

Name:
Roll No:

MTH517A: Time Series Analysis **Quiz #1; Full Marks-20**

[1] Let $\{X_t\}$ be a covariance stationary Gaussian process with mean θ and ACVF $\gamma_X(\cdot)$. Define $U_t = (1 - 0.4B)X_t$, $V_t = (1 - 2.5B)X_t$ and $P_t = a + bt + ct^2 + S_t + X_t$; S_t is a seasonal component with periodicity 3 and a, b and c are constants.

Prove or disprove the following statements:

- (a) $\{U_t\}$ and $\{V_t\}$ are covariance stationary.
- (b) $\{U_t\}$ and $\{V_t\}$ have identical autocorrelation structure.
- (c) $\{U_t\}$ and $\{V_t\}$ are covariance stationary with identical mean and identical autocorrelation structures.
- (d) $\{\nabla_6 P_t\}$ is Gaussian.
- (e) $\{\nabla_6 P_t\}$ is covariance stationary.
- (f) $\{\nabla_3 \nabla_6 P_t\}$ is strict stationary.

1

Solution Quiz #1

$$(a) \quad u_t = x_t - 0.4x_{t-1} \quad v_t = x_t - 2.5x_{t-1}$$

$$E u_t = \theta(1-0.4) \quad E v_t = (1-2.5)\theta$$

$$\text{Cov}(u_t, u_{t+h})$$

$$= \text{Cov}(x_t - 0.4x_{t-1}, x_{t+h} - 0.4x_{t+h-1})$$

$$= \gamma(h) - 0.4\gamma(h-1) - 0.4\gamma(h+1) + (0.4)^2\gamma_h$$

$$= \gamma(h)(1+(0.4)^2) - 0.4(\gamma(h-1) + \gamma(h+1)) = \gamma_u(h)$$

$$\text{Cov}(v_t, v_{t+h})$$

$$= \text{Cov}(x_t - 2.5x_{t-1}, x_{t+h} - 2.5x_{t+h-1})$$

$$= \gamma(h)(1+(2.5)^2) - 2.5(\gamma(h-1) + \gamma(h+1)) = \gamma_v(h)$$

$\Rightarrow \{u_t\}$ & $\{v_t\}$ are covariance stationary

$$(b) \quad \rho_u(h) = \frac{\gamma_u(h)}{\gamma_u(0)}$$

$$\rho_u(h) = \frac{\gamma(h)(1+0.4^2) - 0.4(\gamma(h-1) + \gamma(h+1))}{\gamma(0)(1+0.4^2) - 2 \times 0.4 \gamma(1)}$$

$$\rho_v(h) = \frac{\gamma(h)(1+\frac{1}{0.4^2}) - \frac{1}{0.4}(\gamma(h-1) + \gamma(h+1))}{\gamma(0)(1+\frac{1}{0.4^2}) - 2 \cdot \frac{1}{0.4} \gamma(1)}$$

$$= \frac{\gamma(h)(1+0.4^2) - 0.4(\gamma(h-1) + \gamma(h+1))}{\gamma(0)(1+0.4^2) - 2 \times 0.4 \gamma(1)} = \rho_u(h)$$

$$(c) \quad E u_t \neq E v_t$$

$$\begin{aligned}
 (d) \quad \nabla_6 P_t &= \nabla_6 (a + bt + ct^2 + s_t + x_t) \\
 &= (a + bt + ct^2) - (a + \cancel{bt} + c(t-6)^2) \\
 &\quad + s_t - s_{t-6} + x_t - x_{t-6} \\
 &= ct^2 - (-6b + c(36 + 12t)) + s_t - s_{t-6} + x_t - x_{t-6}
 \end{aligned}$$

$$z_t = \nabla_6 P_t = 6b - 3c + 12ct + x_t - x_{t-6}$$

$$\begin{aligned}
 z_t = \begin{pmatrix} z_{t_1} \\ \vdots \\ z_{t_n} \end{pmatrix} &= \begin{pmatrix} 6b - 3c + 12ct_1 + x_{t_1} - x_{t_1-6} \\ 6b - 3c + 12ct_2 + x_{t_2} - x_{t_2-6} \\ \vdots \\ 6b - 3c + 12ct_n + x_{t_n} - x_{t_n-6} \end{pmatrix} \\
 &= (6b - 3c) \underbrace{\frac{1}{n} \mathbf{1}_n}_{\alpha} + 12c \underbrace{\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}}_{\beta} + \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ - & - & - & - & - \end{pmatrix} \begin{pmatrix} x_t \\ x_{t-6} \\ x_{t_1} \\ x_{t_2} \\ \vdots \\ x_{t_n-6} \\ x_{t_n} \end{pmatrix}
 \end{aligned}$$

$$\alpha \in \mathbb{R}^n$$

$$\underline{x}' z_t = (6b - 3c) \underline{x}' \frac{1}{n} +$$

$$+ 12c \underline{x}' \frac{t}{n} + \underline{\beta}' x_t$$

If there are
no overlappings
of dimension is less,

$$\sim N_1(\cdot, \cdot) \text{ as } \{x_t\} \text{ is Gaussian}$$

$$\Rightarrow \underline{\beta}' x_t \sim N_1 + \underline{\beta} \in \mathbb{R}^{2n}$$

$\Rightarrow \nabla_6 P_t$ is Gaussian

$$(e) \nabla_6 P_t = 6b - \frac{36}{k}c + 12ct + x_t - x_{t-6}$$

$$E(\nabla_6 P_t) = 6b - \frac{36}{k}c + 12ct \leftarrow f^n of t$$

$\Rightarrow \nabla_6 P_t$ is NOT covariance stationary

$$(f) \nabla_3 \nabla_6 P_t = (6b - \frac{36}{k}c + 12ct) - (6b - \frac{36}{k}c + 12c(t-3)) \\ + (x_t - x_{t-3}) - (x_{t-6} - x_{t-9}) \\ = 36c + x_t - x_{t-3} - x_{t-6} + x_{t-9}$$

$\Omega_t = \nabla_3 \nabla_6 P_t$ is Gaussian using similar logic as in (d)

$$E(\nabla_3 \nabla_6 P_t) = 36c$$

$$\text{Cov}(\Omega_t, \Omega_{t+h}) = \text{Cov}(36c + x_t - x_{t-3} - x_{t-6} + x_{t-9}, \\ 36c + x_{t+h} - x_{t+h-3} - x_{t+h-6} + x_{t+h-9}) \\ = (r(h) - r(h-3) - r(h-6) + r(h-9)) \\ + (-r(h+3) + r(h) + r(h-3) - r(h-6)) \\ + (-r(h+6) + r(h+3) + r(h) - r(h-3)) \\ + (r(h+9) - r(h+6) - r(h+3) + r(h))$$

$\rightarrow f^n of h \text{ only}$

$\Rightarrow \Omega_t$ is Gaussian & also covariance stationary

$\Rightarrow \Omega_t = \nabla_3 \nabla_6 P_t$ is strict stationary