Lecture 14

Moving Average Models

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Definition

- A moving average model of order q, abbreviated $MA(q)$, is of the form

$$X_t = W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \ldots + \theta_q W_{t-q}.$$

- Here X_t is stationary, $W_t \sim WN(0, \sigma_W^2)$, and $\theta_1, \theta_2, \ldots, \theta_q$ are constants with $\theta_q \neq 0$.
- $E(X_t) = E(W_t) + \theta_1 E(W_{t-1}) + \ldots + \theta_q E(W_{t-q}) = 0$.
- $\mathrm{Var}(X_t) = \mathrm{Var}(W_t) + \theta_1^2 \mathrm{Var}(W_{t-1}) + \ldots + \theta_q^2 \mathrm{Var}(W_{t-q}) = (1 + \theta_1^2 + \ldots + \theta_q^2)\sigma_W^2$.
- Here we assumed the mean of X_t to be zero. Otherwise, we need to replace by

$$X_t = \alpha + W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \ldots + \theta_q W_{t-q}.$$

Moving Average operator

- We can rewrite the model

$$X_t = W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \ldots + \theta_q W_{t-q}$$

by

$$X_t = W_t + \theta_1 B W_t + \theta_2 B^2 W_t + \ldots + \theta_q B^q W_t,$$

where B is the backshift operator, and hence,

$$X_t = (1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q) W_t = \theta(B) W_t.$$

- Here $\theta(B)$ is called the moving average operator, where

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q.$$

MA(1)

- Consider the MA(1) model $X_t = W_t + \theta W_{t-1}$.
- $E(X_t) = E(W_t) + \theta E(W_{t-1}) = 0 + \theta \times 0 = 0$.
- $\mathrm{Var}(X_t) = \mathrm{Var}(W_t) + \theta^2 \mathrm{Var}(W_{t-1}) = (1 + \theta^2)\sigma_W^2$.
- $\gamma(1) = \mathrm{Cov}(X_{t+1}, X_t) = \mathrm{Cov}(W_{t+1} + \theta W_t, W_t + \theta W_{t-1}) = \theta\sigma_W^2$, $\gamma(2) = 0$.
- $\rho(1) = \frac{\theta}{1+\theta^2}$, $\rho(2) = 0$.

Sample paths of MA(1)

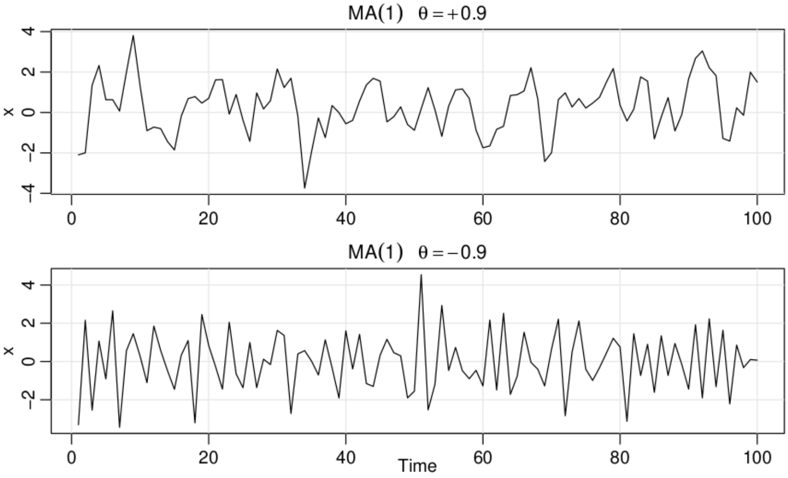

Fig. 3.2. *Simulated MA(1) models:* $\theta = .9$ *(top);* $\theta = -.9$ *(bottom).*

Non-uniqueness of MA models

- Consider the MA(1) model $X_t = W_t + \frac{1}{5} W_{t-1}$ with $W_t \overset{IID}{\sim} N(0, 5^2)$.
- Calculate the means, variances, and autocovariance function.
- Consider another MA(1) model $Y_t = V_t + 5 V_{t-1}$ with $V_t \overset{IID}{\sim} N(0, 1)$.
- Calculate the means, variances, and autocovariance function.
- What is your conclusion?

Invertible process

- Mimicking the criterion of causality for AR models, we will choose the model with an infinite AR representation; it is called an invertible process.
- To discover which model is the invertible model, we can reverse the roles of X_t and W_t and write the MA(1) model as $W_t = X_t - \theta W_{t-1}$.
- If $|\theta| < 1$, then $W_t = \sum_{j=0}^{\infty}(-\theta)^j X_{t-j}$, which is the desired infinite AR representation of the model.
- We will choose the model with $\sigma_W^2 = 25$ and $\theta = 1/5$ because it is invertible.
- If we write $\pi(B)X_t = W_t$, then by matching coefficients, we have

$$\pi(B) = \sum_{j=0}^{\infty}(-\theta)^j B^j.$$

Thank you!