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$7\frac{1}{2}$

MTH517A: Time Series Analysis
Quiz #1; Full Marks-20

$\{\varepsilon_t\},\{\delta_t\}$ are two mutually independent i.i.d. sequence of random variables such that $\varepsilon_t \overset{i.i.d.}{\sim} N(0,\sigma^2)$ and $\delta_t \overset{i.i.d.}{\sim} N(0,\sigma^2)$. Let $\{Y_t\}$ be such that $Y_t = (\alpha+\beta t)S_t^2 + \nabla\varepsilon_t$; S_t is a seasonal component with period 4. Prove or disprove the following statements:

(a) $\nabla_4 Y_t$ is Gaussian.
(b) $\nabla\nabla_4 Y_t$ does not have any trend and seasonal component.
(c) $\nabla_4^2 Y_t$ is covariance stationary.
(d) $Cov(\nabla_4^2 Y_{t+h}, \nabla_4^2 Y_t) = 0;\ \forall |h| > 8$.
(e) $(\nabla\varepsilon_t + \delta_t + \delta_{t-1})$ is white noise.
(f) $(\varepsilon_{2t} - \varepsilon_{3t} + \nabla\delta_t)$ is strict stationary.

$\{\epsilon_t\}$, $\{\delta_t\}$ are mutually independent iid seq of r.v. s.t. $\epsilon_t \overset{iid}{\sim} N(0,\sigma^2)$ and $\delta_t \overset{iid}{\sim} N(0,\sigma^2)$.

Let, $Y_t = (\alpha+\beta t) S_t^2 + \nabla \epsilon_t = (\alpha+\beta t) S_t^2 + \epsilon_t - \epsilon_{t-1}$

S_t is seasonal component with period 4.

$\Rightarrow \quad \ldots = S_{t-8} = S_{t-4} = S_t = S_{t+4} = S_{t+8} = \ldots$

$\Rightarrow \quad \ldots = S_{t-8}^2 = S_{t-4}^2 = S_t^2 = S_{t+4}^2 = S_{t+8}^2 = \ldots \quad \text{——— } (*)$

(a) Let, $Z_t = \nabla_4 Y_t = Y_t - Y_{t-4}$

$$= (\alpha+\beta t) S_t^2 + \epsilon_t - \epsilon_{t-1} - \{\alpha+\beta(t-4)\} S_{t-4}^2 - \epsilon_{t-4} + \epsilon_{t-5}$$

$$= \alpha(S_t^2 - S_{t-4}^2) + \beta t (S_t^2 - S_{t-4}^2) + 4\beta S_{t-4}^2 + \epsilon_t - \epsilon_{t-1} - \epsilon_{t-4} + \epsilon_{t-5}$$

$$\therefore Z_t = 4\beta S_{t-4}^2 + \epsilon_t - \epsilon_{t-1} - \epsilon_{t-4} + \epsilon_{t-5}$$

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MTH517A: Time Series Analysis
Quiz #2; Full Marks-20

Let $\underline{X}_t = \begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix}$, $X_{1,t} = 0.5(1+X_{1,t-1}) + \varepsilon_{1,t}$ and $X_{2,t} = 0.25(1+X_{2,t-1}) + \varepsilon_{2,t}$; $Cov(\varepsilon_{i,t}, X_{i,t-j}) = 0, i = 1,2$ and

$\forall j > 0$; $\underline{\varepsilon}_t = \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{pmatrix}$ is an independent sequence of $N_2(\underline{0}, 2I_2)$. Consider a random sample

$\begin{pmatrix} X_{1,1} \\ X_{2,1} \end{pmatrix}, \begin{pmatrix} X_{1,2} \\ X_{2,2} \end{pmatrix}, \dots, \begin{pmatrix} X_{1,n} \\ X_{2,n} \end{pmatrix}$ from the above bivariate time series.

(1) Prove or disprove the following statements:

A: $\underline{\eta}_t = (\varepsilon_{1,t}, \varepsilon_{2,t}, \varepsilon_{1,t-1})^T \sim VWN$

B: $\underline{X}_t$ can be expressed as a covariance stationary VAR(1)

C: $Cov(\underline{X}_t, \underline{X}_{t+3})$ is a positive definite diagonal matrix

D: $\bar{X}_{1(n)} \left(= n^{-1} \sum_{t=1}^{n} X_{1,t} \right) \xrightarrow{m.s.} 0.5$

$\begin{pmatrix} 0.5 & 0 \\ 0 & 0.25 \end{pmatrix} \begin{matrix} X_{1,t-1} \\ X_{2,t-1} \end{matrix}$

(2) Find $\lim_{n\to\infty} nV(\bar{Y}_{(n)})$; $\bar{Y}_{(n)}$ is the sample mean of $Y_t = X_{1,t} + X_{2,t}$

$\bar{Y}_{(n)} = \frac{1}{n} \sum_{t=1}^{n} (X_{1,t} + X_{2,t})$

2. $\lim_{n\to\infty} n \cdot V(\bar{Y}_{(n)}) = g_Y(1)$ where $g_Y(\cdot)$ is the ACGF

why does this hold here?

of $Y_t = X_{1,t} + X_{2,t}$

$\because$ $g_Y(h) = g_{X_1}(h) + g_{X_2}(h)$, we have,

$g_Y(1) = g_{X_1}(1) + g_{X_2}(1)$. ———— (*).

Now, $X_{1,t} = 0.5 + 0.5 X_{1,t-1} + \epsilon_{1,t}$ $\longrightarrow$ An AR(1) process.

with $\phi_1(B) = 1 - 0.5B$

and, $X_{2,t} = 0.25 + 0.25 X_{2,t-1} + \epsilon_{2,t}$ $\longrightarrow$ An AR(1) process

with $\phi_2(B) = 1 - 0.25B$.

$3\frac{1}{2}$

$\therefore$ $g_{X_1}(1) = \dfrac{\sigma^2}{\phi_1(1)\,\phi_1(1^{-1})} = \dfrac{2}{\frac{1}{2}\times\frac{1}{2}} = 8.$

and, $g_{X_2}(1) = \dfrac{\sigma^2}{\phi_2(1)\,\phi_2(1^{-1})} = \dfrac{2}{\frac{3}{4}\cdot\frac{3}{4}} = \dfrac{32}{9}.$

$g_Y(1) = 8 + \dfrac{32}{9} = \dfrac{104}{9}.$

MTH 517: Time Series Analysis

Mid semester examination; Full Marks-60

Date: September 18, 2018

[1] (a) Let $\{X_t\}$ be a time series given by $X_t = e^{Y} t^2 + \varepsilon_t;\ \varepsilon_t \sim WN(0,\sigma^2)$ and $Y \sim U(0,1)$. If $Z_t = \Delta^2 X_t$, find $Cov(Z_t, Z_{t+h})$ for $h = 0, \pm 1, \pm 2, \ldots$ and verify whether $\{Z_t\}$ is covariance stationary or not.

(b) Consider a Gaussian process $\{X_t\}$ with $E(X_t) = 0\ \forall t$ and $Cov(X_t, X_{t+s}) = e^{-|t-s|}\ \forall t, s$. Let $Y_t = e^{X_t}$. Prove or disprove the following statements:

(i) $\{\Delta X_t\}$ is strict stationary.

(ii) $\{Y_t\}$ is a Gaussian process.

(iii) $\{Y_t\}$ is covariance stationary.

18 (6+12) marks

ARMA(2,1)

[2] Consider a covariance stationary $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t - 0.2\,\varepsilon_{t-1}; \varepsilon_t \sim WN(0,1)$. For the $\{X_t\}$ time series, $\gamma_3/\gamma_1 = 1/4, \gamma_3/\gamma_2 = 1/2, \gamma_3/\gamma_4 = 1/2$ and $\gamma_3/\gamma_5 = 4/7$.

(a) Find explicitly (i.e. **NOT** in terms of other $\gamma_i s$ and $\phi_i s$) γ_6/γ_5.

(b) Is it possible to explicitly calculate γ_0 and γ_1 from the given values of $\gamma_i/\gamma_j\ s$? If yes, find γ_1^{o}. If no, explain why it is not possible.

12 (6+6) marks

[3] (a) Let $\{X_t\}$ be a zero mean covariance stationary complex valued time series given by $X_t = U_t + i\,V_t$. Prove or disprove "$E|X_{t+h} - X_t|^2 = 2\,\mathrm{Re}(\gamma_0 - \gamma_h)\ \forall h$, where $\mathrm{Re}(x)$ = real part of x".

(b) Let $X_t + \varepsilon_{t-1} = \Delta\varepsilon_t;\ \varepsilon_t \sim WN(0,\sigma^2)$ and $Y_t - X_t = \sum_{j=1}^{\infty} (0.5)^j X_{t-j}$. Prove or disprove "$\{Y_t\}$ is a covariance stationary White noise"

12 (6+6) marks

[4] Let $\{X_t\}$ be a Gaussian $WN(0,\sigma^2)$ process. Consider, $\tilde{\gamma}_h = \frac{1}{n}\sum_{t=1}^{n-h} X_t X_{t+h}$ as an estimator for γ_h.

(a) Prove or disprove, "As an estimator for γ_h, $\tilde{\gamma}_h$ is unbiased in the limit".

(b) Find $Cov(\tilde{\gamma}_r, \tilde{\gamma}_s)$.

8 (2+6) marks

[5] $\{X_t\}$ be a covariance stationary $AR(1)$ process $X_t = \phi X_{t-1} + \varepsilon_t, |\phi| < 1,\ \varepsilon_t \sim WN\left(0, \sigma_\varepsilon^2\right)$ and $Y_t = X_t + 0.5\, X_{t-1} + 0.25\, X_{t-2} + \delta_t;\ \delta_t \sim WN\left(0, \sigma_\delta^2\right)$. Further, $\{\varepsilon_t\}$ and $\{\delta_t\}$ are independently distributed.

(a) Find ACGF of $\{Y_t\}$ in terms of ACGFs of $\{\varepsilon_t\}$ and $\{\delta_t\}$.

(b) Using the ACGF of $\{Y_t\}$, derived in 5(a) above, find $Cov(Y_5, Y_6)$.

10 (6+4) marks

MTH 517: Time Series Analysis
End semester examination; Full Marks-100

Date: November 24, 2018

[1] (a) Can there exist a covariance stationary $AR(2)$ process with autocorrelations $\rho_1 = 1/2, \rho_2 = 1/6$ and $\rho_3 = 2/27$?

(b) Let $\{X_t\}$ be a covariance stationary $AR(3)$ process $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + \varepsilon_t$, $\varepsilon_t \sim WN(0,\sigma^2)$ and $Y_t = \sum_{j=0}^{2} X_{t-j}$. Prove or disprove "$\{Y_t\}$ is causal and invertible *ARMA* process".

(c) Let $X_1,\ldots,X_n$ be a random sample from a zero mean covariance stationary time series.

P.Set

Prove or give a counter example "If $Cov(X_n,\bar{X}_n)\to 0$ as $n\to\infty$ then $\sum_{h=-\infty}^{\infty}|\gamma_X(h)|<\infty$".

Counter example

17 (5+6+6) marks

[2] (a) Let $\{X_t\}$ be an *MA(1)* process $X_t = \varepsilon_t + \theta\,\varepsilon_{t-1}$; $\varepsilon_t \sim WN(0,\sigma^2)$, $|\theta|>1$. Define $\{Y_t\}$ as

P. Set

$Y_t = \sum_{j=0}^{\infty}(-\theta)^{-j} X_{t-j}$. Prove or disprove "$\{Y_t\}$ is stationary non-white noise process".

(b) Let $\{X_t\}$ be an $MA(1)$ process $X_t = \mu + \varepsilon_t + \theta\,\varepsilon_{t-1}$, where $|\theta|<1$ and $\varepsilon_t \sim WN(0,\sigma^2)$.

Find $\lim_{k\to\infty} E\left(X_t + \sum_{i=1}^{k}(-\theta)^i X_{t-i} - \varepsilon_t - \mu\sum_{i=1}^{k}(-\theta)^i\right)^2$.

(c) Let $X_1,\ldots,X_n$ be a random sample from a Gaussian invertible $MA(1)$ model $X_t = \mu + \varepsilon_t + \theta\,\varepsilon_{t-1}$, where $|\theta|<1$ and $\varepsilon_t \sim i.i.d. N(0,\sigma^2)$. Prove or disprove "conditional MLE of μ and θ, conditional on ε_0 at it's expected value, can be obtained by minimizing $\sum_{t=1}^{n}\left(\sum_{i=1}^{t}(X_i-\mu)(-\theta)^{t-i}\right)^2$ with respect to μ and θ".

20 (6+7+7) marks

[3] Let $\underset{\sim}{X}_t$ be a 2-variate covariance stationary $VAR(2)$ process,

$\underset{\sim}{X}_t = \Phi_1 \underset{\sim}{X}_{t-1} + \Phi_2 \underset{\sim}{X}_{t-2} + \underset{\sim}{\varepsilon}_t$; $\underset{\sim}{\varepsilon}_t \sim VWN(\underset{\sim}{0},\Sigma), \Sigma>0$. Prove or disprove the following statements:

Statement A: $\begin{pmatrix} \underset{\sim}{X}_t \\ 2\underset{\sim}{X}_{t-1} \\ 3\underset{\sim}{X}_{t-2} \end{pmatrix}$ is a covariance stationary $VAR(1)$ process.

Statement B: $\begin{pmatrix} \underset{\sim}{\varepsilon}_t \\ 2\underset{\sim}{\varepsilon}_{t-1} \\ 3\underset{\sim}{\varepsilon}_{t-2} \end{pmatrix}$ is a covariance stationary VWN process.

12 (8+4) marks

$-\epsilon_{t+h-1}\delta_{t+h}$

$-\epsilon_{t+h-1}$

$v(\epsilon_t\delta_t) = E(\epsilon_t^2\delta_t^2) = \sigma^2\sigma^2 = \sigma^4$

$f_x(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-i\lambda h}\gamma_x(h)$

$\frac{cov(X_t Y_t, X_{t+h}Y_{t+h})}{\sigma^4}$

[4] Let $X_t = \varepsilon_t - \varepsilon_{t-1}$ and $Y_t = \delta_t + \delta_{t-1}$; $\{\varepsilon_t\}$ and $\{\delta_t\}$ are independent white noise $WN(0,\sigma^2)$ processes. Define $Z_t = X_t(1+Y_t)$ and $\underset{\sim}{P}_t = \begin{pmatrix} X_t + Y_t \\ Z_t - X_{t-1} \end{pmatrix}$.

(a) Express the spectral density of $\{Z_t\}$ as function of only spectral densities of $\{X_t\}$ and $\{Y_t\}$ **(without any summation in the expression of the resultant spectral density)** and hence find the value of $f_Z(0)$.

(b) Prove or disprove "$f_{XZ}(\lambda) = f_{ZX}(\lambda)\ \forall \lambda \in [-\pi,\pi]$".

(c) Prove or disprove "$f_{X\varepsilon}(\lambda) = f_{\varepsilon X}(\lambda)\ \forall \lambda \in [-\pi,\pi]$".

(d) Does there exist a K such that $Cov(\underset{\sim}{P}_t, \underset{\sim}{P}_{t+h}) = 0,\ \forall |h| > K$? If yes, find the smallest value of K.

26 (8+6+6+6) marks

[5] Let $\{X_t\}$ and $\{Y_t\}$ be two $AR(1)$ processes given by $X_t = \delta_X + \phi_X X_{t-1} + Z_t$ and $Y_t = \delta_Y + \phi_Y Y_{t-1} + Z_t + U_t$; where, $\{Z_t\} \sim WN(0,\sigma_Z^2)$, $\{U_t\} \sim WN(0,\sigma_U^2)$; $\{Z_t\}$ and $\{U_t\}$ are independent; $|\phi_X| < 1$ and $|\phi_Y| < 1$.

(a) Derive the BLP of X_{t+1} based on Y_t.

(b) Find PACF at lag 2 of the time series $\{Q_t\}$, $Q_t = (1+Z_t)(1-U_t)$.

11 (6+5) marks

[6] Let $X_t = A\cos(\pi t/2) + B\sin(\pi t/2) + C + \varepsilon_t + \varepsilon_{t+2}$; $\varepsilon_t \sim WN(0,\sigma^2)$, A, B, C are three mutually independent random variables with mean 0 and variance σ^2. Further, the sequence $\{\varepsilon_t\}$ and A, B and C are mutually independent.

Find

(a) the spectral distribution function of $\{X_t\}$,

(b) $\gamma_X(0)$ **using the spectral distribution function derived in** (a) and

(c) the continuous and/or discrete spectra associated with spectral distribution function derived in (a).

14 marks