Lecture 29

Dynamic Linear Model: Part 2

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DLM: State equation (Recap)

- DLM in its basic form, employs an order one, p-dimensional vector autoregression as the state equation,

$$\boldsymbol{X}_t = \Phi \boldsymbol{X}_{t-1} + \boldsymbol{W}_t,$$

where $\boldsymbol{W}_t \overset{IID}{\sim} \mathcal{N}_p(\boldsymbol{0}, \boldsymbol{Q})$.

- In the DLM, we assume the process starts with a normal vector $\boldsymbol{X}_0$, such that $\boldsymbol{X}_0 \sim \mathcal{N}_p(\boldsymbol{\mu}_0, \Sigma_0)$.

- Here p is called state dimension.

DLM: Observation equation (Recap)

- We do not observe the state vector $\boldsymbol{X}_t$ directly, but only a linear transformed version of it with noise added, say $\boldsymbol{Y}_t = \boldsymbol{A}_t \boldsymbol{X}_t + \boldsymbol{V}_t$, where $\boldsymbol{A}_t$ is a $q \times p$ measurement or observation matrix; this equation is called the observation equation.

- The observed data vector, $\boldsymbol{Y}_t$, is q-dimensional, which can be larger than or smaller than p, the state dimension. The additive observation noise is $\boldsymbol{V}_t \overset{IID}{\sim} \mathcal{N}_q(\mathbf{0}, \boldsymbol{R})$.

- For simplicity, we initially assume $\boldsymbol{X}_0$, $\{\boldsymbol{W}_t\}$, and $\{V_t\}$ are uncorrelated.

Regression with Autocorrelated Errors (Recap)

- We discuss the modifications to the regression model when the errors are correlated.
- That is, consider the regression model

$$Y_t = \sum_{j=1}^{r} \beta_j u_{tj} + X_t$$

- Here X_t is a process with some covariance function $\gamma_X(s,t)$.
- In ordinary least squares, the assumption is that X_t is white Gaussian noise, in which case $\gamma_X(s,t) = 0$ for $s \neq t$ and $\gamma_X(t,t) = \sigma^2$, independent of t.
- If this is not the case, then weighted least squares should be used.

Multivariate ARMAX models

- Suppose we have a multivariate time series $\boldsymbol{Y}_t, t = 1, 2, \ldots$.
- That is, consider the regression model

$$Y_{it} = \sum_{j=1}^{r} \beta_{ij} u_{tj} + X_{it}, \quad i = 1, \ldots, q.$$

- In vector-matrix notation

$$\boldsymbol{Y}_t = \boldsymbol{B} \boldsymbol{u}_t + \boldsymbol{X}_t.$$

- Here $\boldsymbol{X}_t$ is a multivariate ARMA process with some covariance function matrix $\Gamma_X(s, t)$.
- We have not covered multivariate ARMA but have discussed multivariate time series and univariate ARMA separately; this is a combination of the concepts.

DLM with covariates

- The ARMAX model involves covariates that may enter into the states or into the observations.
- In this case, we suppose we have an $r \times 1$ vector of inputs $\boldsymbol{u}_t$, and write the model as
$$\boldsymbol{X}_t = \Phi \boldsymbol{X}_{t-1} + \gamma \boldsymbol{u}_t + \boldsymbol{W}_t$$
$$\boldsymbol{Y}_t = \boldsymbol{A}_t \boldsymbol{X}_t + \Gamma \boldsymbol{u}_t + \boldsymbol{V}_t$$
- Here γ is $p \times r$ and Γ is $q \times r$; either of these matrices may be the zero matrix.

Example of DLM (Global temperature data)

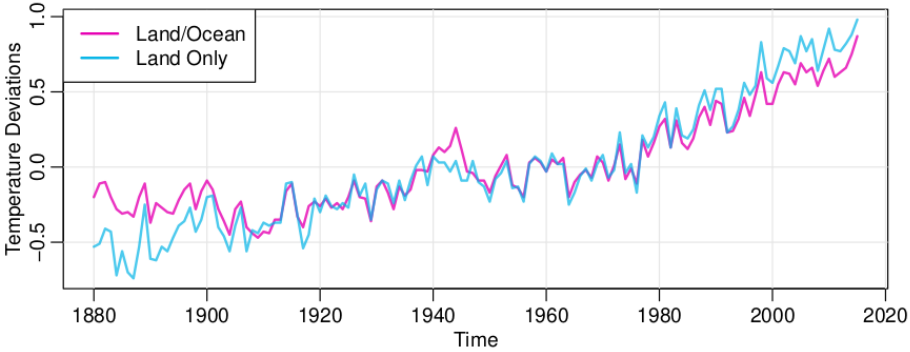

Fig. 6.3. *Annual global temperature deviation series, measured in degrees centigrade, 1880–2015. The series differ by whether or not ocean data is included.*

Example of DLM (Global temperature data, contd.)

- They show two different estimators for the global temperature series from 1880 to 2015.
- First series are the global mean land-ocean temperature index data.
- The second series are the surface air temperature index data using only meteorological station data.
- Conceptually, both series should be measuring the same underlying climatic signal, and we may consider the problem of extracting this underlying signal.

Example of DLM (Global temperature data, contd.)

- We suppose both series are observing the same signal with different noises; that is, $Y_{t1} = X_t + V_{t1}$ and $Y_{t2} = X_t + V_{t2}$ or more compactly as

$$(Y_{t1}, Y_{t2})' = (1, 1)' X_t + (V_{t1}, V_{t2})',$$

  where $\boldsymbol{R} = \mathrm{Var}[(V_{t1}, V_{t2})']$.

- It is reasonable to suppose that the unknown common signal X_t can be modeled as a random walk with drift of the form

$$X_t = \delta + X_{t-1} + W_t,$$

  with $Q = \mathrm{Var}(W_t)$.

- In this example, $p = 1$, $q = 2$, $\Phi = 1$, and $\gamma = \delta$ with $u_t = 1$.

An AR(1) Process with Observational Noise

- Consider a univariate state-space model where the observations are noisy,

$$Y_t = X_t + V_t$$

- The signal (state) is an AR(1) process,

$$X_t = \phi X_{t-1} + W_t$$

- Here $V_t \overset{IID}{\sim} \mathcal{N}(0, \sigma_V^2)$, $W_t \overset{IID}{\sim} \mathcal{N}(0, \sigma_W^2)$, and $X_0 \sim \mathcal{N}(0, (1-\phi^2)^{-1}\sigma_W^2)$

- Besides, X_0, $\{W_t\}$, and $\{V_t\}$ are independent.

An AR(1) Process with Observational Noise (contd.)

- The autocovariance function of X_t is

$$\gamma_X(h) = (1-\phi^2)^{-1}\sigma_W^2\phi^h, \quad h = 0, 1, 2, \ldots$$

- The marginal variance of Y_t is

$$\gamma_Y(0) = \mathrm{Var}(X_t + V_t) = \mathrm{Var}(X_t) + \mathrm{Var}(V_t) = (1-\phi^2)^{-1}\sigma_W^2 + \sigma_V^2$$

- The autocovariance function of Y_t is

$$\gamma_Y(h) = \mathrm{Cov}(X_{t+h}+V_{t+h}, X_t+V_t) = \mathrm{Cov}(X_{t+h}, X_t) = (1-\phi^2)^{-1}\sigma_W^2\phi^h, \;\; h = 1, 2, \ldots$$

- The ACF of Y_t is

$$\rho_Y(h) = \gamma_Y(h)/\gamma_Y(0) = (1+\sigma_V^2/\sigma_W^2(1-\phi^2))^{-1}\phi^h, \;\; h = 1, 2, \ldots$$

- The ACF of Y_t is identical to the ACF of an ARMA(1,1) process.

Things to cover

- We will cover the concepts of
  - prediction
  - filtering
  - smoothing

  state space models and include their derivations.

Thank you!