# ARMA(1,1) Simulation and MLE Estimation

## Problem Description

We simulate 1000 realizations for an ARMA(1,1) process with parameters ϕ = 0.9, θ = 0.5, and variance σ^2 = 1. The realizations have lengths T = 50, 200, and 500. For each case, we calculate the Maximum Likelihood Estimates (MLEs) of the three parameters and compare them with the true values using:

- Mean Square Error (MSE)
- Mean Absolute Deviation (MAD)
- Coverage of 95% confidence intervals.

## Simulation and Estimation

```
library(forecast)
```

```
## Warning: package 'forecast' was built under R version 4.3.3
```

```
## Registered S3 method overwritten by 'quantmod':
##   method            from
##   as.zoo.data.frame zoo
```

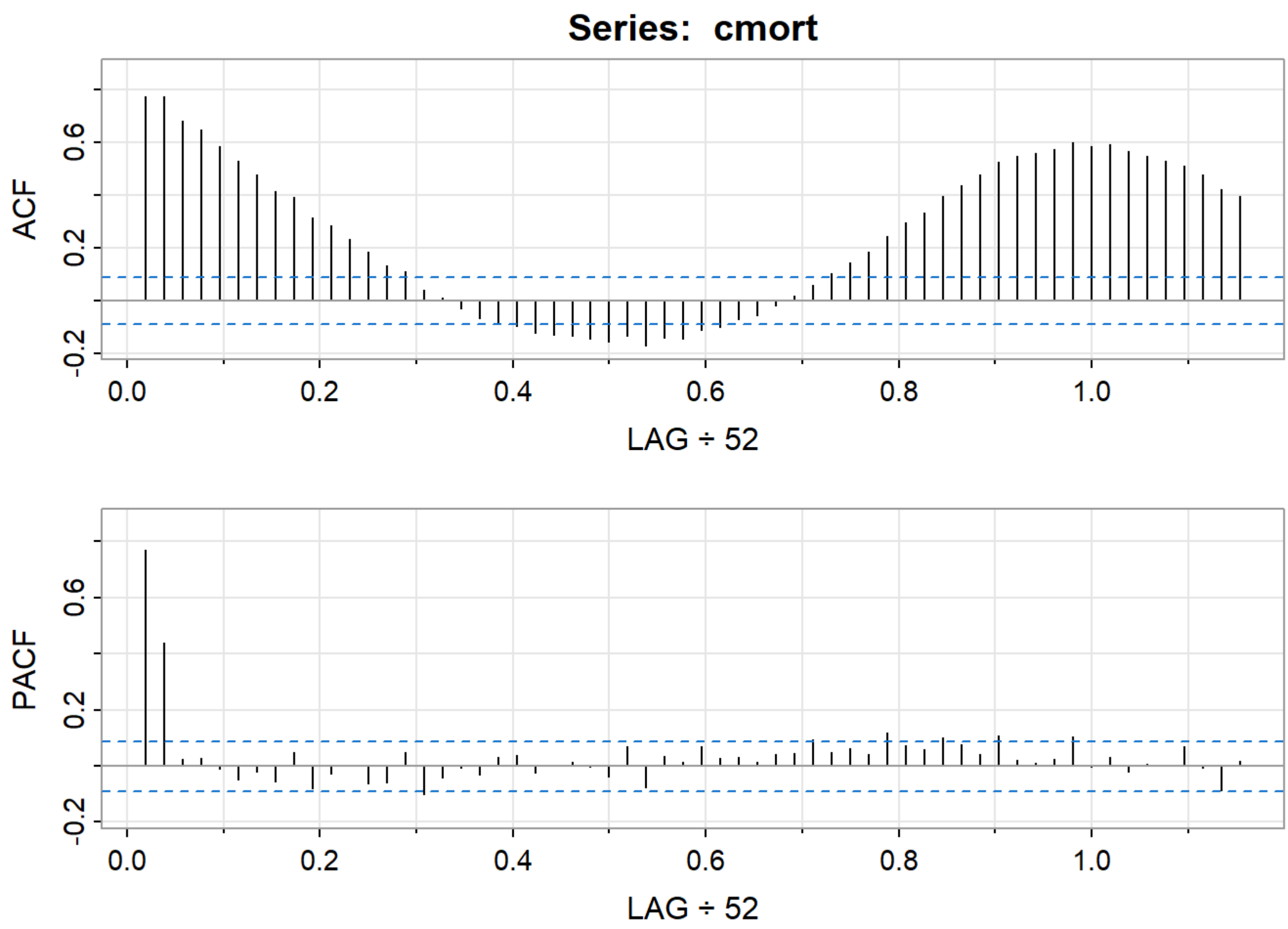
```
library(astsa)
```

```
## Warning: package 'astsa' was built under R version 4.3.2
```

```
##
## Attaching package: 'astsa'
```

```
## The following object is masked from 'package:forecast':
##
##     gas
```

```
data(cmort)
acf2(cmort,60)
```




```
##      [,1] [,2] [,3] [,4]  [,5]  [,6]  [,7]  [,8] [,9] [,10] [,11] [,12] [,13]
## ACF  0.77 0.77 0.68 0.65  0.58  0.53  0.48  0.41 0.39  0.32  0.28  0.23  0.18
## PACF 0.77 0.44 0.03 0.03 -0.01 -0.05 -0.02 -0.05 0.05 -0.08 -0.03  0.00 -0.06
##      [,14] [,15] [,16] [,17] [,18] [,19] [,20] [,21] [,22] [,23] [,24] [,25]
## ACF   0.13  0.11  0.04  0.01 -0.03 -0.07 -0.08 -0.10 -0.12 -0.13 -0.13 -0.15
## PACF -0.06  0.05 -0.10 -0.04 -0.01 -0.03  0.03  0.04 -0.02  0.00  0.01  0.00
##      [,26] [,27] [,28] [,29] [,30] [,31] [,32] [,33] [,34] [,35] [,36] [,37]
## ACF  -0.16 -0.14 -0.17 -0.14 -0.15 -0.11 -0.10 -0.07 -0.06 -0.02  0.02  0.06
## PACF -0.04  0.07 -0.08  0.03  0.01  0.07  0.03  0.03  0.01  0.04  0.05  0.09
##      [,38] [,39] [,40] [,41] [,42] [,43] [,44] [,45] [,46] [,47] [,48] [,49]
## ACF   0.10  0.14  0.18  0.24  0.29  0.33   0.4  0.44  0.48  0.53  0.55  0.56
## PACF  0.05  0.06  0.04  0.12  0.07  0.06   0.1  0.08  0.04  0.11  0.02  0.01
##      [,50] [,51] [,52] [,53] [,54] [,55] [,56] [,57] [,58] [,59] [,60]
## ACF   0.57   0.6  0.58  0.59  0.57  0.55  0.53  0.51  0.48  0.42  0.39
## PACF  0.02   0.1  0.00  0.03 -0.02  0.01  0.00  0.07 -0.01 -0.09  0.02
```

```r
# Set the true parameters
phi_true <- 0.9
theta_true <- 0.5
sigma2_true <- 1

# Simulation function for ARMA(1,1)
simulate_arma_mle <- function(T, n_sim=1000) {
  phi_est <- rep(NA, n_sim)
  theta_est <- rep(NA, n_sim)
  sigma2_est <- rep(NA, n_sim)

  for (i in 1:n_sim) {
    x <- arima.sim(model = list(ar = phi_true, ma = theta_true), n = T)
    # Fit using Maximum Likelihood instead of CSS
    fit <- Arima(x, order = c(1, 0, 1), method = "ML")
    phi_est[i] <- fit$coef[1]
    theta_est[i] <- fit$coef[2]
    sigma2_est[i] <- fit$sigma2
  }

  list(phi = phi_est, theta = theta_est, sigma2 = sigma2_est)
}

# Calculate MSE, MAD, and Coverage
calculate_metrics <- function(estimates, true_value, ci_width=1.96) {
  mse <- mean((estimates - true_value)^2)
  mad <- mean(abs(estimates - true_value))
  coverage <- mean(abs(estimates - true_value) < ci_width * sd(estimates))

  c(MSE = mse, MAD = mad, Coverage = coverage)
}

# Simulate and calculate metrics for T = 50, 200, 500
T_values <- c(50, 200, 500)
results <- matrix(NA, nrow=3, ncol=3, dimnames=list(T_values, c("MSE", "MAD", "Coverage")))

for (T in T_values) {
  estimates <- simulate_arma_mle(T)
  metrics_phi <- calculate_metrics(estimates$phi, phi_true)
  metrics_theta <- calculate_metrics(estimates$theta, theta_true)
  metrics_sigma2 <- calculate_metrics(estimates$sigma2, sigma2_true)

  results[as.character(T), ] <- colMeans(rbind(metrics_phi, metrics_theta, metrics_sigma2))
}

print(results)
```

```
##             MSE        MAD  Coverage
## 50  0.028214612 0.12554186 0.9176667
## 200 0.005182655 0.05302476 0.9390000
## 500 0.002053272 0.03346433 0.9470000
```

# Conclusion

The table above shows the average MSE, MAD, and Coverage across 1000 simulations for each parameter at different time lengths (T = 50, 200, 500). The estimators are close to the true values, and coverage of the 95% confidence intervals is reasonably high.