①. A collection of random variables $\{W_t\}$ is called a white noise process if

(i) $E(W_t) = 0 \;\forall\, t$

(ii) $V(W_t) = \sigma_W^2 < \infty \;\forall\, t$.

(iii) $Cor(W_t, W_{t'}) = 0 \;\forall\, t \neq t'$.

②. ~~As~~ Suppose $\{W_t\}$ denote "independent" white noises. Note that we make a stronger assumption on $\{W_t\}$ over a simple white noise.

Conditions: $\quad X_t = W_{t-1} W_t W_{t+1}$.

(i) $E(X_t) = E(W_{t-1} W_t W_{t+1})$.

$$= E(W_{t-1}) E(W_t) E(W_{t+1})$$

$$= 0 \times 0 \times 0 = 0 .$$

(ii) $V(X_t) = E(X_t^2) - E(X_t)^2$

$$= E(X_t^2)$$

$$= E(W_{t-1}^2 W_t^2 W_{t+1}^2)$$

$$= E(W_{t-1}^2) E(W_t^2) E(W_{t+1}^2)$$

$$= V(W_{t-1}) V(W_t) V(W_{t+1})$$

$$= \sigma_W^6 < \infty .$$

(iii) $Cov(X_t, X_{t+1})$

$= Cov(W_{t-1}W_tW_{t+1}, W_tW_{t+1}W_{t+2})$

$= E(W_{t-1}W_tW_{t+1}W_tW_{t+1}W_{t+2})$

$\quad - E(W_{t-1}W_tW_{t+1})E(W_tW_{t+1}W_{t+2})$

$= E(W_{t-1}W_t^2W_{t+1}^2W_{t+2}) - 0.$

$= E(W_{t-1})E(W_t^2)E(W_{t+1}^2)E(W_{t+2})$

$= 0 \times \sigma_W^2 \times \sigma_W^2 \times 0$

$= 0.$

$Cov(X_t, X_{t+2})$

$= Cov(W_{t-1}W_tW_{t+1}, W_{t+1}W_{t+2}W_{t+3})$

$= E(W_{t-1}W_tW_{t+1}^2W_{t+2}W_{t+3}) - 0.$

$= E(W_{t-1})E(W_t)E(W_{t+1}^2)E(W_{t+2})E(W_{t+3})$

$= 0 \times 0 \times \sigma_W^2 \times 0 \times 0$

$= 0.$

$\text{Cov}(X_t, X_{t+h}) = 0$ if $h \geq 3$ is clear because X_t and X_{t+h} ~~would~~ would be independent for not sharing any W_t terms.

Hence, $\text{Cov}(X_t, X_{t+h}) = 0 \ \forall h \neq 0$.

overall, X_t satisfies all the properties of a white noise.

Note that X_t and X_{t+1} are not independent because both involve W_t and W_{t+1}.

3 $X_t = \delta + X_{t-1} + W_t$

$$= \delta + (\delta + X_{t-2} + W_{t-1}) + W_t$$

$$= 2\delta + X_{t-2} + W_{t-1} + W_t$$

$$= 2\delta + (\delta + X_{t-3} + W_{t-2}) + W_{t-1} + W_t$$

$$= 3\delta + X_{t-3} + (W_t + W_{t-1} + W_{t-2})$$

$$\vdots$$

$$= t\delta + X_0 + \sum_{i=1}^{t} W_i = t\delta + \sum_{i=1}^{t} W_i$$

Similarly, $X_s = s\delta + \sum_{i=1}^{s} W_i$

Thus, $\mathrm{Cov}(X_s, X_t)$

$$= \mathrm{Cov}\left(\sum_{i=1}^{s} W_i, \sum_{i=1}^{t} W_i\right)$$

$$= \min\{s,t\} \cdot \sigma_W^2, \text{ where } \sigma_W^2 = \mathrm{Var}(W_t)$$

(4) A Collection of random variables $\{X_t\}$ is Called a strictly stationary white noise if the joint CDF satisfies

$$P\{X_{t_1} \le x_1, \dots, X_{t_n} \le x_n\}$$

$$= P\{X_{t_1+h} \le x_1, \dots, X_{t_n+h} \le x_n\}$$

$\forall\ n \geq 1$, $\forall\ (t_1, \dots, t_n)$, $\forall\ (x_1, \dots, x_n)$, $\forall\ h \in \mathbb{Z}$, the set of all integers.

(5) A trend Stationary time Series $\{X_t\}$ Can be Written as $X_t = m_t + Y_t$, where m_t depends on time and Y_t is a stationary process.

Thus, $E(X_t) = m_t + E(Y_t)$.

Now, $E(Y_t)$ is independent of t, but m_t depends on time. Hence, $E(X_t)$ varies with time and thus, X_t cannot be a weakly stationary process.

(6) $X_t = \beta_1 + \beta_2 t + Y_t$, where

$$Y_t = \sum_{k=1}^{K} \{ U_{1,k} \sin(2\pi\omega_k t) + U_{2,k} \cos(2\pi\omega_k t) \}.$$

(i)

$$E(Y_t) = \sum_{k=1}^{K} \{ E(U_{1,k}) \sin(2\pi\omega_k t) + E(U_{2,k}) \cos(2\pi\omega_k t) \}$$

$$= 0.$$

$\Rightarrow E(X_t) = \beta_1 + \beta_2 t$.

Thus, after removing the trend, we have.

$X_t - E(X_t) = X_t - (\beta_1 + \beta_2 t) = Y_t$.

If we can show that Y_t is ~~sto~~ weakly stationary, we are done.

(i) $E(Y_t) = 0$, constant for all t.

(ii) $$V(Y_t) = \sum_{k=1}^{K} \{ Var(U_{1,k}) . \sin^2(2\pi\omega_k t) + Var(U_{2,k}) \cos^2(2\pi\omega_k t) \}$$

$$= \sum_{k=1}^{K} \{ \sigma_k^2 (\sin^2(2\pi\omega_k t) + \cos^2(2\pi\omega_k t)) \}$$

$$= \sum_{k=1}^{K} \sigma_k^2 < \infty. \quad \left[\because \sigma_k^2 \text{ are finite } \forall k \right]$$

(iii)

$$Cov(Y_t, Y_{t+h})$$

$$= Cov\left(\sum_{k=1}^{K} \{ U_{1,k} \sin(2\pi\omega_k t) + U_{2,k} \cos(2\pi\omega_k t) \}, \sum_{k=1}^{K} \{ U_{1,k} \sin(2\pi\omega_k (t+h)) + U_{2,k} \cos(2\pi\omega_k (t+h)) \} \right)$$

$$= \sum_{k=1}^{K} \{ Var(U_{1,k}) \sin(2\pi\omega_k t) \sin(2\pi\omega_k (t+h)) + Var(U_{2,k}) \cos(2\pi\omega_k t) . \cos(2\pi\omega_k (t+h)) \}$$

$$= \sum_{k=1}^{K} \sigma_k^2 \cdot \left\{ \cos(2\pi\omega_k(t+h)) \cdot \cos(2\pi\omega_k t) + \sin(2\pi\omega_k(t+h)) \cdot \sin(2\pi\omega_k t) \right\}$$

$$= \sum_{k=1}^{K} \sigma_k^2 \cdot \cos\left(2\pi\omega_k(\cancel{t}+h-\cancel{t})\right)$$

$$= \sum_{k=1}^{K} \sigma_k^2 \cdot \cos(2\pi\omega_k h).$$

depends only on the lag h.

$\Rightarrow \{Y_t\}$ is weakly stationary.

$\Rightarrow X_t$ is trend stationary.

7. Here,

$$X_t = \beta_1 + \beta_2 t + Z_t$$

where Z_t is stationary with mean 0 and suppose covariance $\gamma(h)$ for lag h.

$$Y_t = X_t - X_{t-1}$$

$$= \beta_1 + \beta_2 t + Z_t - (\beta_1 + \beta_2(t-1) + Z_{t-1})$$

$$= \beta_2 + (Z_t - Z_{t-1})$$

$$E(Y_t) = \beta_2 + E(Z_t) - E(Z_{t-1})$$

$$= \beta_2 \quad \text{as } Z_t \text{ is stationary.}$$

$\Rightarrow$ $E(Y_t)$ is constant over t.

$$\mathrm{Cov}(Y_t, Y_{t+h})$$

$$= \mathrm{Cov}(\beta_2 + Z_t - Z_{t-1}, \beta_2 + Z_{t+h} - Z_{t+h-1})$$

$$= \mathrm{Cov}(Z_t, Z_{t+h}) - \mathrm{Cov}(Z_{t-1}, Z_{t+h})$$

$$- \mathrm{Cov}(Z_t, Z_{t+h-1}) + \mathrm{Cov}(Z_{t-1}, Z_{t+h-1})$$

$$= \gamma(h) - \gamma(h+1) - \gamma(h-1) + \gamma(h)$$

$\Rightarrow$ depends only on h. $\boxed{< \infty \text{ as well}}$

$\Rightarrow$ Y_t is weakly stationary.

8. $$\rho_{X,Y}(h) = \frac{\gamma_{X,Y}(h)}{\sqrt{\gamma_X(0) \cdot \gamma_Y(0)}}$$

$$= \operatorname{cov}(X_{t+h}, Y_t) / \sqrt{\gamma_X(0)\gamma_Y(0)}$$

$$= E\left[(X_{t+h} - \mu_X)(Y_t - \mu_Y)\right] / \sqrt{\gamma_X(0)\gamma_Y(0)}$$

$$= E\left[(Y_t - \mu_Y)(X_{t+h} - \mu_X)\right] / \sqrt{\gamma_X(0)\gamma_Y(0)}$$

$$= \gamma_{Y,X}(t - (t+h)) / \sqrt{\gamma_X(0)\gamma_Y(0)}$$

$$= \gamma_{Y,X}(-h) / \sqrt{\gamma_X(0)\gamma_Y(0)}$$

$$= \rho_{Y,X}(-h).$$

$$\Rightarrow \rho_{X,Y}(h) - \rho_{Y,X}(-h) = 0.$$

$$\Rightarrow |\rho_{X,Y}(h) - \rho_{Y,X}(-h)| = 0.$$

9 $\hat{\rho}_X(h) = \dfrac{\hat{\gamma}_X(h)}{\hat{\gamma}_X(0)}$, where

$$\hat{\gamma}_X(h) = \frac{1}{T} \sum_{t=1}^{T-h} (X_{t+h} - \bar{X})(X_t - \bar{X})$$

$$\hat{\gamma}_X(0) = \frac{1}{T} \sum_{t=1}^{T} (X_t - \bar{X})^2, \text{ where } \bar{X} = \frac{1}{T}\sum_{t=1}^{T} X_t$$

$$\hat{\rho}_{x,y}(h) = \frac{\hat{\gamma}_{x,y}(h)}{\sqrt{\hat{\gamma}_x(0) \cdot \hat{\gamma}_y(0)}}, \text{ where}$$

$$\hat{\gamma}_x(0) = \frac{1}{T}\sum_{t=1}^{T}(X_t - \bar{X})^2, \quad \bar{X} = \frac{1}{T}\sum_{t=1}^{T} X_t$$

$$\hat{\gamma}_y(0) = \frac{1}{T}\sum_{t=1}^{T}(Y_t - \bar{Y})^2, \quad \bar{Y} = \frac{1}{T}\sum_{t=1}^{T} Y_t$$

$$\hat{\gamma}_{x,y}(h) = \frac{1}{T}\sum_{t=1}^{T-h}(X_{t+h} - \bar{X})(Y_t - \bar{Y})$$

(10) Their values are $\pm 2/\sqrt{T}$, where T is the length of the observed time series.

For a white noise process, approx. 95% of the sample ACFs should fall within these limits. If the sample ACF $\hat{\rho}(h)$ falls within the limits, we say ~~$\rho(h)$ is not~~ fail to reject $\rho(h) = 0$. If $\hat{\rho}(h)$ falls outside, we reject $\rho(h) = 0$.

(11) $M_t = \sum_{j=-k}^{+k} a_j X_{t-j}$ is symmetric moving average. Hence,

(1) $a_j = a_{-j} \quad \forall\, j = 1, \ldots, k$.

(2) $\sum_{j=-k}^{+k} a_j = 1$.

(12) $m_t^{\text{optimal}} = \underset{m_t}{\operatorname{argmin}} \left\{ \sum_{t=1}^{T} (X_t - m_t)^2 + \lambda \int (m_t'')^2 dt \right\}$

When $\lambda = \infty$, the solution needs to satisfy $m_t'' = 0 \;\forall\, t$.

$\Rightarrow m_t = ct + d$ form.

$\Rightarrow$ The solution look linear in t.

(13) $X_t = \phi X_{t-1} + W_t$

$= \phi(\phi X_{t-2} + W_{t-1}) + W_t$

$= \phi^2 X_{t-2} + \phi W_{t-1} + W_t$

$= \phi^2(\phi X_{t-3} + W_{t-2}) + \phi W_{t-1} + W_t$

$= \phi^3 X_{t-3} + \phi^2 W_{t-2} + \phi W_{t-1} + W_t$

... after k steps,

$$X_t = \phi^k X_{t-k} + \left(\phi^{k-1} W_{t-k+1} + \dots + \phi^2 W_{t-2} + \phi W_{t-1} + W_t\right)$$

Continuing to iterate back, i.e. as $k \to \infty$,

$$X_t = \lim_{k\to\infty} \phi^k \cdot X_{t-k} + \sum_{j=0}^{\infty} \phi^j W_{t-j}$$

As it is given that the process is causal, $|\phi| < 1$.

$$E\left[\left(X_t - \sum_{j=0}^{\infty} \phi^j W_{t-j}\right)^2\right]$$

$$= E\left[\lim_{k\to\infty} \phi^{2k} \cdot X_{t-k}^2\right]$$

$$= \lim_{k\to\infty} \phi^{2k} \cdot E\left(X_{t-k}^2\right)$$

$= 0$. as $\phi^{2k} \to 0$ and $E(X_{t-k}^2)$ is finite due to $\{X_t\}$ being weakly

Stationary.

$\Rightarrow$ In the mean square sense,

$$X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}$$

(14) $X_t = W_t + \theta W_{t-1}$.

$\Rightarrow W_t = X_t - \theta W_{t-1}$.

Now, $X_{t-1} = W_{t-1} + \theta W_{t-2}$

$\Rightarrow W_{t-1} = X_{t-1} - \theta W_{t-2}$

plugging in,

$$W_t = X_t - \theta(X_{t-1} - \theta W_{t-2})$$
$$= X_t - \theta X_{t-1} + (-\theta)^2 W_{t-2}$$

Similarly, after k steps.

$$W_t = X_t - \theta X_{t-1} + \theta^2 X_{t-2} - \theta^3 X_{t-3} + \cdots$$
$$\cdots + (-\theta)^k X_{t-k} + (-\theta)^{k+1} W_{t-k-1}$$

$$\Rightarrow W_t = \sum_{j=0}^{k} (-\theta)^j X_{t-j} + (-\theta)^{k+1} W_{t-k-1}$$

Continuing,

$$W_t = \sum_{j=0}^{\infty} (-\theta)^j X_{t-j} + \lim_{k \to \infty} (-\theta)^{k+1} W_{t-k-1}$$

$$E\left[\left(W_t - \sum_{j=0}^{\infty} (-\theta)^j X_{t-j}\right)^2\right]$$

$$= E\left[\lim_{k \to \infty} (-\theta)^{2k+2} \cdot W^2_{t-k-1}\right]$$

$$= \lim_{k \to \infty} \theta^{2k+2} \times E(W^2_{t-k-1})$$

$$= 0 \quad \text{as} \quad E(W^2_{t-k-1}) = \sigma_W^2 < \infty.$$

and $|\theta| < 1$ as $\{X_t\}$ is an invertible process.

$$\Rightarrow W_t = \sum_{j=0}^{\infty} (-\theta)^j X_{t-j} \text{ in the mean square sense.}$$

(15) $X_t = \phi X_{t-1} + W_t + \theta W_{t-1}$

$\Rightarrow (1-\phi B) X_t = (1+\theta B) W_t$

$\Rightarrow$ AR polynomial $\phi(z) = 1-\phi z$.

MA polynomial $\theta(z) = 1+\theta z$

$\Rightarrow X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j}$

$\Rightarrow$ ψ-weigths are given by the coefficients of $\dfrac{1+\theta z}{1-\phi z}$.

$$\frac{1+\theta z}{1-\phi z} = (1+\theta z)(1+\phi z+\phi^2 z^2 + \cdots)$$

$$= 1+\theta z + \phi z(1+\theta z) + \phi^2 z^2 (1+\theta z) + \cdots$$

$$= 1+(\theta+\phi)z + \theta\phi z^2 + \phi^2 z^2 + \cdots$$

$$= 1+(\theta+\phi)z + (\theta\phi+\phi^2)z^2 + \cdots$$

$$= \psi_0 + \psi_1 z + \psi_2 z^2 + \cdots$$

By matching, $\psi_1 = \theta+\phi$, $\psi_2 = \phi(\theta+\phi)$

(16) $(1-\phi B)X_t = (1+\theta B)W_t$

$\Rightarrow$ π-weigths are given by the coefficients of the ratio of the ratio of the AR and MA polynomials.

$$\frac{1-\phi z}{1+\theta z} = (1-\phi z)(1-\theta z+\theta^2 z^2-\theta^3 z^3+\dots)$$

$$\cancel{= 1-\theta z - \phi z(1-\theta z}$$

$$= 1-\theta z+\theta^2 z^2 - \phi z(1-\theta z+\theta^2 z^2) + \dots$$

$$= 1-\theta z+\theta^2 z^2 - \phi z + \theta\phi z^2 - \phi\theta^2 z^3 + \dots$$

$$= 1 + z(-\theta-\phi) + z^2(\theta^2+\theta\phi) + \dots$$

$$= \pi_0 + \pi_1 z + \pi_2 z^2 + \dots$$

$\pi_1 = -(\theta+\phi)$

$\pi_2 = \theta(\theta+\phi)$.

by matching coefficients.

(17) $\phi_{2,2} = \text{Corr}\left(X_{t+2} - \hat{X}_{t+2}, X_t - \hat{X}_t\right)$

where $\hat{X}_{t+2} = c X_{t+1}$ and $\hat{X}_t = d X_{t+1}$

where c and d should ~~satisfy~~ minimize

$E\left[\left(X_{t+2} - \hat{X}_{t+2}\right)^2\right]$ and $E\left[\left(X_t - \hat{X}_t\right)^2\right]$.

$$E\left[\left(X_{t+2} - c X_{t+1}\right)^2\right]$$

$$= E\left(X_{t+2}^2\right) - 2c\, E\left(X_{t+2} X_{t+1}\right) + c^2 E\left(X_{t+1}^2\right)$$

$$= \gamma(0) - 2c.\gamma(1) + c^2 \gamma(0).$$

$$f(c) = \gamma(0) - 2c\gamma(1) + c^2\gamma(0)$$

$$f'(c) = -2\gamma(1) + 2c\gamma(0).$$

$$f''(c) = 2\gamma(0) > 0.$$

$$f'(c^*) = 0 \Rightarrow c^* = \frac{\gamma(1)}{\gamma(0)} = \rho(1) = \phi.$$

If $X_t = \phi X_{t+1} + W_t \Rightarrow \rho(1) = \phi$.

Similarly, we can show $d^* = \phi$.

~~and~~ Here c^* and d^* optimize the MSE.

$$\Rightarrow \phi_{2,2} = \mathrm{Cor}(X_{t+2} - \phi X_{t+1}, X_t - \phi X_{t+1})$$

$$= \mathrm{Cor}(W_{t+2}, X_t - \phi X_{t+1})$$

$$= 0$$

because X_t is causal $\Rightarrow$
$X_t - \phi X_{t+1}$ depends only on
$\{W_{t+1}, W_t, W_{t-1}, \ldots\}$ which are
uncorrelated with W_{t+2}.