Lecture 5

Measures of Dependence Part 1

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Autocovariance

- The autocovariance function is defined as the second moment product

$$\gamma_x(s,t) = \mathrm{Cov}(X_s, X_t) = E[(X_s - \mu_s)(X_t - \mu_t)].$$

- Note that $\gamma_x(s,t) = \gamma_x(t,s)$ for all time points s and t.
- If X_s and X_t are bivariate normal, $\gamma_x(s,t) = 0$ ensures their independence.
- For $s = t$, the autocovariance reduces to the (assumed finite) variance.

Autocovariance examples

- Autocovariance of white noise: $\gamma_W(s,t) = 0$ if $s \neq t$ and $\gamma_W(s,t) = \sigma_W^2$ if $s = t$.
- Autocovariance of Moving Average Series:
  - $\gamma_V(s,t) = \frac{3}{9}\sigma_W^2$ when $s = t$
  - $\gamma_V(s,t) = \frac{2}{9}\sigma_W^2$ when $|s-t| = 1$
  - $\gamma_V(s,t) = \frac{1}{9}\sigma_W^2$ when $|s-t| = 2$
  - $\gamma_V(s,t) = 0$ when $|s-t| > 2$
- Random Walk with Drift: $\gamma_X(s,t) = \min\{s,t\}\sigma_W^2$.

Other measures of dependence

- The autocorrelation function (ACF) is defined as

$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s) \times \gamma(t,t)}}.$$

- The cross-covariance function between two series, X_t and Y_t, is

$$\gamma_{xy}(s,t) = \mathrm{Cov}(X_s, Y_t) = E[(X_s - \mu_{xs})(Y_t - \mu_{yt})].$$

- The cross-correlation function (CCF) is given by

$$\rho_{xy}(s,t) = \frac{\gamma_{xy}(s,t)}{\sqrt{\gamma_x(s,s) \times \gamma_y(t,t)}}.$$

Multivariate time series

- We may easily extend the above ideas to the case of more than two series, say, $X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(r)}$; that is, multivariate time series with r components.
- The cross-covariance terms would be

$$\gamma_{ij}(s,t) = \mathrm{Cov}(X_s^{(i)}, X_t^{(j)}) = E[(X_s^{(i)} - \mu_s^{(i)})(X_t^{(j)} - \mu_t^{(j)})]$$

  for $i, j = 1, \ldots, r$.
- The cross-correlation terms would be

$$\rho_{ij}(s,t) = \frac{\gamma_{ij}(s,t)}{\sqrt{\gamma_i(s,s) \times \gamma_j(t,t)}}$$

  for $i, j = 1, \ldots, r$.

Stationary Time Series

- A **strictly stationary** time series is one for which

$$F_{t_1,t_2,\ldots,t_n}(c_1,c_2,\ldots,c_n) = F_{t_1+h,t_2+h,\ldots,t_n+h}(c_1,c_2,\ldots,c_n)$$

  for all $n = 1, 2, \ldots$, all time points $t_1, t_2, \ldots, t_n$, all numbers $c_1, c_2, \ldots, c_n$, and all time shifts $h = 0, \pm 1, \pm 2, \ldots$.

- The result also holds for $n = 1$ and thus $F_t(c) = F_{t+h}(c)$ and hence they have same means if they exist.

- The result also holds for $n = 2$ and thus $F_{t_1,t_2}(c_1,c_2) = F_{t_1+h,t_2+h}(c_1,c_2)$ and hence they have same covariances if they exist.

Stationary Time Series

- A **weakly stationary** time series is one for which
  - variance of the process is finite at each time point,
  - the mean value function μ_t is constant and does not depend on t,
  - the autocovariance function, $\gamma(s, t)$ depends on s and t only through their difference $|s - t|$.

- We will use the term stationary to mean weakly stationary; if a process is stationary in the strict sense, we will use the term strictly stationary.

(Weakly) Stationary Time Series

- Because the mean function, $E(X_t) = \mu_t$, of a stationary time series is independent of time t, we will write $\mu_t = \mu$ for all t.

- Let $s = t + h$, where h represents the time shift or lag. Then

$$\gamma(s, t) = \gamma(t + h, t) = \gamma(h, 0).$$

- The autocovariance function of a stationary time series does not depend on the time argument t.

- Henceforth, for convenience, we will drop the second argument of $\gamma(h, 0)$ and write $\gamma(h)$.

Thank you!