3/8/2023

* Fundamental Components of Time Series

1. Deterministic Components :

- Trend component : Long term tendency as a function of time.
- Seasonal component : Distinguishable pattern of annual variation, repeated year after year
- Cyclical component : long range swing above and below some equilibrium level or trend line.

  4 Stages of cyclical component : upswing, peak, down swing, trough

Note: One or more or none can be a part of a time series

- Random Component (or Irregular Component) : Brings the randomness / stochastic nature to the time series.

Additive Model

$$Y_t = \underbrace{m_t}_{\text{trend}} + \underbrace{S_t}_{\text{Seasonal}} + \underbrace{C_t}_{\text{cyclical}} + \underbrace{e_t}_{\text{irregular}}$$

[Assuming all the components are present]

If it is a trend additive model:

$$Y_t = m_t + e_t$$

Multiplicative Model :

$$Y_t = m_t S_t C_t e_t$$

[Assuming all components are present]

Hybrid Model :

$$Y_t = m_t S_t + e_t$$

e.g. $Y_t = (\alpha + \beta t) S_t + e_t$

seasonal factor with period d.

* Preliminary Tests

1. Testing for existence of trend.

(A) Relative Ordering Test

→ Non - Parametric Test (asymptotic)

Null hypothesis :

$\boxed{H_0 : \text{no trend}}$

to be tested against alternative hypothesis

$\boxed{H_A : \text{trend present.}}$

H_0 against H_A.

Now,

$Y_1, Y_2, \ldots, Y_n$ observation set (time series data)

Define,

$$q_{ij} = \begin{cases} 1 ; & Y_i > Y_j \text{ when } i<j \\ 0 ; & o/w \end{cases}$$

$$Q = \sum_{\substack{i \\ i<j}} \sum_{j} q_{ij}$$

[counts the no. of decreasing points in the data]

[$(i, Y_i), (j, Y_j)$; $i<j$, $q_{ij} = 1$ if there is a discordance of direction b/w $i \leftrightarrow j$ & $Y_i \leftrightarrow Y_j$]

∴ also the data : # of discordance pairs]

Under H_0 (of no trend)

$$P(q_{ij} = 0) = \frac{1}{2} = P(q_{ij} = 1) = \frac{1}{2}$$

[Occurence of decreasing points & increasing points are equally likely,] the data is equally likely to occur in any order

Thus, under H_0:

$$E(q_{ij}) = \frac{1}{2}$$

$$E(Q) = \frac{n(n-1)}{4}$$

Q can take minimum value to be 0 ; when all the points for $i<j$, Y_i & Y_j follow increasing trend.

Q can take maximum value to be 1, when all the points for $i<j$, Y_i & Y_j follow decreasing trend.

NOTE :

(1) If obsd $Q \gg E(Q)$ under H_0 ; indicates a decreasing trend.
(2) If obsd $Q \ll E(Q)$ under H_0; indicates a increasing trend.
(3) If Q differs from E(Q) under H_0 "significantly", indicates existence of trend.

Remark : Q is related to Kendall's τ, the rank correlation coeff.

$$\tau = 1 - \frac{4Q}{n(n-1)}$$

Using the standard non-parametric result

$$E(\tau) = 0 \ \& \ V(\tau) = \frac{2(2n+5)}{9n(n-1)} \quad \text{under } H_0.$$

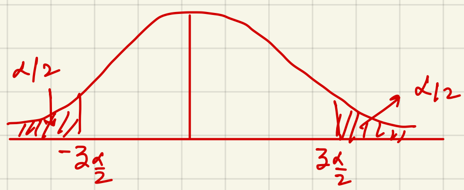
Test Statistic

$$Z = \frac{\tau - E(\tau)}{\sqrt{V(\tau)}} \underset{(n \to \infty)}{\overset{\text{asymp}}{\sim}} N(0,1) \text{ under } H_0.$$

Testing Procedure :

– Reject the null hypothesis H_0 at level of significance α $(0<\alpha<1)$ if obsd

$|z| > z_{\frac{\alpha}{2}}$; where $z_{\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}$th cut off point of $N(0,1)$



If the null hypo is rejected, then we say Q differs "significantly" from $E(Q)$ and trend is present.

Trend estimation

8/08/23

- **Parametric Test for Trend**
  - (underlying assumption : normality)

→ **Existence of Linear Trend**

$Y_t = \alpha + \beta t + \epsilon_t$; ϵ_t is a seq of i.i.d r.vs with $E(\epsilon_t) = 0$ & $V(\epsilon_t) = \sigma^2 < \infty$. s.t. $\epsilon_t \sim N(0, \sigma^2)$

Null hypothesis : $H_0: \beta = 0$

ag. Alternate hypothesis : $H_A: \beta \neq 0$

- usual t or F - test

$t \uparrow t_{n-2}$, $F \downarrow F_{(1, n-2)}$

Estimated trend: $\hat{m}_t = \hat{\alpha} + \hat{\beta} t$

$\hat{\epsilon}_t = Y_t - \hat{\alpha} - \hat{\beta} t$

$(t = 1, 2, \ldots, n)$

- perform test for normality on residual $(\hat{\epsilon}_t)$ [eg. Q-Q Plot]

→ **Quadratic Trend**

$Y_t = \alpha + \beta t + \gamma t^2 + \epsilon_t$; $\epsilon_t \overset{iid}{\sim} N(0, \sigma^2)$

$H_0: \beta = 0, \gamma = 0$

ag. H_A : not H_0

usual F statistic

$\downarrow$

$F_{(2, n-3)}$

- Interest : $H_0: \gamma = 0$ ag. $H_A: \gamma \neq 0$ given $H_0: \beta = 0$ is rejected.

This is on our original data, to see if for $\beta \neq 0$, how does γ performs

usual F statistic

$\downarrow$

$F_{(1, n-3)}$

* Testing for seasonal component

[Non-Parametric Test]

Friedman's trend

- Data is either monthly or quarterly

→ Friedman's test for monthly data

Step 1 : Remove trend, if required
(use detrended data $(Y_t - \hat{m}_t)$)

Step 2 : Rank the detrended data from smallest (1) to highest (12)

M_{ij} : Rank for i^{th} month in j^{th} year (c years of data)
Calculate the monthly totals of ranks i.e. $M_i = \sum_{j=1}^{c} M_{ij}$

Month	Years: 1	2	— — — — — — — —	c	Total
1	M_{11}	M_{12}	— — — — — — — —	M_{1c}	M_1
2	M_{21}	M_{22}	— — — — — — — —	M_{2c}	M_2
⋮	⋮				
12	$M_{12\,1}$	$M_{12\,2}$	— — — — — — — —	M_{12c}	M_{12}

For year j

$$\begin{pmatrix} M_{1j} \\ M_{2j} \\ \vdots \\ M_{12j} \end{pmatrix} \text{ - permutation of } \{1, 2, \ldots, 12\}$$

$\| n = 12 \|$

H_0: no seasonality
ag H_A: not H_0

→ under null hypothesis H_0, all possible permutations are equally likely.

under H_0 ; M_{ij} : $P(M_{ij} = k) = \frac{1}{n}$; $k = 1(1)n$.

$$E(M_{ij}) = \frac{1}{n} \cdot \sum_{k=1}^{n} k = \frac{n+1}{2}.$$

$$E(M_i) = E\left(\sum_{j=1}^{c} M_{ij}\right) = \frac{c(n+1)}{2}.$$

\# expected value of monthly totals under H_0, i.e. no seasonality.

Friedman's test statistic.

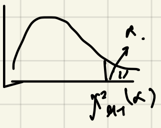
$$X = \frac{12 \sum_{i=1}^{n} \left(M_i - \frac{c(n+1)}{2}\right)^2}{cn(n+1)} \rightarrow$$

- Reject H_0 at α level of significance, if

obsd $(X) > \chi^2_{(n-1)}(\alpha)$

$[\chi^2_{n-1}(\alpha)$ is $P(\chi^2_{n-1} > \chi^2_{n-1}(\alpha)) = \alpha]$

$$X \overset{\text{asymp.}}{\sim} \chi^2_{(n-1)}$$



09/08/23

* Test for Randomness

eg. H_0: the series is purely random.
H_A : not H_0

• Turning Point Test

- A point y_i is a turning pt. if
  $y_i > y_{i-1}$ and $y_i > y_{i+1}$ # local peak $y_{i-1} \nearrow y_i \searrow y_{i+1}$
  or
  $y_i < y_{i-1}$ and $y_i < y_{i+1}$ # local trough $y_{i-1} \searrow y_i \nearrow y_{i+1}$

Define, $U_i = \begin{cases} 1 ; & \text{if } y_i \text{ is a turning point.} \\ 0 ; & o/w \end{cases}$

Total no. of turning points :

$$Q = \sum_{i=2}^{n-1} U_i$$

let $y_{(1)}, y_{(2)}, y_{(3)}$ denote the ordered observations.
(y_{i-1}, y_i, y_{i+1})

$y_{(1)}$: smallest
$y_{(3)}$: largest

(y_{i-1}, y_i, y_{i+1}) can be one of the following 6:

- $(y_{(1)}, y_{(2)}, y_{(3)})$ — ×
- $(y_{(1)}, y_{(3)}, y_{(2)})$ — Turning (peak)
- $(y_{(3)}, y_{(1)}, y_{(2)})$ — Turning (trough)
- $(y_{(3)}, y_{(2)}, y_{(1)})$ — ×
- $(y_{(2)}, y_{(1)}, y_{(3)})$ — Turning (trough)
- $(y_{(2)}, y_{(3)}, y_{(1)})$ — Turning (peak)

Under H_0, that series is purely random, all the 6 possible outcomes are equally likely.

$$E(U_i) \text{ under } H_0 = 1 \times \frac{4}{6} + 0 \times \frac{2}{6} = \frac{4}{6} = \frac{2}{3}$$

$$\left[E(U_i^2) = \frac{2}{3} \right.$$

$$V(U_i) = \frac{2}{3} - \frac{4}{9} = \frac{2}{9}$$

$$E(Q) = E\left(\sum_{i=2}^{n-1} U_i\right) = (n-2)\frac{2}{3}$$

$$V(Q) =$$

$$V(Q) = V\left(\sum_{i=2}^{n-1} U_i\right) = \frac{16n-29}{90}$$

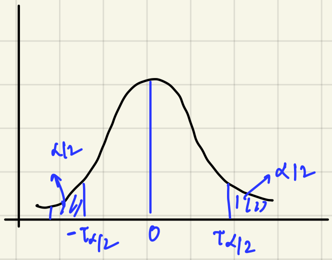
Asymptotic Distbⁿ

$$Z = \frac{Q - E(Q)}{\sqrt{V(Q)}} \xrightarrow{asymp} N(0,1) \text{ under } H_0.$$

- Reject H_0 at level of significance α, if

obsvd $|Z| > \tau_{\alpha/2}$

$\tau_{\alpha/2}$ is $\rightarrow P(X > \tau_{\alpha/2}) = \alpha/2$; $X \sim N(0,1)$



ESTIMATION / ELIMINATION OF TREND

Case 1 : No seasonality

$$Y_t = \underset{\text{Trend}}{m_t} + \underset{\text{Irregular}}{e_t}$$

; e_t is a seq r.v.s
$\rightarrow E(e_t) = 0$

$\rightarrow Cov(e_t, e_s) = \begin{cases} \sigma^2 ; & \text{if } t = s \\ 0 ; & o/w \end{cases}$

Least Squares trend estimation

$$m_t = a_0 + a_1 t + a_2 t^2 + \dots + a_k t^k$$

$$Y_t = a_0 + a_1 t + a_2 t^2 + \dots + a_k t^k + e_t \quad ; \quad t = 1, 2, \dots, n$$

$$\hat{\underset{\sim}{a}}_{LSE} = \arg\min_{\underset{\sim}{a}} \sum_{t=1}^{n} (y_t - a_0 - a_1 t - a_2 t^2 - \dots - a_k t^k)^2$$

$\underset{\sim}{a} = (a_0, a_1, \dots, a_k)$

$$\underset{\sim}{Y} = X\underset{\sim}{a} + \underset{\sim}{e}$$

$\underset{\sim}{Y} = (y_1, y_2, \dots, y_n)'$, $\underset{\sim}{e} = (e_1, \dots, e_n)'$

$\underset{\sim}{a} = (a_1, \dots, a_k)'$

$$X = \begin{pmatrix} 1 & 1 & 1^2 & \dots & 1^k \\ 1 & 2 & 2^2 & \dots & 2^k \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & n & n^2 & & n^k \end{pmatrix}$$

$$\hat{\underset{\sim}{a}}_{LSE} = \arg\min_{\underset{\sim}{a}} (\underset{\sim}{Y} - X\underset{\sim}{a})'(\underset{\sim}{Y} - X\underset{\sim}{a})$$

$$\hat{\underset{\sim}{a}}_{LSE} = (X'X)^{-1} X'Y$$

Estimated Trend : $\hat{m}_t = \hat{a}_0 + \hat{a}_1 t + \hat{a}_2 t^2 + \dots + \hat{a}_k t^k$

Detrended Series : $y_t - \hat{m}_t$

Remark :

- Fit $\hat{m}_1 = \hat{a}_0 + \hat{a}_1 t$ # linear trend
- $\tilde{y}_t^{(1)} = y_t - \hat{m}_1$
- On $(\tilde{y}_1^{(1)}, \dots, \tilde{y}_n^{(1)})$ perform relative ordering test.
- If H_0 is accepted, stop with $\hat{m}_1$. If H_0 is rejected, fit $\hat{m}_2 = \hat{a}_0 + \hat{a}_1 t + \hat{a}_2 t^2$
- $\tilde{y}_t^{(2)} = y_t - \hat{m}_2$
- On $(\tilde{y}_1^{(2)}, \dots, \tilde{y}_n^{(2)})$ perform relative ordering test
- Continue, till H_0 is accepted.

* MOVING AVERAGE estimation of trend

[two sided or one sided]

$(y_1, y_2, \dots, y_n)$

- let q is a non-negative integer
- Two-Sided odd ordered MA

$$\hat{m}_t = \frac{1}{2q+1} \sum_{j=-q}^{q} y_{t+j} \quad ; \quad q+1 \le t \le n-q$$

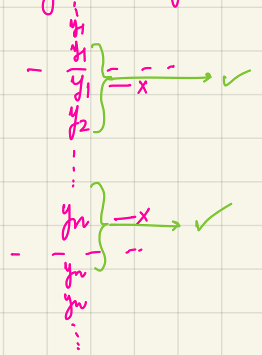
→ $\hat{m}_t$ is based on a simple $\underbrace{\text{(equal weighted)}}_{\frac{1}{2q+1}}$ average of observations $(y_{t-q}, y_{t-q+1}, \dots, y_t, \dots, y_{t+q})$

→ We don't have trend estimates at first q and last q observations.

Padding can be applied to obtain rough estimates

eg. Symmetric Padding or end-point padding.

Symmetric Padding : say $q = 1$



$$\hat{m}_t = \frac{1}{2q+1} \sum_{j=-q}^{q} Y_{t+j}$$

$q=1$

$$\hat{m}_t = \frac{1}{3}(y_{t-1}, y_t, y_{t+1})$$

If $m_t = \alpha + \beta t$ (linear trend)

$Y_t = m_t + \epsilon_t$

$$\hat{m}_t = \frac{1}{3}(3\alpha + \beta(t-1+t+t+1) + 3\epsilon_t)$$

10/08/23

Note: Even order moving averages

- For even order moving averages a simple average of 2 consecutive values are taken to obtain trend at given time points.

$q=2 \to 4$

$y_1, y_2, y_3, y_4, y_5, \ldots, y_n$

$\to \frac{1}{4}(y_1+y_2+y_3+y_4)$

$\to \frac{1}{4}(y_2+y_3+y_4+y_5)$

$$= \frac{1}{2}\left(\frac{1}{4}(y_1+y_2+y_3+y_4) + \frac{1}{4}(y_2+y_3+y_4+y_5)\right) = \frac{1}{4}\left(\frac{1}{2}y_1 + y_2 + \ldots + y_4 + \frac{1}{2}y_5\right)$$

q

$$\hat{m}_t = \frac{1}{2q}\left(\frac{1}{2}y_{t-q} + y_{t-q+1} + \ldots + y_t + \ldots + y_{t+q-1} + \frac{1}{2}y_{t+q}\right)$$

$q+1 \le t \le n-q$

Note: Low-pass filter

Filters out the rapidly fluctuating component and passes less volatile smooth component.

$$\xrightarrow{Y_t} \boxed{\{a_j\}} \longrightarrow \hat{m}_t = \sum a_j Y_{t-j}$$

linear

odd order m.a.

$$a_j = \begin{cases} \frac{1}{2q+1} ; & |j| \le q \\ 0 ; & o/w \end{cases}$$

Note : One sided m.a.

The trend value is associated with the last point in the moving average window.

y_1
y_2
$y_3 \rightarrow \hat{m}_3$
$y_4 \rightarrow \hat{m}_4$
$\vdots$
y_n

Note : unequal weighted m.a.

e.g. Exponential weighted m.a. (EWMA)
or
Exponential Smoothing.

$(y_1, \ldots, y_n) \longrightarrow$

$$\hat{m}_1 = y_1$$

α is fixed, $\alpha \in \left(\frac{1}{2}, 1\right)$, $\forall\, t \geq 2$, $\hat{m}_t = \alpha y_t + (1-\alpha)\hat{m}_{t-1}$

$$\hat{m}_2 = \alpha y_2 + (1-\alpha)\, y_1$$

$$\hat{m}_3 = \alpha y_3 + (1-\alpha)\,\hat{m}_2$$
$$= \alpha y_3 + (1-\alpha)\alpha y_2 + (1-\alpha)^2 y_1$$

involves $\{y_1, y_2, y_2\}$ with $y_1 \to \alpha$, $y_2 \to \alpha(1-\alpha)$, $y_2 \to (1-\alpha)^2$

and so on

$\because\ \underbrace{\alpha > \alpha(1-\alpha) > (1-\alpha)^2}$ decreasing in exponential manner

$\forall\, t \geq 2$, $\hat{m}_t = \alpha y_t + (1-\alpha)\hat{m}_{t-1}$

$$= \alpha y_t + (1-\alpha)\left(\alpha y_{t-1} + (1-\alpha)\hat{m}_{t-2}\right)$$
$$= \alpha y_t + \alpha(1-\alpha)\, y_{t-1} + (1-\alpha)^2\hat{m}_{t-2}$$
$$= \alpha y_t + \alpha(1-\alpha)\, y_t + (1-\alpha)^2\left(\alpha y_{t-2} + (1-\alpha)\,\hat{m}_{t-3}\right)$$
$$= \alpha y_t + \alpha(1-\alpha)\, y_t + (1-\alpha)^2 \alpha y_{t-2} + (1-\alpha)^3\,\hat{m}_{t-3}$$
$$\vdots$$

$$\boxed{\hat{m}_t = \sum_{j=0}^{t-2} \alpha(1-\alpha)^j\, y_{t-j} + (1-\alpha)^{t-1}\, y_1}$$

$\hat{m}_t$: $\{y_t, y_{t-1}, \ldots, y_1\}$

weight : α, $\alpha(1-\alpha)$, $\alpha(1-\alpha)^2, \ldots,$ $(1-\alpha)^{t-1}$

* Elimination of trend by differencing. [without estimation of trend]

- Backward shift operator : B s.t.

$$BY_t = Y_{t-1}$$
$$B^2 Y_t = B(BY_t) = Y_{t-2}.$$
$$B^j Y_t = Y_{t-j}$$

Difference operator : ∇ s.t.

$$\nabla Y_t = Y_t - Y_{t-1} = (1-B)Y_t$$

$$\nabla^j Y_t = \nabla(\nabla^{j-1} Y_t)$$

$$\begin{aligned}\nabla^2 Y_t &= \nabla(\nabla Y_t) \\ &= \nabla(Y_t - Y_{t-1}) \\ &= \nabla Y_t - \nabla Y_{t-1} \\ &= Y_t - Y_{t-1} - Y_{t-1} + Y_{t-2} \\ (1-B)^2 Y_t &= Y_t - 2Y_{t-1} + Y_{t-2}\end{aligned}$$

$Y_t = m_t + e_t$; e_t is s.t. $E(e_t) = 0$, $V(e_t) = \sigma^2$

trend $\rightarrow m_t = \alpha + \beta t$

$$\begin{aligned}(Y_t - Y_{t-1}) = \nabla Y_t &= \nabla(m_t + e_t) \\ &= \nabla m_t + \nabla e_t \\ &= (m_t - m_{t-1}) + (e_t - e_{t-1}) \\ &= (\alpha + \beta t - \alpha - \beta t + \beta) + (e_t - e_{t-1}) \\ Z_t = \nabla Y_t &= \beta + (e_t - e_{t-1}) \rightarrow \text{free from time trend.}\end{aligned}$$

Now, consider

$Y_t = m_t + e_t$

trend $\rightarrow m_t = \alpha + \beta t + \gamma t^2$

$$\nabla m_t = (\alpha + \beta t + \gamma t^2) - (\alpha + \beta(t-1) + \gamma(t-1)^2)$$
$$\nabla^2 m_t = 2\gamma$$

$$\begin{aligned}\nabla^2 Y_t &= \nabla^2 m_t + \nabla^2 e_t \\ &= 2\gamma + (e_t - 2e_{t-1} + e_{t-2}) \rightarrow \text{free from time trend.}\end{aligned}$$

[A suitable order of differencing, can eliminate trend through differencing]

In general,

if $m_t = a_0 + a_1 t + \dots + a_k t^k$

$Y_t = m_t + e_t$

$\nabla^k Y_t = \nabla^k m_t + \nabla^k e_t$

$Z_t = \nabla^k Y_t = (k!)\, a_k + \nabla^k e_t \rightarrow$ free from time trend.

	Δ	Δ^2
y_1		
y_2	$y_2 - y_1$	
$\vdots$	$y_3 - y_2$	$y_3 - 2y_2 + y_1$
$\vdots$	$\vdots$	$\vdots$
y_n	$y_n - y_{n-1}$	$(y_n - y_{n-1}) - (y_{n-1} - y_{n-2})$

→ Once we reach the point, where series is free from time trend, we must stop., because the variability also increases.

16/08/23

→ **Estimation and Elimination of trend and seasonality.**

$Y_t = m_t + s_t + e_t$; $\{e_t\}$ in $\exists$ $E(e_t)=0$, $V(e_t)=\sigma^2<\infty$

s_t : seasonal component with period, say d.

Assume,

$$s_{t-2d} = s_{t-d} = s_t = \dots = s_{t+2d}$$

$$\Rightarrow \sum_{j=1}^{d} s_j = 0$$

Consider, a monthly data with periodicity 12.

Time index t ; as

$$t = 12(j-1)+k$$

j : year number
k : month number

Y_t is $\equiv y_{j,k} = y_{12(j-1)+k}$

* **Slow Trend Method**

→ Trend is constant for a year. i.e. m_j for a particular year is constant.

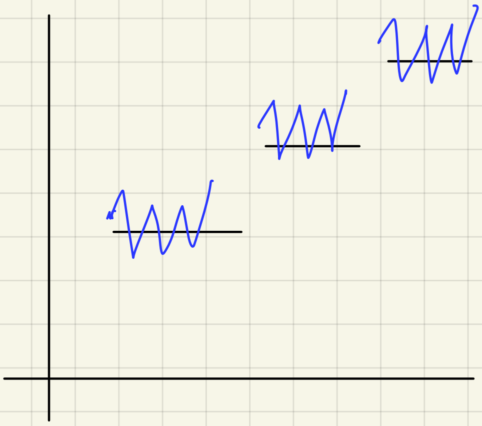
Step 1: estimate trend as :

$$\hat{m}_j = \frac{1}{k}\sum_{k=1}^{12} Y_{j,k} = \frac{1}{12}\sum_{k=1}^{12}(m_j + s_k + e_{j,k})$$

$$= m_j + \frac{1}{12}\sum_{k=1}^{12} e_{j,k}. \qquad \left[\because \sum_{k=1}^{12} s_k = 0\right]$$

Step 2: estimate seasonal factor as,

$$\hat{s}_k = \frac{1}{J}\sum_{j=1}^{J}(Y_{j,k} - \hat{m}_j)$$



logic

data

$$
\begin{array}{c|cccc}
 & \multicolumn{4}{c}{\text{Years}} \\
 & 1 & 2 & 3 \;\cdots & J \\
\hline
\text{Months} \;\; 1 & \left[\begin{matrix} y_{1,1} \\ y_{1,2} \\ \vdots \\ y_{1,12} \end{matrix}\right] & \left[\begin{matrix} y_{2,1} \\ y_{2,2} \\ \vdots \\ y_{2,12} \end{matrix}\right] & \cdots & \left[\begin{matrix} y_{j,1} \\ y_{j,2} \\ \vdots \\ y_{j,12} \end{matrix}\right] \\
2 & & & & \\
\vdots & & & & \\
12 = K & \downarrow & \downarrow & & \downarrow \\
 & \hat{m}_1 & \hat{m}_2 & & \hat{m}_J
\end{array}
$$

de-trended data

$$
\begin{array}{ccc}
\text{Year \#1} & - - - - - - & \text{year \#J} \\
\left(\begin{matrix} Y_{1,1} - \hat{m}_1 \\ Y_{1,2} - \hat{m}_1 \\ \vdots \\ Y_{1,12} - \hat{m}_1 \end{matrix}\right) & & \left(\begin{matrix} Y_{J,1} - \hat{m}_J \\ Y_{J,2} - \hat{m}_J \\ \vdots \\ Y_{J,J} - \hat{m}_J \end{matrix}\right)
\end{array}
$$

Seasonal factor data

↑

Average values, over years for the months are (for K^{th} month), $\hat{s}_K$'s

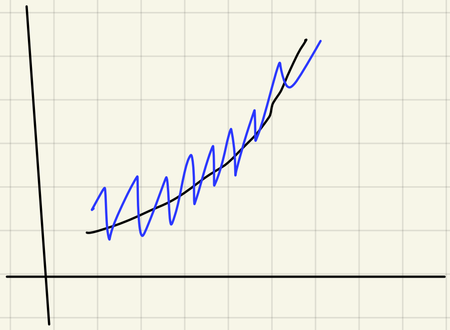
$$\hat{s}_K = \frac{1}{J} \sum_{j=1}^{J} (y_{j,K} - \hat{m}_j)$$

$$\hat{e}_{j,K} = y_{j,K} - \hat{m}_j - \hat{s}_K \; ; \quad \begin{matrix} j = 1(1)J \\ K = 1(1)12 \end{matrix}$$

Note : $\displaystyle\sum_{K=1}^{12} \hat{s}_K = \frac{1}{J} \sum_{K=1}^{12} \sum_{j=1}^{J} (y_{j,K} - \hat{m}_j) = 0$

Method 2 : Fast Trend Method

In case there is significant trend which can not be assumed to be constant for a year, we proceed the following way :



Step 1 : Obtain rough estimates of trend.

Use MA filter, filter coefficients are such that seasonal component is eliminated and noise is dampened (i.e. the output process has lower variance than the original time series).

For a monthly data with period of seasonality 12, use a 12 point moving average to achieve the above

$d = 12 (= 2q)$

$(q = 6)$

$$\hat{m}_t = \frac{1}{12}\left(\frac{1}{2}Y_{t-6} + Y_{t-5} + \dots + Y_t + \dots + Y_{t+5} + \frac{1}{2}Y_{t+6}\right)$$

In general,

$$\hat{m}_t = \frac{1}{2q}\left(\frac{1}{2}Y_{t-q} + Y_{t-q+1} + \dots + Y_t + \dots + Y_{t+q-1} + \frac{1}{2}Y_{t+q}\right)$$

or

$$\hat{m}_t = \frac{1}{2q+1}\left(Y_{t-q} + \dots + Y_t + \dots + Y_{t+q}\right)$$

depending on the period of seasonality (even or odd)

Step 2 : Estimation of seasonal components.

For each month K $(K = 1, 2, \dots, 12)$, compute average (say W_K) of deviations

$$\{Y_{12(j-1)+K} - \hat{m}_{12(j-1)+K} \; ; \; j = 1, 2, \dots, J\}$$

over the J years.

[Basically, $W_K = \frac{1}{J}\sum_{j=1}^{J}\left(Y_{12(j-1)+K} - \hat{m}_{12(j-1)+K}\right)$]

Estimate S_k as;

$$\hat{S}_k = w_k - \frac{1}{12}\sum_{k=1}^{12} w_k . \qquad [\because \sum_{k}^{k} S_k = 0, \text{ now}]$$

$$\& \ \hat{S}_k = \hat{S}_{k-d} \quad \forall \ k > d$$

In general, $\hat{S}_k = w_k - \frac{1}{d}\sum_{k=1}^{d} w_k$.

<u>Remark</u>: Note that W_ks that we obtain here are similar to the estimates of S_ks obtained in method 1. However, the $\{w_k\}$ sequence that we have obtained here are not used as estimates of S_k under the current setup as $\sum_k w_k \neq 0$. With the centering, we have $\sum_{k=1}^{d} S_k = 0$

<u>Step 3</u>: <u>Deseasonalize the data</u>

$$d_t = Y_t - \hat{S}_t \quad ; t = 1, 2, \ldots, n$$

and get $(d_1, \ldots, d_n)$

<u>Step IV</u>

Re-estimate trend using $(d_1, d_2, \ldots, d_n)$ using any of the trend estimation methods

<u>Remark</u>: Iterate, if required.

<u>Method 3</u>: <u>Elimination of trend & seasonality</u>

Using differencing, we can eliminate both trend and seasonality from the data (if they are present)

Define, lag 'd' difference operator

$$\nabla_d Y_t = Y_t - Y_{t-d} = (1 - B^d) Y_t$$

Apply ∇_d to $Y_t = m_t + S_t + e_t$; $E(e_t) = 0$, $V(e_t) = \sigma^2 < \infty$

where d is the present period of seasonality, hence

$$\ldots = S_{t-d} = S_t = S_{t+d} = \ldots$$

$$Z_t = \nabla^d Y_t = \nabla^d m_t + \nabla^d S_t + \nabla^d e_t$$

$$Z_t = \underbrace{(m_t - m_{t-d})}_{\text{deterministic trend comp}} + (\cancel{S_t - S_{t-d}})^{0} + \underbrace{(e_t - e_{t-d})}_{\text{irregular comp.}}$$

[Trend $(m_t - m_{t-d})$ term in Z_t can be eliminated through differencing of appropriate powers of ∇ operator]

17/08/23

In Fact Trend method. $2q = 12 \Rightarrow q = 6$

$$\hat{m}_t = \frac{1}{12}\left(\frac{1}{2} Y_{t-6} + Y_{t-5} + \dots + Y_t + \dots + Y_{t+5} + \frac{1}{2} Y_{t+6}\right)$$

Property of Seasonality

$$\dots = S_{t-d} = S_t = S_{t+d} = \dots$$

$$\sum_{j=1}^{d} S_j = 0$$

$$Y_t = m_t + S_t + e_t$$

S_t :

$$\frac{1}{12}\left(\frac{1}{2} S_{t-6} + S_{t-5} + \dots + S_t + \dots + \frac{1}{2} S_{t+6}\right)$$

same.

$$= \frac{1}{12}\left(S_{t-5} + \cancel{\dots + S_{t+5}} + S_{t+6}\right) \quad 0$$

$\{X_t\}$ – Collection of random variables

T – Countable collection

c.d.f of any finite set of r.v.s from $\{X_t\}$

$(X_{t_1}, X_{t_2}, \dots, X_{t_n})$

$$F_{X_{t_1}, \dots, X_{t_n}}(x_1, \dots, x_n) = P(X_{t_1} \le x_1, \dots, X_{t_n} \le x_n)$$

* **Concept of Stationarity**

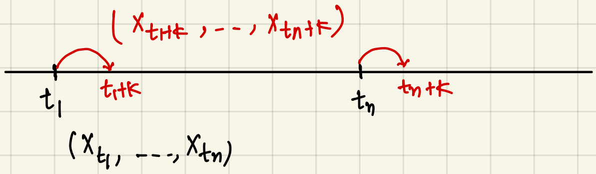
- "Statistical Properties" do not change overtime.
- "Steady State" of statistical equilibrium.

* **Forms of Stationarity**

(a) **Strict Stationarity :**

A time series $\{X_t\}$ is said to be strict stationary or completely stationary if for every n (integer) and any admissible $t_1, t_2, \dots, t_n$ and any integer k, the joint c.d.f of $(X_{t_1}, \dots, X_{t_n})$ is identical with the joint c.d.f of $(X_{t_1+k}, X_{t_2+k}, \dots, X_{t_n+k})$ i.e. $F_{X_{t_1}, \dots, X_{t_k}}(x_1, \dots, x_n) = F_{X_{t_1+k}, \dots, X_{t_n+k}}(x_1, \dots, x_n)$ $\forall k$ $\forall$ admissible $(t_1, t_2, \dots, t_n)$

\# Probability law is indifferent w.r.t time change.



(b) Stationarity upto order m (integer)

A time series $\{X_t\}$ is said to be stationary upto order m, if for any integer n, any admissible $t_1, t_2, ---, t_n$ and any integer k, we have the joint moments upto order m of $(X_{t_1}, --, X_{t_n})$ is identical to the corresponding joint moment of order upto m of $(X_{t_1+k}, ---, X_{t_n+k})$,

i.e. $E\left(X_{t_1}^{m_1} X_{t_2}^{m_2} --- X_{t_n}^{m_n}\right) = E\left(X_{t_1+k}^{m_1} X_{t_2+k}^{m_2} --- X_{t_n+k}^{m_n}\right) \ \forall k \ \forall m_1, --, m_n \ ; m_i \geq 0 ; m_i :$ integers & $\sum_{i=1}^{n} m_i \leq m$

Take $m_2 = m_3 = ... = m_n = 0$

$\Rightarrow E(X_t^{m_1}) = E(X_{t+k}^{m_1}) \ \forall k \ \forall m_1 \leq m$

take $k = -t$

$E(X_t^{m_1}) = E(X_0^{m_1}) \longrightarrow$ # independent of t.

Take $m_3 = m_4 = ... = m_n = 0$

$E(X_t^{m_1} X_s^{m_2}) = E(X_{t+k}^{m_1} X_{s+k}^{m_2}) \ \forall k \ \forall (m_1 \geq 0, m_2 \geq 0)$ s.t. $m_1 + m_2 \leq m$

take $k = -t$

$E(X_t^{m_1} X_s^{m_2}) = \underline{E(X_0^{m_1} X_{s-t}^{m_2})} \longrightarrow$ # depending on time difference of $s-t$

↑ f^n of $(s-t)$ only (neither t nor s)

* Special cases of order m stationarity

(i) Order 1 Stationary / Mean Stationary

$\Rightarrow E(X_t)$ exists and it is independent of t,

i.e. $E(X_t) = \mu \ \forall t$.

mean does not change

(ii) Order 2 Stationary

$\{X_t\}$ is s.t. $E(X_t^2) < \infty$, $E(X_t X_s) < \infty \ \forall t, s$ and

(i) $E(X_t) = \mu \ \forall t$, i.e. independent of t.

(ii) $E(X_t^2) = \mu_2' \ \forall t \Rightarrow V(X_t)$ is independent t

(iii) $E(X_t X_s) = E(X_{t+k} X_{s+k}) \ \forall k, \forall (t,s)$, choose $k = -t \Rightarrow E(X_0 X_{s-t}) = f^n$ of $(s-t)$ only, indep of s & t.

$\Rightarrow Cov(X_t, X_s) = E(X_t X_s) - E(X_t) E(X_s)$
$= f^n$ of $(s-t)$ only

Remark : Order 2 stationarity is also called COVARIANCE stationary or WEAK stationary or stationarity in wide sense.

Result : If $\{X_t\}$ is strict stationary and the second order joint moments of $\{X_t\}$ exists, then $\{X_t\}$ is covariance stationary.

Proof : $n=1$, def^n of strict stationarity.

$\Rightarrow F_{X_t}(x) = F_{X_{t+k}}(x) \quad \forall k$

$\Rightarrow X_t \overset{d}{=} X_{t+k} \quad \forall k$

$\Rightarrow E(X_t) = \mu \quad \forall t \quad$ — (i)

$\Rightarrow V(X_t) = \sigma^2 \quad \forall t \quad$ — (ii)

$n=2$ — def^n of strict stationarity : $F_{X_t, X_s}(x,y) = F_{X_{t+k}, X_{s+k}}(x,y) \quad \forall k$

$\Rightarrow (X_t, X_s) \overset{d}{=} (X_{t+k}, X_{s+k}) \quad \forall k$

$\Rightarrow Cov(X_t, X_s) = Cov(X_{t+k}, X_{s+k}) \quad \forall k$

$\Rightarrow Cov(X_t, X_s)$ is a f^n of $|s-t|$ only

$\Rightarrow \{X_t\}$ is also covariance stationary

Note : Converse of the previous result is not true, in general. 22/08/23

Counter example :

$\{X_t\}$ - time series $\ni$ $\{X_t\}$ is a sequence of independent r.v.

$$X_t \sim \begin{cases} \exp(1), & \text{if } t \text{ is odd.} \\ N(1,1), & \text{if } t \text{ is even.} \end{cases}$$

$X_1, X_2, X_3, X_4, \ldots$

- $E(X_t) = 1 \quad \forall t$
- $Cov(X_t, X_s) = \begin{cases} 0 ; & t \neq s \quad \text{\# Ind.} \\ 1 ; & t = s \end{cases}$

\# Covariance Stationary.

$\Rightarrow \{X_t\}$ is covariance stationary

Now, X_1 & X_2 have diff distributions,

$\Rightarrow \{X_t\}$ is not strict stationary.

[existence of strict stationary, does not always imply existence of weak stationarity, when moments does not exist. eg. Cauchy (iid) : condition of existence of moments is critical.]

Remark : For a special type of time series,

WEAK stationarity $\Longrightarrow$ STRICT stationarity ——— (*)

If $\{X_t\}$ is a Gaussian Process then (*) holds.

Gaussian Process : A time series $\{X_t\}$ is said to be Gaussian Process if for every n and every admissible $t_1, t_2, \ldots, t_n$; $(X_{t_1}, \ldots, X_{t_n})$ follows a multivariate normal distribution.

[admissible $t_1, t_2, \ldots, t_n \Rightarrow$ all the time points inside the time index defined]

Recall :

Def^n : A random vector $\underset{\sim}{X} = (X_1, X_2, \ldots, X_p)'$ with a mean vector $E(\underset{\sim}{X}) = \underset{\sim}{\mu} = \begin{pmatrix} E(X_1) \\ \vdots \\ E(X_n) \end{pmatrix}$ + a covariance matrix $\Sigma = E(\underset{\sim}{X} - \underset{\sim}{\mu})(\underset{\sim}{X} - \underset{\sim}{\mu})'$

$$\Sigma_{p \times p} = \begin{pmatrix} V(X_1) & Cov(X_1, X_2) & \cdots & Cov(X_1, X_p) \\ \vdots & V(X_2) & \cdots & \\ \vdots & & \ddots & \\ \vdots & \cdots & \cdots & V(X_p) \end{pmatrix}$$

is said to follow a multivariate normal distribution ($\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$) iff $\forall\ \underset{\sim}{\alpha} \in \mathbb{R}^p$ $(\underset{\sim}{\alpha} \neq \underset{\sim}{0})$;

$\underset{\sim}{\alpha}'\underset{\sim}{X} \sim N_1$ (univariate normal)

i.e. $\underset{\sim}{X} \sim N_p(\underset{\sim}{\mu}, \Sigma)$ iff $\underset{\sim}{\alpha}'\underset{\sim}{X} \sim N_1\ \forall\ \underset{\sim}{\alpha} \in \mathbb{R}^p$ $(\underset{\sim}{\alpha} \neq \underset{\sim}{0})$

Note: (1) $\underset{\sim}{X}_{p\times 1} \sim N_p(\underset{\sim}{\mu}, \Sigma) \Rightarrow$ every possible sub vector of $\underset{\sim}{X} \sim$ multivariate normal

(2) $\underset{\sim}{X}_{p\times 1} \sim N_p(\underset{\sim}{\mu}, \Sigma) \Rightarrow \underset{\sim}{X}_{p\times 1} \longrightarrow \underset{\sim}{Y}_{q\times 1} = A\underset{\sim}{X}_{p\times 1} + \underset{\sim}{b}_{q\times 1}$

$$\underset{\sim}{Y} \sim N_q(A\underset{\sim}{\mu} + \underset{\sim}{b},\ A\Sigma A')$$

(3) If $\underset{\sim}{X}_{p\times 1} \sim N_p(\underset{\sim}{\mu}, \Sigma)$ and if $\Sigma > 0$ (p.d.), then joint pdf of $\underset{\sim}{X}$ is

$$f_X(x) = \frac{1}{(2\pi)^{p/2}\, |\Sigma|^{1/2}} \exp\left(-\frac{1}{2} (\underset{\sim}{x}-\underset{\sim}{\mu})' \Sigma^{-1} (\underset{\sim}{x}-\underset{\sim}{\mu}) \right) ; \quad \underset{\sim}{x} \in \mathbb{R}^p$$

Result: Suppose $\{X_t\}$ is a Gaussian process and is covariance stationary, then $\{X_t\}$ is strict stationary.

Pf: $\{X_t\}$ is GP

$$\underset{\sim}{Y}_{n\times 1} = (X_{t_1}, \ldots, X_{t_n}) \sim N_n(E\underset{\sim}{Y}, \operatorname{Cov}(\underset{\sim}{Y}))$$

Since, $\{X_t\}$ is covariance stationary,

$$E(X_t) = \mu \quad \forall\ t$$

$$\Rightarrow E(\underset{\sim}{Y}) = \begin{pmatrix} EX_1 \\ \vdots \\ \vdots \\ EX_n \end{pmatrix} = \mu\, \underset{\sim}{1}_n = \underset{\sim}{\mu}$$

- $\operatorname{Cov}(X_{t_i}, X_{t_j}) = f^n$ of only $(t_j - t_i)$ $\forall\ t_j, t_i$; $V(X_{t_i}) = \sigma^2\ \forall\ t_i$

$$\Rightarrow \operatorname{Cov}(\underset{\sim}{Y}) = \underset{n\times n}{\Sigma} = \begin{pmatrix} \sigma^2 & \operatorname{Cov}(X_{t_1}, X_{t_2}) & \cdots & \operatorname{Cov}(X_{t_1}, X_{t_n}) \\ | & \sigma^2 & & | \\ | & & \ddots & | \\ | & \cdots & \cdots & \sigma^2 \end{pmatrix} = \begin{pmatrix} \sigma^2 & f^n(t_2-t_1) & f^n(t_3-t_1) & \cdots & f^n(t_n-t_1) \\ | & \sigma^2 & & & \vdots \\ | & & \ddots & & \vdots \\ | & \cdots & \cdots & \cdots & \sigma^2 \end{pmatrix}$$

$\underset{\sim}{Y} = (X_{t_1}, \ldots, X_{t_n}) \sim N_n(\underset{\sim}{\mu}, \Sigma)$

Consider,

$$\underset{\sim}{Z} = (X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_n+h})$$

$$GP = \underset{\sim}{Z} \sim N_n(\cdot, \cdot)$$

$$E(\underset{\sim}{Z}) = \underset{\sim}{\mu}$$

$$Cov(\underset{\sim}{Z}) = \begin{pmatrix} \sigma^2 & cov(X_{t_1+h}, X_{t_2+h}) & \cdots & cov(X_{t_1+h}, X_{t_n+h}) \\ \vdots & \sigma^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & \cdots & \cdots & \sigma^2 \end{pmatrix} = \begin{pmatrix} \sigma^2 & f^n(t_2+h-t_1-h) & \cdots & f^n(t_n+h-t_1-h) \\ \vdots & \sigma^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & \cdots & \cdots & \sigma^2 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma^2 & f^n(t_2-t_1) & \cdots & f^n(t_n-t_1) \\ \vdots & \sigma^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & \cdots & \cdots & \sigma^2 \end{pmatrix} = \Sigma_{n \times n}.$$

$\Rightarrow \quad \underset{\sim}{Z}_{n\times 1} \sim N_n(\underset{\sim}{\mu}, \Sigma)$

$\Rightarrow (X_{t_1}, \ldots, X_{t_n}) \overset{d}{=} (X_{t_1+h}, \ldots, X_{t_n+h})$

i.e. $\{X_t\}$ is STRICT stationary

(Only in case of GP such conclusion can be drawn)

Examples (Stationary / Non stationary Processes)

(I) $\{X_t\} \ni \{X_t\}$ is a sequence of i.i.d r.v.s.

$\Rightarrow$ $\{X_t\}$ is strict stationarity.

Can't say, anything about weak, since nothing is said about the moments of $\{X_t\}$.

If $\{X_t\}$ is seq of i.i.d r.v. with finite 2^{nd} order moments, then

$\Rightarrow \{X_t\} \rightarrow$ STRICT & WEAK.

(II)

Time series: $Y_t = \alpha + \beta t + \epsilon_t$ # linear trend

$\{\epsilon_t\}$ is seq. of i.i.d r.v with $E(\epsilon_t)=0,\ \forall t$, $var(\epsilon_t)=\sigma^2$; $Cov(\epsilon_t, \epsilon_s) = \begin{cases} 0; & t \neq s \\ \sigma^2; & t=s \end{cases}$

$$\Rightarrow Cov(Y_t, Y_s) = \begin{cases} V(Y_t) = \sigma^2 & ; t=s \\ Cov(Y_t, Y_s) = Cov(\epsilon_t, \epsilon_s) = 0 = f^n(s-t) & ; t \neq s \end{cases}$$

covariance does not change with t.

But $E(Y_t) = \alpha + \beta(t) \rightarrow$ depends on time.

$\Rightarrow \{Y_t\}$ is not covariance stationary.

[Always first look at Exp.]

(III) $Y_t = \alpha + \beta_t + \epsilon_t$

$\{\epsilon_t\}$ is IID R.V. with $E(\epsilon_t)=0$ and $cov(\epsilon_t, \epsilon_s) = \sigma^2 \delta(t=s)$

$$Cov(Y_t, Y_s) = \begin{cases} \sigma^2; & t=s \\ 0 & ; t \neq s \end{cases}$$

$E(Y_t) = \alpha + \beta t$

Example

$Y_t = A\cos\omega t + B\sin\omega t$

A and B are independent R.V. such that

$E(A) = E(B) = 0$

$Var(A) = Var(B) = \sigma^2$

$\omega \in (-\pi, \pi)$, is a fixed constant

$E(Y_t) = 0 \quad \forall\, t$

$$\begin{aligned} Cov(Y_{t+h}, Y_t) &= Cov(A\cos\omega(t+h) + B\sin\omega(t+h),\ A\cos\omega t + B\sin\omega t) \\ &= \sigma^2 (\cos\omega(t+h)\cos\omega t + \sin\omega(t+h)\sin\omega(t)) \\ &= \sigma^2 \cos\omega(t+h-t) \\ &= \sigma^2\cos\omega h \end{aligned}$$

Y_t is a function of h only.

Example

$$Y_t = \sum_{j=1}^{M} (A_j\cos(\omega_j t) + B_j\sin(\omega_j t))$$

$\{A_j\}_1^M$ collection independent $N(0, \sigma^2)$ and $\{A_j\}^M$ is independent of $\{B_j\}_1^M$

Further, $\omega \in (-\pi, \pi)$ are fixed.

$E(Y_t) = 0 \quad \forall\, t$

$$\begin{aligned} Cov(Y_{t+h}, Y_t) &= Cov\left(\sum_{j=1}^{M} A_j\cos w_j(t+h) + B_j\sin w_j(t+h),\ \sum_{j=1}^{M} A_j\cos w_j t + B_j\sin w_j t\right) \\ &= \sigma^2 \sum_{j=1}^{m} \cos(w_j(t+h))\cos(w_j t) + \sin(w_j(t+h))\sin(w_j t)) \\ &= \sigma^2 \sum_{j=1}^{m} \cos(w_j(t+h-t)) \\ &= \sigma^2 \sum_{j=1}^{M} \cos(w_j h) \end{aligned}$$

$\Rightarrow Y_t$ is covariance stationary.

Claim : $\{Y_t\}$ is a Gaussian Process

For any 'n' and any collection $t_1, t_2, ---, t_n$

$$\underset{\sim}{Z} = \begin{pmatrix} Y_{t_1} \\ Y_{t_2} \\ \vdots \\ Y_{t_n} \end{pmatrix}$$

$\forall \alpha \in \mathbb{R}^n$,

$$\alpha^T Z = \alpha_1 y_{t_1} + \alpha_2 y_{t_2} + --- + \alpha_n y_{t_n}$$

$$\alpha^T Z = \alpha_1 \left(\sum_{j=1}^{M} A_j \cos(w_j t_1) + B_j \sin(w_j t_1) \right) + ----$$

$$\alpha^T Z = \beta_1 A_1 + \beta_2 A_2 + --- + \beta_M A_M + \gamma_1 B_1 + \gamma_2 B_2 + --- + \gamma_M B_M$$

where β_i and γ_i are constants.

s.t. $\beta_1 = \alpha_1 \cos w t_1 + \alpha_2 \cos w t_2 + -- + \alpha_n \cos w t_n$

$\alpha^T Z$ is a linear combination of independent $N(0, \sigma^2)$

$\Rightarrow$ $\alpha^T Z$ is a univariate r.v.

$\Rightarrow$ $Z \sim$ Multivariate Normal

$\Rightarrow$ Y_t is a GP.

$$\underset{\sim}{Z} \sim MVN\left(\underset{\sim}{0}, \begin{pmatrix} M\sigma^2 & \sigma^2 \sum_{j=1}^{M} \cos w_j (t_2 - t_1) & \cdots \\ - & \ddots & \\ & - & M\sigma^2 \end{pmatrix}_{n \times n} \right)$$

$$\text{cov}(Y_{t_i}, Y_{t_k}) = \sigma^2 \sum_{j=1}^{M} \cos(w_j \underbrace{(t_i - t_j)}_{t_i - t_k})$$

Example

$Y_t = Z_t + \theta Z_{t-1}$

θ is a fixed constant
$\{Z_t\}$ are iid $E(Z_t) = 0 \ \forall t$ & $V(Z_t) = \sigma^2$

$\Rightarrow E(Y_t) = 0 \ \forall t$

$Cov(Y_{t+h}, Y_t) = Cov(Z_{t+h} + \theta Z_{t+h-1}, Z_t + \theta Z_{t-1})$

covariance is non-zero only when $h = -1, 0, 1$

$$Cov(Y_{t+h}, Y_t) = \begin{cases} \sigma^2(1+\theta^2) & ; h = 0 \\ \theta\sigma^2 & ; h = 1 \\ \theta\sigma^2 & ; h = -1 \\ 0 & ; o/w \end{cases}$$

$\Rightarrow Y_t$ is covariance stationary.

$\rightarrow$ $\{X_t\}$ is a seq of iid $(0, \sigma^2)$ R.V.

s.t. $S_t = \sum_{i=1}^{t} X_i$; $S_t = S_{t-1} + X_t$

$E(S_t) = 0 \ \forall t$
$V(S_t) = E(S_t - E(S_t))^2 = \sigma^2 t$

$Cov(S_{t+h}, S_t) = \sigma^2 t ; h > 0$ $\quad [Cov(\sum_{i=1}^{t+h} X_i, \sum_{i=1}^{t} X_t) = V(\sum_{i=1}^{t} X_i) = \sigma^2 t]$
$= \sigma^2 (t+h) ; h < 0$

Remark

(1) $\{X_t\}$ is covariance stationary.
$\rightarrow Y_t = \alpha X_t$
$\Rightarrow Y_t$ is covariance stationary.

$\rightarrow Y_t = \alpha + X_t$
$\Rightarrow Y_t$ is covariance stationary

$\rightarrow Y_t = \begin{cases} X_t & ; \text{odd} \\ \alpha X_t & ; \text{even} \end{cases}$ is not covariance stationary.

(2) $\{X_t\}$ and $\{Y_t\}$ are uncorrelated covariance stationary process, $Z_t = X_t + Y_t$ is covariance stationary.

* Complex Valued Time Series

$\{U_t$ & $V_t\}$ are real valued time series

$$X_t = U_t + iV_t \quad ; i = \sqrt{-1}$$

$$E(X_t) = E(U_t) + i\,E(V_t)$$

$$\text{Cov}(X_{t+h}, X_t) = E(X_{t+h} - EX_{t+h})^*(X_t - EX_t)$$

X_t is said to be covariance stationary if

EX_t is constant w.r.t t

and $\text{Cov}(X_{t+h}, X_t) = f^{\gamma}(h)$

$\rightarrow \quad Y_t = Ae^{i\omega t} = A\cos\omega t + iA\sin\omega t$

A is a random value s.t. $E(A)=0$, $V(A)=\sigma^2$
$\omega \in (-\pi, \pi)$

$E(Y_t) = 0$, $\text{Cov}(Y_{t+h}, Y_t) = E(Y_{t+h} Y_t)$

$$= E(A^2 e^{i\omega(t+h)} e^{i\omega t})$$

$$=$$

$$= \sigma^2 e^{-i\omega h}$$

This is covariance stationary.

* Auto Covariance Function (ACVF)

Def^n : Let $\{X_t\}$ be a covariance stationary time series, with $EX_t = \mu \ \forall t$. The ACVF of $\{X_t\}$ at lag h is defined as

$$\gamma_X(h) = Cov(X_{t+h}, X_t)$$ # function of h only

$$= E(X_{t+h}-\mu)(X_t-\mu)$$

$$h = 0, \pm 1, \pm 2, \ldots$$

$\{\gamma_X(h)\}$ - ACVF

Now, $\{X_t\}$ - covariance stationary, $E(X_t) = \mu \ \forall t$ & ACVF $\{\gamma_X(h)\}$

Properties :

(i) $\gamma_X(0) \geq 0$

$\quad \overset{\shortparallel}{V(X_t)}$

(ii) $\forall h$, $|\gamma_X(h)| \leq \gamma_X(0)$

Follows from Cauchy - Schwartz Inequality

$$|Cov(X_{t+h}, X_t)| = |E(X_{t+h}-\mu)(X_t-\mu)|$$

$$\overset{\text{c.s. ineq}}{\leq} \left[E(X_{t+h}-\mu)^2 E(X_t-\mu)^2\right]^{1/2}$$

$$\Rightarrow |\gamma_X(h)| \leq \gamma_X(0)$$

(iii) $\gamma_X(\cdot)$ is an even function

$$\gamma_X(h) = E(X_{t+h}-\mu)(X_t-\mu)$$

$$\overset{\text{cov stat}}{=} E(X_{t+h-h}-\mu)(X_{t-h}-\mu)$$ # Since it is a f^n of h only, not t, so translation will not affect.

$$= E(X_t-\mu)(X_{t-h}-\mu)$$

$$= E(X_{t-h}-\mu)(X_t-\mu)$$

$$= \gamma_X(-h)$$

$\forall h \quad \gamma_X(h) = \gamma_X(-h)$

(iv) $\gamma_X(\cdot)$ is a non-negative definite f^n

[**Def^n** : A real valued function f^n, f, defined on the set of integers is non-negative definite if,

$f: \mathbb{Z} \to \mathbb{R} \qquad \sum_{i,j=1}^{n} a_i f(t_i - t_j) a_j \geq 0 \quad \forall$ (+)ve integers n, $\forall \ \underset{\sim}{a} = (a_1, \ldots, a_n)' \in \mathbb{R}^n$

Proof:

$\forall\ \underset{\sim}{a} = (a_1, \ldots, a_n)' \in \mathbb{R}^n$

$\&\ \forall\ \underset{\sim}{t} = (t_1, \ldots, t_n)' \in \mathbb{Z}^n$

Consider,

$$\underset{\sim}{X}_{\underset{\sim}{t}} = (X_{t_1}, \ldots, X_{t_n})'$$

$$E X_{t_i} = \mu$$

$$0 \le V(\underset{\sim}{a}' \underset{\sim}{X}_{\underset{\sim}{t}})$$

$$V(\underset{\sim}{a}'\underset{\sim}{X}_{\underset{\sim}{t}}) = E(\underset{\sim}{a}'\underset{\sim}{X}_{\underset{\sim}{t}} - E(\underset{\sim}{a}'\underset{\sim}{X}_{\underset{\sim}{t}}))(\underset{\sim}{a}'\underset{\sim}{X}_{\underset{\sim}{t}} - E(\underset{\sim}{a}'\underset{\sim}{X}_{\underset{\sim}{t}}))'$$

$$= E(\underset{\sim}{a}'\underset{\sim}{X}_{\underset{\sim}{t}} - \underset{\sim}{a}'\underset{\sim}{\mu})(\underset{\sim}{a}'\underset{\sim}{X}_{\underset{\sim}{t}} - \underset{\sim}{a}'\underset{\sim}{\mu})'$$

$$= E(\underset{\sim}{a}'(\underset{\sim}{X}_{\underset{\sim}{t}} - \underset{\sim}{\mu}))(\underset{\sim}{a}'(\underset{\sim}{X}_{\underset{\sim}{t}} - \mu))'$$

$$= \underset{\sim}{a}'[E(\underset{\sim}{X}_{\underset{\sim}{t}} - \underset{\sim}{\mu})(\underset{\sim}{X}_{\underset{\sim}{t}} - \mu)']\underset{\sim}{a}$$

$$= \underset{\sim}{a}' \mu_n \underset{\sim}{a}$$

$\#\ E(\underset{\sim}{a}'\underset{\sim}{X}_{\underset{\sim}{t}}) = \underset{\sim}{a}' E(\underset{\sim}{X}_{\underset{\sim}{t}})$

$= \underset{\sim}{a}'\underset{\sim}{\mu}$

$$\mu_n = \operatorname{cov}(\underset{\sim}{X}_{\underset{\sim}{t}})$$

$$= \begin{pmatrix} \gamma_X(0) & \gamma_X(t_1-t_2) & \cdots & \gamma_X(t_1-t_n) \\ \vdots & \gamma_X(0) & & \\ \vdots & & \ddots & \\ \gamma_X(t_n-t_1) & \cdots & \cdots & \gamma_X(0) \end{pmatrix}_{n\times n}$$

$$\#\ \underset{\sim}{X}_{\underset{\sim}{t}} = \begin{pmatrix} X_{t_1} \\ X_{t_2} \\ \vdots \\ X_{t_n} \end{pmatrix}$$

i.e. $\underset{\sim}{a}'\mu_n \underset{\sim}{a} \ge 0 \quad \forall\ \underset{\sim}{a},\ \forall t$

i.e. $\sum_{i=1}^{n}\sum_{j=1}^{n} a_i \gamma_X(t_i - t_j) a_j \ge 0$

$\Rightarrow \gamma_X(\cdot)$ is a non-negative definite function.

\# Property (iii) & (iv) are characterizing properties of ACVF i.e.

Remark: If a real valued function defined on integers is even and non-negative def. (n.n.d), then it corresponds to ACVF of some covariance stationary time series.

Characterization of ACVF

A real valued function defined on the set of integers is ACVF of a covariance stationary time series iff it is even and non-negative definite (n.n.d)

* Auto Correlation Function (ACF)

we will see that ACVF and ACF have distinct features w.r.t different models, and helps as a distinguishing factor.

Def^n : $\{X_t\}$ is covariance stationary with $E(X_t) = \mu$ and $\gamma_X(h)$ ACVF,

ACF of $\{X_t\}$ at lag h is defined as:

$$\rho_X(h) = \text{Correl}^n(X_{t+h}, X_t) = \frac{\text{Cov}(X_{t+h}, X_t)}{[V(X_{t+h})\,V(X_t)]^{1/2}} = \frac{\gamma_X(h)}{\gamma_X(0)}$$

$h = 0, \pm 1, \pm 2, \ldots$

Properties

(i) $\rho_X(0) = 1$

(ii) $|\rho_X(h)| \leq 1 \quad \forall h$

(iii) $\rho_X(\cdot)$ is an even function

(iv) $\rho_X(\cdot)$ is non-negative definite (n.n.d) f^n

(v) If X_{t+h} & X_t are independent, $\rho_X(h) = 0$
$[\gamma_X(h) = 0]$

Remark : $\{X_t\}$ & $\{Y_t\}$ are uncorrelated covariance stationary with ACVF $\gamma_X(h)$ & $\gamma_Y(h)$ resp. then,

$Z_t = X_t + Y_t$ is also covariance stationary

$$\boxed{\gamma_Z(h) = \gamma_X(h) + \gamma_Y(h)}$$

ACF does not work like this.

Remark : $\{X_t\}$ is complex valued covariance stationary time series

ACVF,

$$\gamma_X(h) = E(X_{t+h} - \mu)^*(X_t - \mu)$$

$$h = 0, \pm 1, \pm 2, \ldots$$

n.n.d & even f^n properties break down here.

ACF, $\rho_X(h) = \dfrac{\gamma_X(h)}{\gamma_X(0)}$ # not real (except for $h=0$) / # real

* Estimation of μ and $\gamma_X(\cdot)$

$\{X_t\} \to$ covariance stationary with unknown $EX_t = \mu$ and unknown $\gamma_X(h)$ - ACVF

$\{X_1, \dots, X_n\}$ – random sample from $\{X_t\}$

random points from time scale are chosen, $X_1, \dots X_n$ are corresponding realizations, they are not independent but time dependent. Randomness comes from choosing random time points.

Estimation of μ :

Sample mean : $\bar{X}_n = \frac{1}{n} \sum_{t=1}^{n} X_t$

$\bar{X}_n$ is an estimator for μ.

$$E(\bar{X}_n) = \frac{1}{n} \cdot n \cdot \mu = \mu$$

$\Rightarrow \bar{X}_n$ is an unbiased estimator of μ.

30/08/23

Expression for $V(\bar{X}_n)$ [it is not $\frac{\sigma^2}{n}$!!]

$$V(\bar{X}_n) = V\left[\frac{1}{n} \sum_{t=1}^{n} X_t\right]$$

$$= E\left[\frac{1}{n} \sum_{t=1}^{n} X_t - \mu\right]^2$$

$$= E\left[\frac{1}{n} \sum_{t=1}^{n} (X_t - \mu)\right]^2$$

$$= \frac{1}{n^2} E\Big[(X_1-\mu) + \dots + (X_n-\mu)\Big]\Big[(X_1-\mu) + \dots + (X_n-\mu)\Big]$$

$$= \frac{1}{n^2}\Big[(\gamma_0 + \gamma_1 + \dots + \gamma_{n-1}) + (\gamma_1 + \gamma_0 + \gamma_1 + \dots + \gamma_{n-2}) + \dots + (\gamma_{n-1} + \gamma_{n-2} + \dots + \gamma_0)\Big]$$

$$= \frac{1}{n^2}\Big[n\gamma_0 + 2(n-1)\gamma_1 + 2(n-2)\gamma_2 + \dots + 2(n-(n-1))\gamma_{n-1}\Big]$$

$$= \frac{1}{n^2} \sum_{i-j=-n}^{n} (n - |i-j|)\, \gamma_{i-j}$$

i.e. $V(\bar{X}_n) = \frac{1}{n^2} \sum_{h=-n}^{n} (n - |h|)\gamma_h = \frac{1}{n} \sum_{|h|\le n} \left(1 - \frac{|h|}{n}\right)\gamma_h$

Estimation of $\gamma(\cdot)$ and $\rho(\cdot)$

Suppose μ is known, an unbiased estimator of $\gamma(h)$ is

$$\hat{\gamma}^*_\mu(h) = \frac{1}{n-h}\sum_{t=1}^{n-h}(X_t-\mu)(X_{t+h}-\mu) \quad ; h = 0(1)\,n-1$$

as
$$E(\hat{\gamma}^*_\mu(h)) = \frac{1}{n-h}\sum_{t=1}^{n-h} E(X_t-\mu)(X_{t+h}-\mu)$$
$$= \frac{n-h}{n-h}\gamma(h) = \gamma_h$$

An alternate estimator (difference in the divisor only)

$$\hat{\gamma}_\mu(h) = \frac{1}{n}\sum_{t=1}^{n-h}(X_t-\mu)(X_{t+h}-\mu)$$

$$E(\hat{\gamma}_\mu(h)) = \frac{n-h}{n}\gamma_h \neq \gamma_h$$

Bias: $E(\hat{\gamma}_\mu(h)) - \gamma_h = -\frac{h}{n}\gamma_h \to 0$ as $n\to\infty$

$\Rightarrow \hat{\gamma}_\mu(h)$ is unbiased in the limit, although it is a biased estimator.

Note

$$\hat{\mu}_n = \frac{1}{n}TT'$$

where $n\times 2n$ matrix T is given by

$$T = \begin{pmatrix} 0 & \cdots & 0 & Y_1 & Y_2 & \cdots & Y_n \\ 0 & \cdots & Y_1 & Y_2 & \cdots & Y_n & 0 \\ \vdots & & & & & & \vdots \\ 0 & Y_1 & \cdots & Y_n & 0 & \cdots & 0 \end{pmatrix}$$

Thus, $\forall\, \underset{\sim}{a} \in \mathbb{R}^n$; $\underset{\sim}{a}'\,\hat{\mu}_n\,\underset{\sim}{a} \geq 0$

$\Rightarrow \hat{\gamma}(h)$ f^n is n.n.d.

Note: $\hat{\gamma}^*(h)$ f^n is not n.n.d

Standard Models of Time Series

(1) White Noise Process

$$X_t = \varepsilon_t$$

$\{\varepsilon_t\}$: sequence of uncorrelated $(0, \sigma^2)$ random variables.

$$X_t = \varepsilon_t \sim WN(0, \sigma^2)$$

$E(X_t) = 0 \quad \forall t$

ACVF $\quad \gamma_X(h) = \begin{cases} \sigma^2 & ; h = 0 \\ 0 & ; o/w \end{cases}$

ACF $\quad \rho_X(h) = \begin{cases} 1 & ; h = 0 \\ 0 & ; o/w \end{cases}$

(2) Moving Average (MA) process

Suppose $\varepsilon_t \sim WN(0, \sigma^2)$

$\{X_t\}$ is M.A.(q) [q is positive integer] if

$$X_t = \theta_0 \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$$

$$\theta_0 \neq 0, \; \theta_q \neq 0$$

$\theta_0, \theta_1, \dots, \theta_q$ are unknown constants : MA parameters.

Note

W.L.O.G. θ_0 can be taken as 1

otherwise define, $\varepsilon_t' = \theta_0 \varepsilon_t \sim WN(0, \theta_0^2 \sigma^2)$

and $X_t = \varepsilon_t' + \left(\frac{\theta_1}{\theta_0}\right) \varepsilon_{t-1}' + \dots + \left(\frac{\theta_q}{\theta_0}\right) \varepsilon_{t-q}'$

alternate MA(q) representation.

Note : 2-sided MA representation

$$X_t = \sum_{j=-M}^{M} \theta_j \varepsilon_{t-j}.$$

(3) Auto Regressive (AR) Process

Suppose $\varepsilon_t \sim WN(0, \sigma^2)$

$\{X_t\}$ is $AR(p)$ if

$$X_t = \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + \varepsilon_t$$

$$\phi_p \neq 0 \quad , \text{Cov}(\varepsilon_t, X_{t-j}) = 0 \ \forall j > 0$$

$\phi_1, \dots, \phi_p$ are unknown constants : AR parameters.

Note :

W.L.O.G we can take a model without constant term for a stationary process.

Suppose, we take

$$X_t = \delta + \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + \varepsilon_t$$

Since, $\{X_t\}$ is stationary

$$\mu = E X_t = \delta + \phi_1 \mu + \dots + \phi_p \mu$$

$$\Rightarrow \mu(1 - \phi_1 - \phi_2 - \dots - \phi_p) = \delta.$$

- If $(1 - \phi_1 - \dots - \phi_p) = 0$, then $\delta = 0$
- If $(1 - \phi_1 - \dots - \phi_p) \neq 0$, then we can write

$$X_t = \delta + \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + \varepsilon_t$$

as

$$X_t - \mu = \delta + \phi(X_{t-1} - \mu) + \dots + \phi_p (X_{t-p} - \mu) + \varepsilon_t$$

Define, $Y_t = X_t - \mu$

$$Y_t = \phi_1 Y_{t-1} + \dots + \phi_p Y_{t-p} + \varepsilon_t$$

$\{X_t\}$ covariance stationary $\Rightarrow$ $\{Y_t\}$ is also covariance stationary.

with identical ACVF & ACF as $\{X_t\}$.

5/09/23

* MA (q)

$\epsilon_t \sim WN(0,\sigma^2)$

$X_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \dots + \theta_q \epsilon_{t-q}$

$Cov(\epsilon_t, X_{t-j}) = 0 \quad \forall j > 0$ # MA(q) is given, then this condition follows

it is the covariance b/w white noise at instant t, and time series at t-j i.e. some past value, so it is right also that it is 0.
[i.e. the white noise process is not correlated among its past values, ∵ it would also imply that WN is not random]

AR (p)

$x_t = \phi_1 x_{t-1} + \dots + \phi_p x_{t-p} + \epsilon_t$

$\phi_p \neq 0$, $Cov(\epsilon_t, x_{t-j}) = 0 \quad \forall j > 0$ # AR(p) this needs to be specified.

Special case of AR(p) : Random walk

AR(1) : $X_t = X_{t-1} + \epsilon_t$; $\epsilon_t \sim WN(0,\sigma^2)$

Initialization : $X_0 = 0$

$X_1 = \epsilon_1$
$X_2 = X_1 + \epsilon_2 = \epsilon_1 + \epsilon_2$
$\vdots$
$X_t = \epsilon_1 + \dots + \epsilon_t = \sum_{i=1}^{t} \epsilon_i$

$V(X_t) = t\sigma^2 \Rightarrow \{X_t\}$ is not stationary.

AR processes are not always stationary.

* Auto Regressive Moving Average (ARMA)

$\epsilon_t \sim WN(0,\sigma^2)$

ARMA (p,q)

$X_t = \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \dots + \theta_q \epsilon_{t-q}$.

$\phi_p \neq 0$, $\theta_q \neq 0$; $Cov(\epsilon_t, X_{t-j}) = 0 \quad \forall j > 0$

→ $\phi_1, \dots, \phi_p$; AR parameters of ARMA (p,q)
$\theta_1, \dots, \theta_q$; MA parameters of ARMA (p,q)

Time Domain Properties of the models.

(I) $WN(0,\sigma^2)$

→ ACVF : $\gamma_x(h) = \begin{cases} \sigma^2 ; & h=0 \\ 0 ; & h \neq 0 \end{cases}$

Always Stationary

[If variance depends on time, then not stationary. If variance is not finite eg. cauchy then also not covariance stationary]

(II) MA(q)

MA(1) $\quad X_t = \epsilon_t + \theta_1 \epsilon_{t-1} \; ; \epsilon_t \sim WN(0,\sigma^2)$

$E(X_t) = 0 \quad \forall\, t$

ACVF : $\gamma_x(h) = Cov(X_{t+h}, X_t) = Cov(\epsilon_{t+h} + \theta_1 \epsilon_{t+h-1}, \epsilon_t + \theta_1 \epsilon_{t-1})$

$$= \begin{cases} \sigma^2(1+\theta^2) & ; h=0 \\ \theta_1 \sigma^2 & ; h = \pm 1 \\ 0 & ; o/w \end{cases}$$

f^n of h only $\Rightarrow$ $\{X_t\}$ is always $(\forall \theta)$ covariance stationary.

ACF : $\rho_x(h) = \dfrac{\gamma_x(h)}{\sqrt{V(X_{t+h})}\sqrt{V(X_t)}} = \begin{cases} 1 & ; h=0 \\ \dfrac{\theta_1\sigma^2}{\sigma^2(1+\theta^2)} = \dfrac{\theta}{1+\theta^2} & ; h = \pm 1 \\ 0 & ; o/w \end{cases}$

Note

(1) $\{X_t\}$ is always $(\forall \theta)$ covariance Stationary.

(2) $X_t = \epsilon_t + \theta \epsilon_{t-1} \rightarrow \rho_x(1) = \dfrac{\theta}{1+\theta^2}$

$X_t = \epsilon_t + \dfrac{1}{\theta} \epsilon_{t-1} \rightarrow \rho_x(1) = \dfrac{\theta}{1+\theta^2}$

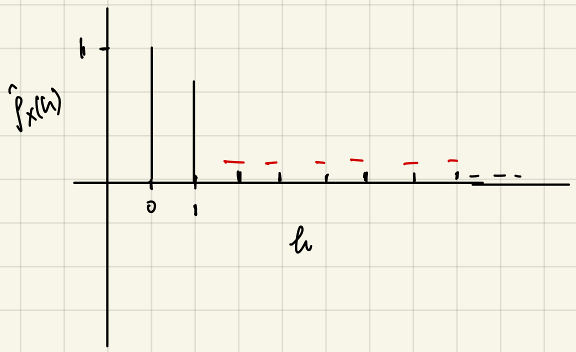
two different models have same properties / identical covariance structure for MA(1) procedure.

$\max\limits_{\theta} \; \rho_x(1) = \dfrac{1}{2}$ at $\theta = 1$

$\min\limits_{\theta} \; \rho_x(1) = \dfrac{-1}{2}$ at $\theta = -1$

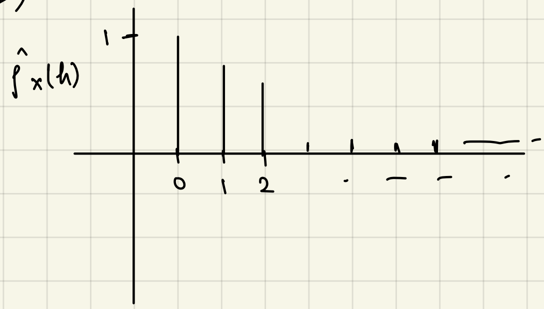
for every value of $\rho_x(h)$ between $\frac{-1}{2}$ and $\frac{1}{2}$, we have two MA(1) models.

if MA (1) is the best possible fit for the data then the graph would look like:



we generally define a band, s.t. if the value is less than the band then it is not significantly from zero.

MA(2)



Now, general

MA (q) :

$$X_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \dots + \theta_q \epsilon_{t-q}$$

$$\theta_q \neq 0 \quad ; \epsilon_t \sim WN(0, \sigma^2)$$

$$E X_t = 0 \ \forall t$$

$$\mathrm{Cov}(X_{t+h}, X_t) = \mathrm{cov}\left(\theta_0 \epsilon_{t+h} + \theta_1 \epsilon_{t+h-1} + \dots + \theta_q \epsilon_{t+h-q}, \ \theta_0 \epsilon_t + \theta_1 \epsilon_{t-1} + \dots + \theta_q \epsilon_{t-q}\right) \qquad \# \ \theta_0 = 1$$

$$\forall \ |h| > q \ ; \ \mathrm{Cov}(X_{t+h}, X_t) = 0$$

$$0 \le h \le q$$

$$\gamma_X(1) = \sigma^2(\theta_0\theta_1 + \theta_1\theta_2 + \dots + \theta_{q-1}\theta_q)$$

$$\gamma_X(h) = \sigma^2(\theta_0\theta_h + \dots + \theta_{q-h}\theta_q)$$

$$= \sigma^2 \sum_{h=0}^{q-h} \theta_j \theta_{j+h}$$

$\because$ it is ACVF : $\gamma_X(h) = \gamma_X(-h)$

$$\text{ACVF} : \ \gamma_X(h) = \begin{cases} \sigma^2\left(1 + \sum_{j=1}^{q} \theta_j^2\right) & ; h = 0 \\ \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|} & ; |h| \le q \\ 0 & ; o/w \end{cases}$$

ACF ;

$$\rho_X(h) = \begin{cases} 1 & ; h=0 \\ \left(\dfrac{\sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|}}{1+\sum_{j=1}^{q} \theta_j^2}\right) & ; |h| \le q \\ 0 & ; o/w. \end{cases}$$

the ACVF and ACF, hence become a model identification methods. [Plot and see]

Note

(1) MA processes of finite orders are always covariance stationary.

(2) ACF/ACVFs beyond the order of MA processes are all 0s., it is referred to as cut off property of MA processes.

MA (∞)

$\epsilon_t \sim WN(0, \sigma^2)$

$X_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$ # infinite no. of lags.

$\{\psi_j\}_{j=0}^{\infty} \to$ seq. of real no..

$\to V(X_t) = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 < \infty$; $EX_t = 0 \ \forall t$

(i) if $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ then we have MA(∞) to be covariance stationary.

(ii) if $\sum_{j=0}^{\infty} |\psi_j| < \infty$ # stricter cond^n and also implies it is covariance stationary.

If (i) & (ii) holds then $\{X_t\}$ is covariance stationary.

If $\{X_t\}$ is covariance stationary, then $h \geq 0$

$$\gamma_X(h) = \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$$

* Lag Operator Representation of MA(q)

$\epsilon_t \sim WN(0, \sigma^2)$; $x_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \dots + \theta_q \epsilon_{t-q}$

$$\boxed{X_t = \theta(B)\epsilon_t}$$

$\theta(B) = (1 + \theta_1 B + \theta_2 B^2 + \dots + \theta_q B^q)$

$\theta(B) \Rightarrow$ MA polynomial

* AR processes

AR(1) : [or MARKOV PROCESS]

$\epsilon_t \sim WN(0, \sigma^2)$

$$x_t = \phi x_{t-1} + \epsilon_t$$

such processes are not always covariance stationary. in such a way we have the same variable as we are modelling.

→ $\phi = 1 \Rightarrow$ Random walk model, which is not covariance stationary.

So, we consider difference eqⁿ s.t.

$$(1 - \phi B) x_t = \epsilon_t \quad ; \phi(B) x_t = \epsilon_t$$

$|\phi| > 1$ − series explodes

$$x_t = \phi x_{t-1} + \epsilon_t$$
$$= \phi(\phi x_{t-2} + \epsilon_{t-2}) + \epsilon_t$$
$$= \phi^2 x_{t-2} + \phi \epsilon_{t-1} + \phi \epsilon_t$$
$$\vdots$$
$$x_t = \phi^t x_0 + \phi^{t-1}\epsilon_1 + \dots + \epsilon_t$$

∵ it is ϕ^t, if $\phi > 1$ will be very high, if it is negative and less than −1, it will fluctuate b/w +ve & −ve.

$\Rightarrow |\phi| > 1$; X_t explodes $\Rightarrow X_t$ can't be stationary.

∴ We rule out regions $|\phi| > 1$ and $\phi = 1$.

Region of stationarity : $|\phi|<1$

- If $\{X_t\}$ is covariance stationary

$$X_t = \phi X_{t-1} + \epsilon_t$$

$$\sigma_x^2 = V(X_t) = V(\phi X_{t-1} + \epsilon_t)$$

$$= \phi^2 V(X_{t-1}) + V(\epsilon_t)$$

$$= \phi^2 \sigma_x^2 + \sigma^2$$

$cov(\epsilon_t, X_{t-j}) = 0 \ \forall j>0$

$$\Rightarrow \sigma_x^2 = \frac{\sigma^2}{1-\phi^2}$$

$\Rightarrow |\phi|<1$ is a valid region

Alternate formulation of stationarity

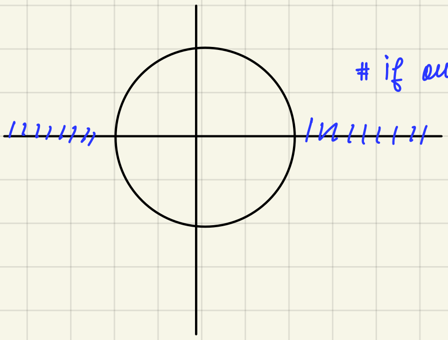
$$\phi(B) X_t = \epsilon_t$$

Root of $\phi(z)=0$ .i.e. $1-\phi z = 0$

$$\Rightarrow z = \frac{1}{\phi}$$

$|\phi|<1 \iff \underbrace{\text{root of } \phi(z)=0 \ > 1}$

root is outside the circle $\Rightarrow$ covariance stationary



if outside the unit circle in complex plane then stationarity is satisfied : $|\phi|<1$, if inside then not stationary.

similary for higher order AR processes, the no of roots (real or complex) should lie outside the unit circle.

$\{x_t\}$ – covariance stationary AR(1)

$$x_t = \phi x_{t-1} + \epsilon_t$$

$$\mu_x = E(x_t) = \phi \mu_x + 0$$

i.e. $(1-\phi)\mu_x = 0$

$1-\phi \neq 0$ ($\because$ cov stationary)

$\Rightarrow \mu_x = 0$

$$x_t = \delta + \phi x_{t-1} + \epsilon_t$$
$$\mu_x = \delta + \phi \mu_x$$
$$(1-\phi)\mu_x = \delta$$
$$\mu_x = \frac{\delta}{1-\phi}$$

<u>ACVF</u>

$$\gamma_1 = Cov(x_{t+1}, x_t)$$
$$= E(x_{t+1} \cdot x_t)$$
$$= E(\phi x_t + \epsilon_{t+1}) x_t$$

$$\left[E(x_t \epsilon_{t+1}) = Cov(x_t, \epsilon_{t+1}) = 0 \right] \quad \because Cov(\epsilon_t, x_{t-j}) = 0 \ \forall j > 0$$
$\because$ (ϵ_t at time point t & x at a past time point)

$$\boxed{\gamma_1 = \phi \frac{\sigma^2}{1-\phi^2}} = \gamma_{-1}$$

$$\gamma_2 = E(x_{t+2} \cdot x_t)$$
$$= E(\phi x_{t+1} + \epsilon_{t+2}) x_t$$
$$= \phi \gamma_1 = \phi^2 \frac{\sigma^2}{1-\phi^2}$$

unlike MA(1), they are not zero

$$\boxed{\gamma_2 = \phi^2 \frac{\sigma^2}{1-\phi^2}} = \gamma_{-2}$$

it is not going to die down:

<u>$h>0$</u>

$$x_{t+h} = \phi x_{t+h-1} + \epsilon_{t+h}$$
$$= \phi(\phi x_{t+h-2} + \epsilon_{t+h-1}) + \epsilon_{t+h}$$
$$= \vdots$$
$$= \phi^h x_t + \phi^{h-1}\epsilon_{t+1} + \dots + \epsilon_{t+h}$$

$$\gamma_h = E(x_{t+h} \cdot x_t)$$
$$= E(\phi^h x_t + \phi^{h-1}\epsilon_{t+1} + \dots + \epsilon_{t+h}) x_t$$

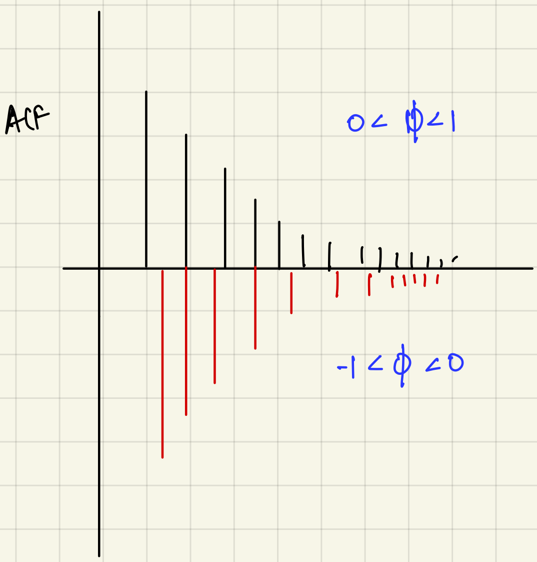
$$\boxed{\gamma_h = \phi^h \frac{\sigma^2}{1-\phi^2}} = \gamma_{-h}$$

$\Rightarrow$ ACVF : $\boxed{\gamma_h = \phi^{|h|} \frac{\sigma^2}{1-\phi^2}}$

$\Rightarrow$ ACF $= \rho_h = \frac{\gamma_h}{\gamma_0}$

$\Rightarrow \boxed{ACF = \rho_h = \phi^{|h|}}$

for AR(1) remember $|\phi| < 1$, so it decreases



cut off doesn't happen, but keeps on decreasing i.e. dies down.

* <u>Yule - Walker eq^n</u>

$\forall\ h > 0$

$\gamma_h = Cov(X_{t+h}, X_t)$

$= Cov(\phi X_{t+h-1} + \epsilon_{t+h}, X_t)$

Y-W eq^n $\boxed{\gamma_h = \phi \gamma_{h-1}}$ # same as the data equation ; $X_t = \phi X_{t-1} + \epsilon_t$ keeping aside the noise.

AR(2) [or Yule Process]

$\epsilon_t \sim WN(0, \sigma^2)$

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \epsilon_t$$

$$\phi(B) x_t = \epsilon_t$$

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2$$

Condition for stationarity

AR(2) is stationary if the roots of $\phi(z) = 0$ all lie outside the unit circle.

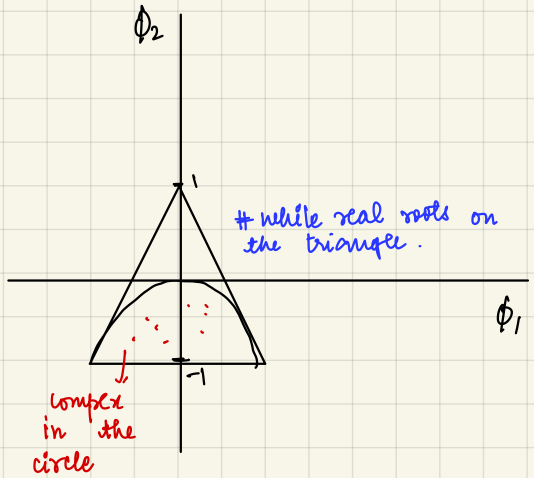
.i.e. roots of $(1 - \phi_1 z - \phi_2 z^2 = 0)$ all lie outside the unit circle.

[# in case of complex roots unit circle makes sense, in terms of real roots the roots should be strictly greater than 1 on axis.]

Region of stationarity in $\phi_1 - \phi_2$ plane

$$1 - \phi_1 - \phi_2 > 0$$
$$1 - \phi_2 + \phi_1 > 0$$

$$\Rightarrow |\phi_2| < 1$$



see notes

AR(2) :

12/09/23

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$$

$$Cov(\epsilon_t, X_{t-j}) = 0 \quad \forall j>0 \quad ; \epsilon_t \sim WN(0,\sigma^2)$$

If $\{X_t\}$ is covariance stationary,

$$\mu_x = E(X_t) = \phi_1 \mu_x + \phi_2 \mu_x$$

$$\Rightarrow (1-\phi_1-\phi_2)\mu_x = 0$$

$$1-\phi_1-\phi_2 \neq 0 \quad (\because \text{covariance stationary})$$

$$\Rightarrow \mu_x = 0$$

$\phi(B) = 1-\phi_1 B - \phi_2 B^2$

$\phi(z)=0$ has roots outside the unit circle

$\Rightarrow \phi(1) \neq 0$

$$V(X_t) = \sigma_x^2 = \phi_1^2\sigma_x^2 + \phi_2^2\sigma_x^2 + \sigma^2 + 2\phi_1\phi_2\gamma_1$$

i.e. $\sigma_x^2 = \phi_1^2\sigma_x^2 + \phi_2^2\sigma_x^2 + \sigma^2 + 2\rho_1\sigma_x^2\phi_1\phi_2$

$\rho_1 = corr(X_{t+1}, X_t) = \dfrac{\gamma_1}{\sigma_x^2}$

$$\Rightarrow \sigma_x^2 = \frac{\sigma^2}{(1-\phi_1^2-\phi_2^2-2\rho_1\phi_1\phi_2)}$$

$$\gamma_1 = Cov(X_{t+1}, X_t)$$
$$= E(\phi_1 X_t + \phi_2 X_{t-1} + \epsilon_{t+1})X_t$$
$$= \phi_1\gamma_0 + \phi_2\gamma_1 + 0$$

$$\Rightarrow \rho_1 = \phi_1 + \phi_2\rho_1$$

$$\Rightarrow \rho_1 = \frac{\phi_1}{(1-\phi_2)}$$

$$\Rightarrow \sigma_x^2 = \frac{\sigma^2}{\left(1-\phi_1^2-\phi_2^2-2\phi_1\phi_2\left(\frac{\phi_1}{1-\phi_2}\right)\right)} = \frac{\sigma^2(1-\phi_2)}{(1-\phi_1-\phi_2)(1-\phi_2+\phi_1)(1+\phi_2)}$$

* **Yule Walker eqn**

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$$

$\forall h>0$; $\gamma_h = Cov(X_{t+h}, X_t)$

$$= Cov[\phi_1 X_{t+h-1} + \phi_2 X_{t+h-2} + \epsilon_{t+h}, X_t]$$

$$\boxed{\gamma_h = \phi\gamma_{h-1} + \phi_2\gamma_{h-2}.} \quad \text{———} \quad (*)$$

for $h=0$, $Cov(X_t, \epsilon_t) = \sigma^2$

$$\boxed{\gamma_0 = \phi_1\gamma_{-1} + \phi_2\gamma_{-2} + \sigma^2}$$

γ_1 and γ_2 from Y-W add σ^2, we will get σ_x^2 only.

(*) is the Y-W eq^n

Y-W eq^n for ACF sequence

$\forall\, h>0$, $\rho_h = \phi_1 \rho_{h-1} + \phi_2 \rho_{h-2}$.

$h=1$; $\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_1$

$h=2$; $\rho_2 = \phi_1 \rho_1 + \phi_2 \rho_0$

$$\Rightarrow \boxed{\rho_1 = \frac{\phi_1}{1-\phi_2}}$$

$$\rho_2 = \phi_1 \left(\frac{\phi_1}{1-\phi_2}\right) + \phi_2 \rho_0$$

$\rho_h = \frac{\gamma_h}{\gamma_0}$

$$\rho_2 = \phi_1 \left(\frac{\phi_1}{1-\phi_2}\right) + \phi_2 \frac{\gamma_0}{\gamma_0}$$

$$\boxed{\rho_2 = \phi_1 \left(\frac{\phi_1}{1-\phi_2}\right) + \phi_2}$$

$h=3$, $\rho_3 = \phi_1 \rho_2 + \phi_2 \rho_1$

$$= \phi_1 \left(\phi_1 \left(\frac{\phi_1}{1-\phi_2}\right) + \phi_2\right) + \phi_2 \left(\frac{\phi_1}{1-\phi_2}\right)$$

If we see for ρ_1, ρ_2, we have.

$$\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$h=1$, $h=2$ $\Rightarrow$ $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}$

$(Y_1, Y_2, \ldots, Y_n)$
we can get $(\hat{\gamma}_0, \hat{\gamma}_1, \hat{\gamma}_2, \ldots)$ and hence $(\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_n)$ then we can plug it in and get model parameters.

$h=2$, $h=3$ $\Rightarrow$ $\rho_2 = \phi_1 \rho_1 + \phi_2$

$\rho_3 = \phi_1 \rho_2 + \phi_2 \rho_1$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \rho_1 & 1 \\ \rho_2 & \rho_1 \end{pmatrix}^{-1} \begin{pmatrix} \rho_2 \\ \rho_3 \end{pmatrix}$$

for different Y-W eq^n, we will get different possible estimates.

AR(p)

$$X_t = \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + \epsilon_t$$

$$\epsilon_t \sim WN(0, \sigma^2) \; ; \; \text{cov}(X_{t-j}, \epsilon_t) = 0 \quad \forall j > 0$$

$$\phi(B) X_t = \epsilon_t$$

AR polynomial : $\phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p$

* Condition for stationarity

$\{X_t\}$ is covariance stationary if all the roots of $\phi(z) = 0$ i.e. $1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p = 0$ lie outside the unit circle.
i.e. all roots $y^p + \phi_1 y^{p-1} - \dots - \phi_p = 0$ lie inside the unit circle. $\left(y = \frac{1}{z}\right)$

$$E X_t = 0$$

Yule - Walker eq^n

$$\gamma_h = \text{cov}(X_{t+h}, X_t)$$

$\forall h > 0$

$$= \text{Cov}(\phi_1 X_{t+h-1} + \phi_2 X_{t+h-2} + \dots + \phi_p X_{t+h-p} + \epsilon_{t+h}, X_t)$$

$$\gamma_h = \phi_1 \gamma_{h-1} + \phi_2 \gamma_{h-2} + \dots + \phi_p \gamma_{h-p} \quad \text{—— (*) Y-W}$$

for $h = 0$

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \dots + \phi_p \gamma_p + \sigma^2 \quad ; \quad \text{cov}(X_t, \epsilon_t) = \sigma^2$$

Y-W eq^n for ACF seq

$$\rho_h = \phi_1 \rho_{h-1} + \dots + \phi_p \rho_{h-p}$$

$h=1$ $\quad \rho_1 = \phi_1 \rho_0 + \phi_2 \rho_1 + \dots + \phi_p \rho_{p-1}$

$\vdots$

$h=p$ $\quad \rho_p = \phi_1 \rho_{p-1} + \dots + \phi_p \rho_0$

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_p \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \rho_1 & \cdots & \rho_{p-1} \\ \vdots & \ddots & & \vdots \\ & & \ddots & \vdots \\ \rho_{p-1} & \cdots & \cdots & 1 \end{pmatrix}}_{A_p} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_p \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_p \end{pmatrix} = A_p^{-1} \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_p \end{pmatrix}$$

take any set of p equations, the estimators will be different.

since, it gets 0 after the lag points (cut off)

* MA can be recognised by ACVF graph, AR can not since all kind of go so diff to ~~of~~ figure out order.

* ARMA (p, q)

$$\epsilon_t \sim WN(0, \sigma^2)$$

$$X_t = \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \dots + \theta_q \epsilon_{t-q}$$

$$\operatorname{Cov}(\epsilon_t, X_{t-j}) = 0 \ \forall j > 0$$

$$\underset{\substack{\uparrow \\ \text{AR poly.}}}{\phi(B)} X_t = \underset{\substack{\uparrow \\ \text{MA poly.}}}{\theta(B)} \epsilon_t$$

* Condition for stationarity

$\{X_t\}$ is covariance stationary, if the roots of $\phi(z) = 0$ i.e. $1 - \phi_1(z) - \dots - \phi_p z^p = 0$ all lie outside the unit circle.

$\Leftrightarrow$ roots of $y^p - \phi_1 y^{p-1} - \dots - \phi_p = 0$ all lie inside the unit circle.

$E X_t = 0.$