

MTH 442: Time Series Analysis

Problem Set#5

1. Let $\{X_t\}$ and $\{Y_t\}$ be two covariance stationary processes. Define a bivariate process

$Z_t = (X_t, Y_t)'$. Prove or give a counter example " Z_t is covariance stationary".

2. Let $\{X_t\}$ be a covariance stationary process with mean μ and ACVF $\gamma_X(h)$. Define

$\{Y_t\}$ as $Y_t = (X_t, X_{\alpha+\beta t})'$. Find the values of α and β for which $\{Y_t\}$ is stationary.

3. Let $X_t = \phi X_{t-1} + \mu_t$ and $Y_t = \phi X_{t-2} + \delta_t$, where $|\phi| < 1$, $\{\mu_t\}$ and $\{\delta_t\}$ are independent $WN(0, \sigma^2)$ sequences with $Cov(\mu_t, X_{t-j}) = 0$ for all $j > 0$. Find the cross correlation $\rho_{XY}(k); k = 0, 1, 2, \dots$

4. Let $\{X_t\}$ be a stationary time series with mean zero and ACVF $\gamma_X(h) = \left(\frac{1}{2}\right)^{|h|}$ and $\{Y_t\}$ be

a $WN(0, \sigma_Y^2)$ independent of $\{X_t\}$. Define a bivariate time series

$$Z_t = \begin{pmatrix} X_t(Y_t + Y_{t-1}) \\ X_t Y_{t-2} \end{pmatrix}$$

Show that Z_t is a covariance stationary vector process. Further, find k such that

$Cov(Z_t, Z_{t+h})$ is a null matrix for all $|h| > k$.

5. A and B are constants and $\theta \sim U(0, 2\pi)$, verify whether

$X_t = (A \cos(t + \theta), B \cos(2t + \theta))'$ is covariance stationary bivariate process.

6. Let $\{X_t\}$ be a sequence of independent and identically distributed $N(0, 1)$ random

variables. Define $Y_t = (-1)^t + X_t + X_{t-1} + X_{t-2}$ and $Z_t = (-1)^t X_t$. Verify whether $\begin{pmatrix} Y_t \\ Z_t \end{pmatrix}$

is covariance stationary bivariate process.

7. Suppose $\{\varepsilon_{1,t}\}$ and $\{\varepsilon_{2,t}\}$ are two independent $WN(0, \sigma_\varepsilon^2)$ processes. Define

$$X_t = 0.5 X_{t-1} + \varepsilon_{1,t}$$

$$Y_t = 0.5 X_{t-2} + \varepsilon_{2,t}$$

Find the cross-covariance between X_t and Y_t at lags 2 and 3.

8. Let $\{X_t\}$ be zero mean stationary process with absolutely summable ACVF $\gamma_X(h)$.

Define a vector process Z_t as

$$Z_t = \begin{pmatrix} \sum_{h=-\infty}^{\infty} \psi_{1,h} B^h X_t \\ \sum_{h=-\infty}^{\infty} \psi_{2,h} B^h X_{a+bt} \end{pmatrix}$$

where, $\sum_{h=-\infty}^{\infty} |\psi_{i,h}| < \infty; i=1,2$. Find the values of a and b such that the vector process

\underline{Z}_t is covariance stationary. Further, consider \underline{Z}_t with $a=1, b=1$ and $X_t \sim WN(0, \sigma^2)$.

Prove or disprove “ \underline{Z}_t is a vector white noise process”.

[9] Suppose $\underline{\epsilon}_t \sim VWN(0, \Sigma), \Sigma > 0$ and let $\underline{Y}_t = \begin{pmatrix} \underline{\epsilon}_t \\ 2\underline{\epsilon}_{2t+3} \end{pmatrix}$. Verify whether or not $\underline{Y}_t \sim VWN$.

[10] Suppose $\epsilon_t \sim WN(0,1); X_t = 2\epsilon_t + \epsilon_{t-1} + \epsilon_{t+1}$ and $Y_t = 2 + \epsilon_t - \epsilon_{t-1}$.

Define $\underline{Z}_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}$.

(a) Verify whether or not \underline{Z}_t is covariance stationary.

(b) Find the smallest integer k such that $Cov(\underline{Z}_t, \underline{Z}_{t+h})$ is a null matrix for $|h| \geq k$.

[11] Let $\{e_t\}$ be a sequence of i.i.d. $N(0,1)$ random variables. Define

$$X_t = e_1 + e_2 \cos t + e_3 \sin t$$

$$Y_t = t + e_1 \cos t + e_2 \sin t$$

$$Z_t = (-1)^t e_4 + e_5 \cos t + e_6 \sin t$$

Verify whether the bivariate processes $\underline{P}_t = \begin{pmatrix} X_t \\ Y_t - t \end{pmatrix}$ and $\underline{Q}_t = \begin{pmatrix} X_t \\ Z_t \end{pmatrix}$ are covariance stationary.

[12] Consider the 3-variate stationary $VAR(1)$, $\underline{X}_t = \Phi \underline{X}_{t-1} + \underline{\epsilon}_t$; where, $\{\underline{\epsilon}_t\} \sim VWN(0, \Sigma)$.

Define $\underline{Y}_t = \begin{pmatrix} \underline{X}_t \\ \alpha \underline{X}_{t-1} \end{pmatrix}, \alpha \in \mathbb{R}$. Prove or disprove “ \underline{Y}_t is covariance stationary VAR

$\forall \alpha \in \mathbb{R}$ ”.

[13] Consider a n -dimensional stationary $VAR(p)$ process

$$\underline{X}_t = \Phi_1 \underline{X}_{t-1} + \Phi_2 \underline{X}_{t-2} + \dots + \Phi_p \underline{X}_{t-p} + \underline{\epsilon}_t; \underline{\epsilon}_t \sim VWN(0, \Sigma).$$

Define a new np -dimensional vector process $\underline{Z}_t = (\underline{X}_t' \underline{X}_{t-1}' \dots \underline{X}_{t-p+1}')'$. Let us define

$$\underset{np \times np}{F} = \begin{bmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \dots & \Phi_{p-1} & \Phi_p \\ I_n & 0 & 0 & \dots & 0 & 0 \\ 0 & I_n & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & I_n & 0 \end{bmatrix} \text{ and } \underset{np \times 1}{\underline{\eta}_t} = \begin{pmatrix} \underline{\epsilon}_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Using F and $\underline{\eta}_t$, express \underline{Z}_t as a VAR process. Further, if $E(\underline{Z}_t \underline{Z}_t') = \Omega$, prove that

$$\Omega = F\Omega F' + Q; \text{ where } E(\underline{\eta}_t \underline{\eta}_t') = Q.$$

[14] Consider the following 2-variate $VAR(2)$ process;

$$\underline{X}_t = \underline{\epsilon}_t - \underline{X}_{t-1} - \frac{1}{4} \underline{X}_{t-2};$$

where, $\{\varepsilon_t\} \sim VWN(0, \Sigma), \Sigma > 0$. Is the given $VAR(2)$ process covariance stationary? In case the process is causal, obtain its $VMA(\infty)$ representation.

- [15] Consider a k -dimensional stationary $VAR(2)$ process

$\underline{X}_t = \Phi_1 \underline{X}_{t-1} + \Phi_2 \underline{X}_{t-2} + \varepsilon_t; \varepsilon_t \sim VWN(0, \Sigma)$. Prove or disprove “ $\underline{Z}_t = \begin{pmatrix} \underline{X}_t \\ \underline{X}_{t-1} \end{pmatrix}$ is a stationary $VAR(1)$ process”.

- [16] Let $\{\underline{X}_t\}$ be a bivariate vector $ARMA(1, 1)$ process:

$$\underline{X}_t = \Phi \underline{X}_{t-1} + \varepsilon_t + \Theta \varepsilon_{t-1}; \{\varepsilon_t\} \sim VWN(0, I_2), \text{ where, } \Phi = \Theta = \begin{pmatrix} a & a \\ 0 & a \end{pmatrix}.$$

- Find the value(s) of the constant a such that $\{\underline{X}_t\}$ is causal and invertible.
 - Using an appropriate value of a (obtained in (1)), find the $VMA(\infty)$ and $VAR(\infty)$ representation of $\{\underline{X}_t\}$.
 - Verify whether the sequence of matrices, say $\{\Psi_j\}_{j=0}^{\infty}$, associated with the $VMA(\infty)$ representation is absolutely summable or not.
 - Find the impulse response of the two component variables of $\{\underline{X}_t\}$ with respect to shocks in the other variable.
- [17] Let \underline{X}_t be a 2-variate $VAR(1)$ process $\underline{X}_t = \Phi \underline{X}_{t-1} + \varepsilon_t, \varepsilon_t \sim VWN(0, \Sigma); \Sigma > 0, \Phi = \begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix}$. Which of the following statements is (are) CORRECT?
- For $a = 0.5$ and $b = 0.5$, \underline{X}_t is covariance stationary
 - For $a = 0.5$ and $b = 1$, \underline{X}_t is covariance stationary
 - For all $a, b \in \mathbb{R}$, \underline{X}_t is covariance stationary
- [18] Suppose the $k \times 1$ random vector $\varepsilon_t \sim VWN(0, \Sigma); \Sigma > 0$ and $\underline{X}_t = \varepsilon_t + \Theta \varepsilon_{t-1}$, Θ is a $k \times k$ matrix of constants. Define $\underline{Z}_t = \begin{pmatrix} \underline{X}_t \\ \varepsilon_{t-1} \end{pmatrix}$. Prove or disprove the following statements.
- \underline{Z}_t is covariance stationary for all Θ
 - $\underline{Z}_t \sim VWN$
 - \underline{Z}_t has a $VMA(1)$ representation