Name: ____________________

Roll Number: _______________

Practice Final Answers

MTH301A - Analysis I

(Odd Semester 2023/24, IIT Kanpur)

INSTRUCTIONS

1. Write your **Name** and **Roll number** above.

2. This exam contains **6 + 1** questions and is worth **60%** of your grade.

3. Answer **ALL** questions.

Question 1. [5 × 2 Points]

For each of the following statements, determine whether it is **true or false**. No justification required.

(i) Every compact metric space is separable.

(ii) There exists a function $f : \mathbb{R} \to \mathbb{R}$ which is everywhere differentiable and $f'(x)$ is discontinuous at every irrational x.

(iii) If $f : [a, b] \to \mathbb{R}$ is a bounded Riemann integrable function and $\displaystyle\int_a^b |f(x)|dx = 0$, then f is identically zero on $[a, b]$.

(iv) The sequence of functions $\langle nx^n : n \geq 1\rangle$ uniformly converges to 0 on $[0, 0.9]$.

(v) The set $\{p(x^2) : p(x) \text{ is a polynomial with rational coefficients}\}$ is dense in $C([-1, 1], d_\infty)$.

Solution

(i) **True**. HW 67(b).

(ii) **False**. f' is a Baire class one function. Hence the set of points of continuity of f' is a countable intersection of open dense subets of $[a, b]$ which is not equal to $[a, b] \setminus \mathbb{Q}$

(iii) **False**. Let $f : [0, 1]$ be the characteristic function of $\{0.5\}$.

(iv) **True**.

(v) **False**. Consider $f(x) = \sin x$ (or any function that is not even on $[-1, 1]$).

Question 2. [10 Points]

(a) [**5 Points**] Let $f : [0,1] \to \mathbb{R}$ be a bounded Riemann integrable function. Let $g : [0,1] \to \mathbb{R}$ be a function such that $\{x \in [0,1] : f(x) \neq g(x)\}$ is finite. Show that g is Riemann integrable and $\int_0^1 f(x)dx = \int_0^1 g(x)dx$.

(b) [**5 Points**] Suppose $f : [-1,1] \to \mathbb{R}$ is continuous. Assume that for every $-1 \leq a < b \leq 1$, if $b - a < 2^{-10}$, then $\int_a^b f(x)dx = 0$. Show that f is identically zero on $[-1,1]$.

Solution

(a) For $c \in [0,1]$, let $h_c : [0,1] \to \mathbb{R}$ be defined by $h_c(x) = 1$ if $x = c$ and 0 otherwise. Show that h_c is Riemann integrable and $\int_0^1 h_c(x)dx = 0$. Let $\{c_1, \cdots, c_N\}$ list $\{x \in [0,1] : f(x) \neq g(x)\}$. Put $a_k = g(c_k) - f(c_k)$ for $1 \leq k \leq N$. Then $g = f + \sum_{k=1}^{N} a_k h_{c_k}$. Since f and each h_{c_k} is Riemann integrable, we get g is Riemann integrable. Also

$$\int_0^1 g(x)dx = \int_0^1 f(x)dx + \sum_{k=1}^{N} \int_0^1 a_k h_{c_k}(x)dx = \int_0^1 f(x)dx + \sum_{k=1}^{N} 0 = \int_0^1 f(x)dx.$$

$\square$

(b) (Compare with HW 55) Suppose not and fix $c \in [-1,1]$ such that $f(c) \neq 0$. Assume $f(c) = h > 0$ (the other case is similar). As f is continuous at c, we can find an interval $[a_1, b_1]$ around c such that $f(x) \geq h/2$ for every $x \in [a_1, b_1]$. Let $[a,b] \subseteq [a_1, b_1]$ be an interval around c of length $< 2^{-10}$. It follows that

$$0 = \int_a^b f(x)dx \geq \int_a^b h/2dx = h(b-a)/2 > 0$$

which is a contradiction. $\square$

Question 3. [10 Points]

Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \cos\left(\frac{1+x}{n(\pi + x^4)}\right).$$

(a) [**4 Points**] Show that $\langle f_n : n \geq 1 \rangle$ is uniformly convergent on $\mathbb{R}$.

(b) [**6 Points**] Compute $\lim\limits_{n\to\infty} \int_0^1 e^x f_n(x)dx$. Justify all of your steps.

Solution

(a) We will show that $f_n \xrightarrow{\text{uniformly}} 1$ on $\mathbb{R}$.

Let $\varepsilon > 0$. Since $\cos x$ is continuous at $x = 0$, we can find $\delta > 0$ such that

$$|x| < \delta \implies |\cos x - \cos 0| = |\cos x - 1| < \varepsilon$$

Next observe that $|1 + x| \leq 1 + |x| \leq 2 + x^4$ for every $x \in \mathbb{R}$. So $\left|\frac{1+x}{n(\pi + x^4)}\right| \leq \frac{1}{n}$ for every $x \in \mathbb{R}$. Fix $N \geq 1$ such that $1/N < \delta$. It follows that for every $n \geq N$ and $x \in \mathbb{R}$,

$$|f_n(x) - 1| = \left|\cos\left(\frac{1+x}{n(\pi + x^4)}\right) - 1\right| < \varepsilon$$

Hence $f_n \xrightarrow{\text{uniformly}} 1$ on $\mathbb{R}$. □

(b) As e^x is bounded on $[0, 1]$ and f_n's uniformly converge to 1 on $[0, 1]$, it follows that $e^x f_n(x)$ uniformly converges to $(e^x)(1) = e^x$ on $[0, 1]$. Therefore we can interchange limit and integral to get

$$\lim_{n\to\infty} \int_0^1 e^x f_n(x)dx = \int_0^1 e^x dx = e - 1.$$

□

Question 4. [10 Points]

Consider the series

$$\sum_{n=0}^{\infty} \frac{n^3}{3^n} \cos(nx)$$

(a) [**2 Points**] Show that this series is absolutely convergent for every $x \in \mathbb{R}$.

(b) [**2 Points**] Show that this series is uniformly convergent on $\mathbb{R}$.

(c) [**6 Points**] Define $f(x) = \sum_{n=0}^{\infty} \frac{n^3}{3^n} \cos(nx)$. Show that f is everywhere differentiable and

$$f'(x) = \sum_{n=1}^{\infty} \frac{-n^4}{3^n} \sin(nx)$$

Solution

(a) Let $a_n = \frac{n^3}{3^n}$. Then $\limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{n^{3/n}}{3} = 1/3 < 1$. So by the root test, we get $\sum_{n=0}^{\infty} |a_n| < \infty$. It follows that for every $x \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} \left| \frac{n^3}{3^n} \cos(nx) \right| \leq \sum_{n=0}^{\infty} \left| \frac{n^3}{3^n} \right| < \infty.$$

□

(b) Put $s_n = \sum_{k=0}^{n} \frac{k^3}{3^k} \cos(kx)$. We need to show that $\langle s_n : n \geq 1 \rangle$ is uniformly convergent on $\mathbb{R}$. Let $\varepsilon > 0$. Choose $N \geq 1$ such that $\sum_{n=N}^{\infty} \frac{n^3}{3^n} < \varepsilon$. Then for every $m, n \geq N$ and $x \in \mathbb{R}$, we have

$$|s_n(x) - s_m(x)| \leq \sum_{k=N}^{\infty} \left| \frac{k^3}{3^k} \cos(kx) \right| \leq \sum_{k=N}^{\infty} \left| \frac{k^3}{3^k} \right| < \varepsilon.$$

It follows that $\langle s_n : n \geq 1 \rangle$ is uniformly convergent on $\mathbb{R}$. □

(c) Put $s_n = \sum_{k=0}^{n} \frac{k^3}{3^k} \cos(kx)$. We already showed that $\langle s_n : n \geq 1 \rangle$ is uniformly convergent on $\mathbb{R}$. We need to check that s_n' is also uniformly convergent on $\mathbb{R}$. This is similar to the proof of (b) using the following

$$\frac{d}{dx}\left(\frac{n^3}{3^n} \cos(nx) \right) = \frac{-n^4}{3^n} \sin(nx)$$

$$\left| \frac{-n^4}{3^n} \sin(nx) \right| \leq \frac{n^4}{3^n} \text{ and}$$

$$\sum_{n=1}^{\infty} \frac{n^4}{3^n} < \infty \text{ (Use root test).}$$

□

Question 5. [10 Points]

Consider $C([a,b])$ under uniform metric $d_\infty(f,g) = ||f-g||_\infty$.

(a) [**2 Points**] State Weierstrass approximation theorem.

(b) [**3 Points**] Let $D = \{f \in C([a,b]) : f \text{ is differentiable on } [a,b]\}$. Is D dense in $(C([a,b]), d_\infty)$. Why?

(c) [**5 Points**] Give an example of a uniformly bounded sequence of functions $\langle f_n : n \geq 1\rangle$ in $C([0,1])$ such that $\lim_{n\to\infty} \int_0^1 |f_n(x)|dx = 0$ but $\lim_{n\to\infty} f_n(x)$ diverges for every $x \in [0,1]$.

Solution

(a) For every $f \in C([a,b])$ and $\varepsilon > 0$, there exists a polynomial $p = p(x)$ such that

$$||f - p||_\infty < \varepsilon.$$

It follows that the set of all polynomials is dense in $(C([a,b]), d_\infty)$. □

(b) Since every polynomial is in D, it follows that D is also dense $(C([a,b]), d_\infty)$. □

(c) Let $\langle J_n : n \geq 1\rangle$ be an enumeration of all intervals of the form $\{[k2^{-n}, (k+1)2^{-n}] : n \geq 1 \text{ and } 0 \leq k < n-1\}$ in decreasing order of length. So

$$J_1 = [0, 1/2], J_2 = [1/2, 1], J_3 = [0, 1/8], \cdots$$

For each $n \geq 1$, say $J_n = [a_n - r_n, a_n + r_n] \subseteq [0,1]$. Define $f_n : [0,1] \to [0,1]$ to be a piecewise linear function such that $f_n(x) = 1$ if $|x - a_n| \leq r_n$ and $f_n(x) = 0$ if $|x - a_n| \geq 2r_n$. It is easy to see that $\{f_n : n \geq 1\}$ is uniformly bounded by 1 and $\int_0^1 |f_n(x)|dx \leq 2 \,\mathsf{length}(J_n) \to 0$. Now check that for every $x \in [0,1]$, $\liminf_{n\to\infty} f_n(x) = 0$ and $\limsup_{n\to\infty} f_n(x) = 1$. □

Question 6. [10 Points]

(a) [**3 Points**] State Arzela-Ascoli theorem.

(b) [**7 Points**] Let $f_n : [0,1] \to \mathbb{R}$ be a bounded Riemann integrable function for each $n \geq 1$. Define $F_n : [0,1] \to \mathbb{R}$ by

$$F_n(x) = \int_0^x f_n(t)dt$$

Assume that there is a constant $M > 0$ such that for every $n \geq 1$, $\int_0^1 (f_n(x))^2 dx < M$. Show that $\langle F_n : n \geq 1 \rangle$ has a uniformly convergent subsequence.

Solution

(a) Suppose (X, d) is a compact metric space and $\langle f_n : n \geq 1 \rangle$ is a uniformly bounded sequence of continuous functions from X to $\mathbb{R}$. Suppose that the family $\{f_n : n \geq 1\}$ is uniformly equicontinuous on X. Then $\langle f_n : n \geq 1 \rangle$ has a uniformly convergent subsequence. □

(b) Since $[a, b]$ is compact, it suffices to check that $\langle F_n : n \geq 1 \rangle$ the hypotheses of Arzela-Ascoli theorem hold. First note that for any $a \geq 0$, $a \leq 1 + a^2$. Hence for any $x \in [0, 1]$,

$$|F_n(x)| \leq \int_0^x |f_n(t)|dt \leq \int_0^x (1 + (f_n(t))^2)dt \leq \int_0^1 (1 + (f_n(t))^2)dt \leq 1 + M$$

So $\langle F_n : n \geq 1 \rangle$ is uniformly bounded. Next fix $\varepsilon > 0$ and $0 \leq x < y \leq 1$. Then

$$|F_n(x) - F_n(y)| \leq \int_x^y |f_n(t)|dt \leq \left(\int_x^y (1)^2 dt \right)^{1/2} \left(\int_x^y (f_n(t))^2)dt \right)^{1/2} \leq |x - y|^{1/2} M^{1/2}$$

where we used the following (see HW 58):

$$\int_x^y |g(t)h(t)|dt \leq \left(\int_x^y (g(t))^2 dt \right)^{1/2} \left(\int_x^y (h(t))^2 dt \right)^{1/2}$$

Choose $\delta = \varepsilon^2/4M$. Then $|x - y| < \delta \implies |F_n(x) - F_n(y)| \leq \varepsilon/2 < \varepsilon$. Therefore $\{F_n : n \geq 1\}$ is uniformly equicontinuous on $[0, 1]$ and we are done. □

Bonus Question [5 Points]

Let $f_n : [0,1] \to [0,1]$ be monotonically increasing for $n \geq 1$. Show that there is a subsequence $\langle f_{n_k} : k \geq 1 \rangle$ of $\langle f_n : n \geq 1 \rangle$ and a function $f : [0,1] \to [0,1]$ such that $f_n \xrightarrow{\text{pointwise}} f$.

Solution sketch

Let D_0 be the set of all rationals in $[0,1]$. For $n \geq 1$, let D_n be the set of points of discontinuity of f_n. By HW 9, each D_n is countable. Put $D = \bigcup_{n \geq 0} D_n$. Then D is a countable dense subset of $[0,1]$.

By a Lemma proved in class, there is a subsequence $\langle f_{n_k} : k \geq 1 \rangle$ of $\langle f_n : n \geq 1 \rangle$ such that for every $r \in D$, $\lim\limits_{k \to \infty} f_{n_k}(r)$ exists (call it $h(r)$). Define $f : [0,1] \to [0,1]$ by $f(x) = \sup(\{h(r) : r \in D \text{ and } r \leq x\})$. Now check that for every $x \in [0,1]$, $\lim\limits_{n \to \infty} f_n(x) = f(x)$. $\square$

End of Exam