Lecture 21

Forecasting Part 2

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One-step-ahead prediction (Recap)

- The BLP of X_{T+1} is of the form

$$X_{T+1}^{T} = \phi_{T,1}X_T + \phi_{T,2}X_{T-1} + \ldots + \phi_{T,T}X_1 = \sum_{j=1}^{T} \phi_{T,j}X_{T+1-j}.$$

- The coefficients $\{\phi_{T,1}, \phi_{T,2}, \ldots, \phi_{T,T}\}$ satisfy

$$\sum_{j=1}^{T} \phi_{T,j}\, \gamma(k-j) = \gamma(k), \quad k = 1, \ldots, T.$$

- In matrix notations, $\mathbf{\Gamma}_T\, \boldsymbol{\phi}_T = \boldsymbol{\gamma}_T$ and hence, $\boldsymbol{\phi}_T = \mathbf{\Gamma}_T^{-1}\boldsymbol{\gamma}_T$ and $X_{T+1}^{T} = \boldsymbol{\phi}_T' \tilde{\boldsymbol{X}}$, where $\tilde{\boldsymbol{X}} = (X_T, X_{T-1}, \ldots, X_1)$.

Mean square prediction error (Recap)

- The mean square one-step-ahead prediction error is

$$P_{T+1}^{T} = E[(X_{T+1} - X_{T+1}^{T})^2] = \gamma(0) - \gamma_T' \mathbf{\Gamma}_T^{-1} \gamma_T.$$

- For a causal AR(2) model $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t$, we have

$$\phi_{T,1} = \phi_1, \phi_{T,2} = \phi_2, \phi_{T,3} = \ldots = \phi_{T,T} = 0,$$

as $E[(X_{T+1} - \phi_1 X_T - \phi_2 X_{T-1}) X_k] = E[W_{T+1} X_k] = 0$ for $k = T, T-1, \ldots, 1$.

- By similar logic, for a causal AR(p) process $X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + W_t$, we have

$$\phi_{T,1} = \phi_1, \ldots, \phi_{T,p} = \phi_p, \phi_{T,p+1} = \ldots = \phi_{T,T} = 0,$$

for $T \geq p$.

Fast solution of ϕ_T and P_{T+1}^T

- For ARMA models in general, the prediction equations will not be as simple as the pure AR case.
- In addition, for T large, the use of $\phi_T = \mathbf{\Gamma}_T^{-1}\gamma_T$ is prohibitive because it requires the inversion of a large matrix.
- There are, however, iterative solutions that do not require any matrix inversion.
- In this context, we discuss the Durbin-Levinson Algorithm.

Durbin-Levinson Algorithm

- Equations $\phi_T = \mathbf{\Gamma}_T^{-1}\gamma_T$ and $P_{T+1}^T = \gamma(0) - \gamma_T' \mathbf{\Gamma}_T^{-1}\gamma_T$ can be solved iteratively as follows:

$$\phi_{0,0} = 0, \quad P_1^0 = \gamma(0),$$

for $T \geq 1$,

$$\phi_{T,T} = \frac{\rho(T) - \sum_{k=1}^{T-1} \phi_{T-1,k}\, \rho(T-k)}{1 - \sum_{k=1}^{T-1} \phi_{T-1,k}\, \rho(k)}$$

and

$$P_{T+1}^T = P_T^{T-1}(1 - \phi_{T,T}^2),$$

where, for $T \geq 2$,

$$\phi_{T,k} = \phi_{T-1,k} - \phi_{T,T}\, \phi_{T-1,T-k}, \quad k = 1, 2, \ldots, T-1.$$

Use of Durbin-Levinson Algorithm

- For $T = 1$?
- For $T = 2$?
- For $T = 3$?
- In general,
$$P_{T+1}^{T} = \gamma(0) \prod_{j=1}^{T} [1 - \phi_{j,j}^2].$$
- Iterative Solution for the PACF: The PACF of a stationary process X_t can be obtained iteratively via Durbin-Levinson Algorithm as $\phi_{T,T}$, for $T = 1, 2, \ldots$.

PACF of AR(p) process

- We have seen

$$X_{T+1}^{T} = \phi_{T,1}X_T + \phi_{T,2}X_{T-1} + \ldots + \phi_{T,T}X_1 = \sum_{j=1}^{T} \phi_{T,j}X_{T+1-j}.$$

- Putting $T = p$, we have

$$X_{p+1}^{p} = \phi_{p,1}X_p + \phi_{p,2}X_{p-1} + \ldots + \phi_{p,p}X_1.$$

- At the same time, from BLP equations, we have seen

$$X_{p+1}^{p} = \phi_1 X_p + \phi_2 X_{p-1} + \ldots + \phi_p X_1.$$

- Hence, $\phi_{p,p} = \phi_p$.
- Calculate the PACF values $\phi_{1,1}$, $\phi_{2,2}$, and $\phi_{3,3}$ of an AR(2) model.

m-step-ahead prediction

- The BLP of X_{T+m} is of the form

$$X_{T+m}^{T} = \phi_{T,1}^{(m)} X_T + \phi_{T,2}^{(m)} X_{T-1} + \ldots + \phi_{T,T}^{(m)} X_1 = \sum_{j=1}^{T} \phi_{T,j}^{(m)} X_{T+1-j}.$$

- The coefficients $\{\phi_{T,1}^{(m)}, \phi_{T,2}^{(m)}, \ldots, \phi_{T,T}^{(m)}\}$ satisfy

$$E\left[\left(X_{T+m} - \sum_{j=1}^{T} \phi_{T,j}^{(m)} X_{T+1-j}\right) X_{T+1-k}\right] = 0, \quad k = 1, \ldots, T.$$

- This implies

$$\sum_{j=1}^{T} \phi_{T,j}^{(m)} \, \gamma(k-j) = \gamma(m+k-1), \quad k = 1, \ldots, T.$$

- In matrix notations, $\mathbf{\Gamma}_T \, \phi_T^{(m)} = \gamma_T^{(m)}$ and hence, $\phi_T^{(m)} = \mathbf{\Gamma}_T^{-1} \gamma_T^{(m)}$ and $X_{T+m}^{T} = \phi_T^{(m)\prime} \tilde{\boldsymbol{X}}$, where $\tilde{\boldsymbol{X}} = (X_T, X_{T-1}, \ldots, X_1)$.

Mean square m-step-ahead prediction error

- Show that the mean square m-step-ahead prediction error is

$$P_{T+m}^{T} = E[(X_{T+m} - X_{T+m}^{T})^2] = \gamma(0) - \boldsymbol{\gamma}_T^{(m)\prime} \boldsymbol{\Gamma}_T^{-1} \boldsymbol{\gamma}_T^{(m)}.$$

The Innovations Algorithm

- The one-step-ahead predictors, X^T_{T+1}, and their mean-squared errors, P^T_{T+1}, can be calculated iteratively as

$$X^0_1 = 0, \quad P^0_1 = \gamma(0),$$

for $t \geq 1$,

$$X^t_{t+1} = \sum_{j=1}^{t} \theta_{t,j}(X_{t+1-j} - X^{t-j}_{t+1-j})$$

and

$$P^t_{t+1} = \gamma(0) - \sum_{j=0}^{t-1} \theta^2_{t,t-j} P^j_{j+1},$$

where, for $j = 0, \ldots, t-1$,

$$\theta_{t,t-j} = [\gamma(t-j) - \sum_{k=0}^{j-1} \theta_{j,j-k}\theta_{t,t-k} P^k_{k+1}]/P^j_{j+1}.$$

Use of Innovations Algorithm for MA(1) process

- Consider an MA(1) model, $X_t = W_t + \theta W_{t-1}$. Recall that $\gamma(0) = (1+\theta^2)\sigma_w^2$, $\gamma(1) = \theta\sigma_w^2$, and $\gamma(h) = 0$ for $h > 1$.
- $\theta_{T,1} = \theta\sigma_w^2 / P_T^{T-1}$
- $\theta_{T,j} = 0, \;\; j = 2, \ldots, T$
- $P_1^0 = (1+\theta^2)\sigma_w^2$
- $P_{T+1}^T = (1+\theta^2 - \theta\theta_{T,1})\sigma_w^2$
- Finally, $X_{T+1}^T = \theta(X_T - X_T^{T-1})\sigma_w^2 / P_T^{T-1}$

Thank you!