MTH 441: Problems set 2

P 1. Prove that
$$\sum_{i=1}^{n}(y_i-\bar{y})^2=\sum_{i=1}^{n}(\widehat{y}_i-y_i)^2+\sum_{i=1}^{n}(\widehat{y}_i-\bar{y})^2 \tag{1}$$

P 2. Express (1) as
$$\mathbf{y}'\mathbf{y}=\mathbf{y}'\mathrm{A}_1\mathbf{y}+\mathbf{y}'\mathrm{A}_2\mathbf{y}+\mathbf{y}'\mathrm{A}_3\mathbf{y}, \tag{2}$$
where $\mathbf{y}'\mathrm{A}_1\mathbf{y}=\sum_{i=1}^{n}(\widehat{y}_i-y_i)^2=RSS$, $\mathbf{y}'\mathrm{A}_2\mathbf{y}=\sum_{i=1}^{n}(\widehat{y}_i-\bar{y})^2=SS_{\mathrm{R}}$ and $\mathrm{A}_i's$ are appropriate matrices for $i=1,2,3$.

P 3. Verify the Cochran theorem for (2) and deduce the distributions of RSS/σ^2 and SS_{R}/σ^2.

P 4. Show that
$$S_{(i)}^2=\frac{(n-p)MS_{\mathrm{Res}}-e_i^2/(1-h_{ii})}{n-p-1}.$$

P 5. Prove that
$$(1-h_{ii})^2+\sum_{j\neq i}h_{ij}=(1-h_{ii}).$$

P 6. Consider a full rank model with usual assumptions
$$y_i=\beta_0+\beta_1x_{i1}+\cdots+\beta_{p-1}x_{i,p-1}+\epsilon_i \qquad i=1,\ldots,n.$$
Suppose that $\sum_i x_{ij}=0$ and $\sum_i x_{ij}^2=c$ for $j=1,\ldots,p-1$. Prove that
$$\frac{1}{p}\sum_{j=0}^{p-1}\mathrm{Var}[\widehat{\beta}_j]$$
is minimized when the columns of X are mutually orthogonal.

P 7. Given $\mathbf{y}=\boldsymbol{\theta}+\boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon}\sim\mathcal{N}(\mathbf{0},\sigma^2\mathrm{I}_4)$ and $\sum_{i=1}^{4}\theta_i=0$, show that the F–statistics for testing $H:\theta_1=\theta_3$ is
$$\frac{2(y_1-y_3)^2}{(\sum_{i=1}^{4}y_i)^2}.$$

P 8. Suppose that $\beta_1=\cdots=\beta_{p-1}=0$. Find the distribution of R^2 and hence prove that
$$\mathbb{E}[R^2]=\frac{p-1}{n-1}.$$