MTH442 Assignment 5

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Q1

Given system process :

$$X_t = \phi X_{t-2} + W_t, \quad t = 1, 2, \ldots, T$$

- $X_0 \sim N(0, \sigma_0^2)$ and $X_{-1} \sim N(0, \sigma_1^2)$ are initial condition.
- W_t is gaussian white noise with var.σ_W^2, i.e., $W_t \sim N(0, \sigma_W^2)$.

process X_t is observed with measurement noise:

$$Y_t = X_t + V_t$$

- V_t is gaussian white noise with var.σ_V^2, i.e., $V_t \sim N(0, \sigma_V^2)$.
- $X_0, X_{-1}, \{W_t\}$, and $\{V_t\}$ are all mutually independent.

(a)

we have to express system in standard state space form, comprising state and observation eqn.s.

1. State-Space Model Formulation

to represent process as state-space model, i define state vector that includes all dependencies in recursive eqn.
current state X_t depends on X_{t-2}, soi need to keep track of both X_t and X_{t-1} as part of our state vector.
i define state vector (i am representing it in bold letter so don't get confused) at time t :

$$\mathbf{X}_t = \begin{bmatrix} X_t \\ X_{t-1} \end{bmatrix}$$

here:

- X_t is current state of process.
- X_{t-1} is needed to reference X_{t-2} in future time steps.

2. deriving State Eqn

from given process,

$$X_t = \phi X_{t-2} + W_t$$

using this eqn. and point 1 , express $\mathbf{X}_t$ in terms of $\mathbf{X}_{t-1}$ using these relations i rewrite evolution of state vector:

$$\mathbf{X}_t = \begin{bmatrix} X_t \\ X_{t-1} \end{bmatrix} = \begin{bmatrix} \phi X_{t-2} + W_t \\ X_{t-1} \end{bmatrix} = \begin{bmatrix} 0 & \phi \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_{t-1} \\ X_{t-2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} W_t$$

in general form:

$$\mathbf{X}_t = \mathbf{A}\mathbf{X}_{t-1} + \mathbf{B}W_t$$

$$\mathbf{A} = \begin{bmatrix} 0 & \phi \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- matrix $\mathbf{A}$ defines transition dynamics of state vector.
- matrix $\mathbf{B}$ shows how process noise W_t affect current state.

3. Observation Eqn

observed process is:

$$Y_t = X_t + V_t$$

as X_t is first element of state vector $\mathbf{X}_t$, i express Y_t in term of $\mathbf{X}_t$ using point 1 as:

$$Y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_t \\ X_{t-1} \end{bmatrix} + V_t$$

in general form:

$$Y_t = \mathbf{H}\mathbf{X}_t + V_t$$

here

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\mathbf{X}_t = \begin{bmatrix} X_t \\ X_{t-1} \end{bmatrix}$$

- matrix $\mathbf{H}$ extracts relevant part of state vector (i.e., X_t) for observation eqn.
- V_t is observation noise with var. σ_V^2.

4.complete state space model

- **State eqn.**:

$$\mathbf{X}_t = \begin{bmatrix} 0 & \phi \\ 1 & 0 \end{bmatrix} \mathbf{X}_{t-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} W_t$$

  here $W_t \sim N(0, \sigma_W^2)$.

- **Observation Eqn**:

$$Y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{X}_t + V_t$$

  here $V_t \sim N(0, \sigma_V^2)$.

this formulation can be used to apply state estimation technique to estimate X_t from observation Y_t, considering both process and observation noise.

Q1(b)

using relation btw Y_t, X_t.i can say for Y_t to be stationary, X_t must be stationary. i derive conditions for X_t to satisfy stationarity.

Process

process X_t is :

$$X_t = \Phi X_{t-1} + W_t,$$

1. find X_{2t-1}

for $t = 1$:

$$X_1 = \phi X_{-1} + W_1$$

for $t = 3$:

$$X_3 = \phi X_1 + W_3$$

$$X_3 = \phi(\phi X_{-1} + W_1) + W_3$$

$$X_3 = \phi^2 X_{-1} + \phi W_1 + W_3$$

for $t = 5$:

$$X_5 = \phi X_3 + W_5$$

$$X_5 = \phi(\phi^2 X_{-1} + \phi W_1 + W_3) + W_5$$

$$X_5 = \phi^3 X_{-1} + \phi^2 W_1 + \phi W_3 + W_5$$

i can generalize this for t=0,1,2,...by recursively putting previous values (t here is index ,dont be confused in notation):

$$X_{2t-1} = \phi^t X_{-1} + \sum_{j=0}^{t-1} \phi^j W_{2t-1-2j}$$

i can also prove same by induction

Base Case ($t = 1$):

$$X_1 = \phi X_{-1} + W_1.$$

it matches form:

$$X_{2t-1} = \phi^t X_{-1} + \sum_{j=0}^{t-1} \phi^j W_{2t-1-2j},$$

as $t = 1 \Rightarrow \phi^1 X_{-1} + W_1$. so base case holds.

Inductive Hypothesis:

Assume form holds for $t = k$:

$$X_{2k-1} = \phi^k X_{-1} + \sum_{j=0}^{k-1} \phi^j W_{2k-1-2j}.$$

Inductive Step ($t = k + 1$):

From recurrence relation:

$$X_{2(k+1)-1} = X_{2k+1} = \phi X_{2k-1} + W_{2k+1}.$$

Substitute inductive hypothesis:

$$X_{2k+1} = \phi \left(\phi^k X_{-1} + \sum_{j=0}^{k-1} \phi^j W_{2k-1-2j} \right) + W_{2k+1}.$$

$$X_{2k+1} = \phi^{k+1} X_{-1} + \phi \sum_{j=0}^{k-1} \phi^j W_{2k-1-2j} + W_{2k+1}.$$

$$X_{2k+1} = \phi^{k+1} X_{-1} + \sum_{j=0}^{k} \phi^j W_{2k+1-2j}.$$

so, formula holds for $t = k + 1$. by induction, formula holds.

2. find X_{2t}

similarly

$$X_2 = \phi X_0 + W_2$$

$$X_4 = \phi X_2 + W_4$$
$$X_4 = \phi(\phi X_0 + W_2) + W_4$$
$$X_4 = \phi^2 X_0 + \phi W_2 + W_4$$

$$X_6 = \phi X_4 + W_6$$
$$X_6 = \phi(\phi^2 X_0 + \phi W_2 + W_4) + W_6$$
$$X_6 = \phi^3 X_0 + \phi^2 W_2 + \phi W_4 + W_6$$

Generalize:

$$X_{2t} = \sum_{j=0}^{t-1} \phi^j W_{2t-2j} + \phi^t X_0$$

3. independent:

using Ques. $X_0, X_{-1}, \{W_t\}$, and $\{V_t\}$ are all mutually independent.

Independence of Noise Terms:

$$\{W_{2t}, W_{2t-2}, \ldots, W_2\} \cap \{W_{2t-1}, W_{2t-3}, \ldots, W_1\} = \emptyset,$$

$$W_t \text{ are i.i.d. random variables.}$$

$$\sum_{j=0}^{t-1} \phi^j W_{2t-2j} \perp \sum_{j=0}^{t-1} \phi^j W_{2t-1-2j}.$$

Independence of Deterministic Terms using Ques. :

$$X_0 \perp X_{-1}.$$

so:

$$X_{2t}, X_{2t-1}. \text{ are independent}$$

4. making X_t stationary

process:

$$X_t = \phi X_{t-1} + W_t,$$

i claim that setting $X_0 = \frac{W_0}{\sqrt{1-\phi^2}}$ and $X_{-1} = \frac{W_{-1}}{\sqrt{1-\phi^2}}$ ensures that process X_t becomes stationary.

Expectation of X_t

as:

$$X_{2t} = \sum_{j=0}^{t-1} \phi^j W_{2t-2j} + \phi^t X_0$$

$$X_{2t-1} = \phi^t X_{-1} + \sum_{j=0}^{t-1} \phi^j W_{2t-1-2j}$$

expectation $E(X_t)$ is:

$$E(X_t) = E(X_{2t} + X_{2t-1}).$$

substituting X_{2t} and X_{2t-1} and Using linearity of expectation:

$$E(X_{2t}) = \sum_{j=0}^{t-1} \phi^j E(W_{2t-2j}) + \phi^t E(X_0),$$

$$E(X_{2t-1}) = \phi^t E(X_{-1}) + \sum_{j=0}^{t-1} \phi^j E(W_{2t-1-2j}).$$

as W_t has zero mean, $E(W_t) = 0$,also from Q. $E(X_0) = 0$ and $E(X_{-1}) = 0$,i get:

$$E(X_{2t}) = 0,$$

$$E(X_{2t-1}) = 0.$$

so , expectation of X_t is:

$$E(X_t) = E(X_{2t}) + E(X_{2t-1}) = 0 + 0 = 0.$$

so mean of Xt remains constant over time.

Variance of X_t

Variance of X_{2t}

$$X_{2t} = \sum_{j=0}^{t-1} \phi^j W_{2t-2j} + \phi^t X_0$$

Variance of X_{2t} is:

$$\mathrm{Var}(X_{2t}) = \mathrm{Var}\left(\sum_{j=0}^{t-1} \phi^j W_{2t-2j}\right) + \mathrm{Var}(\phi^t X_0)$$

For first term:

$$\mathrm{Var}\left(\sum_{j=0}^{t-1} \phi^j W_{2t-2j}\right) = \sigma_w^2 \sum_{j=0}^{t-1} \phi^{2j}$$

Using geometric series sum formula:

$$\sum_{j=0}^{t-1} \phi^{2j} = \frac{1-\phi^{2t}}{1-\phi^2}$$

so:

$$\mathrm{Var}\left(\sum_{j=0}^{t-1} \phi^j W_{2t-2j}\right) = \sigma_w^2 \frac{1-\phi^{2t}}{1-\phi^2}$$

For second term:

$$\mathrm{Var}(\phi^t X_0) = (\phi^t)^2 \sigma_0^2$$

Combining both:

$$\mathrm{Var}(X_{2t}) = \sigma_w^2 \frac{1-\phi^{2t}}{1-\phi^2} + (\phi^t)^2 \sigma_0^2$$

Variance of X_{2t-1}

$$X_{2t-1} = \phi^t X_{-1} + \sum_{j=0}^{t-1} \phi^j W_{2t-1-2j}$$

Variance of X_{2t-1} is:

$$\mathrm{Var}(X_{2t-1}) = \mathrm{Var}(\phi^t X_{-1}) + \mathrm{Var}\left(\sum_{j=0}^{t-1} \phi^j W_{2t-1-2j}\right)$$

For first term:

$$\mathrm{Var}(\phi^t X_{-1}) = (\phi^t)^2 \sigma_{-1}^2$$

For second term:

$$\mathrm{Var}\left(\sum_{j=0}^{t-1} \phi^j W_{2t-1-2j}\right) = \sigma_w^2 \sum_{j=0}^{t-1} \phi^{2j}$$

Using ssame geometric series formula:

$$\sum_{j=0}^{t-1} \phi^{2j} = \frac{1-\phi^{2t}}{1-\phi^2}$$

so:

$$\mathrm{Var}\left(\sum_{j=0}^{t-1} \phi^j W_{2t-1-2j}\right) = \sigma_w^2 \frac{1-\phi^{2t}}{1-\phi^2}$$

Combining both:

$$\mathrm{Var}(X_{2t-1}) = (\phi^t)^2 \sigma_{-1}^2 + \sigma_w^2 \frac{1-\phi^{2t}}{1-\phi^2}$$

Variance of X_t

$$\mathrm{Var}(X_t) = \mathrm{Var}(X_{2t}) + \mathrm{Var}(X_{2t-1}) + 2\,\mathrm{Cov}(X_{2t}, X_{2t-1})$$

using that both are independent from point 3 , $\mathrm{Cov}(X_{2t}, X_{2t-1}) = 0$:

$$\mathrm{Var}(X_t) = \mathrm{Var}(X_{2t}) + \mathrm{Var}(X_{2t-1})$$

Substituting:

$$\mathrm{Var}(X_t) = \sigma_w^2 \frac{1-\phi^{2t}}{1-\phi^2} + (\phi^t)^2\sigma_0^2 + (\phi^t)^2\sigma_{-1}^2 + \sigma_w^2 \frac{1-\phi^{2t}}{1-\phi^2}$$

$$\mathrm{Var}(X_t) = 2\sigma_w^2 \frac{1-\phi^{2t}}{1-\phi^2} + (\phi^t)^2(\sigma_0^2 + \sigma_{-1}^2)$$

now using claim that $\mathrm{X}_0 = W_0/\sqrt{1-\phi^2}$ and $X_{-1} = W_{-1}/\sqrt{1-\phi^2}$
so

$$\mathrm{Var}(X_0) = \sigma_0^2 = \frac{\sigma_w^2}{1-\phi^2}, \quad \mathrm{Var}(X_{-1}) = \sigma_{-1}^2 = \frac{\sigma_w^2}{1-\phi^2}$$

substitue

$$\mathrm{Var}(X_t) = 2\sigma_w^2 \frac{1-\phi^{2t}}{1-\phi^2} + (\phi^t)^2(\sigma_0^2 + \sigma_{-1}^2)$$

$$\mathrm{Var}(X_t) = 2\sigma_w^2 \frac{1-\phi^{2t}}{1-\phi^2} + (\phi^t)^2 \frac{2\sigma_w^2}{1-\phi^2}$$

$$\mathrm{Var}(X_t) = \frac{2\sigma_w^2}{1-\phi^2}\left(1-\phi^{2t}+\phi^{2t}\right)$$

$$\mathrm{Var}(X_t) = \frac{2\sigma_w^2}{1-\phi^2}$$

so variance is constant

Autocovariance of X_t and Stationarity

let h is always even.

$$\mathrm{Cov}(X_{2t+h}, X_{2t}) = \mathrm{Cov}\left(\phi^{(t+h/2)}X_0 + \sum_{j=0}^{t+h/2-1} \phi^j W_{2t+h-2j}, \phi^t X_0 + \sum_{j=0}^{t-1}\phi^j W_{2t-2j}\right).$$

Using property of covar.,

$$\mathrm{Cov}(X_{2t+h}, X_{2t}) = \mathrm{Cov}\left(\phi^{(t+h/2)}X_0, \phi^t X_0\right) + \mathrm{Cov}\left(\phi^{(t+h/2)}X_0, \sum_{j=0}^{t-1}\phi^j W_{2t-2j}\right)$$
$$+\mathrm{Cov}\left(\sum_{j=0}^{t+h/2-1} \phi^j W_{2t+h-2j}, \phi^t X_0\right) + \mathrm{Cov}\left(\sum_{j=0}^{t+h/2-1} \phi^j W_{2t+h-2j}, \sum_{j=0}^{t-1}\phi^j W_{2t-2j}\right).$$

i solve each part separate covar.of X_0 term:

$$\mathrm{Cov}\left(\phi^{(t+h/2)}X_0, \phi^t X_0\right) = \phi^{(t+h/2)}\phi^t \mathrm{Var}(X_0) = \phi^{(2t+h/2)}\mathrm{Var}(X_0).$$

covariance between X_0 and W_t term: from Ques.X_0 is independent of W_t, these covar.s are zero:

$$\mathrm{Cov}\left(\phi^{(t+h/2)}X_0, \sum_{j=0}^{t-1}\phi^j W_{2t-2j}\right) = 0$$

$$\mathrm{Cov}\left(\sum_{j=0}^{t+h/2-1} \phi^j W_{2t+h-2j}, \phi^t X_0\right) = 0.$$

covariance btw W_t term: For W_t terms, covar.is:

$$\mathrm{Cov}\left(\sum_{j=0}^{t+h/2-1}\phi^j W_{2t+h-2j}, \sum_{j=0}^{t-1}\phi^j W_{2t-2j}\right) = \sum_{j=0}^{t-1}\phi^j\phi^{j+h/2}\mathrm{Cov}(W_{2t-2j}, W_{2t+h-2j}).$$

as W_t is white noise,i have $\mathrm{Cov}(W_i, W_j) = \sigma_W^2$ when $i = j$, and 0 otherwise. so covar.simplifies to:

$$\mathrm{Cov}(X_{2t+h}, X_{2t}) = \phi^{(2t+h/2)}\mathrm{Var}(X_0) + \sum_{j=0}^{t-1}\phi^j\phi^{j+h/2}\sigma_W^2.$$

Stationarity Condn:
autocovariance should not depend on t, variance of X_0 must be constant

$$\mathrm{Cov}(X_{2t+h}, X_{2t}) = \phi^{(2t+h/2)}\frac{\sigma_W^2}{1-\phi^2} + \sum_{j=0}^{t-1}\phi^j\phi^{j+h/2}\sigma_W^2.$$

For autocovariance to be constant,i need summation term to be independent of t. By stationarity, value of $\sigma_0^2 = \frac{\sigma_w^2}{1-\phi^2}$,
similarly

$$\mathrm{Cov}(X_{2t+h-1}, X_{2t-1}) = \mathrm{Cov}\left(\phi^{(t+h/2-1)}X_{-1} + \sum_{j=0}^{t+h/2-2}\phi^j W_{2t+h-2j-1}, \phi^{(t-1)}X_{-1} + \sum_{j=0}^{t-2}\phi^j W_{2t-2j-1}\right).$$

$$\mathrm{Cov}(X_{2t+h-1}, X_{2t-1}) = \mathrm{Cov}\left(\phi^{(t+h/2-1)}X_{-1}, \phi^{(t-1)}X_{-1}\right) + \mathrm{Cov}\left(\phi^{(t+h/2-1)}X_{-1}, \sum_{j=0}^{t-2}\phi^j W_{2t-2j-1}\right)$$

$$+\mathrm{Cov}\left(\sum_{j=0}^{t+h/2-2}\phi^j W_{2t+h-2j-1}, \phi^{(t-1)}X_{-1}\right) + \mathrm{Cov}\left(\sum_{j=0}^{t+h/2-2}\phi^j W_{2t+h-2j-1}, \sum_{j=0}^{t-2}\phi^j W_{2t-2j-1}\right).$$

Covariance of X_{-1} terms: covar.between X_{-1} terms is:

$$\mathrm{Cov}\left(\phi^{(t+h/2-1)}X_{-1}, \phi^{(t-1)}X_{-1}\right) = \phi^{(t+h/2-1)}\phi^{(t-1)}\mathrm{Var}(X_{-1}) = \phi^{(2t+h/2-2)}\mathrm{Var}(X_{-1}).$$

Covariance between X_{-1} and W_t terms: as X_{-1} is independent of W_t, these covar.s are zero:

$$\mathrm{Cov}\left(\phi^{(t+h/2-1)}X_{-1}, \sum_{j=0}^{t-2}\phi^j W_{2t-2j-1}\right) = 0$$

$$\mathrm{Cov}\left(\sum_{j=0}^{t+h/2-2}\phi^j W_{2t+h-2j-1}, \phi^{(t-1)}X_{-1}\right) = 0.$$

Covariance between W_t terms: For covar.between W_t terms,i have:

$$\mathrm{Cov}\left(\sum_{j=0}^{t+h/2-2}\phi^j W_{2t+h-2j-1}, \sum_{j=0}^{t-2}\phi^j W_{2t-2j-1}\right) = \sum_{j=0}^{t-2}\phi^j\phi^{j+h/2-1}\mathrm{Cov}(W_{2t-2j-1}, W_{2t+h-2j-1}).$$

as W_t is white noise,i know that $\mathrm{Cov}(W_i, W_j) = \sigma_W^2$ if $i = j$, and zero otherwise. so, covar.simplifies to:

$$\mathrm{Cov}(X_{2t+h-1}, X_{2t-1}) = \phi^{(2t+h/2-2)}\mathrm{Var}(X_{-1}) + \sum_{j=0}^{t-2}\phi^j\phi^{j+h/2-1}\sigma_W^2.$$

Stationarity Condition
To ensure autocovariance does not depend on t, var of X_{-1} must be constant. soi set:

$$\mathrm{Var}(X_{-1}) = \sigma_{-1}^2 = \frac{\sigma_W^2}{1-\phi^2}.$$

setting initial variance of X_{-1} to $\frac{\sigma_W^2}{1-\phi^2}$ ensures that autocovariance is stationary, and process X_t becomes stationary.
Results
$\mathrm{Var}(X_0) = \sigma_0^2 = \frac{\sigma_W^2}{1-\phi^2}$,
$\mathrm{Var}(X_{-1}) = \sigma_{-1}^2 = \frac{\sigma_W^2}{1-\phi^2}$.
it ensures Xt is stationary which then implies Yt is stationary so results hold for given system process.

Q 2

given state-space model :

$$Y_t = X_t + V_t,$$
$$X_t = \phi X_{t-1} + W_t,$$

here:

- $V_t \sim$ IID $N(0, \sigma_V^2)$ (observation noise),
- $W_t \sim$ IID $N(0, \sigma_W^2)$ (process noise),
- $X_0 \sim N\left(0, \frac{\sigma_W^2}{1-\phi^2}\right)$ (initial state).

variables X_0, $\{W_t\}$, and $\{V_t\}$ are independent of each other.

$$X_t^{t-1} = \mathbb{E}(X_t | Y_{t-1}, \ldots, Y_1),$$
$$P_t^{t-1} = \mathbb{E}\left[(X_t - X_t^{t-1})^2\right].$$

term X_t^{t-1} represents best linear estimate of X_t given past observations, and P_t^{t-1} is estimation error var.for X_t given $Y_{t-1}, \ldots, Y_1$.

innovation sequence (or residuals) is defined as:

$$\varepsilon_t = Y_t - Y_t^{t-1},$$

$$Y_t^{t-1} = \mathbb{E}(Y_t | Y_{t-1}, \ldots, Y_1).$$

as $Y_t = X_t + V_t$,i can express ε_t as:

$$\varepsilon_t = (X_t + V_t) - Y_t^{t-1}.$$

expression for Y_t^{t-1}
from notesi know

$$Y_t^{t-1} = \mathbb{E}(Y_t | Y_{t-1}, \ldots, Y_1) = \mathbb{E}(X_t | Y_{t-1}, \ldots, Y_1) + \mathbb{E}(V_t | Y_{t-1}, \ldots, Y_1).$$

as V_t is independent of past observations and has zero mean, $\mathbb{E}(V_t | Y_{t-1}, \ldots, Y_1) = 0$. so ,

$$Y_t^{t-1} = X_t^{t-1}.$$

so, innovation sequence becomes:

$$\varepsilon_t = Y_t - Y_t^{t-1} = (X_t + V_t) - X_t^{t-1} = (X_t - X_t^{t-1}) + V_t.$$

Case 1: s < t

Without Loss of Generality, let $s < t$.

i expand covar.of ϵ_s and ϵ_t: Substituting $\epsilon_s = Y_s - E(Y_s | Y_{s-1}, \ldots, Y_1)$

$$\mathrm{Cov}(\epsilon_s, \epsilon_t) = \mathrm{Cov}(Y_s - E(Y_s | Y_{s-1}, \ldots, Y_1), \epsilon_t).$$

Substituting $Y_s = X_s + V_s$,i can rewrite:

$$\mathrm{Cov}(\epsilon_s, \epsilon_t) = \mathrm{Cov}(Y_s - E(X_s + V_s | Y_{s-1}, \ldots, Y_1), \epsilon_t).$$

conditional expectation $E(X_s + V_s | Y_{s-1}, \ldots, Y_1)$ simplifies to X_s^{s-1}, which is projection of X_s onto $Y_1, \ldots, Y_{s-1}$. so:

$$\mathrm{Cov}(\epsilon_s, \epsilon_t) = \mathrm{Cov}(Y_s - X_s^{s-1}, \epsilon_t).$$

Expanding covar.:

$$\mathrm{Cov}(\epsilon_s, \epsilon_t) = \mathrm{Cov}(Y_s, \epsilon_t) - \mathrm{Cov}(X_s^{s-1}, \epsilon_t).$$

For $\text{Cov}(Y_s, \epsilon_t)$, as $s < t$, Y_s and ϵ_t are independent, leading to:

$$\text{Cov}(Y_s, \epsilon_t) = 0.$$

Now, consider $\text{Cov}(X_s^{s-1}, \epsilon_t)$. By definition, X_s^{s-1} is linear function of $Y_1, \ldots, Y_{s-1}$:

$$X_s^{s-1} = \sum_{i=1}^{s-1} \beta_i Y_i.$$

Substituting X_s^{s-1} into covar.:

$$\text{Cov}(X_s^{s-1}, \epsilon_t) = \text{Cov}\left(\sum_{i=1}^{s-1} \beta_i Y_i, \epsilon_t\right).$$

Using linearity of covar.:

$$\text{Cov}(X_s^{s-1}, \epsilon_t) = \sum_{i=1}^{s-1} \beta_i \text{Cov}(Y_i, \epsilon_t).$$

For $i < t$, $\text{Cov}(Y_i, \epsilon_t) = 0$, as Y_i and ϵ_t are uncorrelated. so :

$$\text{Cov}(X_s^{s-1}, \epsilon_t) = 0.$$

Substituting these results back into original expression for $\text{Cov}(\epsilon_s, \epsilon_t)$:

$$\text{Cov}(\epsilon_s, \epsilon_t) = 0 - \text{Cov}(X_s^{s-1}, \epsilon_t) = 0.$$

so, for $s < t$,i conclude:

$$\text{Cov}(\epsilon_s, \epsilon_t) = 0.$$

Case 2: s = t

Now,i calculate $\text{Cov}(\epsilon_t, \epsilon_t)$, it is equal to var. of ϵ_t.

defn. of ϵ_t:

$$\epsilon_t = Y_t - E(Y_t | Y_{t-1}, \ldots, Y_1).$$

so, variance is:

$$\text{Var}(\epsilon_t) = \text{Cov}(\epsilon_t, \epsilon_t) = \text{Var}(Y_t - E(Y_t | Y_{t-1}, \ldots, Y_1)).$$

Now,i can express Y_t as:

$$Y_t = X_t + V_t.$$

so,i have:

$$\text{Var}(\epsilon_t) = \text{Var}(X_t + V_t - E(Y_t | Y_{t-1}, \ldots, Y_1)).$$

We can simplify this:

$$\text{Var}(\epsilon_t) = \text{Var}(X_t - X_t^{t-1}) + \text{Var}(V_t).$$

as X_t and X_t^{t-1} are related by conditional expectation,i have:

$$\text{Var}(X_t - X_t^{t-1}) = P_t^{t-1},$$

where P_t^{t-1} is conditional variance of X_t.

so , variance of ϵ_t becomes:

$$\text{Var}(\epsilon_t) = P_t^{t-1} + \sigma_V^2.$$

so, covar.of innovation sequence is:

$$\text{Cov}(\epsilon_t, \epsilon_t) = P_t^{t-1} + \sigma_V^2.$$

so For $s < t$:

$$\text{Cov}(\epsilon_s, \epsilon_t) = 0.$$

For $s = t$:

$$\text{Cov}(\epsilon_t, \epsilon_t) = P_t^{t-1} + \sigma_V^2.$$

Q3(a)

given in Ques.

$$X_0 = W_0,$$
$$X_t = X_{t-1} + W_t,$$
$$Y_t = X_t + V_t, \quad t = 1, 2, \ldots$$

- $W_t \sim$ IID $N(0, \sigma_W^2)$ is process noise,
- $V_t \sim$ IID $N(0, \sigma_V^2)$ is observation noise,

W_t and V_t are independent.

i'll show that Y_t follows IMA$(1,1)$, so from classnotes defn. of IMA(1,1) i want to show that $\nabla Y_t = Y_t - Y_{t-1}$ follows an MA(1) process.

1. i define differenced Observation Process ∇Y_t

$$\nabla Y_t = Y_t - Y_{t-1}.$$

Substitute $Y_t = X_t + V_t$ and $Y_{t-1} = X_{t-1} + V_{t-1}$ to get:

$$\nabla Y_t = (X_t + V_t) - (X_{t-1} + V_{t-1}).$$

$$\nabla Y_t = (X_t - X_{t-1}) + (V_t - V_{t-1}).$$

2: Substitute State eqn.
From state eqn.,i have:

$$X_t = X_{t-1} + W_t.$$

so,

$$X_t - X_{t-1} = W_t.$$

i substitute this into ∇Y_t:

$$\nabla Y_t = W_t + (V_t - V_{t-1}).$$

3:i then analyze ∇Y_t

$$\nabla Y_t = W_t + (V_t - V_{t-1}).$$

as $W_t \sim N(0, \sigma_W^2)$ and $V_t \sim N(0, \sigma_V^2)$ are both white noise processes, difference $V_t - V_{t-1}$ will be MA(1) process with moving average parameter -1.

4: Covar. of ∇Y_t
to confirm ∇Y_t follows MA(1) model,i need to examine its covar.structure.
Mean of ∇Y_t: as both W_t and $V_t - V_{t-1}$ have zero mean, it follows that:

$$\mathbb{E}[\nabla Y_t] = 0.$$

var.of ∇Y_t is:

$$\mathrm{Var}(\nabla Y_t) = \mathrm{Var}(W_t) + \mathrm{Var}(V_t - V_{t-1}).$$

as $W_t \sim N(0, \sigma_W^2)$ and $V_t - V_{t-1} \sim N(0, 2\sigma_V^2)$,i have:

$$\mathrm{Var}(\nabla Y_t) = \sigma_W^2 + 2\sigma_V^2.$$

Autocovar.of ∇Y_t: For $k \geq 2$, autocovar.$\gamma(k) = \mathrm{Cov}(\nabla Y_t, \nabla Y_{t-k}) = 0$, because W_t and V_t are white noise and independent.
For $k = 1$:

$$\gamma(1) = \mathrm{Cov}(\nabla Y_t, \nabla Y_{t-1}).$$

as $\nabla Y_t = W_t + V_t - V_{t-1}$ and $\nabla Y_{t-1} = W_{t-1} + V_{t-1} - V_{t-2}$, only non-zero covar.term is $-\sigma_V^2$ from V_{t-1}. so,

$$\gamma(1) = -\sigma_V^2.$$

so, autocovar.structure of ∇Y_t matches that of an MA(1) model with parameter $\theta = -1$.

$$\nabla Y_t = W_t + V_t - V_{t-1}$$

$$\gamma(h) = \text{Cov}(\nabla Y_t, \nabla Y_{t-h})$$

For h = 0:

$$\gamma(0) = \text{Var}(\nabla Y_t) = \sigma_W^2 + 2\sigma_V^2$$

For h = 1:

$$\gamma(1) = \text{Cov}(W_t + V_t - V_{t-1}, W_{t-1} + V_{t-1} - V_{t-2})$$
$$\gamma(1) = -\sigma_V^2$$

$$\text{For } |h| > 1:$$
$$\gamma(h) = 0$$

Autocorrelation function:

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

$$\rho(h) = \begin{cases} 1 & \text{if } h = 0, \\ \frac{-\sigma_V^2}{\sigma_W^2 + 2\sigma_V^2} & \text{if } h = \pm 1, \\ 0 & \text{if } |h| > 1. \end{cases}$$

i can also write it as

$$\rho(h) = \frac{-\sigma_V^2}{\sigma_W^2 + 2\sigma_V^2}\delta_1^h \quad \text{for } h = 1, 2, \ldots,$$

δ_1^h is an indicator function that is 1 if $h = 1$ and 0 otherwise. here $|\rho(1)| \leq 0.5$ for all $\sigma_W^2 \geq 0$ and $\sigma_V^2 > 0$.

so,i showed that $\text{Y}_t follows \text{IMA}(1,1)$ model.

Q 4

We have state-space model :

$$X_t = \Phi X_{t-1} + W_t$$
$$Y_t = A_t X_t + V_t$$

where: - $W_t \sim \mathcal{N}(0, \sigma_W^2)$ and $V_t \sim \mathcal{N}(0, \sigma_V^2)$ are independent Gaussian white noise processes, - A_t is known design matrix independent of parameters Θ, - $X_0 \sim \mathcal{N}(0, \sigma_{X_0}^2)$ is initial state.

gradient of log-likelihood with respect to Θ_i is given by:

$$\frac{\partial \log L_Y(\Theta)}{\partial \Theta_i} = \sum_{t=1}^{T} \left\{ \epsilon_t' \Sigma_t^{-1} \frac{\partial \epsilon_t}{\partial \Theta_i} - 0.5 \epsilon_t' \Sigma_t^{-1} \frac{\partial \Sigma_t}{\partial \Theta_i} \Sigma_t^{-1} \epsilon_t + 0.5 \, \text{tr} \left(\Sigma_t^{-1} \frac{\partial \Sigma_t}{\partial \Theta_i} \right) \right\}$$

where: - $\epsilon_t = Y_t - Y_t^{t-1}$ is prediction error, - $Y_t^{t-1} = E(Y_t | Y_{t-1}, \ldots, Y_1)$ is one-step-ahead prediction of Y_t, - Σ_t is covar.of prediction error ϵ_t.

(a)

Innovation sequence defined as:

$$\epsilon_t = Y_t - E(Y_t | Y_{t-1}, \ldots, Y_1)$$

Substitute $Y_t = A_t X_t + V_t$:

$$\epsilon_t = Y_t - E(A_t X_t + V_t | Y_{t-1}, \ldots, Y_1)$$
$$\epsilon_t = Y_t - A_t X_t^{t-1}$$

Differentiate with respect to i:

$$\partial_i \epsilon_t = \partial_i Y_t^{t-1} - A_t \partial_i X_t^{t-1}$$

Using $\partial_i Y_t^{t-1} = 0$, simplify:

$$\partial_i \epsilon_t = -A_t \partial_i X_t^{t-1}$$

(b)

Recursive relationship for X_t^{t-1}:

$$X_t^{t-1} = \Phi X_{t-1}^{t-2} + K_{t-1}\epsilon_{t-1}.$$

Differentiating with respect to Θ_i:

$$\partial_i X_t^{t-1} = \partial_i(\Phi X_{t-1}^{t-2}) + \partial_i(K_{t-1}\epsilon_{t-1}),$$

where:

$$\partial_i(\Phi X_{t-1}^{t-2}) = \partial_i \Phi X_{t-1}^{t-2} + \Phi \partial_i X_{t-1}^{t-2},$$

and:

$$\partial_i(K_{t-1}\epsilon_{t-1}) = \partial_i K_{t-1}\epsilon_{t-1} + K_{t-1}\partial_i \epsilon_{t-1}.$$

Combining these terms:

$$\partial_i X_t^{t-1} = \partial_i \Phi X_{t-1}^{t-2} + \Phi \partial_i X_{t-1}^{t-2} + \partial_i K_{t-1}\epsilon_{t-1} + K_{t-1}\partial_i \epsilon_{t-1}.$$

(c)

The covariance matrix Σ_t is:

$$\Sigma_t = A_t P_t^{t-1} A_t' + R.$$

Differentiating with respect to Θ_i:

$$\partial_i \Sigma_t = \partial_i(A_t P_t^{t-1} A_t') + \partial_i R.$$

Since A_t is independent of Θ, only P_t^{t-1} contributes:

$$\partial_i(A_t P_t^{t-1} A_t') = A_t \partial_i P_t^{t-1} A_t'.$$

Thus:

$$\partial_i \Sigma_t = A_t \partial_i P_t^{t-1} A_t' + \partial_i R.$$

(d)

The Kalman gain K_t is:

$$K_t = P_t^{t-1} A_t' \Sigma_t^{-1}.$$

Differentiating with respect to Θ_i:

$$\partial_i K_t = \partial_i \left(P_t^{t-1} A_t' \Sigma_t^{-1} \right).$$

Using the product rule:

$$\partial_i K_t = \left[\partial_i P_t^{t-1} A_t' + P_t^{t-1} \partial_i A_t' \right] \Sigma_t^{-1} + P_t^{t-1} A_t' \partial_i(\Sigma_t^{-1}).$$

For $\partial_i(\Sigma_t^{-1})$, we use:

$$\partial_i \Sigma_t^{-1} = -\Sigma_t^{-1}(\partial_i \Sigma_t)\Sigma_t^{-1}.$$

Substituting:

$$\partial_i K_t = \left[\partial_i P_t^{t-1} A_t' + P_t^{t-1} \partial_i A_t' \right] \Sigma_t^{-1} - P_t^{t-1} A_t' \Sigma_t^{-1}(\partial_i \Sigma_t)\Sigma_t^{-1}.$$

(e)

The covariance update equation is:

$$P_t^{t-1} = \Phi P_{t-2}^{t-1} \Phi' + Q - K_t \Sigma_t K_t'.$$

Differentiating with respect to Θ_i:

$$\partial_i P_t^{t-1} = \partial_i \left(\Phi P_{t-2}^{t-1} \Phi' \right) + \partial_i Q - \partial_i \left(K_t \Sigma_t K_t' \right).$$

Using the product rule for $\Phi P_{t-2}^{t-1} \Phi'$:

$$\partial_i(\Phi P_{t-2}^{t-1} \Phi') = \partial_i \Phi P_{t-2}^{t-1} \Phi' + \Phi \partial_i P_{t-2}^{t-1} \Phi' + \Phi P_{t-2}^{t-1} \partial_i \Phi'.$$

For $\partial_i(K_t \Sigma_t K_t')$, use the product rule:

$$\partial_i(K_t \Sigma_t K_t') = \partial_i K_t \Sigma_t K_t' + K_t \partial_i \Sigma_t K_t' + K_t \Sigma_t \partial_i K_t'.$$

Substituting these into the equation:

$$\begin{gathered}\partial_i P_t^{t-1} = \partial_i \Phi P_{t-2}^{t-1} \Phi' + \Phi \partial_i P_{t-2}^{t-1} \Phi' + \Phi P_{t-2}^{t-1} \partial_i \Phi' + \partial_i Q \\ -\partial_i K_t \Sigma_t K_t' - K_t \partial_i \Sigma_t K_t' - K_t \Sigma_t \partial_i K_t'.\end{gathered}$$