(1) $f(\lambda) = \begin{cases} 100, & \pi/6 - 0.01 \le \lambda \le \pi/6 + 0.01 \\ 0, & o/w \end{cases}$

$f(\lambda) = f(-\lambda)$ on $[-\pi, 0]$

$$\gamma_X(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\, d\lambda$$

$$\gamma_X(0) = \int_{-\pi/6 - 0.01}^{-\pi/6 + 0.01} 100\, d\lambda + 100 \int_{\pi/6 - 0.01}^{\pi/6 + 0.01} d\lambda = ----$$

$$\gamma_X(1) = 100 \int_{-\pi/6 - 0.01}^{-\pi/6 + 0.01} e^{i\lambda}\, d\lambda + 100 \int_{\pi/6 - 0.01}^{\pi/6 + 0.01} e^{i\lambda}\, d\lambda = ----.$$

(2) $f(\lambda) = \lambda^2; \quad \lambda \in [-\pi, \pi]$

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\, d\lambda$$

$$\gamma(0) = \int_{-\pi}^{\pi} \lambda^2\, d\lambda = \frac{2\pi^3}{3}$$

$h \neq 0$ $$\gamma(h) = \int_{-\pi}^{\pi} \lambda^2 e^{ih\lambda}\, d\lambda$$

$$= \lambda^2 \frac{e^{ih\lambda}}{ih} - \int 2\lambda \frac{e^{ih\lambda}}{ih}\, d\lambda$$

$\vdots$

(3) $Z_t = X_t + Y_t$

$$\gamma_Z(h) = \gamma_X(h) + \gamma_Y(h)$$

$$f_Z(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma_Z(h)$$

$$= \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\left(\gamma_X(h)+\gamma_Y(h)\right)$$

$$= f_X(\lambda)+f_Y(\lambda).$$

(4). $X_t = \phi X_{t-1} + \epsilon_t \; ; \epsilon_t \sim WN(0,\sigma^2) \quad |\phi|<1$

$$Y_t = \frac{1}{3}\left(X_{t-1}+X_t+X_{t+1}\right)$$

$$= \sum_{-1}^{1} a_j X_{t-j} \; ; \quad a_{-1}=a_0=a_1=\frac{1}{3}$$

$$f_Y(\lambda) = f_X(\lambda)\left(\sum_{-1}^{1}\frac{1}{3}e^{ij\lambda}\right)\left(\frac{1}{3}\sum_{-1}^{1}e^{-ij\lambda}\right)$$

$$= f_X(\lambda)\left(\frac{1}{9}\left(e^{-i\lambda}+1+e^{i\lambda}\right)\left(e^{-i\lambda}+1+e^{i\lambda}\right)\right)$$

$$= \frac{1}{9}f_X(\lambda)\left(1+2\cos\lambda\right)^2$$

$$f_X(\lambda) = \frac{\sigma^2}{2\pi}\frac{1}{(1-\phi e^{i\lambda})(1-\phi e^{-i\lambda})} = \frac{\sigma^2}{2\pi}\cdot\frac{1}{1+\phi^2-2\phi\cos\lambda}$$

$$f_Y(\lambda) = \frac{\sigma^2}{18\pi}\frac{(1+2\cos\lambda)^2}{(1+\phi^2-2\phi\cos\lambda)}.$$

(5) $Z_t \sim WN(0,1)$ $\qquad \alpha_1=\frac{1}{8}, \alpha_2=\frac{3}{4}, \alpha_3=\frac{3}{2}; \alpha_4=1$

$$X_t = \sum_{j=1}^{4}\alpha_j Z_{t-j+1}Z_{t-j} \; ;$$

$$X_t = \alpha_1 Z_t Z_{t-1} + \alpha_2 Z_{t-1}Z_{t-2} + \alpha_3 Z_{t-2}Z_{t-3} + \alpha_4 Z_{t-3}Z_{t-4}$$

$$= \alpha_1 U_t + \alpha_2 U_{t-1} + \alpha_3 U_{t-2} + \alpha_4 U_{t-3}$$

$U_t = Z_t Z_{t-1}$; $E U_t = 0 \quad E U_t^2 = 1$

$E U_t U_s = 0 \;\; \forall t \neq s \qquad U_t \sim WN(0,1)$

$$X(t) = \theta(B)\,U_t \;;\quad \theta(B) = (\alpha_1 + \alpha_2 B + \alpha_3 B^2 + \alpha_4 B^3)$$

$$f_X(\lambda) = f_U(\lambda)\Big(\sum_{j=-\infty}^{\infty} \alpha_j e^{ij\lambda}\Big)\Big(\sum_{j=-\infty}^{\infty} \alpha_j e^{-ij\lambda}\Big).$$

$$= \frac{\sigma^2}{2\pi}\, \theta(e^{i\lambda})\, \theta(e^{-i\lambda}).$$

(6) $$f_X(\lambda) = \frac{\sigma^2}{2\pi}\left(\theta_0 + \theta_1 e^{-i\lambda} + \theta_2 e^{-2i\lambda} + \cdots + \theta_q e^{-qi\lambda}\right)$$

$$\left(\theta_0 + \theta_1 e^{i\lambda} + \theta_2 e^{2i\lambda} + \cdots + \theta_q e^{qi\lambda}\right)$$

$$= \frac{\sigma^2}{2\pi}\Big[\overbrace{(\theta_0^2 + \cdots + \theta_q^2)}^{\gamma_0}$$

$$+ e^{i\lambda}\overbrace{(\theta_0\theta_1 + \cdots + \theta_{q-1}\theta_q)}^{\gamma(-1)}$$

$$+ e^{-i\lambda}\underbrace{(\theta_0\theta_1 + \cdots + \theta_{q-1}\theta_q)}_{\gamma(1)}$$

$$\vdots$$

$$+ \underbrace{\theta_0\theta_q e^{qi\lambda}}_{\gamma(-q)} + \underbrace{\theta_0\theta_q e^{-qi\lambda}}_{\gamma(q)}\Big]$$

$$= \frac{1}{2\pi}\sum_{h=-q}^{q} e^{-i\lambda h}\gamma(h)$$

$$\gamma(K) = \begin{cases} \sigma^2(\theta_0\theta_K + \cdots + \theta_{q-K}\theta_q); & K = 0, \pm 1, \ldots, \pm q \\ 0, & o/w. \end{cases}$$

(7) An absolutely summable $\gamma(.)$ is ACVF iff it is even and

$$f(\lambda) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h) \geq 0 \quad \forall \lambda$$

$$\gamma(h) = \begin{cases} 1, & h = 0 \\ -.5 & h = \pm 2 \\ -.25 & h = \pm 3 \\ 0 & o/w. \end{cases}$$

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-3}^{3} e^{-ih\lambda} \gamma(h)$$

$$f(\lambda) = \frac{1}{2\pi} \left[-.25\, e^{3i\lambda} - .5\, e^{2i\lambda} + 1 - .5\, e^{-2i\lambda} - .25\, e^{-3i\lambda} \right]$$

$$\Rightarrow f(0) = \frac{1}{2\pi} \left[-.5 - 1 + 1 \right] = -\frac{1}{4\pi} < 0$$

$\Rightarrow \gamma(.)$ is not ACVF

(b) $$\gamma(h) = \begin{cases} 1 & |h| \le 1 \\ 0 & \text{o/w.} \end{cases}$$

$$f(\lambda) = \frac{1}{2\pi} \left[e^{i\lambda} + 1 + e^{-i\lambda} \right]$$

$$= \frac{1}{2\pi} \left[1 + 2\cos\lambda \right]$$

$f(\pi) = -\frac{1}{2\pi} < 0 \Rightarrow \gamma(.)$ is not ACVF.

(8) $$\gamma(h) = \begin{cases} 1, & h = 0 \\ a, & |h| = 1 \\ 0 & \text{o/w} \end{cases}$$

$$f(\lambda) = \frac{1}{2\pi} \left(a e^{i\lambda} + 1 + a e^{-i\lambda} \right)$$

$$= \frac{1}{2\pi} \left(1 + 2a\cos\lambda \right)$$

$f(\lambda) \ge 0 \ \forall \lambda$ iff $1 + 2a\cos\lambda \ge 0 \ \forall \lambda$

$\lambda = 0 \quad 1 + 2a \ge 0 \Rightarrow a \ge -\frac{1}{2}$

$\lambda = \pi \quad 1 - 2a \ge 0 \Rightarrow a \le \frac{1}{2}$,

$\Rightarrow \forall\ |a| \le \frac{1}{2}$ $\gamma(.)$ is ACVF

(9)

$$\{Z_t\} \xrightarrow{g} \{X_t\} \xrightarrow{g} \{Y_t\}$$

$$f_Z(\lambda) = 1 \quad \forall\ \lambda \in [-\pi, \pi]$$

$$\gamma_X(h) = e^{-|h|}.$$

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} e^{-|h|}$$

$$= \frac{1}{2\pi}\left[1 + \sum_{h=1}^{\infty} e^{-ih\lambda} e^{-h} + \sum_{h=1}^{\infty} e^{ih\lambda} e^{-h}\right]$$

$$= \frac{1}{2\pi}\left[1 + \frac{e^{-(i\lambda+1)}}{1-e^{-(i\lambda+1)}} + \frac{e^{i\lambda-1}}{1-e^{i\lambda-1}}\right]$$

$$= \frac{1}{2\pi}\left[\frac{(1+\cancel{e^{-2}} - \cancel{e^{-i\lambda-1}} - \cancel{e^{i\lambda-1}}) + (\cancel{e^{-i\lambda-1}} - \cancel{e^{-2}}) + (\cancel{e^{i\lambda-1}} - e^{-2})}{(1-e^{-(i\lambda+1)})(1-e^{i\lambda-1})}\right]$$

$$= \frac{1}{2\pi} \cdot \frac{1-e^{-2}}{(1-e^{-i\lambda-1})(1-e^{i\lambda-1})}$$

$$Y_t = \sum_{j=-\infty}^{\infty} g_j X_{t-j}$$

$$f_Y(\lambda) = f_X(\lambda)\,|g(e^{-i\lambda})|^2$$

$$X_t = \sum_{j=-\infty}^{\infty} g_j Z_{t-j}$$

$$f_X(\lambda) = f_Z(\lambda)\,|g(e^{-i\lambda})|^2$$

i.e. $f_X(\lambda) = |g(e^{-i\lambda})|^2$ — (*)

i.e. $f_Y(\lambda) = (f_X(\lambda))^2$ (using *)

$$\Rightarrow\ f_Y(\lambda) = \left(\frac{1}{2\pi} \cdot \frac{1-e^{-2}}{(1-e^{-i\lambda-1})(1-e^{i\lambda-1})}\right)^2.$$

(10)

$$X_t = \epsilon_t + \theta \epsilon_{t-1}; \quad \epsilon_t \sim WN(0,1)$$

$$Y_t \sim WN(0,1) \quad ; \{Y_t\} \,\&\, \{\epsilon_t\} \text{ indep}$$

$$Z_t = \sum_{j=-\infty}^{\infty} \psi_j X_{t-j} + \sum_{j=-\infty}^{\infty} \theta_j Y_{t-j}$$

$$\psi_j = \begin{cases} 0.5, & j = \pm 1 \\ 0, & \text{o/w} \end{cases} \qquad \theta_j = \begin{cases} 1, & j = 0, 2 \\ 0, & \text{o/w} \end{cases}$$

$$Z_t = P_t + Q_t \; ; \{P_t\} \,\&\, \{Q_t\} \text{ are indep}$$

$$f_Z(\lambda) = f_P(\lambda) + f_Q(\lambda)$$

$$P_t = \sum_j \psi_j X_{t-j}$$

$$f_P(\lambda) = \left(\sum \psi_j e^{-ij\lambda}\right)\left(\sum \psi_k e^{ik\lambda}\right) f_X(\lambda)$$

$$\text{i.e. } f_P(\lambda) = \left(\tfrac{1}{2} e^{i\lambda} + \tfrac{1}{2} e^{-i\lambda}\right)\left(\tfrac{1}{2} e^{-i\lambda} + \tfrac{1}{2} e^{i\lambda}\right) \cdot \frac{1}{2\pi} (1+\theta e^{-i\lambda})(1+\theta e^{i\lambda})$$

$$f_P(\lambda) = \frac{1}{8\pi} (2\cos\lambda)^2 (1+\theta^2+2\theta\cos\lambda).$$

$$Q_t = \sum_j \theta_j Y_{t-j}$$

$$f_Q(\lambda) = \left(\sum \theta_j e^{-ij\lambda}\right)\left(\sum \theta_k e^{ik\lambda}\right) f_Y(\lambda)$$

$$= (1+e^{-2i\lambda})(1+e^{2i\lambda}) \frac{1}{2\pi}$$

$$= (2 + 2\cos 2\lambda) \frac{1}{2\pi} = \frac{1+\cos 2\lambda}{\pi}$$

$$\therefore f_Z(\lambda) = \frac{1}{2\pi} (\cos\lambda)^2 (1+\theta^2+2\theta\cos\lambda)$$

$$+ \frac{1+\cos 2\lambda}{\pi}$$

$$\therefore f_Z(\pi) = \frac{(1-\theta)^2}{2\pi} + \frac{2}{\pi}$$

(11) $X_t = A\cos(\pi/4\, t) + B\sin(\pi/4\, t) + Y_t$

$Y_t = \frac{1}{2}\epsilon_t$ A & B u.c. r.v.s mean 0 and var σ^2

ϵ_t u.c with A & B

$$F_X(\lambda) = F_Z(\lambda) + F_Y(\lambda).$$

Now $\gamma_X(h) = \sigma^2 \cos(\pi/4\, h) = \int_{-\pi}^{\pi} e^{i\lambda h}\, dF_Z(\lambda)$

$$\Rightarrow F_Z(\lambda) = \begin{cases} 0, & -\pi \le \lambda < -\pi/4 \\ \sigma^2/2, & -\pi/4 \le \lambda < \pi/4 \\ \sigma^2, & \pi/4 \le \lambda \le \pi. \end{cases}$$

$Y_t = \frac{1}{2}\epsilon_t \Rightarrow f_Y(\lambda) = f_\epsilon(\lambda)\, |g(e^{-i\lambda})|^2 \quad \left(g(B) = \frac{1}{2}\right).$

i.e. $f_Y(\lambda) = \frac{\sigma^2}{2\pi} \cdot \frac{1}{4} = \frac{\sigma^2}{8\pi}$

$$\Rightarrow F_Y(\lambda) = \int_{-\pi}^{\lambda} f_Y(\lambda)\, d\lambda = \frac{\sigma^2}{8\pi}(\lambda + \pi)$$

$$\Rightarrow F_X(\lambda) = \begin{cases} \frac{\sigma^2}{8\pi}(\lambda+\pi), & -\pi \le \lambda < -\frac{\pi}{4} \\ \frac{\sigma^2}{8\pi}(\lambda+\pi) + \sigma^2/2, & -\pi/4 \le \lambda < \pi/4 \\ \sigma^2 + \frac{\sigma^2}{8\pi}(\lambda+\pi), & \pi/4 \le \lambda \le \pi. \end{cases}$$

(12) $X_t = \alpha_1 \cos t + \alpha_2 \sin t + Y_t$

i.e. $X_t = Z_t + Y_t \; ; \; Z_t = \alpha_1 \cos t + \alpha_2 \sin t$

$\cancel{Y_t =}$ - -

$Y_t = \epsilon_t - \epsilon_{t-1}$

$$f_y(\lambda) = \frac{\sigma_\epsilon^2}{2\pi}\left(1 - e^{i\lambda}\right)\left(1 - e^{-i\lambda}\right)$$

$$= \frac{\sigma^2}{\pi}\left(1 - \cos\lambda\right)$$

$$F_y(\lambda) = \frac{\sigma^2}{\pi}\int_{-\pi}^{\lambda}(1 - \cos\lambda)\, d\lambda$$

$$= \frac{\sigma^2}{\pi}\left(\lambda - \sin\lambda \Big|_{-\pi}^{\lambda}\right)$$

$$= \frac{\sigma^2}{\pi}\left(\pi + \lambda - \sin\lambda\right)$$

$$\gamma_z(h) = \sigma^2 \cos h$$

$$F_z(\lambda) = \begin{cases} 0, & -\pi \le \lambda < -1 \\ \sigma^2/2, & -1 \le \lambda < 1 \\ \sigma^2 & 1 \le \lambda \le \pi \end{cases}$$

$\underline{\sigma^2 = 3}$

$$\Rightarrow \quad F_x(\lambda) = F_z(\lambda) + F_y(\lambda)$$

$$= \begin{cases} \frac{3}{\pi}\left(\pi + \lambda - \sin\lambda\right), & -\pi \le \lambda < -1 \\ \frac{3}{2} + \frac{3}{\pi}\left(\pi + \lambda - \sin\lambda\right), & -1 \le \lambda < 1 \\ 3 + \frac{3}{\pi}\left(\pi + \lambda - \sin\lambda\right), & 1 \le \lambda \le \pi \end{cases}$$

$$\gamma_x(0) = F_x(\pi) = 3 + \frac{3}{\pi}\left(\pi + \pi - \sin\pi\right) = \underline{\underline{9}}$$