MTH 441: Problems set 1

P 1. Let A and B are square matrices. Prove that if $AB = I$ then $BA = I$.

P 2. Show that for an idempotent matrix A, $Rank(A) = trace(A)$.

P 3. Let $\boldsymbol{X} = (X_1, X_2, X_3)'$ with variance

$$\text{Var}[\boldsymbol{X}] = \begin{pmatrix} 5 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix}$$

(a) Find the variance of $X_1 - 2X_2 + X_3$.

(b) Find the variance matrix of $\boldsymbol{Y} = (Y_1, Y_2)'$ where $Y_1 = X_1 + X_2$ and $Y_2 = X_1 + X_2 + X_3$.

P 4. Let $X_1, \dots, X_n$ be a random sample from a standard normal distribution. Show that $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$ and $\sum_{i=1}^{n}(X_i - \bar{X})^2$ are independent.

P 5. Let $X_1, \dots, X_n$ be a random sample from normal distribution with mean μ and variance σ^2. Show that $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$ and $\sum_{i=1}^{n-1}(X_i - X_{i+1})^2$ are independent.

P 6. If $\boldsymbol{Y} \sim \mathcal{N}_2(\boldsymbol{0}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma} = (\sigma_{ij})$, prove that

$$(\boldsymbol{Y}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y} - \frac{Y_1^2}{\sigma_{11}}) \sim \chi_1^2.$$

P 7. Consider a simple linear regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ with Gauss–Markov assumptions. Prove that the LSEs of β_0 and β_1 are uncorrelated if and only if $\bar{x} = 0$.

P 8. **(Centering)** Consider the following regression model:

$$y_i = \beta_0 + \beta_1(x_{i1} - \bar{x}_1) + \cdots + \beta_{p-1}(x_{i,p-1} - \bar{x}_{p-1}) + \epsilon_i,$$

where $\bar{x}_j = \sum_i x_{ij}/n$. Prove that the LSE of β_0 and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{p-1})'$ are

$$\widehat{\beta}_0 = \bar{y} = \sum_{i=1}^{n} y_i/n$$

and

$$\widehat{\boldsymbol{\beta}} = (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\boldsymbol{y},$$

where

$$\tilde{X} = \begin{bmatrix} x_{11} - \bar{x}_1 & \cdots & x_{1,p-1} - \bar{x}_{p-1} \\ \vdots & \vdots & \vdots \\ x_{n1} - \bar{x}_1 & \cdots & x_{n,p-1} - \bar{x}_{p-1} \end{bmatrix}. \tag{1}$$

P 9. **(Scaling)** Consider the following regression model:

$$y_i = \beta_0 + \beta_1(x_{i1} - \bar{x}_1)/s_1 + \cdots + \beta_{p-1}(x_{1,p-1} - \bar{x}_{p-1})/s_{p-1} + \epsilon_i,$$

where $\bar{x}_j = \sum_i x_{ij}/n$ and $s_j^2 = \sum_{i=1}^{n}(x_{ij} - \bar{x}_j)^2$ for $i = 1, \dots, n$ and $j = 1, \dots, p-1$. Find the variance of the LSE of $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{p-1})'$.

P 10. Let

$$\begin{aligned} Y_1 &= \alpha_1 + \epsilon_1 \\ Y_2 &= 2\alpha_1 - \alpha_2 + \epsilon_2 \\ Y_3 &= \alpha_1 + 2\alpha_2 + \epsilon_3, \end{aligned}$$

where $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \epsilon_2)^{\mathrm{T}} \sim \mathcal{N}(0, \sigma^2 \mathrm{I}_3)$. Derive the F–statistics for testing $H_0 : \alpha_1 = \alpha_2$ in terms of RSS and $(\widehat{\alpha}_1, \widehat{\alpha}_2)$.