MTH 517: Time Series Analysis
Quiz #1; Full Marks 20

Date: September 06, 2019

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Let $\{\varepsilon_t\}$ be a sequence of i.i.d. $N\left(0,\sigma^2\right)$ and s_t be a seasonal component of periodicity 12. Define $X_t = \left(a_0 + a_1 t\right)e^{-s_t} + \varepsilon_t + \varepsilon_{t-12}$ and $Y_t = \sum_{j=0}^{t-1}\phi^j \varepsilon_{t-j}, |\phi| < 1.$

Prove or disprove the following statements:

(a) $\{\nabla X_t\}$ does not contain any time trend component

(b) $\{\nabla_{12} X_t\}$ does not contain any seasonal component

(c) $\{\nabla_{12} X_t\}$ is a Gaussian process

(d) $\{\nabla_{12}^2 X_t\}$ is strict stationary

(e) $\{Y_t\}$ is a covariance stationary process

(f) $\{Y_t\}$ is a Gaussian process

(g) $\{e^{s_t+\varepsilon_t}\}$ is a Gaussian process

a) $\nabla X_t = X_t - X_{t-1}$ $\quad [X_t = m_t e^{-S_t} + \varepsilon_t + \varepsilon_{t-12}, m_t$

$= (a_0 + a_1 t)e^{-S_t} + \varepsilon_t + \varepsilon_{t-12} - \{a_0 + a_1(t-1)\}e^{-S_{t-1}} -$

$= a_0\left(e^{-S_t} - e^{-S_{t-1}}\right) + a_1\left(te^{-S_t} - (t-1)e^{-S_{t-1}}\right) + \varepsilon_t -$

$+ \varepsilon_{t-12} - \varepsilon_{t-13}$

$2\frac{1}{2}$

As seasonal components have periodicity 12 so $\left(e^{-S_t} - e^{-S_{t-1}}\right)$ do not vanish and hence the component remain in the expression of $\{\nabla X_t\}$.

$\therefore \{\nabla X_t\}$ contains time trend component.

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MTH517A: Time Series Analysis
Mid semester examination: Full Marks 60

[1] Let $\{X_t\}$ be an $MA(1)$ process $X_t = \varepsilon_t + \varepsilon_{t-1}; \{\varepsilon_t\}$ is a sequence of independently and identically distributed $N(0,\sigma^2)$ random variables. Consider the exponentially weighted moving average obtained from $\{X_t\}$ as $Y_1 = X_1$ and for $2 \le t \le n, Y_t = \alpha X_t + (1-\alpha) Y_{t-1}$ with $\alpha = 3/4$.

(a) Find the joint distribution of (Y_1, Y_2, Y_3).

(b) Is $\{Y_t : t \ge 1\}$ a Gaussian process?

(c) Is $\{Y_t : t \ge 1\}$ a strict stationary process?

12 marks

[2] Let $\{\varepsilon_t\}$ and $\{\delta_t\}$ be two mutually independent sequences of independently and identically distributed $N(0,\sigma^2/2)$ random variables. Let $\{Y_t\}$ be a complex valued time series defined as $Y_t = \varepsilon_{2t-1} e^{i\omega t} + (\varepsilon_t + i\delta_t); \omega \in (0,\pi)$ is a fixed constant and $i = \sqrt{-1}$. Prove or disprove the following statements:

(a) $\{Y_t\}$ is covariance stationary.

(b) $Z_t = \varepsilon_t + \delta_t + \varepsilon_{2t+1}$ is white noise process.

8 marks

[3] Consider the $AR(2)$ process $X_t = 0.5X_{t-1} - 0.25X_{t-2} + \varepsilon_t,\ \varepsilon_t \sim WN(0,\sigma^2)$ and define $Y_t = \sum_{k=0}^{1} (k+1) X_{t-k}$. Prove or disprove the following statements:

(a) $\{Y_t\}$ is a causal $ARMA$ process.

(b) $\{Y_t\}$ is stationary and invertible $ARMA$ process.

8 marks

[4] Let $\{X_t\}$ be a causal and invertible $ARMA(1,1)$ process

$$X_t = \phi X_{t-1} + \delta + \varepsilon_t + \theta \varepsilon_{t-1},\ |\phi| < 1, |\theta| < 1\ \varepsilon_t \sim WN(0,\sigma^2)$$

Find $\lim_{N\to\infty} E\left(X_t - \varepsilon_t - \delta \sum_{j=0}^{N} (-\theta)^j - (\theta+\phi) \sum_{j=1}^{N} (-\theta)^{j-1} X_{t-j} \right)^2$.

8 marks

Name:
Roll No.

[5] Let $\{X_t\}$ and $\{Y_t\}$ be two covariance stationary ARMA processes given by

$$X_t = \phi X_{t-1} + \varepsilon_t - \alpha^{-1}\varepsilon_{t-1} \text{ and } Y_t = \alpha Y_{t-1} + \delta_t - \left(\phi + \phi^{-1}\right)\delta_{t-1} + \delta_{t-2};\ |\phi| < 1, |\alpha| < 1$$

$\{\varepsilon_t\}$ and $\{\delta_t\}$ be two mutually independent sequences of independently and identically distributed $N\left(0,\sigma^2\right)$ random variables. Define $\{P_t\}$, $\{Q_t\}$ and $\{R_t\}$ as:

$P_t = \left(1-\alpha^{-1}B\right)\left(1-\phi^{-1}B\right)\left(\varepsilon_t + \delta_t\right);$

$Q_t = (1-\phi B)(1-\alpha B)\left(X_t + Y_t\right)$ and $(1-\phi B)(1-\alpha B)R_t = \left(1-\alpha^{-1}B\right)\left(1-\phi^{-1}B\right)\varepsilon_t$.

(a) Express ACGF of $\{P_t\}$ in terms of ACGFs of $\{X_t\}$ and $\{Y_t\}$.

(b) Using the ACGF of $\{P_t\}$, obtained in (a), find $\gamma_P(1)$.

(c) Does there exist a finite *k*, such that $\gamma_Q(h) = 0, \forall |h| > k$?

(d) Prove or disprove: "$\{R_t\}$ is a white noise process".

Note: Appropriate conditions for existence of ACGFs may be assumed to hold.

16 marks

[6] $\{\varepsilon_t\}$, $\{\delta_t\}$ and $\{\gamma_t\}$ be three mutually independent white noise $WN\left(0,\sigma^2\right)$ processes. Define

$$X_t = \varepsilon_t + \delta_t \cos(\pi t/4) + \gamma_t \sin(\pi t/4); Y_t = \delta_{t-1} + \gamma_{t-1}\cos(\pi t/4) + \varepsilon_{t-1}\sin(\pi t/4).$$

Prove or disprove the following statements:

(a) $\begin{pmatrix} X_t \\ Y_t \end{pmatrix}$ is covariance stationary.

(b) $\begin{pmatrix} \varepsilon_t \\ \delta_t \\ \varepsilon_{t-3} \end{pmatrix} \sim VWN.$

8 marks

MTH 517: Time Series Analysis
Quiz #2; Full Marks 20

Date: September 06, 2019

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Let $\{\varepsilon_t\}$, $\{\delta_t\}$ and $\{\eta_t\}$ be three mutually independent sequence of i.i.d. $N(0,\sigma^2)$. Define $X_t = X_{t-1} + 0.5\,Y_{t-1} + \varepsilon_t$; $Y_t = 0.5 + 0.6\,Y_{t-1} + \delta_t$ and $Z_t = 0.5\,Z_{t-1} + 0.4\,Y_{t-1} + \eta_t$ such that $\forall\, j>0, Cov(\varepsilon_t, X_{t-j}) = Cov(\delta_t, X_{t-j}) = Cov(\eta_t, X_{t-j}) = Cov(\varepsilon_t, Y_{t-j}) = Cov(\delta_t, Y_{t-j}) = Cov(\eta_t, Y_{t-j}) = Cov(\varepsilon_t, Z_{t-j}) = Cov(\delta_t, Z_{t-j}) = Cov(\eta_t, Z_{t-j}) = 0$.

(a) Prove or disprove: $(X_t, Y_t)^T$ is a causal $VAR(1)$ process.

(b) Prove or disprove: $(Y_t, Z_t, 2Y_{t-1}, 2Z_{t-1})^T$ is a stationary $VAR(1)$ process.

(c) Prove or disprove: $(\varepsilon_t, \eta_t, 2\varepsilon_{t-1}, 2\eta_{t-1})^T$ is a VWN process.

(d) If $(Y_1, \ldots, Y_n)$ is a sample of size n, then find the distribution of $\bar{Y}_n = n^{-1}\sum_{t=1}^{n} Y_t$.

(e) Find the BLP of Y_{n+1} based on Y_n and Y_{n-1}.

a) Let $\underline{U}_t = \begin{bmatrix} X_t \\ Y_t \end{bmatrix}$

Now $X_t = X_{t-1} + 0.5\,Y_{t-1} + \varepsilon_t$

$Y_t = 0.5 + 0.6\,Y_{t-1} + \delta_t$

$$\underline{U}_t = \begin{bmatrix} X_t \\ Y_t - 0.5 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0.5 \\ 0 & 0.6 \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} X_{t-1} \\ Y_{t-1} \end{bmatrix}}_{U_{t-1}} + \underbrace{\begin{bmatrix} \varepsilon_t \\ \delta_t \end{bmatrix}}_{\eta_t}$$

or, $\underline{U}_t = \Phi\, \underline{U}_{t-1} + \underline{\eta}_t$ will be VAR(

MTH 517: Time Series Analysis
End semester examination; Full Marks-100

Date: November 27, 2019

[1] (a) Let X_1, X_2, X_3 be random sample from a causal AR(1) process $X_t = \mu_X(1-\phi) + \phi X_{t-1} + \varepsilon_t;\ \mu_X = E(X_t), |\phi| < 1, \varepsilon_t \sim WN(0,\sigma^2)$. Suppose $\delta_1 = (X_1 + X_2)/2$ and $\delta_2 = (X_1 + X_2 + X_3)/3$. **Prove or disprove:** "$V(\delta_1) \geq V(\delta_2)\ \forall |\phi| < 1$".

(b) $\{X_t\}$ is a covariance stationary $AR(1)$ process; $X_t = 0.5\, X_{t-1} + \varepsilon_t;\ \varepsilon_t \sim WN(0,1)$ and $Y_t = X_t + \eta_t;\ \eta_t \sim WN(0,1)$, ε_t and η_t are independently distributed.

?

$t = \phi X_{t-1} + \varepsilon_t$, $|\phi| < 1$

$\phi = 0.5$

(i) **Prove or disprove:** "In the BLP of X_{t+2} based on Y_t and Y_{t-1}, the coefficient of Y_t is greater than the coefficient of Y_{t-1}".

(ii) Find the PACF at lag 2 of $\{Y_t\}$.

20 (8+6+6) marks

[2] Let $X_1, \ldots, X_n$ be a random sample from a Gaussian invertible $MA(1)$ model $X_t = \varepsilon_t + \theta\,\varepsilon_{t-1}, |\theta| < 1$ and $\varepsilon_t \sim i.i.d.N(0,\sigma^2)$.

(a) **Prove or disprove:** "conditional LSE of θ, conditional on given ε_0 at it's expected value is $\hat{\theta}_{CLSE} = \arg\min_{\theta} \sum_{t=1}^{n} \left(X_t - \theta \left(\sum_{k=0}^{t-2} (-\theta)^k X_{t-k-1} \right) \right)^2$".

(b) **Prove or disprove:** "conditional MLE of θ, conditional on given ε_0 at it's expected value is $\hat{\theta}_{CMLE} = \arg\min_{\theta} \sum_{t=1}^{n} \left(\sum_{i=1}^{t} (-\theta)^{t-i-1} X_i \right)^2$".

16 (8+8) marks

[3] Let $\{X_t\}$ and $\{Y_t\}$ be 2 independent 0 mean covariance stationary time series processes with absolutely summable ACVF $\gamma_X(h)$ and $\gamma_Y(h)$, respectively. Define $Z_t = (1 - X_t)Y_t$.

(a) Express the spectral density function of $\{Z_t\}, f_Z(\lambda)$, as $f_Z(\lambda) = \int_{-\pi}^{\pi} \psi(\lambda,\omega)\, d\omega$; where $\psi(\lambda,\omega)$ is a function **ONLY** of the spectral densities of $\{X_t\}$ and $\{Y_t\}$.

?

(b) Suppose $X_t = \delta_t - \delta_{t-2}$ and $Y_t = \varepsilon_t$, $\{\varepsilon_t\}$ and $\{\delta_t\}$ are independent $WN(0,\sigma^2)$ processes. Using the spectral density function of $\{Z_t\}$, **derived in (a)** **(and NOT using** $\gamma_Z(h)$**)**, find the value of $f_Z(0)$.

16 (8+8) marks

[4] (a) Consider the following $ARMA(3,3)$ representation of $\{X_t\}$

?

$$\left(1-\frac{5}{6}B+\frac{B^2}{6}\right)\left(1-\frac{B}{4}\right)X_t=\left(1-5B+6B^2\right)\left(1-4B\right)\varepsilon_t,$$

$\varepsilon_t \sim WN\left(0,\sigma^2\right)$. Find $\gamma_X(5)$.

(b) Let $\{\underset{\sim}{X}_t\}$ be a k-variate covariance stationary $VAR(1)$ process $\underset{\sim}{X}_t=\Phi\underset{\sim}{X}_{t-1}+\underset{\sim}{\varepsilon}_t$;

? $\underset{\sim}{\varepsilon}_t \sim VWN\left(\underset{\sim}{0},\Sigma\right),\Sigma>0$. **Prove or disprove:**

"$\underset{\sim}{Z}_t=\begin{pmatrix}\underset{\sim}{X}_t\\ 3\underset{\sim}{X}_{t-3}\end{pmatrix}$ is a covariance stationary $VAR(3)$ process".

16 (8+8) marks

[5] (a) Let $\{X_t\}$ be a linear covariance stationary time series with mean μ and ACVF

?

$$\gamma(h)=(0.6)^{|h|}+2(0.3)^{|h|}+(0.1)^{|h|}$$

Using the asymptotic distribution of $\bar{X}_n$, find the smallest n such that $P\left(\bar{X}_n-0.49\le\mu\le\bar{X}_n+0.49\right)\ge 0.95$.

(b) Let $\underset{\sim}{X}_t=\begin{pmatrix}X_{1,t}\\ \vdots\\ X_{k,t}\end{pmatrix}$ be a k-variate covariance stationary process such that $E\left(\underset{\sim}{X}_t\right)=\underset{\sim}{0}\ \forall t$;

? ACVF of $\{X_i\}$, $\gamma_{X_i}(h)$ is absolutely summable $\forall i=1(1)k$ and the cross-covariance between $\{X_i\}$ and $\{X_j\}$, $\gamma_{X_iX_j}(h)$ is absolutely summable $\forall i\ne j;\ i,j=1(1)k$. If $C_{X_iX_j}(\lambda)$ denote the co-spectrum between X_i and X_j, **prove or disprove:**

$$\text{“}C_{X_iX_j}(\lambda)=\frac{1}{2\pi}\left(\gamma_{X_iX_j}(0)+\sum_{h=1}^{\infty}\left(\gamma_{X_iX_j}(h)-\gamma_{X_jX_i}(h)\right)\cos\lambda h\right)\text{”}$$

14 (8+6) marks

[6] Let $X_t=\sum_{j=1}^{2}\left(A_j\cos(\pi jt/4)+B_j\sin(\pi jt/4)+j\,\varepsilon_{t-j}\right)$, A_1,A_2,B_1,B_2 are independent random variables with mean 0 and variance 1, $\varepsilon_t\sim WN\left(0,\sigma^2\right)$. Further, $\{\varepsilon_t\}$ is independent of A_1,A_2,B_1,B_2.

(a) Find the spectral distribution function of $\{X_t\}$,

(b) **Using the spectral distribution function derived in (a),** find $\gamma_X(0)$.

(c) Find the continuous and/or discrete spectra associated with spectral distribution function derived in (a).

18 (9+4+5) marks

Useful data: $Z\sim N(0,1)$, $P(Z>1.96)=0.025$