PROBLEM SET-3

1) $R^2_{1,adj}$ denotes adjusted R^2 after removing x_1.

$$H_0: \cancel{R^2}\ \beta_1 = 0 \quad \text{vs.} \quad H_1: \beta_1 > 0$$

R^2_{adj} is the adjusted R^2 for the full model.

Then, $R^2_{adj} = 1 - \dfrac{RSS}{TSS} \times \dfrac{n-1}{n-p} = 1 - (1-R^2) \cdot \dfrac{n-1}{n-p}$

and $R^2 = 1 - RSS/TSS$

The F-statistic is given by -

$$F = \frac{RSS_1 - RSS}{RSS} \cdot \frac{n-p}{1}$$

$$= \frac{RSS_1/TSS - RSS/TSS}{RSS/TSS} \cdot (n-p)$$

$$= \frac{(1-R_1^2) - (1-R^2)}{1-R^2} \cdot (n-p)$$

$$= \left(\frac{1-R_1^2}{1-R^2} - 1\right) \cdot (n-p)$$

$$= \left[\frac{1-R_1^2}{1-R^2_{adj}} \cdot \frac{n-1}{n-p} - 1\right] (n-p).$$

Now,

$$F \geq 1 \iff \left[\frac{1-R_1^2}{1-R^2_{adj}} \cdot \frac{n-1}{n-p} - 1\right](n-p) \geq 1.$$

$$\iff \frac{1-R_1^2}{1-R^2_{adj}} \cdot \frac{n-1}{n-p} \geq 1 + \frac{1}{n-p}$$

$$\iff \frac{1-R_1^2}{1-R^2_{adj}} \geq \frac{n-p+1}{n-1}$$

Now for the reduced model,

$$R_{1,,\,adj}^2 = 1-(1-R_1^2)\cdot\frac{n-1}{n-(p-1)}$$

$$\Rightarrow 1-R_j^2 = (1-R_j^2{}_{adj})\cdot\frac{n-(p-1)}{n-1}$$

~~For~~ Then, $F \geq 1 \iff \frac{1-R_j^2{}_{adj}}{1-R^2{}_{adj}} \cdot \frac{n-(p-1)}{n-1} \geq \frac{n-(p-1)}{n-1}$

$$\iff 1-R_j^2{}_{adj} \geq 1-R^2{}_{adj}$$

$$\iff R_j^2{}_{adj} \leq R^2{}_{adj} \text{ [Proved]}.$$

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27. $Y = X\beta + \epsilon$

The ridge estimator is given by -

$$\hat{\beta}_R = (X'X+kI)^{-1}X'\underline{Y}$$

$$X_A = \begin{pmatrix} X \\ \sqrt{k}I_p \end{pmatrix} \quad y_A = \begin{pmatrix} \underline{y} \\ \underline{0}_p \end{pmatrix}.$$

$$y_A = X_A\beta_R + \epsilon$$

The LSE of this model is -

$$\hat{\beta}_R^* = (X_A'X_A)^{-1}X_A'\underline{Y}.$$

$$(X_A'X_A) = (X' \;\; \sqrt{k}I_p)\begin{pmatrix} X \\ \sqrt{k}I_p \end{pmatrix}$$

$$= X'X + kI_p$$

$$X_A'Y = (X' \;\; \sqrt{k}I_p)\begin{pmatrix} \underline{y} \\ \underline{0}_p \end{pmatrix} = X'\underline{Y}$$

$$\therefore \hat{\beta}_R^* = (X'X+kI_p)^{-1}X'\underline{Y}$$

$$= \hat{\beta}_R \text{ (Proved)}$$

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37 $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$

$var(\epsilon_i) = \sigma^2 x_i^2$

(a) The transformation is given by -

$y' = y/x \ , \ x' = \frac{1}{x}$.

The ~~⊟~~ new model is -

$$\cancel{y_i^*} \quad \frac{y_i}{x_i} = \frac{\beta_0}{x_i} + \beta_1 \frac{x_i}{x_i} + \frac{\epsilon_i}{x_i}$$

$$\Rightarrow y_i' = \cancel{\beta_0 x_i} \ \beta_1 + \beta_0 x_i' + \epsilon_i'$$

$$= \beta_0^* + \beta_1^* x_i' + \epsilon_i'$$

$$var(\epsilon_i') = \frac{1}{x_i^2} \cdot var(\epsilon_i) = \frac{1}{x_i^2} \cdot \sigma^2 x_i^2 = \underbrace{\sigma^2}_{\text{constant}}$$

Hence, it is a variance stabilizing transformation.

(b) The slope parameter of new ~~parameter~~ model is β_1^* where

$\beta_1^* = \beta_0$.

The intercept parameter of new model is β_0^* where

$\beta_0^* = \beta_1$.

(c)

$w_i = 1/x_i^2$ Then $var(\epsilon_i) = \frac{\sigma^2}{w_i}$

$$= \frac{\sigma^2}{1/x_i^2} = \sigma^2 x_i^2$$

and the transformation is done as -

$$y_i^* = y_i \sqrt{w_i}$$

$$= y_i \sqrt{\frac{1}{x_i^2}} = y_i / x_i$$

$\Rightarrow$ The method is equivalent to the transformation

47. $y_{ij} = \mu + \tau_i + \epsilon_{ij}$ $\quad i = 1, 2, 3$

$j = 1, 2, \ldots, n.$

$\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$

$y_{ij} = \beta_0 + \beta_1 x_{1j} + \beta_2 x_{2j} + \epsilon_{ij}$

where, $x_{1j} = \begin{cases} 1, & \text{if obs. is from trt. 1} \\ -1, & \text{if obs. is from trt. 2} \\ 0, & \text{o/w.} \end{cases}$

$x_{2j} = \begin{cases} 1, & \text{if obs. is from trt. 2} \\ -1, & \text{if obs. " " " 3} \\ 0, & \text{o/w.} \end{cases}$

For ~~trt. 1 the ANOVA model is~~ –

$\cancel{y_{1j} = \mu + \tau_1 + \epsilon_{1j} = \mu_1} + \epsilon_{1j}$

For ~~trt. 1 the~~

Treatment	ANOVA Model.	Regression model
1	$y_{1j} = \mu + \tau_1 + \epsilon_{1j} = \mu_1 + \epsilon_{1j}$	$y_{1j} = \beta_0 + \beta_1 + \epsilon_{1j}$
2	$y_{2j} = \mu + \tau_2 + \epsilon_{2j} = \mu_2 + \epsilon_{2j}$	$y_{2j} = \beta_0 - \beta_1 + \beta_2 + \epsilon_{2j}$
3	$y_{3j} = \mu + \tau_3 + \epsilon_{3j} = \mu_3 + \epsilon_{3j}$	$y_{3j} = \beta_0 - \beta_2 + \epsilon_{2j}$

Therefore, $\left.\begin{aligned} \mu_1 &= \beta_0 + \beta_1 \\ \mu_2 &= \beta_0 - \beta_1 + \beta_2 \\ \mu_3 &= \beta_0 - \beta_2 . \end{aligned}\right\}$

$\mu_1 = \bar{\mu} + \beta_1$

$\Rightarrow \boxed{\beta_1 = \mu_1 - \bar{\mu}}$

$\mu_3 = \bar{\mu} - \beta_2 \Rightarrow \beta_2 = \bar{\mu} - \mu_3$

$\therefore 3\beta_0 = \mu_1 + \mu_2 + \mu_3$

$\Rightarrow \boxed{\beta_0 = \frac{\mu_1 + \mu_2 + \mu_3}{3} = \bar{\mu}}.$

$$\mu_2 = \beta_0 - \beta_1 + \beta_2$$

$$\Rightarrow \mu_2 = \bar{\mu} - \mu_1 + \bar{\mu} + \beta_2$$

$$\Rightarrow \underbrace{\beta_2 = \mu_1 + \mu_2 - 2\bar{\mu}}$$

$$\mu_3 = \beta_0 - \beta_2.$$

$$= \bar{\mu} - \beta_2$$

$$\Rightarrow \beta_2 = \underbrace{\bar{\mu} - \mu_3}$$

or

(b) $\underset{\sim}{y} = (y_{11}, \ldots, y_{1n}, y_{21}, \ldots, y_{2n}, y_{31}, \ldots, y_{3n})'$

$$X = \begin{bmatrix} 1 & 1 & 0 \\ \vdots & & \\ 1 & 1 & 0 \\ 1 & -1 & 1 \\ & \vdots & \\ 1 & -1 & 1 \\ 1 & 0 & -1 \\ & \vdots & \\ 1 & 0 & -1 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n \text{ rows} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n \text{ rows} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n \text{ rows.} \end{matrix}$$

(c) $H_0: \tau_1 = \tau_2 = \tau_3 = 0$

$$X^T X = \begin{bmatrix} 1 & & 1 & 1 & & 1 & 1 & & 1 \\ 1 & \ldots & 1 & -1 & \ldots & -1 & 0 & \ldots & 0 \\ 0 & & 0 & 1 & & 1 & -1 & & -1 \end{bmatrix} \begin{bmatrix} \cdots \end{bmatrix}$$

$$= \begin{bmatrix} 3n & 0 & 0 \\ 0 & 2n & -n \\ 0 & -n & 2n \end{bmatrix}$$

$$(X^T X)^{-1} = \frac{1}{n} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1}$$

$$= \frac{1}{n} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 2/3 & 1/3 \\ 0 & 1/3 & 2/3 \end{bmatrix}$$

$$X^T \underline{Y} = \begin{bmatrix} \sum_{i=1}^{3}\sum_{j=1}^{n} y_{ij} \\ \sum_{j=1}^{n} y_{1j} - \sum_{j=1}^{n} y_{2j} \\ \sum_{j=1}^{n} y_{2j} - \sum_{j=1}^{n} y_{3j} \end{bmatrix}$$

$$\hat{\beta} = (X^T X)^{-1} X^T \underline{Y}$$

$$= \cancel{3n \quad 0 \quad 0} \begin{bmatrix} 1/3n & 0 & 0 \\ 0 & 2/3n & 1/3n \\ 0 & 1/3n & 2/3n \end{bmatrix} \begin{bmatrix} \sum\sum y_{ij} \\ \sum y_{1j} - \sum y_{2j} \\ \sum y_{2j} - \sum y_{3j} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3n} \sum_{i=1}^{3}\sum_{j=1}^{n} y_{ij} \\ \frac{2}{3n}\left[\sum_{j=1}^{n} y_{1j} - \sum_{j=1}^{n} y_{2j}\right] + \frac{1}{3n}\left[\sum_{j=1}^{n} y_{2j} - \sum_{j=1}^{n} y_{3j}\right] \\ \frac{1}{3n}\left[\sum y_{1j} - \sum y_{2j}\right] + \frac{2}{3n}\left[\sum_{j=1}^{n} y_{2j} - \sum_{j=1}^{n} y_{3j}\right] \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3n} \sum\sum y_{ij} \\ \frac{2}{3n}\sum y_{1j} - \frac{1}{3n}\sum y_{2j} - \frac{1}{3n}\sum y_{3j} \\ \frac{1}{3n}\sum y_{1j} + \frac{1}{3n}\sum y_{2j} - \frac{2}{3n}\sum y_{3j} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{y} \\ \frac{1}{3}\left[2\bar{y}_1 - \bar{y}_2 - \bar{y}_3\right] \\ \frac{1}{3}\left[\bar{y}_1 + \bar{y}_2 - 2\bar{y}_3\right] \end{bmatrix}$$

$$\hat{y} = X\hat{\beta} = \begin{bmatrix} \left(\bar{y} + \frac{1}{3}\left[2\bar{y}_1 - \bar{y}_2 - \bar{y}_3\right)\right) \underline{1}_n \\ \left[\bar{y} - \frac{1}{3}(2\bar{y}_1 - \bar{y}_2 - \bar{y}_3) + \frac{1}{3}(\bar{y}_1 + \bar{y}_2 - 2\bar{y}_3)\right] \underline{1}_n \\ \left[\bar{y} - \frac{1}{3}(\bar{y}_1 + \bar{y}_2 - 2\bar{y}_3)\right] \underline{1}_n \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\bar{y}_1 + \bar{y}_2 + \bar{y}_3}{3} + \frac{2\bar{y}_1}{3} - \frac{\bar{y}_2}{3} - \frac{\bar{y}_3}{n}\right) \underline{1}_n \\ \left[\frac{\bar{y}_1 + \bar{y}_2 + \bar{y}_3}{3} - \frac{2\bar{y}_1}{3} + \frac{\bar{y}_2}{3} + \frac{\bar{y}_3}{3} + \frac{\bar{y}_1}{3} + \frac{\bar{y}_2}{3} - \frac{2\bar{y}_3}{3}\right] \underline{1}_n \\ \left[\frac{\bar{y}_1 + \bar{y}_2 + \bar{y}_3}{3} - \frac{1}{3}\bar{y}_1 - \frac{1}{3}\bar{y}_2 + \frac{2\bar{y}_3}{3}\right] \underline{1}_n \end{bmatrix}$$

$$= \begin{bmatrix} \bar{y}_1 \underline{1}_n \\ \bar{y}_2 \underline{1}_n \\ \bar{y}_3 \underline{1}_n \end{bmatrix}$$

Now, the estimated $\hat{y}_i$'s are same as the $\hat{y}_i$ with the usual one-way ANOVA setup.

So, SS due to Treatment is same with the usual setup.

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$$\left.\begin{aligned}
y_{11} &= \mu + \alpha_1 + \epsilon_{11}\\
&\;\vdots\\
y_{1n_1} &= \mu + \alpha_1 + \epsilon_{1n_1}\\
&\;\vdots\\
y_{k1} &= \mu + \alpha_k + \epsilon_{nk}\\
&\;\vdots\\
y_{kn_k} &= \mu + \alpha_k + \epsilon_{kn_k}.
\end{aligned}\right.$$

Define $n = \sum n_i$

In matrix form,

$$Y = \underbrace{\begin{bmatrix} 1_{n_1} & 1_{n_1} & 0 & 0 & \cdots & 0 \\ 1_{n_2} & 0 & 1_{n_2} & 0 & \cdots & 0 \\ \vdots & & & \vdots & & \\ 1_{n_k} & 0 & 0 & \cdots & & 1_{n_k} \end{bmatrix}_{n \times \overline{k+1}}}_{X} \underbrace{\begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix}_{k+1\times 1}}_{\beta}$$

$$X^T X = \begin{bmatrix} n & n_1 & n_2 & \cdots & n_k \\ n_1 & n_1 & 0 & \cdots & 0 \\ n_2 & 0 & n_2 & \cdots & 0 \\ \vdots & & \vdots & & \\ n_k & 0 & 0 & \cdots & n_k \end{bmatrix}$$

$$(X^T X)^- = \begin{bmatrix} 0 & 0 & 0 & \cdots & \\ 0 & 1/n_1 & 0 & \cdots & 0 \\ \vdots & 0 & 1/n_2 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1/n_k \end{bmatrix}_{\overline{k+1} \times \overline{k+1}}$$

[Generalized inverse].

$$X^T Y = \begin{bmatrix} 1_{n_1}^T & 1_{n_2}^T & \cdots & 1_{n_k}^T \\ 1_{n_1}^T & 0 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & 1_{n_k}^T \end{bmatrix} \begin{bmatrix} y_{11} \\ \vdots \\ y_{1n_1} \\ \vdots \\ y_{k1} \\ \vdots \\ y_{kn_k} \end{bmatrix}$$

$$= \begin{bmatrix} \sum\sum y_{ij} \\ \sum y_{1j} \\ \vdots \\ \sum y_{kj} \end{bmatrix}$$

$$\hat{\beta} = (X^TX)^- X^TY.$$

$$= \begin{bmatrix} 0 \\ \bar{y}_1 \\ \vdots \\ \bar{y}_k \end{bmatrix} . \left\{ \begin{array}{l} \hat{\mu} = 0. \\ \hat{\alpha}_i = \bar{y}_i \\ \quad = \dfrac{1}{n_i} \displaystyle\sum_{j=1}^{n_i} y_{ij} \end{array} \right.$$

Now, $SS_{Res} = Y^T \left[I - X(X^TX)^- X^T \right] Y.$

$\downarrow$

check whether this is idempotent or not.

$$\left[I - X(X^TX)^- X^T \right]\left[I - X(X^TX)^- X^T \right]$$

$$= I - X(X^TX)^- X^T - X(X^TX)^- X^T + X(X^TX)^- X^T X (X^TX)^- X^T \quad \text{— (★)}$$

$$\cancel{= I - 2X(X^TX)^- X^T + X(X^TX)^- X^T}$$

Now, $(X^TX)^-(X^TX)(X^TX)^-$

$$= \begin{bmatrix} 0 & \underline{0} & & \\ \vdots & 1/n_1 & & \\ \underline{0} & & \ddots & \\ & & & 1/n_k \end{bmatrix} \begin{bmatrix} n & n_1 & \cdots & n_k \\ n_1 & n_1 & & \\ \vdots & & n_2 & \\ n_k & & \cdots & n_k \end{bmatrix} \begin{bmatrix} 0 & \underline{0} & & \\ \underline{0} & 1/n_1 & & \\ & & \ddots & \\ & & & 1/n_k \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ 1 & 0 & 1 \cdots & 0 \\ \vdots & & & \\ 1 & 0 & 0 \cdots & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1/n_1 & \cdots & 0 \\ 0 & 0 & 1/n_2 \cdots & 0 \\ \vdots & & & \\ 0 & 0 & 0 \cdots & 1/n_k \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1/n_1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1/n_k \end{bmatrix} = (X^TX)^-$$

$$\text{(★)} = I - X(X^TX)^- X^T - \cancel{X(X^TX)^- X^T} + \cancel{X(X^TX)^- X^T}$$

$$= I - X(X^TX)^- X^T \quad \text{[Idempotent].}$$

Now, $\underset{\sim}{Y} \sim MVN\left[\underbrace{\begin{bmatrix}(\mu+\alpha_1)\underline{1}_{n_1}\\ \vdots \\ (\mu+\alpha_k)\underline{1}_{n_k}\end{bmatrix}}_{\underline{\mu}}, \sigma^2 I\right]$

$\lambda = \left((\mu+\alpha_1)\underline{1}_{n_1}' \dots (\mu+\alpha_k)\underline{1}_{n_k}'\right)\left[I - X(X^TX)^- X^T\right]\begin{pmatrix}(\mu+\alpha_1)\underline{1}_{n_1}\\ \vdots \\ (\mu+\alpha_k)\underline{1}_{n_k}\end{pmatrix}.$

$= \left[n_1(\mu+\alpha_1)^2 + n_2(\mu+\alpha_2)^2 + \dots + n_k(\mu+\alpha_k)^2\right]$

$- (\_\_\_\_\_\_)X(X^TX)^- X^T(\ \ ).$

Now, $X(X^TX)^- X^T = \begin{bmatrix}\frac{1}{n_1}\underline{1}_{n_1}\underline{1}_{n_1}^T & & & \\ & \frac{1}{n_2}\underline{1}_{n_2}\underline{1}_{n_2}^T & & \\ & & \ddots & \\ & & & \frac{1}{n_3}\underline{1}_{n_3}\underline{1}_{n_3}^T\end{bmatrix}$

$\therefore\ \underline{\mu}'\ X(X^TX)^- X^T\ \underline{\mu}.$

$= \left[(\mu+\alpha_1)\underline{1}_{n_1}'\ (\mu+\alpha_2)\underline{1}_{n_2}'\ (\mu+\alpha_3)\underline{1}_{n_3}'\right] \cdot \begin{bmatrix}(\mu+\alpha_1)\underline{1}_{n_1}\\ (\mu+\alpha_2)\underline{1}_{n_2}\\ (\mu+\alpha_3)\underline{1}_{n_3}\end{bmatrix}$

$$\lambda = \left[n_1(\mu+\alpha_1)^2 + \dots + n_k(\mu+\alpha_k)^2\right]$$
$$- \left[n_1(\mu+\alpha_1)^2 + \dots + n_k(\mu+\alpha_k)^2\right]$$
$$= 0$$

~~Hence, by the Theorem~~

Now, $tr\left(X(X^TX)^- X^T\right)$.

$$= \frac{1}{n_1} \times n_1 + \frac{1}{n_2} \times n_2 + \dots + \frac{1}{n_k} \times n_k$$

$$= k.$$

$$tr\left(I - X(X^TX)^- X^T\right)$$

$$= n - k.$$

$\therefore$ $\boxed{\cancel{\frac{SS_{Reg}}{\sigma^2} \sim \chi^2}}$ $\qquad \dfrac{SS_{Res}}{\sigma^2} \sim \chi^2_{n-k}$

$$SS_{Res} = \underline{Y}^T \left[I - X(X^TX)^- X^T\right] \underline{Y}$$

$$= \underline{Y}^T \begin{bmatrix} I_{n_1} - \frac{1}{n_1}\underline{1}_{n_1}\underline{1}_{n_1}^T & & & \\ & I_{n_2} - \frac{1}{n_2}\underline{1}_{n_2}\underline{1}_{n_2}^T & & \\ & & \ddots & \\ & & & I_{n_k} - \frac{1}{n_k}\underline{1}_{n_k}\underline{1}_{n_k}^T \end{bmatrix} \underline{Y}$$