MTH441A – PROBLEM SET 2

17. $\sum_{i=1}^{n} (y_i - \bar{y})^2$

$= \sum_{i=1}^{n} (y_i - \hat{y}_i + \hat{y}_i - \bar{y})^2$

$= \sum_{i=1}^{n} [(y_i - \hat{y}_i)^2 + \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 + 2 \sum_{i=1}^{n} (y_i - \hat{y}_i)(\hat{y}_i - \bar{y})$

$\cancel{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}$

Now, $\sum_{i=1}^{n} (y_i - \hat{y}_i)(\hat{y}_i - \bar{y})$

$\cancel{\sum_{i=1}^{n} (y_i \hat{y}_i - y_i \bar{y}}$

$= \sum_{i=1}^{n} \hat{y}_i (y_i - \hat{y}_i) - \bar{y} \sum_{i=1}^{n} (y_i - \hat{y}_i)$

$= \sum_{i=1}^{n} \hat{y}_i (y_i - \hat{y}_i) - 0$ [since $\sum_{i=1}^{n} (y_i - \hat{y}_i) = \sum_{i=1}^{n} e_i = 0$]

$= \sum_{i=1}^{n} e_i \hat{y}_i$

$= \sum_{i=1}^{n} e_i (\hat{\beta}_0 + \hat{\beta}_1 x_{1i} + \hat{\beta}_2 x_{2i} + \dots + \hat{\beta}_p x_{pi})$

$= \hat{\beta}_0 \sum e_i + \hat{\beta}_1 \sum e_i x_{1i} + \dots + \hat{\beta}_p \sum e_i x_{pi}$

$= 0.$

Hence, $\boxed{\sum (y_i - \bar{y})^2 = \sum (y_i - \hat{y}_i)^2 + \sum (\hat{y}_i - \bar{y})^2}$. [Proved].

2)

$$\sum(y_i-\bar{y})^2 = \sum y_i^2 - n\bar{y}^2$$

$$= \sum y_i^2 - \frac{1}{n}\left(\sum y_i\right)^2$$

$$= \underset{\sim}{y}'\underset{\sim}{y} - \frac{1}{n}\left(\sum_i\sum_j y_i y_j\right).$$

$$= \underset{\sim}{y}'\underset{\sim}{y} - \frac{1}{n}\, \underset{\sim}{y}'\underline{1}\,\underline{1}^T\underset{\sim}{y}.$$

$$= \underset{\sim}{y}'\underset{\sim}{y} - \underset{\sim}{y}'\left(\frac{1}{n}\underline{1}\,\underline{1}^T\right)\underset{\sim}{y}.$$

$$\boxed{A_3 = \frac{1}{n}\underset{\sim}{1}\,\underset{\sim}{1}^T}.$$

Now $\sum(y_i-\hat{y}_i)^2 = (\underset{\sim}{y}-\hat{\underset{\sim}{y}})^T(\underset{\sim}{y}-\hat{\underset{\sim}{y}})$

$$= (\underset{\sim}{y}-X\hat{\underset{\sim}{\beta}})^T(\underset{\sim}{y}-X\hat{\underset{\sim}{\beta}}).$$

Now, $(\underset{\sim}{y}-X\hat{\underset{\sim}{\beta}}) = \underset{\sim}{y} - X(X^TX)^{-1}X\underset{\sim}{y}$

$$= (I - X(X^TX)^{-1}X)\underset{\sim}{y}.$$

$$= H\underset{\sim}{y}.$$

$\therefore \sum(y_i-\hat{y}_i)^2 = (H\underset{\sim}{y})^T H\underset{\sim}{y}$

$$= \underset{\sim}{y}(H^TH)\underset{\sim}{y}.$$

$$= \underset{\sim}{y}(H.H)\underset{\sim}{y}. \quad [H^T=H]$$

$$= \underset{\sim}{y}(H^2)\underset{\sim}{y} \quad [H^2=H]$$

$$= \underset{\sim}{y}H\underset{\sim}{y}.$$

$$\therefore \boxed{A_1 = H = (I - X(X^TX)^{-1}X)}.$$

$$\sum(\hat{y}_i-\bar{y})^2 \quad \left[\sum_{i=1}^{n}(y_i-\bar{y})^2-\sum_{i=1}^{n}(y_i-\hat{y}_i)^2\right]$$

$$=\sum_{i=1}^{n}\hat{y}_i^2-n\bar{y}^2$$

$\underset{\sim}{y}'\underset{\sim}{y}$

$$\left[\bar{\hat{y}}=\frac{1}{n}\sum_{i=1}^{n}\hat{y}_i=\frac{1}{n}\sum_{i=1}^{n}(y_i-e_i)=\bar{y}-\bar{e}=\bar{y}\right]$$

$$=(\hat{\underset{\sim}{y}}'\hat{\underset{\sim}{y}})-n\bar{y}^2$$

$$=\underset{\sim}{y}'(I-H)\underset{\sim}{y}-\underset{\sim}{y}'\left(\frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)\underset{\sim}{y}$$

$$=\underset{\sim}{y}'\left(I-H-\frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)\underset{\sim}{y}.$$

$$\therefore \boxed{A_2=\left(I-H-\frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)}$$

$$\left[\because \hat{\underset{\sim}{y}}'\hat{\underset{\sim}{y}}\right.$$
$$=(X\hat{\underset{\sim}{\beta}})'(X\hat{\underset{\sim}{\beta}})$$
$$=\underset{\sim}{y}'(I-H)(I-H)\underset{\sim}{y}$$
$$=\underset{\sim}{y}'(I-2H+H^2)\underset{\sim}{y}$$
$$=\underset{\sim}{y}'(I-2H+H)\underset{\sim}{y}$$
$$\left.=\underset{\sim}{y}'(I-H)\underset{\sim}{y}\right]$$

37. $A_1=H$

$$A_2=I-H-\frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T$$

$$A_3=\frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T$$

$$\therefore \boxed{A_1+A_2+A_3=I}$$

$$A_1^2=H^2=H=A_1$$

$$A_3^2=\frac{1}{n^2}\underset{\sim}{1}\underset{\sim}{1}^T\cdot\underset{\sim}{1}\cdot\underset{\sim}{1}^T=\frac{1}{n^2}\cdot n\cdot\underset{\sim}{1}\underset{\sim}{1}^T=\frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T=A_3.$$

$$A_2^2=\left(I-H-\frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)^2$$

$$=I+H^2+\frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T-2H-\frac{2}{n}\underset{\sim}{1}\underset{\sim}{1}^T+\frac{2}{n}H\cdot\underset{\sim}{1}\underset{\sim}{1}^T$$

$$=I-H-\frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T+\frac{2}{n}\left(I-X(X^TX)^{-1}X^T\right)\underset{\sim}{1}\underset{\sim}{1}^T\cdot$$

$$= I - H - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T + \frac{2}{n}\left[\underset{\sim}{1}\underset{\sim}{1}^T - \underset{\sim}{1}\underset{\sim}{1}^T\right] \quad \left[\text{Since } X(X^TX)^{-1}X^T\underset{\sim}{1} = \underset{\sim}{1}\right]$$

$$= I - H - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T$$

Hence, $tr(A_1) = r_1 = tr(H)$

$$= tr\left(I - X(X^TX)^{-1}X^T\right)$$

$$= n - tr\left(X(X^TX)^{-1}X^T\right)$$

$$= n - tr\left((X^TX)(X^TX)^{-1}\right)$$

$$= n - tr(I_p)$$

$$= n - p.$$

$$tr(A_2) = r_2 = tr\left(I - H - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)$$

$$= n - (n-p) - tr\left(\frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)$$

$$= p - \frac{1}{n}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ & & \vdots & \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

$$= p - \frac{1}{n} \times n$$

$$= p - 1$$

$$Y^TY = Y^TA_1Y + Y^TA_2Y + Y^TA_3Y$$

$$tr(A_3) = r_3 = 1.$$

Now, $Y^TA_1Y = (Y - X\beta)^T A_1 (Y - X\beta)$ (check).

$$r_1 + r_2 + r_3 = n - p + p - 1 + 1 = n$$

$$\therefore \frac{RSS}{\sigma^2} \sim \chi^2_{n-p} \quad \& \quad \frac{SS_R}{\sigma^2} \sim \chi^2_{p-1}.$$

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47 $(n-p-1) S_{(i)}^2 = \sum_{\substack{j=1 \\ j \neq i}}^{n} (y_j - \hat{y}_{j(i)})^2$.

$$= \sum_{j=1}^{n} [y_j - \underset{\sim}{x_j}' \hat{\beta}_{(i)}]^2 \quad \text{— } \circledast$$

Now, $\hat{\beta}_{(i)} = (X_{(i)}' X_{(i)})^{-1} X_{(i)}' \underset{\sim}{y}_{(i)}$.

$$(X_{(i)}' X_{(i)})^{-1} = (X'X)^{-1} + \frac{(X'X)^{-1} \underset{\sim}{x_i} \underset{\sim}{x_i}' (X'X)^{-1}}{1-h_{ii}} \quad \text{[Derived in class]}.$$

$$\hat{\beta}_{(i)} = \left[(X'X)^{-1} + \frac{(X'X)^{-1} \underset{\sim}{x_i} \underset{\sim}{x_i}' (X'X)^{-1}}{1-h_{ii}} \right] (X'\underset{\sim}{y} - y_i \underset{\sim}{x_i})$$

$$= \hat{\beta} - (X'X)^{-1} y_i \underset{\sim}{x_i} + \frac{(X'X)^{-1} \underset{\sim}{x_i} \overbrace{\underset{\sim}{x_i}' \hat{\underset{\sim}{\beta}}}^{\hat{y}_i}}{1-h_{ii}} - y_i \frac{[(X'X)^{-1} \underset{\sim}{x_i} x_i' (X'X)^{-1} \underset{\sim}{x_i}]}{1-h_{ii}}$$

$$= \hat{\beta} - (X'X)^{-1} y_i \underset{\sim}{x_i} + \frac{(X'X)^{-1} \underset{\sim}{x_i} \hat{y}_i}{1-h_{ii}} - \frac{y_i (X'X)^{-1} \underset{\sim}{x_i} h_{ii}}{1-h_{ii}}$$

$$= \hat{\beta} - (X'X)^{-1} y_i \underset{\sim}{x_i} + \frac{(X'X)^{-1} \underset{\sim}{x_i}}{1-h_{ii}} (\hat{y}_i - h_{ii} y_i)$$

$$= \hat{\beta} + \frac{(X'X)^{-1} \underset{\sim}{x_i}}{1-h_{ii}} \hat{y}_i - (X'X)^{-1} \underset{\sim}{x_i} y_i \left[1 + \frac{h_{ii}}{1-h_{ii}} \right]$$

$$= \hat{\beta} - \frac{(X'X)^{-1} \underset{\sim}{x_i}}{1-h_{ii}} (\underbrace{y_i - \hat{y}_i}_{e_i}).$$

$$\Rightarrow \hat{\beta} - \hat{\beta}_{(i)} = \frac{(X'X)^{-1} x_i e_i}{1-h_{ii}} \quad \text{— } (\cdot)$$

Using (·) in (*), $= \sum_{\substack{j=1 \\ j\neq i}}^{n} \left[y_j - \underset{\sim}{x_j}'\underset{\sim}{\hat{\beta}} + \frac{\underset{\sim}{x_j}'(X'X)^{-1}x_i e_i}{1-h_{ii}}\right]^2$

$$= \sum_{\substack{j=1 \\ j\neq i}}^{n} \left[y_j - \underset{\sim}{x_j}'\hat{\beta} + \frac{h_{ij}\, e_i}{1-h_{ii}}\right]^2$$

$$= \sum_{j=1}^{n} \left[y_j - \underset{\sim}{x_j}'\underset{\sim}{\hat{\beta}} + \frac{h_{ij}\, e_i}{1-h_{ii}}\right]^2 - \left(e_i + \frac{h_{ii}e_i}{1-h_{ii}}\right)^2$$

$$= \sum_{j=1}^{n} \left[y_j - \underset{\sim}{x_j}'\hat{\beta} + \frac{h_{ij}\, e_i}{1-h_{ii}}\right]^2 - \frac{e_i^2}{(1-h_{ii})^2}$$

$$= \sum_{\substack{j=1 \\ i\neq j}}^{n} [\cancel{y_j - x_j'\hat{\beta}}\,. \left(e_j + \frac{h_{ij}\, e_i}{1-h_{ii}}\right)^2 - \frac{e_i^2}{(1-h_{ii})^2}$$

$$= \sum_{j=1}^{n} e_j^2 + \frac{e_i^2}{(1-h_{ii})^2}\sum_{j=1}^{n} h_{ij}^2 + \frac{2e_i}{1-h_{ii}}\sum_{j=1}^{n} e_j h_{ij} - \frac{e_i^2}{(1-h_{ii})^2}$$

[Now, $H^2 = H \Rightarrow \sum_{j=1}^{n} h_{ij}^2 = h_{ii}$]

$$= \sum_{j=1}^{n} e_j^2 + \frac{e_i^2}{(1-h_{ii})^2}\cdot h_{ii} - \frac{e_i^2}{(1-h_{ii})^2}.$$

[Since, $H\underset{\sim}{e} = H(Y - X\hat{\beta})$

$= H(Y-(I-H)Y).$

$= H(I-H)Y.$

$= (H-H^2)Y = (H-H)Y = 0$]

$$= RSS - \frac{e_i^2}{(1-h_{ii})^2}(1-h_{ii})$$

$$= RSS - \frac{e_i^2}{1-h_{ii}}$$

$\therefore\ s_{(i)}^2 = \left[(n-p)RSS - e_i^2/(1-h_{ii})\right] / (n-p-1)$ [Proved].

PROBLEM SET - 2

(1-4 previous copy)

5) We know, $H^2 = H$ $[H = X(X'X)^{-1}X']$

$$(I-H)^2 = I - 2H + H^2$$
$$= I - 2H + H = I - H$$

i.e. $\sum_{j=1}^{n} (e_{ij} - h_{ij})(e_{ji} - h_{ji}) = 1 - h_{ii}$ [Comparing (i,i)th element from both side]

$\Rightarrow \sum_{j=1}^{n} (e_{ij} - h_{ij})^2 = 1 - h_{ii}$ [Since $(I-H)^T = I - H$]

$\Rightarrow \sum_{j=1}^{n} e_{ij}^2 + \sum_{j=1}^{n} h_{ij}^2 - 2\sum_{j=1}^{n} e_{ij} h_{ij} = 1 - h_{ii}$

$\Rightarrow 1 - 2h_{ii} + h_{ii}^2 + \sum_{i \neq j} h_{ij}^2 = 1 - h_{ii}$

$\Rightarrow (1 - h_{ii})^2 + \sum_{i \neq j} h_{ij}^2 = 1 - h_{ii}$ [Proved]

...

7) $Y_i = \theta_i + \epsilon_i$, $\epsilon_i \overset{IID}{\sim} N(0, \sigma^2)$, $i = 1(1)4$
$i = 1(1)4$

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 0$$
$$\Rightarrow \theta_4 = -(\theta_1 + \theta_2 + \theta_3).$$

$$\therefore \begin{cases} Y_1 = \theta_1 + \epsilon_1 \\ Y_2 = \theta_2 + \epsilon_2 \\ Y_3 = \theta_3 + \epsilon_3 \\ Y_4 = -(\theta_1 + \theta_2 + \theta_3) + \epsilon_4 \end{cases} \qquad X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

$H_0: \theta_1 - \theta_3 = 0 \qquad A = [1 \quad 0 \quad -1]$

$$X'X = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = I_3 + \underline{1}\,\underline{1}^T$$

$$(X'X)^{-1} = I_3 - \frac{\underline{1}\,\underline{1}^T}{1 + \underline{1}^T \underline{1}}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$RSS = Y^T H Y$$

$$H = X(X'X)^{-1}X'$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

$$SSE = y^T H y.$$

$$= y^T \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \underline{y}$$

$$= \frac{1}{4} \underline{y}^T \underline{1}^T \underline{1} \underline{y}$$

$$= \frac{1}{4} (\underline{1}' \underline{y})^T \underline{1} \underline{y}$$

$$= \frac{1}{4} \left(\Sigma y_i^2 \right).$$

$$\hat{\beta} = (X'X)^{-1}X'y$$

$$= \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \end{bmatrix} \underset{\sim}{y}.$$

$$A(X^TX)^{-1}A^T$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 2$$

$$RSS - SS_R = (A\hat{\beta})^T (A(X^TX)^{-1}A^T)^{-1} A\hat{\beta}$$

$$= \hat{\beta}^T A^T \cdot \frac{1}{2} \cdot A\hat{\beta}$$

$$= \underset{\sim}{y}^T \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} [1 \; 0 \; -1] \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \end{bmatrix} \underset{\sim}{y}$$

$$= \frac{1}{32} \underset{\sim}{y}^T \begin{bmatrix} 4 \\ 0 \\ -4 \\ 0 \end{bmatrix} [4 \; 0 \; -4 \; 0] \underset{\sim}{y}.$$

$$= \frac{1}{2} [y_1 \; y_2 \; y_3 \; y_4] \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} [1 \; 0 \; -1 \; 0] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$= \frac{1}{2} (y_1 - y_3)^2$$

$$F = \frac{\frac{(y_1-y_3)^2}{2} / 1}{\frac{\sum_{i=1}^{4} y_i^2}{4} / 1} = \frac{2(y_1-y_3)^2}{\sum y_i^2}.$$

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b) $X = [\underset{\sim}{1} \; \underset{\sim}{x_1} \; \ldots \; \underset{\sim}{x_{p-1}}]$

$$X^T X = \begin{bmatrix} \underset{\sim}{1}^T \\ \underset{\sim}{x_1}^T \\ \vdots \\ \underset{\sim}{x_{p-1}}^T \end{bmatrix} [\underset{\sim}{1} \; \underset{\sim}{x_1} \; \ldots \; \underset{\sim}{x_{p-1}}] \Rightarrow (X^T X)^{-1} = \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & C^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} n & 0 \\ 0 & C \end{bmatrix} \quad \text{where } c_{ii} = c \;\forall i, \quad c_{ij} = \underline{x_i}^T \underline{x_j}$$

Now, $\frac{1}{p} \sum_{j=0}^{p-1} var(\hat{\beta_j}) = \frac{1}{p} \sigma^2 \, tr(X'X)^{-1}$

$$= \frac{1}{p} \sigma^2 \left[\frac{1}{n} + trace(C^{-1})\right]$$

Let, $\lambda_1, \ldots, \lambda_{p-1}$ are eigen values of C.

$$trace(C^{-1}) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_{p-1}}$$

Now $\Sigma \lambda_i = trace(C) = c(p-1)$

Now, $\min \sum_{i=1}^{p-1} \frac{1}{\lambda_i}$ given $\Sigma \lambda_i = (p-1)c$.

$$f(\underset{\sim}{\lambda}) = \sum_{i=1}^{p-1} \frac{1}{\lambda_i} + \alpha \left[\Sigma \lambda_i - c(p-1)\right]$$

$$\frac{\partial f(\underset{\sim}{\lambda})}{\partial \lambda_i} = -\frac{1}{\lambda_i^2} + \alpha = 0 \quad \forall i = 1(1)p-1.$$

$$\Rightarrow \alpha = \frac{1}{\lambda_i^2} \Rightarrow \lambda_i^2 = \frac{1}{\alpha} = \text{const.}$$

Say all λ_i's are equal. ($= \lambda$ (say))

$\Rightarrow \sum \lambda_i = (p-1)c$

$\Rightarrow (p-1)\lambda = (p-1)c \Rightarrow \lambda = c.$

Then,

$C = T'DT = c I_{p-1}.$ [C was a symmetric matrix]

[Diagonilization].

and $D = c I_{p-1}.$

and $T'T = I_{p-1}$ [since T is orthogonal]

then $\underline{x_i}' \underline{x_j} = 0 \; \forall i, j$

i.e. columns of x are orthogonal.

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8) $H_0: \beta_1 = \beta_2 = \dots = \beta_{p-1} = 0$

$$R^2 = \frac{SS_{Reg}}{TSS}$$

~~For RSSR - RSS~~.

$$RSS_R = \sum_{i=1}^{n} (y_i - \hat{y}_{i(R)})^2.$$

$$= \sum_{i=1}^{n} (y_i - \hat{\beta}_0)^2$$

$$= \sum_{i=1}^{n} (y_i - \bar{y})^2 = TSS = \underset{\sim}{y}' (I - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T) \underset{\sim}{y}.$$

$$\cancel{= \underset{\sim}{y}' A_1 \underset{\sim}{y}}.$$

$$RSS = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

$$RSS_R - RSS = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 = SS_{Reg}.$$

$$= \underset{\sim}{y}' (I - H - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T) \underset{\sim}{y}.$$

$$\cancel{= \underset{\sim}{y}' A_2 \underset{\sim}{y}}.$$

Then,

$$\frac{\frac{RSS_R - RSS}{q}}{\frac{RSS}{n-p}} \sim F_{q, n-p}$$

Here $q = \text{rank}(I - H - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{1}^T) = p - 1.$

$$\therefore \frac{n-p}{p-1} \frac{RSS_R - RSS}{RSS} \sim F_{q, n-p}$$

$$\Rightarrow \frac{n-p}{p-1} \cdot \frac{SS_{Reg}}{RSS} \sim F_{q, n-p}.$$

Now, $R^2 = \frac{SS_{Reg}}{TSS}$ $\quad \frac{R^2}{1-R^2} = \frac{SS_{Reg}}{RSS}.$

$$\frac{n-p}{p-1} \cdot \frac{SS_{Reg}}{RSS} \sim F_{r,n-p} \quad [r = p-1]$$

$$\Rightarrow \frac{SS_{Reg}}{RSS} \sim \text{Beta}_{II}\left(\frac{p-1}{2}, \frac{n-p}{2}\right).$$

$(= X_0, \text{say})$

Now, if $X_0 \sim \text{Beta}_{II}\left(\frac{p-1}{2}, \frac{n-p}{2}\right)$

$$\Rightarrow \frac{1}{1+X_0} \sim \text{Beta}_{I}\left(\frac{n-p}{2}, \frac{p-1}{2}\right).$$

$$\Rightarrow 1-R^2 \sim \text{Beta}_{I}\left(\frac{n-p}{2}, \frac{p-1}{2}\right)$$

$$\Rightarrow \boxed{R^2 \sim \text{Beta}_{I}\left(\frac{p-1}{2}, \frac{n-p}{2}\right).}$$

$$E(R^2) = \frac{p-1}{p-1+n-p} = \frac{p-1}{n-1}$$

[Proved]