PS3

① $R^2_{1;adj}$ denotes adjusted R^2 after removing x_1

$H_0: \beta_1 = 0$, vs $H_1: \beta_1 > 0$. $R^2_{1;adj} = 1-(1-R_1^2)\frac{n-1}{n-p+1}$

$= 1 - \frac{RSS_1}{TSS} \times \frac{n-1}{n-p+1}$

R^2_{adj} is adjusted R^2 for the full model

Then, $1 - R^2_{adj} = \frac{RSS}{TSS} \times \frac{n-1}{n-p}$

$= (1-R^2)\frac{n-1}{n-p}$ $\left[\because R^2 = 1 - \frac{RSS}{TSS}\right]$

$$F = \frac{RSS_1 - RSS}{RSS} \cdot \frac{n-p}{1} = \frac{\frac{RSS_1}{TSS} - \frac{RSS}{TSS}}{\frac{RSS}{TSS}} \cdot (n-p)$$

$$= \frac{(1-R_1^2) - (1-R^2)}{(1-R^2)} (n-p)$$

$$= \left(\frac{1-R_1^2}{1-R^2} - 1\right)(n-p) = \left(\frac{1-R_1^2}{1-R^2_{adj}} \cdot \frac{n-1}{n-p} - 1\right)(n-p)$$

Now, $F \geq 1 \Leftrightarrow \left(\frac{1-R_1^2}{1-R^2_{adj}} \cdot \frac{n-1}{n-p} - 1\right)(n-p) \geq 1$

$$\Leftrightarrow \frac{1-R_1^2}{1-R^2_{adj}} \frac{n-1}{n-p} \geq 1 + \frac{1}{n-p} = \frac{n-p+1}{n-p}$$

$$\Leftrightarrow \frac{1-R_1^2}{1-R^2_{adj}} \geq \frac{n-p+1}{n-1} \Leftrightarrow (1-R_1^2)\left(\frac{n-1}{n-p+1}\right) \geq 1 - R^2_{adj}$$

$$\Leftrightarrow 1 - R^2_{1,adj} \geq 1 - R^2_{adj} \Leftrightarrow \boxed{R^2_{adj} \geq R^2_{1,adj}}$$

②

(2) $\underline{y} = X\underline{\beta} + \underline{\varepsilon}$

LSE of $\underline{\beta}$ is $\rightarrow$ $\hat{\underline{\beta}} = (X'X)^{-1}X'\underline{y}$.

Ridge estimate of $\hat{\underline{\beta}}(k) = (X'X + kI_p)^{-1}X'\underline{y}$

[k is the ~~[illegible]~~ penalty term]

Now, for the linear regression

$$\underline{y}_A = X_A\underline{\beta}_R + \underline{\epsilon} \quad \text{— (*)}$$

$\boxed{\text{LSE of } \beta_R \text{ is}}$

$$\hat{\beta}_R = (X_A'X_A)^{-1}X_A'\underline{y}_A$$

$$= (X'X + kI_p)^{-1}X'\underline{y} = \hat{\beta}(k)$$

So LSE of (*) model is same as Ridge estimate with penalty k.

$$X_A'X_A$$
$$= \begin{bmatrix} X' & \sqrt{k}I_p \end{bmatrix}\begin{bmatrix} X \\ \sqrt{k}I_p \end{bmatrix}$$
$$= X'X + kI_p.$$

$$X_A'\underline{y}_A = \begin{bmatrix} X' & \sqrt{k}I_p \end{bmatrix}\begin{bmatrix} \underline{y} \\ \underline{0}_p \end{bmatrix}$$
$$= X'\underline{y}$$

(3) (a) Given $f(y_i) = \frac{y_i}{x_i}$ $\Rightarrow f(\mu) = \frac{\mu}{x_i}$

So, we have $V(y_i) = \sigma^2 x_i^2 \Rightarrow V\left(\frac{y_i}{x_i}\right) = \sigma^2$

$\Rightarrow V(f(y_i)) = \sigma^2$

$=$ constant

Our transformed model is –

$$\frac{y_i}{x_i} = \beta_0^* + \beta_1^* \frac{1}{x_i} + \varepsilon_i^*$$

$V(\varepsilon_i^*) = V\left(\frac{y_i}{x_i}\right) = \sigma^2 =$ constant

$$\boxed{\begin{aligned} V(f(y_i)) &\approx \left(\frac{\partial f}{\partial \mu}\right)^2 V(y_i) \\ &= \left(\frac{1}{x_i}\right)^2 \sigma^2 x_i^2 \\ &= \sigma^2 = \text{constant} \end{aligned}}$$

So, $f(y_i) = y_i' = \frac{y_i}{x_i}$ is a variance stabilizing transform

(b) Our transformed model is –

$$f(y_i) = \frac{y_i}{x_i} = \beta_0^* + \frac{1}{x_i}\beta_1^* + \varepsilon_i^*, \quad \varepsilon_i^* = \frac{\varepsilon_i}{x_i}$$

Our original model is $\rightarrow$ $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$

(LSE can't be applied) $\rightarrow$

$\Rightarrow \frac{y_i}{x_i} = \frac{\beta_0}{x_i} + \beta_1 + \frac{\varepsilon_i}{x_i}$

(LSE can be applied) $\rightarrow$

comparing we get, $\boxed{\beta_0^* = \beta_1, \quad \beta_1^* = \beta_0}$

Also, comparing LSE's of $\hat{\beta}_0^*, \hat{\beta}_1^*$ with $\hat{\beta}_0, \hat{\beta}_1$

$$\hat{\beta}_1^* = \frac{cov\left(\frac{y}{x}, \frac{1}{x}\right)}{V\left(\frac{1}{x}\right)} \qquad \hat{\beta}_0 = \frac{cov\left(\frac{y}{x}, \frac{1}{x}\right)}{V\left(\frac{1}{x}\right)}$$

$$\hat{\beta}_0^* = \overline{\left(\frac{y}{x}\right)} - \overline{\left(\frac{1}{x}\right)}\hat{\beta}_1^* \qquad \hat{\beta}_1 = \overline{\left(\frac{y}{x}\right)} - \overline{\left(\frac{1}{x}\right)}\hat{\beta}_0$$

comparing LSEs $\hat{\beta}_0^* = \hat{\beta}_1$, $\hat{\beta}_1^* = \hat{\beta}_0$

(c) Now, we will use WLS with $w_i = \frac{1}{x_i^2}$

Let $f(\cdot)$ be the transformation corresponding to the weights $w_i = \frac{1}{x_i^2}$.

$$V(f(y_i)) = w_i V(y_i) = \frac{1}{x_i^2} \sigma^2 x_i^2 = \sigma^2$$

$$\Rightarrow V(f(y_i)) = V(\sqrt{w_i}\, y_i) = V\left(\frac{y_i}{x_i}\right) = \sigma^2 = \text{constant.}$$

So, $f(y_i) = \frac{y_i}{x_i}$. the transformation introduced is same as part (a)

Also, WLS model is –

$$\sqrt{w_i}\, y_i = \sqrt{w_i}\, \beta_0 + \sqrt{w_i}\, \beta_1 x_i + \sqrt{w_i}\, \epsilon_i$$

$$\Rightarrow v_i = \beta_0^* + u_i \beta_1^* + z_i$$

$$\hat{\beta}_1^* = \frac{\text{cov}(u, v)}{V(u)} = \frac{\text{cov}\left(\frac{y}{x}, \frac{1}{x}\right)}{V\left(\frac{1}{x}\right)}$$

$$\hat{\beta}_0^* = \overline{\left(\frac{y}{x}\right)} - \overline{\left(\frac{1}{x}\right)}\, \hat{\beta}_1^*$$

$v_i = \sqrt{w_i}\, y_i = \frac{y_i}{x_i}$

$u_i = \sqrt{w_i} = \frac{1}{x_i}$

$\beta_1^* = \beta_1 x_i$

$\beta_0^* = \sqrt{w_i}\, \beta_0 = \frac{\beta_0}{x_i}$

$z_i = \sqrt{w_i}\, \epsilon_i = \frac{\epsilon_i}{x_i}$

which are same as the transformed model.

(4) (a) $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$, $i = 1, 2, 3$, $j = 1, 2, \ldots n$

$\varepsilon_{ij} \sim N(0, \sigma^2)$

$y_{ij} = \beta_0 + \beta_1 x_{1j} + \beta_2 x_{2j} + \varepsilon_{ij}$

$$x_{1j} = \begin{cases} 1 & \text{for Treatment 1} \\ -1 & \text{for Treatment 2} \\ 0 & \text{o/w.} \end{cases}$$

$$x_{2j} = \begin{cases} 1 & \text{for Treatment 2} \\ -1 & \text{for Treatment 3} \\ 0 & \text{o/w.} \end{cases}$$

Treatment	ANOVA model	Regression Model
1	$y_{1j} = \mu + \tau_1 + \varepsilon_{1j} = \mu_1 + \varepsilon_{1j}$	$y_{1j} = \beta_0 + \beta_1 + \varepsilon_{1j}$
2	$y_{2j} = \mu + \tau_2 + \varepsilon_{2j} = \mu_2 + \varepsilon_{2j}$	$y_{2j} = \beta_0 - \beta_1 + \beta_2 + \varepsilon_{2j}$
3	$y_{3j} = \mu + \tau_3 + \varepsilon_{3j} = \mu_3 + \varepsilon_{3j}$	$y_{3j} = \beta_0 - \beta_2 + \varepsilon_{3j}$

Thus, $\mu_1 = \beta_0 + \beta_1$

$\mu_2 = \beta_0 - \beta_1 + \beta_2$

$\mu_3 = \beta_0 - \beta_2$.

Now, $3\beta_0 = \mu_1 + \mu_2 + \mu_3 \Rightarrow \boxed{\beta_0 = \frac{\mu_1 + \mu_2 + \mu_3}{3}} \boxed{= \bar{\mu}}$

$\mu_1 = \bar{\mu} + \beta_1 \Rightarrow \boxed{\beta_1 = \mu_1 - \bar{\mu}}$

$\mu_3 = \beta_0 - \beta_2 = \bar{\mu} - \beta_2$

$\Rightarrow \beta_2 = \bar{\mu} - \mu_3$

$\beta_2 = \mu_2 - \bar{\mu} + \mu_1 - \bar{\mu}$

$= \mu_1 + \mu_2 - 2\bar{\mu}$

$= \bar{\mu} - \mu_3$.

(b) $\underline{y} = (y_{11}, \ldots, y_{1n}, y_{21}, \ldots, y_{2n}, y_{31}, \ldots, y_{3n})'$

$$X = \begin{bmatrix} 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 0 \\ 1 & -1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & -1 & 1 \\ 1 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{bmatrix}_{3n \times 3} \begin{array}{l} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \to n \text{ rows} \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \to n \text{ rows} \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} \to n \text{ rows} \end{array}$$

(c) $H_0: \tau_1 = \tau_2 = \tau_3 = 0 \Leftrightarrow \beta_1 = \beta_1 + \beta_2 = \beta_2 = 0$

$\Rightarrow \beta_1 = \beta_2 = 0$

To find RSS,

$$\frac{\partial}{\partial \beta_0} \sum_i \sum_j (y_{ij} - \beta_0 - \beta_1 x_{1j} - \beta_2 x_{2j})^2 = 0$$

$$\Rightarrow -2 \sum_i \sum_j (y_{ij} - \beta_0 - \beta_1 x_{1j} - \beta_2 x_{2j}) = 0$$

$$\Rightarrow \sum_i \sum_j y_{ij} - 3n\beta_0 - \beta_1 (n-n) - \beta_2 (n-n) = 0.$$

$$\Rightarrow \hat{\beta}_0 = \frac{1}{3n} \sum_i \sum_j y_{ij} = \bar{y}_{00}$$

$$\boxed{\begin{aligned} &\sum_{i=1}^{3} \sum_j x_{ij} \\ &= \sum_{j=1}^{n} x_{ij}\cdot(1) + \sum_{j=1}^{n} x_{ij}(-1) \\ &\quad\ \ (\text{for } i=1) \qquad (\text{for } i=2) \\ &+ \sum_{j=1}^{n} x_{ij}(0) \\ &\quad\ \ (\text{for } i=3) \end{aligned}}$$

$$\frac{\partial}{\partial \beta_1} \sum_i \sum_j (y_{ij} - \beta_0 - \beta_1 x_{1j} - \beta_2 x_{2j})^2 = 0$$

$$\Rightarrow -2 \sum_i \sum_j x_{1j} (y_{ij} - \beta_0 - \beta_1 x_{1j} - \beta_2 x_{2j}) = 0$$

$$\Rightarrow \sum_{j=1}^{n} y_{1j} - \sum_{j=1}^{n} y_{2j} - \beta_0 (n-n) - \beta_1 (2n) - \beta_2 \sum_i \sum_j x_{1j} x_{2j} = 0$$

$$\Rightarrow n\bar{y}_{10} - n\bar{y}_{20} - 2n\beta_1 + n\beta_2 = 0$$

$$\Rightarrow 2\beta_1 - \beta_2 = \bar{y}_{10} - \bar{y}_{20}.$$

$$\boxed{x_{1j} x_{2j} = \begin{cases} 1 & \text{for } i=1 \\ -1 & \text{for } i=2 \\ -1 & \text{for } i=3 \end{cases}}$$

$$\frac{\partial}{\partial \beta_2} \sum_i \sum_j (y_{ij} - \beta_0 - \beta_1 x_{1j} - \beta_2 x_{2j})^2 = 0$$

$$\Rightarrow -2 \sum_i \sum_j x_{2j} (y_{ij} - \beta_0 - \beta_1 x_{1j} - \beta_2 x_{2j}) = 0$$

$$\Rightarrow \sum_{j=1}^{n} y_{2j} - \sum_{j=1}^{n} y_{3j} - \beta_0 (n-n) - \beta_1 (-n) - \beta_2 (2n) = 0$$

$$\Rightarrow \beta_1 - 2\beta_2 = \bar{y}_{30} - \bar{y}_{20}$$

$$3\hat{\beta}_2 = -2\bar{y}_{30} + 2\bar{y}_{20} + \bar{y}_{10} - \bar{y}_{20}$$

$$\Rightarrow \hat{\beta}_2 = \frac{1}{3}\left(\bar{y}_{10} + \bar{y}_{20} - 2\bar{y}_{30}\right)$$

$$\hat{\beta}_1 = \bar{y}_{30} - \bar{y}_{20} + \frac{2}{3}\bar{y}_{10} + \frac{2}{3}\bar{y}_{20} - \frac{4}{3}\bar{y}_{30}$$

$$\Rightarrow \hat{\beta}_1 = \frac{1}{3}\left(2\bar{y}_{10} - \bar{y}_{20} - \bar{y}_{30}\right)$$

$$RSS = \sum_j \left(y_{1j} - \bar{y}_{00} - \frac{2}{3}\bar{y}_{10} + \frac{1}{3}\bar{y}_{20} + \frac{1}{3}\bar{y}_{30}\right)^2$$

$$+ \sum_j \left(y_{2j} - \bar{y}_{00} + \frac{2}{3}\bar{y}_{10} - \frac{1}{3}\bar{y}_{20} - \frac{1}{3}\bar{y}_{30} - \frac{1}{3}\bar{y}_{10} - \frac{1}{3}\bar{y}_{20} + \frac{2}{3}\bar{y}_{30}\right)^2$$

$$+ \sum_j \left(y_{3j} - \bar{y}_{00} + \frac{1}{3}\bar{y}_{10} + \frac{1}{3}\bar{y}_{20} - \frac{2}{3}\bar{y}_{30}\right)^2$$

$$\boxed{\bar{y}_{00} = \frac{1}{3}\bar{y}_{10} + \frac{1}{3}\bar{y}_{20} + \frac{1}{3}\bar{y}_{30}}$$

$$= \sum_j \left(y_{1j} - \bar{y}_{10}\right)^2 + \sum_j \left(y_{2j} - \bar{y}_{20}\right)^2 + \sum_j \left(y_{3j} - \bar{y}_{30}\right)^2$$

$$= \sum_i \sum_j \left(y_{ij} - \bar{y}_{i0}\right)^2 \sim \sigma^2 \chi^2_{3n-3}$$

$\Leftrightarrow$ which is same as ANOVA (one way model)

$$SSE_{H_0} = \sum_i \sum_j \left(y_{ij} - \hat{\beta}_0\right)^2 = \sum_i \sum_j \left(y_{ij} - \bar{y}_{00}\right)^2$$

$$= \sum_i \sum_j \left(y_{ij} - \bar{y}_{i0} + \bar{y}_{i0} - \bar{y}_{00}\right)^2$$

$$= \sum_i \sum_j \left(y_{ij} - \bar{y}_{i0}\right)^2 + \sum_{i=1}^{3} n\left(\bar{y}_{i0} - \bar{y}_{00}\right)^2 \sim \sigma^2 \chi^2_{3n-1}.$$

$$SS_{Reg} = SSE_{H_0} - RSS = \sum_{i=1}^{3} n\left(\bar{y}_{i0} - \bar{y}_{00}\right)^2 \sim \sigma^2 \chi^2_{3-1}$$

$$F = \frac{SS_{Reg}/2}{RSS/3n-3} \sim F_{2,3n-3}.$$

SS_{Reg} = SS due to treatment which is same as usual one way ANOVA.

(5) $y_{ij} = \mu + \alpha_i + \epsilon_{ij}$, $i = 1 .. k$, $j = 1, -- n_i$

One way ANOVA. $\boxed{\sum_{i=1}^{k} n_i \alpha_i = 0}$ $\epsilon_{ij} \sim N(0, \sigma^2)$

(a) The linear model for this case is —

$$\underline{y} = X\underline{\beta} + \underline{\epsilon}$$

$$\underline{\beta} = (\mu, \alpha_1, -- \alpha_k)'$$

$n \times \overline{k+1}$

$$X = \begin{bmatrix} \underline{1}_{n_1} & \underline{1}_{n_1} & \underline{0} & \cdots & \underline{0} \\ \underline{1}_{n_2} & \underline{0} & \underline{1}_{n_2} & \cdots & \underline{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \underline{1}_{n_k} & \underline{0} & \underline{0} & \cdots & \underline{1}_{n_k} \end{bmatrix}$$

we know, rank $(X) = k$.

so, X is not full column rank.

Thus estimable parametric function is —

$$\phi = c_0\mu + \sum_{i=1}^{k} c_i \alpha_i$$

$X \begin{pmatrix} 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} = \underline{0} \Rightarrow \begin{pmatrix} 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} \in N(X)$.

As, rank $(X) = \#$ column $-1 = (k+1)-1$

so, we have one linear restriction so that $\underline{c}^T \in R(X)$.

$$(c_0\ c_1 \ldots\ c_k)\begin{pmatrix} 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} = 0 \Rightarrow c_0 = \sum_{i=1}^{k} c_i$$

Every row X follows linear restriction, $\underline{x}_{i0}' \begin{pmatrix} 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} = \underline{0}$

→ Also, any $\sum_{i=1}^{k} c_i \alpha_i$ is estimable if $\sum_{i=1}^{k} c_i = 0$

↳ Those are called <u>contrasts</u>. ■

→ To find residual sum of squares (RSS):-

$$\frac{\partial}{\partial \mu} \sum_i \sum_j \varepsilon_{ij}^2 = 0$$

$$\Rightarrow \frac{\partial}{\partial \mu} \sum_i \sum_j (y_{ij} - \mu - \alpha_i)^2 = 0 \Rightarrow -2 \sum_i \sum_j (y_{ij} - \mu - \alpha_i) = 0$$

$$\Rightarrow \sum_i \sum_j y_{ij} - n\mu - \underbrace{\sum_i n_i \alpha_i}_{=0} = 0 \Rightarrow \boxed{\hat{\mu} = \frac{1}{n} \sum_i \sum_j y_{ij} = \bar{y}_{00}}$$

$$\frac{\partial}{\partial \alpha_i} \sum_i \sum_j \varepsilon_{ij}^2 = 0$$

$$\Rightarrow \frac{\partial}{\partial \alpha_i} \sum_i \sum_j (y_{ij} - \mu - \alpha_i)^2 = 0 \Rightarrow -2 \sum_i \sum_j (y_{ij} - \mu - \alpha_i) = 0$$

$$\Rightarrow \sum_j y_{ij} - n_i\mu - n_i \alpha_i = 0 \Rightarrow \hat{\alpha}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij} - \hat{\mu}$$

$$\Rightarrow \boxed{\hat{\alpha}_i = \bar{y}_{i0} - \bar{y}_{00}}$$

$$\boxed{\text{where, } \bar{y}_{i0} = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}}$$

Now, $RSS = \min_{\mu, \alpha_i} \sum_i \sum_j \varepsilon_{ij}^2 = \min_{\mu, \alpha_i} \sum \sum (y_{ij} - \mu - \alpha_i)^2$

$$\Rightarrow RSS = \sum_i \sum_j (y_{ij} - \hat{\mu} - \hat{\alpha}_i)^2 = \sum_i \sum_j (y_{ij} - \bar{y}_{00} - \bar{y}_{i0} + \bar{y}_{00})^2$$

$$\Rightarrow \boxed{RSS = \sum_i \sum_j (y_{ij} - \bar{y}_{i0})^2}$$

(b) We have, $RSS = \sum_i \sum_j (y_{ij} - \bar{y}_{i0})^2$

$$= \sum_i \sum_j \left(\mu + \alpha_i + \varepsilon_{ij} - \frac{1}{n_i} \sum_j (\mu + \alpha_i + \varepsilon_{ij}) \right)^2$$

$$= \sum_i \sum_j \left(\varepsilon_{ij} - \frac{1}{n_i} \sum_j \varepsilon_{ij} \right)^2 = \sum_i \sum_j (\varepsilon_{ij} - \bar{\varepsilon}_{i0})^2$$

We have, $\varepsilon_{ij} \sim N(0, \sigma^2) \Rightarrow \bar{\varepsilon}_{i0} \sim N(0, \frac{\sigma^2}{n_i})$

$$\sum_{j=1}^{n_i} (\varepsilon_{ij} - \bar{\varepsilon}_{i0})^2 \sim \chi^2_{n_i - 1} \sigma^2 \text{ — Indeptly}$$

$$\Rightarrow \frac{1}{\sigma^2}\sum_{j=1}^{n_i}(\varepsilon_{ij}-\bar{\varepsilon}_{i0})^2 \sim \chi^2_{n_i-1}$$

$$n=\sum_{i=1}^{k} n_i$$

$$\Rightarrow \frac{1}{\sigma^2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(\varepsilon_{ij}-\bar{\varepsilon}_{i0})^2 \sim \chi^2_{\sum_{i=1}^{k}(n_i-1)} \equiv \chi^2_{n-k}$$

$$\Rightarrow \boxed{\frac{RSS}{\sigma^2} \sim \chi^2_{n-k}}.$$

(6) $H_0: \alpha_1=\alpha_2=\cdots=\alpha_k=0$

vs $H_1: \alpha_i \neq \alpha_j$ for atleast one (i,j).

$$\cancel{SS_{Reg} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij} - \bar{y}_{00}}$$

$$SS_{Reg} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\hat{y}_{ij}-\bar{y})^2 \qquad \boxed{\bar{y}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij} = \bar{y}_{i0}}$$

$$\left(\hat{y}_{ij} = \hat{\mu}+\hat{\alpha}_i = \bar{y}_{i0},\ \bar{y} = \frac{1}{n}\sum_{i=1}^{k}\sum_{j=1}^{n_i} y_{ij} = \bar{y}_{00}\right)$$

So, we have, $SS_{Reg} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\bar{y}_{i0}-\bar{y}_{00})^2$

$$= \sum_{i=1}^{k} n_i(\bar{y}_{i0}-\bar{y}_{00})^2.$$

Now, calculating SSE_{H_0}, under $H_0: \alpha_i = 0\ \forall i$.

$$SSE_{H_0} = \min_{\mu}\sum_i\sum_j \varepsilon_{ij}^2 = \min_{\mu}(y_{ij}-\mu)^2$$

$$\frac{\partial}{\partial\mu}\sum_i\sum_j(y_{ij}-\mu)^2 = 0 \Rightarrow -2\sum_i\sum_j(y_{ij}-\mu)=0$$

$$\Rightarrow \boxed{\hat{\mu} = \frac{1}{n}\sum_i\sum_j y_{ij} = \bar{y}_{00}}$$

$$SSE_{H_0} = \sum_i \sum_j (y_{ij} - \hat{\mu})^2 = \sum_i \sum_j (y_{ij} - \bar{y}_{00})^2 = \sum_i \sum_j (\epsilon_{ij} - \bar{\epsilon}_{00})^2$$

$$= \sum_i \sum_j (y_{ij} - \bar{y}_{i0} + \bar{y}_{i0} - \bar{y}_{00})^2$$

$$= \sum_i \sum_j (y_{ij} - \bar{y}_{i0})^2 + \sum_i n_i (\bar{y}_{i0} - \bar{y}_{00})^2$$

$$= RSS + SS_{Reg}$$

$\cancel{SSE_{H_0} - RSS}$

$\cancel{\bar{y}_{i0} \sim N\left(0, \frac{\sigma^2}{n_i}\right) \ \forall\, i = 1, \ldots, k}$

$\cancel{\bar{y}_{00} = \frac{1}{k} \sum_{i=1}^{k} \bar{y}_{i0} \sim N\left(0, \frac{\sigma^2}{n}\right)}$

$$\epsilon_{ij} \sim N(0, \sigma^2), \quad \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \epsilon_{ij} \sim N\left(0, \frac{\sigma^2}{n}\right)$$

$$(= \bar{\epsilon}_{00})$$

and $\sum_i \sum_j (\epsilon_{ij} - \bar{\epsilon}_{00})^2 \sim \sigma^2 \chi^2_{n-1}$

$$\Rightarrow \frac{1}{\sigma^2} \sum_i \sum_j (y_{ij} - \bar{y}_{00})^2 = \frac{SSE_{H_0}}{\sigma^2} \sim \chi^2_{n-1}$$

$$\frac{SS_{Reg}}{\sigma^2} = \frac{SSE_{H_0} - RSS}{\sigma^2} \sim \chi^2_{(n-1)-(n-k)} \overset{P}{\equiv} \chi^2_{k-1}$$

~~A/S as, $H_0 : \alpha_1 = \ldots = \alpha_k \Leftrightarrow L\underline{\beta} = \underline{0}$~~

$\cancel{L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & -1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \end{bmatrix}}$

so, we have $\frac{SS_{Reg}}{\sigma^2} \sim \chi^2_{k-1}$

$$F = \frac{\frac{1}{\sigma^2}(SS_{H_0} - RSS)/(k-1)}{\frac{1}{\sigma^2} RSS/(n-k)} = \frac{SS_{Reg}/(k-1)}{RSS/(n-k)} \sim F_{(k-1),(n-k)}$$