Name:
Roll No:

MTH517A: Time Series Analysis
Quiz #1; Full Marks-20

[1] Let $\{X_t\}$ be a covariance stationary Gaussian process with mean θ and ACVF $\gamma_X(.)$. Define $U_t=(1-0.4B)X_t$, $V_t=(1-2.5B)X_t$ and $P_t=a+bt+ct^2+S_t+X_t$; S_t is a seasonal component with periodicity 3 and a, b and c are constants.

Prove or disprove the following statements:

(a) $\{U_t\}$ and $\{V_t\}$ are covariance stationary.

(b) $\{U_t\}$ and $\{V_t\}$ have identical autocorrelation structure.

(c) $\{U_t\}$ and $\{V_t\}$ are covariance stationary with identical mean and identical autocorrelation structures.

(d) $\{\nabla_6 P_t\}$ is Gaussian.

(e) $\{\nabla_6 P_t\}$ is covariance stationary.

(f) $\{\nabla_3 \nabla_6 P_t\}$ is strict stationary.

Solution Quiz #1

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(a) $U_t = X_t - 0.4 X_{t-1}$ $\qquad V_t = X_t - 2.5 X_{t-1}$

$E U_t = \theta(1-0.4)$ $\qquad E V_t = (1-2.5)\theta$

$$\begin{aligned}
&\operatorname{Cov}(U_t, U_{t+h}) \\
&= \operatorname{Cov}(X_t - 0.4 X_{t-1}, X_{t+h} - 0.4 X_{t+h-1}) \\
&= \gamma(h) - 0.4\gamma(h-1) - 0.4\gamma(h+1) + (0.4)^2 \gamma_h \\
&= \gamma(h)(1+(0.4)^2) - 0.4(\gamma(h-1)+\gamma(h+1)) = \gamma_U(h)
\end{aligned}$$

$$\begin{aligned}
&\operatorname{Cov}(V_t, V_{t+h}) \\
&= \operatorname{Cov}(X_t - 2.5 X_{t-1}, X_{t+h} - 2.5 X_{t+h-1}) \\
&= \gamma(h)(1+(2.5)^2) - 2.5(\gamma(h-1)+\gamma(h+1)) = \gamma_V(h)
\end{aligned}$$

$\Rightarrow \{U_t\}$ & $\{V_t\}$ are covariance stationary

(b) $$\rho_U(h) = \frac{\gamma_U(h)}{\gamma_U(0)}$$

$$\rho_U(h) = \frac{\gamma(h)(1+0.4^2) - 0.4(\gamma(h-1)+\gamma(h+1))}{\gamma(0)(1+.4^2) - 2\times 0.4\,\gamma(1)}$$

$$\rho_V(h) = \frac{\gamma(h)\left(1+\frac{1}{.4^2}\right) - \frac{1}{.4}(\gamma(h-1)+\gamma(h+1))}{\gamma(0)\left(1+\frac{1}{.4^2}\right) - 2\,\frac{1}{.4}\,\gamma(1)}$$

$$= \frac{\gamma(h)(1+.4^2) - 0.4(\gamma(h-1)+\gamma(h+1))}{\gamma(0)(1+.4^2) - 2\times 0.4\,\gamma(1)} = \rho_U(h)$$

(c) $E U_t \neq E V_t$

(d) $\nabla_6 P_t = \nabla_6 \left(a + bt + ct^2 + S_t + X_t \right)$

$$= (\alpha + b\cancel{t} + ct^2) - \left(\alpha + \underset{b\,\overline{\cancel{t}-1}}{\cancel{\text{bt}}} + c(t-1)^2\right)$$

$$+ S_t - S_{t-6} + X_t - X_{t-6}$$

$$= c\cancel{t^2} - \left(-b + c(\cancel{t^2} + 1 - 2t)\right) + \cancel{S_t} - \cancel{S_{t-6}} + X_t - X_{t-6}$$

$$Z_t = \nabla_6 P_t = b - c + 2ct + X_t - X_{t-6}$$

$$\underset{\sim}{Z}_t = \begin{pmatrix} Z_{t_1} \\ \vdots \\ Z_{t_n} \end{pmatrix} = \begin{pmatrix} b - c + 2ct_1 + X_{t_1} - X_{t_1 - 6} \\ b - c + 2ct_2 + X_{t_2} - X_{t_2 - 6} \\ \vdots \\ b - c + 2ct_n + X_{t_n} - X_{t_n - 6} \end{pmatrix}$$

$$= (b - c)\underset{\sim}{1}_n + 2c \underbrace{\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}}_{\underset{\sim}{t}} + \underbrace{\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots \\ & & \cdots & & \end{pmatrix}}_{n \times 2n} \overbrace{\begin{pmatrix} X_{t_1-6} \\ X_{t_1} \\ X_{t_2-6} \\ X_{t_2} \\ \vdots \\ X_{t_n-6} \\ X_{t_n} \end{pmatrix}}^{\underset{\sim}{X}_t}$$

If there are no overlappings of dimension is less,

$\underset{\sim}{\alpha} \in \mathbb{R}^n$

$$\underline{\alpha}' \underset{\sim}{Z}_t = (b-c)\underline{\alpha}'\underset{\sim}{1}_n + 2c\,\underline{\alpha}'\underset{\sim}{t} + \underline{\beta}'\underset{\sim}{X}_t$$

$\sim N_1(\cdot,\cdot)$ as $\{X_t\}$ is Gaussian

$\Rightarrow \underline{\beta}' \underset{\sim}{X}_t \sim N_1 \;\forall \beta \in \mathbb{R}^{2n}$

$\Rightarrow \nabla_6 P_t$ is Gaussian

(e)

$$\nabla_6 P_t = b - c + 2ct + X_t - X_{t-6}$$

$$E(\nabla_6 P_t) = b - c + 2ct \leftarrow f^n \text{ of } t$$

$\Rightarrow \nabla_6 P_t$ is NOT covariance stationary

(f)

$$\nabla_3 \nabla_6 P_t = (\cancel{b} - \cancel{c} + 2\cancel{c}t) - (\cancel{b} - \cancel{c} + 2c(\cancel{t}-3)) + (X_t - X_{t-3}) - (X_{t-6} - X_{t-9})$$

$$= 6c + X_t - X_{t-3} - X_{t-6} + X_{t-9}$$

$g_t = \nabla_3 \nabla_6 P_t$ is Gaussian using similar logic as in (d)

$$E(\nabla_3 \nabla_6 P_t) = 6c$$

$$Cov(g_t, g_{t+h}) = Cov(6c + X_t - X_{t-3} - X_{t-6} + X_{t-9},\; 6c + X_{t+h} - X_{t+h-3} - X_{t+h-6} + X_{t+h-9})$$

$$= (\gamma(h) - \gamma(h-3) - \gamma(h-6) + \gamma(h-9))$$
$$+ (-\gamma(h+3) + \gamma(h) + \gamma(h-3) - \gamma(h-6))$$
$$+ (-\gamma(h+6) + \gamma(h+3) + \gamma(h) - \gamma(h-3))$$
$$+ (\gamma(h+9) - \gamma(h+6) - \gamma(h+3) + \gamma(h))$$

$\longrightarrow f^n$ of h only

$\Rightarrow g_t$ is Gaussian & also covariance stationary

$\Rightarrow g_t = \nabla_3 \nabla_6 P_t$ is strict stationary