Lecture 13

Autoregressive Models Part 2

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Definition (Recap)

- An autoregressive model of order p, abbreviated $AR(p)$, is of the form

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + W_t.$$

- Here X_t is stationary, $W_t \sim WN(0, \sigma_W^2)$, and $\phi_1, \phi_2, \ldots, \phi_p$ are constants with $\phi_p \neq 0$.
- Here we assumed the mean of X_t to be zero. Otherwise, we need to replace all X_s terms by $X_s - \mu$, i.e.,

$$X_t - \mu = \phi_1 (X_{t-1} - \mu) + \phi_2 (X_{t-2} - \mu) + \ldots + \phi_p (X_{t-p} - \mu) + W_t.$$

- Equivalently, for $\alpha = (1 - \phi_1 - \ldots - \phi_p)\mu$, we can write

$$X_t = \alpha + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + W_t.$$

Autoregressive operator (Recap)

- We can rewrite the model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + W_t$$

by

$$X_t = \phi_1 B X_t + \phi_2 B^2 X_t + \ldots + \phi_p B^p X_t + W_t,$$

where B is the backshift operator, and hence,

$$\phi(B) X_t = (1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p) X_t = W_t.$$

- Here $\phi(B)$ is called the autoregressive operator, where

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p.$$

- For a AR(1) process $X_t = \phi X_{t-1} + W_t$, we have $\phi(B) = 1 - \phi B$

Autoregressive process as a linear process (Recap)

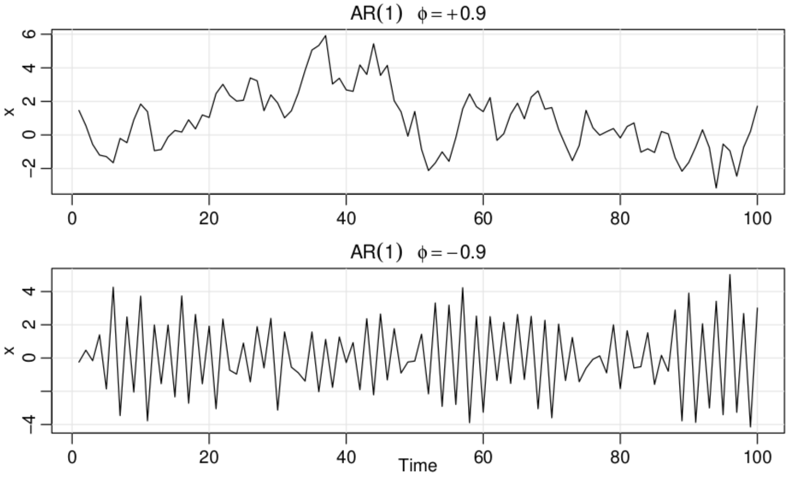
- We can represent a AR(1) process $X_t = \phi X_{t-1} + W_t$ as a linear process

$$X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}.$$

- $E(X_t) = E\left(\sum_{j=0}^{\infty} \phi^j W_{t-j}\right) = \sum_{j=0}^{\infty} \phi^j E(W_{t-j}) = 0.$
-

$$\begin{aligned}
\gamma(h) &= \mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}\left(\sum_{j=0}^{\infty} \phi^j W_{t+h-j}, \sum_{j=0}^{\infty} \phi^j W_{t-j}\right) \\
&= \mathrm{Cov}\left(\sum_{j'=-h}^{\infty} \phi^{j'+h} W_{t-j'}, \sum_{j=0}^{\infty} \phi^j W_{t-j}\right) \quad \text{(replacing } j' = j - h\text{)} \\
&= \phi^h \sum_{j=0}^{\infty} \phi^{2j} \mathrm{Var}(W_t) = \phi^h \sigma_W^2 (1-\phi^2)^{-1}
\end{aligned}$$

Sample paths of AR(1)



Explosive AR Models and Causality

- We have seen that $X_t = X_{t-1} + W_t$ is not stationary.
- We might wonder whether there is a stationary AR(1) process with $|\phi| > 1$.
- Show that the process can be written as

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j} W_{t+j}$$

- When a process does not depend on the future, such as the AR(1) when $|\phi| < 1$, we will say the process is causal.

Explosive AR Models and linear process representation

- We have $X_t = \phi X_{t-1} + W_t, \;\; X_{t+1} = \phi X_t + W_{t+1}, \;\; X_{t+2} = \phi X_{t+1} + W_{t+2}, \ldots$.
- This implies

$$
\begin{aligned}
X_t &= \phi^{-1} X_{t+1} - \phi^{-1} W_{t+1} \\
&= \phi^{-1}(\phi^{-1} X_{t+2} - \phi^{-1} W_{t+2}) - \phi^{-1} W_{t+1} = \phi^{-2} X_{t+2} - \phi^{-2} W_{t+2} - \phi^{-1} W_{t+1} \\
& \ldots \\
&= \phi^{-k} X_{t+k} - \phi^{-k} W_{t+k} - \ldots - \phi^{-2} W_{t+2} - \phi^{-1} W_{t+1} \\
&= \phi^{-k} X_{t+k} - \sum_{j=1}^{k} \phi^{-j} W_{t+j}.
\end{aligned}
$$

- Letting $k \uparrow \infty$ and assuming X_t has finite variance, we have

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j} W_{t+j}.$$

Mean and covariance function of explosion

- $E(X_t) = E\left(-\sum_{j=1}^{\infty} \phi^{-j} W_{t+j}\right) = -\sum_{j=1}^{\infty} \phi^{-j} E(W_{t+j}) = 0.$
-

$$\begin{aligned}
\gamma(h) &= \mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}\left(-\sum_{j=1}^{\infty} \phi^{-j} W_{t+h+j}, -\sum_{j=1}^{\infty} \phi^{-j} W_{t+j}\right) \\
&= \mathrm{Cov}\left(-\sum_{j'=1+h}^{\infty} \phi^{-(j'-h)} W_{t+j'}, -\sum_{j=1}^{\infty} \phi^{-j} W_{t+j}\right) \quad \text{(replacing } j' = j+h\text{)} \\
&= \phi^h \sum_{j=1+h}^{\infty} \phi^{-2j} \mathrm{Var}(W_t) = \phi^h \sigma_W^2 \sum_{j=1+h}^{\infty} \phi^{-2j} \\
&= \phi^h \sigma_W^2 \phi^{-2(h+1)} (1 - \phi^{-2})^{-1} \\
&= \sigma_W^2 \phi^{-2} \phi^{-h} (1 - \phi^{-2})^{-1}.
\end{aligned}$$

Every Explosion Has a Cause

- If $X_t = \phi X_{t-1} + W_t$ with $|\phi| > 1$ and $W_t \overset{IID}{\sim} N(0, \sigma^2_W)$, then $\{X_t\}$ is a non-causal stationary Gaussian process with $E(X_t) = 0$.

- The causal process defined by $Y_t = \phi^{-1} Y_{t-1} + V_t$ where $W_t \overset{IID}{\sim} N(0, \sigma^2_W \phi^{-2})$ is stochastically equal to the X_t process.

- For example, if $X_t = 2X_{t-1} + W_t$ with $\sigma^2_W = 1$, then $Y_t = 0.5 Y_{t-1} + V_t$ with $\sigma^2_V = 1/4$ is an equivalent causal process.

General case

- The representation of the AR(1) process X_t as a linear process is possible via iteration, but it is difficult to do the same for higher orders.
- A general technique is that of matching coefficients.
- Suppose the linear process representation is $X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j}$. We should have $\phi(B)\psi(B)W_t = W_t$.
- Here $\phi(B) = 1 - \phi B$. Show that $\psi(B) = 1 + \phi B + \phi^2 B^2 + \ldots + \phi^j B_j + \ldots$

Proof

- Suppose $X_t = \sum_{j=0}^{\infty} \phi_j W_{t-j} = \psi(B) W_t$.
- We need to show $\phi_0 = 1, \phi_1 = \phi, \phi_2 = \phi^2, \ldots$.
- We have

$$
\begin{aligned}
\phi(B)\psi(B)W_t &= \phi(B)[\sum_{j=0}^{\infty} \phi_j W_{t-j}] = \sum_{j=0}^{\infty} \phi_j \phi(B) W_{t-j} \\
&= \sum_{j=0}^{\infty} \phi_j (1 - \phi B) W_{t-j} = \sum_{j=0}^{\infty} \phi_j (W_{t-j} - \phi W_{t-j-1}) \\
&= \phi_0 W_t - \phi_0 \phi W_{t-1} + \phi_1 W_{t-1} - \phi\phi_1 W_{t-2} + \phi_2 W_{t-2} - \phi\phi_2 W_{t-3} \ldots
\end{aligned}
$$

- Matching the coefficients, we get $\phi_0 = 1$, $\phi_1 = \phi_0 \phi = \phi$, and $\phi_2 = \phi_1 \phi = \phi^2$ and so on.

Thank you!