Lecture 16

ARMA Models Part 2

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ARMA models (Recap)

- A autoregressive moving average model of order (p, q), abbreviated $ARMA(p, q)$, is of the form

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \ldots + \theta_q W_{t-q}.$$

- Here X_t is stationary, $W_t \sim WN(0, \sigma_W^2)$, and $\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q$ are constants with $\phi_p, \theta_q \neq 0$.

- We represent the ARMA(p, q) model using $\phi(B)X_t = \theta(B)W_t$.

Three problems with ARMA models (Recap)

1. Parameter redundant models
2. Stationary AR models that depend on the future
3. MA models that are not unique

Solutions to the problems (Recap)

- Solution to Problem 1: The AR polynomial $\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p$ and MA polynomial $\theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q$ have no common factors.
- Solution to Problem 2: An ARMA(p, q) model is causal if and only if $\phi(z)$ does not have any root z_0 for $|z_0| \leq 1$.
- Solution to Problem 3: An ARMA(p, q) model is invertible if and only if $\theta(z)$ does not have any root z_0 for $|z_0| \leq 1$.

MA and AR representation of ARMA(p, q)

- A causal ARMA(p, q) model $\{X_t; t = 0, \pm 1, \pm 2, \ldots\}$ can be written as a one-sided linear process:

$$X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j} = \psi(B) W_t,$$

  where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$, and $\sum_{j=0}^{\infty} |\psi_j| < \infty$; we set $\psi_0 = 1$.

- An invertible ARMA(p, q) model $\{X_t; t = 0, \pm 1, \pm 2, \ldots\}$ can be written as

$$\pi(B) X_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} = W_t,$$

  where $\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j$, and $\sum_{j=0}^{\infty} |\pi_j| < \infty$; we set $\pi_0 = 1$.

Coefficients of $\psi(z)$ and $\pi(z)$

- An ARMA(p, q) model is defined by $\phi(B)X_t = \theta(B)W_t$.
- An ARMA(p, q) model is said to be causal, if $X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j} = \psi(B)W_t$.
- An ARMA(p, q) model is said to be invertible, if $\pi(B)X_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} = W_t$.
- The coefficients ψ_j's can be determined by solving $\psi(z) = \theta(z)/\phi(z)$, where $|z| \leq 1$.
- The coefficients π_j's can be determined by solving $\pi(z) = \phi(z)/\theta(z)$, where $|z| \leq 1$.

Illustrations

- Consider the process

$$X_t = 0.4X_{t-1} + 0.45X_{t-2} + W_t + W_{t-1} + 0.25W_{t-2}$$

- Despite the process appears to be ARMA(2,2), show that it is ARMA(1,1).
- Check whether the model is causal or not.
- Check whether the model is invertible or not.
- Calculate the coefficients of $\psi(z)$ and $\pi(z)$.
- For an AR(2) model $(1 - \phi_1 B - \phi_2 B^2)X_t = W_t$, show that process is causal if $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$, and $|\phi_2| < 1$.

Difference equation: Motivation

- Consider the AR(1) model $X_t = \phi X_{t-1} + W_t$ with $|\phi| < 1$.
- We can represent it as a linear process $X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}$.
- We showed that

$$\begin{aligned}\gamma(h) &= \mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}\left(\sum_{j=0}^{\infty} \phi^j W_{t+h-j}, \sum_{j=0}^{\infty} \phi^j W_{t-j}\right) \\ &= \phi^h \sum_{j=0}^{\infty} \phi^{2j} \mathrm{Var}(W_t) = \phi^h \sigma_W^2 (1-\phi^2)^{-1}\end{aligned}$$

- Clearly, $\gamma(h-1) = \phi^{h-1}\sigma_W^2(1-\phi^2)^{-1}$ and thus, $\gamma(h) = \phi\gamma(h-1)$.
- Dividing by $\gamma(0)$, we get $\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi\frac{\gamma(h-1)}{\gamma(0)} = \phi\rho(h-1)$.

Difference equation

- Suppose we have a sequence of numbers $u_0, u_1, u_2, \ldots$ such that

$$u_n - \alpha u_{n-1} = 0, \;\; \alpha \neq 0, \;\; n = 1, 2, \ldots$$

- The equation represents a homogeneous difference equation of order 1.
- To solve it, we write: $u_1 = \alpha u_0, u_2 = \alpha u_1 = \alpha^2 u_0, \ldots, u_n = \alpha u_{n-1} = \alpha^n u_0$.
- Given an initial condition $u_0 = c$, we have $u_n = \alpha^n c$.
- We can write $u_n - \alpha u_{n-1} = 0$ as $(1 - \alpha B) u_n = 0$.
- The root z_0 of the associated polynomial $\alpha(z) = 1 - \alpha z$ is $z_0 = 1/\alpha$ and we can write the final solution also as $u_n = (z_0^{-1})^n c = z_0^{-n} c$.

Difference equation of higher orders: Motivation

- Suppose $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t$ is a causal AR(2) process.
- Multiply each side of the model by X_{t-h} for $h > 0$, and take expectation:

$$E(X_t X_{t-h}) = \phi_1 E(X_{t-1} X_{t-h}) + \phi_2 E(X_{t-2} X_{t-h}) + E(W_t X_{t-h}).$$

- The result is $\gamma(h) = \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2), h = 1, 2, \ldots$.
- Dividing by $\gamma(0)$, we have $\rho(h) = \phi_1 \rho(h-1) + \phi_2 \rho(h-2)$.
- Let z_1 and z_2 be the roots of the polynomial $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$. Then, when z_1 and z_2 are real and distinct, then check that

$$\rho(h) = c_1 z_1^{-h} + c_2 z_2^{-h}$$

is a solution.

Difference equation: General result for order 2

- Suppose the sequence $u_0, u_1, u_2, \ldots$ satisfies

$$u_n - \alpha_1 u_{n-1} - \alpha_2 u_{n-2} = 0, \quad \alpha_2 \neq 0, \quad n = 2, 3, \ldots .$$

- This equation is a homogeneous difference equation of order 2.
- The corresponding polynomial is $\alpha(z) = 1 - \alpha_1 z - \alpha_2 z^2$, which has two roots, say, z_1 and z_2.
- If $z_1 \neq z_2$, the general solution is $u_n = c_1 z_1^{-n} + c_2 z_2^{-n}$, where c_1 and c_2 depend on the initial conditions.
- When $z_1 = z_2 (= z_0)$, a general solution is $u_n = z_0^{-n}(c_1 + c_2 n)$.
- Overall, $u_n = z_1^{-n} \times (\text{a polynomial in n of degree } m_1 - 1) + z_2^{-n} \times (\text{a polynomial in n of degree } m_2 - 1)$ where m_i is the multiplicity of z_i for $i = 1, 2$.
- For repeated root, the solution is $u_n = z_0^{-n} \times (\text{a polynomial in n of degree } m_0 - 1)$, where $m_0 = \text{multiplicity}(z_0)$.

Difference equation: General result for order p

- These results generalize to the homogeneous difference equation of order p:

$$u_n - \alpha_1 u_{n-1} - \ldots - \alpha_p u_{n-p} = 0, \quad \alpha_p \neq 0, \quad n = p, p+1, \ldots.$$

- The associated polynomial is $\alpha(z) = 1 - \alpha_1 z - \ldots - \alpha_p z^p$.
- Suppose $\alpha(z)$ has r distinct roots, z_i with multiplicity m_i for $i = 1, \ldots, r$, such that $\sum_{i=1}^r m_i = p$.
- The general solution is

$$u_n = z_1^{-n} P_1(n) + z_2^{-n} P_2(n) + \ldots + z_r^{-n} P_r(n),$$

where $P_j(n)$, for $j = 1, 2, \ldots, r$, is a polynomial in n, of degree $m_j - 1$.

Thank you!