METRIC SPACES

2.15 Definition A set X, whose elements we shall call *points*, is said to be a *metric space* if with any two points p and q of X there is associated a real number $d(p, q)$, called the *distance* from p to q, such that

(*a*) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$;
(*b*) $d(p, q) = d(q, p)$;
(*c*) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

Any function with these three properties is called a *distance function*, or a *metric*.

2.17 Definition By the *segment* (a, b) we mean the set of all real numbers x such that $a < x < b$.

2.18 Definition Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X.

(*a*) *A neighborhood* of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$, for some $r > 0$. The number r is called the *radius* of $N_r(p)$.
(*b*) A point p is a *limit point* of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$.
(*c*) If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E.
(*d*) E is *closed* if every limit point of E is a point of E.
(*e*) A point p is an *interior* point of E if there is a neighborhood N of p such that $N \subset E$.
(*f*) E is *open* if every point of E is an interior point of E.
(*g*) The *complement* of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.
(*h*) E is *perfect* if E is closed and if every point of E is a limit point of E.
(*i*) E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.
(*j*) E is *dense in* X if every point of X is a limit point of E, or a point of E (or both).

2.19 Theorem *Every neighborhood is an open set.*

2.20 Theorem *If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.*

Corollary *A finite point set has no limit points.*

2.22 Theorem *Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α. Then*

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c). \tag{20}$$

2.23 Theorem *A set E is open if and only if its complement is closed.*

Corollary *A set F is closed if and only if its complement is open.*

2.24 Theorem

(*a*) *For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.*
(*b*) *For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.*
(*c*) *For any finite collection $G_1, \dots, G_n$ of open sets, $\bigcap_{i=1}^n G_i$ is open.*
(*d*) *For any finite collection $F_1, \dots, F_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.*

2.26 Definition If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X, then the *closure* of E is the set $\bar{E} = E \cup E'$.

2.27 Theorem *If X is a metric space and $E \subset X$, then*

(*a*) *$\bar{E}$ is closed,*
(*b*) *$E = \bar{E}$ if and only if E is closed,*
(*c*) *$\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.*

By (*a*) and (*c*), $\bar{E}$ is the *smallest* closed subset of X that contains E.

2.28 Theorem *Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.*

2.30 Theorem *Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.*

COMPACT SETS

2.31 Definition By an *open cover* of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$.

2.32 Definition A subset K of a metric space X is said to be *compact* if every open cover of K contains a *finite* subcover.

More explicitly, the requirement is that if $\{G_\alpha\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \ldots, \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}.$$

2.33 Theorem *Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.*

2.34 Theorem *Compact subsets of metric spaces are closed.*

2.35 Theorem *Closed subsets of compact sets are compact.*

Corollary *If F is closed and K is compact, then $F \cap K$ is compact.*

2.36 Theorem *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.*

Corollary *If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ $(n = 1, 2, 3, \ldots)$, then $\bigcap_1^\infty K_n$ is not empty.*

2.37 Theorem *If E is an infinite subset of a compact set K, then E has a limit point in K.*

2.38 Theorem *If $\{I_n\}$ is a sequence of intervals in R^1, such that $I_n \supset I_{n+1}$ $(n = 1, 2, 3, \ldots)$, then $\bigcap_1^\infty I_n$ is not empty.*

2.39 Theorem *Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}$ $(n = 1, 2, 3, \ldots)$, then $\bigcap_1^\infty I_n$ is not empty.*

2.40 Theorem *Every k-cell is compact.*

2.41 Theorem *If a set E in R^k has one of the following three properties, then it has the other two:*

(*a*) *E is closed and bounded.*
(*b*) *E is compact.*
(*c*) *Every infinite subset of E has a limit point in E.*

2.42 Theorem (Weierstrass) *Every bounded infinite subset of R^k has a limit point in R^k.*

PERFECT SETS

2.43 Theorem *Let P be a nonempty perfect set in R^k. Then P is uncountable.*

Corollary *Every interval $[a, b]$ $(a < b)$ is uncountable. In particular, the set of all real numbers is uncountable.*

CONNECTED SETS

2.45 Definition Two subsets A and B of a metric space X are said to be *separated* if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty, i.e., if no point of A lies in the closure of B and no point of B lies in the closure of A.

A set $E \subset X$ is said to be *connected* if E is *not* a union of two nonempty separated sets.

2.46 Remark Separated sets are of course disjoint, but disjoint sets need not be separated. For example, the interval $[0, 1]$ and the segment $(1, 2)$ are *not* separated, since 1 is a limit point of $(1, 2)$. However, the segments $(0, 1)$ and $(1, 2)$ *are* separated.

The connected subsets of the line have a particularly simple structure:

2.47 Theorem *A subset E of the real line R^1 is connected if and only if it has the following property: If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.*