MTH442 Assignment 5

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Q1 (a)

Given a system process defined by:

$$X_t = \phi X_{t-2} + W_t, \quad t = 1, 2, \ldots, T$$

where:

- $X_0 \sim N(0, \sigma_0^2)$ and $X_{-1} \sim N(0, \sigma_1^2)$ are initial conditions with known variances σ_0^2 and σ_1^2.
- W_t represents Gaussian white noise with variance σ_W^2, i.e., $W_t \sim N(0, \sigma_W^2)$.

The process X_t is observed with measurement noise, where:

$$Y_t = X_t + V_t$$

Here:

- V_t is Gaussian white noise with variance σ_V^2, i.e., $V_t \sim N(0, \sigma_V^2)$.
- $X_0, X_{-1}, \{W_t\}$, and $\{V_t\}$ are all mutually independent.

Our objective is to express the system in the standard state-space form, comprising state and observation equations.

State-Space Model Formulation

To represent this process as a state-space model, we must define a state vector that includes all dependencies in the recursive equation. Notice that the current state X_t depends on X_{t-2}, so we need to keep track of both X_t and X_{t-1} as part of our state vector.

Define the state vector at time t as:

$$\mathbf{X}_t = \begin{bmatrix} X_t \\ X_{t-1} \end{bmatrix}$$

where:

- The first element X_t represents the current state of the process.
- The second element X_{t-1} stores the previous state, which is needed to reference X_{t-2} in future time steps.

Given this state vector, we derive the state equation by expressing $\mathbf{X}_t$ in terms of $\mathbf{X}_{t-1}$ and the process noise W_t.

Deriving the State Equation

From the given process, $X_t = \phi X_{t-2} + W_t$. To express $\mathbf{X}_t$ in terms of $\mathbf{X}_{t-1}$, observe that:

$$X_t = \phi X_{t-2} + W_t$$

and

$$X_{t-1} = X_{t-1}$$

Using these relations, we can rewrite the evolution of the state vector as:

$$\mathbf{X}_t = \begin{bmatrix} X_t \\ X_{t-1} \end{bmatrix} = \begin{bmatrix} \phi X_{t-2} + W_t \\ X_{t-1} \end{bmatrix} = \begin{bmatrix} \phi & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_{t-1} \\ X_{t-2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} W_t$$

In general form, this can be expressed as:

$$\mathbf{X}_t = \mathbf{A}\mathbf{X}_{t-1} + \mathbf{B}W_t$$

where:

$$\mathbf{A} = \begin{bmatrix} \phi & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Here:

- The matrix $\mathbf{A}$ defines the transition dynamics of the state vector.
- The matrix $\mathbf{B}$ specifies how the process noise W_t influences the current state.

Observation Equation

The observed process is given by:

$$Y_t = X_t + V_t$$

Since X_t is the first element of the state vector $\mathbf{X}_t$, we can express Y_t in terms of $\mathbf{X}_t$ as:

$$Y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{X}_t + V_t$$

or, in general form:

$$Y_t = \mathbf{H}\mathbf{X}_t + V_t$$

where:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

In this representation:

- The matrix $\mathbf{H}$ extracts the relevant part of the state vector (i.e., X_t) for the observation equation.
- V_t represents the observation noise with variance σ_V^2.

Complete State-Space Model

Thus, the state-space model for the given process is:

- **State Equation**:

$$\mathbf{X}_t = \begin{bmatrix} \phi & 0 \\ 1 & 0 \end{bmatrix} \mathbf{X}_{t-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} W_t$$

  where $W_t \sim N(0, \sigma_W^2)$.

- **Observation Equation**:

$$Y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{X}_t + V_t$$

  where $V_t \sim N(0, \sigma_V^2)$.

Key Points and Assumptions

- $X_0 \sim N(0, \sigma_0^2)$ and $X_{-1} \sim N(0, \sigma_1^2)$ serve as initial conditions for the process.
- The process noise $W_t \sim N(0, \sigma_W^2)$ introduces randomness into the state transitions.
- The observation noise $V_t \sim N(0, \sigma_V^2)$ adds uncertainty to the measurements Y_t.
- X_0, X_{-1}, W_t, and V_t are assumed to be independent.

This formulation enables us to apply state estimation techniques, such as the Kalman filter, to estimate X_t from observations Y_t, accounting for both process and observation noise.

Q1(b) Solution

To make the observations Y_t stationary, we need the system state X_t to be stationary. We are given the following:
1. $X_t = \phi X_{t-2} + W_t$, where $W_t \sim N(0, \sigma_W^2)$. 2. $Y_t = X_t + V_t$, where $V_t \sim N(0, \sigma_V^2)$.

Step 1: Define the Stationary Variance of X_t

For X_t to be stationary, its variance $\text{Var}(X_t)$ should remain constant over time. Let us denote this stationary variance as γ_0.
The state equation can be written as:

$$X_t = \phi X_{t-2} + W_t$$

where W_t is Gaussian white noise with variance σ_W^2.
Since W_t is independent of X_{t-2}, we calculate $\text{Var}(X_t)$ as follows:

$$\text{Var}(X_t) = \text{Var}(\phi X_{t-2} + W_t)$$

Expanding this expression, we get:

$$\text{Var}(X_t) = \phi^2 \,\text{Var}(X_{t-2}) + \text{Var}(W_t)$$

For stationarity, we set $\text{Var}(X_t) = \text{Var}(X_{t-2}) = \gamma_0$, leading to:

$$\gamma_0 = \phi^2 \gamma_0 + \sigma_W^2$$

Solving for γ_0, we find:

$$\gamma_0(1 - \phi^2) = \sigma_W^2$$

$$\gamma_0 = \frac{\sigma_W^2}{1 - \phi^2}$$

Thus, the stationary variance of X_t is:

$$\gamma_0 = \frac{\sigma_W^2}{1 - \phi^2}$$

This result holds if $|\phi| < 1$, which ensures the process is stationary.

Step 2: Set Initial Variances σ_0^2 and σ_1^2

To ensure stationarity, the initial variances σ_0^2 and σ_1^2 (for X_0 and X_{-1}) should match the stationary variance γ_0.

Since $X_0 \sim N(0, \sigma_0^2)$ and $X_{-1} \sim N(0, \sigma_1^2)$, set:

$$\sigma_0^2 = \gamma_0 = \frac{\sigma_W^2}{1 - \phi^2}$$

and similarly,

$$\sigma_1^2 = \gamma_0 = \frac{\sigma_W^2}{1 - \phi^2}$$

These values for σ_0^2 and σ_1^2 ensure that the initial states X_0 and X_{-1} are consistent with the stationary variance of X_t, making Y_t stationary as well.

Q 2

We are given a state-space model defined by:

$$Y_t = X_t + V_t,$$
$$X_t = \phi X_{t-1} + W_t,$$

where:

- $V_t \sim$ IID $N(0, \sigma_V^2)$ (observation noise),
- $W_t \sim$ IID $N(0, \sigma_W^2)$ (process noise),
- $X_0 \sim N\left(0, \frac{\sigma_W^2}{1-\phi^2}\right)$ (initial state).

The variables X_0, $\{W_t\}$, and $\{V_t\}$ are independent of each other.

Definitions and Setup

Let:

$$X_t^{t-1} = \mathbb{E}(X_t | Y_{t-1}, \ldots, Y_1),$$
$$P_t^{t-1} = \mathbb{E}\left[(X_t - X_t^{t-1})^2\right].$$

The term X_t^{t-1} represents the best linear estimate of X_t given past observations, and P_t^{t-1} is the estimation error variance for X_t given $Y_{t-1}, \ldots, Y_1$.

The innovation sequence (or residuals) is defined as:

$$\varepsilon_t = Y_t - Y_t^{t-1},$$

where:

$$Y_t^{t-1} = \mathbb{E}(Y_t | Y_{t-1}, \ldots, Y_1).$$

Since $Y_t = X_t + V_t$, we can express ε_t as:

$$\varepsilon_t = (X_t + V_t) - Y_t^{t-1}.$$

Step 1: Expression for Y_t^{t-1}

We know that:
$$Y_t^{t-1} = \mathbb{E}(Y_t|Y_{t-1}, \dots, Y_1) = \mathbb{E}(X_t|Y_{t-1}, \dots, Y_1) + \mathbb{E}(V_t|Y_{t-1}, \dots, Y_1).$$
Since V_t is independent of the past observations and has zero mean, $\mathbb{E}(V_t|Y_{t-1}, \dots, Y_1) = 0$. Therefore,
$$Y_t^{t-1} = X_t^{t-1}.$$

Thus, the innovation sequence becomes:
$$\varepsilon_t = Y_t - Y_t^{t-1} = (X_t + V_t) - X_t^{t-1} = (X_t - X_t^{t-1}) + V_t.$$

Step 2: Covariance $\text{Cov}(\varepsilon_s, \varepsilon_t)$
We aim to calculate $\text{Cov}(\varepsilon_s, \varepsilon_t)$ for two cases: $s = t$ and $s \neq t$.
Case 1: $s \neq t$
For $s \neq t$, we have:
$$\text{Cov}(\varepsilon_s, \varepsilon_t) = \text{Cov}((X_s - X_s^{s-1} + V_s), (X_t - X_t^{t-1} + V_t)).$$

Since V_s and V_t are independent for $s \neq t$, we have $\text{Cov}(V_s, V_t) = 0$. Similarly, $X_s - X_s^{s-1}$ and $X_t - X_t^{t-1}$ are also uncorrelated for $s \neq t$ because they are based on independent noise terms W_s and W_t. Hence,
$$\text{Cov}(\varepsilon_s, \varepsilon_t) = 0.$$

Case 2: $s = t$
For $s = t$, we are interested in calculating the variance of ε_t, which is given by:
$$\text{Var}(\varepsilon_t) = \text{Var}(X_t - X_t^{t-1} + V_t).$$

Using the fact that $X_t - X_t^{t-1}$ and V_t are independent, we can separate the variances:
$$\text{Var}(\varepsilon_t) = \text{Var}(X_t - X_t^{t-1}) + \text{Var}(V_t).$$

We know that:
$$\text{Var}(X_t - X_t^{t-1}) = P_t^{t-1} \quad \text{and} \quad \text{Var}(V_t) = \sigma_V^2.$$

Thus,
$$\text{Var}(\varepsilon_t) = P_t^{t-1} + \sigma_V^2.$$

Final Answer
We can now summarize the covariance of the innovation sequence $\{\varepsilon_t\}$ as follows:
$$\text{Cov}(\varepsilon_s, \varepsilon_t) = \begin{cases} 0 & \text{if } s \neq t, \\ P_t^{t-1} + \sigma_V^2 & \text{if } s = t. \end{cases}$$

This completes the derivation of the covariance of the innovation sequence $\{\varepsilon_t\}$ in terms of X_t^{t-1} and P_t^{t-1}.

Q3(a)

We are given a univariate state-space model with the following state and observation equations:
$$\begin{aligned} X_0 &= W_0, \\ X_t &= X_{t-1} + W_t, \\ Y_t &= X_t + V_t, \quad t = 1, 2, \dots \end{aligned}$$

where:

- $W_t \sim \text{IID } N(0, \sigma_W^2)$ is the process noise,
- $V_t \sim \text{IID } N(0, \sigma_V^2)$ is the observation noise,

and W_t and V_t are independent.

Our objective is to show that Y_t follows an Integrated Moving Average (IMA) model of order $(1, 1)$, that is, we want to show that $\nabla Y_t = Y_t - Y_{t-1}$ follows an MA(1) process.
Step 1: Define the Differenced Observation Process ∇Y_t
Define the differenced observation process:
$$\nabla Y_t = Y_t - Y_{t-1}.$$

Substitute $Y_t = X_t + V_t$ and $Y_{t-1} = X_{t-1} + V_{t-1}$ to get:

$$\nabla Y_t = (X_t + V_t) - (X_{t-1} + V_{t-1}).$$

This simplifies to:

$$\nabla Y_t = (X_t - X_{t-1}) + (V_t - V_{t-1}).$$

Step 2: Substitute the State Equation
From the state equation, we have:

$$X_t = X_{t-1} + W_t.$$

Thus,

$$X_t - X_{t-1} = W_t.$$

Substitute this result into the expression for ∇Y_t:

$$\nabla Y_t = W_t + (V_t - V_{t-1}).$$

Step 3: Analyze ∇Y_t
Now, ∇Y_t is expressed as:

$$\nabla Y_t = W_t + (V_t - V_{t-1}).$$

Since $W_t \sim N(0, \sigma_W^2)$ and $V_t \sim N(0, \sigma_V^2)$ are both white noise processes, the difference $V_t - V_{t-1}$ will be an MA(1) process with moving average parameter -1.

Therefore, ∇Y_t can be written as an MA(1) process:

$$\nabla Y_t = W_t + V_t - V_{t-1}.$$

Step 4: Covariance Structure of ∇Y_t
To confirm that ∇Y_t follows an MA(1) model, we need to examine its covariance structure.
1. Mean of ∇Y_t: Since both W_t and $V_t - V_{t-1}$ have zero mean, it follows that:

$$\mathbb{E}[\nabla Y_t] = 0.$$

2. Variance of ∇Y_t: The variance of ∇Y_t is:

$$\mathrm{Var}(\nabla Y_t) = \mathrm{Var}(W_t) + \mathrm{Var}(V_t - V_{t-1}).$$

Since $W_t \sim N(0, \sigma_W^2)$ and $V_t - V_{t-1} \sim N(0, 2\sigma_V^2)$, we have:

$$\mathrm{Var}(\nabla Y_t) = \sigma_W^2 + 2\sigma_V^2.$$

3. Autocovariance of ∇Y_t: For $k \geq 2$, the autocovariance $\gamma(k) = \mathrm{Cov}(\nabla Y_t, \nabla Y_{t-k}) = 0$, because W_t and V_t are white noise and independent.

For $k = 1$:

$$\gamma(1) = \mathrm{Cov}(\nabla Y_t, \nabla Y_{t-1}).$$

Since $\nabla Y_t = W_t + V_t - V_{t-1}$ and $\nabla Y_{t-1} = W_{t-1} + V_{t-1} - V_{t-2}$, the only non-zero covariance term is $-\sigma_V^2$ from V_{t-1}. Thus,

$$\gamma(1) = -\sigma_V^2.$$

Hence, the autocovariance structure of ∇Y_t matches that of an MA(1) model with parameter $\theta = -1$.
Conclusion
We have shown that $\nabla Y_t = Y_t - Y_{t-1}$ follows an MA(1) model with:

$$\nabla Y_t = W_t + V_t - V_{t-1},$$

where $W_t \sim N(0, \sigma_W^2)$ and $V_t - V_{t-1} \sim N(0, 2\sigma_V^2)$. Therefore, Y_t follows an IMA(1,1) model.

Q 4

We have the state-space model defined by:

$$X_t = \Phi X_{t-1} + W_t$$

$$Y_t = A_t X_t + V_t$$

where: - $W_t \sim \mathcal{N}(0, \sigma_W^2)$ and $V_t \sim \mathcal{N}(0, \sigma_V^2)$ are independent Gaussian white noise processes, - A_t is a known design matrix independent of the parameters Θ, - $X_0 \sim \mathcal{N}(0, \sigma_{X_0}^2)$ is the initial state.

The gradient of the log-likelihood with respect to Θ_i is given by:

$$\frac{\partial \log L_Y(\Theta)}{\partial \Theta_i} = \sum_{t=1}^{T} \left\{ \epsilon_t' \Sigma_t^{-1} \frac{\partial \epsilon_t}{\partial \Theta_i} - 0.5 \epsilon_t' \Sigma_t^{-1} \frac{\partial \Sigma_t}{\partial \Theta_i} \Sigma_t^{-1} \epsilon_t + 0.5 \operatorname{tr} \left(\Sigma_t^{-1} \frac{\partial \Sigma_t}{\partial \Theta_i} \right) \right\}$$

where: - $\epsilon_t = Y_t - Y_t^{t-1}$ is the prediction error, - $Y_t^{t-1} = E(Y_t | Y_{t-1}, \ldots, Y_1)$ is the one-step-ahead prediction of Y_t, - Σ_t is the covariance of the prediction error ϵ_t.

(a)

Show that:

$$\frac{\partial \epsilon_t}{\partial \Theta_i} = -A_t \frac{\partial X_t^{t-1}}{\partial \Theta_i}$$

Solution:
1. Define ϵ_t:

$$\epsilon_t = Y_t - Y_t^{t-1}$$

where $Y_t = A_t X_t + V_t$ and $Y_t^{t-1} = A_t X_t^{t-1}$ because V_t has mean zero.
2. Substitute for Y_t and Y_t^{t-1}:

$$\epsilon_t = (A_t X_t + V_t) - A_t X_t^{t-1}$$

3. Simplify ϵ_t:

$$\epsilon_t = A_t (X_t - X_t^{t-1}) + V_t$$

4. Take the derivative with respect to Θ_i: Since V_t does not depend on Θ_i, we have:

$$\frac{\partial \epsilon_t}{\partial \Theta_i} = A_t \frac{\partial (X_t - X_t^{t-1})}{\partial \Theta_i}$$

5. Express $X_t - X_t^{t-1}$: Noting that $X_t - X_t^{t-1}$ is just the prediction error in the state, we get:

$$\frac{\partial \epsilon_t}{\partial \Theta_i} = -A_t \frac{\partial X_t^{t-1}}{\partial \Theta_i}$$

Thus, we have shown that:

$$\frac{\partial \epsilon_t}{\partial \Theta_i} = -A_t \frac{\partial X_t^{t-1}}{\partial \Theta_i}$$

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(b)

Show that:

$$\frac{\partial X_t^{t-1}}{\partial \Theta_i} = \frac{\partial \Phi}{\partial \Theta_i} X_{t-2}^{t-1} + \Phi \frac{\partial X_{t-2}^{t-1}}{\partial \Theta_i} + \frac{\partial K_{t-1}}{\partial \Theta_i} \epsilon_{t-1} + K_{t-1} \frac{\partial \epsilon_{t-1}}{\partial \Theta_i}$$

Solution:
1. Express X_t^{t-1} using the state transition:

$$X_t^{t-1} = \Phi X_{t-1}^{t-1}$$

2. Differentiate X_t^{t-1} with respect to Θ_i: Taking the derivative, we get:

$$\frac{\partial X_t^{t-1}}{\partial \Theta_i} = \frac{\partial \Phi}{\partial \Theta_i} X_{t-1}^{t-1} + \Phi \frac{\partial X_{t-1}^{t-1}}{\partial \Theta_i}$$

3. Express X_{t-1}^{t-1}: We can express X_{t-1}^{t-1} in terms of X_{t-2}^{t-1} as:

$$X_{t-1}^{t-1} = \Phi X_{t-2}^{t-1} + K_{t-1}\epsilon_{t-1}$$

4. Differentiate X_{t-1}^{t-1}: Taking the derivative with respect to Θ_i, we get:

$$\frac{\partial X_{t-1}^{t-1}}{\partial \Theta_i} = \frac{\partial \Phi}{\partial \Theta_i} X_{t-2}^{t-1} + \Phi \frac{\partial X_{t-2}^{t-1}}{\partial \Theta_i} + \frac{\partial K_{t-1}}{\partial \Theta_i}\epsilon_{t-1} + K_{t-1}\frac{\partial \epsilon_{t-1}}{\partial \Theta_i}$$

5. Combine Results: Substitute back to obtain:

$$\frac{\partial X_{t}^{t-1}}{\partial \Theta_i} = \frac{\partial \Phi}{\partial \Theta_i} X_{t-2}^{t-1} + \Phi \frac{\partial X_{t-2}^{t-1}}{\partial \Theta_i} + \frac{\partial K_{t-1}}{\partial \Theta_i}\epsilon_{t-1} + K_{t-1}\frac{\partial \epsilon_{t-1}}{\partial \Theta_i}$$

(c)

Show that:

$$\frac{\partial \Sigma_t}{\partial \Theta_i} = A_t \frac{\partial P_t^{t-1}}{\partial \Theta_i} A_t' + \frac{\partial R}{\partial \Theta_i}$$

Solution:
1. Define Σ_t:

$$\Sigma_t = A_t P_t^{t-1} A_t' + R$$

where P_t^{t-1} is the covariance matrix of the predicted state X_t^{t-1} and R is the variance of the observation noise V_t.
2. Differentiate Σ_t with respect to Θ_i:

$$\frac{\partial \Sigma_t}{\partial \Theta_i} = \frac{\partial}{\partial \Theta_i}\left(A_t P_t^{t-1} A_t' + R\right)$$

3. Apply the Product Rule: Since A_t is independent of Θ_i,

$$\frac{\partial \Sigma_t}{\partial \Theta_i} = A_t \frac{\partial P_t^{t-1}}{\partial \Theta_i} A_t' + \frac{\partial R}{\partial \Theta_i}$$

Thus,

$$\frac{\partial \Sigma_t}{\partial \Theta_i} = A_t \frac{\partial P_t^{t-1}}{\partial \Theta_i} A_t' + \frac{\partial R}{\partial \Theta_i}$$

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(d)

Show that:

$$\frac{\partial K_t}{\partial \Theta_i} = \left[\frac{\partial P_t^{t-1}}{\partial \Theta_i} A_t' + \Phi \frac{\partial P_{t-1}^{t-1}}{\partial \Theta_i} A_t' - K_t \frac{\partial \Sigma_t}{\partial \Theta_i}\right] \Sigma_t^{-1}$$

Solution:
1. Define K_t:

$$K_t = P_t^{t-1} A_t' \Sigma_t^{-1}$$

2. Differentiate K_t with respect to Θ_i: Using the product rule,

$$\frac{\partial K_t}{\partial \Theta_i} = \frac{\partial (P_t^{t-1} A_t' \Sigma_t^{-1})}{\partial \Theta_i}$$

3. Expand using Product Rule:

$$\frac{\partial K_t}{\partial \Theta_i} = \frac{\partial P_t^{t-1}}{\partial \Theta_i} A_t' \Sigma_t^{-1} + P_t^{t-1} A_t' \frac{\partial \Sigma_t^{-1}}{\partial \Theta_i}$$

4. Substitute the derivative of Σ_t^{-1}:

$$\frac{\partial \Sigma_t^{-1}}{\partial \Theta_i} = -\Sigma_t^{-1} \frac{\partial \Sigma_t}{\partial \Theta_i} \Sigma_t^{-1}$$

5. Combine terms:

$$\frac{\partial K_t}{\partial \Theta_i} = \left(\frac{\partial P_t^{t-1}}{\partial \Theta_i} A_t' - K_t \frac{\partial \Sigma_t}{\partial \Theta_i}\right) \Sigma_t^{-1}$$

Thus,

$$\frac{\partial K_t}{\partial \Theta_i} = \left[\frac{\partial P_t^{t-1}}{\partial \Theta_i} A_t' + \Phi \frac{\partial P_{t-1}^{t-1}}{\partial \Theta_i} A_t' - K_t \frac{\partial \Sigma_t}{\partial \Theta_i}\right] \Sigma_t^{-1}$$

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(e)

Show that:

$$\frac{\partial P_t^{t-1}}{\partial \Theta_i} = \Phi \frac{\partial P_{t-2}^{t-1}}{\partial \Theta_i} \Phi' + \Phi \frac{\partial P_{t-2}^{t-1}}{\partial \Theta_i} + \frac{\partial Q}{\partial \Theta_i} - \frac{\partial K_t}{\partial \Theta_i} \Sigma_t K_t' - K_t \frac{\partial \Sigma_t}{\partial \Theta_i} K_t'$$

Solution:
1. Define P_t^{t-1}:

$$P_t^{t-1} = \Phi P_{t-1}^{t-1} \Phi' + Q$$

2. Differentiate P_t^{t-1} with respect to Θ_i:

$$\frac{\partial P_t^{t-1}}{\partial \Theta_i} = \frac{\partial (\Phi P_{t-1}^{t-1} \Phi' + Q)}{\partial \Theta_i}$$

3. Apply Product Rule:

$$\frac{\partial P_t^{t-1}}{\partial \Theta_i} = \Phi \frac{\partial P_{t-1}^{t-1}}{\partial \Theta_i} \Phi' + \Phi \frac{\partial P_{t-2}^{t-1}}{\partial \Theta_i} + \frac{\partial Q}{\partial \Theta_i}$$

4. Subtract the terms involving K_t:

$$\frac{\partial P_t^{t-1}}{\partial \Theta_i} = \Phi \frac{\partial P_{t-2}^{t-1}}{\partial \Theta_i} \Phi' + \Phi \frac{\partial P_{t-2}^{t-1}}{\partial \Theta_i} + \frac{\partial Q}{\partial \Theta_i} - \frac{\partial K_t}{\partial \Theta_i} \Sigma_t K_t' - K_t \frac{\partial \Sigma_t}{\partial \Theta_i} K_t'$$

This completes the derivation of the solution.

Q3 b

Introduction

In this analysis, we fit a moving average model (MA(1)) to the differenced logarithm of the glacial varve series data. This analysis aims to show that the differenced series of `Y_t` follows an MA(1) process. We will apply the model to glacial varve data, available in the `astsa` package as `varve`, to assess its fit and interpret the results.

Mathematical Derivation

Model Setup

Given a univariate state-space model:

$$Y_t = X_t + V_t$$

$$X_t = X_{t-1} + W_t$$

where $W_t \sim \mathcal{N}(0, \sigma_W^2)$ and $V_t \sim \mathcal{N}(0, \sigma_V^2)$ are Gaussian white noise processes, and both are independent.

Differencing the Observed Series

The observed data Y_t can be transformed into a stationary series by differencing:

$$\nabla Y_t = Y_t - Y_{t-1} = (X_t + V_t) - (X_{t-1} + V_{t-1})$$

Expanding, we get:

$$\nabla Y_t = (X_t - X_{t-1}) + (V_t - V_{t-1}) = W_t + \nabla V_t$$

Since W_t is white noise and $\nabla V_t = V_t - V_{t-1}$ is a first-order moving average process, ∇Y_t follows an MA(1) model:

$$\nabla Y_t = W_t + \theta V_{t-1}$$

Interpretation of Parameters

The MA(1) model is defined by:

$$\nabla Y_t = W_t + \theta W_{t-1}$$

where: - θ is the MA(1) coefficient. - W_t is white noise with variance σ_W^2.

Our goal is to fit this MA(1) model to the differenced series and interpret the results.

Data Preparation

We start by loading and transforming the glacial varve series data.

```r
data(varve, package = "astsa")
log_varve <- log(varve)
diff_log_varve <- diff(log_varve)
```

Model Fitting

We fit an MA(1) model to the differenced log-transformed series.

```r
# Fit an MA(1) model to the differenced series
model <- Arima(diff_log_varve, order = c(0, 0, 1))
summary(model)
```

```
## Series: diff_log_varve
## ARIMA(0,0,1) with non-zero mean
##
## Coefficients:
##           ma1     mean
##       -0.7710  -0.0013
## s.e.   0.0341   0.0044
##
## sigma^2 = 0.236:  log likelihood = -440.68
## AIC=887.36   AICc=887.39   BIC=900.71
##
## Training set error measures:
##                        ME      RMSE       MAE  MPE MAPE      MASE      ACF1
## Training set 0.0004021077 0.4850623 0.3823767 -Inf  Inf 0.4963925 0.1200001
```

Observations

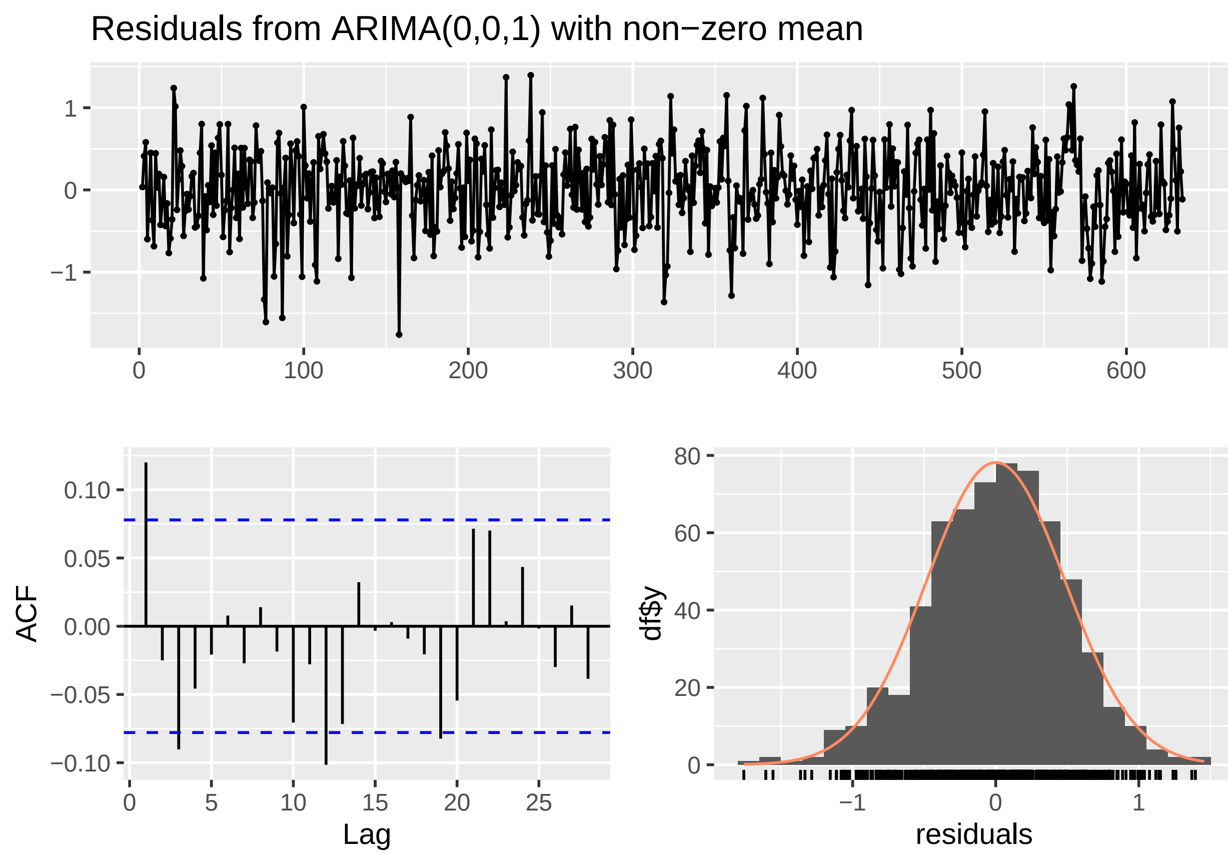
From the model output, we obtain: - The estimated MA(1) coefficient θ. - The variance σ^2 of the residuals. - Information criteria (AIC, BIC) that assess the model fit.

These values provide insights into how well an MA(1) process fits the differenced series.

Diagnostic Checks

After fitting the model, we examine the residuals to ensure they resemble white noise, as required by the MA(1) assumption.

```r
# Plot residuals and perform diagnostic checks
checkresiduals(model)
```




```
##
##  Ljung-Box test
##
## data:  Residuals from ARIMA(0,0,1) with non-zero mean
## Q* = 20.441, df = 9, p-value = 0.01538
##
## Model df: 1.   Total lags used: 10
```

Interpretation of Diagnostics

The diagnostic plot includes:

1. **Residual Time Series Plot**: Checks if residuals appear randomly distributed around zero.
2. **ACF Plot of Residuals**: Residual autocorrelations should lie within the confidence bounds, indicating no significant correlation.
3. **Residual Density Plot**: Ensures normality of residuals.
4. **Ljung-Box Test**: A formal test for residual autocorrelation. A high p-value supports the null hypothesis of no autocorrelation.

If these diagnostics are satisfactory, it suggests the MA(1) model is appropriate for the data.

Summary of Findings

The fitted MA(1) model and diagnostics indicate:

- **Model Fit**: The MA(1) model seems appropriate for the differenced log-transformed varve series.
- **Coefficient Interpretation**: The estimated MA(1) parameter indicates the extent of correlation between Y_t and Y_{t-1} after differencing.
- **Residual Analysis**: Diagnostic checks confirm that residuals approximate white noise, validating the model's suitability.

In conclusion, an MA(1) model provides a reasonable fit to the differenced series, and this model can help understand the underlying process in the glacial varve data. “‘