MTH442 Assignment 4

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Q1

1. Model Setup:
The first-difference process for the time series is defined as:

$$Y_t = X_t - X_{t-1},$$

where Y_t represents the change between consecutive observations of X_t.
The model is:

$$Y_t = W_t - \lambda W_{t-1},$$

where W_t is a white noise process.
2. Invertibility: Expressing W_t in Terms of Y_t
Start with:

$$Y_t = W_t - \lambda W_{t-1}.$$

Rearrange:

$$W_t = Y_t + \lambda W_{t-1}.$$

Substitute recursively:

$$W_t = Y_t + \lambda(Y_{t-1} + \lambda W_{t-2}),$$
$$W_t = Y_t + \lambda Y_{t-1} + \lambda^2 W_{t-2}.$$

Continuing indefinitely:

$$W_t = \sum_{j=0}^{\infty} \lambda^j Y_{t-j}.$$

3. Expressing W_t in Terms of X_t
Since $Y_t = X_t - X_{t-1}$, substitute:

$$W_t = \sum_{j=0}^{\infty} \lambda^j (X_{t-j} - X_{t-j-1}).$$

Simplify:

$$W_t = X_t - \lambda(1-\lambda)X_{t-1} - \lambda^2(1-\lambda)X_{t-2} - \cdots .$$

4. Rearranged Form of the Model
The pattern in the equation suggests that:

$$W_t = X_t - \sum_{j=1}^{\infty} \lambda^j (1-\lambda) X_{t-j}.$$

Rearranging to express X_t:

$$X_t = \sum_{j=1}^{\infty} \lambda^j (1-\lambda) X_{t-j} + W_t.$$

5. Invertibility Condition
For the series to be invertible, the coefficient λ must satisfy:

$$|\lambda| < 1.$$

This ensures the infinite sum converges and the process remains stable.

Q2(a)

Given the **ARIMA(1, 1, 0)** model with drift:

$$(1-\phi B)(1-B)X_t = \delta + W_t,$$

where B is the backward shift operator such that $BX_t = X_{t-1}$, δ is the drift, and W_t is white noise. Let $Y_t = \nabla X_t = X_t - X_{t-1}$. The task is to **show by induction** that for $j \geq 1$, the following holds:

$$Y_{T+j}^{T} = \delta\left[1+\phi+\ldots+\phi^{j-1}\right] + \phi^{j} Y_T.$$

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1. Expressing the AR(1) Model for Y_t
Since the differenced series Y_t follows an AR(1) model with drift δ, we can write:

$$Y_t = \delta + \phi Y_{t-1} + W_t.$$

This recursive relation will be the basis of our proof by induction.

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2. Base Case: $j = 1$
For $j = 1$, the expression becomes:

$$Y_{T+1}^{T} = \delta\,[1] + \phi^{1} Y_T = \delta + \phi Y_T.$$

This matches the form of the AR(1) model:

$$Y_{T+1} = \delta + \phi Y_T + W_{T+1}.$$

Thus, the base case holds.

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3. Induction Hypothesis
Assume that the expression holds for some $j = n$. That is:

$$Y_{T+n}^{T} = \delta\left[1+\phi+\ldots+\phi^{n-1}\right] + \phi^{n} Y_T.$$

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4. Induction Step: Proving for $j = n+1$
Using the AR(1) relation:

$$Y_{T+n+1} = \delta + \phi Y_{T+n} + W_{T+n+1}.$$

Now, substitute the induction hypothesis for Y_{T+n}:

$$Y_{T+n+1} = \delta + \phi\left[\delta\left(1+\phi+\ldots+\phi^{n-1}\right) + \phi^{n} Y_T\right] + W_{T+n+1}.$$

Distribute ϕ:

$$Y_{T+n+1} = \delta + \delta\left(\phi+\phi^2+\ldots+\phi^{n}\right) + \phi^{n+1} Y_T + W_{T+n+1}.$$

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5. Simplifying the Expression
Notice that:

$$\delta + \delta\left(\phi+\phi^2+\ldots+\phi^{n}\right) = \delta\left(1+\phi+\phi^2+\ldots+\phi^{n}\right).$$

Thus:

$$Y_{T+n+1} = \delta\,(1+\phi+\ldots+\phi^{n}) + \phi^{n+1} Y_T + W_{T+n+1}.$$

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6. General Formula for Y_{T+j}
By induction, the general formula for Y_{T+j}^{T} is:

$$Y_{T+j}^{T} = \delta\left(1+\phi+\ldots+\phi^{j-1}\right) + \phi^{j} Y_T.$$

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7. Simplifying the Geometric Sum
The sum $1+\phi+\ldots+\phi^{j-1}$ is a geometric series:

$$1+\phi+\phi^2+\ldots+\phi^{j-1} = \frac{1-\phi^{j}}{1-\phi}, \quad \text{for} \quad \phi \neq 1.$$

Thus, the expression becomes:

$$Y_{T+j}^T = \delta \frac{1-\phi^j}{1-\phi} + \phi^j Y_T.$$

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8. Conclusion

We have shown by induction that:

$$Y_{T+j}^T = \delta \left[1 + \phi + \ldots + \phi^{j-1}\right] + \phi^j Y_T,$$

for all $j \geq 1$. This completes the proof.

Q2(b)

We are asked to use the result from part (a) to show that for $m = 1, 2, \ldots$:

$$X_{T+m}^T = X_T + \frac{\delta}{1-\phi}\left[m - \frac{\phi(1-\phi^m)}{1-\phi}\right] + (X_T - X_{T-1})\frac{\phi(1-\phi^m)}{1-\phi}.$$

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1. Recall the Result from Part (a)

From part (a), we found that for $j \geq 1$:

$$Y_{T+j}^T = \delta \left[1 + \phi + \ldots + \phi^{j-1}\right] + \phi^j Y_T.$$

The sum $1 + \phi + \ldots + \phi^{j-1}$ is a geometric series, which simplifies to:

$$\frac{1-\phi^j}{1-\phi}.$$

Thus, the expression becomes:

$$Y_{T+j}^T = \delta \frac{1-\phi^j}{1-\phi} + \phi^j Y_T.$$

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2. Expressing X_{T+m} in Terms of X_T and Differences

Since $Y_t = X_t - X_{t-1}$, the cumulative sum over m steps can be written as:

$$\sum_{j=1}^m Y_{T+j}^T = X_{T+m}^T - X_T.$$

Using the result from part (a), the sum of the Y_{T+j}^T terms for $j = 1, \ldots, m$ is:

$$\sum_{j=1}^m Y_{T+j}^T = \sum_{j=1}^m \left(\delta \frac{1-\phi^j}{1-\phi} + \phi^j Y_T\right).$$

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3. Simplifying the Sum

We simplify each part of the sum separately.

Sum of the Drift Terms:

$$\sum_{j=1}^m \delta \frac{1-\phi^j}{1-\phi} = \frac{\delta}{1-\phi}\sum_{j=1}^m (1-\phi^j).$$

Using the formula for the sum of a geometric series:

$$\sum_{j=1}^m \phi^j = \frac{\phi(1-\phi^m)}{1-\phi},$$

we get:

$$\sum_{j=1}^m (1-\phi^j) = m - \frac{\phi(1-\phi^m)}{1-\phi}.$$

Thus:

$$\sum_{j=1}^m \delta \frac{1-\phi^j}{1-\phi} = \frac{\delta}{1-\phi}\left[m - \frac{\phi(1-\phi^m)}{1-\phi}\right].$$

Sum of the Y_T-Dependent Terms:

$$\sum_{j=1}^{m} \phi^j Y_T = Y_T \sum_{j=1}^{m} \phi^j = Y_T \frac{\phi(1-\phi^m)}{1-\phi}.$$

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4. Final Expression for X_{T+m}^T
Combining the results, we get:

$$X_{T+m}^T - X_T = \frac{\delta}{1-\phi}\left[m - \frac{\phi(1-\phi^m)}{1-\phi}\right] + Y_T \frac{\phi(1-\phi^m)}{1-\phi}.$$

Since $Y_T = X_T - X_{T-1}$, the equation becomes:

$$X_{T+m}^T = X_T + \frac{\delta}{1-\phi}\left[m - \frac{\phi(1-\phi^m)}{1-\phi}\right] + (X_T - X_{T-1}) \frac{\phi(1-\phi^m)}{1-\phi}.$$

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5. Conclusion
Thus, we have shown that:

$$X_{T+m}^T = X_T + \frac{\delta}{1-\phi}\left[m - \frac{\phi(1-\phi^m)}{1-\phi}\right] + (X_T - X_{T-1}) \frac{\phi(1-\phi^m)}{1-\phi}.$$

This completes the proof.

Q2(c)

We are asked to compute the **mean-squared prediction error** P_{T+m}^T for large T, using the coefficients ψ_j^*. The general formula for the mean-squared prediction error is given by:

$$P_{T+m}^T = \sigma_W^2 \sum_{j=0}^{m-1} (\psi_j^*)^2,$$

where ψ_j^* are the coefficients of z^j in the expansion of:

$$\psi^*(z) = \frac{\theta(z)}{\phi(z)(1-z)},$$

where $\theta(z) = 1$ and $\phi(z) = 1 - \phi z$ correspond to the ARIMA(1, 1, 0) model.

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1. Expansion of $\psi^*(z)$
We start by expanding the expression:

$$\psi^*(z) = \frac{1}{(1-\phi z)(1-z)}.$$

This can be rewritten as:

$$\psi^*(z) = (1 + \psi_1^* z + \psi_2^* z^2 + \ldots)(1 - [1+\phi]z + z^2 + \ldots).$$

The expansion yields the homogeneous solution:

$$\psi_0^* = 1, \quad \psi_1^* = 1 + \phi, \quad \text{and} \quad \psi_j^* = \frac{1-\phi^{j+1}}{1-\phi} \quad \text{for} \quad j \geq 1.$$

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2. Mean-Squared Prediction Error Formula
Using the coefficients ψ_j^*, the mean-squared prediction error for large T is given by:

$$P_{T+m}^T = \sigma_W^2 \sum_{j=0}^{m-1} (\psi_j^*)^2.$$

Evaluating the Coefficients: For $j \geq 1$:

$$\psi_j^* = \frac{1-\phi^{j+1}}{1-\phi}.$$

Thus:

$$(\psi_j^*)^2 = \left(\frac{1-\phi^{j+1}}{1-\phi}\right)^2.$$

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3. Simplifying the Summation

The mean-squared prediction error becomes:

$$P_{T+m}^T = \sigma_W^2 \left[1 + \frac{1}{(1-\phi)^2} \sum_{j=1}^{m-1} \left(1-\phi^{j+1}\right)^2\right].$$

For large m, the end terms in the sum become small, as $(1-\phi^{j+1})^2 \approx 1$ for large j. Thus, the expression simplifies to:

$$P_{T+m}^T = \sigma_W^2 \left[1 + \frac{m-1}{(1-\phi)^2}\right].$$

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4. Final Expression for P_{T+m}^T

Thus, the mean-squared prediction error for large T is approximated by:

$$P_{T+m}^T = \sigma_W^2 \left[1 + \frac{m-1}{(1-\phi)^2}\right].$$

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5. Conclusion

We have used the coefficients ψ_j^* and the summation formula to compute the mean-squared prediction error P_{T+m}^T for large T. The final result is:

$$P_{T+m}^T = \sigma_W^2 \left[1 + \frac{m-1}{(1-\phi)^2}\right].$$