Lecture 39

The Spectral Density: Part 1

Arnab Hazra

Periodic process (recap)

- We first define a cycle as one complete period of a sine or cosine function defined over a unit time interval.
- We consider the periodic process

$$X_t = A\cos(2\pi\omega_0 t + \phi)$$

  for $t = 0, \pm 1, \pm 2, \ldots$, where ω_0 is a fixed frequency index.
- Here A determines the height or amplitude of the function and ϕ, called the phase, determining the start point of the cosine function.
- We can introduce random variation in this time series by allowing the amplitude and phase to vary randomly.

Periodic process (contd., recap)

- For purposes of data analysis, it is easier to write X_t as

$$X_t = U_1 \cos(2\pi\omega_0 t) + U_2 \sin(2\pi\omega_0 t),$$

  where $U_1 = A\cos(\phi)$ and $U_2 = -A\sin(\phi)$.
- We then often take U_1 and U_2 to be normally distributed.
- The amplitude is $A = \sqrt{U_1^2 + U_2^2}$ and the phase is $\phi = \tan^{-1}(-U_2/U_1)$.
- Here, A and ϕ are independent random variables if U_1 and U_2 are independent standard normal random variables.
- Then $A^2 \sim \chi_2^2$ and $\phi \sim \mathrm{Unif}(-\pi, \pi)$.
- Straightforward Jacobian calculations show that the reverse is also true.

Moments of X_t

If we assume that U_1 and U_2 are uncorrelated random variables with mean 0 and variance σ^2, then

- Mean $E(X_t) = 0$.
- Covariance

$$\mathrm{Cov}(X_{t+h}, X_t) = \sigma^2 \cos(2\pi\omega_0 h) = \frac{\sigma^2}{2} \exp[\iota\, 2\pi\omega_0 h] + \frac{\sigma^2}{2} \exp[-\iota\, 2\pi\omega_0 h].$$

- Note that we can we write $\mathrm{Cov}(X_{t+h}, X_t)$ as $\int_{-1/2}^{1/2} \exp(\iota\, 2\pi\omega h) dF(\omega)$.
- Here $F(\cdot)$ is the function defined by $F(\omega) = 0$ if $\omega < -\omega_0$, $F(\omega) = \frac{\sigma^2}{2}$ if $-\omega_0 \leq \omega < \omega_0$, and $F(\omega) = \sigma^2$ if $\omega \geq \omega_0$.
- Basically two jumps of $\sigma^2/2$ at $-\omega_0$ and ω_0.

Interpretation of $F(\cdot)$

- The function $F(\cdot)$ behaves like a cumulative distribution function for a discrete random variable, except that $F(\infty) = \sigma^2 = \mathrm{Var}(X_t)$ instead of one.

- In fact, $F(\cdot)$ is a cumulative distribution function, not of probabilities, but rather of variances, with $F(\infty)$ being the total variance of the process X_t.

- Hence, we term $F(\cdot)$ the spectral distribution function.

- A representation of $X_t = U_1 \cos(2\pi\omega_0 t) + U_2 \sin(2\pi\omega_0 t)$ always exists for a stationary process.

Spectral Representation of an Autocovariance Function

- If $\{X_t\}$ is stationary with autocovariance $\gamma(h) = \mathrm{Cov}(X_{t+h}, X_t)$, there exists a unique monotonically increasing function $F(\cdot)$, called the spectral CDF.

- $F(-\infty) = F(-1/2^-) = 0$, and $F(\infty) = F(1/2) = \gamma(0)$ such that

$$\gamma(h) = \int_{-1/2}^{1/2} \exp[\iota\, 2\pi\omega h] dF(\omega).$$

- An important situation we use repeatedly is the case when the autocovariance function is absolutely summable.

- In that case the spectral distribution function is absolutely continuous with $dF(\omega) = f(\omega)d\omega$, and we can talk about spectral density.

Spectral density

If the autocovariance function $\gamma(h)$ of a stationary process satisfies $\sum_{h=0}^{\infty} |\gamma(h)| < \infty$,

- Then it has the representation

$$\gamma(h) = \int_{-1/2}^{1/2} \exp(\iota\, 2\pi\omega h) f(\omega) d\omega, \quad h = 0, \pm 1, \pm 2, \ldots$$

- Then it can be written as the inverse transform of the spectral density

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) \exp(-\iota\, 2\pi\omega h), \quad -1/2 \leq \omega \leq 1/2.$$

Spectral density: properties

- This spectral density is the analogue of the probability density function.
- The fact that $\gamma(h)$ is non-negative definite ensures $f(\omega) \geq 0$ for all ω.
- It follows immediately that $f(\omega) = f(-\omega)$ verifying the spectral density is an even function.
- Because of the evenness, we will typically only plot $f(\omega)$ for $0 \leq \omega \leq 1/2$.
- In addition, putting $h = 0$ yields $\gamma(0) = \mathrm{Var}(X_t) = \int_{-1/2}^{1/2} f(\omega) d\omega$.
- This expresses the total variance as the integrated spectral density over all of the frequencies.

Example: White noise

- The autocovariance function is $\gamma_W(h) = \sigma_W^2$ for $h = 0$, and zero, otherwise.
- Hence, $f_W(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) \exp(-\iota\, 2\pi\omega h) = \sigma_W^2, \quad -1/2 \leq \omega \leq 1/2$.
- Hence the process contains equal power at all frequencies.
- This property is seen in the realization, which seems to contain all different frequencies in a roughly equal mix.
- In fact, the name white noise comes from the analogy to white light, which contains all frequencies in the color spectrum at the same level of intensity.

Thank you!