MTH441

Sol 1.

Block \ ~~Treat~~	Treatment			
a	1	2	3	4
b	2	1	4	3
c	3	4	2	1
d	4	3	1	2

$y_{ij} = \mu + \alpha_i + B_j + \epsilon_{ij}$ $i = a, b, c, d$
$j = 1, 2, 3, 4.$

$\Sigma \alpha_i = 0$ & $\Sigma B_j = 0$

Block	Treatment				
a	3	2	5	6	16
b	2	4	7	8	21
c	9	2	1	5	17
d	5	6	4	3	18
					72

Least Square estimates of $\mu, \alpha_i, B_i \ \forall i, j.$

$$\sum_j \sum_{i=1}^{4} \epsilon_{ij}^2 = \sum_j \sum_{i=1}^{4} (y_{ij} - \mu - \alpha_i - B_j)^2$$

$$\frac{\partial \epsilon_{ij}^2}{\partial \mu} = 2\sum\sum_{i=1}^{4} (y_{ij} - \mu - \alpha_i - B_j)(-1) \overset{set}{=} 0$$

$$= \sum\sum y_{ij} - \sum\sum \mu - \underbrace{\sum\sum \alpha_i}_{=0} - \underbrace{\sum\sum B_j}_{=0} = 0$$

$$4 \times 4 \times \mu = \sum_{j=1}^{4} \sum_{i=1}^{4} y_{ij}$$

$$\hat{\mu}_{LSE} = \frac{\sum_i \sum_j y_{ij}}{4 \times 4} = \bar{y}_{..} = \frac{72}{4 \times 4} = \frac{9}{2} = 4.5.$$

$$\cancel{\frac{\sum \epsilon_{ij}^2}{}}$$

Let $\epsilon' = \sum_{j=1}^{4}\sum_{i=1}^{4} \epsilon_{ij}^2$

$$\frac{\partial^2 \epsilon}{\partial \mu^2} = +4\times 4 \geq 0 \Rightarrow \cancel{\text{max}}.\ \text{min}.$$

$$\frac{\partial \epsilon}{\partial \alpha_i} = 2\sum_{j=1}^{4}(y_{ij} - \mu - \alpha_i - \beta_j)(-1) \overset{\text{set.}}{=} 0.$$

$$\frac{\partial^2 \epsilon}{\partial \alpha_i^2} = 2\times 4 > 0 \Rightarrow \text{min}.$$

$$\Rightarrow \sum_{j=1}^{4} y_{ij} - \sum_{j=1}^{4}\mu - \sum_{j=1}^{4}\alpha_i - \underbrace{\sum_{j=1}^{4}\beta_j}_{=0} = 0$$

$$= \sum_{j=1}^{4} y_{ij} - 4\mu = 4\alpha_i$$

$$= \hat{\alpha}_i = \frac{\sum_{j=1}^{4} y_{ij}}{4} - \mu.$$

$$= \bar{y}_{i\circ} - \bar{y}_{\circ\circ}$$

$$\boxed{\alpha_a = \frac{16}{4} - 4.5 = -0.5}$$

Find all α_i's similarly.

Similarly

$$\hat{\beta}_j = \bar{y}_{\circ j} - \bar{y}_{\circ\circ}$$

$$\hat{\beta}_1 = \frac{19}{4} - 4.5 = 4.75 - 4.5 = 0.25.$$

Solⁿ. 2.

$$\eta_i = \sum_{j=1}^{p} x_{ij}\beta_j \quad \begin{matrix} i=1,\dots n \\ j=1,\dots p. \end{matrix}$$

Prob. mass function

$$f(y_i, \mu_i) = \frac{e^{-\mu_i}\mu_i^{y_i}}{y_i!}$$

(a)

$$f_y(y_i) = \exp\{[y_i\gamma_i - b(\gamma_i)]/\tau^2 - c(y_i - \tau)$$

$$\eta(x) = \beta_0 + \beta_1 x_1 + \cdots + \beta_{p-1} x_{p-1}$$

$$\boxed{\log[E[y_i]] = \eta_i(x)} \longrightarrow \text{Canonical link function}$$

(a)

$$\log E[f(y_i, \mu_i)] = \eta_i$$

$$\Rightarrow \boxed{\mu_i = e^{\sum_{j=1}^{p} x_{ij}\beta_j}} \quad \text{Canonical link function for Possion regression.}$$

(b)

$$f(y_i, \mu_i) = \frac{e^{-\mu_i}\mu_i^{y_i}}{y_i!}$$

$$= e^{-\mu_i} e^{\log \mu_i^{y_i}} \cdot e^{-y_i!}$$

$$= \exp\{[\underbrace{-\mu_i}_{b(\gamma_i)} + \underbrace{y_i \log\mu_i}_{y_i\gamma_i} - \underbrace{y_i!}_{c(y_i-\tau)}]$$

~~where~~

$$\gamma_i = \log\mu_i \qquad \tau^2 = 1$$

$$b(\gamma_i) = \mu_i \qquad c(y_i - \tau) = y_i!$$

where $E[y_i] = b'(\gamma_i)$

$$\mathrm{Var}(y_i) = \tau^2 \cdot b''(\gamma_i)$$

So, $E[y_i] = b'(\gamma_i) = \frac{d e^{\gamma_i}}{d\gamma_i} = e^{\gamma_i} = \mu_i$

$$\mathrm{Var}[y_i] = \tau^2 b''(\gamma_i) = \frac{d}{d\gamma_i}(b'(\gamma_i))$$

$$= \frac{d e^{\gamma_i}}{d\gamma_i} = \mu_i$$

(c) $\mathrm{Var}(\hat{\beta})$ by C.R. lowerbound.

$$\cancel{\theta} = \frac{E[g'(\theta)]^2}{E\left[-\frac{\partial^2 \log L}{\partial \theta^2}\right]}$$

$$= \frac{1}{E\left[-\frac{\partial^2 L}{\partial \beta_u \beta_v}\right]}$$

$$L = \prod_{i=1}^{n} \frac{e^{-\mu_i} \mu_i^{y_i}}{y_i!}$$

$$\log L = \sum_{i=1}^{n} \left[-\mu_i + y_i \log \mu_i - \log y_i!\right]$$

$$\text{Now, } \mu_i = e^{\sum x_{ij}\beta_j} \quad \text{from (a)}$$

$$\log L = \sum_{i=1}^{n} \left[-e^{\sum_{j=1}^{p} x_{ij}\beta_j} + y_i \left(\sum_{j=1}^{p} x_{ij}\beta_j\right) - \log y_i!\right]$$

$$\frac{\partial L}{\partial \beta_j} = \sum_{i=1}^{n} \left[-e^{\sum_{j=1}^{p} x_{ij}\beta_j}(x_{ij}) + y_i(x_{ij})\right]$$

$$\frac{\partial^2 L}{\partial \beta_j \partial \beta_{j'}} = \sum_{i=1}^{n} \left[-e^{\sum_{j=1}^{p} x_{ij}\beta_j}(x_{ij})(x_{ij'}) + \cancel{0}\right]$$

$$-\frac{\partial^2 \log L}{\partial B_j \partial B_{j'}} = \sum_{i=1}^{n} \left[x_{ij'} e^{\sum_{j=1}^{p} x_{ij} B_j} x_{ij} \right]$$

$\forall j \ \& \ j'$ ~~s.t. j ≠ j'~~

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & & & \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{bmatrix}$$

$$W = \begin{bmatrix} e^{\eta_1} & 0 & & 0 \\ 0 & e^{\eta_2} & & 0 \\ & & \ddots & 0 \\ 0 & 0 & & e^{\eta_n} \end{bmatrix}$$

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ \vdots & & & \\ \vdots & & & \end{bmatrix}$$

$$= X^T W X$$

So, $Var(\hat{\beta}) = (X^T W X)^{-1}$

3(i)

$$y_i = \beta_0 + \beta_1(x_{i1} - \bar{x}_1) + \cdots + \beta_{p-1}(x_{i,p-1} - \bar{x}_{p-1}) + \varepsilon_i \quad ; i = 1, \dots, n$$

$$\bar{x}_j = \sum_i x_{ij}/n$$

β_{0LSE} & β_{LSE} for $\beta = (\beta_1, \dots, \beta_{p-1})'$

$$y_i = \beta_0 + \underset{\sim}{x}^T \beta + \varepsilon_i$$

$$\underset{\sim}{x}_i = \begin{bmatrix} x_{i1} - \bar{x}_1 \\ x_{i2} - \bar{x}_2 \\ \vdots \\ x_{i,p-1} - \bar{x}_{p-1} \end{bmatrix} \qquad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_{p-1} \end{bmatrix}$$

$$\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} (y_i - \beta_0 - x_i^T \beta)^2$$

$$\frac{d}{d\beta_0} = n\bar{y} - n\beta_0 - \sum_{i=1}^{n} x_i^T \beta = 0 \quad -① $$

$$\frac{d}{d\beta} = \sum_{i=1}^{n} (y_i - \beta_0 - x_i^T \beta) x_i \quad -②$$

Do in matrix format.

get $\hat{\beta}_0$ & $\hat{\beta}_1$

4.

$$y_A = X_A \beta + \varepsilon$$

$$\varepsilon^T \varepsilon = (y_A - X_A\beta)^T (y_A - X_A\beta)$$

$$Var(\hat{\beta}_j) = \sigma^2 (X_A^T X)^{-1}$$

$$\hat{\beta}_j = (X_A^T X_A)^{-1} X_A^T y.$$

$$Var(\hat{\beta}_j) = Var((X_A^T X_A)^{-1} X_A^T y)$$

$$= \cancel{(X_A^T X_A)^T} X_A^T X_A (X_A^T X_A)^{-1} \sigma^2 I_n.$$

$$= (X_A^T X^T)^{-1} \sigma^2.$$

$$\sum_{j=1}^{n} Var(\hat{\beta}_j) = \sigma^2 tra (X_A^T X^T)^{-1}$$

$$X_A = \begin{pmatrix} X_{n\times p} \\ \sqrt{k} I_p \end{pmatrix}$$

$$X_A^T X_A = [X_{p\times n}^T \quad \sqrt{k} I_p] \begin{bmatrix} X_{np} \\ \sqrt{k} I_p \end{bmatrix}$$

$$= X^T_{p\times n} X_{n\times p} + k I_{p\times p}$$

$$= [X^T X + kI]_{p\times p}$$

$$= \begin{bmatrix} 1-k & 0 & \cdots & 0 \\ 0 & 2-k & & \\ \vdots & & \ddots & \\ 0 & & & p-k \end{bmatrix} + \begin{bmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & & & 0 \\ & 2 & & \\ & & \ddots & \\ 0 & & & p \end{bmatrix} = X_A^T X$$

$$= \cancel{X_A^T X} \quad \cancel{\operatorname{tra}(X_A^T X)}$$

$$(X^T X)^{-1} = \begin{bmatrix} 1 & & & 0 \\ & \frac{1}{2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{p} \end{bmatrix}$$

$$\operatorname{tra}(X^T X)^{-1} = \sum_{j=1}^{p} \frac{1}{j}$$

$$\sum_{j=1} \operatorname{Var}(\hat{\beta}_j) = \sigma^2 \sum_{j=1}^{p} \frac{1}{j}$$