Solutions

1/. $f(\omega) = \sigma^2, \quad \omega \in [-1/2, 1/2]$

$$\gamma(0) = \int_{-1/2}^{1/2} e^{i.2\pi.\omega.0} f(\omega)\, d\omega$$

$$= \int_{-1/2}^{1/2} f(\omega)\, d\omega$$

$$= \int_{-1/2}^{+1/2} \sigma^2 . d\omega$$

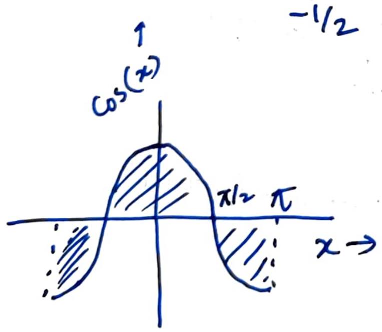
$$= \sigma^2 . \left(1/2 - (-1/2)\right)$$

$$= \sigma^2 .$$

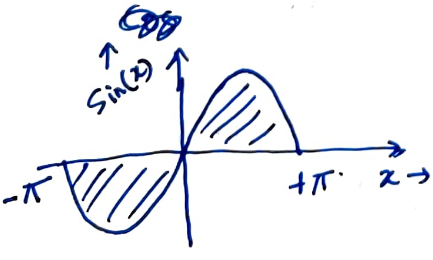
$$\gamma(1) = \int_{-1/2}^{+1/2} e^{i.2\pi\omega.1} f(\omega)\, d\omega$$

$$= \sigma^2 . \int_{-1/2}^{+1/2} \left[\cos(2\pi\omega) + i.\sin(2\pi\omega)\right] d\omega$$

$$= \sigma^2 \int_{-1/2}^{1/2} \cos(2\pi\omega)\, d\omega + i.\sigma^2 . \int_{-1/2}^{1/2} \sin(2\pi\omega)\, d\omega$$



$$\int_{-1/2}^{1/2} \cos(2\pi\omega)\, d\omega = 0 .$$



$$\int_{-1/2}^{1/2} \sin(2\pi\omega)\, d\omega = 0 .$$

$$\Rightarrow \gamma(1) = 0 .$$

2/. $X_t = \phi X_{t-1} + W_t \Rightarrow W_t = X_t - \phi X_{t-1}$

$$\gamma_W(h) = Cov(W_{t+h}, W_t)$$
$$= Cov(X_{t+h} - \phi X_{t+h-1}, X_t - \phi X_{t-1})$$
$$= Cov(X_{t+h}, X_t) - \phi Cov(X_{t+h-1}, X_t) - \phi Cov(X_{t+h}, X_{t-1}) + \phi^2 Cov(X_{t+h-1}, X_{t-1})$$

$$= \gamma_X(h) - \phi \gamma_X(h-1) - \phi \gamma_X(h+1) + \phi^2 \cdot \gamma_X(h).$$

$$f_W(\omega) = \sum_{h=-\infty}^{+\infty} \gamma_W(h) \cdot e^{-i \cdot 2\pi\omega h}.$$

$$= \sum_{h=-\infty}^{+\infty} \{\gamma_X(h) - \phi \gamma_X(h-1) - \phi \gamma_X(h+1) + \phi^2 \gamma_X(h)\} e^{-i \cdot 2\pi\omega h}$$

$$= \sum_{h=-\infty}^{+\infty} \gamma_X(h) \cdot e^{-i \cdot 2\pi\omega h} - \phi \sum_{h=-\infty}^{+\infty} \gamma_X(h-1) \cdot e^{-i \cdot 2\pi\omega h} - \phi \sum_{h=-\infty}^{+\infty} \gamma_X(h+1) e^{-i \cdot 2\pi\omega h} + \phi^2 \sum_{h=-\infty}^{+\infty} \gamma_X(h) e^{-i \cdot 2\pi\omega h}$$

$$= (1+\phi^2) f_X(\omega) - \phi \sum_{h=-\infty}^{+\infty} \gamma_X(h-1) \cdot e^{-i \cdot 2\pi\omega(h-1+1)} - \phi \sum_{h=-\infty}^{+\infty} \gamma_X(h+1) \cdot e^{-i \cdot 2\pi\omega(h+1-1)}$$

$$= (1+\phi^2) f_X(\omega) - \phi \cdot e^{-i \cdot 2\pi\omega} \cdot \sum_{h=-\infty}^{+\infty} \gamma_X(h-1) \cdot e^{-i \cdot 2\pi\omega(h-1)} - \phi \cdot e^{i \cdot 2\pi\omega} \cdot \sum_{h=-\infty}^{+\infty} \gamma_X(h+1) \cdot e^{-i \cdot 2\pi\omega(h+1)}$$

$$= (1+\phi^2) f_X(\omega) - \phi \cdot (e^{-i \cdot 2\pi\omega} + e^{i \cdot 2\pi\omega}) \cdot f_X(\omega)$$

$= (1+\phi^2) f_X(\omega) - \phi . \{ \cos(2\pi\omega) - i . \cancel{\sin(2\pi\omega)} + \cos(2\pi\omega) + i . \cancel{\sin(2\pi\omega)} \} f_X(\omega)$

$= (1+\phi^2) f_X(\omega) - 2\phi . \cos(2\pi\omega) . f_X(\omega)$

$= (1+\phi^2 - 2\phi . \cos(2\pi\omega)) f_X(\omega)$.

But, we know, $f_W(\omega) = \sigma^2 \; \forall \omega \in [-1/2, 1/2]$

$\Rightarrow f_X(\omega) = \sigma^2 . \dfrac{1}{1+\phi^2 - 2\phi \cos(2\pi\omega)} \; . \; \forall \omega \in [-1/2 . 1/2]$.

3/. The scaled periodogram is symmetric around 0.5.
Hence, $P(0.7) = P(0.3) = 8$, and $P(0.9) = P(0.1) = 2$.

4/. $R_t \sim$ ARCH(1). with $\alpha_0 = 0.5$, $\alpha_1 = 0.5$,
i.e., $R_t | R_{t-1} \sim N(0, \alpha_0 + \alpha_1 R_{t-1}^2)$.

$E(R_t) = E(E[R_t | R_{t-1}]) = E(0) = 0$.

$Cov(R_{t+h}, R_t) = E(R_{t+h} R_t) = E(E(R_{t+h} R_t | R_{t+h-1}))$

$= E(R_t . E(R_{t+h} | R_{t+h-1})) = E(R_t \cdot 0) = 0$.

$V(R_t) = E(R_t^2) = E(\alpha_0 + \alpha_1 R_{t-1}^2) = \alpha_0 + \alpha_1 E(R_{t-1}^2)$

$= \alpha_0 + \alpha_1 V(R_{t-1})$.

However, because $V(R_t)$ is constant W.R.T. t.

$\Rightarrow V(R_t) = V(R_{t-1}) = V$, (say)

Then, $V = \alpha_0 + \alpha_1 V \Rightarrow V = \dfrac{\alpha_0}{1-\alpha_1} = \dfrac{0.5}{0.5} = 1$.

5/ $Var[Var(R_t | R_{t-1}) | R_{t-1}]$

$= Var(\alpha_0 + \alpha_1 R_{t-1}^2 | R_{t-1})$.

$= 0$.

6/. $\rho(h) \sim h^{2d-1}$.

Hence, $\sum_{h=-\infty}^{+\infty} |\rho(h)| \sim \sum_{h=-\infty}^{+\infty} |h|^{2d-1}$.

We call it a long-memory process, when this sum is unbounded, which happens when

$2d-1 > -1 \Rightarrow 2d > 0 \Rightarrow d > 0$.

We define fractional differencing for $|d| < 0.5$.

Hence, here the range is $(0, 0.5)$.

7/. Consider a stochastic process $X_t = \phi X_{t-1} + W_t$, where W_t is a white noise. Here, if $\phi = 1$, it becomes a random walk. If $|\phi| < 1$, then the process is causal. Hence, to test whether the a process is causal or a random walk, the unit root test performs the hypothesis testing $H_0: \phi = 1$ vs $H_1: |\phi| < 1$.

The test statistic used here is $T(\hat{\phi} - 1)$ and it is called Dickey-Fuller (DF) statistic.

8/ No, the Y_t's are not conditionally independent.

$$\pi(Y_1,\dots,Y_T) = \int \pi(Y_1,\dots,Y_T, X_1,\dots,X_T)\,dX_1\cdots dX_T$$

$$= \int \pi(Y_1,\dots,Y_T \mid X_1,\dots,X_T)\,.\,\pi(X_1,\dots,X_T)\,dX_1\cdots dX_T$$

$$= \int \Big\{\prod_{t=1}^{T} \pi(Y_t \mid X_t)\Big\}.\,\pi(X_1)\,.\prod_{t=2}^{T} \pi(X_t \mid X_{t-1})\,dX_1\cdots dX_T$$

$$\neq \int \Big\{\prod_{t=1}^{T} \pi(Y_t \mid X_t)\Big\}\,\pi(X_1)\,.\prod_{t=2}^{T} \pi(X_t)\,dX_1\cdots dX_T$$

Unless $\phi = 0$.

$$= \int \prod_{t=1}^{T} \pi(Y_t \mid X_t)\,.\prod_{t=1}^{T} \pi(X_t)\,dX_1\cdots dX_T.$$

$$= \prod_{t=1}^{T} \Big\{\int \pi(Y_t \mid X_t)\,\pi(X_t)\,dX_t\Big\}$$

$$= \prod_{t=1}^{T} \pi(Y_t).$$

Because. $\pi(Y_1,\dots,Y_T) \neq \prod_{t=1}^{T} \pi(Y_t)$, Y_t's are not independent.

9/. $$X_t^{t-1} = E(X_t \mid Y_1,\dots,Y_{t-1}) = E(\phi X_{t-1} + W_t \mid Y_1,\dots,Y_{t-1})$$

$$= \phi E(X_{t-1} \mid Y_1,\dots,Y_{t-1}) + E(W_t \mid Y_1,\dots,Y_{t-1})$$

$$= \phi . X_{t-1}^{t-1}, \quad \text{where } E(W_t \mid Y_1,\dots,Y_{t-1}) = E(W_t) = 0$$

through projection theorem.

$$P_t^{t-1} = Var(X_t \mid Y_1,\dots,Y_{t-1}) = var(\phi X_{t-1} + W_t \mid Y_1,\dots,Y_{t-1})$$

$$= Var(\phi X_{t-1} \mid Y_1,\dots,Y_{t-1}) + Var(W_t \mid Y_1,\dots,Y_{t-1})$$

$$= \phi^2 .\, var(X_{t-1} \mid Y_1,\dots,Y_{t-1}) + var(W_t) = \phi^2 . P_{t-1}^{t-1} + \sigma_W^2.$$

10/. $\gamma_Y(h) = Cov(Y_{t+h}, Y_t)$

$$= Cov(X_{t+h} + V_{t+h}, X_t + V_t)$$

$$= Cov(X_{t+h}, X_t) + Cov(V_{t+h}, V_t)$$

$$= (1-\phi^2)^{-1} \sigma_W^2 \cdot \phi^h + \sigma_V^2 I(h=0)$$

$$\gamma_Y(0) = (1-\phi^2)^{-1} \sigma_W^2 \cancel{\phi} + \sigma_V^2$$

$$\gamma_Y(1) = (1-\phi^2)^{-1} \sigma_W^2 \cdot \phi$$

$$\gamma_Y(2) = (1-\phi^2)^{-1} \sigma_W^2 \phi^2 .$$

$$\Rightarrow \frac{\gamma_Y(2)}{\gamma_Y(1)} = \frac{(1-\phi^2)^{-1} \sigma_W^2 \cdot \phi^2}{(1-\phi^2)^{-1} \sigma_W^2 \phi} = \phi .$$

$$\Rightarrow \frac{\gamma_Y(2)/\gamma_Y(0)}{\gamma_Y(1)/\gamma_Y(0)} = \phi$$

$$\Rightarrow \frac{\rho_y(2)}{\rho_y(1)} = \phi .$$

Hence, we can calculate Sample ACF with $\hat{\rho}_Y(2)$ and $\hat{\rho}_Y(1)$ and choose $\phi^{(0)} = \frac{\hat{\rho}_Y(2)}{\hat{\rho}_Y(1)}$.

We can also calculate sample autocovariance.

$\hat{\gamma}_Y(1) = (1-\phi^2)^{-1} \sigma_W^2 \cdot \phi$.

Plugging in $\phi^{(0)}$, we get. $\hat{\gamma}_Y(1) = (1-\phi^{(0)2})^{-1} \sigma_W^{2(0)} \cdot \phi^{(0)}$

$\Rightarrow \sigma_W^{2(0)} = \hat{\gamma}_Y(1) \cdot (1-\phi^{(0)2})/\phi^{(0)}$.

We can also calculate $\hat{\gamma}_Y(0)$, the Sample autocovariance at lag 0.

$\hat{\gamma}_Y(0) = (1-\phi^{(0)2})^{-1} \sigma_W^{2(0)} + \sigma_V^{2(0)} \Rightarrow \sigma_V^{2(0)} = \hat{\gamma}_Y^{(0)} - (1-\phi^{(0)2})^{-1} \sigma_W^{2(0)}$

11/ $\mathcal{L}_{x,y}(\Theta) \propto \pi(x_0)\prod_{t=1}^{T}\pi(x_t|x_{t-1}) \times \prod_{t=1}^{T}\pi(y_t|x_t)$.

$$\propto \frac{1}{\sqrt{2\pi.(1-\phi^2)^{-1}\sigma_w^2}} e^{-\frac{x_0^2}{2(1-\phi^2)^{-1}\sigma_w^2}} \times \prod_{t=1}^{T}\frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{1}{2\sigma_w^2}(x_t-\phi x_{t-1})^2}$$
$$\times \prod_{t=1}^{T}\frac{1}{\sqrt{2\pi\sigma_v^2}} e^{-\frac{1}{2\sigma_v^2}(y_t-x_t)^2}$$

$$\propto (1-\phi^2)^{1/2}(\sigma_w^2)^{-1/2} \times (\sigma_w^2)^{-T/2} \times (\sigma_v^2)^{-T/2}$$
$$\times e^{-\frac{1}{2}\left\{\frac{(1-\phi^2)x_0^2}{\sigma_w^2} + \frac{1}{\sigma_w^2}\sum_{t=1}^{T}(x_t-\phi x_{t-1})^2 + \frac{1}{\sigma_v^2}\sum_{t=1}^{T}(y_t-x_t)^2\right\}}$$

$$\log L_{x,y}(\Theta) = -\frac{1}{2}\log(1-\phi^2) - \frac{T+1}{2}\log(\sigma_w^2) - \frac{T}{2}\log(\sigma_v^2)$$
$$-\frac{1}{2}\left\{\frac{(1-\phi^2)x_0^2}{\sigma_w^2} + \frac{1}{\sigma_w^2}\sum_{t=1}^{T}(x_t-\phi x_{t-1})^2 + \frac{1}{\sigma_v^2}\sum_{t=1}^{T}(y_t-x_t)^2\right\}$$

$$-2\log L_{x,y}(\Theta) = \log(1-\phi^2) + (T+1)\log(\sigma_w^2) + T\log(\sigma_v^2)$$
$$+ \frac{(1-\phi^2)x_0^2}{\sigma_w^2} + \frac{1}{\sigma_w^2}\sum_{t=1}^{T}(x_t-\phi x_{t-1})^2 + \frac{1}{\sigma_v^2}\sum_{t=1}^{T}(y_t-x_t)^2$$

12/ $Q(\Theta|\Theta^{(j-1)}) = E(-2\log L_{x,y}(\Theta | y_{1:T}, \Theta^{(j-1)}))$

$$E(-2\log L_{x,y}(\Theta) | y_{1:T}) = \log(1-\phi^2) + (T+1)\log(\sigma_w^2) + T\log(\sigma_v^2)$$
$$+ \frac{1-\phi^2}{\sigma_w^2}\cdot E(x_0^2 | y_{1:T}) + \frac{1}{\sigma_w^2}\sum_{t=1}^{T}E([x_t-\phi x_{t-1}]^2 | y_{1:T})$$
$$+ \frac{1}{\sigma_v^2}\sum_{t=1}^{T}E((y_t-x_t)^2 | y_{1:T})$$

$$E(x_0^2|y_{1:T}) = E(x_0|y_{1:T})^2 + V(x_0|y_{1:T})$$
$$= (x_0^T)^2 + P_0^T$$

$$E(y_t^2 - 2y_t x_t + x_t^2 | y_{1:T}) = E(y_t^2|y_{1:T}) - 2E(y_t x_t|y_{1:T}) + E(x_t^2|y_{1:T})$$
$$= y_t^2 - 2y_t E(x_t|y_{1:T}) + \{E(x_t|y_{1:T})^2 + V(x_t|y_{1:T})\}$$

$$= y_t^2 - 2y_t x_t^T + (x_t^T)^2 + P_t^T$$

$$E\left[(x_t - \phi x_{t-1})^2 \mid y_{1:T}\right]$$

$$\cancel{= E\left(\left[x_t - x_t^T + x_t^T - \phi(x_{t-1} - x_{t-1}^T + x_{t-1}^T)\right]^2 \mid y_{1:T}\right)}$$

$$= E\left[(x_t^2 - 2\phi x_t x_{t-1} + \phi^2 x_{t-1}^2) \mid y_{1:T}\right]$$

$$= E(x_t^2 \mid y_{1:T}) - 2\phi E(x_t x_{t-1} \mid y_{1:T}) + \phi^2 E(x_{t-1}^2 \mid y_{1:T}).$$

$$E(x_t^2 \mid y_{1:T}) = E(x_t \mid y_{1:T})^2 + V(x_t \mid y_{1:T})$$

$$= (x_t^T)^2 + P_t^T.$$

$$E(x_{t-1}^2 \mid y_{1:T}) = E(x_{t-1} \mid y_{1:T})^2 + V(x_{t-1} \mid y_{1:T})$$

$$= \cancel{\phi}(x_{t-1}^T)^2 + P_{t-1}^T.$$

$$E(x_t x_{t-1} \mid y_{1:T}) = E\left((x_t - x_t^T + x_t^T)(x_{t-1} - x_{t-1}^T + x_{t-1}^T) \mid y_{1:T}\right)$$

$$= E\left[(x_t - x_t^T)(x_{t-1} - x_{t-1}^T) \mid y_{1:T}\right]$$

$$+ x_t^T E\left[(x_{t-1} - x_{t-1}^T) \mid y_{1:T}\right]$$

$$+ x_{t-1}^T E\left[(x_t - x_t^T) \mid y_{1:T}\right] + x_t^T \cdot x_{t-1}^T$$

$$= P_{t,t-1}^T + 0 + 0 + x_t^T x_{t-1}^T$$

$$= P_{t,t-1}^T + x_t^T \cdot x_{t-1}^T$$

$$\Rightarrow E\left[(x_t - \phi x_{t-1})^2 \mid y_{1:T}\right] = \left[x_t^T\right]^2 + P_t^T + \phi^2\left(\left[x_{t-1}^T\right]^2 + P_{t-1}^T\right)$$

$$- 2\phi\left(P_{t,t-1}^T + x_t^T x_{t-1}^T\right).$$

By plugging in all the terms, we obtain the final expression of $Q(\Theta \mid \Theta^{(j-1)}) = E\left(-2\log L_{x,y}(\Theta \mid y_{1:T}), \Theta^{(j-1)}\right)$

$$= \log(1-\phi^2) + (T+1)\log(\sigma_W^2) + T\log(\sigma_V^2)$$
$$+ \frac{1-\phi^2}{\sigma_W^2}\left([X_0^T]^2 + P_0^T\right) + \frac{1}{\sigma_V^2}\sum_{t=1}^{T}\left(Y_t^2 - 2Y_tX_t^T + [X_t^T]^2 + P_t^T\right)$$
$$+ \frac{1}{\sigma_W^2}\sum_{t=1}^{T}\left\{[X_t^T]^2 + P_t^T + \phi^2\left([X_{t-1}^T]^2 + P_{t-1}^T\right) - 2\phi\left(P_{t,t-1}^T + X_t^TX_{t-1}^T\right)\right\}$$

where ϕ, σ_W^2, and σ_V^2 are plugged in as $\phi^{(j-1)}$, $\sigma_W^{2(j-1)}$ and $\sigma_V^{2(j-1)}$.

13/ the terms involving σ_V^2 are.

$$g(\sigma_V^2) \equiv T\log(\sigma_V^2) + \frac{1}{\sigma_V^2}\sum_{t=1}^{T}\left(Y_t^2 - 2Y_tX_t^T + [X_t^T]^2 + P_t^T\right)$$

$$g'(\sigma_V^2) = T/\sigma_V^2 - \frac{1}{(\sigma_V^2)^2}\cdot\sum_{t=1}^{T}\left(Y_t^2 - 2Y_tX_t^T + [X_t^T]^2 + P_t^T\right) = 0$$

$$\Rightarrow \sigma_V^{2(j)} = \frac{1}{T}\sum_{t=1}^{T}\left(Y_t^2 - 2Y_tX_t^T + [X_t^T]^2 + P_t^T\right).$$

the terms involving σ_W^2 are.

$$h(\sigma_W^2) = (T+1)\log(\sigma_W^2) + \frac{(1-\phi^2)\left\{[X_0^T]^2 + P_0^T\right\} + \sum_{t=1}^{T}\left\{[X_t^T]^2 + P_t^T + \phi^2\left([X_{t-1}^T]^2 + P_{t-1}^T\right) - 2\phi\left(P_{t,t-1}^T + X_t^TX_{t-1}^T\right)\right\}}{\sigma_W^2}$$

$$h'(\sigma_W^2) = 0$$

$$\Rightarrow \hat{\sigma}_W^{2\,(j)} = \frac{1}{T+1}\left\{(1-\phi^2)\left\{[X_0^T]^2 + P_0^T\right\} + \sum_{t=1}^{T}\left\{[X_t^T]^2 + P_t^T + \phi^2\left([X_{t-1}^T]^2 + P_{t-1}^T\right) - 2\phi\left(P_{t,t-1}^T + X_t^TX_{t-1}^T\right)\right\}\right\}$$

RHS calculated based on $\phi^{(j-1)}$.

The terms involving ϕ are

$$K(\phi) = \log(1-\phi^2) + \frac{1-\phi^2}{\sigma_w^2}\left([X_0^T]^2 + P_0^T\right)$$
$$+ \frac{1}{\sigma_W^2}\sum_{t=1}^{T}\left\{[X_t^T]^2 + P_t^T + \phi^2\left([X_{t-1}^T]^2 + P_{t-1}^T\right) - 2\phi\left(P_{t,t-1}^T + X_t^T X_{t-1}^T\right)\right\}$$

$$K'(\phi) = \frac{-2\phi}{1-\phi^2} - \frac{2\phi}{\sigma_W^2}\left([X_0^T]^2 + P_0^T\right) + \frac{1}{\sigma_W^2}\sum_{t=1}^{T}\left\{2\phi\left([X_{t-1}^T]^2 + P_{t-1}^T\right) - 2\left(P_{t,t-1}^T + X_t^T X_{t-1}^T\right)\right\}$$

We can solve $K'(\phi) = 0$. and Solve numerically.
whatever Solution we get, ~~Supp~~ we denote by $\phi^{(j)}$.