# Lecture 15

# ARMA Models: Part 1

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# Definition of ARMA models

- A autoregressive moving average model of order $(p, q)$, abbreviated $ARMA(p, q)$, is of the form

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \ldots + \theta_q W_{t-q}.$$

- Here $X_t$ is stationary, $W_t \sim WN(0, \sigma_W^2)$, and $\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q$ are constants with $\phi_p, \theta_q \neq 0$.
- The parameters $p$ and $q$ are called the autoregressive and the moving average orders, respectively.
- If $X_t$ has a nonzero mean $\mu$, we set $\alpha = (1 - \phi_1 - \ldots - \phi_p)\mu$ and write the model as

$$X_t = \alpha + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \ldots + \theta_q W_{t-q}.$$

# Concise representation

- A autoregressive moving average model of order $(p, q)$, abbreviated $ARMA(p, q)$, is of the form

$$X_t - \phi_1 X_{t-1} - \ldots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \ldots + \theta_q W_{t-q}.$$

- When $q = 0$, the model is AR($p$). We represent it using $\phi(B)X_t = W_t$.
- When $p = 0$, the model is MA($q$). We represent it using $X_t = \theta(B)W_t$.
- We thus represent the ARMA($p, q$) model using $\phi(B)X_t = \theta(B)W_t$.
- We should not unnecessarily disturb this to obtain $\eta(B)\phi(B)X_t = \eta(B)\theta(B)W_t$.
- Representing $X_t = W_t$ by $(1 - 0.5B)X_t = (1 - 0.5B)W_t$ can make the white noise process appear like ARMA$(1, 1)$.

# Three problems with ARMA models

1. Parameter redundant models
2. Stationary AR models that depend on the future
3. MA models that are not unique

They need to be resolved! We need restrictions to ensure they do not happen!

# Solution to Problem 1

- The AR and MA polynomials are defined as

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p, \quad \phi_p \neq 0,$$

  and

$$\theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q, \quad \theta_q \neq 0,$$

  respectively, where $z$ is a complex number.

- Along with its original definition, we will also require that $\phi(z)$ and $\theta(z)$ have no common factors.

- This ensures that the model $\eta(B)\phi(B)X_t = \eta(B)\theta(B)W_t$ is simplified to $\phi(B)X_t = \theta(B)W_t$, where $\phi(z)$ and $\theta(z)$ have no common factors.

# Solution to Problem 2

- An ARMA($p, q$) model is said to be causal, if the time series $\{X_t; t = 0, \pm 1, \pm 2, \ldots\}$ can be written as a one-sided linear process:

$$X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j} = \psi(B) W_t,$$

  where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$, and $\sum_{j=0}^{\infty} |\psi_j| < \infty$; we set $\psi_0 = 1$.
- To ensure this, we can look at the root of the AR polynomial.
- Consider the model $X_t = \phi X_{t-1} + W_t$; this process is causal when $|\phi| < 1$.
- The root of the AR polynomial $\phi(z) = 1 - \phi z$ is $z_0 = 1/\phi$; thus, $|z_0| = |\phi|^{-1} > 1$.
- Property: An ARMA($p, q$) model is causal if and only if $\phi(z)$ does not have any root $z_0$ for $|z_0| \leq 1$.

# Solution to Problem 3

- An ARMA($p, q$) model is said to be invertible, if the time series $\{X_t; t = 0, \pm 1, \pm 2, \ldots\}$ can be written as

$$\pi(B)X_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} = W_t,$$

  where $\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j$, and $\sum_{j=0}^{\infty} |\pi_j| < \infty$; we set $\pi_0 = 1$.
- To ensure this, we can look at the root of the MA polynomial.
- Consider the model $X_t = W_t + \theta W_{t-1}$; this process is invertible when $|\theta| < 1$.
- The root of the MA polynomial $\theta(z) = 1 + \theta z$ is $z_0 = -1/\theta$; thus, $|z_0| = |\theta|^{-1} > 1$.
- An ARMA($p, q$) model is invertible if and only if $\theta(z)$ does not have any root $z_0$ for $|z_0| \leq 1$.

Thank you!