Mid Sem: MTH441

Question 1: Consider the model

$$y_i = x_i\beta + \epsilon_i$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_n)^T$ are independently distributed with mean 0 and variance $\mathrm{Var}(\epsilon) = V = \mathrm{Diag}(1/\omega_1, \ldots, 1/\omega_n)$.

Find:

(i) Find a general formula for the weighted least-squares estimator of β. (2 marks)

(ii) What is the variance of the weighted least-squares estimator? (2 marks)

(iii) Suppose that $\mathrm{Var}(y_i) = \sigma_i^2$. Using the results of parts (i) and (ii), find the weighted least-squares estimator of β and the variance of this estimator. (3 marks)

(iv) Suppose that $\mathrm{Var}(y_i) = cx_i^2$. Using the results of parts (i) and (ii), find the weighted least-squares estimator of β and the variance of this estimator. (3 marks)

Question 2: Consider the following multiple linear regression model

$$y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_{p-1}x_{i,p-1} + \epsilon_i \quad \text{for } i = 1, \ldots, n$$

where $\{\epsilon_1, \ldots, \epsilon_n\}$ are independently distributed with mean 0 and variance σ^2. Let the regressors be transformed as follows:

$$x_{ij}^* = \frac{x_{ij} - \bar{x}_j}{\sqrt{\sum_{i=1}^{n}(x_{ij} - \bar{x}_j)^2}} \quad \text{for } i = 1, \ldots, n \text{ and } j = 1, \ldots, p-1$$

The transformed model is:

$$y_i = \alpha_0 + \alpha_1 x_{i1}^* + \cdots + \alpha_{p-1}x_{i,p-1}^* + \epsilon_i \quad \text{for } i = 1, \ldots, n$$

Let $\hat{\alpha}_0$ be the least square estimate of α_0. Prove or disprove that:

$$\lim_{n\to\infty} \mathrm{Var}(\hat{\alpha}_0) = 0$$

(5 marks)

Question 3:

If $Y = (Y_1, Y_2)^T \sim N_2(0, \Sigma)$, where

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

prove or disprove that:

$$\left(Y^T \Sigma^{-1} Y - \frac{Y_2^2}{\sigma_{22}} \right) \sim \chi_1^2$$

(5 marks)

Question 4: Consider the following regression model with usual assumptions

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \epsilon_i \quad \text{for } i = 1, \ldots, n$$

(i) Write an appropriate test statistics for testing $H_0 : T\beta = 0$ versus $H_1 : T\beta \neq 0$, where

$$T = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(3 marks)

(ii) What is the distribution of the test statistics you proposed? (2 marks)