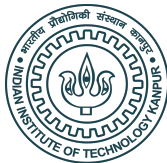


Lecture 15

ARMA Models: Part 1

Arnab Hazra



Definition of ARMA models

- ▶ A autoregressive moving average model of order (p, q) , abbreviated $ARMA(p, q)$, is of the form

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots + \phi_p X_{t-p} + W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \dots + \theta_q W_{t-q}.$$

- ▶ Here X_t is stationary, $W_t \sim WN(0, \sigma_W^2)$, and $\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$ are constants with $\phi_p, \theta_q \neq 0$.
- ▶ The parameters p and q are called the autoregressive and the moving average orders, respectively.
- ▶ If X_t has a nonzero mean μ , we set $\alpha = (1 - \phi_1 - \dots - \phi_p)\mu$ and write the model as

$$X_t = \alpha + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots + \phi_p X_{t-p} + W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \dots + \theta_q W_{t-q}.$$

Concise representation

- ▶ A autoregressive moving average model of order (p, q) , abbreviated $ARMA(p, q)$, is of the form

$$X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \dots + \theta_q W_{t-q}.$$

- ▶ When $q = 0$, the model is $AR(p)$. We represent it using $\phi(B)X_t = W_t$.
- ▶ When $p = 0$, the model is $MA(q)$. We represent it using $X_t = \theta(B)W_t$.
- ▶ We thus represent the $ARMA(p, q)$ model using $\phi(B)X_t = \theta(B)W_t$.
- ▶ We should not unnecessarily disturb this to obtain $\eta(B)\phi(B)X_t = \eta(B)\theta(B)W_t$.
- ▶ Representing $X_t = W_t$ by $(1 - 0.5B)X_t = (1 - 0.5B)W_t$ can make the white noise process appear like $ARMA(1, 1)$.

Three problems with ARMA models

- 1 Parameter redundant models
- 2 Stationary AR models that depend on the future
- 3 MA models that are not unique

They need to be resolved! We need restrictions to ensure they do not happen!

Solution to Problem 1

- ▶ The AR and MA polynomials are defined as

$$\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p, \quad \phi_p \neq 0,$$

and

$$\theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q, \quad \theta_q \neq 0,$$

respectively, where z is a complex number.

- ▶ Along with its original definition, we will also require that $\phi(z)$ and $\theta(z)$ have no common factors.
- ▶ This ensures that the model $\eta(B)\phi(B)X_t = \eta(B)\theta(B)W_t$ is simplified to $\phi(B)X_t = \theta(B)W_t$, where $\phi(z)$ and $\theta(z)$ have no common factors.

Solution to Problem 2

- ▶ An ARMA(p, q) model is said to be causal, if the time series $\{X_t; t = 0, \pm 1, \pm 2, \dots\}$ can be written as a one-sided linear process:

$$X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j} = \psi(B) W_t,$$

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$, and $\sum_{j=0}^{\infty} |\psi_j| < \infty$; we set $\psi_0 = 1$.

- ▶ To ensure this, we can look at the root of the AR polynomial.
- ▶ Consider the model $X_t = \phi X_{t-1} + W_t$; this process is causal when $|\phi| < 1$.
- ▶ The root of the AR polynomial $\phi(z) = 1 - \phi z$ is $z_0 = 1/\phi$; thus, $|z_0| = |\phi|^{-1} > 1$.
- ▶ Property: An ARMA(p, q) model is causal if and only if $\phi(z)$ does not have any root z_0 for $|z_0| \leq 1$.

Solution to Problem 3

- ▶ An ARMA(p, q) model is said to be invertible, if the time series $\{X_t; t = 0, \pm 1, \pm 2, \dots\}$ can be written as

$$\pi(B)X_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} = W_t,$$

where $\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j$, and $\sum_{j=0}^{\infty} |\pi_j| < \infty$; we set $\pi_0 = 1$.

- ▶ To ensure this, we can look at the root of the MA polynomial.
- ▶ Consider the model $X_t = W_t + \theta W_{t-1}$; this process is invertible when $|\theta| < 1$.
- ▶ The root of the MA polynomial $\theta(z) = 1 + \theta z$ is $z_0 = -1/\theta$; thus, $|z_0| = |\theta|^{-1} > 1$.
- ▶ An ARMA(p, q) model is invertible if and only if $\theta(z)$ does not have any root z_0 for $|z_0| \leq 1$.

Thank you!