Lecture 3

Time Series Statistical Models

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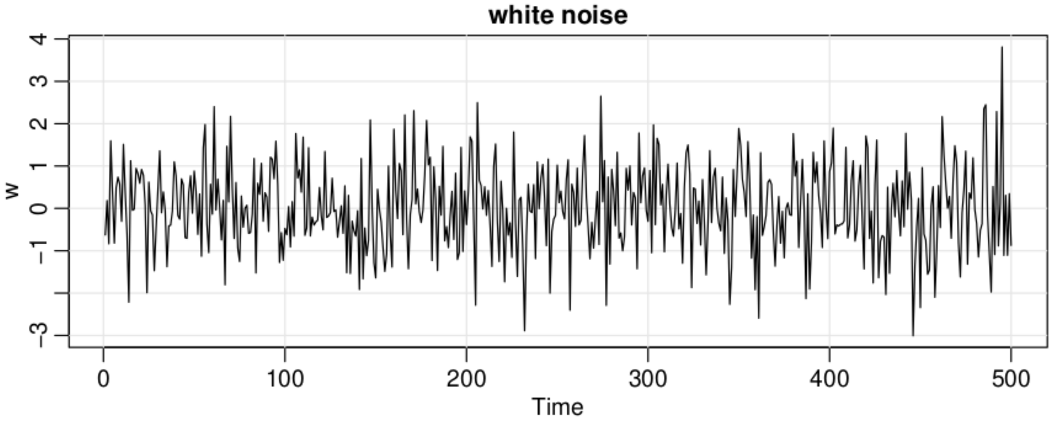
Time Series

- We assume a time series can be defined as a collection of random variables indexed according to the order they are obtained in time.
- For example, we may consider a time series as a sequence of random variables, $X_1, X_2, X_3, \ldots$, where the random variable X_1 denotes the value taken by the series at the first time point, the variable X_2 denotes the value for the second time period, and so on.
- In general, a collection of random variables, $\{X_t\}$, indexed by t is referred to as a stochastic process.
- In this text, t will typically be discrete and vary over the integers $t = 0, \pm 1, \pm 2, \ldots$, or some subset of the integers.
- The observed values of a stochastic process are referred to as a realization of the stochastic process.

White Noise

- A simple kind of generated series might be a collection of uncorrelated random variables, W_t, with mean 0 and finite variance σ_W^2; we shall denote this process as $W_t \sim \mathrm{WN}(0, \sigma_W^2)$.
- The designation white originates from the analogy with white light and indicates that all possible periodic oscillations are present with equal strength.
- We will sometimes require the noise to be IID. We then denote them by $W_t \overset{IID}{\sim} \mathcal{WN}(0, \sigma_W^2)$.
- A particularly useful white noise series is Gaussian white noise, wherein $W_t \overset{IID}{\sim} \mathrm{N}(0, \sigma_W^2)$.
- If the stochastic behavior of all time series could be explained in terms of the white noise model, classical statistical methods would suffice.

White Noise



- The resulting series bears a slight resemblance to the explosion.

Moving Average

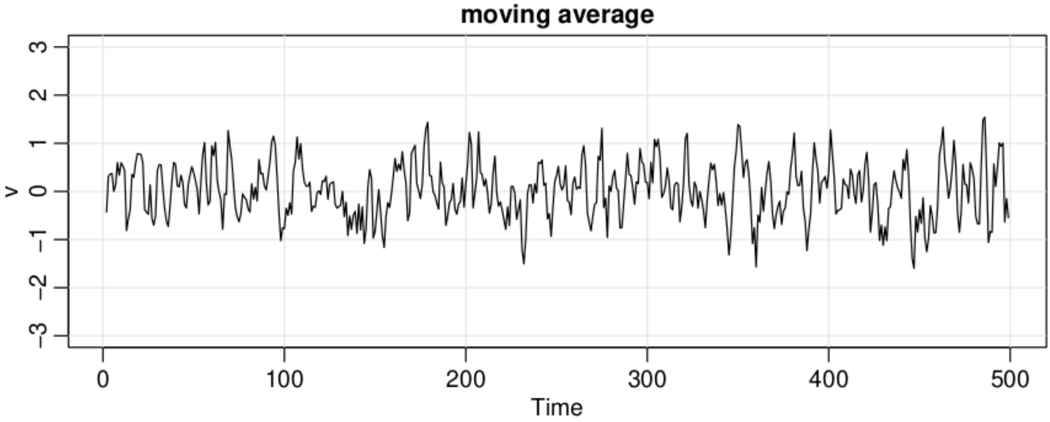
- We might replace the white noise series W_t by a moving average that smooths the series.

- For example, consider replacing W_t by an average of its current value and its immediate neighbors in the past and future. That is, let

$$V_t = \frac{1}{3}[W_{t-1} + W_t + W_{t+1}].$$

- Inspecting the series shows a smoother version of the first series, reflecting the fact that the slower oscillations are more apparent and some of the faster oscillations are taken out.

Moving Average



- The resulting series bears a slight resemblance to the SOI or perhaps, to some of the fMRI series.

Autoregression

- Suppose we consider W_t's as input and calculate the output using the second-order equation

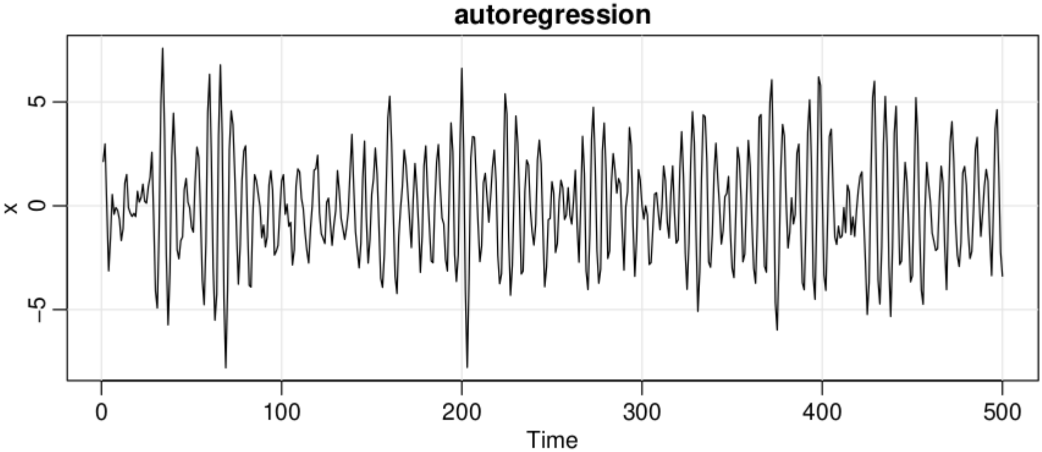
$$X_t = X_{t-1} - 0.9X_{t-2} + W_t$$

  successively for $t = 1, 2, \ldots, 500$.

- This represents a regression or prediction of X_t as a function of the past two values of the series, and, hence, the term autoregression is suggested for this model.

- A problem with startup values exists here because the model also depends on the initial conditions X_0 and X_{-1}, but assuming we have the values, we generate the succeeding values.

Autoregression



- The resulting series bears a slight resemblance to the speech.

Random Walk with Drift

- A model for analyzing trend such as seen in the global temperature data is the random walk with drift model given by

$$X_t = \delta + X_{t-1} + W_t$$

for $t = 1, 2, \ldots$, with initial condition $X_0 = 0$, and where W_t is white noise.

- The constant δ is called the drift, and when $\delta = 0$, the model is called simply a random walk.
- Note that we may rewrite the model as a cumulative sum of white noise variates. That is,

$$X_t = \delta t + \sum_{j=1}^{t} W_j$$

Random Walk with Drift

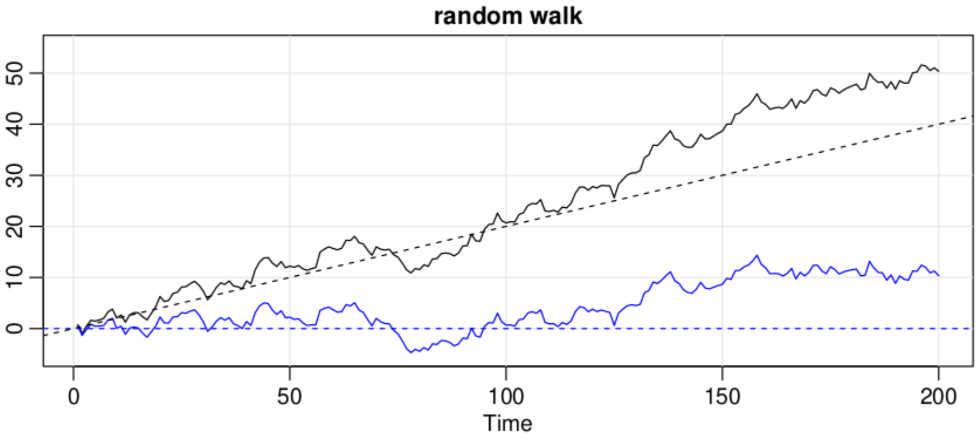


Fig. 1.10. *Random walk, $\sigma_w = 1$, with drift $\delta = .2$ (upper jagged line), without drift, $\delta = 0$ (lower jagged line), and straight (dashed) lines with slope δ.*

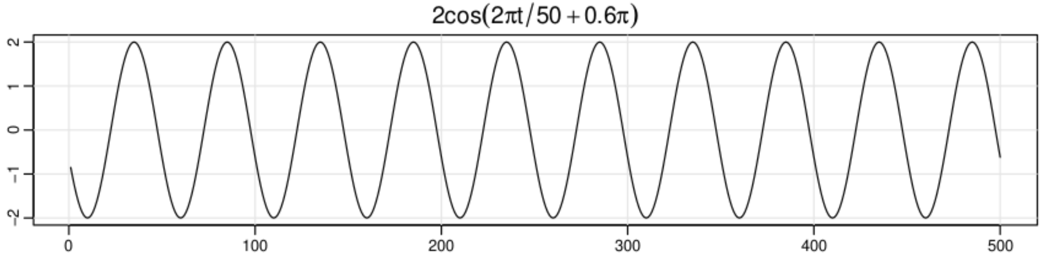
Noise in Signals

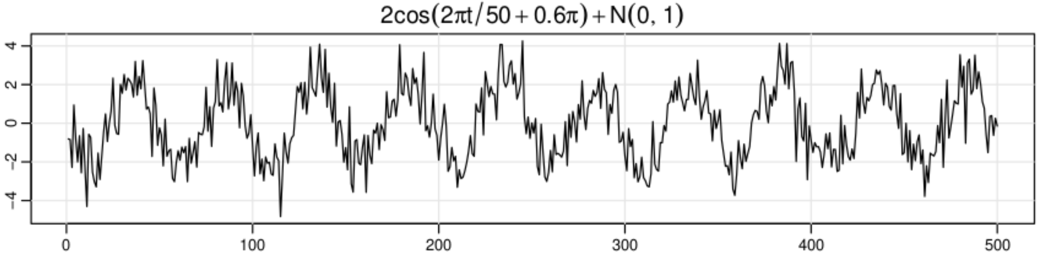
- Many realistic models for generating time series assume an underlying signal with some consistent periodic variation, contaminated by adding a noise.
- For example, it is easy to detect the regular cycle fMRI series.
- Consider the model

$$xt = 2\cos(2\pi\frac{t+15}{50}) + W_t$$

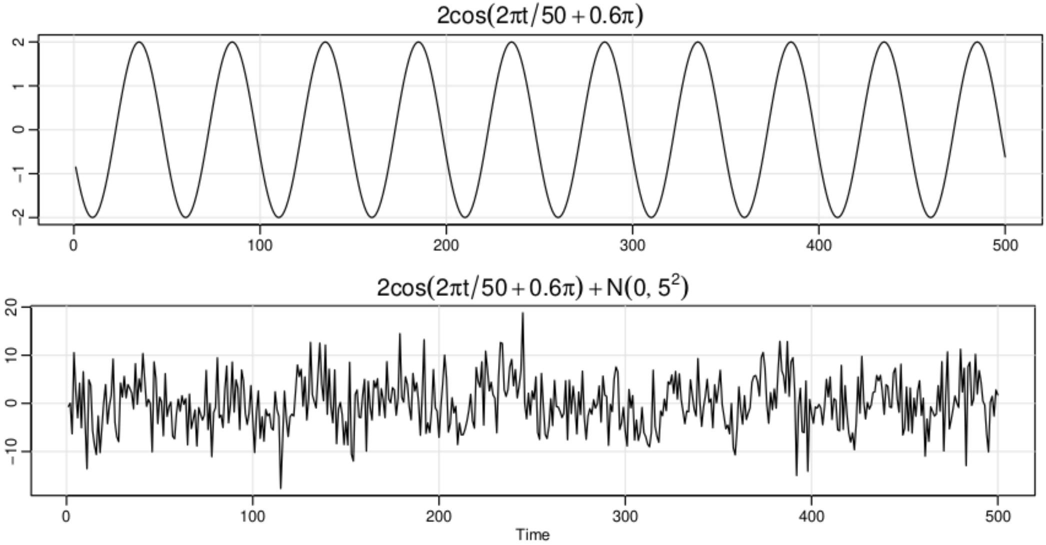
  for $t = 1, 2, \ldots, 500$.
- A sinusoidal waveform can be written as $A\cos(2\pi\omega t + \phi)$ where A is the amplitude, ω is the frequency of oscillation, and ϕ is a phase shift.
- Of course, the degree to which the signal is obscured depends on the amplitude of the signal and the size of σ_W. The ratio of the amplitude of the signal to σ_W (or some function of the ratio) is sometimes called the signal-to-noise ratio (SNR)
- The larger the SNR, the easier it is to detect the signal.

Noise in Signals





Noise in Signals



Thank you!