Lecture 17

Difference equation

Arnab Hazra

Difference equation: Motivation

- Consider the AR(1) model $X_t = \phi X_{t-1} + W_t$ with $|\phi| < 1$.
- We can represent it as a linear process $X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}$.
- We showed that

$$\begin{aligned}
\gamma(h) &= \mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}\left(\sum_{j=0}^{\infty} \phi^j W_{t+h-j}, \sum_{j=0}^{\infty} \phi^j W_{t-j}\right) \\
&= \phi^h \sum_{j=0}^{\infty} \phi^{2j} \mathrm{Var}(W_t) = \phi^h \sigma_W^2 (1-\phi^2)^{-1}
\end{aligned}$$

- Clearly, $\gamma(h-1) = \phi^{h-1}\sigma_W^2(1-\phi^2)^{-1}$ and thus, $\gamma(h) = \phi\gamma(h-1)$.
- Dividing by $\gamma(0)$, we get $\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi\frac{\gamma(h-1)}{\gamma(0)} = \phi\rho(h-1)$.

Difference equation

- Suppose we have a sequence of numbers $u_0, u_1, u_2, \ldots$ such that

$$u_n - \alpha u_{n-1} = 0, \quad \alpha \neq 0, \; n = 1, 2, \ldots$$

- The equation represents a homogeneous difference equation of order 1.
- To solve it, we write: $u_1 = \alpha u_0, u_2 = \alpha u_1 = \alpha^2 u_0, \ldots, u_n = \alpha u_{n-1} = \alpha^n u_0$.
- Given an initial condition $u_0 = c$, we have $u_n = \alpha^n c$.
- We can write $u_n - \alpha u_{n-1} = 0$ as $(1 - \alpha B)u_n = 0$.
- The root z_0 of the associated polynomial $\alpha(z) = 1 - \alpha z$ is $z_0 = 1/\alpha$ and we can write the final solution also as $u_n = (z_0^{-1})^n c = z_0^{-n} c$.

Difference equation of higher orders: Motivation

- Suppose $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t$ is a causal AR(2) process.
- Multiply each side of the model by X_{t-h} for $h > 0$, and take expectation:

$$E(X_t X_{t-h}) = \phi_1 E(X_{t-1} X_{t-h}) + \phi_2 E(X_{t-2} X_{t-h}) + E(W_t X_{t-h}).$$

- The result is $\gamma(h) = \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2), h = 1, 2, \ldots$.
- Dividing by $\gamma(0)$, we have $\rho(h) = \phi_1 \rho(h-1) + \phi_2 \rho(h-2)$.
- Let z_1 and z_2 be the real and distinct roots of $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$. Then,

$$\rho(h) = c_1 z_1^{-h} + c_2 z_2^{-h}$$

is a solution.

Difference equation: General result for order 2

- Suppose the sequence $u_0, u_1, u_2, \ldots$ satisfies

$$u_n - \alpha_1 u_{n-1} - \alpha_2 u_{n-2} = 0, \quad \alpha_2 \neq 0, \quad n = 2, 3, \ldots.$$

- This equation is a homogeneous difference equation of order 2.
- The corresponding polynomial is $\alpha(z) = 1 - \alpha_1 z - \alpha_2 z^2$, which has two roots, say, z_1 and z_2.
- If $z_1 \neq z_2$, the general solution is $u_n = c_1 z_1^{-n} + c_2 z_2^{-n}$, where c_1 and c_2 depend on the initial conditions.
- When $z_1 = z_2 (= z_0)$, a general solution is $u_n = z_0^{-n}(c_1 + c_2 n)$.
- Overall, $u_n = z_1^{-n} \times (\text{a polynomial in n of degree } m_1 - 1) + z_2^{-n} \times (\text{a polynomial in n of degree } m_2 - 1)$ where m_i is the multiplicity of z_i for $i = 1, 2$.
- For repeated root, the solution is $u_n = z_0^{-n} \times (\text{a polynomial in n of degree } m_0 - 1)$, where $m_0 = \text{multiplicity}(z_0)$.

Difference equation: General result for order p

- These results generalize to the homogeneous difference equation of order p:

$$u_n - \alpha_1 u_{n-1} - \ldots - \alpha_p u_{n-p} = 0, \quad \alpha_p \neq 0, \quad n = p, p+1, \ldots.$$

- The associated polynomial is $\alpha(z) = 1 - \alpha_1 z - \ldots - \alpha_p z^p$.
- Suppose $\alpha(z)$ has r distinct roots, z_i with multiplicity m_i for $i = 1, \ldots, r$, such that $\sum_{i=1}^{r} m_i = p$.
- The general solution is

$$u_n = z_1^{-n} P_1(n) + z_2^{-n} P_2(n) + \ldots + z_r^{-n} P_r(n),$$

where $P_j(n)$, for $j = 1, 2, \ldots, r$, is a polynomial in n, of degree $m_j - 1$.

AR(2) with Complex Roots

- Suppose $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t$ is a causal AR(2) process.
- We have $\rho(h) = \phi_1 \rho(h-1) + \phi_2 \rho(h-2)$.
- When z_1 and z_2 be the real and distinct roots of $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$. Then, $\rho(h) = c_1 z_1^{-h} + c_2 z_2^{-h}$ is a solution.
- When z_1 and $\bar{z}_2$ are a complex conjugate pair, then $c_2 = \bar{c}_1$ (because $\rho(h)$ is real), and
$$\rho(h) = c_1 z_1^{-h} + \bar{c}_1 \bar{z}_1^{-h}.$$
- Writing $z_1 = |z_1| \exp[\iota\theta]$ in the polar representation, the solution has the form
$$\rho(h) = a|z_1|^{-h} \cos(h\theta + b),$$
where a and b are determined by the initial conditions.
- Example: $X_t = 1.5X_{t-1} - 0.75X_{t-2} + W_t$, with $\sigma_w^2 = 1$.

Solving ψ-weights of ARMA(p, q)

- A causal ARMA(p, q) model $\{X_t; t = 0, \pm 1, \pm 2, \ldots\}$ can be written as a one-sided linear process:

$$X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j} = \psi(B) W_t,$$

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$, and $\sum_{j=0}^{\infty} |\psi_j| < \infty$; we set $\psi_0 = 1$.

- To solve for the ψ-weights in general, we must match the coefficients in $\psi(z)\psi(z) = \theta(z)$:

$$(1 - \phi_1 z - \phi_2 z^2 - \ldots)(\psi_0 + \psi_1 z + \psi_2 z^2 + \ldots) = (1 + \theta_1 z + \theta_2 z^2 + \ldots)$$

Thank you!