

MTH441A - PROBLEM SET 2

$$17. \sum_{i=1}^n (y_i - \bar{y})^2.$$

$$= \sum_{i=1}^n (y_i - \hat{y}_i + \hat{y}_i - \bar{y})^2.$$

$$= \sum_{i=1}^n \left\{ (y_i - \hat{y}_i)^2 + \sum_{i=1}^n (\hat{y}_i - \bar{y})^2 + 2 \sum_{i=1}^n (y_i - \hat{y}_i)(\hat{y}_i - \bar{y}) \right\}.$$

$$\cancel{\sum_{i=1}^n (y_i - \hat{y}_i)^2}$$

$$\text{Now, } \sum_{i=1}^n (y_i - \hat{y}_i)(\hat{y}_i - \bar{y}).$$

$$\cancel{\sum_{i=1}^n (y_i - \hat{y}_i - y_i + \bar{y})}$$

$$= \sum_{i=1}^n \hat{y}_i (y_i - \hat{y}_i) - \bar{y} \sum_{i=1}^n (y_i - \hat{y}_i).$$

$$= \sum_{i=1}^n \hat{y}_i (y_i - \hat{y}_i) - 0 \quad \left[ \text{since } \sum_{i=1}^n (y_i - \hat{y}_i) = \sum_{i=1}^n e_i = 0 \right]$$

$$= \sum_{i=1}^n e_i \hat{y}_i$$

$$= \sum_{i=1}^n e_i (\hat{\beta}_0 + \hat{\beta}_1 x_{1i} + \hat{\beta}_2 x_{2i} + \dots + \hat{\beta}_p x_{pi}).$$

$$= \hat{\beta}_0 \sum e_i + \hat{\beta}_1 \sum x_{1i} e_i + \dots + \hat{\beta}_p \sum x_{pi} e_i$$

$$= 0.$$

$$\text{Hence, } \boxed{\sum (y_i - \bar{y})^2 = \sum (y_i - \hat{y}_i)^2 + \sum (\hat{y}_i - \bar{y})^2}. \quad [\text{Proved}].$$

$$\begin{aligned}
 27 \quad \sum (y_i - \bar{y})^2 &= \sum y_i^2 - n \bar{y}^2 \\
 &= \sum y_i^2 - \frac{1}{n} (\sum y_i)^2 \\
 &= \underline{y}' \underline{y} - \frac{1}{n} \left( \sum y_i y_i \right) \\
 &= \underline{y}' \underline{y} - \frac{1}{n} \underline{y}' \underline{1} \underline{1}^T \underline{y} \\
 &= \underline{y}' \underline{y} - \underline{y}' \left( \frac{1}{n} \underline{1} \underline{1}^T \right) \underline{y}
 \end{aligned}$$

$$\boxed{A_3 = \frac{1}{n} \underline{1} \underline{1}^T}$$

$$\begin{aligned}
 \text{Now } \sum (y_i - \hat{y}_i)^2 &= (\underline{y} - \hat{\underline{y}})^T (\underline{y} - \hat{\underline{y}}) \\
 &= (\underline{y} - \underline{X} \hat{\underline{\beta}})^T (\underline{y} - \underline{X} \hat{\underline{\beta}})
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } (\underline{y} - \underline{X} \hat{\underline{\beta}}) &= \underline{y} - \underline{X} (\underline{X}^T \underline{X})^{-1} \underline{X}^T \underline{y} \\
 &= (\underline{I} - \underline{X} (\underline{X}^T \underline{X})^{-1} \underline{X}^T) \underline{y} \\
 &= \underline{H} \underline{y}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \sum (y_i - \hat{y}_i)^2 &= (\underline{H} \underline{y})^T \underline{H} \underline{y} \\
 &= \underline{y}^T (\underline{H}^T \underline{H}) \underline{y} \quad \text{R.} \\
 &= \underline{y}^T (\underline{H} \cdot \underline{H}) \underline{y} \quad [\underline{H}^T = \underline{H}] \\
 &= \underline{y}^T (\underline{H}^2) \underline{y} \quad [\underline{H}^2 = \underline{H}] \\
 &= \underline{y}^T \underline{H} \underline{y}
 \end{aligned}$$

$$\therefore \boxed{A_1 = \underline{H} = (\underline{I} - \underline{X} (\underline{X}^T \underline{X})^{-1} \underline{X}^T)}$$

$$\sum (\hat{y}_i - \bar{y})^2 = \sum_{i=1}^n \hat{y}_i^2 - n\bar{y}^2$$

$$= \sum_{i=1}^n (y_i - \bar{y})^2 - \sum_{i=1}^n (y_i - \hat{y}_i)^2$$

$$[\bar{y} = \frac{1}{n} \sum_{i=1}^n \hat{y}_i = \frac{1}{n} \sum_{i=1}^n (y_i - e_i) = \bar{y} - \bar{e} = \bar{y}]$$

$$= (\hat{y}' \hat{y}) - n\bar{y}^2$$

$$= y'(I-H)y - y'(\frac{1}{n} \mathbf{1} \mathbf{1}^T) y$$

$$= y'(I-H - \frac{1}{n} \mathbf{1} \mathbf{1}^T) y$$

$$\therefore \boxed{A_2 = (I-H - \frac{1}{n} \mathbf{1} \mathbf{1}^T)}$$

$$\hat{y}' \hat{y} = (X\hat{\beta})'(X\hat{\beta}) = y'(I-H)(I-H)y = y'(I-2H+H^2)y = y'(I-2H+H)y = y'(I-H)y$$

37. ~~the~~  $A_1 = H$   
 $A_2 = I-H - \frac{1}{n} \mathbf{1} \mathbf{1}^T$   
 $A_3 = \frac{1}{n} \mathbf{1} \mathbf{1}^T$

$$\therefore \boxed{A_1 + A_2 + A_3 = I}$$

$$A_1^2 = H^2 = H = A_1$$

$$A_3^2 = \frac{1}{n^2} \mathbf{1} \mathbf{1}^T \cdot \mathbf{1} \mathbf{1}^T = \frac{1}{n^2} \cdot n \cdot \mathbf{1} \mathbf{1}^T = \frac{1}{n} \mathbf{1} \mathbf{1}^T = A_3$$

$$A_2^2 = (I-H - \frac{1}{n} \mathbf{1} \mathbf{1}^T)^2$$

$$= I + H^2 + \frac{1}{n} \mathbf{1} \mathbf{1}^T - 2H - \frac{2}{n} \mathbf{1} \mathbf{1}^T + \frac{2}{n} H \cdot \mathbf{1} \mathbf{1}^T$$

$$= I - H - \frac{1}{n} \mathbf{1} \mathbf{1}^T + \frac{2}{n} (I - X(X^T X)^{-1} X^T) \mathbf{1} \mathbf{1}^T$$

$$= I - H - \frac{1}{n} \mathbf{1} \mathbf{1}^T + \frac{2}{n} \left[ \mathbf{1} \mathbf{1}^T - \mathbf{1} \mathbf{1}^T \right] \left[ \begin{array}{l} \text{Since} \\ \mathbf{x} (\mathbf{x}^T \mathbf{x})^{-1} \mathbf{x}^T \mathbf{1} = \mathbf{1} \end{array} \right]$$

$$= I - H - \frac{1}{n} \mathbf{1} \mathbf{1}^T$$

Hence,  $\text{tr}(A_1) = \pi_1 = \text{tr}(H)$ .

$$= \text{tr}(I - \mathbf{x} (\mathbf{x}^T \mathbf{x})^{-1} \mathbf{x}^T)$$

$$= n - \text{tr}(\mathbf{x} (\mathbf{x}^T \mathbf{x})^{-1} \mathbf{x}^T)$$

$$= n - \text{tr}((\mathbf{x}^T \mathbf{x}) (\mathbf{x}^T \mathbf{x})^{-1})$$

$$= n - \text{tr}(I_p)$$

$$= n - p.$$

$$\text{tr}(A_2) = \pi_2 = \text{tr}\left(I - H - \frac{1}{n} \mathbf{1} \mathbf{1}^T\right)$$

$$= n - (n - p) - \text{tr}\left(\frac{1}{n} \mathbf{1} \mathbf{1}^T\right)$$

$$= p - \frac{1}{n} \begin{bmatrix} 1 & 1 & \dots & 1 \\ & & \ddots & \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

$$= p - \frac{1}{n} \times n$$

$$= p - 1$$

$$\mathbf{y}^T \mathbf{y} = \mathbf{y}^T A_1 \mathbf{y}$$

$$+ \mathbf{y}^T A_2 \mathbf{y}$$

$$+ \mathbf{y}^T A_3 \mathbf{y}$$

$$\text{tr}(A_3) = \pi_3 = 1.$$

Now,  $\mathbf{y}^T A_1 \mathbf{y}$

$$= (\mathbf{y} - \mathbf{x} \beta)^T A_1 (\mathbf{y} - \mathbf{x} \beta)$$

(check).

$$\pi_1 + \pi_2 + \pi_3 = n - p + p - 1 + 1 = n$$

$$\therefore \frac{RSS}{\sigma^2} \sim \chi^2_{n-p} \quad \& \quad \frac{SSR}{\sigma^2} \sim \chi^2_{p-1}.$$

...



$$47 \quad (n-p-1) S_{ei}^2 = \sum_{\substack{j=1 \\ j \neq i}}^n (y_j - \hat{y}_j(\omega))^2.$$

$$= \sum_{j=1}^n [y_j - \tilde{x}_j' \hat{\beta}(\omega)]^2 \quad \text{--- } \textcircled{*}$$

Now,  $\hat{\beta}(\omega) = (x_{(i)}' x_{(i)})^{-1} x_{(i)}' y_{(i)}.$

$$(x_{(i)}' x_{(i)})^{-1} = (x'x)^{-1} + \frac{(x'x)^{-1} \tilde{x}_i \tilde{x}_i' (x'x)^{-1}}{1 - h_{ii}} \quad \text{[Deriva in class].}$$

$$\hat{\beta}(\omega) = \left[ (x'x)^{-1} + \frac{(x'x)^{-1} \tilde{x}_i \tilde{x}_i' (x'x)^{-1}}{1 - h_{ii}} \right] (x'y - y_i \tilde{x}_i)$$

$$= \hat{\beta} - (x'x)^{-1} y_i \tilde{x}_i + \frac{(x'x)^{-1} \tilde{x}_i \tilde{x}_i' \hat{\beta}}{1 - h_{ii}}$$

$$- \frac{y_i [(x'x)^{-1} \tilde{x}_i \tilde{x}_i' (x'x)^{-1} \tilde{x}_i \hat{\beta}]}{1 - h_{ii}}$$

$$= \hat{\beta} - (x'x)^{-1} y_i \tilde{x}_i + \frac{(x'x)^{-1} \tilde{x}_i \hat{y}_i}{1 - h_{ii}}$$

$$- \frac{y_i (x'x)^{-1} \tilde{x}_i h_{ii}}{1 - h_{ii}}$$

$$= \hat{\beta} - (x'x)^{-1} y_i \tilde{x}_i + \frac{(x'x)^{-1} \tilde{x}_i}{1 - h_{ii}} (\hat{y}_i - h_{ii} y_i)$$

$$= \hat{\beta} + \frac{(x'x)^{-1} \tilde{x}_i}{1 - h_{ii}} \hat{y}_i - (x'x)^{-1} \tilde{x}_i y_i \left[ 1 + \frac{h_{ii}}{1 - h_{ii}} \right]$$

$$= \hat{\beta} - \frac{(x'x)^{-1} \tilde{x}_i}{1 - h_{ii}} (y_i - \hat{y}_i).$$

$$\Rightarrow \hat{\beta} - \hat{\beta}(\omega) = \frac{(x'x)^{-1} \tilde{x}_i e_i}{1 - h_{ii}} \quad \text{--- } (*)$$

$$\begin{aligned}
\text{Using } (*) \text{ in } (*), &= \sum_{j=1}^n \sum_{j \neq i} [y_j - x_j' \hat{\beta} + \frac{x_j' (x'x)^{-1} x_i e_i}{1-h_{ii}}]^2 \\
&= \sum_{j=1}^n \sum_{j \neq i} \left[ y_j - x_j' \hat{\beta} + \frac{h_{ij} e_i}{1-h_{ii}} \right]^2 \\
&= \sum_{j=1}^n \left[ y_j - x_j' \hat{\beta} + \frac{h_{ij} e_i}{1-h_{ii}} \right]^2 - \left( e_i + \frac{h_{ii} e_i}{1-h_{ii}} \right)^2 \\
&= \sum_{j=1}^n \left[ y_j - x_j' \hat{\beta} + \frac{h_{ij} e_i}{1-h_{ii}} \right]^2 - \frac{e_i^2}{(1-h_{ii})^2} \\
&= \sum_{j=1}^n \left[ \cancel{y_j - x_j' \hat{\beta}} \cdot \left( e_i + \frac{h_{ij} e_i}{1-h_{ii}} \right)^2 - \frac{e_i^2}{(1-h_{ii})^2} \right] \\
&= \sum_{j=1}^n e_j^2 + \frac{e_i^2}{(1-h_{ii})^2} \sum_{j=1}^n h_{ij}^2 + \frac{2e_i}{1-h_{ii}} \sum_{j=1}^n e_j h_{ij} - \frac{e_i^2}{(1-h_{ii})^2} \\
\left[ \text{Now, } H^2 = H \Rightarrow \sum_{j=1}^n h_{ij}^2 = h_{ii} \right] & \\
\cdot \cancel{\sum_{j=1}^n e_j^2 + \frac{e_i^2}{(1-h_{ii})^2} \cdot h_{ii} - \frac{e_i^2}{(1-h_{ii})^2}} & \\
= \sum_{j=1}^n e_j^2 + \frac{e_i^2}{(1-h_{ii})^2} \cdot h_{ii} - \frac{e_i^2}{(1-h_{ii})^2} & \\
\left[ \text{Since, } H \tilde{e} = H(Y - X\hat{\beta}) \right. & \\
= H(Y - (X'X)^{-1} X'Y) & \\
= H(I - H)Y & \\
= (H - H^2)Y = (H - H)Y = 0 & \\
= RSS - \frac{e_i^2}{(1-h_{ii})^2} (1-h_{ii}) & \\
= RSS - \frac{e_i^2}{1-h_{ii}} & \\
\therefore s(e_i^2) = \left[ (n-p) RSS - e_i^2 / (1-h_{ii}) \right] / (n-p-1) \text{ (Proved)} &
\end{aligned}$$

## PROBLEM SET-2

(1-4 previous copy)

5) We know,  $H^2 = H$   $[H = X(X'X)^{-1}X']$

$$\begin{aligned}(I-H)^2 &= I - 2H + H^2 \\ &= I - 2H + H = I - H\end{aligned}$$

i.e.  $\sum_{j=1}^n (e_{ij} - h_{ij})(e_{ji} - h_{ji}) = 1 - h_{ii}$  [Comparing  $(i,i)$ th element from both sides]

$$\Rightarrow \sum_{j=1}^n (e_{ij} - h_{ij})^2 = 1 - h_{ii} \quad [\text{Since } (I-H)^T = I-H]$$

$$\Rightarrow \sum_{j=1}^n e_{ij}^2 + \sum_{j=1}^n h_{ij}^2 - 2 \sum_{j=1}^n e_{ij} h_{ij} = 1 - h_{ii}$$

$$\Rightarrow 1 - 2h_{ii} + \overset{1}{h_{ii}^2} + \sum_{i \neq j} h_{ij}^2 = 1 - h_{ii}$$

$$\Rightarrow (1 - h_{ii})^2 + \overset{2}{\sum_{i \neq j} h_{ij}^2} = 1 - h_{ii} \quad [\text{Proved}]$$

...

7)  $Y_i = \theta_i + \epsilon_i$ ,  $\epsilon_i \stackrel{iid}{\sim} N(0, \sigma^2)$ ,  $i=1(1)4$

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 0$$

$$\Rightarrow \theta_4 = -(\theta_1 + \theta_2 + \theta_3)$$

$$\therefore \begin{cases} Y_1 = \theta_1 + \epsilon_1 \\ Y_2 = \theta_2 + \epsilon_2 \\ Y_3 = \theta_3 + \epsilon_3 \\ Y_4 = -(\theta_1 + \theta_2 + \theta_3) + \epsilon_4 \end{cases}$$

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

$$H_0: \theta_1 - \theta_3 = 0 \quad A = [1 \ 0 \ -1]$$

$$X'X = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = I_3 + \underline{1}\underline{1}^T$$

$$(X'X)^{-1} = I_3 - \frac{\underline{1}\underline{1}^T}{1 + \underline{1}^T \underline{1}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$RSS = Y^T H Y$$

$$H = X(X'X)^{-1}X'$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$$



$$SSE = y^T H y.$$

$$= y^T \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \underline{y}$$

$$= \frac{1}{4} \underline{y}^T \underline{1}^T \underline{1} \underline{y}$$

$$= \frac{1}{4} (\underline{1}' \underline{y})^T \underline{1} \underline{y}$$

$$= \frac{1}{4} (\sum y_i^2).$$

~~81~~

$$\hat{\beta} = (X'X)^{-1} X'y$$

$$= \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \end{bmatrix} y$$

$$A(X'X)^{-1} A^T$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 2$$

$$RSS - SS_R = (A\hat{\beta})^T (A(X^T X)^{-1} A^T)^{-1} A\hat{\beta}$$

$$= \hat{\beta}^T A^T \cdot \frac{1}{2} \cdot A\hat{\beta}$$

$$= \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} [1 \ 0 \ -1] \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \frac{1}{2}$$

$$= \frac{1}{32} \cdot \frac{1}{2} \cdot \begin{bmatrix} 4 \\ 0 \\ -4 \\ 0 \end{bmatrix} [4 \ 0 \ -4 \ 0] \frac{1}{2}$$

$$= \frac{1}{2} \cdot \frac{1}{2} [y_1 \ y_2 \ y_3 \ y_4] \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} [1 \ 0 \ -1 \ 0] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$= \frac{1}{2} (y_1 - y_3)^2$$

$$F = \frac{\frac{(y_1 - y_3)^2}{2} / 1}{\frac{\sum_{i=1}^4 y_i^2}{4} / 1} = \frac{2(y_1 - y_3)^2}{\sum y_i^2}$$

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$$b) X = \begin{bmatrix} 1 & \underline{x}_1 & \dots & \underline{x}_{p-1} \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 1^T & \underline{x}_1^T & \dots & \underline{x}_{p-1}^T \end{bmatrix} \begin{bmatrix} 1 & \underline{x}_1 & \dots & \underline{x}_{p-1} \end{bmatrix} \Rightarrow (X^T X)^{-1} = \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & C^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} n & 0 \\ 0 & C \end{bmatrix} \quad \text{where } c_{ii} = c \quad \forall i$$

$$c_{ij} = \underline{x}_i^T \underline{x}_j$$

$$\begin{aligned} \text{Now, } \frac{1}{p} \sum_{j=0}^{p-1} \text{var}(\hat{\beta}_j) &= \frac{1}{p} \sigma^2 \text{tr}(X^T X)^{-1} \\ &= \frac{1}{p} \sigma^2 \left[ \frac{1}{n} + \text{trace}(C^{-1}) \right] \end{aligned}$$

Let,  $\lambda_1, \dots, \lambda_{p-1}$  are eigen values of  $C$ .

$$\text{trace}(C^{-1}) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \dots + \frac{1}{\lambda_{p-1}}$$

$$\text{Now } \sum \lambda_i = \text{trace}(C) = c(p-1)$$

$$\text{Now, } \min \sum_{i=1}^{p-1} \frac{1}{\lambda_i} \quad \text{given } \sum \lambda_i = (p-1)c.$$

$$f(\lambda) = \sum_{i=1}^{p-1} \frac{1}{\lambda_i} + \alpha [\sum \lambda_i - c(p-1)]$$

$$\frac{\partial f(\lambda)}{\partial \lambda_i} = -\frac{1}{\lambda_i^2} + \alpha = 0 \quad \forall i = 1(1)p-1.$$

$$\Rightarrow \alpha = \frac{1}{\lambda_i^2} \Rightarrow \lambda_i^2 = \frac{1}{\alpha} = \text{const.}$$



Say <sup>all</sup>  $\lambda_i$ 's are equal. ( $= \lambda$  say)

$$\Rightarrow \sum \lambda_i = (p-1)c$$

$$\Rightarrow (p-1)\lambda = (p-1)c \Rightarrow \lambda = c.$$

Then,

$$C = T' D T = c I_{p-1}.$$

[Diagonalization].

[C was a symmetric matrix]  
and  $D = c I_{p-1}$ .

and  $T' T = I_{p-1}$  [since T is orthogonal]

$$\text{then } \underline{x_i}' \underline{x_j} = 0 \quad \forall i, j$$

i.e. columns of X are orthogonal.  
...

$$8) H_0: \beta_1 = \beta_2 = \dots = \beta_{p-1} = 0$$

~~For RSS~~

$$R^2 = \frac{SS_{Reg}}{TSS}$$

$$RSS_R = \sum_{i=1}^n (y_i - \hat{y}_{i(R)})^2$$

$$= \sum_{i=1}^n (y_i - \beta_0)^2$$

$$= \sum_{i=1}^n (y_i - \bar{y})^2 = TSS = y' (I - \frac{1}{n} \mathbf{1}\mathbf{1}^T) y$$

$$RSS = \sum_{i=1}^n (y_i - \hat{y}_i)^2$$

$$= y' A_1 y$$

$$RSS_R - RSS = \sum_{i=1}^n (\hat{y}_i - \bar{y})^2 = SS_{Reg}$$

$$= \sum_{i=1}^n y' (I - H - \frac{1}{n} \mathbf{1}\mathbf{1}^T) y$$

$$= y' A_2 y$$

Then,

$$\frac{\frac{RSS_R - RSS}{r}}{\frac{RSS}{n-p}} \sim F_{r, n-p}$$

$$\text{Here } r = \text{rank} \left( I - H - \frac{1}{n} L L^T \right) = p - 1.$$

$$\therefore \frac{n-p}{p-1} \cdot \frac{RSS_R - RSS}{RSS} \sim F_{r, n-p}$$

$$\Rightarrow \frac{n-p}{p-1} \cdot \frac{SS_{\text{Reg.}}}{RSS} \sim F_{r, n-p}.$$

$$\text{Now, } R^2 = \frac{SS_{\text{Reg.}}}{TSS} \quad \frac{R^2}{1-R^2} = \frac{SS_{\text{Reg.}}}{RSS}.$$

$$\frac{n-p}{p-1} \cdot \frac{SS_{\text{Reg}}}{RSS} \sim F_{p-1, n-p} \quad [r=p-1]$$

$$\Rightarrow \frac{SS_{\text{Reg}}}{RSS} \sim \text{Beta}_{\text{II}} \left( \frac{p-1}{2}, \frac{n-p}{2} \right).$$

(=X<sub>0</sub>, say)

now, if  $X_0 \sim \text{Beta}_{\text{II}} \left( \frac{p-1}{2}, \frac{n-p}{2} \right)$

$$\Rightarrow \frac{1}{1+X_0} \sim \text{Beta}_{\text{I}} \left( \frac{n-p}{2}, \frac{p-1}{2} \right)$$

$$\Rightarrow 1-R^2 \sim \text{Beta}_{\text{I}} \left( \frac{n-p}{2}, \frac{p-1}{2} \right)$$

$$\Rightarrow \boxed{R^2 \sim \text{Beta}_{\text{I}} \left( \frac{p-1}{2}, \frac{n-p}{2} \right)}.$$

$$E(R^2) = \frac{p-1}{p-1+n-p} = \frac{p-1}{n-1}$$

[Proved]