MTH 441A: Problems set 3

P 1. Let $R^2_{1,Adj}$ denotes the adjusted R^2 when a regressor say x_1 is removed from the model consist of $p-1$ regressors. Let F be the test statistics for testing the hypothesis $H_0 : \beta_1 = 0$ versus $\beta_1 \neq 0$. Show that $R^2_{Adj} \geq R^2_{1,Adj}$ if and only if $F \geq 1$ where R^2_{Adj} denotes the adjusted R^2 for the full model.

P 2. Consider the following regression model

$$\boldsymbol{y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{1}$$

Define

$$X_A = \begin{pmatrix} X \\ \sqrt{k}\mathrm{I}_p \end{pmatrix} \qquad \boldsymbol{y}_A = \begin{pmatrix} \boldsymbol{y} \\ \mathbf{0}_p \end{pmatrix}$$

where I_p is a square matrix of order $p \times p$ and $\mathbf{0}_p$ is a $p \times 1$ vector of zeros. Show that the ridge estimator is the LSE for the following model

$$\boldsymbol{y}_A = X_A\boldsymbol{\beta}_R + \boldsymbol{\epsilon}.$$

P 3. Consider the simple linear regression model $y_i = \beta_0 + \beta_1 x + \epsilon_i$, where $Var(\epsilon_i) = \sigma^2 x_i^2$.

(a) Suppose that we use the transformations $y' = y/x$ and $x' = 1/x$. Is this a variance–stabilizing transformation?

(b) What are the relationships between the parameters in the original and transformed models?

(c) Suppose we use the method of weighted least squares with $w_i = 1/x_i^2$. Is this equivalent to the transformation introduced in part (a)?

P 4. Consider the following one–way ANOVA model

$$y_{ij} = \mu + \tau_i + \epsilon_{ij}; \quad i = 1,2,3, \text{ and } j = 1,\ldots,n.$$

where μ is a parameter common to all 3 treatments (usually called the grand mean), τ_i is a parameter that represents the effect of the ith treatment, and ϵ_{ij} is an NID$(0, \sigma^2)$ error component. The associated regression model is

$$y_{ij} = \beta_0 + \beta_1 x_{1j} + \beta_2 x_{2j} + \epsilon_{ij}; \quad i = 1,2,3, \text{ and } j = 1,\ldots,n.$$

where

$$x_{1j} = \begin{cases} 1 & \text{if observation is from treatment 1} \\ -1 & \text{if observation is from treatment 2} \\ 0, & \text{otherwise} \end{cases}$$

and

$$x_{2j} = \begin{cases} 1 & \text{if observation is from treatment 2} \\ -1 & \text{if observation is from treatment 3} \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that the relationship between the parameters in the regression and analysis - of - variance models is

$$\begin{aligned} \beta_0 &= \frac{\mu_1 + \mu_2 + \mu_3}{3} = \bar{\mu} \\ \beta_1 &= \mu_1 - \bar{\mu},\ \beta_2 = \mu_2 - \bar{\mu}, \end{aligned}$$

where $\mu_i = \mu + \tau_i$.

(b) Write down the vector $\boldsymbol{y}$ and matrix X.

(c) Develop an appropriate sum of squares for testing the hypothesis $H_0 : \tau_1 = \tau_2 = \tau_3 = 0$. Is this the usual treatment sum of squares in the one- way analysis of variance?

P 5. Consider the one–way ANOVA model

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij},\ i = 1, \ldots, K,\ j = 1, \ldots, n_i \tag{2}$$

where ϵ_{ij} are i.i.d. $N(0, \sigma^2)$.

(a) Find the residual sum of squares (RSS) for the model (2).

(b) Find the distribution of RSS/σ^2.

P 6. Consider the ANOVA model (2). The regression sum of squares SS_{Reg} is defined as

$$SS_{\text{Reg}} = \sum_{i=1}^{K} \sum_{j=1}^{n_i} (\widehat{y}_{ij} - \bar{y}), \tag{3}$$

where $\widehat{y}_{ij} = \widehat{\mu} + \widehat{\alpha}_i$, $\widehat{\mu}$ and $\widehat{\alpha}_i$ are least squares estimators of μ and α_i, $\bar{y} = \sum_{i=1}^{K} n_i \bar{y}_i / \sum_{i=1}^{K} n_i$ and $\bar{y}_i = \sum_{j=1}^{n_i} y_{ij}$ for $i = 1, \ldots, K$. Find the distribution of SS_{Reg}/σ^2.

Remark: The hypothesis

$$H_0 : \alpha_1 = \cdots = \alpha_K \text{ versus } H_1 : \alpha_i \neq \alpha_j \text{ for at least one pair of } (i, j)$$

can tested using the statistics

$$F = \frac{RSS/df_1}{SS_{\text{Reg}}/df_2} \sim F_{df_1, df_2},$$

where df_1 and df_2 are degrees of freedoms associated to the distributions of RSS and SS_{Reg} respectively.