|| PROBLEM SET - 1 ||

1) $AB = I \Rightarrow \text{rank}(AB) = \text{rank}(I) = n$

Now, $\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\} \le n$
$= n$.

[as A & B are $n \times n$ matrices]

$\therefore \text{rank}(A) = n = \text{rank}(B)$.

$\therefore A$ & B are invertible.

$AB = I \Rightarrow B = A^{-1}$

$\therefore BA = A^{-1}.A = I$.

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2) Given: $A^2 = A$

Say, $\text{rank}(A) = r$.

$A^{n\times n} = B^{n\times r} C^{r\times n}$. where, B has left inverse & C has right inverse.

$A^2 = A.A = BC.BC = BC$

$\Rightarrow CBC = C$. [Pre Multiplying with left inverse of B]

$\Rightarrow CB = I_r$. [Post Multiplying with right inverse of C]

$\text{trace}(A) = \text{trace}(BC) = \text{trace}(CB) = \text{trace}(I_r) = r$.

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3) $X = (x_1, x_2, x_3)'$ $\quad var(X) = \begin{pmatrix} 5 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix}$.

(a) $\cdot\ x_1 - 2x_2 + x_3 = (1. \ -2 \ \ 1)' x$

$= l'x$

$var(l'x) = l'\, var(x)\, l.$

$= (1 \ -2 \ \ 1)' \begin{pmatrix} 5 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$

$= (1 \ -2 \ \ 1) \begin{pmatrix} 4 \\ -4 \\ 6 \end{pmatrix}$

$= 18.$

(b) $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$= AX$

$var(Y) = var(AX)$

$= A\, var(x)\, A'$

$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

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4) $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \left[\frac{1}{n}\ \frac{1}{n}\ \frac{1}{n} \cdots \frac{1}{n}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$= \underset{\sim}{l}' \underset{\sim}{x} \qquad \left[\underset{\sim}{l} = \left(\frac{1}{n}\ \frac{1}{n} \cdots \frac{1}{n}\right)' \quad \underset{\sim}{x} = (x_1, x_2, \ldots, x_n)' \right]$

$q = \sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$

$= \sum_{i=1}^{n} x_i . x_i - \frac{1}{n}(\Sigma x_i)(\Sigma x_i)$

$= \sum_{i=1}^{n} x_i^2 - \frac{1}{n} \sum_{i,j}\sum x_i x_j$

$= \sum_{i=1}^{n} \left(1 - \frac{1}{n}\right) x_i^2 - \frac{1}{n} \sum_{i \neq j}\sum x_i x_j$

$= \underset{\sim}{x}^T A \underset{\sim}{x}$

where $A = \begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ & & \vdots & & \\ -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} & \cdots & 1-\frac{1}{n} \end{bmatrix}$

A is symmetric

$= I - \frac{1}{n} \underset{\sim}{1}\underset{\sim}{1}^T$

$A^2 = \left(I - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)\left(I - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T\right)$

$= \left(I - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T + \frac{1}{n^2} \cdot n \underset{\sim}{1}\underset{\sim}{1}^T\right)$

$= I - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T$

$= A$.

$$A\underset{\sim}{l} = \left\{ \begin{aligned} &(I - \tfrac{1}{n}\underset{\sim}{1}\underset{\sim}{1}^T)\cdot \tfrac{1}{n}\cdot \underset{\sim}{1} \\ &= \tfrac{1}{n}\underset{\sim}{1} - \tfrac{1}{n^2}\cdot \underset{\sim}{1}\cdot n \end{aligned} \right.$$

$$= 0.$$

$\Rightarrow \bar{X}$ & $\Sigma(x_i - \bar{x})^2$ are independent

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57 $X_1, \ldots, X_n \overset{IID}{\sim} N(\mu, \sigma^2)$

$$\bar{x} = l'\underset{\sim}{x} \qquad l' = (\tfrac{1}{n}\ \tfrac{1}{n} \cdots \tfrac{1}{n})'$$

$$\sum_{i=1}^{n-1} (x_i - x_{i+1})^2$$

$$= (x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_{n-1} - x_n)^2$$

$$= x_1^2 + 2x_2^2 + \cdots + 2x_{n-1}^2 + x_n^2 - 2x_1x_2 - 2x_2x_3 - \cdots - 2x_{n-1}x_n.$$

$$= \underset{\sim}{x}^T A \underset{\sim}{x}$$

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

A is symmetric.

$$A^2 = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

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6) $Y \sim N_2(0, \Sigma)$. $\Sigma = (\sigma_{ij})$. $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$

$\sigma_{12} = \sigma_{21}$

$$Y'\Sigma^{-1}Y - Y_1^2/\sigma_{11}$$

$$= (Y_1\ Y_2) \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}^{-1} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} - Y_1^2/\sigma_{11}$$

$$= (Y_1\ Y_2) \frac{1}{\Delta} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} - \frac{Y_1^2}{\sigma_{11}} \quad [\Delta = \sigma_{11}\sigma_{22} - \sigma_{12}^2]$$

$$= Y_1^2 \left(\frac{\sigma_{22}}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} - \frac{1}{\sigma_{11}} \right) + \frac{\sigma_{11}}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} Y_2^2$$

$$- \frac{2\sigma_{12}}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} Y_1 Y_2$$

$$= Y_1^2 \left(\frac{\cancel{\sigma_{11}\sigma_{22}} - \cancel{\sigma_{11}\sigma_{22}} + \sigma_{12}^2}{\sigma_{11}(\sigma_{11}\sigma_{22} - \sigma_{12}^2)} \right) + \frac{\sigma_{11}}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} Y_2^2$$

$$- \frac{2\sigma_{12}}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} Y_1 Y_2.$$

$$A = \frac{1}{\Delta} \begin{bmatrix} \frac{\sigma_{12}^2}{\sigma_{11}} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix}.$$

$$A\Sigma = \frac{1}{\Delta} \begin{bmatrix} \sigma_{12}^2/\sigma_{11} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} 0 & \frac{\sigma_{12}^3}{\sigma_{11}} - \sigma_{12}\sigma_{11} \\ 0 & -\sigma_{12}^2 + \sigma_{11}\sigma_{22} \end{bmatrix} = \begin{bmatrix} 0 & -\sigma_{12}/\sigma_{11} \\ 0 & 1 \end{bmatrix}$$

$$(A\Sigma)^2 = \begin{bmatrix} 0 & -\sigma_{12}/\sigma_{11} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\sigma_{12}/\sigma_{11} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\sigma_{12}/\sigma_{11} \\ 0 & 1 \end{bmatrix} = A\Sigma$$

$\therefore A\Sigma$ is idempotent.

Now, $\underset{\sim}{Y} \sim N(0, \Sigma)$

& $A\Sigma$ is idempotent.

$Y'AY \sim \chi^2_{\text{rank}(A),\, \lambda}$

Now, $\text{rank}(A) = 1$

and $\lambda = \mu^T A \mu$.

$= 0 \; [\mu = 0]$

$\left[\begin{array}{l} \det(A) = 0. \\ \therefore \text{rank}(A) < 2 \\ \Rightarrow \text{rank}(A) = 1 \quad [\text{As}, A \neq 0] \end{array} \right]$

$\therefore Y'AY \sim \chi^2_1$ [Proved]

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8) 7) $\text{cov}(\hat{\beta}_0, \hat{\beta}_1)$

$\text{var}(\hat{\beta}_1) = \dfrac{\sigma^2}{S_{xx}}$

$= \text{cov}(\bar{y} - \hat{\beta}_1 \bar{x}, \hat{\beta}_1)$

$= \text{cov}(\bar{y}, \hat{\beta}_1) - \bar{x}\,\text{cov}(\hat{\beta}_1, \hat{\beta}_1)$

$= \text{cov}(\bar{y}, \hat{\beta}_1) - \bar{x}\,\text{var}(\hat{\beta}_1)$

$\text{cov}(\bar{y}, \hat{\beta}_1) = \dfrac{1}{n}\text{cov}\left(\sum_{i=1}^{n} y_i, \sum_{i=1}^{n} c_i y_i\right)$ $\quad c_i = \dfrac{x_i - \bar{x}}{\sum(x_i - \bar{x})^2}$

$= \dfrac{1}{n} \sum_{i=1}^{n}\sum_{j=1}^{n} c_j \,\text{cov}(y_i, y_j)$

$= \dfrac{1}{n} \sum_{i=1}^{n} c_i \,\text{var}(y_i)$ $\quad [\text{cov}(y_i, y_j) = 0 \ \forall i \neq j]$

$= \dfrac{1}{n} \sum_{i=1}^{n} \dfrac{x_i - \bar{x}}{\sum(x_i - \bar{x})^2} \cdot \sigma^2$

$= \dfrac{\sigma^2}{n} \cdot 0 = 0$

$\text{cov}(\hat{\beta}_0, \hat{\beta}_1) = \cancel{\text{cov}} \; -\bar{x} \cdot \text{var}(\hat{\beta}_1)$

$= -\bar{x} \cdot \dfrac{\sigma^2}{S_{xx}} = 0$

$\iff \bar{x} = 0$ [Proved]

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8). $y_i = \beta_0 + \beta_1(x_{i1} - \bar{x}_1) + \dots + \beta_{p-1}(x_{ip-1} - \bar{x}_{p-1}) + \epsilon_i$

$\underset{\sim}{y} = \beta_0 \underset{\sim}{1} + \tilde{X} \underset{\sim}{\beta} + \underset{\sim}{\epsilon}.$

$\underset{\sim}{\epsilon} = (\underset{\sim}{y} - \beta_0 \underset{\sim}{1} - \tilde{X} \underset{\sim}{\beta})$

$S = \underset{\sim}{\epsilon}^T \underset{\sim}{\epsilon} = (\underset{\sim}{y} - \beta_0 \underset{\sim}{1} - \tilde{X}\beta)^T (\underset{\sim}{y} - \beta_0 \underset{\sim}{1} - \tilde{X}\underset{\sim}{\beta})$

$$= \underset{\sim}{y}^T \underset{\sim}{y} - \beta_0 \underset{\sim}{1}^T \underset{\sim}{y} - \beta^T \tilde{X}^T \underset{\sim}{y}$$

$$- \beta_0 \underset{\sim}{y}^T \underset{\sim}{1} + \beta_0^2 \underset{\sim}{1}^T \underset{\sim}{1} + \beta^T \tilde{X}^T \beta_0 \underset{\sim}{1}$$

$$- \cancel{\beta^T\tilde{X}} \underset{\sim}{y}^T \tilde{X} \underset{\sim}{\beta} + \beta_0 1^T X \beta$$

$$+ \beta^T \tilde{X}^T \tilde{X} \beta.$$

$$\frac{\partial S}{\partial \beta_0} = -\underset{\sim}{1}^T \underset{\sim}{y} - \underset{\sim}{y}^T \underset{\sim}{1} + 2\beta_0 \underset{\sim}{1}^T \underset{\sim}{1} + \beta^T \tilde{X}^T \underset{\sim}{1} + 1^T X \underset{\sim}{\beta}$$

$$= -2\underset{\sim}{y}^T \underset{\sim}{1} + 2n\beta_0 + 2(\tilde{X}\beta)^T \underset{\sim}{1} = 0$$

$$\Rightarrow \hat{\beta}_0 = \frac{1}{n}\left[\underset{\sim}{y}^T \underset{\sim}{1} - (\tilde{X}\beta)^T \underset{\sim}{1}\right]$$

$$= \bar{y} \quad \left[\text{Since, column sum of } \tilde{X}\beta \text{ is } 0\right]$$

$\frac{\partial}{\partial x} a'x = a$

$$\frac{\partial S}{\partial \beta} = -2\tilde{X}^T \underset{\sim}{y} + 2\beta_0 \tilde{X}^T \underset{\sim}{1} + 2\tilde{X}^T \tilde{X} \beta = 0$$

$$\Rightarrow -2\tilde{X}^T \underset{\sim}{y} + \cancel{2\tilde{X}^T} 2\tilde{X}^T \underset{\sim}{X} \beta = 0 \quad [\tilde{X}^T \underset{\sim}{1} = \underset{\sim}{0}]$$

$$\Rightarrow \hat{\beta} = (\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T \underset{\sim}{y}.$$

97. $y_i = \beta_0 + \beta_1 (x_{1i} - \bar{x}_1)/s_1 + \cdots + \beta_{p-1} (x_{(p-1)i} - \bar{x}_{p-1})/s_{p-1} + \epsilon_i$

$$Y = \beta_0 1 + \cancel{\beta}\tilde{X} \beta + \epsilon.$$

$$\tilde{X} = \begin{bmatrix} \frac{x_{11} - \bar{x}_1}{s_1} & \cdots & \frac{x_{(p-1)i} - \bar{x}_{p-1}}{s_{p-1}} \\ \vdots & & \\ \frac{x_{n1} - \bar{x}_1}{s_1} & \cdots & \frac{x_{(p-1)i} - \bar{x}_{p-1}}{s_{p-1}} \end{bmatrix}$$

$$S = \epsilon^T \epsilon = (y - \beta_0 1 - \tilde{X}\beta)^T (y - \beta_0 1 - \tilde{X}\beta).$$

$$\frac{\partial S}{\partial \beta_0} = 0 \Rightarrow \hat{\beta}_0 = \frac{1}{n}\left[y^T 1 - (X\beta)^T 1\right]$$

$$= \bar{y} - \frac{1}{n}(X\hat{\beta})^T 1$$

$$= \bar{y}$$

$$\frac{\partial S}{\partial \beta} = 0 \Rightarrow -\tilde{X}^T y + \beta_0 \tilde{X}^T 1 + \tilde{X}^T \tilde{X} \beta = 0$$

$$\Rightarrow \hat{\beta} = (\tilde{X}^T \tilde{X})^{-1} \tilde{X} Y.$$

$$\Rightarrow -\tilde{X}^T y + \left(\frac{1}{n} y^T 1 - \frac{1}{n}(X\hat{\beta})^T 1\right) \tilde{X}^T 1 + \tilde{X}^T \tilde{X} \hat{\beta} = 0$$

$$\Rightarrow -\tilde{X}^T y + \frac{1}{n} 1^T y \cdot \tilde{X}^T 1 - \frac{1}{n} \beta^T \tilde{X}^T 1 \cancel{- \frac{1}{n} 1^T \beta \tilde{X} \beta \tilde{X}^T 1} + \tilde{X}^T \tilde{X} \hat{\beta} = 0$$

$$\operatorname{var}(\hat{\beta}) = \sigma^2 (\tilde{X}^T \tilde{X})^{-1}$$

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107. $Y_1 = \alpha_1 + \epsilon_1$

$Y_2 = 2\alpha_1 - \alpha_2 + \epsilon_2$

$Y_3 = \alpha_1 + 2\alpha_2 + \epsilon_3$

$$\underset{\sim}{Y} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \underset{\sim}{\epsilon}$$

$$= X \underset{\sim}{\beta} + \underset{\sim}{\epsilon}.$$

$$\hat{\beta} = (X^T X)^{-1} X' \underset{\sim}{Y}.$$

$$X^T X = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}$$

$$(X^T X)^{-1} = \cancel{\frac{1}{30}\begin{pmatrix} 5 & 0 \\ 0 & 6 \end{pmatrix}} = \begin{pmatrix} 1/6 & 0 \\ 0 & 1/5 \end{pmatrix}.$$

$$\hat{\beta} = (X^T X)^{-1} X' Y = \begin{pmatrix} 1/6 & 0 \\ 0 & 1/5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1/6 & 1/3 & 1/6 \\ 0 & -1/5 & 2/5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{pmatrix} = \begin{pmatrix} 1/6\, y_1 + 1/3\, y_2 + 1/6\, y_3 \\ -1/5\, y_2 + 2/5\, y_3 \end{pmatrix}$$

$H_0 : \alpha_1 = \alpha_2 \Rightarrow \alpha_1 - \alpha_2 = 0$

$$\cancel{A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \hat{\beta} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \quad C =}$$

$A = [1 \ -1]$, $\hat{\beta} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$, $C = 0$

$$\hat{\beta}_R = \hat{\beta} + (X'X)^{-1} A' [A(X'X)^{-1} A']^{-1} [C - A\hat{\beta}]$$

$$\cancel{= \begin{bmatrix} \frac{1}{6} y_1 + \frac{1}{3} y_2 + 1/6\, y_3 \\ -1/5\, y_2 \end{bmatrix}}$$

$$= \hat{\beta} + \begin{bmatrix} 1/6 & 0 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left[[1 \ -1] \begin{bmatrix} 1/6 & 0 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right]^{-1} (0 - [1 \ -1]\hat{\beta})$$

$$\hat{\sigma}_R = \frac{\|Y - X\hat{\beta}_R\|^2}{n} \qquad \Delta = \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_R^2}\right)^{n/2}$$

$$\hat{\sigma} = \frac{\|Y - X\hat{\beta}\|^2}{n}$$

~~LR~~.

Test statistic $= \left(\frac{3-2}{1}\right)\left(\Delta^{2/n} - 1\right)$

$= \Delta^{2/n} - 1$