Lecture 18

ACF of AR, MA, and ARMA models

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ACF of an MA(q) process

- Consider the ACF of an MA(q) process, $X_t = \theta(B)W_t$, where
$$\theta(B) = 1 + \theta_1 B + \ldots + \theta_q B^q.$$
- $E(X_t) = 0$
- $\mathrm{Var}(X_t) = \sigma_W^2(1 + \theta_1^2 + \ldots + \theta_q^2)$
- $\gamma(h) = \mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}(\sum_{j=0}^q \theta_j W_{t+h-j}, \sum_{j=0}^q \theta_j W_{t-j})$
- $\rho(h) = \gamma(h)/\gamma(0)$

Difference equation: General result for order p (Recap)

- These results generalize to the homogeneous difference equation of order p:

$$u_n - \alpha_1 u_{n-1} - \ldots - \alpha_p u_{n-p} = 0, \quad \alpha_p \neq 0, \quad n = p, p+1, \ldots.$$

- The associated polynomial is $\alpha(z) = 1 - \alpha_1 z - \ldots - \alpha_p z^p$.

- Suppose $\alpha(z)$ has r distinct roots, z_i with multiplicity m_i for $i = 1, \ldots, r$, such that $\sum_{i=1}^{r} m_i = p$.

- The general solution is

$$u_n = z_1^{-n} P_1(n) + z_2^{-n} P_2(n) + \ldots + z_r^{-n} P_r(n),$$

where $P_j(n)$, for $j = 1, 2, \ldots, r$, is a polynomial in n, of degree $m_j - 1$.

ACF of an AR(p) process

- Suppose $X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + W_t$ is a causal AR(p) process.
- Multiply each side of the model by X_{t-h} for $h \geq p$, and take expectation:

$$E(X_t X_{t-h}) = \phi_1 E(X_{t-1} X_{t-h}) + \ldots + \phi_p E(X_{t-p} X_{t-h}) + E(W_t X_{t-h}).$$

- The result is $\gamma(h) = \phi_1 \gamma(h-1) + \ldots + \phi_p \gamma(h-p), h = p, p+1, \ldots$.
- Dividing by $\gamma(0)$, we have $\rho(h) = \phi_1 \rho(h-1) + \ldots + \phi_p \rho(h-p)$.
- Suppose $\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p$ has r distinct roots, z_i with multiplicity m_i for $i = 1, \ldots, r$, such that $\sum_{i=1}^r m_i = p$.
- The general solution is

$$\rho(h) = z_1^{-h} P_1(h) + z_2^{-h} P_2(h) + \ldots + z_r^{-h} P_r(h), \quad h = p, p+1, \ldots$$

where $P_j(h)$, for $j = 1, 2, \ldots, r$, is a polynomial in h, of degree $m_j - 1$.

ACF of an ARMA(p, q) process

- A causal ARMA(p, q) model $\{X_t; t = 0, \pm 1, \pm 2, \ldots\}$ can be written as a one-sided linear process $X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j} = \psi(B) W_t$.
- Show that

$$\gamma(h) = \mathrm{Cov}(X_{t+h}, X_t) = \sum_{j=1}^{p} \phi_j \gamma(h-j) + \sigma_w^2 \sum_{j=h}^{q} \theta_j \psi_{j-h}, \quad h \geq 0.$$

- From there, we can write $\gamma(h) - \sum_{j=1}^{p} \phi_j \gamma(h-j) = 0, \quad h \geq \max\{p, q+1\}$, with initial conditions

$$\gamma(h) - \sum_{j=1}^{p} \phi_j \gamma(h-j) - \sigma_w^2 \sum_{j=h}^{q} \theta_j \psi_{j-h} = 0, \quad 0 \leq h < \max\{p, q+1\}.$$

Example: ACF of an ARMA$(1,1)$ process

- Consider the model: $X_t = \phi X_{t-1} + \theta W_{t-1} + W_t$, where $|\phi| < 1$.
- We can obtain $\gamma(h) - \phi\gamma(h-1) = 0,\ h \geq 2$, which implies $\gamma(h) = c\phi^h, h \geq 2$.
- Initial conditions can be solved as: $\gamma(0) = \phi\gamma(1) + \sigma_W^2[1 + \theta\phi + \theta^2]$ and $\gamma(1) = \phi\gamma(0) + \sigma_W^2\theta$.
- The final solution is $\rho(h) = \dfrac{(1+\theta\phi)(\phi+\theta)}{1+2\theta\phi+\theta^2}\phi^{h-1},\quad h \geq 1$.
- Thus, the dominating terms in $\rho(h)$ for AR(1) and ARMA(1,1) are the exponentially decaying terms ϕ^h.
- As a result, it is not possible to distinguish between them based on ACF.
- We need a tool to solve this issue and PACF will be the savior!

Thank you!