MTH 442: Time Series Analysis
Problem Set # 2

[1] Let $\{Y_t\}$ be a stationary process with mean zero, finite variance σ^2 and ACVF $\gamma_Y(h)$. Define $X_t = (\alpha + \beta t)s_t + Y_t$, where α, β are constants and s_t is a seasonal component with period 4. Applying appropriate lag difference operators on $\{X_t\}$ to reduce $\{X_t\}$ to a stationary process and express the ACVF of the resulting stationary process in terms of that of $\{Y_t\}$.

[2] Suppose the time series $\{X_t\}$ is given by $X_t = A + Bt$, where A and B are random variables such that $E(A) = E(B) = 0$ and have the covariance matrix $\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$. Is $\{X_t\}$ covariance stationary?

[3] The time series $\{X_t\}$ is such that $X_t = \mu + X_{t-1} + \varepsilon_t$, suppose $X_1 = \mu + \varepsilon_1$ with $\varepsilon_t \sim N(0, \sigma^2)$.

(a) Is $\{X_t\}$ a mean stationary or/and a covariance stationary process?

(b) Consider the special case that $\mu = 0$, does the conclusion of (a) change under this special case?

[4] Consider the time series $\{X_t\}$ given by $X_t = \varepsilon_t \cos(\omega_0 t) + \varepsilon_{t-1} \sin(\omega_0 t) + \varepsilon_{t-2}$; where, ω_0 is a fixed constant, $\{\varepsilon_t\}$ is a sequence of independent and identically distributed $N(0, \sigma^2)$ random variables.

(a) Find $Cov(X_{t+h}, X_t); h = 0, \pm 1, \pm 2, \ldots$

(b) Is $\{X_t\}$ mean stationary?

(c) Is $\{X_t\}$ covariance stationary?

(d) Is $\{X_t\}$ a Gaussian time series?

(e) Is $\{X_t\}$ strict stationary?

[5] Let $\{X_t\}$ be a time series given by $X_t = \alpha + \beta t + S_t + Y_t$; where, α and β are real constants, S_t is a seasonal component with period 6 and $Y_t = \varepsilon_t - \varepsilon_{t-1}$; $\{\varepsilon_t\}$ is a sequence of independent and identically distributed $N(0, \sigma^2)$ random variables.

Prove or disprove the following statements:

(a) $Cov(X_t, X_{t+h}) = 0; \forall |h| \geq 2$

(b) $\{X_t\}$ is covariance stationary

(c) $\{X_t\}$ is a Gaussian time series

(d) $\{\nabla X_t\}$ is covariance stationary

(e) $Cov(\nabla_6 X_t, \nabla_6 X_{t+h}) = 0; h = \pm 2, \pm 3, \pm 4$

(f) $\{\nabla_6 X_t\}$ is covariance stationary

(g) $\{\nabla_6 X_t\}$ is strict stationary

[6] Let $\{X_t\}$ be a stationary process with mean μ_X and ACVF $\gamma_X(h)$. Consider the following process generated by $\{X_t\}$ $Z_t = X_t - X_{t-1}$. Verify whether $\{Z_t\}$ is covariance stationary.

[7] A covariance stationary time series $\{Z_t\}$ is given by $Z_t = X_t + Y_t$. Prove or give a counter example "$\{X_t\}$ and $\{Y_t\}$ are also covariance stationary".

[8] Let $\{\varepsilon_t\}$ be a sequence of independent normal random variables, each with mean 0 and variance 1, and let a, b and c be constants. Verify whether the following processes are covariance stationary or not.

(i) $X_t = a + b\varepsilon_t + c\varepsilon_{t-2}$

(ii) $X_t = \varepsilon_t \cos(at) + \varepsilon_{t-1}\sin(at)$

(iii) $X_t = \varepsilon_t \varepsilon_{t-1}$

[9] Let $\{X_t\}$ be a time series given by

$$X_t = A\cos(\omega_0 t) + B\sin(\omega_0 t) + Z_t;\ t=1,\ldots,n$$

(a) Suppose A, B, ω_0 are fixed constants $\left(A, B \in \Re;\ \omega_0 \in (0,\pi)\right)$ and $\{Z_t\}$ be a sequence of independent $\left(0,\sigma^2\right)$ random variables. Is $\{X_t\}$ covariance stationary?

(b) Suppose A and B are independently distributed random variable $\left(0,\sigma_0^2\right)$ and $\{Z_t\}$ be a sequence of independent $\left(0,\sigma^2\right)$ random variables, independent of A and B. ω_0 is a fixed constant $\in (0,\pi)$. Is $\{X_t\}$ covariance stationary?

Imp [10] Identify the stationary time series $\{X_t\}$ for which $Cov(X_{t+h}, X_t) = (-1)^{|h|} + cos\left(\frac{\pi}{4}h\right)$.

[11] Let $\{X_t\}$ be time series defined by $X_t = \varepsilon_{t-1}\,\varepsilon_t\,\varepsilon_{t+1}$, where $\{\varepsilon_t\}$ is a sequence of independently and identically distributed $N\left(0,\sigma^2\right)$ random variables. Verify whether $\{X_t\}$ is covariance stationary.

[12] Let $X_t = Y_t\left(\cos(\omega_0 t + \theta)\right) + Z_t$; where $\{Y_t\}$ and $\{Z_t\}$ are independent covariance stationary processes with auto covariance functions $\gamma_Y(h)$ and $\gamma_Z(h)$ respectively, ω_0 is a fixed constant. Verify covariance stationarity of $\{X_t\}$ when $\theta \sim U(-\pi,\pi)$ (a continuous uniform distribution on $(-\pi,\pi)$) and is independent of $\{Y_t\}$ and $\{Z_t\}$.

[13] Let $\{X_t\}$ be a covariance stationary time series with zero mean and auto covariance function, $\gamma_X(h)$; and define $Y_t = I_t X_t + (1 - I_t)X_{t-1}$, where $\{I_t\}$ is an i.i.d. sequence, independent of $\{X_t\}$, with $P(I_t = 1) = 1 - p = 1 - P(I_t = 0)$. Find $Cov(Y_t, Y_{t+h}), h = 0, \pm1, \pm2, \ldots$, and verify whether $\{Y_t\}$ is covariance stationary.

[14] Let $\{X_t\}$ be a time series given by $X_t = e^Y t^2 + \varepsilon_t$; $\varepsilon_t \sim WN(0,\sigma^2)$ and $Y \sim U(0,1)$. Y and ε_t are independently distributed. If $Z_t = \nabla^2 X_t$, find $Cov(Z_t, Z_{t+h})$ for $h = 0, \pm 1, \pm 2, \ldots$. and verify whether $\{Z_t\}$ is covariance stationary or not.

[15] Consider a Gaussian process $\{X_t\}$ with $E(X_t) = 0\ \forall t$ and $Cov(X_t, X_{t+s}) = e^{-|t-s|}\ \forall t, s$. Let $Y_t = e^{X_t}$. Prove or disprove the following statements:

(i) $\{\nabla X_t\}$ is strict stationary.

(ii) $\{Y_t\}$ is a Gaussian process.

(iii) $\{Y_t\}$ is covariance stationary.

[16] Let $\{X_t\}$ be an $MA(1)$ process $X_t = \varepsilon_t + \varepsilon_{t-1}$; $\{\varepsilon_t\}$ is a sequence of independently and identically distributed $N(0,\sigma^2)$ random variables. Consider the exponentially weighted moving average obtained from $\{X_t\}$ as $Y_1 = X_1$ and for $2 \le t \le n, Y_t = \alpha X_t + (1-\alpha) Y_{t-1}$ with $\alpha = 3/4$.

(a) Find the joint distribution of (Y_1, Y_2, Y_3).

(b) Is $\{Y_t : t \ge 1\}$ a Gaussian process?

(c) Is $\{Y_t : t \ge 1\}$ a strict stationary process?