

$$(1) \quad z_t = \begin{cases} \epsilon_t + it\epsilon_t, & t \text{ odd} \\ \epsilon_t - it\epsilon_t, & t \text{ even} \end{cases} \quad E(z_t) = 0 \quad \forall t$$

$\epsilon_t \sim i.i.d (0, \sigma^2)$

$$\text{Cov}(z_{t+h}, z_t)$$

Case 1:  $t$  &  $h$  even

$$\begin{aligned} \text{Cov}(z_{t+h}, z_t) &= E((\epsilon_{t+h} - i(t+h)\epsilon_{t+h})^*(\epsilon_t - it\epsilon_t)) \\ &= E(\epsilon_{t+h} + i(t+h)\epsilon_{t+h})(\epsilon_t - it\epsilon_t) \\ &= 0 \quad \forall h \neq 0 \end{aligned}$$

Sly for other cases  $\text{Cov}(z_{t+h}, z_t) = 0 \quad \forall h \neq 0$

$t$  even

$$\begin{aligned} V(Y_t) &= E((\epsilon_t - it\epsilon_t)^*(\epsilon_t - it\epsilon_t)) \\ &= E(\epsilon_t + it\epsilon_t)(\epsilon_t - it\epsilon_t) \\ &= \sigma^2(1+t^2) \end{aligned}$$

$t$  odd

$$\begin{aligned} V(Y_t) &= E((\epsilon_t + it\epsilon_t)^*(\epsilon_t - it\epsilon_t)) \\ &= \sigma^2(1+t^2) \end{aligned}$$

$\{z_t\}$  is not covariance stationary

$$(2) \quad r_x(h) = E(x_{t+h} - \mu)^* (x_t - \mu) \quad \begin{matrix} x_t \text{ is covariance} \\ \text{stationary} \end{matrix} \quad (2)$$

$$= E(x_{t+h-h} - \mu)^* (x_{t-h} - \mu)$$

$$= E(x_t - \mu)^* (x_{t-h} - \mu)$$

$$= (E(x_{t-h} - \mu)^* (x_t - \mu))^*$$

$$= (r_x(-h))^*$$

$$r_x(h)^* = r_x(-h) + h$$

(3) By Weak law of large numbers

$$\frac{1}{n} \sum_{t=1}^n z_t \xrightarrow{P} E z_1 = \mu$$

i.e.  $\bar{z}_n \xrightarrow{P} \mu$ .

$$(4) \quad \sum_{t=1}^{n-h} (z_t - \mu)(z_{t+h} - \mu)$$

$$= \sum_{t=1}^{n-h} (z_t - \bar{z}_n + \bar{z}_n - \mu)(z_{t+h} - \bar{z}_n + \bar{z}_n - \mu)$$

$$= \sum_{t=1}^{n-h} (z_t - \bar{z}_n)(z_{t+h} - \bar{z}_n) + (n-h)(\bar{z}_n - \mu)^2$$

$$+ (\bar{z}_n - \mu) \sum_{t=1}^{n-h} (z_{t+h} - \bar{z}_n)$$

$$+ (\bar{z}_n - \mu) \sum_{t=1}^{n-h} (z_t - \bar{z}_n)$$

Using the given approximation

$$\sum_{t=1}^{n-h} (z_t - \bar{z}_n) \approx \sum_{t=1}^{n-h} (z_{t+h} - \bar{z}_n) \approx \sum_{t=1}^n (z_t - \bar{z}_n) = 0$$

The last 2 terms are  $\approx 0$

Thus

$$\begin{aligned} & \sum_{t=1}^{n-h} (z_t - \mu)(z_{t+h} - \mu) \\ & \approx \sum_{t=1}^{n-h} (z_t - \bar{z}_n)(z_{t+h} - \bar{z}_n) + (n-h)(\bar{z}_n - \mu)^2 \\ \Rightarrow E \left( \sum_{t=1}^{n-h} (z_t - \mu)(z_{t+h} - \mu) \right) & \approx \sum_{t=1}^{n-h} E(z_t - \bar{z}_n)(z_{t+h} - \bar{z}_n) \\ & \quad + (n-h)V(\bar{z}_n) \end{aligned}$$

i.e. (approx)

$$i.e. (n-h) \hat{\gamma}_2(h) \approx (n-h) E(\hat{\gamma}_2^*(h)) + (n-h)V(\bar{z}_n)$$

$$\left( \hat{\gamma}_2^*(h) = \frac{1}{n-h} \sum_{t=1}^{n-h} (z_t - \bar{z}_n)(z_{t+h} - \bar{z}_n). \right)$$

$$\text{bias } E(\hat{\gamma}_2^*(h)) - \gamma_2(h) \approx -V(\bar{z}_n).$$

(4)

(5)

$$X_t = \phi X_{t-1} + \epsilon_t \quad \epsilon_t \sim WN(0, 1)$$

$$(a) V\left(\frac{1}{4}(x_1 + x_2 + x_3 + x_4)\right)$$

$$= \frac{1}{16} \left( 4\gamma_0 + 2 \sum_{i < j} \text{cov}(x_i, x_j) \right)$$

$$= \frac{1}{16} \left( 4\gamma_0 + 6\gamma_1 + 4\gamma_2 + 2\gamma_3 \right)$$

$$\gamma_X(k) = \frac{\sigma^2}{1-\phi^2} \cdot \phi^{|k|}; \quad \phi = 0.8$$

$$(b) Y_t = \sum_{i=1}^t x_i$$

$$Y_1 = x_1, \quad Y_2 = x_1 + x_2$$

$$V(Y_1) = \gamma_0 \neq V(Y_2) = 2\gamma_0 + 2\gamma_1; \quad \phi = 0.8$$

$$(6) \quad X_t = \begin{cases} z_t, & t \text{ even} \\ (z_{t-1}^2 - 1)/\sqrt{2}, & t \text{ odd} \end{cases} \quad z_t \stackrel{i.i.d.}{\sim} N(0, 1)$$

$$E X_t = 0 \quad \forall t \left( = \begin{cases} E z_t, & t \text{ even} \\ \frac{1}{\sqrt{2}}(E z_{t-1}^2 - 1), & t \text{ odd} \end{cases} \right).$$

$$\underline{t \text{ even}} \quad V(X_t) = E z_t^2 = 1$$

$$\underline{t \text{ odd}} \quad V(X_t) = \frac{1}{2} \left( E z_{t-1}^4 + 1 - 2 E z_{t-1}^2 \right)$$

$$= \frac{1}{2} (3 + 1 - 2) = 1$$

$t$  even &  $h$  even

$$\text{Cov}(x_{t+h}, x_t) = E(z_{t+h} z_t) \geq 0 \quad \forall h \neq 0$$

$t$  even,  $h$  odd

$$\begin{aligned} \text{Cov}(x_{t+h}, x_t) &= \frac{1}{\sqrt{2}} E(z_{t-1+h}^2 - 1) z_t \\ &= 0 \quad \forall h \neq 0 \quad \begin{pmatrix} h=1 & \text{odd order} \\ h \neq 1 & \text{moment} \\ & \text{indep} \end{pmatrix} \end{aligned}$$

By other cases.

$$\Rightarrow \{x_t\} \sim WN(0, 1)$$

(7)  
(a)  $V(x_t) = \sigma^2 + c^2 \sigma^2 \sum_1^\infty 1 \quad \neq \infty$

$\Rightarrow \{x_t\}$  is not covariance stationary

(b)  $x_t = \epsilon_t + c \epsilon_{t-1} + c \epsilon_{t-2} + \dots$

$$x_{t-1} = \epsilon_{t-1} + c \epsilon_{t-2} + c \epsilon_{t-3} + \dots$$

$$\nabla x_t = x_t - x_{t-1}$$

$$z_t = \nabla x_t = \epsilon_t + (c-1) \epsilon_{t-1} \leftarrow MAC(1)$$

$\Rightarrow \{z_t\}$  is covariance stationary

$$(8) \quad Y_t = \nabla X_t = \frac{(\epsilon_t + \frac{1}{2}\epsilon_{t-1}) - (\epsilon_{t-1} + \frac{1}{2}\epsilon_{t-2})}{MA(2)}$$

$$(9) \quad v_t = (1 - z_t) x_t + y_t$$

$$\begin{aligned} x_t &= \epsilon_t + \epsilon_{t-1} & \epsilon_t &\sim WN(0, 1) \\ && y_t &\sim WN(0, 1) \\ && z_t &\sim WN(0, 1) \end{aligned} \quad \left. \begin{array}{l} \text{mutually} \\ \text{indep} \end{array} \right\}$$

$$E v_t = 0$$

$$\begin{aligned} \text{Cov}(v_t, v_{t+h}) &= E((1-z_t)x_t + y_t)((1-z_{t+h})x_{t+h} + y_{t+h}) \\ &= E(1-z_t)(1-z_{t+h}) E(x_t x_{t+h}) \\ &\quad + E(1-z_t) E \cancel{x_t} E(y_{t+h})^0 \\ &\quad + E y_t E \cancel{(1-z_{t+h})} \cancel{x_{t+h}}^0 \\ &\quad + E(y_t y_{t+h}) \\ &= (1 + \gamma_x(h)) \gamma_x(h) + \gamma_y(h) \end{aligned}$$

$$\begin{aligned} \gamma_v(1) &= (1 + \gamma_z(1)) \gamma_x(1) + \gamma_y(1) \\ &= \gamma_x(1) \neq 0 \end{aligned}$$

$$\Rightarrow v_t \not\sim WN$$

(7)

$$(10) \quad X_t \sim WN(0, \sigma_x^2) \quad Y_t \sim WN(0, \sigma_y^2) \quad \xrightarrow{\text{indep}}$$

$$Z_t = X_t + Y_t$$

$$E Z_t = 0 \neq t$$

$$\begin{aligned} \text{Cov}(Z_t, Z_s) &= E Z_t Z_s \\ &= E(X_t + Y_t)(X_s + Y_s) \\ &= E(X_t X_s) + E(Y_t Y_s) \\ &= \begin{cases} \sigma_x^2 + \sigma_y^2, & t=s \\ 0, & \text{if } t \neq s \end{cases} \end{aligned}$$

$$\Rightarrow Z_t \sim WN(0, \sigma_x^2 + \sigma_y^2)$$

$$\text{Let } X_t = \epsilon_t + \epsilon_{t-1}, \quad \epsilon_t \sim WN(0, 1)$$

$$Y_t = \delta_t - \delta_{t-1}, \quad \delta_t \sim WN(0, 1)$$

$$\gamma_X(h) = \begin{cases} 2, & h=0 \\ 1, & h=\pm 1 \\ 0, & \text{if } h \neq 0, \pm 1 \end{cases} \quad \{\epsilon_t\} \text{ & } \{\delta_t\} \text{ are indep.}$$

$$\gamma_Y(h) = \begin{cases} 2, & h=0 \\ -1, & h=\pm 1 \\ 0, & \text{if } h \neq 0, \pm 1 \end{cases}$$

$$Z_t = X_t + Y_t ; \{X_t\} \text{ & } \{Y_t\} \text{ are indep.}$$

$$\gamma_2(h) = \gamma_x(h) + \gamma_y(h)$$

$$= \begin{cases} 4, & h=0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow Z_t \sim WN(0, 4)$$

(11)

$$x_t = \mu + \epsilon_t + \epsilon_{t-1} + \phi \epsilon_{t-2}; \quad \epsilon_t \stackrel{\text{i.i.d}}{\sim} N(0, \sigma^2)$$

$$(a) \quad \delta_1 = \frac{2x_1 + x_3}{3} \quad \delta_2 = \frac{x_3 + x_4 + x_5}{3}$$

$$E x_t = \mu \quad \forall t$$

$$E(\delta_1) = \mu \quad E \delta_2 = \mu$$

both u.e.

$$(b) \quad \text{Cov}(x_t, x_{t+h}) = \begin{cases} (2+\phi^2)\sigma^2, & h=0 \\ (1+\phi)\sigma^2, & h=\pm 1 \\ \phi\sigma^2, & h=\pm 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} V(\delta_1) &= \frac{1}{9} (4V(x_1) + V(x_3) + 4\text{Cov}(x_1, x_3)) \\ &= \frac{1}{9} (4\gamma_0 + \gamma_0 + 4\gamma_2) = \frac{1}{9} (5\gamma_0 + 4\gamma_2) \end{aligned}$$

$$V \delta_2 = \frac{1}{9} (3\gamma_0 + 4\gamma_1 + 2\gamma_2)$$

$$V \delta_1 - V \delta_2 = \frac{1}{9} (2\gamma_0 - 4\gamma_1 + 2\gamma_2)$$

i.e.

$$\begin{aligned} V\delta_1 - V\delta_2 &= \frac{2\sigma^2}{q} \left( (2+\phi^2) - 2(1+\phi) + \phi \right) \\ &= \frac{2\sigma^2}{q} (\phi^2 - \phi) \end{aligned}$$

Let  $g(\phi) = \phi^2 - \phi$

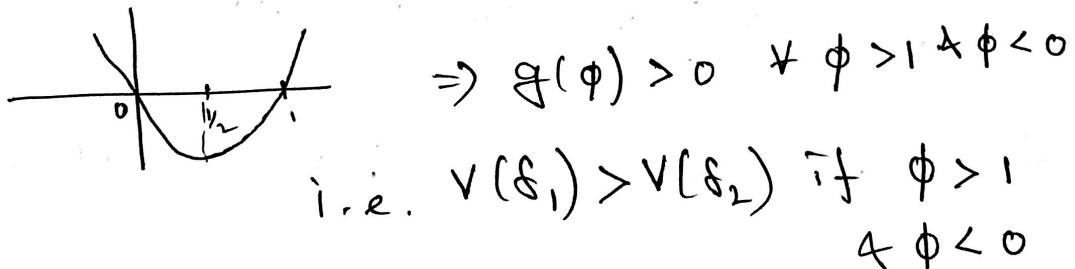
$$g''(\phi) = 2 > 0 \neq \phi$$

$$g'(\phi) = 0 \Rightarrow \phi = \frac{1}{2}$$

$$g\left(\frac{1}{2}\right) = -\frac{1}{4} < 0$$

$$g(\phi) = 0 \Rightarrow \phi = 0, 1$$

$g$  is convex with min at  $\phi = \frac{1}{2}$  &  $g(\phi) = 0$  at 0 & 1



(c)  $\tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$\tilde{x} \in \mathbb{R}^n (\tilde{x} \neq 0)$$

$$\begin{aligned} \tilde{x}' \tilde{x} &= \text{lin comb of indep } N_i \sim N_1 \neq \tilde{x} \in \mathbb{R}^n \\ &\quad (\tilde{x} \neq 0) \\ \Rightarrow \tilde{x} &\sim N_n \end{aligned}$$

$$E(\tilde{x}) = \mu \mathbf{1}_n$$

$$\text{Cov}(\tilde{x}) = \begin{pmatrix} r_0 & r_1 & r_2 & & & \\ r_1 & \ddots & \ddots & \ddots & & 0 \\ r_2 & \ddots & \ddots & \ddots & \ddots & r_2 \\ & \ddots & \ddots & \ddots & \ddots & r_1 \\ & & \ddots & \ddots & \ddots & r_2 \\ 0 & & & & r_2 & r_1 & r_0 \end{pmatrix} = \Sigma$$

$$\tilde{x} \sim N_n(\mu \mathbf{1}_n, \Sigma)$$

$\{x_t\}$  is MA(2) - Cov stationary

$\{x_t\}$  is Gaussian

Hence  $\{x_t\}$  is strict stationary

$$(12) \quad x_t = \phi x_{t-1} + \epsilon_t; \quad |\phi| < 1 \quad \epsilon_t \sim WN(0, \sigma^2)$$

$$y_t = x_t - \frac{1}{\phi} x_{t-1}$$

$$E y_t = 0 \neq t$$

$$\begin{aligned} \text{Cov}(y_{t+h}, y_t) &= E(x_{t+h} - \frac{1}{\phi} x_{t+h-1})(x_t - \frac{1}{\phi} x_{t-1}) \\ &= r_x(h) - \frac{1}{\phi} r_x(h+1) - \frac{1}{\phi} r_x(h-1) + \frac{1}{\phi^2} r_x(h) \end{aligned}$$

$$\text{For } h \geq 1 \quad = \frac{\sigma^2}{1-\phi^2} (\phi^h - \phi^{h-1} - \phi^{h-2} - \phi^{h-3}) = 0$$

$$\text{Sly } h \leq -1 \quad = 0$$

$$\text{i.e. } \gamma_y(h) = 0 \quad \forall h \neq 0$$

$$\gamma_y(0) = \gamma_x(0) \left(1 + \frac{1}{\phi^2}\right) - \frac{2}{\phi} \gamma_x(1)$$

$$Y_T \sim WN(0, \gamma_y(0))$$

$$(13) \quad T_1 = \frac{x_4+x_5}{2} \quad V(T_1) = \frac{1}{4} (2r_0 + 2r_1) = \frac{1}{2} (1 + \phi^2 + \phi)$$

$$V(T_2) = \frac{1}{9} (3r_0 + 4r_1)$$

$$T_2 = \frac{x_3+x_4+x_5}{3} = \frac{1}{9} (3(1 + \phi^2) + 4\phi)$$

$$V(T_1) - V(T_2) = \frac{9(1 + \phi^2 + \phi) - 2(3 + 3\phi^2 + 4\phi)}{18}$$

$$= \frac{1}{18} (3 + 3\phi^2 + \phi)$$

$$g(\phi) = 3\phi^2 + \phi + 3$$

$$g''(\phi) = 6 > 0 \quad \forall \phi$$

$$g'(0) = 0 \Rightarrow \phi = -\frac{1}{6}$$

$$g\left(-\frac{1}{6}\right) = 3 \cdot \frac{1}{6^2} - \frac{1}{6} + 3 > 0$$

$$g(\phi) > 0 \quad \forall \phi$$

$$\Rightarrow V(T_1) > V(T_2) \quad \forall \phi$$

(14)

$$y_t = \sum_{j=0}^{\infty} \left( \frac{1}{\theta} \right)^j x_{t-j}; |\theta| > 1$$

$$\begin{aligned} X_t &= \epsilon_t + \theta \epsilon_{t-1}; \quad \epsilon_t \sim WN(0, \sigma^2) \\ &\rightarrow X_t = \theta(B) \epsilon_t; \quad \theta(B) = 1 + \theta B \\ \left(1 + \frac{1}{\theta} B\right) y_t &= X_t \end{aligned}$$

ACGF of L.H.S.

$$\left(1 + \frac{z}{\theta}\right) \left(1 + \frac{z^{-1}}{\theta}\right) g_y(z)$$

ACGF of R.H.S.

$$\sigma^2 \theta(z) \theta(z')$$

i.e.  $\left(1 + \frac{z}{\theta}\right) \left(1 + \frac{z^{-1}}{\theta}\right) g_y(z) = \sigma^2 (1 + \theta z)(1 + \theta z')$ .

$$g_y(z) = \sigma^2 \frac{(1 + \theta z)(1 + \theta \frac{1}{z})}{\left(1 + \frac{z}{\theta}\right) \left(1 + \frac{1}{\theta} z\right)}$$

i.e.  $g_y(z) = \theta^2 \sigma^2 \leftarrow \text{Const ACF}$

$$\Rightarrow y_t \sim WN(0, \theta^2 \sigma^2)$$

Note: The problem can be solved without using ACF but that would be involving laborious calculations ...

$$(15) \quad \phi(B) \gamma_t = f_t ; \quad f_t \sim WN(0, \sigma^2)$$

$$\phi(B) = 1 - \phi B - \phi^2 B^2$$

$$\text{Roots of } \phi(z) = 0 \quad z_1 = \frac{-1-\sqrt{5}}{2\phi} ; \quad z_2 = \frac{-1+\sqrt{5}}{2\phi}$$

$$\begin{aligned} |z_1| > 1 & \Leftrightarrow |\phi| < \frac{\sqrt{5}+1}{2} \\ |z_2| > 1 & \Leftrightarrow |\phi| < \frac{\sqrt{5}-1}{2} \end{aligned} \quad \left. \begin{array}{l} \Rightarrow \{x_t\} \text{ is stationary for} \\ |\phi| < \frac{\sqrt{5}-1}{2} \end{array} \right\}$$

(16) Similar to (15); such problem can be solved using triangular stationarity region also

Note that for

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + f_t$$

conditions for stationarity

$$1 - \phi_1 - \phi_2 > 0, \quad 1 - \phi_2 + \phi_1 > 0 \quad \& \quad |\phi_2| < 1. \quad (*)$$

For  $\phi_1 = 1$  &  $\phi_2 = c$   $(*)$  is

$$\left. \begin{array}{l} -c > 0 \\ 2 > c \\ |c| < 1 \end{array} \right\} \Rightarrow -1 < c < 0 \text{ is the region of stationarity}$$

(17) Yule-Walker eq<sup>n</sup>

$$Y_h = \phi_1 Y_{h-1} + \phi_2 Y_{h-2} \quad \forall h > 0$$

$$h=1 ; \quad Y_1 = \phi_1 Y_0 + \phi_2 Y_1$$

$$h=2 ; \quad Y_2 = \phi_1 Y_1 + \phi_2 Y_0$$

$$\Rightarrow \frac{Y_1}{Y_0} = \phi_1 + \phi_2 \frac{Y_0}{Y_1}$$

$$\& \frac{Y_2}{Y_0} = \phi_1 + \phi_2 \frac{Y_0}{Y_1}$$

$$\Rightarrow \left. \begin{array}{l} \frac{1}{2} = \phi_1 + \frac{1}{2} \phi_2 \\ \frac{1}{4} = \phi_1 + 2 \phi_2 \end{array} \right\} \text{Solve} \quad \begin{array}{l} \phi_2 = -\frac{1}{6} \\ \phi_1 = \frac{7}{12} \end{array}$$

Also for  $h=3$ , Y-W eq<sup>n</sup> is

$$Y_3 = \phi_1 Y_2 + \phi_2 Y_1$$

$$\begin{aligned} \text{i.e. } \frac{Y_3}{Y_2} &= \phi_1 + \phi_2 \frac{Y_1}{Y_2} \\ &= \frac{7}{12} + \left(-\frac{1}{6}\right) 4 = \dots \end{aligned}$$

(18) Yule-Walker eq<sup>n</sup> for AR(1)

$$Y_h = \phi Y_{h-1}$$

$$\text{l.h.s} = Y_{-h} \quad \& \quad \text{r.h.s} = \phi Y_{-(h-1)} = \phi Y_{-h+1}$$

$$\text{i.e. } Y_{-h} = \phi Y_{-h+1} \quad \rightarrow$$

(19)

$$(1 - \phi_1^{(1)} B) X_t = (1 + \theta_1^{(1)} B + \theta_2^{(1)} B^2 + \theta_3^{(1)} B^3) \epsilon_t$$

$$\& (1 - \phi_1^{(2)} B) Y_t = (1 + \theta_1^{(2)} B + \theta_2^{(2)} B^2) \delta_t$$

$$\epsilon_t \sim WN(0, \sigma^2)$$

$$\delta_t \sim WN(0, \sigma^2) > \text{indep}$$

$\{X_t\}$  &  $\{Y_t\}$  are covariance stationary and independent

(a)

$$Z_t = (1 - \phi_1^{(1)} B) (1 - \phi_1^{(2)} B) X_t$$

$$Z_t = (1 - \phi_1^{(2)} B) \underline{(1 - \phi_1^{(1)} B) X_t}$$

$$\text{i.e. } Z_t = (1 - \phi_1^{(2)} B) (1 + \theta_1^{(1)} B + \theta_2^{(1)} B^2 + \theta_3^{(1)} B^3) \epsilon_t$$

$$\text{i.e. } Z_t = (1 + \alpha_1 B + \alpha_2 B^2 + \alpha_3 B^3 + \alpha_4 B^4) \epsilon_t$$

$$\Rightarrow Z_t \sim MA(4) \text{ covariance stationary}$$

(b)

$$U_t = (1 - \phi_1^{(1)} B) (1 - \phi_1^{(2)} B) (X_t + Y_t)$$

$$U_t = (1 - \phi_1^{(1)} B) (1 - \phi_1^{(2)} B) X_t$$

$$+ (1 - \phi_1^{(1)} B) (1 - \phi_1^{(2)} B) Y_t$$

$$\text{i.e. } U_t = (1 - \phi_1^{(2)} B) \underline{(1 - \phi_1^{(1)} B) X_t}$$

$$+ (1 - \phi_1^{(1)} B) \underline{(1 - \phi_1^{(2)} B) Y_t}$$

$$\text{i.e. } U_t = (1 - \phi_1^{(2)} B) (1 + \theta_1^{(1)} B + \theta_2^{(1)} B^2 + \theta_3^{(1)} B^3) \epsilon_t$$

$$+ (1 - \phi_1^{(1)} B) (1 + \theta_1^{(2)} B + \theta_2^{(2)} B^2) \delta_t$$

$$= A_t + B_t, \text{ say}$$

$$A_t \sim MA(4) \Rightarrow r_A(h) = 0 \quad \forall |h| \geq 5$$

$$\& \quad B_t \sim MA(3) \Rightarrow r_B(h) = 0 \quad \forall |h| \geq 4$$

Further,  $\{A_t\}$  &  $\{B_t\}$  are independent as  $\{\epsilon_t\}$  &  $\{\delta_t\}$  are indep

$$\Rightarrow r_u(h) = r_A(h) + r_B(h)$$

$$\Rightarrow r_u(h) = 0 \quad \forall |h| \geq 5$$

$$(20) \quad X_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}; \quad \epsilon_t \sim WN(0, \sigma^2)$$

$$\{\psi_j\} \Rightarrow \sum_{j=0}^{\infty} |\psi_j| < \infty$$

$$r_h = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$$

$$|r_h| \leq \sigma^2 \sum_{j=0}^{\infty} |\psi_j| |\psi_{j+h}|$$

$$\& \sum_{h=0}^{\infty} |r_h| \leq \sigma^2 \sum_{h=0}^{\infty} \sum_{j=0}^{\infty} |\psi_j| |\psi_{j+h}|$$

Note that

$$\sum_{h=0}^{\infty} \sum_{j=0}^{\infty} |\psi_j| |\psi_{j+h}|$$

$$= \sum_{h=0}^{\infty} (|\psi_0| |\psi_h| + |\psi_1| |\psi_{h+1}| + |\psi_2| |\psi_{h+2}| + \dots)$$

$$= |\psi_0| \sum_{h=0}^{\infty} |\psi_h| + |\psi_1| \sum_{h=0}^{\infty} |\psi_{h+1}| + |\psi_2| \sum_{h=0}^{\infty} |\psi_{h+2}| + \dots$$

$$\leq |\psi_0| \left( \sum_{h=0}^{\infty} |\psi_h| \right) + |\psi_1| \left( \sum_{h=0}^{\infty} |\psi_h| \right) + |\psi_2| \left( \sum_{h=0}^{\infty} |\psi_h| \right) + \dots$$

$$= \left( \sum_{h=0}^{\infty} |\psi_h| \right)^2 < \infty \quad \text{as } \{\psi_j\} \text{ is absolutely summable}$$

$$\Rightarrow \sum_{h=0}^{\infty} |r_h| < \infty \Rightarrow \{r_h\} \text{ is also absolutely summable}$$