Lecture 35

GARCH Models Part 1

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Dow Jones Industrial Average (recap)

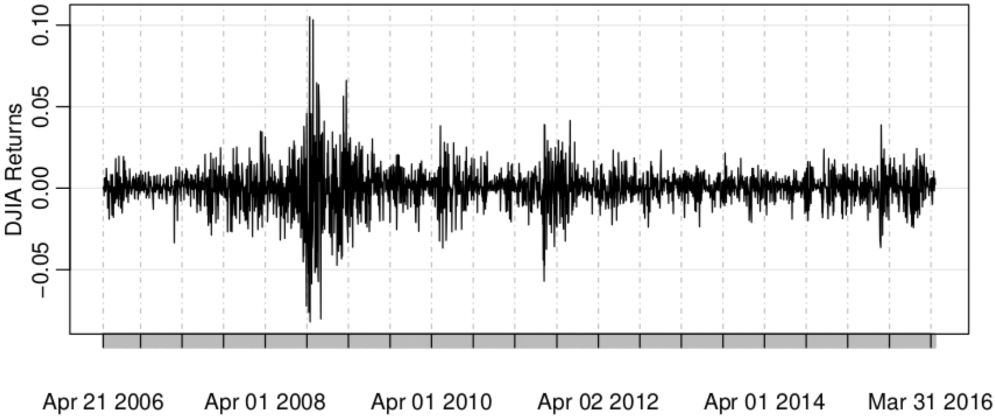

Fig. 1.4. *The daily returns of the Dow Jones Industrial Average (DJIA) from April 20, 2006 to April 20, 2016.*

Dow Jones Industrial Average (recap)

- It is easy to spot the financial crisis of 2008 in the figure.
- The data shown here are typical of return data ($R_t = \frac{X_t - X_{t-1}}{X_{t-1}}$).
- If the return represents a small (in magnitude) percentage change then $\nabla \log(X_t) \approx R_t$. Either value, $\nabla \log(X_t)$ or $\frac{X_t - X_{t-1}}{X_{t-1}}$ are called the return.
- The mean of the series appears to be stable with an average return of nearly zero, however, highly volatile periods tend to be clustered together.
- A problem in the analysis of these type of financial data is to forecast the volatility of future returns.
- Models such as ARCH and GARCH models and stochastic volatility models have been developed to handle these problems.

ARCH(1) model

- If R_t follows an AR(1) process, $\mathrm{Var}(R_t | R_{t-1}, R_{t-2}, \ldots) = \mathrm{Var}(R_t | R_{t-1}) = \sigma_W^2$.
- Typically, for financial series, R_t does not have a constant conditional variance, and highly volatile periods tend to be clustered together.
- The simplest ARCH model, the ARCH(1), models the return as

  $$R_t = \sigma_t \varepsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 R_{t-1}^2$$

  where ε_t's are IID standard Gaussian white noise.
- With ARMA models, we must impose some constraints ($\alpha_0, \alpha_1 \geq 0$) on the model parameters to obtain desirable properties.
- The conditional distribution $R_t | R_{t-1} \sim \mathrm{N}(0, \alpha_0 + \alpha_1 R_{t-1}^2)$.

AR(1)-type representation of ARCH(1) model

- We can write the ARCH(1) model as a non-Gaussian AR(1) model for R_t^2.
- First, we write as
$$R_t^2 = \sigma_t^2 \varepsilon_t^2, \quad \alpha_0 + \alpha_1 R_{t-1}^2 = \sigma_t^2.$$
- We subtract the two equations to obtain
$$R_t^2 - (\alpha_0 + \alpha_1 R_{t-1}^2) = \sigma_t^2 \varepsilon_t^2 - \sigma_t^2 \overset{\textit{Notation}}{=} V_t.$$
- Here $V_t = \sigma_t^2(\varepsilon_t^2 - 1)$. Because ε_t^2 is the square of a N(0, 1) random variable, $\varepsilon_t^2 - 1$ is a shifted (to have mean-zero), χ_1^2 random variable.
- Overall, $R_t^2 = \alpha_0 + \alpha_1 R_{t-1}^2 + V_t$ where $V_t | R_{t-1} \sim (\alpha_0 + \alpha_1 R_{t-1}^2) \times (\chi_1^2 - 1)$.

Digression: Martingale

- A basic definition of a discrete-time martingale is a discrete-time stochastic process $\{X_1, X_2, X_3, \ldots\}$ that satisfies for any time T,

$$E(|X_t|) < \infty,$$

$$E(X_{T+1}|X_1, \ldots, X_T) = X_T.$$

- We can define $X_t^* = X_t - E(X_t|X_{t-1}, X_{t-2}, \ldots, X_1)$.

- Here, clearly,

$$E(X_t^*|X_{t-1}^*, X_{t-2}^*, \ldots, X_1^*) = E(X_t|X_{t-1}, X_{t-2}, \ldots, X_1) - E(X_t|X_{t-1}, X_{t-2}, \ldots, X_1) = 0.$$

- Here X_t^* is called martingale difference.

Properties of GARCH

- We define $\mathcal{R}_s = \{R_s, R_{s-1}, \ldots\}$.
- Because $E(R_t|\mathcal{R}_{t-1}) = 0$, the process R_t is said to be a martingale difference.
- Because R_t is a martingale difference, it is also an uncorrelated sequence.
- Therefore, $E(R_t^2)$ and $\mathrm{Var}(R_t^2)$ must be constant with respect to time t.

Properties of GARCH (contd.)

- $$E(R_t) = EE(R_t|\mathcal{R}_{t-1}) = EE(R_t|R_{t-1}) = 0$$

- $$\mathrm{Cov}(R_{t+h}, R_t) = E(R_t R_{t+h}) = EE(R_t R_{t+h}|R_{t+h-1}) = ER_t E(R_{t+h}|R_{t+h-1}) = 0$$

- $$E(R_t^2) = \mathrm{Var}(R_t) = \frac{\alpha_0}{1-\alpha_1}$$

- $$E(R_t^4) = \frac{3\alpha_0^2}{(1-\alpha_1)^2} \times \frac{1-\alpha_1^2}{1-3\alpha_1^2}$$

Thank you!