A Bounded Derivative That Is Not Riemann Integrable

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An important goal for those of us who teach elementary real analysis is to help students view broader mathematical vistas. Presenting examples of functions that exhibit strange behavior is one way to achieve this goal. For instance, the existence of continuous functions that are nowhere differentiable can be quite a surprise for students. As the title of this note indicates, we are interested in differentiable functions whose derivatives are not Riemann integrable. The existence of such functions explains the need for a seemingly extra hypothesis in the statement of the fundamental theorem of calculus and illustrates that a more powerful integration process is necessary for higher level analysis.

The fundamental theorem of calculus

In simple, but not precise, language, the fundamental theorem of calculus states that differentiation and integration are inverse processes. Since the two operations can occur in either order, there are two parts of the theorem. For our purposes, we are interested in the following statement.

- If the function f is Riemann integrable on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then $\int_a^b f = F(b) - F(a)$.

By incorporating the definition of antiderivative into the hypothesis, we obtain

- If F is differentiable on $[a, b]$ and the function F' is Riemann integrable on $[a, b]$, then $\int_a^b F' = F(b) - F(a)$.

The second part of the hypothesis seems redundant. Since integration is (informally) the inverse operation of differentiation, aren't all derivatives integrable? (The reader may wish to consult [**4**] for an insightful discussion of the history behind the statement of the fundamental theorem of calculus.) However, once we recall that Riemann integrable functions must be bounded, an example of a derivative that is not Riemann integrable is close at hand. For example, the derivative of the function F defined by $F(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $F(0) = 0$ exists at all points, but the function F' is not bounded on [0, 1]. (For the record, the function F is not absolutely continuous on [0, 1], so F' is not even Lebesgue integrable on [0, 1].) After seeing examples such as this, we might hope that at least the following statement is valid.

Math. Mag. **89** (2016) 364–370. doi:10.4169/math.mag.89.5.364.
MSC: Primary 26A24

- If F is differentiable on $[a, b]$ and F' is bounded on $[a, b]$, then $\int_a^b F' = F(b) - F(a)$.

Unfortunately, this version of the fundamental theorem of calculus is not valid for the Riemann integral; there is no guarantee that bounded derivatives are Riemann integrable. (It does follow, however, that all bounded derivatives are Lebesgue integrable.) As we might expect, creating such an unusual function is rather involved. However, with some care, the ideas are accessible to students in an elementary real analysis course.

Brief historical background

As a start, consider the function G defined by $G(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $G(0) = 0$. The function G is differentiable at every point, and G' is bounded on any interval of the form $[a, b]$. In addition, the function G' is continuous everywhere except at $x = 0$. As a result, it is Riemann integrable on $[0, 1]$ and the fundamental theorem of calculus holds. By slightly modifying the function G and placing its analogues in the open intervals constituting the complement of a perfect nowhere dense set of positive measure, Volterra [**13**] was able to construct a bounded derivative that was not Riemann integrable. The details tend to be rather tedious, so very few elementary real analysis texts include this function. In fact, other than the text [**9**, p. 183], I did not find any texts at this level that mention the existence of such functions. The more advanced books [**10**], [**11**, p. 490], and [**12**, p. 312] mention Volterra's function, with the latter having the details as a list of exercises. The book by Bressoud [**5**] presents a lengthy and helpful historical discussion, as well as many of the details, behind Volterra's function. (A summary of a talk given by Bressoud on this topic is available at [**3**].) See also [**8**, p. 35] for another source that presents the computational details necessary to show that this function has the desired properties.

A short note by Goffman in the March 1977 issue of the *American Mathematical Monthly* (see [**7**]) presents a collection of simpler examples of bounded derivatives that are not Riemann integrable. Since these functions seem to have attracted very little notice (the only reference I found was [**6**, p. 79], and it essentially recopies Goffman's work), we have decided to give a more careful treatment here of a slight variation on Goffman's examples. Our goal is to include as many of the details as possible, keeping the perspective of a typical student in an elementary real analysis course in mind. Having said this, however, there are still a fair number of unavoidable prerequisite ideas needed.

A bounded function f that is not Riemann integrable

Let $O = \bigcup_{n=1}^{\infty} (a_n, b_n)$ be an open dense set in the interval $(0, 1)$, where the intervals (a_n, b_n) are pairwise disjoint and the inequality $\sum_{n=1}^{\infty} (b_n - a_n) \leq \frac{1}{2}$ is satisfied. Without loss of generality, we may assume that $a_i \neq b_j$ for all positive integers i and j. An explanation of the terms in the opening sentence of this paragraph, as well as a way to create such a set, can be found in a later section. For each positive integer n, choose points c_n and d_n so that

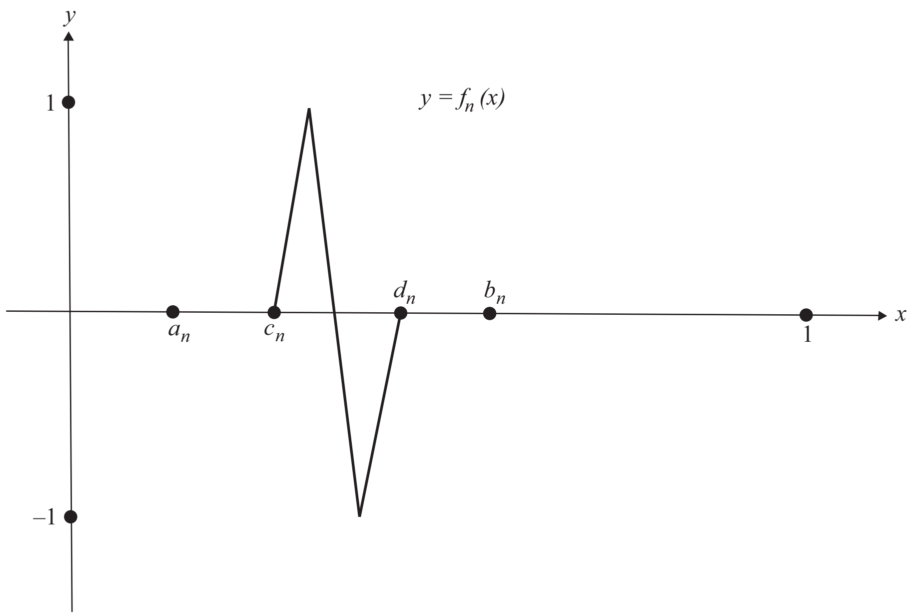
$$a_n < c_n < d_n < b_n, \quad \frac{c_n + d_n}{2} = \frac{a_n + b_n}{2}, \quad \text{and} \quad d_n - c_n = (b_n - a_n)^2.$$

In other words, the interval $[c_n, d_n]$ is a subset of (a_n, b_n) with the same center and having a length equal to the square of the length of $[a_n, b_n]$. For future reference, note that

$$c_n - a_n = \frac{(b_n - a_n) - (b_n - a_n)^2}{2} = \frac{b_n - a_n}{2}\Big(1 - (b_n - a_n)\Big) > \frac{b_n - a_n}{4},$$

where we have used the fact that $b_n - a_n < \frac{1}{2}$ for each n.

For each positive integer n, define a continuous function $f_n\colon [0,1] \to [-1,1]$ so that f_n is 0 on the intervals $[0, c_n]$ and $[d_n, 1]$, there is a point $t_n \in (c_n, d_n)$ such that $|f(t_n)| = 1$, and $\int_{c_n}^{d_n} f_n = 0$. There are, of course, many possibilities for f_n, but perhaps the simplest example is the piecewise linear function graphed below (the graph is not to scale).



(We note in passing that a graph of the derivative of the modified version of $x^2 \sin(1/x)$ needed for Volterra's example cannot be fully drawn on its corresponding interval so our f_n functions are more visually accessible.) Now define a bounded function $f\colon [0,1] \to [-1,1]$ by $f(x) = \sum_{n=1}^{\infty} f_n(x)$; the function f is well-defined since at most one term in the series is nonzero for each $x \in [0,1]$. It is a good exercise for students to show that f is continuous at each point of the set O and that f is not continuous at each point of the set $E = [0,1] \setminus O$. Since the set E has positive measure, the function f is not continuous almost everywhere and thus not Riemann integrable on $[0,1]$. Since most courses in elementary real analysis do not prove that a function is Riemann integrable on $[a,b]$ if and only if it is bounded and continuous almost everywhere on $[a,b]$, we present a more elementary proof that f is not Riemann integrable on $[0,1]$.

Since there are several ways to define the Riemann integral, we need to specify the version that we consider here. Let h be a bounded function defined on an interval $[a,b]$. The oscillation of h on $[a,b]$ is defined by

$$\omega(h, [a,b]) = \sup\{h(x) : x \in [a,b]\} - \inf\{h(x) : x \in [a,b]\}.$$

Using this definition, we obtain the following criterion for Riemann integrability.

- A bounded function h is Riemann integrable on $[a, b]$ if and only if for each positive number ϵ there exists a partition $P = \{x_i : 0 \le i \le n\}$ of $[a, b]$ such that $\sum_{i=1}^{n} \omega(h, [x_{i-1}, x_i])(x_i - x_{i-1}) < \epsilon$.

This result is either a theorem (when using Riemann sums to define the integral) or a definition (when using upper and lower sums to define the integral); see [**2**, p. 173], [**6**, p. 77], or [**9**, p. 237].

To verify that f is not Riemann integrable on $[0, 1]$, let $\{x_i : 0 \le i \le q\}$ be a partition of $[0, 1]$. Define S to be the set of all indices i such that $1 \le i \le q$ and the set $(x_{i-1}, x_i) \cap E$ is empty, and let $T = \{i : 1 \le i \le q\} \setminus S$. Note that

i) if $i \in S$, then $(x_{i-1}, x_i) \subseteq O$;

ii) if $i \in T$, then $(a_n, b_n) \subseteq (x_{i-1}, x_i)$ for some n and thus $\omega(f, [x_{i-1}, x_i]) \ge 1$.

To verify observation (ii), note that the set $(x_{i-1}, x_i) \cap E$ cannot contain just one point; if so, then $a_i = b_j$ for some i and j. Suppose that $u < v$ are two points in $(x_{i-1}, x_i) \cap E$. Since O is dense in $(0, 1)$, there is a point $t \in (u, v) \cap O$. If $t \in (a_n, b_n)$, then $(a_n, b_n) \subseteq (u, v) \subseteq (x_{i-1}, x_i)$. From these two observations and the fact that the intervals $[x_{i-1}, x_i]$ are nonoverlapping, it follows that

$$\sum_{i\in S}(x_i - x_{i-1}) \le \sum_{n=1}^{\infty}(b_n - a_n) \le \frac{1}{2} \qquad \text{and thus} \qquad \sum_{i\in T}(x_i - x_{i-1}) \ge \frac{1}{2}.$$

We then find that

$$\begin{aligned}\sum_{i=1}^{q} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) &\ge \sum_{i\in T} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1})\\ &\ge \sum_{i\in T}(x_i - x_{i-1}) \ge \frac{1}{2}.\end{aligned}$$

Since this inequality holds for every partition of $[0, 1]$, the function f is not Riemann integrable on $[0, 1]$.

Our function f is a derivative

Now define a function F on $[0, 1]$ by $F(x) = \sum_{k=1}^{\infty} \int_0^x f_k$. (Note once again that at most one term of this series is nonzero for each value of x.) We claim that F is differentiable on $[0, 1]$ and that $F'(x) = f(x)$ for all $x \in [0, 1]$. For each $x \in (a_n, b_n)$, we find that $F(x) - F(a_n) = \int_{a_n}^x f_n$. Since f_n is a continuous function, the fundamental theorem of calculus (the other version!) reveals that $F'(x) = f_n(x) = f(x)$ for all $x \in (a_n, b_n)$. We conclude that $F'(x) = f(x)$ for all $x \in O$. We now prove that

$$\lim_{y\to x^+} \frac{F(y) - F(x)}{y - x} = 0$$

for each $x \in E \setminus \{1\}$. If $x = a_k$ for some positive integer k, then the result is trivial since F is 0 on the interval $[a_k, c_k]$. Assume that $x \ne a_k$ for all k and fix a value of y so that $x < y < 1$. There are two cases to consider.

I. Suppose that $y \notin \bigcup_{k=1}^{\infty}(c_k, d_k)$. Then $F(y) - F(x) = 0 - 0 = 0$.

II. Suppose that $y \in \bigcup_{k=1}^{\infty}(c_k, d_k)$ and choose an index p so that $y \in (c_p, d_p)$. Using both the inequalities $x < a_p < c_p < y$ and $b_p - a_p < 4(c_p - a_p)$ (see the definition of the intervals $[c_n, d_n]$), we have

$$|F(y) - F(x)| = \left| \int_{c_p}^{y} f_p \right| \le \int_{c_p}^{y} |f_p| \le d_p - c_p$$
$$= (b_p - a_p)^2 < 16(c_p - a_p)^2 \le 16(y - x)^2.$$

We have thus shown that $|F(y) - F(x)| \le 16(y - x)^2$ for each $y \in (x, 1)$ and hence

$$\lim_{y \to x^+} \frac{F(y) - F(x)}{y - x} = 0.$$

A similar argument reveals that

$$\lim_{y \to x^-} \frac{F(y) - F(x)}{y - x} = 0$$

for each $x \in E \setminus \{0\}$. We conclude that $F'(x) = f(x)$ for all $x \in E$ and thus $F' = f$ on $[0, 1]$. Therefore, the function F' is a bounded derivative that is not Riemann integrable.

Goffman defines a continuous function $f_n \colon [0, 1] \to [0, 1]$ so that f_n is 0 on the intervals $[0, c_n]$ and $[d_n, 1]$, and $f_n((c_n + d_n)/2) = 1$. A proof that the corresponding function F satisfies $F' = f$ has a few extra details, but the reasoning is essentially the same. Discovering the necessary changes to the proof is a good (but nontrivial) exercise for students. For the record, Goffman's function F is increasing on $[0, 1]$. In our situation, the function $F(x) + x$ is increasing on $[0, 1]$; this function still has a bounded derivative that is not Riemann integrable on $[0, 1]$.

Construction of the open set O

We now make a few remarks concerning the set O. A set A of real numbers is open if for each $x \in A$ there exists $r > 0$ such that $(x - r, x + r) \subseteq A$. The simplest example of an open set is an open interval. It is easy to see that any union of open sets is an open set and that the intersection of two open sets is an open set. With a little more work, it can be shown that every open set is a countable union of disjoint open intervals (see [**1**, p. 66], [**2**, p. 350], or [**9**, p. 296]. A set D is dense in an interval $[a, b]$ if each open interval that intersects $[a, b]$ contains a point of D. The set $\mathbb{Q} \cap (a, b)$ of rational numbers in (a, b) is dense in the interval $[a, b]$. Since the set $\mathbb{Q}$ is countably infinite, the set $\mathbb{Q} \cap (0, 1)$ can be written as a sequence $\{r_k\}$ of distinct points. For example, we could write

$$\{r_k\} = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \ldots,$$

and continue the pattern of increasing denominators. For each positive integer k, let

$$I_k = \left(r_k - \frac{1}{2^{k+2}}, r_k + \frac{1}{2^{k+2}}\right) \cap (0, 1).$$

The set $O = \bigcup_{k=1}^{\infty} I_k$ is an open set, and since $\mathbb{Q} \cap (0, 1) \subseteq O$, the set O is dense in $(0, 1)$. We can then write $O = \bigcup_{n=1}^{\infty} (a_n, b_n)$, where the intervals are pairwise disjoint and the number of open intervals is countably infinite (see below). For each positive integer n, let S_n be the set $\{k \in \mathbb{Z}^+ : I_k \subseteq (a_n, b_n)\}$. Since the sets S_n partition $\mathbb{Z}^+$ and $(a_n, b_n) = \bigcup_{k \in S_n} I_k$ for each n, we find that

$$\sum_{n=1}^{\infty}(b_n - a_n) \leq \sum_{n=1}^{\infty}\sum_{k \in S_n} \ell(I_k) = \sum_{k=1}^{\infty} \ell(I_k) \leq \sum_{k=1}^{\infty} \frac{2}{2^{k+2}} = \frac{1}{2},$$

where $\ell(I_k)$ is the length of the interval I_k. This inequality, coupled with the fact that O is dense, shows that the number of intervals (a_n, b_n) is infinite. Hence, the set O has the properties needed for the construction of our function F.

For some people, it makes more sense to obtain the set O in the following manner. Choose an open interval $I_1 \subseteq (0, 1)$ that contains $1/2$ and has length less than $1/4$. Move down the list of rational numbers given above until you reach a number that is not in the interval I_1; call this term r_{k_1}. Choose an open interval $I_2 \subseteq (0, 1) \setminus I_1$ that contains r_{k_1} and has length less than $1/8$. Once again, move down the list of rational numbers until you reach a number that is not in the set $I_1 \cup I_2$; call this term r_{k_2}. Choose an open interval $I_3 \subseteq (0, 1) \setminus (I_1 \cup I_2)$ that contains r_{k_2} and has length less than $1/16$. Continuing in this fashion generates a sequence I_k of disjoint open intervals and the set $O = \bigcup_{k=1}^{\infty} I_k$ has the desired properties.

The set O can also be obtained as a complement of a set formed in much the same fashion as the well-known Cantor set. The set used in Volterra's example is often referred to as the Smith–Volterra–Cantor set (see [**3**]) and has Lebesgue measure $1/2$. Although these Cantor-like sets are quite interesting, they do introduce a new level of abstraction. The previous discussion illustrates one way to obtain a set O without bringing in these new ideas. In other words, a description of these Cantor-like sets is not essential to the construction of our example.

Comments for readers with a more advanced analysis background

We make one final comment for readers with more background in analysis. Using the function f that formed the basis for our example, let

$$V = \{x \in [0, 1] : f(x) \neq 0\} \subseteq \bigcup_{n=1}^{\infty}(c_n, d_n);$$

$$W = \{x \in [0, 1] : f(x) = 0\} = [0, 1] \setminus V.$$

It can be shown that each point of E is a point of density of the set W. To verify this, it is easiest to show that x is a point of dispersion of the set V. The details behind such an argument are essentially the same as those presented here (with the few extra details needed for Goffman's functions) to show that $F'(x) = 0$ for points in the set E. We just need to change the function f_n from the continuous function used earlier to the characteristic function of the interval (c_n, d_n). We leave the details to the interested reader. Consequently, the function f is approximately continuous at each point of E. Since f is continuous at each point of O, it follows that f is approximately continuous on [0, 1]. If F is then defined as $\int_0^x f$, where the integral is a Lebesgue integral, we find that $F' = f$ on [0, 1] by a standard theorem in the theory of Lebesgue integration (see [**8**, p. 227]).

Stromberg [**12**] states "Lebesgue said in his thesis (1902) that this example of Volterra's motivated him to devise a method of integration by which functions with bounded derivatives can be reconstructed from their derivatives." Lebesgue did in fact succeed in devising such an integration process. However, there are still derivatives that are not Lebesgue integrable; the function involving $x^2 \sin(1/x^2)$ on the interval [0, 1] provides one example. A search for an integration process that integrates all derivatives leads to some interesting ideas; see [**8**] for a discussion of the integrals that arise from this investigation.

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Summary. We present an example, different from Volterra's, of a bounded derivative that is not Riemann integrable. The existence of such functions was one of the motivations for Lebesgue to devise a stronger integration process. The goal of the presentation is to keep the ideas at a level appropriate for an undergraduate real analysis student.

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