

# **Introduction to Statistical Time Series**

## **WILEY SERIES IN PROBABILITY AND STATISTICS**

**Established by WALTER A. SHEWHART and SAMUEL S. WILKS**

**Editors: *Vic Barnett, Ralph A. Bradley, Nicholas I. Fisher, J. Stuart Hunter, J. B. Kadane, David G. Kendall, David W. Scott, Adrian F. M. Smith, Jozef L. Teugels, Geoffrey S. Watson***

**A complete list of the titles in this series appears at the end of this volume**

# Introduction to Statistical Time Series

Second Edition

WAYNE A. FULLER

Iowa State University



A Wiley-Interscience Publication

JOHN WILEY & SONS, INC.

New York • Chichester • Brisbane • Toronto • Singapore

# A NOTE TO THE READER

This book has been electronically reproduced from digital information stored at John Wiley & Sons, Inc. We are pleased that the use of this new technology will enable us to keep works of enduring scholarly value in print as long as there is a reasonable demand for them. The content of this book is identical to previous printings.

This text is printed on acid-free paper.

Copyright © 1996 by John Wiley & Sons, Inc.

All rights reserved. Published simultaneously in Canada.

No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, electronic, mechanical, photocopying, recording, scanning or otherwise, except as permitted under Sections 107 or 108 of the 1976 United States Copyright Act, without either the prior written permission of the Publisher, or authorization through payment of the appropriate per-copy fee to the Copyright Clearance Center, 222 Rosewood Drive, Danvers, MA 01923, (978) 750-8400, fax (978) 750-4470. Requests to the Publisher for permission should be addressed to the Permissions Department, John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030, (201) 748-6011, fax (201) 748-6008, E-Mail PERMREQ@WILEY.COM.

To order books or for customer service please, call 1(800)-CALL-WILEY (225-5945).

## *Library of Congress Cataloging in Publication Data:*

Fuller, Wayne A.

Introduction to statistical time series / Wayne A. Fuller. -- 2nd ed.

p. cm. --- (Wiley series in probability and statistics)

"A Wiley-Interscience publication."

Includes bibliographical references and index.

ISBN 0-471-55239-9 (cloth : alk. paper)

1. Time-series analysis. 2. Regression analysis.

I. Title

II. Series.

QA280.F84 1996

519.5'5--dc20

95-14875

10 9 8 7 6 5 4 3

To Evelyn

# Contents

<b>Preface to the First Edition</b>	<b>xi</b>
<b>Preface to the Second Edition</b>	<b>xiii</b>
<b>List of Principal Results</b>	<b>xv</b>
<b>List of Examples</b>	<b>xxi</b>
<b>1. Introduction</b>	<b>1</b>
1.1 Probability Spaces	1
1.2 Time Series	3
1.3 Examples of Stochastic Processes	4
1.4 Properties of the Autocovariance and Autocorrelation Functions	7
1.5 Complex Valued Time Series	12
1.6 Periodic Functions and Periodic Time Series	13
1.7 Vector Valued Time Series	15
References	17
Exercises	17
<b>2. Moving Average and Autoregressive Processes</b>	<b>21</b>
2.1 Moving Average Processes	21
2.2 Absolutely Summable Sequences and Infinite Moving Averages	26
2.3 An Introduction to Autoregressive Time Series	39
2.4 Difference Equations	41
2.5 The Second Order Autoregressive Time Series	54
2.6 Alternative Representations of Autoregressive and Moving Average Processes	58
2.7 Autoregressive Moving Average Time Series	70
2.8 Vector Processes	75
2.9 Prediction	79
2.10 The Wold Decomposition	94
	<b>vii</b>

2.11	Long Memory Processes	98
	References	101
	Exercises	101
<b>3.</b>	<b>Introduction to Fourier Analysis</b>	<b>112</b>
3.1	Systems of Orthogonal Functions—Fourier Coefficients	112
3.2	Complex Representation of Trigonometric Series	130
3.3	Fourier Transform—Functions Defined on the Real Line	132
3.4	Fourier Transform of a Convolution	136
	References	139
	Exercises	139
<b>4.</b>	<b>Spectral Theory and Filtering</b>	<b>143</b>
4.1	The Spectrum	143
4.2	Circulants—Diagonalization of the Covariance Matrix of Stationary Process	149
4.3	The Spectral Density of Moving Average and Autoregressive Time Series	155
4.4	Vector Processes	169
4.5	Measurement Error—Signal Detection	181
4.6	State Space Models and Kalman Filtering	187
	References	205
	Exercises	205
<b>5.</b>	<b>Some Large Sample Theory</b>	<b>214</b>
5.1	Order in Probability	214
5.2	Convergence in Distribution	227
5.3	Central Limit Theorems	233
5.4	Approximating a Sequence of Expectations	240
5.5	Estimation for Nonlinear Models	250
	5.5.1 Estimators that Minimize an Objective Function	250
	5.5.2 One-Step Estimation	268
5.6	Instrumental Variables	273
5.7	Estimated Generalized Least Squares	279
5.8	Sequences of Roots of Polynomials	290
	References	299
	Exercises	299

<b>6. Estimation of the Mean and Autocorrelations</b>	<b>308</b>
6.1 Estimation of the Mean	308
6.2 Estimators of the Autocovariance and Autocorrelation Functions	313
6.3 Central Limit Theorems for Stationary Time Series	320
6.4 Estimation of the Cross Covariances	339
References	348
Exercises	348
<b>7. The Periodogram, Estimated Spectrum</b>	<b>355</b>
7.1 The Periodogram	355
7.2 Smoothing, Estimating the Spectrum	366
7.3 Other Estimators of the Spectrum	380
7.4 Multivariate Spectral Estimates	385
References	400
Exercises	400
<b>8. Parameter Estimation</b>	<b>404</b>
8.1 First Order Autoregressive Time Series	404
8.2 Higher Order Autoregressive Time Series	407
8.2.1 Least Squares Estimation for Univariate Processes	407
8.2.2 Alternative Estimators for Autoregressive Time Series	413
8.2.3 Multivariate Autoregressive Time Series	419
8.3 Moving Average Time Series	421
8.4 Autoregressive Moving Average Time Series	429
8.5 Prediction with Estimated Parameters	443
8.6 Nonlinear Processes	451
8.7 Missing and Outlier Observations	458
8.8 Long Memory Processes	466
References	471
Exercises	471
<b>9. Regression, Trend, and Seasonality</b>	<b>475</b>
9.1 Global Least Squares	476
9.2 Grafted Polynomials	480
9.3 Estimation Based on Least Squares Residuals	484
9.3.1 Estimated Autocorrelations	484
9.3.2 Estimated Variance Functions	488



9.4	Moving Averages—Linear Filtering	497
9.4.1	Moving Averages for the Mean	497
9.4.2	Moving Averages of Integrated Time Series	502
9.4.3	Seasonal Adjustment	504
9.4.4	Differences	507
9.5	Structural Models	509
9.6	Some Effects of Moving Average Operators	513
9.7	Regression with Time Series Errors	518
9.8	Regression Equations with Lagged Dependent Variables and Time Series Errors	530
	References	538
	Exercises	538
<b>10.</b>	<b>Unit Root and Explosive Time Series</b>	<b>546</b>
10.1	Unit Root Autoregressive Time Series	546
10.1.1	The Autoregressive Process with a Unit Root	546
10.1.2	Random Walk with Drift	565
10.1.3	Alternative Estimators	568
10.1.4	Prediction for Unit Root Autoregressions	582
10.2	Explosive Autoregressive Time Series	583
10.3	Multivariate Autoregressive Processes with Unit Roots	596
10.3.1	Multivariate Random Walk	596
10.3.2	Vector Process with a Single Unit Root	599
10.3.3	Vector Process with Several Unit Roots	617
10.4	Testing for a Unit Root in a Moving Average Model	629
	References	638
	Exercises	638
10.A	Percentiles for Unit Root Distributions	641
10.B	Data Used in Examples	653
	<b>Bibliography</b>	<b>664</b>
	<b>Index</b>	<b>689</b>

# Preface to the First Edition

This textbook was developed from a course in time series given at Iowa State University. The classes were composed primarily of graduate students in economics and statistics. Prerequisites for the course were an introductory graduate course in the theory of statistics and a course in linear regression analysis. Since the students entering the course had varied backgrounds, chapters containing elementary results in Fourier analysis and large sample statistics, as well as a section on difference equations, were included in the presentation.

The theorem-proof format was followed because it offered a convenient method of organizing the material. No attempt was made to present the most general results available. Instead, the objective was to give results with practical content whose proofs were generally consistent with the prerequisites. Since many of the statistics students had completed advanced courses, a few theorems were presented at a level of mathematical sophistication beyond the prerequisites. Homework requiring application of the statistical methods was an integral part of the course.

By emphasizing the relationship of the techniques to regression analysis and using data sets of moderate size, most of the homework problems can be worked with any of a number of statistical packages. One such package is SAS (Statistical Analysis System, available through the Institute of Statistics, North Carolina State University). SAS contains a segment for periodogram computations that is particularly suited to this text. The system also contains a segment for regression with time series errors compatible with the presentation in Chapter 9. Another package is available from International Mathematical and Statistical Library, Inc.; this package has a chapter on time series programs.

There is some flexibility in the order in which the material can be covered. For example, the major portions of Chapters 1, 2, 5, 6, 8, and 9 can be treated in that order with little difficulty. Portions of the later chapters deal with spectral matters, but these are not central to the development of those chapters. The discussion of multivariate time series is positioned in separate sections so that it may be introduced at any point.

I thank A. R. Gallant for the proofs of several theorems and for the repair of others: J. J. Goebel for a careful reading of the manuscript that led to numerous substantive improvements and the removal of uncounted mistakes; and D. A.

Dickey, M. Hidioglou, R. J. Klemm, and G. H. K. Wang for computing examples and for proofreading. G. E. Battese, R. L. Carter, K. R. Crouse, J. D. Cryer, D. P. Hasza, J. D. Jobson, B. Macpherson, J. Mellon, D. A. Pierce and K. N. Wolter also read portions of the manuscript. I also thank my colleagues, R. Groeneveld, D. Isaacson, and O. Kempthorne, for useful comments and discussions. I am indebted to a seminar conducted by Marc Nerlove at Stanford University for the organization of some of the material on Fourier analysis and spectral theory. A portion of the research was supported by joint statistical agreements with the U.S. Bureau of the Census.

I thank Margaret Nichols for the repeated typings required to bring the manuscript to final form and Avonelle Jacobson for transforming much of the original illegible draft into typescript.

WAYNE A. FULLER

*Ames, Iowa*  
*February 1976*

## Preface to the Second Edition

Considerable development in statistical time series has occurred since the first edition was published in 1976. Notable areas of activity include nonstationary models, nonlinear estimation, multivariate models, state space representations and empirical model identification. The second edition attempts to incorporate new results and to respond to recent emphases while retaining the basic format of the first edition.

With the exception of new sections on the Wold decomposition, partial autocorrelation, long memory processes, and the Kalman filter, Chapters one through four are essentially unchanged from the first edition. Chapter 5 has been enlarged, with additional material on central limit theorems for martingale differences, an expanded treatment of nonlinear estimation, a section on estimated generalized least squares, and a section on the roots of polynomials. Chapter 6 and Chapter 8 have been revised using the asymptotic theory of Chapter 5. Also, the discussion of estimation methods has been modified to reflect advances in computing. Chapter 9 has been revised and the material on the estimation of regression equations has been expanded.

The material on nonstationary autoregressive models is now in a separate chapter, Chapter 10. New tests for unit roots in univariate processes and in vector processes have been added.

As with the first edition, the material is arranged in sections so that there is considerable flexibility to the order in which topics can be covered.

I thank David Dickey and Heon Jin Park for constructing the tables of Chapter 10. I thank Anthony An, Rohit Deo, David Hasza, N. K. Nagaraj, Sastry Pantula, Heon Jin Park, Savas Papadopoulos, Sahadeb Sarkar, Dongwan Shin, and George H. K. Wang for many useful suggestions. I am particularly indebted to Sastry Pantula who assisted with the material of Chapters 5, 8, 9, and 10 and made substantial contributions to other parts of the manuscript, including proofs of several results. Sahadeb Sarkar contributed to the material on nonlinear estimation of Chapter 5, Todd Sanger contributed to the discussion of estimated generalized least squares, Yasuo Amemiya contributed to the section on roots of polynomials, Rohit Deo contributed to the material on long memory processes, Sastry Pantula, Sahadeb Sarkar and Dongwan Shin contributed to the material on the limiting

distribution of estimators for autoregressive moving averages, and Heon Jin Park contributed to the sections on unit root autoregressive processes. I thank Abdoulaye Adam, Jay Breidt, Rohit Deo, Kevin Dodd, Savas Papadopoulos, and Anindya Roy for computing examples. I thank SAS Institute, Cary, NC, for providing computing support to Heon Jin Park for the construction of tables for unit root tests. The research for the second edition was partly supported by joint statistical agreements with the U.S. Bureau of the Census.

I thank Judy Shafer for the extensive word processing required during preparation of the second edition.

WAYNE A. FULLER

*Ames, Iowa*  
*November 1995*

# List of Principal Results

<i>Theorem</i>	<i>Topic</i>
1.4.1	Covariance function is positive semidefinite, 7
1.4.2	Covariance function is even, 8
1.4.3	Correlation function on real line is a characteristic function, 9
1.4.4	Correlation function on integers is a characteristic function, 9
1.5.1	Covariance function of complex series is positive semidefinite, 13
2.2.1	Weighted average of random variables, where weights are absolutely summable, defines a random variable, 31
2.2.2	Covariance of two infinite sums of random variables, 33
2.2.3	Convergence in mean square of sum of random variables, 35
2.4.1	Order of a polynomial is reduced by differencing, 46
2.4.2	Jordan canonical form of a matrix, 51
2.6.1	Representation of autoregressive process as an infinite moving average, 59
2.6.2	Representation of invertible moving average as an infinite autoregression, 65
2.6.3	Moving average representation of a time series based on covariance function, 66
2.6.4	Canonical representation of moving average time series, 68
2.7.1	Representation of autoregressive moving average as an infinite moving average, 72
2.7.2	Representation of autoregressive moving average as an infinite autoregression, 74
2.8.1	Representation of vector autoregression as an infinite moving average, 77
2.8.2	Representation of vector moving average as an infinite autoregression, 78
2.9.1	Minimum mean square error predictor, 80
2.9.2	Durbin–Levinson algorithm for constructing predictors, 82

- 2.9.3 Predictors as a function of previous prediction errors, 86
- 2.9.4 Limit of prediction error is a moving average, 89
- 2.10.1 Limit of one period prediction error, 94
- 2.10.2 Wold decomposition, 96
- 3.1.1 Sine and cosine functions form an orthogonal basis for  $N$  dimensional vectors, 112
- 3.1.2 Sine and cosine functions are orthogonal on  $[-\pi, \pi]$ , 116
- 3.1.3 Bessel's inequality, 118
- 3.1.4 Fourier coefficients are zero if and only if function is zero, 119
- 3.1.5 If Fourier coefficients are zero, integral of function is zero, 119
- 3.1.6 Integral of function defined in terms of Fourier coefficients, 120
- 3.1.7 Pointwise representation of a function by Fourier series, 123
- 3.1.8 Absolute convergence of Fourier series for a class of functions, 125
- 3.1.9 The correlation function defines a continuous spectral density, 127
- 3.1.10 The Fourier series of a continuous function is Cesàro summable, 129
- 3.3.1 Fourier integral theorem, 133
- 3.4.1 Fourier transform of a convolution, 137
- 4.2.1 Approximate diagonalization of covariance matrix with orthogonal sine-cosine functions, 154
- 4.3.1 Spectral density of an infinite moving average of a time series, 156
- 4.3.2 Moving average representation of a time series based on covariance function, 162
- 4.3.3 Moving average representation of a time series based on continuous spectral density, 163
- 4.3.4 Autoregressive representation of a time series based on continuous spectral density, 165
- 4.3.5 Spectral density of moving average with square summable coefficients, 167
- 4.4.1 Spectral density of vector process, 179
- 4.5.1 Linear filter for time series observed subject to measurement error, 183
- 5.1.1 Chebyshev's inequality, 219
- 5.1.2 Common probability limits, 221
- 5.1.3 Convergence in  $r$ th mean implies convergence in probability, 221
- 5.1.4 Probability limit of a continuous function, 222
- 5.1.5 The algebra of  $O_p$ , 223
- 5.1.6 The algebra of  $o_p$ , 224
- 5.1.7 Taylor expansion about a random point, 226
- 5.2.1 Convergence in law when difference converges in probability, 228

- 5.2.2 Helly–Bray Theorem, 230
- 5.2.3 Joint convergence of distribution functions and characteristic functions, 230
- 5.2.4 Convergence in distribution of continuous functions, 230
- 5.2.5 Joint convergence in law of independent sequences, 230
- 5.2.6 Joint convergence in law when one element converges to a constant, 232
- 5.3.1 Lindeberg central limit theorem, 233
- 5.3.2 Liapounov central limit theorem, 233
- 5.3.3 Central limit theorem for vectors, 234
- 5.3.4 Central limit theorem for martingale differences, 235
- 5.3.5 Functional central limit theorem, 236
- 5.3.6 Convergence of functions of partial sums, 237
- 5.3.7 Multivariate functional central limit theorem, 238
- 5.3.8 Almost sure convergence of martingales, 239
- 5.4.1 Moments of products of sample means, 242
- 5.4.2 Approximate expectation of functions of means, 243
- 5.4.3 Approximate expectation of functions of vector means, 244
- 5.4.4 Bounded integrals of functions of random variables, 247
- 5.5.1 Limiting properties of nonlinear least squares estimator, 256
- 5.5.2 Limiting distribution of estimator defined by an objective function, 260
- 5.5.3 Consistency of nonlinear estimator with different rates of convergence, 262
- 5.5.4 Limiting properties of one-step Gauss–Newton estimator, 269
- 5.6.1 Limiting properties of instrumental variable estimators, 275
- 5.7.1 Convergence of estimated generalized least squares estimator to generalized least squares estimator, 280
- 5.7.2 Central limit theorem for estimated generalized least squares, 284
- 5.7.3 Estimated generalized least squares with finite number of covariance parameters, 286
- 5.7.4 Estimated generalized least squares based on simple least squares residuals, 289
- 5.8.1 Convergence of roots of a sequence of polynomials, 295
- 5.8.2 Differentials of roots of determinantal equation, 298
- 6.1.1 Convergence in mean square of sample mean, 309
- 6.1.2 Variance of sample mean as function of the spectral density, 310
- 6.1.3 Limiting efficiency of sample mean, 312
- 6.2.1 Covariances of sample autocovariances, 314
- 6.2.2 Covariances of sample autocovariances, mean estimated, 316



- 6.2.3 Covariances of sample correlations, 317
- 6.3.1 Central limit theorem for  $m$ -dependent sequences, 321
- 6.3.2 Convergence in probability of mean of an infinite moving average, 325
- 6.3.3 Central limit theorem for mean of infinite moving average, 326
- 6.3.4 Central limit theorem for linear function of infinite moving average, 329
- 6.3.5 Consistency of sample autocovariances, 331
- 6.3.6 Central limit theorem for autocovariances, 333
- 6.4.1 Covariances of sample autocovariances of vector time series, 342
- 6.4.2 Central limit theorem for cross covariances one series independent  $(0, \sigma^2)$ , 345
- 7.1.1 Expected value of periodogram, 359
- 7.1.2 Limiting distribution of periodogram ordinates, 360
- 7.2.1 Covariances of periodogram ordinates, 369
- 7.2.2 Limiting behavior of weighted averages of periodogram ordinates, 372
- 7.3.1 Limiting behavior of estimated spectral density based on weighted autocovariances, 382
- 7.4.1 Diagonalization of covariance matrix of bivariate process, 387
- 7.4.2 Distribution of sample Fourier coefficients for bivariate process, 389
- 7.4.3 Distribution of smoothed bivariate periodogram, 390
- 8.2.1 Limiting distribution of regression estimators of parameters of  $p$ th order autoregressive process, 408
- 8.2.2 Equivalence of alternative estimators of parameters of autoregressive process, 418
- 8.2.3 Limiting distribution of estimators of parameter of  $p$ th order vector autoregressive process, 420
- 8.3.1 Limiting distribution of nonlinear estimator of parameter of first order moving average, 424
- 8.4.1 Limiting distribution of nonlinear estimator of vector of parameters of autoregressive moving average, 432
- 8.4.2 Equivalence of alternative estimators for autoregressive moving average, 434
- 8.5.1 Order of error in prediction due to estimated parameters, 444
- 8.5.2 Expectation of prediction error, 445
- 8.5.3 Order  $n^{-1}$  approximation to variance of prediction error, 446
- 8.6.1 Polynomial autoregression, 452
- 8.8.1 Maximum likelihood estimators for long memory processes, 470
- 9.1.1 Limiting distribution of simple least squares estimated parameters of regression model with time series errors, 478

- 9.1.2 Spectral representation of covariance matrix of simple least squares estimator and of generalized least squares estimator, 479
- 9.1.3 Asymptotic relative efficiency of simple least squares and generalized least squares, 480
- 9.3.1 Properties of autocovariances computed from least squares residuals, 485
- 9.4.1 Centered moving average estimator of polynomial trend, 501
- 9.4.2 Trend moving average removes autoregressive unit root, 502
- 9.4.3 Effect of a moving average for polynomial trend removal, 504
- 9.7.1 Asymptotic equivalence of generalized least squares and estimated generalized least squares for model with autoregressive errors, 521
- 9.7.2 Limiting distribution of maximum likelihood estimator of regression model with autoregressive errors, 526
- 9.8.1 Limiting distribution of least squares estimator of model with lagged dependent variables, 530
- 9.8.2 Limiting distribution of instrumental variable estimator of model with lagged dependent variables, 532
- 9.8.3 Limiting properties of autocovariances computed from instrumental variable residuals, 534
- 10.1.1 Limiting distribution of least squares estimator of first order autoregressive process with unit root, 550
- 10.1.2 Limiting distribution of least squares estimator of  $p$ th order autoregressive process with a unit root, 556
- 10.1.3 Limit distribution of least squares estimator of first order unit root process with mean estimated, 561
- 10.1.4 Limiting distribution of least squares estimator of  $p$ th order process with a unit root and mean estimated, 563
- 10.1.5 Limiting distribution of least squares estimator of unit root autoregression with drift, 566
- 10.1.6 Limiting distribution of least squares estimator of unit root autoregression with time in fitted model, 567
- 10.1.7 Limiting distribution of symmetric estimator of first order unit root autoregressive process, 570
- 10.1.8 Limiting distribution of symmetric estimators adjusted for mean and time trend, 571
- 10.1.9 Limiting distribution of symmetric test statistics for  $p$ th order autoregressive process with a unit root, 573
- 10.1.10 Limiting distribution of maximum likelihood estimator of unit root, 573
- 10.1.11 Order of error in prediction of unit root process with estimated parameters, 582

- 10.2.1 Limiting distribution of explosive autoregressive estimator, 585
- 10.2.2 Limiting distribution of vector of least squares estimators for  $p$ th order autoregression with an explosive root, 589
- 10.3.1 Limiting distribution of least squares estimator of vector of estimators of equation containing the lagged dependent variable with unit coefficient, 600
- 10.3.2 Limiting distribution of least squares estimator when one of the explanatory variables is a unit root process, 603
- 10.3.3 Limiting distribution of least squares estimator of coefficients of vector process with unit roots, 610
- 10.3.4 Limiting distribution of maximum likelihood estimator for multivariate process with a single root, 613
- 10.3.5 Limiting distribution of maximum likelihood estimator for multivariate process with  $g$  unit roots, 619

# List of Examples

<i>Number</i>	<i>Topic</i>
1.3.1	Finite index set, 4
1.3.2	White noise, 5
1.3.3	A nonstationary time series, 5
1.3.4	Continuous time series, 6
2.1.1	First order moving average time series, 23
2.1.2	Second order moving average time series, 24
2.3.1	First order autoregressive time series, 40
2.5.1	Correlogram of time series, 57
2.9.1	Prediction for unemployment rate, 90
2.9.2	Prediction for autoregressive moving average, 91
4.1.1	Spectral distribution function composed of jumps, 147
4.1.2	Spectral distribution function of series with white noise component, 148
4.5.1	Filter for time series observed subject to measurement error, 184
4.6.1	Kalman Filter for Des Moines River data, 192
4.6.2	Kalman filter, Des Moines River, mean unknown, 193
4.6.3	Kalman filter, autoregressive unit root, 195
4.6.4	Kalman filter for missing observations, 198
4.6.5	Predictions constructed with the Kalman filter, 202
5.1.1	Taylor expansion about a random point, 227
5.4.1	Approximation to expectation, 248
5.5.1	Transformation of model with different rates of convergence, 265
5.5.2	Failure of convergence of second derivative of nonlinear model, 267
5.5.3	Gauss–Newton estimation, 272
5.6.1	Instrumental variable estimation, 277
6.3.1	Sample autocorrelations and means of unemployment rate, 336

- 6.4.1 Autocorrelations and cross correlations of Boone–Saylorville data, 345
- 7.1.1 Periodogram for wheat yield data, 363
- 7.2.1 Periodogram of autoregression, 375
- 7.2.2 Periodogram of unemployment rate, 379
- 7.4.1 Cross spectrum computations, 394
- 8.2.1 Autoregressive fit to unemployment rate, 412
- 8.3.1 Estimation of first order moving average, 427
- 8.4.1 Autoregressive moving average fit to artificial data, 439
- 8.4.2 Autoregressive moving average fit to unemployment rate, 440
- 8.5.1 Prediction with estimated parameters, 449
- 8.6.1 Nonlinear models for lynx data, 455
- 8.7.1 Missing observations, 459
- 8.7.2 Outlier observations, 462
- 9.2.1 Grafted quadratic fit to U.S. wheat yields, 482
- 9.3.1 Variance as a function of the mean, 490
- 9.3.2 Stochastic volatility model, 495
- 9.5.1 Structural model for wheat yields, 510
- 9.7.1 Regression with autoregressive errors, spirit consumption, 522
- 9.7.2 Nonlinear estimation of trend in wheat yields with autoregressive error, 527
- 9.7.3 Nonlinear estimation for spirit consumption model, 528
- 9.8.1 Regression with lagged dependent variable and autoregressive errors, 535
- 10.1.1 Estimation and testing for a unit root, ordinary least squares, 564
- 10.1.2 Testing for a unit root, symmetric and likelihood procedures, 577
- 10.1.3 Estimation for process with autoregressive root in  $(-1, 1]$ , 581
- 10.1.4 Prediction for a process with a unit root, 583
- 10.2.1 Estimation with an explosive root, 593
- 10.2.2 Prediction for an explosive process, 596
- 10.3.1 Estimation of regression with autoregressive explanatory variable, 606
- 10.3.2 Estimation and testing for vector process with unit roots, 624
- 10.4.1 Test for a moving average unit root, 635

# Introduction to Statistical Time Series