilon > 0$, $\exists \delta > 0$ st

$d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$

kmn

(X, d_X) & (Y, d_Y)

$$f : X \to Y$$

f is cts at point $a \in X$ iff $\forall \varepsilon > 0 \ \exists \delta > 0$ st $\forall x \in B_X(a, \delta)$, $f(x) \in B_Y(f(a), \varepsilon)$

If $U \subseteq Y$ is an open set $\Rightarrow$ $f^{-1}(U)$ is open.

Take an $a \in f^{-1}(U)$

$\therefore f(a) \in U$ is open

$\exists \varepsilon > 0$ st

$$d_Y(f(a), f(x)) < \varepsilon \quad \forall \ d_X(x, a) < \delta$$

$$B_Y(f(a), \varepsilon) \subseteq U$$

$$x \in \boxed{B_X(a, \delta) \subseteq f^{-1}(U)}$$

$$\Downarrow$$

Open [Boo hoo]

<u>Lemma</u> $f : (X, d_X) \to (Y, d_Y)$ is cts iff $\forall$ open U of Y, $f^{-1}(U)$ is open.

Lemma → (X, d_X) is a metric space and $E \subseteq X$. E is compact. Let $f: (X, d_X) \to (Y, d_Y)$ be a cts function, then $f(E)$ is compact

Proof → Even Pancakes can do it.

(Use f^{-1} is open)

Lemma → If E is compact, $E \subseteq X$, then $f|E$ is uniformly cts.

Ex:- Show that
$f: X \to Y$, (X, d_1) & (Y, d_2) metric spaces,
the following are equivalent.
(1) f is uniformly cts on X.
(2) For all sequences $\langle (x_n, y_n) : n \geq 1 \rangle$ of pairs in X, $d_1(x_n, y_n) \to 0 \Rightarrow d_2(f(x_n), f(y_n)) \to 0$

$\Downarrow$

Can use this to show that $y = x^2$ not uniform.

$\langle n, n + \frac{1}{n} : n \geq 1 \rangle$

Thm → (X, d_1) compact, (Y, d_2) metric space & $f: X \to Y$ cts $\Rightarrow$
f is uniformly cts on X

Proof → Fix $\varepsilon > 0$. Find $\delta > 0$ st

$$(\forall x, y \in X)(d_1(x,y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon)$$

For each $x \in X$, fix $\delta_x > 0$ st

$$(\forall y \in B_X(x, \delta_x))\left(d_2(f(y), f(x)) < \frac{\varepsilon}{2}\right)$$

$\mathcal{F} = \{ B_X(x, \frac{\delta_x}{2}) : x \in X \}$

$\mathcal{F}$ covers X. Open cover

Since X is compact

$\therefore$ $\mathcal{E}_1$ finite st

$\mathcal{E}_1 = \{ B_X(x_1, \frac{\delta_{x_1}}{2}), B_X(x_2, \frac{\delta_{x_2}}{2}), \ldots, B_X(x_n, \frac{\delta_{x_n}}{2}) \}$

$\downarrow$

$x_1, x_2, \ldots x_n \in X$

n is finite

$$\bigcup \mathcal{E}_1 \supset X$$

$$\boxed{\delta = \min\left(\frac{\delta_{x_1}}{2}, \ldots \frac{\delta_{x_n}}{2}\right) > 0}$$

Proof that $\frac{\delta}{4}$ works.

Let $x, y \in X$ & suppose $d_1(x,y) < \frac{\delta}{4}$

Claim, for some $k \in \{1, 2, 3, \ldots, n\}$,
both $x, y \in B_X(x_k, \delta_{x_k})$.

$\exists k \in \{1, 2, \ldots, n\}$ st
$x \in B_X(x_k, \frac{\delta_{x_k}}{2})$

Claim: $y \in B_X(x_n, \delta_{x_n})$

If not,

~~$d(y, x_n) > \delta_{x_n} + d_1(x, x_n) \geq d_1(y, x_n)$~~

$d_1(x, x_n) + d_1(x, y) \geq d_1(x_n, y)$

$d_1(x_n, y) \leq \frac{\delta_{x_n}}{2} + \underbrace{\frac{\delta}{4}}_{< \frac{\delta_{x_n}}{2}}$

$d_1(x_n, y) \leq \delta_{x_n}$

$\therefore \; (x, y) \in B_X(x_n, \delta_{x_n})$

$$d_2(f(x), f(y)) \leq \underbrace{d_2(f(x), f(x_n))}_{< \frac{\varepsilon}{2}} + \underbrace{d_2(f(y), f(x_n))}_{< \frac{\varepsilon}{2}}$$

$\therefore \; \boxed{d_2(f(x), f(y)) < \varepsilon}$

Ex:- $E \subseteq \mathbb{R}$ compact & non empty $\Rightarrow$ Both $\sup(E), \inf(E) \in E$

Connected Sets

Let (X, d) be metric space.

① $E \subseteq X$ is clopen in E iff E is both closed & open in E.
② X is connected iff the only clopen subset of X are $\emptyset$ and X.
③ X is disconnected iff its not connected

(i) $X = (0,1) \cup (2,3)$ clopen sets in X = $\emptyset, X, (0,1), (2,3)$

$(0,1)$ is open in X

$(0,1)$ is closed in X

(ii) $X = [0,1) \cup (1,2]$ clopen sets in X = $\emptyset, X, [0,1), (1,2]$

Thm → ① (X, d_1) connected, (Y, d_2) metric spaces

$f : X \to Y$ continuous

$\Rightarrow$ range (f) is connected

② $E \subseteq \mathbb{R}$ is connected iff E is an interval

$\to (\forall x, y \in E)$
$(x < y \Rightarrow [x, y] \subseteq E)$

Proof → (1) Image(open) = open
Image (closed) = Closed] Use this

(2) E is connected.
Take E is not an interval,
take $x < y \in E$, $x < z < y$ $z \notin E$
Check $A = \{w \in E : w < z\}$ is clopen in E & $A \notin \{\emptyset, E\}$
If E is connected, it is an interval.
$\boxed{\Rightarrow}$

E is interval
Take E is not connected.
Fix $A \subseteq E$ st A is clopen in E and
$A \notin \{\emptyset, E\}$

Fix $a \in A$ and $b \in E \setminus A$

Note that $[a,b] \subseteq E$

Put $W = A \cap [a,b]$. Then $a \in W$ & $b \notin W$

Let $\sup(W) = x$. $a \leq x \leq b$,

So $x \in E$.

x either in W or limit point of W.

$x \in cl(W)$

W is closed in E [A is clopen]

$\therefore$ $x \in W$ [$x \neq b$, $x < b$]

Also, $(x, b] \not\subseteq W$

$x \in ([a,b] \setminus W)'$

Hence, $x \in (E \setminus A)'$ (As $[a,b] \setminus W \subseteq E \setminus A$)

$x \in E \setminus A$

$\therefore$ $x \in A$ & $x \in E \setminus A$.

$\therefore$ E is connected.

HW 23. Let $\{a_n : n \geq 1\}$ be a one-to-one list of the set of all rationals $\mathbb{Q}$.
For each $n \geq 1$, define

$$U_n = \bigcup_{k \geq 1} \left(a_k - 2^{-(n+k)}, a_k + 2^{-(n+k)} \right)$$

Is $\bigcap_{n \geq 1} U_n = \mathbb{Q}$?

A. Fact: Let $\langle U_n : n \geq 1 \rangle$ be a sequence of open sets in $\mathbb{R}$ st each U_n is dense in $\mathbb{R}$.

(for each $a < b$, $(a,b) \cap U_n \neq \emptyset$)

Then $\bigcap U_n$ is uncountable.

Proof of fact → Let $E_n = \mathbb{R} \setminus U_n$. E_n is closed & Nowhere dense in $\mathbb{R}$. (Why?)

Let (a,b) be any open interval in $\mathbb{R}$

As U_n is dense in $\mathbb{R}$, $\exists x \in (a,b) \cap U_n$

As U_n is open in $\mathbb{R}$, $\exists r > 0$ st

$(x-r, x+r) \subseteq U_n \Rightarrow$

$E_n \cap (x-r, x+r) = \emptyset$

$$\bigcap U_n = \mathbb{R} \setminus \bigcup E_n$$

Suppose $\bigcap U_n$ is countable.

$$\bigcap_{n \geq 1} U_n = \{a_1, a_2, \ldots\}$$

Let $\mathcal{F}$ be the family of:

$E_n \ \forall n \geq 1$, $\{a_k\}$, $\forall k \geq 1$

$\mathcal{F}$ is countable, and each member of $\mathcal{F}$ is nowhere dense

By Baire Category Theorem,

$\mathbb{R} \setminus \cup f$ is dense in $\mathbb{R}$

But $\mathbb{R} \setminus \cup f$ is $\emptyset$

$$\cup f = \left(\bigcup_{n\geq 1} E_n\right) \cup \left(\bigcup_{k\geq 1} \{a_n\}\right)$$

$$= \left(\bigcup_{n\geq 1} E_n\right) \cup \left(\bigcap_{n\geq 1} U_n\right)$$

$$= \left(\bigcup_{n\geq 1} E_n\right) \cup \left(\mathbb{R} \setminus \bigcup_{n\geq 1} E_n\right)$$

$$= \mathbb{R}$$

$\therefore$ $\boxed{\bigcap U_n \text{ is uncountable}}$

Cantor Set

$C_0 = [0,1]$

$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$\vdots$

$\cancel{C_k = \bigsqcup_{0 \leq k \leq 2^k - 1}}$

C_k - Disjoint union of 2^k intervals of length $\frac{1}{3^k}$

$C \overset{\text{defn}}{=} \bigcap_{n \geq 1} C_n$

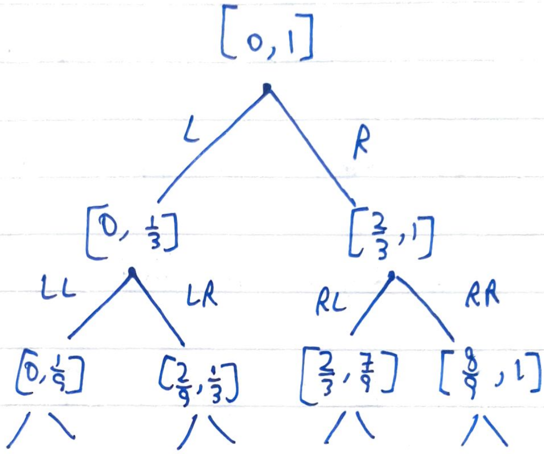
(1) C is closed in $\mathbb{R}$.

(2) $|C| = |2^{\mathbb{N}}| = |\mathbb{R}|$

(3) $x \in C \iff \exists \langle a_n : n \geq 1 \rangle$ each $a_n \in \{0, 2\}$ &

$$x = \sum \frac{a_n}{2^n}$$

Tree Representation



$x \in C$ iff $x \in C_k, \forall k \geq 0$

If $x \neq y$, their path is different.
$\therefore$ It is one-to-one

No. of x's $= |\{L, R\}^{\mathbb{N}}| = |2^{\mathbb{N}}| = \mathbb{R}$

$\therefore$ C is uncountable

$F: 2^{\mathbb{N}} \leftrightarrow C$

$F: \{0, 2\}^{\mathbb{N}} \leftrightarrow C$

(4) C is compact

(5) C is nowhere dense in $\mathbb{R}$.

Subsequence

Let $\langle x_n : n \geq 1 \rangle$ & $\langle y_n : n \geq 1 \rangle$ be sequences.

$\langle y_n : n \geq 1 \rangle$ is a subsequence of $\langle x_n : n \geq 1 \rangle$ iff for some $n_1 < n_2 < \ldots$

$$\langle y_k : k \geq 1 \rangle = \langle x_{n_k} : k \geq 1 \rangle$$

Theorem (X, d) compact metric space

Every sequence in (X, d) has a convergent subsequence

Corollary Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Proof of corollary

Let $\langle x_n : n \geq 1 \rangle$ be a bounded seq in $\mathbb{R}^n$

Choose $K > 0$ st each $x_n \in [-K, K] \times [-K, K] \ldots [-K, K]$

Proof of Thm

Let $\langle x_n : n \geq 1 \rangle$ be a seq in X.
Let $A = \{ x_{n_k} : k \geq 1 \}$

Case 1 → A is finite. Easy

Case 2 → A is infinite
By a previous theorem, $A' \neq \emptyset$
Fix $x \in A'$. We'll construct a subsequence of $\langle x_n : n \geq 1 \rangle$ that converges to x.

For each $k \geq 1$, choose n_k st
(a) $x_{n_k} \in B(x, \frac{1}{k})$

(b) $n_k < n_{k+1}$

$$x_{n_1} \in B(x, 1)$$

$$x_{n_2} \in B(x, \tfrac{1}{2})$$

Observe that $A \cap B(x, \frac{1}{k})$ is infinite.
So, we can choose $n_{k+1} > n_k$ st

$$x_{n_{k+1}} \in B(x, \tfrac{1}{k})$$

$$\lim_{k \to \infty} d(x_{n_k}, x) = 0 \qquad \left(d < \tfrac{1}{k} \right)$$

$$\lim_{k \to \infty} x_{n_k} = x$$

Yo mum gae, lol :)

Real Sequences

$\langle a_n : n \geq 1 \rangle$ be a sequence in $\mathbb{R}$

Define $S_n = \sup(\{a_k : k \geq n\})$
$t_n = \inf(\{a_k : k \geq n\})$

$S_1 \geq S_2 \geq S_3 \ldots$
$t_1 \leq t_2 \leq t_3 \ldots$

Define $\lim_{n\to\infty} \sup(a_n) = \lim_{n\to\infty} S_n = \inf(\{S_n : n \geq 1\})$

$\geq$

$\lim_{n\to\infty} \inf(a_n) = \lim_{n\to\infty} t_n = \sup(\{t_n : n \geq 1\})$

Converges if $=$ [#thatdude]

Thm $\langle a_n : n \geq 1 \rangle$ bounded seq in $\mathbb{R}$ $\langle a_n : n \geq 1 \rangle$ converges iff
$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \inf a_n$

Proof $(\Rightarrow)$

Fix $a \in \mathbb{R}$ st

$\lim_{n\to\infty} a_n = a$

Let $\varepsilon > 0$ be any positive no. $\exists N \geq 1$ st
$(\forall n \geq N)(|a_n - a| < \varepsilon)$

$a - \varepsilon < a < a + \varepsilon \quad , \quad \forall n \geq N$

$$a - \varepsilon \leq t_n \leq S_n \leq a + \varepsilon \qquad [\forall \varepsilon > 0]$$

$$\downarrow \qquad\qquad \downarrow$$

$$\lim_{n\to\infty} \inf a_n \qquad \lim_{n\to\infty} \sup a_n$$

$$\downarrow$$

$$\therefore \quad \lim_{n\to\infty} \inf(a_n) = \lim_{n\to\infty} \sup(a_n)$$

$(\Leftarrow)$ do it, bitch.

Series

$\langle a_n : n \geq 1 \rangle$ seq in $\mathbb{R}$

" $\sum_{n\geq 1}^{\infty} a_n$ " denotes $\langle S_n : n \geq 1 \rangle$ where $S_n = \sum_{k\geq 1}^{n} a_k$

$\sum_{n\geq 1}^{\infty} a_n$ converges to s iff $\lim_{n\to\infty} S_n = s$

$\sum a_n$ converges absolutely iff $\sum |a_n|$ converges

Facts from Calculus

(1) $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$

(2) $\sum \frac{1}{n^p}$ converges if $1 < p < \infty$

diverges if $-\infty < p \leq 1$

(3) Leibnitz test for Alternating Series

$a_1 \geq a_2 \geq a_3 \ldots$ is sequence,

$a_n \geq 0$

Then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges iff $\lim_{n\to\infty} a_n = 0$

(4) $e = \sum \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$

Fact e is irrational.

Take $e = \frac{m}{n}$

$m(n-1)! = \text{Integer} + \frac{n!}{(n+1)!} + \frac{n!}{\cdots} \cdots$

$\downarrow$

Integer

$\frac{1}{n+1} + \frac{1}{(n+1)(n+2)}$

$\downarrow$

Geometric Series

$\downarrow$

$0 < s < 1$

$\therefore$ Fraction

$\therefore$ Contradiction

Thm → Let $a>0$ and $n \geq 1 \in \mathbb{Z}$
Then, $\exists$ unique $x>0$ st
$x^n = a$
$x = a^{\frac{1}{n}}$

Proof → If $x \neq x'$, say
$0 < x < x'$, then
$x^n < (x')^n$

Existance → Put $E = \{y>0 : y^n < a\}$

$E \neq \emptyset$ $\left[\frac{a}{1+a} \in E\right]$
$\left[b = \frac{a}{1+a} \right.$

$\left. b^n < b < a \right]$

E is bounded from above by $(1+a)$

So, $\sup(E) = x > 0$ exists

~~Claim $x^n = a$~~

Claim $x^n = a$.

Case 1 → $x^n < a$

We claim $\exists h>0$ st
$(x+h)^n \in E$
Not possible as $x = \sup(E)$
$(x+h)^n - x^n \leq nh(x+h)^{n-1} < a - x^n$
$[(x+h)^n - x^n = ((x+h) - x)((x+h)^{n-1}$
$+ (x+h)^{n-2}x^1 + \ldots)]$

~~Case 2 →~~

for small enough h.
$(x+h)^n < a$
So, $x+h \in E$,
So, cowabunga.

Case 2 → $x^n > a$

DIY

Lemma → (1) $a > 0 \Rightarrow \lim_{n \to \infty} a^{\frac{1}{n}} = 1$

(2) $\lim_{n \to \infty} n^{\frac{1}{n}} = 1$

Proof → (1) $\cancel{x_n = a^{\frac{1}{n}} - 1 > 0}$

Assume $a > 1$,

then $a^{\frac{1}{n}} > 1$.

$$x_n = a^{\frac{1}{n}} - 1$$

$$a = (1 + x_n)^n \geq 1 + n x_n$$

$$0 < x_n \leq \frac{a-1}{n} \to 0 \text{ as } n \to \infty$$

$$\therefore x_n \to 0$$

$$a^{\frac{1}{n}} \to 1, \quad n \to \infty$$

(2) $x_n = n^{\frac{1}{n}} - 1 \geq 0$

$$\binom{n}{2} x_n^2 < (1 + x_n)^n = n$$

$$n > \frac{n(n-1)}{2} x_n^2$$

$$x_n^2 < \frac{2}{n-1}, \quad n \geq 2$$

$$0 < x_n < \sqrt{\frac{2}{n-1}} \to 0 \ , n \to \infty$$

$$\therefore x_n \to 0$$

$$\boxed{n^{\frac{1}{n}} \to 1}$$

Tests

Ratio Test: $\sum_{n=1}^{\infty} a_n \ , |a_n| > 0 \ ,$ $\alpha = \limsup_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}$

① $\alpha < 1$, Absolutely Convergent

② $\alpha \geq 1$, nothing can be said.

~~Ratio T~~

Root Test: $\alpha = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}}$

① $\alpha < 1$, Ab Con

② $\alpha > 1$, DiV

③ $\alpha = 1$,

Power Series

$\sum_{n=0}^{\infty} a_n x^n$, $\alpha = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}}$

Define $R = \begin{cases} \infty, & \alpha = 0 \\ 0, & \alpha = \infty \\ \frac{1}{\alpha}, & \text{otherwise} \end{cases}$

If $|x|<R$, conv

$|x|>R$, div

$|x|=R$, no clue

Proof of Ratio Test

(1) $\alpha < 1$.

Let $\beta = \alpha + \frac{1-\alpha}{2}$

$\alpha < \beta < 1$

$\alpha = \lim_{n\to\infty} \sup \left| \frac{a_{n+1}}{a_n} \right| < \beta,$

$\exists N \in \mathbb{N}^+$ st $(\forall n \geq N) \left(\left| \frac{a_{n+1}}{a_n} \right| < \beta \right)$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|$

$\downarrow$

$\leq |a_N| + |a_N| \beta + |a_N| \beta^2 + \ldots$

$\leq \left(\sum \right) + \frac{|a_N|}{1-\beta}$

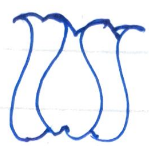
Proof of Root Test

(i) $\alpha < 1$

$$\beta = \alpha + \left(\frac{1-\alpha}{2}\right)$$

$$(\exists N \in \mathbb{N}^+)(\forall n \geq N)\left(|a_n|^{\frac{1}{n}} < \beta\right)$$

$$a_n < \beta^n$$



$\alpha > 1$

$$\beta = \alpha - \frac{\alpha - 1}{2}$$

$$\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = \alpha > \beta > 1$$

$$\cancel{(\forall N)(\exists n > N)(}$$

$$\cancel{(\exists N)(\forall n > N)(|a_n|^{\frac{1}{n}} > \beta)}$$

$$(\forall N)(\exists n > N)(X_n > \beta > 1)$$

$$\therefore \lim_{n\to\infty} a_n \neq 0$$

Lemma: • $a_n \neq 0 \quad \forall n \geq 1$

$$\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} \leq \lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right|$$

Riemann Rearrangement Theorem

Fact → (1) $\sum |a_n|$ converges ⟹ $\forall$ bijection $f: \mathbb{N}^+ \to \mathbb{N}^+$

$$\sum a_n = \sum a_{f(n)}$$

↑ Rearrangement

(2) $\sum |a_n|$ diverges, but $\sum a_n$ converges.

$\sum a_n$ is conditionally convergent.

$\forall \alpha \in \mathbb{R}, \quad \exists f : \mathbb{N}^+ \leftrightarrow \mathbb{N}^+,$

st $\sum a_{f(n)} = \alpha$

Proof (1) $\forall N \in \mathbb{N}$

$$\sum^{N} |a_{f(n)}| \le \sum^{\infty} |a_n| = A$$

So $\lim\limits_{N \to \infty} \sum^{N} |a_{f(n)}| \le A$

So, $\sum a_{f(n)}$ is abs conv.

~~So, $\sum |a_{f(n)}|$~~

Put $\alpha = \sum_{n=1}^{\infty} a_n$

Choose $N_1 \geq 1$, st $\forall n \geq N_1$,

$$\left| \sum_{k=1}^{n} a_k - \alpha \right| < \frac{\varepsilon}{2} \quad \& \quad \sum_{k > N_1}^{\infty} |a_k| < \frac{\varepsilon}{2}$$

Choose $N_2 \geq N_1$ st

$\{1, 2, 3, \dots N_1\} \subseteq \{f(1), f(2), \dots, f(N_2)\}$

If $n \geq N_2$,

$$\sum_{k=1}^{n} a_{f(k)} - \alpha = \sum_{k=1}^{N_1} a_k - \alpha + \sum^{\text{finite}} a_k$$

$$\left| \sum_{k=1}^{n} a_{f(k)} - \alpha \right| \leq \left| \sum_{k=1}^{N_1} a_k - \alpha \right| + \sum_{k \geq N_2}^{\infty} |a_k|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

$$\therefore \sum_{k=1}^{n} a_{f(k)} \to \alpha$$

Proof (2)

$\alpha \to$ You know

$\infty \to$ Add (+) to 10, (−) less than 9.

(+) till 100, (−) less than 99, and so on..