Q4:

Claim: Partial Autocorrelation Function (PACF) ϕ_{hh} can be written as:

$$\phi_{hh} = \frac{\rho(h) - \tilde{\rho}'_{h-1} R_{h-1}^{-1} \rho_{h-1}}{1 - \tilde{\rho}'_{h-1} R_{h-1}^{-1} \tilde{\rho}_{h-1}}.$$

Proof:

1. **Prediction Equations:** prediction equations are:

$$\Gamma_h \phi_h = \gamma_h.$$

2. **Divide by $\gamma(0)$:** Dividing both sides by $\gamma(0)$ gives:

$$\frac{\Gamma_h \phi_h}{\gamma(0)} = \frac{\gamma_h}{\gamma(0)}.$$

We denote R_h and ρ_h as:

$$R_h = \frac{\Gamma_h}{\gamma(0)},$$

$$\rho_h = \frac{\gamma_h}{\gamma(0)}.$$

Thus, we can express equation as:

$$R_h \phi_h = \rho_h.$$

- R_h is normalized autocorrelation matrix.
- $\phi_h = (\phi_{h1}, \ldots, \phi_{hh})'$ is vector of coefficients.
- ρ_h is normalized vector of lagged autocorrelations.

3. **partitioning equations:** we can partition equation write ϕ_h as:

$$\phi_h = \begin{pmatrix} \phi_{h-1} \\ \phi_{hh} \end{pmatrix}.$$

this expression indicates that ϕ_h consists of previous partition and new value both. from point 2 and .

$$R_h = \begin{pmatrix} R_{h-1} & \tilde{\rho}_{h-1} \\ \tilde{\rho}'_{h-1} & 1 \end{pmatrix}, \quad \phi_h = \begin{pmatrix} \phi_{h-1} \\ \phi_{hh} \end{pmatrix}, \quad \rho_h = \begin{pmatrix} \rho_{h-1} \\ \rho(h) \end{pmatrix}$$

I write corresponding matrix equation as:

$$\begin{pmatrix} R_{h-1} & \tilde{\rho}_{h-1} \\ \tilde{\rho}'_{h-1} & 1 \end{pmatrix} \begin{pmatrix} \phi_{h-1} \\ \phi_{hh} \end{pmatrix} = \begin{pmatrix} \rho_{h-1} \\ \rho(h) \end{pmatrix}.$$

4. **system of equations:** now we just do matrix multiply to get equation:

$$R_{h-1}\phi_{h-1} + \tilde{\rho}_{h-1}\phi_{hh} = \rho_{h-1}, \quad (i)$$

$$\tilde{\rho}'_{h-1}\phi_{h-1} + \phi_{hh} = \rho(h). \quad (ii)$$

5. **solving equation (i) for ϕ_{h-1}:** arranging equation (i):

$$R_{h-1}\phi_{h-1} = \rho_{h-1} - \tilde{\rho}_{h-1}\phi_{hh}.$$

multiply by R_{h-1}^{-1}:

$$\phi_{h-1} = R_{h-1}^{-1}(\rho_{h-1} - \tilde{\rho}_{h-1}\phi_{hh}).$$

6. **substituting into (ii):** substitute ϕ_{h-1} from point 5 into (ii)equation:

$$\tilde{\rho}'_{h-1} R_{h-1}^{-1}(\rho_{h-1} - \tilde{\rho}_{h-1}\phi_{hh}) + \phi_{hh} = \rho(h).$$

$\implies \phi_{hh}(1 - \tilde{\rho}'_{h-1} R_{h-1}^{-1} \tilde{\rho}_{h-1}) = \rho(h) - \tilde{\rho}'_{h-1} R_{h-1}^{-1} \rho_{h-1}.$

solve for ϕ_{hh}:

$$\phi_{hh} = \frac{\rho(h) - \tilde{\rho}'_{h-1} R_{h-1}^{-1} \rho_{h-1}}{1 - \tilde{\rho}'_{h-1} R_{h-1}^{-1} \tilde{\rho}_{h-1}}. \quad (iii)$$

7. **PACF:** we have to also show that PACF can be written in form of (iii) PACF is defined as:

$$\phi_{hh} = \frac{E(\epsilon_t \delta_{t-h})}{\sqrt{E(\epsilon_t^2)E(\delta_{t-h}^2)}},$$

let $X = (X_{t-1}, \ldots, X_{t-h+1})'$.

8. **Regression Def.:**
The regression of X_t on X is given by:

$$(\Gamma_{h-1}^{-1}\gamma_{h-1})'X.$$

The regression of X_{t-h} on X is given by:

$$(\Gamma_{h-1}^{-1}\tilde{\gamma}_{h-1})'X.$$

so,

$$\epsilon_t = x_t - \gamma_{h-1}'\Gamma_{h-1}^{-1}x$$

$$\delta_{t-h} = x_{t-h} - \tilde{\gamma}_{h-1}'\Gamma_{h-1}^{-1}x$$

$$E(\epsilon_t\delta_{t-h}) = E\left[(x_t - \gamma_{h-1}'\Gamma_{h-1}^{-1}x)(x_{t-h} - \tilde{\gamma}_{h-1}'\Gamma_{h-1}^{-1}x)\right]$$

$$= E(x_t x_{t-h}) - E\left(\gamma_{h-1}'\Gamma_{h-1}^{-1}x x_{t-h}\right) - E\left(\tilde{\gamma}_{h-1}'\Gamma_{h-1}^{-1}x x_t\right) + E\left(\gamma_{h-1}'\Gamma_{h-1}^{-1}\tilde{\gamma}_{h-1}'\Gamma_{h-1}^{-1}x\right)$$

$$= E(x_t x_{t-h}) - \gamma_{h-1}'\Gamma_{h-1}^{-1}E(x_{t-h}) - \tilde{\gamma}_{h-1}'\Gamma_{h-1}^{-1}E(x_t) + \gamma_{h-1}'\Gamma_{h-1}^{-1}\tilde{\gamma}_{h-1}'\Gamma_{h-1}^{-1}E(x)$$

$$= \gamma(h) - \gamma_{h-1}'\Gamma_{h-1}^{-1}\gamma_{h-1}$$

$$E(\epsilon_t\delta_{t-h}) = \operatorname{cov}(\epsilon_t, \delta_{t-h}) = \gamma(h) - \tilde{\gamma}_{h-1}'\Gamma_{h-1}^{-1}\gamma_{h-1}$$

9. **Expectation Calculation:** We calculate the expectation as follows:

$$E(\epsilon_t\delta_{t-h}) = \operatorname{cov}(\epsilon_t, \delta_{t-h}) = \gamma(h) - \tilde{\gamma}_{h-1}'\Gamma_{h-1}^{-1}\gamma_{h-1}.$$

10. **Variance of δ_{t-h}:** The variance of δ_{t-h} is given by:

$$E(\delta_{t-h}^2) = \operatorname{var}(\delta_{t-h}) = \gamma(0) - \tilde{\gamma}_{h-1}'\Gamma_{h-1}^{-1}\tilde{\gamma}_{h-1}.$$

11. **Regression Error Analysis:** The error of the regression of X_t on $\tilde{X}$ can be expressed as:

$$\tilde{X} = (X_{t-h+1}, \ldots, X_{t-1})', \quad \tilde{\epsilon}_t = X_t - \left(\Gamma_{h-1}^{-1}\tilde{\gamma}_{h-1}\right)'\tilde{X}.$$

12. **Variance of Regression Error:** The variance of the regression error $\tilde{\epsilon}_t$ can be computed as:

$$E(\tilde{\epsilon}_t^2) = \operatorname{var}(\tilde{\epsilon}_t) = \gamma(0) - \tilde{\gamma}_{h-1}'\Gamma_{h-1}^{-1}\tilde{\gamma}_{h-1} = \gamma(0) - \tilde{\gamma}_{h-1}'\Gamma_{h-1}^{-1}\tilde{\gamma}_{h-1}.$$

13. **Conclusion of Error Factorization:** This result consolidates the expectation and variance calculations:

$$E(\tilde{\epsilon}_t^2) = \gamma(0) - \tilde{\gamma}_{h-1}'\Gamma_{h-1}^{-1}\tilde{\gamma}_{h-1},$$

confirms model's predictions and demonstrating factorization of $\gamma(0)$ in both numerator and denominator.

Final Form of PACF: Substituting into PACF expression yields:

$$\phi_{hh} = \frac{\gamma(h) - \tilde{\gamma}_{h-1}'R_{h-1}^{-1}\gamma_{h-1}}{\sqrt{(\gamma(0) - \tilde{\gamma}_{h-1}'R_{h-1}^{-1}\tilde{\gamma}_{h-1})^2}}.$$

Factoring out $\gamma(0)$:

$$\phi_{hh} = \frac{\rho(h) - \tilde{\rho}_{h-1}'R_{h-1}^{-1}\rho_{h-1}}{1 - \tilde{\rho}_{h-1}'R_{h-1}^{-1}\tilde{\rho}_{h-1}}.$$

hence proved