4) $\varepsilon_t = X_t - \sum_{i=1}^{n-1} a_i X_{t-i}$

a_i's will be so choosen that

$E(\varepsilon_t^2)$ is minimized

The Normal eqn will be

$$\frac{\partial}{\partial a_i} E\left(X_t - \sum_{i=1}^{n-1} a_i X_{t-i}\right)^2 = 0 \quad \forall\, i = 1(1)n-1$$

$$\Rightarrow E\left[\left(X_t - \sum_{i=1}^{n-1} a_i X_{t-i}\right) X_{t-i}\right] = 0$$

This boils down to

$$\begin{bmatrix} \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(n-1) \end{bmatrix} = \begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-2) \\ \gamma(1) & \gamma(2) & \cdots & \gamma(n-3) \\ \vdots & & & \\ \gamma(n-2) & \gamma(n-3) & \cdots & \gamma(0) \end{bmatrix} \underset{\sim}{a}$$

dividing both side by $\gamma(0)$ we get

$$\Rightarrow \underset{\sim}{a} = R_{n-1}^{-1} P_n$$

$\delta_{t-n} = X_{t-n} - \sum_{j=1}^{n-1} b_j X_{t-j}$, b_j's will be so chosen that

$E(\delta_{t-n}^2)$ is minimized

So,

$$\begin{pmatrix} \gamma(n-1) \\ \gamma(n-2) \\ \vdots \\ \gamma(1) \end{pmatrix} = R_{n-1} \underset{\sim}{b}$$

$$\underset{\sim}{b} = R_{n-1}^{-1} \hat{\rho}_{n-1}$$

S)

$$\phi_{n,n} = \frac{E(\varepsilon_t \delta_{t-n})/\gamma(0)}{\sqrt{E(\varepsilon_t^2)\, E(\delta_{t-n}^2)/\gamma(0)}}$$

$$\frac{E(\varepsilon_t \delta_{t-n})}{\gamma(0)} = \frac{1}{\gamma(0)} E\left[(X_t - \underset{\sim}{a}'\underset{\sim}{x})(X_{t-n} - \underset{\sim}{b}'\underset{\sim}{x})\right]$$

$$= \frac{1}{\gamma(0)} \left[\gamma(n) - \hat{\rho}_{n-1}' R_{n-1}^{-1} \hat{\rho}_{n-1} \gamma(0) - \rho_n' R_{n-1}^{-1} \hat{\rho}_{n-1} \gamma(0) + Cov(\underset{\sim}{a}'\underset{\sim}{x}, \underset{\sim}{b}'\underset{\sim}{x}) \right]$$

$$= \left[\rho(n) - 2\hat{\rho}_{n-1}' R_{n-1}^{-1} \rho_{n-1} + \rho_{n-1} R_{n-1}^{-1} R_{n-1} R_{n-1}^{-1} \hat{\rho}_{n-1} \right]$$

$$= \rho(n) - \hat{\rho}_{n-1}' R_{n-1}^{-1} \rho_{n-1}$$

[as $\rho_{n-1}' R_{n-1}^{-1} \hat{\rho}_{n-1} = \hat{\rho}_{n-1} R_{n-1}^{-1} \rho_{n-1}$, Scalar ~~Scalar~~]

Now,

$$\frac{1}{\gamma(0)} E(\varepsilon_t^2) = V\Big(X_t - \sum_{i=1}^{h-1} a_i X_{t-i}\Big)$$

$$= \frac{1}{\gamma(0)} \Big[V(X_t) + \underset{\sim}{a}' \gamma(0) R_{h-1} \underset{\sim}{a} - 2\,\text{Cov}(X_t, \textstyle\sum a_i X_{t-i})\Big]$$

$$= \frac{1}{\gamma(0)} \Big[\gamma(0) + \underset{\sim}{P}_{h-1}' \gamma(0) R_{h-1}^{-1} P_{h-1} - 2\, \underset{\sim}{a}' P_{h-1}\gamma(0)\Big]$$

$$= \Big[1 - P_{h-1}' R_{h-1}^{-1} P_{h-1}\Big]$$

Now, similarly $\frac{1}{\gamma(0)} E(\delta_{t-h}^2) = \Big[1 - \tilde{P}_{h-1}' R_{h-1}^{-1} \tilde{P}_{h-1}\Big]$

Now,

$$\tilde{P}_{h-1} = E\, P_h \quad \text{where}$$

is a Elementary matrix for which j th row and $(h-1-j+1)$ row will be interchanged

Now,

$$1 - \tilde{P}_{h-1}' R_{h-1}^{-1} \tilde{P}_{h-1} = 1 - P_{h-1}' E R_{h-1}^{-1} E P_{h-1}$$

Now, E is symmetric, R_{h-1}^{-1} is symmetric, Hence Pre-multiplication and Post multiplication with same elementary mat to a symmetric Matrix will generate the same matrix

Hence

$$1-\tilde{\rho}_{n-1}' R_{n-1}^{-1} \tilde{\rho}_{n-1} = 1-\rho_{n-1}' R_{n-1}^{-1} \rho_{n-1}$$

so,

$$\phi_{n,n} = \frac{\rho(n) - \tilde{\rho}_{n-1}' R_{n-1}^{-1} \rho_{n-1}}{1-\tilde{\rho}_{n-1}' R_{n-1}^{-1} \tilde{\rho}_{n-1}}$$

Now, In 1 setep ahead Forecasting

$$\cancel{X_t^n = \sum_{j=1}^{n} \alpha_{n,j} X_{t-j}}$$

$$X_t^n = \sum_{j=1}^{n} \alpha_{n,j} X_{t-j}$$

By Yule walker equation

$$\begin{pmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(n-1) \\ \rho(n) \end{pmatrix} = \frac{1}{\gamma(0)} \begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-2) & \gamma(n-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-3) & \gamma(n-2) \\ \vdots & & & & \\ \gamma(n-2) & \gamma(n-3) & \cdots & \gamma(0) & \gamma(1) \\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(1) & \gamma(0) \end{bmatrix} \underset{\sim}{\alpha}$$

$$\begin{pmatrix} P_{n-1} \\ p(n) \end{pmatrix} = \begin{bmatrix} R_{n-1} & \tilde{P}_{n-1} \\ \tilde{P}_{n-1}' & P(0) \end{bmatrix} \begin{pmatrix} \underset{\sim}{\alpha}_{n,n-1} \\ \alpha_{nn} \end{pmatrix}$$

Where

$$\underset{\sim}{\alpha}_{n-1} = (\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,n-1})$$

Then

$$P_{n-1} = R_{n-1} \underset{\sim}{\alpha}_{n,n-1} + \tilde{P}_{n-1} \alpha_{n,n}$$

$$\Rightarrow R_{n-1}^{-1}(P_{n-1} - \tilde{P}_{n-1} \alpha_{n,n}) = \underset{\sim}{\alpha}_{n,n-1}$$

Again

$$p(n) = \tilde{P}_{n-1}' \underset{\sim}{\alpha}_{n,n-1} + \alpha_{n,n}$$

$$\Rightarrow p(n) = \tilde{P}_{n-1}' R_{n-1}^{-1} P_{n-1} - \tilde{P}_{n-1}' R_{n-1}^{-1} \tilde{P}_{n-1} \alpha_{n,n} + \alpha_{n,n}$$

$$\Rightarrow \frac{P(n) - \tilde{P}_{n-1}' R_{n-1}^{-1} P_{n-1}}{1 - \tilde{P}_{n-1}' R_{n-1}^{-1} \tilde{P}_{n-1}} = \alpha_{n,n}$$