Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

Time Series Analysis (MTH442)

Mid-semester examination, Date: Sep 17, 2024, Tuesday

Time: 6:00–7:30 PM Max point: 25

Carrying mobile phones during the exam, even in silent mode, is not allowed. It should be kept in your bag in silent mode or switched off and the bag should be kept in front of the invigilator only. A maximum of only one bio-break will be given during the exam for each student and the duration of leaving the room to coming back cannot exceed 5 minutes overall. Anyone asked to change seat during the exam should do that immediately without any further argument.

All answers should be to the point. Any unnecessary extra sentences will be penalized. Please fold your answer sheet horizontally in the middle and a maximum of half page can be used per one point (exceptions for Q2 and Q6). If any rough work is necessary, you can use the end of the answer booklet. Calculation mistakes will be penalized. No extra time will be provided. Answers should be written in a neat and clean handwriting. No point will be given for unreadble answers.

1. Write down the three conditions of a white noise process. (0.5+0.5+0.5=1.5 points, no partial credit)

2. Let W_t, for $t = 0, 1, 2, \ldots$ be an independent white noise process and consider the series $X_t = W_{t-1}W_tW_{t+1}$ for $t = 1, 2, \ldots$. Check whether X_t is a white noise process or not, i.e., check the three conditions of Question 1. (1+1+1 = 3 points)

3. Consider the the random walk with drift model given by $X_t = \delta + X_{t-1} + W_t$ for $t = 1, 2, \ldots$, with initial condition $X_0 = 0$, and where W_t is white noise. Calculate $\text{Cov}(X_s, X_t)$ for some finite s and t. (1 point)

4. Write down the definition of a strictly stationary time series. (1 point, no partial credit)

5. Is trend stationary model a special case of a weakly stationary model? Justify. No point for a wrong justification, even if the answer to the first part is correct. (1 point)

6. A time series with K periodic components can be constructed from

$$X_t = \beta_1 + \beta_2 t + \sum_{k=1}^{K} \{U_{1,k} \sin(2\pi\omega_k t) + U_{2,k} \cos(2\pi\omega_k t)\},$$

   where β_1 and β_2 are known constants, ω_k are known frequencies, $U_{1,k}$ and $U_{2,k}$ are independent random variables (across k as well) with zero means and $E(U_{1,k}^2) = E(U_{2,k}^2) = \sigma_k^2$ for $k = 1, \ldots, K$. Show that X_t series is a trend stationary process (check three conditions). (1+1+1=3 points)

7. Is the differenced process $Y_t = \nabla X_t = X_t - X_{t-1}$ a weakly stationary process? Justify. No point for a wrong justification, even if the answer to the first part is correct. (1 point)

8. For a bivariate time series $(X_t, Y_t), t = 1, 2, \ldots$, calculate the absolute difference of CCFs $|\rho_{X,Y}(h) - \rho_{Y,X}(-h)|$. (1 point)

9. For two time series $\{X_t\}$ and $\{Y_t\}$ observed for $t = 1, \ldots, T$, write down the definitions of sample ACF $\hat{\rho}_X(h)$ and sample CCF $\hat{\rho}_{X,Y}(h)$. (0.5+0.5=1 point, no partial credit)

10. Along with actual sample ACF values for different lags, an ACF plot generated in `R` has two horizontal dashed lines. What are the Y-axis values of them and what do they represent? (0.5+0.5=1 point, no partial credit)

11. If X_t represents the observations, then for $M_t = \sum_{j=-k}^{k} a_j X_{t-j}$ being a symmetric moving average, what are the two conditions on a_j's? (0.5+0.5=1 point, no partial credit)

12. Smoothing splines minimize a compromise between the fit and the degree of smoothness given by $\sum_{t=1}^{T}(X_t - m_t)^2 + \lambda \int (m_t^{''})^2 dt$. When $\lambda = \infty$, how does the solution look like? Justify. No point for a wrong justification, even if the answer to the first part is correct. (1 point)

13. Represent a causal AR(1) model $X_t = \phi X_{t-1} + W_t$ as a linear process in the limiting mean square sense. (1.5 points)

14. For an invertible MA(1) model $X_t = W_t + \theta W_{t-1}$, represent W_t as a linear process of X_t's in the limiting mean square sense. (1.5 points)

15. A causal ARMA(p, q) model $\{X_t; t = 0, \pm 1, \pm 2, \ldots\}$ can be written as a one-sided linear process $X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j} = \psi(B) W_t$, where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$, and $\sum_{j=0}^{\infty} |\psi_j| < \infty$; we set $\psi_0 = 1$. For an ARMA(1,1) model $X_t = \phi X_{t-1} + W_t + \theta W_{t-1}$, calculate the ψ-weights ψ_j for $j = 1, 2$. (1+1=2 points)

16. An ARMA(p, q) model is said to be invertible, if the time series $\{X_t; t = 0, \pm 1, \pm 2, \ldots\}$ can be written as $\pi(B) X_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} = W_t$, where $\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j$, and $\sum_{j=0}^{\infty} |\pi_j| < \infty$; we set $\pi_0 = 1$. For an ARMA(1,1) model $X_t = \phi X_{t-1} + W_t + \theta W_{t-1}$, calculate the π-weights π_j for $j = 1, 2$. (1+1=2 points)

17. Prove that the Partial ACF for an AR(1) process is zero for lag 2. (1.5 points)