Linear orderings $(X, <)$ is a linear ordering iff

(1) $<$ is a rel^n from $X \times X$

(2) $(\forall a \in X)$ $\neg(a < a)$ (Irreflexive)

(3) $(\forall a, b, c \in X)$ $(a<b) \& (b<c) \rightarrow (a<c)$
transitive

(4) $(\forall a, b \in X)$ ($a=b$ or $a<b$ or $a>b$)
(only one)

Ordered fields: $(F, +, \cdot, <)$

Field axioms.

$+ : F \times F \rightarrow F$ & $\cdot : F \times F \rightarrow F$ with A, M & D axioms

(Rudin)

Prop 1.14

(a) if $x+y = x+z$ then $y=z$

(b) $x+y = x$ then $y=0$

(c) $x+y=0$ then $y=-x$

(d) $-(-x) = x$

Prop 1.15

(a) $x \neq 0$ & $xy = xz$ then $y=z$

(b) $x \neq 0$ $xy = x$ then $y=1$

(c) $x \neq 0$ $xy = 1$ then $y = 1/x$

(d) $x \neq 0$ $\frac{1}{(1/x)} = x$

1.16

(a) $0 \cdot x = 0$

(b) $x \neq 0$ & $y \neq 0$ then $xy \neq 0$

(c) $(-x)y = -(xy) = x(-y)$

(d) $(-x)(-y) = xy$

★ F is a linear ordering also

(a) $x<y \Rightarrow x+a < y+a$ $\forall x, y, a \in F$

(b) $x>0, y>0 \Rightarrow x \cdot y > 0$

1.18

(a) $x>0$ then $-x<0$

(b) $x>0$ & $y<z$ then $xy<xz$

(c) $x<0$ & $y<z$ then $xy>xz$

(d) $x\neq 0$ then $x^2>0$ i.e. $1>0$

(e) $0<x<y$ then $0<\frac{1}{y}<\frac{1}{x}$

complete ordered field:

$(\forall X\subseteq F)\ [(X\neq\emptyset$ & X is bounded from above in $(F,<)$ $\Rightarrow (\exists a\in F)(\sup(X)=a)]$

* Archimidean property is true for a complete ordered field:

* $\mathbb{R}$ is complete ordered field while $\mathbb{Q}$ isn't.

* If A in countable, then the set of all finite sequences in A are countable.

* X is countable & $Y\subseteq X \Rightarrow Y$ is countable

Sequence & Series

$\langle a_n : n \geq 1 \rangle$

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} S_n = \inf(\{S_n : n \geq 1\})$$

$$S_n = \sup(\{a_k : k \geq n\}),\ t_n = \inf(\{a_k : k \geq n\})$$

So, $\quad S_1 \geq S_2 \geq S_3 \ldots\ldots$

$\quad t_1 \leq t_2 \leq t_3 \ldots\ldots$

Thm: $\langle a_n : n \geq 1 \rangle$ converges iff $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$

Facts: ① $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$

② $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $1 < p < \infty$

③ (Leibnitz test)

if $\langle a_n : n \geq 1 \rangle$ has $a_n \geq 0\ \forall n$ &

$\underbrace{a_n \geq a_{n+1}\ \forall n}$ Then

$\sum_{n=1}^{\infty} (-1)^n a_n$ converges iff $\lim_{n\to\infty} a_n = 0$

④ $e = \sum_{n=0}^{\infty} \frac{1}{n!}$

Thm: $a > 0$ & $n \geq 1$ (n integer) Then

$\exists$ unique $x > 0$ s.t. $x^n = a$.

Lemma: ① $a > 0 \Rightarrow \lim_{n\to\infty} a^{1/n} = 1$

② $\lim_{n\to\infty} n^{1/n} = 1$

Ratio test $\quad \alpha = \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

(1) $\alpha < 1 \Rightarrow \sum_{n=1}^{\infty} |a_n| < \infty$

(2) $\alpha \geq 1 \Rightarrow$ Nothing can be said

Root test $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$

(1) $\alpha < 1 \Rightarrow \sum_{n=1}^{\infty} |a_n| < \infty$

(2) $\alpha > 1 \Rightarrow \lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges

(3) $\alpha = 1 \Rightarrow$ Inconclusive

Power Series

$\sum_{n=0}^{\infty} a_n x^n$, Let $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$

$$R = \begin{cases} \infty & \alpha = 0 \\ 0 & \alpha = \infty \\ 1/\alpha & \text{otherwise} \end{cases}$$

Then (1) $|x| < R \Rightarrow \sum_{n=0}^{\infty} |a_n x^n| < \infty$

(2) $|x| > R \Rightarrow \sum_{n=0}^{\infty} a_n x^n$ diverges

(3) $|x| = R \Rightarrow$ Inconclusive

* $\limsup_{n\to\infty} a_n < \beta \Rightarrow (\exists N \geq 1)(\forall n \geq N)(a_n < \beta)$

Lemma: $a_n \neq 0, \forall n \geq 1$

$$\limsup_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

Riemann rearrangement theorem

(1) $\sum_{n=1}^{\infty} |a_n| < \infty \Rightarrow \forall$ bijection $f: \mathbb{N} \to \mathbb{N}$

$$\sum_{n=1}^{\infty} a_n = \underbrace{\sum_{n=1}^{\infty} a_{f(n)}}$$

↳ rearrangement of $\sum_{n=1}^{\infty} a_n$ via f.

* (2) $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n| = \infty$

So, $\sum_{n=1}^{\infty} a_n$ is conditionally convergent

Then $\forall \alpha \; -\infty \leq \alpha \leq \infty \; \exists$ bijection $f: \mathbb{N}^+ \to \mathbb{N}^+$ s.t.

$$\sum_{n=1}^{\infty} a_{f(n)} = \alpha.$$

Cauchy criterion for summability

for a sequence $\langle a_n : n \geq 1 \rangle$ of real nos.

$\lim_{n\to\infty} a_n$ exists iff $(\forall \varepsilon>0)(\exists N \geq 1)(\forall m,n \geq N)(|a_m - a_n| < \varepsilon)$

New, $\langle a_n : n \geq 1 \rangle$ is cauchy iff

$$\lim_{n\to\infty} \operatorname{diam}(\{a_k : k \geq 1\}) = 0$$

Rudin

A sequence $\{p_n\}$ in metric space X is said to converge if $\exists p \in X$ s.t.
$(\forall \varepsilon>0)(\exists N \geq 1)(\forall n \geq N)(d(p_n, p) < \varepsilon)$

Thm: if $E \subset X$ & p is a limit pt. of E then there is a sequence $\{p_n\}$ in E s.t. $p = \lim_{n\to\infty} p_n$

Go for $d(p_n, p) < \frac{1}{n}$ i.e. $B(p, \frac{1}{n}) \cap E \setminus \{p\} \neq \phi$

Some thms related to compactness

Thm: Let (X,d) be metric space $E \subseteq X$. The E is compact in X iff E is compact in E.

Thm: (X,d) metric space & $E \subseteq X$ if for every open cover of $\mathcal{F}$ of E s.t. each member of $\mathcal{F}$ is an open ball in X, there is a finite $\mathcal{E}_1 \subseteq \mathcal{F}$ s.t. $\cup \mathcal{E}_1 \supseteq E$, then E is compact.

$\boxed{E \subseteq X \text{ is finite} \Rightarrow E \text{ is compact}}$

- A closed interval in $\mathbb{R}$ are compact.

llly $[a,b] \times [c,d] \times \ldots n$ intervals are compact in $\mathbb{R}^n$

Thm: (1) (X,d) is a metric space, then
Assume E is compact, then
(a) E is closed in X
(b) E is bounded in X
(c) $\forall$ infinite $A \subseteq E$,
$\exists\, x \in E$ s.t. x is a limit pt. of A in E.

(2) If X is compact & E is closed in X then E is compact.

Thm: Let $E \subseteq \mathbb{R}^n$, then
(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)
(i) E is closed in $\mathbb{R}^n$ & bounded
(ii) E is compact
(iii) $\forall$ infinite $A \subseteq E$, A has a limit point in E.

(iii) $\rightarrow$ (i) samajh nahi aaya proof.

(X,d) is a metric space
let $f: X \to \mathbb{R}$
$a \in X$

$$\lim_{x \to a} f(x) = C$$

means that $\forall \varepsilon > 0$, $\exists\, \delta_a > 0$ s.t.

$$d(x,a) < \delta_a \Rightarrow |f(x) - C| < \varepsilon$$

$\boxed{\text{OR}}$.

$\forall \varepsilon > 0$ $\exists\, \delta_a > 0$ s.t. $\forall x \in B_X(a, \delta)$ we have

$$|f(x) - C| < \varepsilon$$

* f is continuous at a iff $\lim_{x \to a} f(x) = f(a)$

★ f is uniformly continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d_X(x,y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$

If $f: X \to Y$ where (X, d_X) & (Y, d_Y) are metric spaces

f is continuous at pt. $a \in X$ iff $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in B_X(a, \delta)$,

$$f(x) \in B_Y(f(a), \varepsilon)$$

★ If $U \subseteq Y$ is an open set $\Rightarrow f^{-1}(U)$ is open considering $f: X \to Y$ is continuous.

Lemma: $f: (X, d_X) \to (Y, d_Y)$ is cts. iff $\forall$ open U of Y, $f^{-1}(U)$ is open

Lemma: (X, d_X) is a metric space & $E \subseteq X$. E is compact. Let $f: (X, d_X) \to (Y, d_Y)$ be a cts. function, then $f(E)$ is compact.

Lemma: If E is compact, $E \subseteq X$, then

$F \restriction E$ is uniformly continuous.

Imp: (Ex) Show that

$f: X \to Y$, (X, d_1) & (Y, d_2) metric spaces, the following are equivalent.

① f is uniformly conti. on X

② For all sequences $\langle (x_n, y_n) : n \geq 1 \rangle$ of pairs in X, $d_1(x_n, y_n) \to 0 \Rightarrow d_2(f(x_n), f(y_n)) \to 0$

$y = x^2$ is not uniformly continuous.

$\langle n, n+\frac{1}{n} : n \geq 1 \rangle$

Thm: (X, d_1) compact, (Y, d_2) metric space &

$f: X \to Y$ cts. $\Rightarrow$ f is uniformly cts. on X.

ex: $E \subseteq \mathbb{R}$ compact & non-empty

$\Downarrow$

Both $\sup(E), \inf(E) \in E$

Connected: X is connected iff the only clopen subset of X are $\emptyset$ & X.

Thm: (1) (X, d_1) is connected, (Y, d_2) metric space

$f: X \to Y$ cont.

$\Rightarrow$ $f[X] = \text{range}(f)$ is connected

(2) $E \subseteq \mathbb{R}$ is connected iff E is an interval.

Cantor set: $C \stackrel{\text{def}^n}{=} \bigcap_{k \geq 1} C_k$

$C_0 = [0, 1]$

$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

C_k = disj union of 2^k intervals of length $\frac{1}{3^k}$

(1) C is compact in $\mathbb{R}$

(2) $|C| = |2^{\mathbb{N}}| = |\mathbb{R}|$

(3) $x \in C \iff \exists \langle a_n : n \geq 1 \rangle$ where each $a_n \in \{0, 2\}$ & $x = \sum_{n \geq 1} \frac{a_n}{3^n}$

(4) C is nowhere dense in $\mathbb{R}$.

for (3) $F : \{0,2\}^{\mathbb{N}} \longrightarrow C$

$$F(\langle a_n : n \geq 1 \rangle) = \sum_{n \geq 1} \frac{a_n}{3^n}$$

Subsequence

$\langle y_n : n \geq 1 \rangle$ is called subsequence of $\langle x_n : n \geq 1 \rangle$ iff. for some $n_1 < n_2 < \cdots$

$$\langle y_k : k \geq 1 \rangle = \langle x_{n_k} : k \geq 1 \rangle$$

Thm: (X, d) be a compact metric space. Every sequence in (X, d) has a convergent subsequence

Cor: Every bounded seq. in $\mathbb{R}^n$ has a convergent subsequence.

If p is a prime $\nexists$ no $x \in \mathbb{Q}$ s.t. $x^n = p$

$(F, +, \cdot, <)$

atleast 2 elem

Lineer

$x < y$

α is a cut in $(\mathbb{Q}, <)$ $\alpha \subseteq \mathbb{Q}$ satisfying.

(1) $\alpha \neq \Phi, \mathbb{Q}$

(2) $x \in \alpha$, $y < x$ then $y \in \alpha$

(3) α doesn't have a largest rational no.

$x \in \alpha_r$

* E is dense in X iff $cl(E) = X$

Facts : Let (X,d) be a metric space

(0) ϕ, X clopen in X

(1) $B(y,r)$ is open in X $\forall y \in X$ & $r>0$

(2) $[y \in X$ & $E \subseteq X$ & $y \in E'] \Rightarrow (\forall r>0)$ $(B(y,r) \cap E)$ is infinite.

(3) E is open iff $X \setminus E$ is closed

(4) (a) Union of any family of open sets is open

(b) Intersection of any family of closed sets is closed

(c) Intersection of finite no. of open sets is open

(d) Union of finite closed sets is closed.

* $\langle x_n : n \geq 1 \rangle$ conv. to x iff $\lim_{n\to\infty} d(x, x_n) = 0$

* $\text{diam}(E) = \sup(\{d(x,y) : x, y \in E\})$

* $\langle x_n : n \geq 1 \rangle$ is cauchy iff $\lim_{n\to\infty} \text{diam}(\langle x_n : n \geq k \rangle) = 0$

* if $\langle x_n : n \geq 1 \rangle$ converges to x $\Rightarrow$ it should be cauchy.

* A metric space (X,d) is complete iff every cauchy sequency in (X,d) converges to

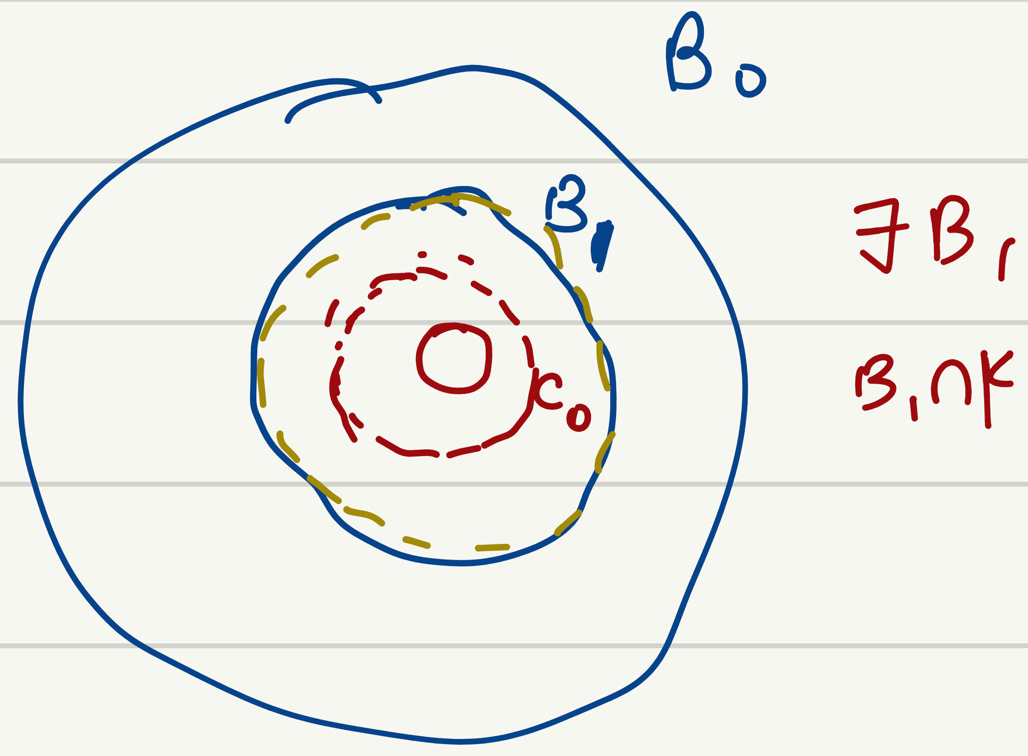
some pt. in X.

Baire category thm:

(X, d) is a complete metric space. E_n is nowhere dense in (X,d) $\forall\, n \geq 1$

Then $X \setminus \bigcup_{n\geq 1} E_n \neq \phi$

$\left(X \setminus \bigcup_{n\geq 1} E_n \right)$ is dense in X.



$(Y, d \restriction (Y \times Y))$

Lemma: (X,d) be a metric space $Y \subseteq X$ non-empty.

(1) For each $y \in Y$, $B_Y(y, r) = B_X(y,r) \cap Y$

(2) $E \subseteq Y$ is open in $Y \Rightarrow \exists$ open $U \subseteq X$ st. $E = U \cap Y$

(3) $U \subseteq X$ is open in $X \Rightarrow U \cap Y$ is open Y