Problem Set #1

(1) $$y_t = m_t + \epsilon_t \; ; \; \epsilon_t \text{ i.i.d. } N(0, \sigma^2)$$

$$\nabla y_t = \nabla m_t + \nabla \epsilon_t = (m_t - m_{t-1}) + (\epsilon_t - \epsilon_{t-1})$$

$$\nabla^2 y_t = (m_t - m_{t-1}) - (m_{t-1} - m_{t-2})$$
$$+ (\epsilon_t - \epsilon_{t-1}) - (\epsilon_{t-1} - \epsilon_{t-2})$$

i.e $$\nabla^2 y_t = m_t - 2m_{t-1} + m_{t-2}$$
$$+ \epsilon_t - 2\epsilon_{t-1} + \epsilon_{t-2}$$

$$\nabla_2 y_t = \nabla_2 m_t + \nabla_2 \epsilon_t = (m_t - m_{t-2}) + (\epsilon_t - \epsilon_{t-2})$$

(a) $$m_t = a + bt$$

$$\nabla^2 y_t = \left[(a+bt) - 2(a + b(t-1)) + (a + b(t-2))\right]$$
$$+ \epsilon_t - 2\epsilon_{t-1} + \epsilon_{t-2}$$

i.e $$\nabla^2 y_t = \epsilon_t - 2\epsilon_{t-1} + \epsilon_{t-2}$$

$$E(\nabla^2 y_t) = 0; \quad V(\nabla^2 y_t) = 6\sigma^2$$

$\nabla^2 y_t$ is free from trend

$$\nabla_2 Y_t = [(a+bt) - (a+b(t-2))] + \epsilon_t - \epsilon_{t-2}$$

$$\nabla_2 Y_t = 2b + \epsilon_t - \epsilon_{t-2}$$

$$E\,\nabla_2 Y_t = 2b \quad V(\nabla_2 Y_t) = 2\sigma^2$$

$\nabla_2 Y_t$ is free from time trend (so is ∇Y_t)

(b) $m_t = a + bt + ct^2$

$$\nabla Y_t = [(\cancel{a} + bt + ct^2) - (\cancel{a} + b(\cancel{t}-1) + c(t-1)^2)] + \epsilon_t - \epsilon_{t-1}$$

$$= [\cancel{ct^2} + b - c(\cancel{t^2} + 1 - 2t)] + \epsilon_t - \epsilon_{t-1}$$

$$\nabla Y_t = (b + 2ct - c) + \epsilon_t - \epsilon_{t-1}$$

$$\nabla^2 Y_t = \nabla(\nabla Y_t)$$

$$= 2c + \epsilon_t - 2\epsilon_{t-1} + \epsilon_{t-2}$$

$$E(\nabla^2 Y_t) = 2c \qquad V(\nabla^2 Y_t) = 6\sigma^2$$

$$\nabla_2 Y_t = m_t - m_{t-2} + \epsilon_t - \epsilon_{t-2}$$

$$= (a + bt + ct^2) - (a + b(t-2) + c(t-2)^2)$$

$$+ \epsilon_t - \epsilon_{t-2}$$

$$= 2b + 4ct - 4c + \epsilon_t - \epsilon_{t-2}$$

$$E(\nabla_2 Y_t) = 2b + 4ct - 4c \; ; \; V(\nabla_2 Y_t) = 2\sigma^2$$

(2) $Y_t = m_t + S_t + \epsilon_t$

ϵ_t i.i.d. $N(0, \sigma^2)$

$m_t = a + bt$

S_t : seasonal comp with period 4

(a) $\nabla_4 Y_t = \nabla_4 m_t + \nabla_4 S_t + \nabla_4 \epsilon_t$

$= \big(a+bt) - (a+b(t-4)\big) + \underbrace{S_t - S_{t-4}}_{=0} + \epsilon_t - \epsilon_{t-4}$

$= 4b + \epsilon_t - \epsilon_{t-4}$

$\Rightarrow$ $\nabla_4 Y_t$ is free from time trend, i.e. trend is eliminated

Also $\nabla_4 Y_t$ does not have a seasonal component

$V(Y_t) = \sigma^2$

$V(\nabla_4 Y_t) = 2\sigma^2$

$\Rightarrow$ $\nabla_4 Y_t$ does not dampen the noise

(b) $Z_t = \nabla_4 Y_t \sim N(4b, 2\sigma^2)$

Is the seq $\{Z_t\}$ indep seq of r.v.s ??

$\nabla \nabla_4 Y_t = (\epsilon_t - \epsilon_{t-1}) - (\epsilon_{t-4} - \epsilon_{t-5})$

$\sim N(0, 4\sigma^2)$ – indep ??

Sly ∇Y_t

(3)

(i) $m_t = c_0 + c_1 t$

$$\sum_{j=-q}^{q} a_j \, m_{t-j} = \frac{1}{2q+1} \sum_{j=-q}^{q} \left(c_0 + c_1 (t-j) \right)$$

equal weighted MA $\Rightarrow a_j = \frac{1}{2q+1}$

$$= \frac{1}{2q+1} \left(\sum_{-q}^{q} c_0 + c_1 t (2q+1) - c_1 \sum_{j=-q}^{q} j \right).$$

$$= c_0 + c_1 t = m_t$$

$\Rightarrow$ linear trend passes un distorded

(ii) $A_t = \sum_{j=-q}^{q} a_j z_{t-j}$ $\quad \{z_t\}$ ind r.v. seq $(0, \sigma^2)$

$$E(A_t) = 0$$

$$V(A_t) = \sum_{j=-q}^{q} a_j^2 \sigma^2 = \frac{\sigma^2}{2q+1} \downarrow 0 \text{ as } q \uparrow$$

$\Rightarrow$ higher q gives smoother series. close to expected value.

(4) $Y_t = m_t + S_t + \epsilon_t$

$m_t = \alpha + \beta t + \gamma t^2$

S_t : seasonal comp of period 3

$\ldots = S_{t-3} = S_t = S_{t+3} = \ldots \qquad \sum_{t=1}^{3} S_t = 0$

$$Z_t = \sum_{j=-1}^{1} a_j Y_{t+j} = \sum_{-1}^{1} a_j m_{t+j} + \sum_{-1}^{1} a_j S_{t+j} + \sum_{-1}^{1} a_j \epsilon_{t+j}$$

i.e. $Z_t = \frac{1}{3}\Big[(m_{t-1} + m_t + m_{t+1}) + \cancel{(S_{t-1} + S_t + S_{t+1})}$
$\qquad + (\epsilon_{t-1} + \epsilon_t + \overset{0}{\epsilon}_{t+1})\Big]$

$$= \frac{1}{3}\Big[(\alpha + \beta(t-1) + \gamma(t-1)^2) + (\alpha + \beta t + \gamma t^2)$$
$$+ (\alpha + \beta(t+1) + \gamma(t+1)^2$$
$$+ (\epsilon_{t-1} + \epsilon_t + \epsilon_{t+1})\Big]$$

i.e $Z_t = \frac{1}{3}(3\alpha + 3\beta t + 3\gamma t^2 + 2\gamma) + \frac{1}{3}\sum_{-1}^{1} \epsilon_{t+j}$

$$Z_t = m_t + \frac{2}{3}\gamma + \frac{1}{3}\sum_{-1}^{1} \epsilon_{t+j}$$

$\Rightarrow$ Seasonality is removed; trend is distorted

as m_t is changed to $m_t + \frac{2}{3}\gamma$

$\left(E Y_t = \underset{\substack{\uparrow \\ \text{trend}}}{m_t} + \underset{\substack{\uparrow \\ \text{seasonality}}}{S_t} ; \ E Z_t = \underbrace{m_t + \tfrac{2}{3}\gamma}_{\text{trend}} . \right)$

(5) Spencer 15-pt filter

$[a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7]$

$$= \frac{1}{320}[74, 67, 46, 21, 3, -5, -6, -3]$$

$$a_i = 0 \quad \forall \ |i| > 7$$
$$= a_{-i} \quad |i| \le 7$$

This will result in unequal weighted MA filtering

$$Y_t = m_t + \epsilon_t \qquad m_t = a + bt$$

$$\sum_{j=-7}^{7} a_j m_{t+j} = \frac{1}{320}\left[-3(a+b(t-7)) - 6(a+b(t-6)) + \cdots - 6(a+b(t+6)) - 3(a+b(t+7))\right]$$

$$= a + bt$$

$$\Rightarrow \sum_{-7}^{7} a_j Y_{t+j} = m_t + \sum_{-7}^{7} a_j \epsilon_{t+j}$$

$\Rightarrow$ passes trend without distortion.

(6) $[a_{-2}, a_{-1}, a_0, a_1, a_2] = \frac{1}{9}[-1, 4, 3, 4, -1]$

$Y_t = m_t + s_t + \epsilon_t$

$m_t = a + bt$

s_t : seasonal comp with period 3

$\cdots = s_{t-3} = s_t = s_{t+3} = \cdots$

Sum of 3 consecutive $s_t = 0$

$$\sum_{j=-2}^{2} a_j Y_{t+j} = \sum_{-2}^{2} a_j m_{t+j} + \sum_{-2}^{2} a_j s_{t+j} + \sum_{-2}^{2} a_j \epsilon_{t+j}$$

$$\sum_{-2}^{2} a_j m_{t+j} = \frac{1}{9}\Big[-(a+b(t-2)) + (a+b(t-1)) \times 4 + 3(a+bt) + 4(a+b(t+1)) - (a+b(t+2))\Big]$$

$$= a + bt$$

$$\sum_{-2}^{2} a_j s_{t+j} = \frac{1}{9}\Big[-s_{t-2} + 4s_{t-1} + 3s_t + 4s_{t+1} - s_{t+2}\Big]$$

$$= \frac{1}{9}\Big[3s_t + 3s_{t+1} + 3s_{t+2}\Big]$$

$$= 0.$$

$$\Rightarrow \sum_{-2}^{2} a_j Y_{t+j} = m_t + \sum_{-2}^{2} a_j \epsilon_{t+j}$$

Trend undistorted & seasonality eliminated

(7) $m_t = a + bt + ct^2$; $t = 0, \pm 1, \pm 2, \ldots$

$\{a_j\}$ & $\{b_j\}$ are 2 symmetric filters.

(a) trivial calculations once again to show that

$$\sum_{j=-2}^{2} a_j m_{t+j} = m_t$$

& $$\sum_{j=-3}^{3} b_j m_{t+j} = m_t$$

(b) $Y_t = m_t + \epsilon_t$; $\{\epsilon_t\}$ i.i.d $N(0, \sigma^2)$

$$U_t = \sum_{-2}^{2} a_j Y_{t+j} = m_t + \sum_{j=-2}^{2} a_j \epsilon_{t+j}$$

$$V_t = \sum_{-3}^{3} b_j Y_{t+j} = m_t + \sum_{-3}^{3} b_j \epsilon_{t+j}$$

$$E(U_t) = E(V_t) = m_t$$

$$V(U_t) = \sigma^2 \sum_{-2}^{2} a_j^2 = \sigma^2 \frac{1}{35^2}\left[2 \times 3^2 + 2 \times 12^2 + 17^2\right]$$
$$\simeq .49\sigma^2$$

$$V(V_t) = \sigma^2 \sum_{-3}^{3} b_j^2 = \frac{\sigma^2}{21^2}\left[2 \times 2^2 + 2 \times 3^2 + 2 \times 6^2 + 7^2\right]$$
$$\simeq .33\sigma^2$$

7 (b)

$$Cov(U_t, U_{t+1}) = E(U_t - E U_t)(U_{t+1} - E(U_{t+1}))$$

$$= E\left(\sum_{-2}^{2} a_j \epsilon_{t+j}\right)\left(\sum_{-2}^{2} a_j \epsilon_{t+1+j}\right)$$

$\epsilon_{t-2} \cdots \epsilon_{t+2}$ $\quad \epsilon_{t-1}, \cdots \epsilon_{t+3}$

$$= \sigma^2 (a_{-1}a_{-2} + a_0 a_{-1} + a_1 a_0 + a_2 a_1)$$

$$Cov(V_t, V_{t+1}) = E\left(\sum_{-3}^{3} b_j \epsilon_{t+j}\right)\left(\sum_{-3}^{3} b_j \epsilon_{t+1+j}\right)$$

$$= \sigma^2 (b_{-2}b_{-3} + b_{-1}b_{-2} + b_0 b_{-1} + b_1 b_0 + b_2 b_1 + b_3 b_2)$$

$$Cov(U_t, V_t) = \cdots$$

(8) $\quad Y_t = (a_1 + a_2 t) S_t + X_t$

S_t : seasonal comp of period 4

$E X_t = 0 \qquad Cov(X_t, X_s) = \begin{cases} \sigma^2, & t = s \\ 0, & o/w \end{cases}$

$$\nabla_4 Y_t = \nabla_4 \left((a_1 + a_2 t) S_t\right) + \nabla_4 X_t$$

$$= (a_1 + a_2 t) S_t - (a_1 + a_2(t-4)) S_{t-4} + X_t - X_{t-4}$$

$$= a_1 S_t + a_2 t S_t - [a_1 S_{t-4} + a_2 t S_{t-4} - 4 a_2 S_{t-4}]$$

$$+ X_t - X_{t-4}$$

i.e. $\nabla_4 Y_t = 4a_2 \underline{\underline{S_{t-4}}} + X_t - X_{t-4}$

$(S_t = S_{t-4})$

seasonal comp.
trend eliminated

$\nabla_4^2 Y_t = \underbrace{4a_2 S_{t-4} - 4a_2 S_{t-8}}_{=0} + \nabla_4 (X_t - X_{t-4})$

$\underline{\underline{\nabla_4^2 Y_t}} = \nabla_4 (X_t - X_{t-4})$

Series with no trend and seasonal comp

Random comp seq is no longer uncorrelated.

(9) $X_t = \alpha + \beta t + \gamma t^2 + S_t + Y_t$

$X_t \rightarrow Z_t$ by differencing $\ni$

Z_t is free from trend & seasonality