MTH442 Assignment 4

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Q1

1. Model Setup:

first difference process for time series is :

$$Z_t = X_t - X_{t-1},$$

(here Z_t is change between consecutive observations of X_t.)

now given that

X$_t = X_{t-1} + W_t - \lambda W_{t-1}$,

X$_{t-1} = X_{t-2} + W_{t-1} - \lambda W_{t-2}$.

substitut in $\therefore X_t - X_{t-1} = W_t - \lambda W_{t-1}$,

so model is:

$$Z_t = W_t - \lambda W_{t-1},$$

here W_t is white noise process.

2. invertibile (express W_t in term of Z_t)

from point 1

$$Z_t = W_t - \lambda W_{t-1}.$$

rearrange:

$$W_t = Z_t + \lambda W_{t-1}.$$

substituting recursively:

W$_t = Z_t + \lambda W_{t-1}$,

$$W_t = Z_t + \lambda(Z_{t-1} + \lambda W_{t-2}),$$
$$W_t = Z_t + \lambda Z_{t-1} + \lambda^2 W_{t-2}.$$

continue substituting indefinitely:

W$_t = Z_t + \lambda Z_{t-1} + \lambda^2 W_{t-2} + \dots$,

i did not include negative index because in Ques. it is given that X$_t = 0, \quad$ for all $t < 0$

$$W_t = \sum_{j=0}^{\infty} \lambda^j Z_{t-j}.$$

from classnotes invertibility condition

for series to be invertible, coefficient λ must satisfy:

$$|\lambda| < 1.$$

it ensures infinite sum converges and process remains stable.

3. write W_t in term of X_t

from point 2

W$_t = \sum_{j=0}^{\infty} \lambda^j Z_{t-j}$,

as Z$_t = X_t - X_{t-1}$,

Z$_{t-j} = X_{t-j} - X_{t-j-1}$,

so

$$W_t = \sum_{j=0}^{\infty} \lambda^j (X_{t-j} - X_{t-j-1}).\text{(approx for large t)}$$

W$_t = \lambda^0(X_t - X_{t-1}) + \lambda^1(X_{t-1} - X_{t-2}) + \lambda^2(X_{t-2} - X_{t-3}) + \dots$,

4. rearrange form of model

pattern in equation is :

W$_t = (X_t - X_{t-1}) + \lambda(X_{t-1} - X_{t-2}) + \lambda^2(X_{t-2} - X_{t-3}) + \dots$

$\mathrm{W}_t = X_t - X_{t-1} + \lambda X_{t-1} - \lambda X_{t-2} + \lambda^2 X_{t-2} - \lambda^2 X_{t-3} + \dots$
$\mathrm{W}_t = X_t + (-1+\lambda)X_{t-1} + (-\lambda+\lambda^2)X_{t-2} + (-\lambda^2+\lambda^3)X_{t-3} + \dots$
$\mathrm{W}_t = X_t - \lambda(1-\lambda)X_{t-1} - \lambda^2(1-\lambda)X_{t-2} - \dots$
so as an approximation for large t,

$$W_t = X_t - \sum_{j=1}^{\infty} \lambda^j (1-\lambda) X_{t-j}.$$

rearrange :

$$X_t = \sum_{j=1}^{\infty} \lambda^j (1-\lambda) X_{t-j} + W_t. \text{ hence proved}$$

Q2(a)

given ARIMA(1, 1, 0) model with drift:

$$(1-\phi B)(1-B)X_t = \delta + W_t,$$

here B is backward shift operator s.t. $BX_t = X_{t-1}$, δ is drift, and W_t is white noise. $Y_t = \nabla X_t = X_t - X_{t-1}$.

1.
now from given

$$(1-\phi B)(1-B)X_t = \delta + W_t$$

$$(1-\phi B)(X_t - X_{t-1}) = \delta + W_t$$

$$X_t - X_{t-1} - \phi(X_{t-1} - X_{t-2}) = \delta + W_t$$

as $Y_t = X_t - X_{t-1}$put it in above eqn.

$$Y_t - \phi Y_{t-1} = \delta + W_t$$

Y_t follows AR(1) model with drift δ so:

$$Y_t = \delta + \phi Y_{t-1} + W_t.$$

forecast of Y_{T+1} based on value at time T:

$$Y^T_{T+1} = E_T[Y_{T+1}]$$

$$Y^T_{T+1} = E_T[\delta + \phi Y_T + W_{T+1}]$$

$$Y^T_{T+1} = \delta + \phi Y_T + E_T[W_{T+1}]$$

as

$$E_T[W_{T+1}] = 0$$

$$Y^T_{T+1} = \delta + \phi Y_T$$

(basis of induction is this recursive relation)

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2. now i will show by induction that for $j \geq 1$:

$$Y^T_{T+j} = \delta\left[1 + \phi + \ldots + \phi^{j-1}\right] + \phi^j Y_T.$$

base case of induction: $j = 1$
for $j = 1$:

$$Y^T_{T+1} = \delta\,[1] + \phi^1 Y_T = \delta + \phi Y_T.$$

it is already true from point 1, so base case holds.

3. induction from point 2

$$Y_{T+1}^{T} = \delta + \phi Y_T$$

for $j = 2$:

$$Y_{T+2}^{T} = E_T[Y_{T+2}]$$

$$Y_{T+2}^{T} = E_T[\delta + \phi Y_{T+1} + W_{T+2}]$$

$$Y_{T+2}^{T} = \delta + \phi E_T[Y_{T+1}] \quad (\text{as } E_T[W_{T+2}] = 0)$$

$$Y_{T+2}^{T} = \delta + \phi Y_{T+1}^{T}$$

$$Y_{T+2}^{T} = \delta + \phi(\delta + \phi Y_T)$$

$$Y_{T+2}^{T} = \delta + \phi\delta + \phi^2 Y_T$$

$$Y_{T+2}^{T} = \delta(1 + \phi) + \phi^2 Y_T$$

for $j = 3$:

$$Y_{T+3}^{T} = \delta + \phi Y_{T+2}^{T}$$

substitute $Y_{T+2}^{T} = \delta(1 + \phi) + \phi^2 Y_T$:

$$Y_{T+3}^{T} = \delta + \phi \left(\delta(1 + \phi) + \phi^2 Y_T\right)$$

$$Y_{T+3}^{T} = \delta + \phi\delta(1 + \phi) + \phi^3 Y_T$$

$$Y_{T+3}^{T} = \delta(1 + \phi + \phi^2) + \phi^3 Y_T$$

by continuing this i can write for general j:

$$Y_{T+j}^{T} = \delta(1 + \phi + \cdots + \phi^{j-1}) + \phi^j Y_T$$

or i can use induction hypothesis

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4 Induction Hypothesis
assume that for some $j = k$, following holds:

$$Y_{T+k}^{T} = \delta \left[1 + \phi + \ldots + \phi^{k-1}\right] + \phi^k Y_T.$$

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5. induction step for $j = k + 1$
prove for $j = k + 1$. using AR(1) forecast relation:

$$Y_{T+k+1}^{T} = \delta + \phi Y_{T+k}^{T}$$

—
substitute

$$Y_{T+k}^{T} = \delta \left[1 + \phi + \ldots + \phi^{k-1}\right] + \phi^k Y_T$$

into forecast equation:

$$Y_{T+k+1}^{T} = \delta + \phi \left(\delta \left[1 + \phi + \ldots + \phi^{k-1}\right] + \phi^k Y_T\right)$$

—

$$Y_{T+k+1}^{T} = \delta \left[1 + \phi + \ldots + \phi^{k}\right] + \phi^{k+1} Y_T$$

so eqn. holds for $j = k + 1$.

6. general for Y_{T+j}
so by induction, i proved for Y^T_{T+j} :

$$Y^T_{T+j} = \delta\left[1 + \phi + \ldots + \phi^{j-1}\right] + \phi^j Y_T,$$

for all $j \geq 1$. hence proved.

Q2(b)

we have to show that for $m = 1, 2, \ldots$:

$$X^T_{T+m} = X_T + \frac{\delta}{1-\phi}\left[m - \frac{\phi(1-\phi^m)}{1-\phi}\right] + (X_T - X_{T-1})\frac{\phi(1-\phi^m)}{1-\phi}.$$

1. from Part (a)
for $j \geq 1$:

$$Y^T_{T+j} = \delta\left(1 + \phi + \ldots + \phi^{j-1}\right) + \phi^j Y_T.$$

sum $1 + \phi + \ldots + \phi^{j-1}$ is geometric series:

$$1 + \phi + \phi^2 + \ldots + \phi^{j-1} = \frac{1-\phi^j}{1-\phi}, \quad \text{for} \quad \phi \neq 1.$$

so

$$Y^T_{T+j} = \delta\frac{1-\phi^j}{1-\phi} + \phi^j Y_T.$$

2. cumulative sum
as $Y_t = X_t - X_{t-1}$, the cumulative sum over m steps is:

$$\sum_{j=1}^{m} Y^T_{T+j} = \sum_{j=1}^{m}\left(X^T_{T+j} - X^T_{T+j-1}\right).$$

telescoping property of sums:

$$\sum_{j=1}^{m}\left(X^T_{T+j} - X^T_{T+j-1}\right) = X^T_{T+m} - X_T.$$

now, i substitute expression for Y^T_{T+j} from point 1:

$$\sum_{j=1}^{m} Y^T_{T+j} = \sum_{j=1}^{m}\left(\delta\frac{1-\phi^j}{1-\phi} + \phi^j Y_T\right).$$

3. calculate the summation
distribute sum:

$$\sum_{j=1}^{m} Y^T_{T+j} = \sum_{j=1}^{m}\frac{\delta(1-\phi^j)}{1-\phi} + \sum_{j=1}^{m}\phi^j Y_T.$$

3.1 first sum

$$\sum_{j=1}^{m}\frac{\delta(1-\phi^j)}{1-\phi} = \frac{\delta}{1-\phi}\sum_{j=1}^{m}(1-\phi^j).$$

use geometric series sum:

$$\sum_{j=1}^{m}(1-\phi^j) = m - \frac{1-\phi^{m+1}}{1-\phi},$$

put back in :

$$\sum_{j=1}^{m} \frac{\delta(1-\phi^j)}{1-\phi} = \frac{\delta}{1-\phi}\left(m - \frac{1-\phi^{m+1}}{1-\phi}\right).$$

3.2 second sum

$$\sum_{j=1}^{m} \phi^j Y_T = Y_T \sum_{j=1}^{m} \phi^j = Y_T \frac{\phi(1-\phi^m)}{1-\phi}.$$

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4. substituting results
i substitute both sum from point 3:

$$\sum_{j=1}^{m} Y_{T+j}^T = \frac{\delta}{1-\phi}\left(m - \frac{1-\phi^{m+1}}{1-\phi}\right) + Y_T \frac{\phi(1-\phi^m)}{1-\phi}.$$

using telescoping property:

$$X_{T+m}^T - X_T = \frac{\delta}{1-\phi}\left(m - \frac{1-\phi^{m+1}}{1-\phi}\right) + Y_T \frac{\phi(1-\phi^m)}{1-\phi}.$$

as

$$Y_T = X_T - X_{T-1}.$$

substitute in eqn:

$$X_{T+m}^T - X_T = \frac{\delta}{1-\phi}\left(m - \frac{1-\phi^{m+1}}{1-\phi}\right) + (X_T - X_{T-1}) \frac{\phi(1-\phi^m)}{1-\phi}.$$

6.
rearrange X_{T+m}^T:

$$X_{T+m}^T = X_T + \frac{\delta}{1-\phi}\left[m - \frac{\phi(1-\phi^m)}{1-\phi}\right] + (X_T - X_{T-1}) \frac{\phi(1-\phi^m)}{1-\phi}. \qquad \text{hence prove}$$

Q2(c)

I have to compute mean squared prediction error P_{T+m}^T for large T, using coefficients ψ_j^*:

$$P_{T+m}^T = \sigma_W^2 \sum_{j=0}^{m-1} (\psi_j^*)^2,$$

where ψ_j^* are coefficients of z^j in the expansion of:

$$\psi^*(z) = \frac{\theta(z)}{\phi(z)(1-z)^d},$$

now $\theta(z) = 1$ and $\phi(z) = 1 - \phi z$ correspond to ARIMA(1, 1, 0) model given in Ques..

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1. first i expand $\psi^*(z)$
by expanding expression:

$$\psi^*(z) = \frac{1}{(1-\phi z)(1-z)}.$$

first expand denominator:

$$(1-\phi z)(1-z) = 1 - (1+\phi)z + \phi z^2.$$

rewrite:

$$\psi^*(z) = \frac{1}{1-(1+\phi)z+\phi z^2}.$$

use geometric series expansion:

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n, \quad u = (1+\phi)z - \phi z^2,$$

we get:

$$\psi^*(z) = \sum_{n=0}^{\infty} \left[(1+\phi)z - \phi z^2\right]^n.$$

$$n = 0: \quad \left[(1+\phi)z - \phi z^2\right]^0 = 1$$

$$n = 1: \quad \left[(1+\phi)z - \phi z^2\right]^1 = (1+\phi)z - \phi z^2$$

$$n = 2: \quad \left[(1+\phi)z - \phi z^2\right]^2 = (1+\phi)^2 z^2 - 2\phi(1+\phi)z^3 + \phi^2 z^4 \qquad \text{so on ...}$$

so

$$\psi^*(z) = 1 + (1+\phi)z + \left[(1+\phi)^2 - \phi\right] z^2 + \dots$$

as ψ_j^* are coefficients of z^j in the expansion of $\psi^*(z)$

$$\psi^*(z)(1-\phi z)(1-z) = \left(1 + \psi_1^* z + \psi_2^* z^2 + \dots\right)\left(1 - [1+\phi]z + z^2\right) = 1$$

$$1 \cdot (1 - [1+\phi]z + z^2) + \psi_1^* z \cdot (1 - [1+\phi]z + z^2) + \psi_2^* z^2 \cdot (1 - [1+\phi]z + z^2) + \dots = 1.$$

i compare coeffs. from both sides:

Collect terms by powers of z:

for z^0:

$$\psi_0^* = 1.$$

for z^1:

$$-(1+\phi) + \psi_1^* = 0 \implies \psi_1^* = 1 + \phi.$$

similarly for z^j (for $j \geq 2$):

$$\psi_j^* = \frac{1-\phi^{j+1}}{1-\phi}.$$

so homogeneous solution is:

$$\psi_0^* = 1, \quad \psi_j^* = \frac{1-\phi^{j+1}}{1-\phi} \quad \text{for} \quad j \geq 1.$$

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2. mean squared prediction error

mean-squared prediction error for large T is given by:

$$P_{T+m}^T = \sigma_W^2 \sum_{j=0}^{m-1} (\psi_j^*)^2.$$

i use coeffs ψ_j^* from point 1,

$$(\psi_0^*)^2 = 1\ , (\psi_j^*)^2 = \left(\frac{1-\phi^{j+1}}{1-\phi}\right)^2. \quad \text{for} \quad j \geq 1.$$

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3. simplifying Summation

from 2 mean-squared prediction error becomes:

$$P_{T+m}^T = \sigma_W^2 \left[1 + \frac{1}{(1-\phi)^2} \sum_{j=1}^{m-1} \left(1 - \phi^{j+1}\right)^2\right].$$

for large m, end terms in sum become small, as $(1-\phi^{j+1})^2 \approx 1$ for large j. so expression for mean-squared prediction error for large T is approximated by:

$$P_{T+m}^T = \sigma_W^2 \left[1 + \frac{m-1}{(1-\phi)^2}\right].$$