Lecture 6

Measures of Dependence Part 2

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Stationary Time Series (Recap)

- A **weakly stationary** time series is one for which
  - variance of the process is finite at each time point,
  - the mean value function μ_t is constant and does not depend on t,
  - the autocovariance function, $\gamma(s, t)$ depends on s and t only through their difference $|s - t|$.

- We will use the term stationary to mean weakly stationary; if a process is stationary in the strict sense, we will use the term strictly stationary.

(Weakly) Stationary Time Series

- Because the mean function, $E(X_t) = \mu_t$, of a stationary time series is independent of time t, we will write $\mu_t = \mu$ for all t.

- Let $s = t + h$, where h represents the time shift or lag. Then

$$\mathrm{Cov}(X_s, X_t) = \gamma(s, t) = \gamma(t + h, t) = \gamma(h, 0).$$

- The autocovariance function of a stationary time series does not depend on the time argument t.

- Henceforth, for convenience, we will drop the second argument of $\gamma(h, 0)$ and write $\gamma(h)$.

(Weakly) Stationary Time Series

- The autocovariance function of a stationary time series will be written as

$$\gamma(h) = \mathrm{Cov}(X_{t+h}, X_t) = E[(X_{t+h} - \mu)(X_t - \mu)].$$

- The autocorrelation function (ACF) of a stationary time series will be written as

$$\rho(h) = \frac{\gamma(t+h, t)}{\sqrt{\gamma(t+h, t+h) \times \gamma(t, t)}} = \frac{\gamma(h)}{\gamma(0)}.$$

- Stationarity of White Noise: $\gamma_W(h) = ?$, $\rho_W(h) = ?$
- Stationarity of a Moving Average: $\gamma_V(h) = ?$, $\rho_V(h) = ?$
- The autocovariance function is symmetric, i.e.,

$$\gamma(h) = \mathrm{Cov}(X_{t+h}, X_t) = E[(X_{t+h} - \mu)(X_t - \mu)] = \mathrm{Cov}(X_t, X_{t+h}) = \gamma(-h).$$

Autocorrelation function (ACF)

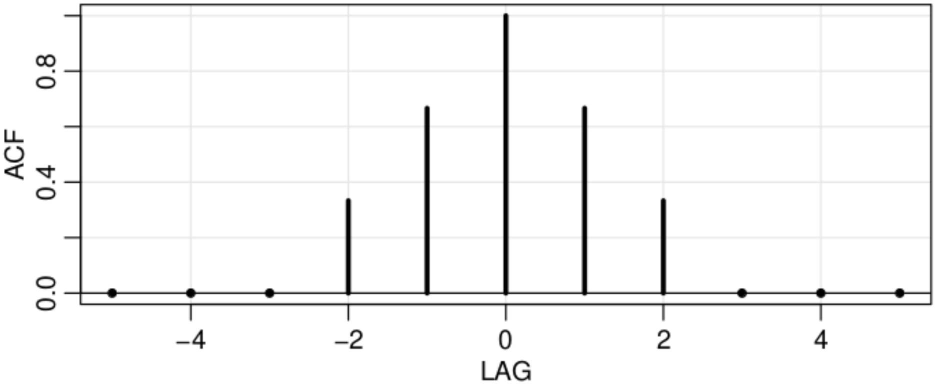

Fig. 1.12. *Autocorrelation function of a three-point moving average.*

Random walk

- A model for analyzing trend such as seen in the global temperature data is the random walk with drift model given by

$$X_t = \delta + X_{t-1} + W_t.$$

  for $t = 1, 2, \ldots$, with initial condition $X_0 = 0$, and where W_t is white noise.

- Note that we may rewrite the model as a cumulative sum of white noise variates. That is, $X_t = \delta t + \sum_{j=1}^{t} W_j$.
- Is random walk stationary? Mean? Autocovariance?
- Is random walk with drift stationary? Mean? Autocovariance?
- We can also think of trend stationarity.

Trend stationary models

- Suppose the observed time series is a realization of the process

$$X_t = \beta_0 + \beta_1 t + Y_t, \;\; t = 1, \ldots$$

  where Y_t is stationary.

- If $\mu_Y = E[Y_t]$, $\mu_{X,t} = E[X_t] = ?$

- If $\sigma_Y^2 = \mathrm{Var}(Y_t)$, then $\sigma_{X,t}^2 = \mathrm{Var}[X_t] = ?$

- If $\gamma_Y(h) = \mathrm{Cov}(Y_{t+h}, Y_t)$, then $\gamma_X(h) = ?$

Properties of $\gamma(\cdot)$

- For any $n \geq 1$, and constants $a_1, \ldots, a_n$,

$$0 \leq \mathrm{Var}(a_1 X_{t_1} + \ldots + a_n X_{t_n}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \gamma(t_i - t_j).$$

- For $h = 0$, $\gamma(0) = E[(X_t - \mu)^2]$, and and the Cauchy-Schwarz inequality implies

$$|\gamma(h)| \leq \gamma(0).$$

- The autocovariance function of a stationary series is symmetric around the origin, i.e.,

$$\gamma(h) = \gamma(-h).$$

Joint stationarity

- Two time series X_t and Y_t are said to be jointly stationary if they are each stationary, and the cross-covariance function

$$\gamma_{X,Y}(h) = \mathrm{Cov}(X_{t+h}, Y_t) = E[(X_{t+h} - \mu_X)(Y_t - \mu_Y)]$$

is a function only of lag h.

- The cross-correlation function (CCF) of jointly stationary time series X_t and Y_t is defined as

$$\rho_{X,Y}(h) = \frac{\gamma_{X,Y}(h)}{\sqrt{\gamma_X(0) \times \gamma_Y(0)}}.$$

- Show that $\rho_{X,Y}(h) = \rho_{Y,X}(h)$ is not necessarily true but $\rho_{X,Y}(h) = \rho_{Y,X}(-h)$.
- Example: $X_t = W_t + W_{t-1}$ and $Y_t = W_t - W_{t-1}$.

Thank you!