Q5

linear prediction function $g(x) = a + bx$ to minimizes (MSE):

$$\text{MSE} = E[(Y - g(X))^2],$$

we are given in Q.: X and Y are random variables with a joint distribution described by density function $f(x, y)$.

$Y = X^2 + Z$, where X and Z are independent random variables.

Both X and Z are zero-mean normal variables with var 1.

1: Prediction Equation

We want to find $g(x)$ to minimize:

$$E[Y - g(X)]^2.$$

i can write this as:

$$E[E(Y - g(X))^2 | X].$$

now to minimize inner expectation, take derivative wrt $g(x)$:

$$\frac{\partial E(Y - g(X))^2 | X}{\partial g(x)} = 2[E(Y|X) - g(x)] = 0.$$

$$E(Y|X) - g(x) = 0 \quad \Rightarrow \quad g(x) = E(Y|X).$$

so best predictor of Y given X is conditional expectation $E(Y|X)$.

2: now i can apply on Model

$$Y = X^2 + Z,$$

we have:

$$g(x) = E(Y|X) = E(X^2 + Z|X) = E(X^2|X) + E(Z|X) = X^2 + E(Z) = X^2.$$

Now, calculating MSE:

$$\text{MSE} = E[Y - g(X)]^2 = E[Y - X^2]^2 = E[(X^2 + Z - X^2)^2].$$

so

$$\text{MSE} = E[Z^2],$$

given in Ques. Z is a zero-mean normal variable with variance 1:

$$E[Z^2] = \text{var}(Z) = 1.$$

$$\text{MSE} = 1.$$

3: Solve for a and b

$g(x) = a + bx$.

from notes prediction equations are:

i. $E[Y - g(x)] = 0$

ii. $E[(Y - g(x))X] = 0$

solve (i):

$$E[Y] = E[a + bx] \implies E[Y] = a + bE(X).$$

given $E(X) = 0 \;\; E(Y) = 1$:

$$a = 1.$$

now, from (ii):

$$E[XY] = E[(a + bx)X] \implies E[Y] = aE(X) + bE(X^2).$$

so:

$$E[Y] = 0 + bE(X^2).$$

as$E(X^2) = 1$:

$$E[Y] = b.$$

Now we need to find $E[XY]$:

$$E[XY] = E[X(X^2 + Z)] = E[X^3] + E[XZ].$$

as Z is independent of X and Z has mean 0:

$$E[XZ] = E[X]E[Z] = 0.$$

also, $E[X^3] = 0$ (as it is an odd moment for a standard normal variable):

$$E[XY] = 0.$$

as both term arre 0 so:

$$b = 0.$$

4: solving MSE

as we calculated in previous points, prediction function becomes:

$$g(x) = 1 + 0 \cdot x = 1.$$

calculate MSE:

$$\text{MSE} = E[Y - 1]^2 = E[Y^2] - 2E[Y] + E[1].$$

from prev point:

$$E[Y] = 1.$$

Now i find $E[Y^2]$:

$$E[Y^2] = E[(X^2 + Z)^2] = E[X^4] + 2E[X^2]E[Z] + E[Z^2].$$

Using properties of normal distribution:

- $E[Z^2] = 1$,
- For $E[X^4]$, fourth moment of a standard normal variable is 3.

so:

$$E[Y^2] = 3 + 0 + 1 = 4.$$

Now substituting into MSE equation:

$$\text{MSE} = 4 - 2(1) + 1 = 4 - 2 + 1 = 3.$$

so, best linear predictor has 3 times error of optimal predictor (conditional expectation).