Lecture 7

Estimation of Correlation Part 1

Arnab Hazra

Linear process

- A linear process X_t is defined to be a linear combination of white noise variates W_t, and is given by

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty$$

- Autocovariance is given by $\gamma_X(h) = \sigma_W^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}$

- Check that $\gamma_X(h) = \gamma_X(-h)$

- What is the condition for finite variances?

Gaussian process

- A process X_t is said to be a Gaussian process if the n-dimensional vectors $\boldsymbol{X} = (X_{t_1}, X_{t_2}, \ldots, X_{t_n})$, for every collection of distinct time points $t_1, t_2, \ldots, t_n$, and every positive integer n, have a multivariate normal distribution.

- Suppose $\boldsymbol{\mu} = [\mu_{t_1}, \ldots, \mu_{t_n}]$ and $\Gamma = [\gamma(t_i, t_j), i, j = 1, \ldots, n]$. Then the joint density is

$$f(\boldsymbol{x}) = (2\pi)^{-n/2} |\Gamma|^{-1/2} \exp\left[-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})' \Gamma^{-1} (\boldsymbol{x} - \boldsymbol{\mu})\right].$$

- If $\mu_t = \mu$ for all t and $\gamma(t_i, t_j) = \gamma(t_i - t_j)$, the series X_t depend only on time lag and not on the actual times, and hence the series must be strictly stationary.

- If a time series is Gaussian, then it is a linear process with $W_t \overset{IID}{\sim} N(0, \sigma_W^2)$.

Mean estimation

- If a time series is stationary, the mean function $\mu_t = \mu$ is constant.
- If the observable random variables are $X_1, \ldots, X_T$, we can estimate μ by the sample mean

$$\hat{\mu} = \bar{X} = \frac{1}{T} \sum_{t=1}^{T} X_t.$$

- The estimator is unbiased as

$$E(\hat{\mu}) = E(\bar{X}) = E\left(\frac{1}{T} \sum_{t=1}^{T} X_t \right) = \frac{1}{T} \sum_{t=1}^{T} E(X_t) = \mu.$$

- What about the standard error?

Sample autocovariance function

- Suppose the realizations are $x_1, \ldots, x_T$.
- The sample autocovariance function is defined as

$$\hat{\gamma}(h) = \frac{1}{T} \sum_{t=1}^{T-h} (x_{t+h} - \bar{x})(x_t - \bar{x})$$

with $\hat{\gamma}(-h) = \hat{\gamma}(h)$ for $h = 0, 1, \ldots, T-1$.

- Why not just divide by $T - h$ instead of T?
- Hint: Ensure that $\widehat{\mathrm{Var}}(a_1 X_1 + \ldots + a_T X_T)$ is also non-negative.

Thank you!