Lecture 40

The Spectral Density: Part 2

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Spectral Representation of $\gamma(h)$ (recap)

- If $\{X_t\}$ is stationary with autocovariance $\gamma(h) = \mathrm{Cov}(X_{t+h}, X_t)$, there exists a unique monotonically increasing function $F(\cdot)$, called the spectral CDF.

- $F(-\infty) = F(-1/2^-) = 0$, and $F(\infty) = F(1/2) = \gamma(0)$ such that

$$\gamma(h) = \int_{-1/2}^{1/2} \exp[\iota\, 2\pi\omega h] dF(\omega).$$

- An important situation we use repeatedly is the case when the autocovariance function is absolutely summable.

- In that case the spectral distribution function is absolutely continuous with $dF(\omega) = f(\omega)d\omega$, and we can talk about spectral density.

Spectral density (recap)

If the autocovariance function $\gamma(h)$ of a stationary process satisfies $\sum_{h=0}^{\infty} |\gamma(h)| < \infty$,

- Then it has the representation

$$\gamma(h) = \int_{-1/2}^{1/2} \exp(\iota\, 2\pi\omega h) f(\omega) d\omega, \quad h = 0, \pm 1, \pm 2, \ldots$$

- Then it can be written as the inverse transform of the spectral density

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) \exp(-\iota\, 2\pi\omega h), \;\; -1/2 \leq \omega \leq 1/2.$$

Spectral density: properties (recap)

- This spectral density is the analogue of the probability density function.
- The fact that $\gamma(h)$ is non-negative definite ensures $f(\omega) \geq 0$ for all ω.
- It follows immediately that $f(\omega) = f(-\omega)$ verifying the spectral density is an even function.
- Because of the evenness, we will typically only plot $f(\omega)$ for $0 \leq \omega \leq 1/2$.
- In addition, putting $h = 0$ yields $\gamma(0) = \mathrm{Var}(X_t) = \int_{-1/2}^{1/2} f(\omega) d\omega$.
- This expresses the total variance as the integrated spectral density over all of the frequencies.

Example: White noise (recap)

- The autocovariance function is $\gamma_W(h) = \sigma_W^2$ for $h = 0$, and zero, otherwise.
- Hence, $f_W(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) \exp(-\iota\, 2\pi\omega h) = \sigma_W^2, \quad -1/2 \leq \omega \leq 1/2.$
- Hence the process contains equal power at all frequencies.
- This property is seen in the realization, which seems to contain all different frequencies in a roughly equal mix.
- In fact, the name white noise comes from the analogy to white light, which contains all frequencies in the color spectrum at the same level of intensity.

Linear process (recap)

- A zero-mean linear process X_t is defined to be a linear combination of white noise variates W_t, and is given by

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j W_{t-j}, \quad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty$$

- Autocovariance is given by $\gamma_X(h) = \sigma_W^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}$
- In general, a linear filter uses a set of specified coefficients, say a_j, for $j = 0, \pm 1, \pm 2, \ldots$, to transform an input series, X_t, producing an output series

$$Y_t = \sum_{j=-\infty}^{\infty} a_j X_{t-j}, \quad \sum_{j=-\infty}^{\infty} |a_j| < \infty$$

Output Spectrum of a Filtered Stationary Series

- The coefficients $\{a_j\}$ are collectively called the impulse response function.
- The Fourier transform

$$A(\omega) = \sum_{j=-\infty}^{\infty} a_j \exp[-\iota\, 2\pi\omega j]$$

  is called the frequency response function.
- If X_t has spectral density $f_X(\omega)$, then the spectrum of the filtered output, Y_t, say $f_Y(\omega)$, is related to the spectrum of the input X_t by

$$f_Y(\omega) = |A(\omega)|^2 f_X(\omega).$$

- Proof: First represent $\gamma_Y(h)$ in terms of $\gamma_X(\cdot)$ and then use its spectral representation.

Proof

$$
\begin{aligned}
\gamma_Y(h) &= \mathrm{Cov}(Y_{t+h}, Y_t) = \mathrm{Cov}\left(\sum_{j=-\infty}^{\infty} a_j X_{t+h-j}, \sum_{k=-\infty}^{\infty} a_k X_{t-k}\right) \\
&= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_j \mathrm{Cov}\left(X_{t+h-j}, X_{t-k}\right) a_k = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_j \gamma_X(h-j+k) a_k \\
&= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_j a_k \int_{-1/2}^{1/2} \exp(\iota\, 2\pi\omega[h-j+k]) f_X(\omega) d\omega \\
&= \int_{-1/2}^{1/2} \left[\sum_{j=-\infty}^{\infty} a_j \exp(-\iota\, 2\pi\omega j)\right]\left[\sum_{k=-\infty}^{\infty} a_k \exp(\iota\, 2\pi\omega k)\right] \exp(\iota\, 2\pi\omega h) f_X(\omega) d\omega \\
&= \int_{-1/2}^{1/2} |A(\omega)|^2 \exp(\iota\, 2\pi\omega h) f_X(\omega) d\omega = \int_{-1/2}^{1/2} f_Y(\omega) d\omega
\end{aligned}
$$

By uniqueness of spectral densities, $f_Y(\omega) = |A(\omega)|^2 f_X(\omega)$.

Spectrum for an ARMA(p, q) model

- If X_t is ARMA(p, q), $\phi(B)X_t = \theta(B)W_t$, its spectral density is given by

$$f_X(\omega) = \sigma_W^2 \frac{|\theta(\exp[-\iota\ 2\pi\omega])|^2}{|\phi(\exp[-\iota\ 2\pi\omega])|^2}.$$

- Here $\phi(\cdot)$ and $\theta(\cdot)$ are AR and MA polynomials.
- The proof follows from $X_t = \psi(B)W_t = \sum_{j=0}^{\infty} \psi_j W_{t-j}$, with $\psi(z) = \theta(z)/\phi(z)$.
- The corresponding frequency response function based on $\{\psi_j,\ j = 0, 1, \dots\}$ is $\Psi(\omega) = \sum_{j=0}^{\infty} \psi_j \exp[-\iota\ 2\pi\omega j]$.
- Hence, $f_X(\omega) = |\Psi(\omega)|^2 f_W(\omega) = \sigma_W^2 |\Psi(\omega)|^2$.
- Calculate the spectral densities for $X_t = W_t + 0.5W_{t-1}$ and $X_t - X_{t-1} + 0.9X_{t-2} = W_t$.

Example: Moving average $X_t = W_t + 0.5W_{t-1}$

- Here $\gamma_X(0) = \sigma_W^2(1 + 0.5^2) = 1.25\sigma_W^2$ and $\gamma(\pm 1) = 0.5\sigma_W^2$.
- Hence,

$$\begin{aligned}
f_X(\omega) &= \sum_{h=-\infty}^{\infty} \gamma_X(h) \exp(-\iota\, 2\pi\omega h) \\
&= 1.25\sigma_W^2 + 0.5\sigma_W^2[\exp(-\iota\, 2\pi\omega) + \exp(\iota\, 2\pi\omega)] \\
&= 1.25\sigma_W^2 + \sigma_W^2 \cos(2\pi\omega) \\
&= \sigma_W^2[1.25 + \cos(2\pi\omega)].
\end{aligned}$$

- Now $|\theta(h)| = 1 + 0.5h$. Hence,

$$\begin{aligned}
\sigma_W^2|\theta(\exp[-\iota\, 2\pi\omega])|^2 &= \sigma_W^2|1 + 0.5(\exp[-\iota\, 2\pi\omega])|^2 \\
&= \sigma_W^2|[1 + 0.5\cos(2\pi\omega)] + \iota 0.5\sin(2\pi\omega)|^2 \\
&= \sigma_W^2[1 + \cos(2\pi\omega) + 0.25\cos^2(2\pi\omega) + 0.25\sin^2(2\pi\omega)] \\
&= \sigma_W^2[1.25 + \cos(2\pi\omega)]
\end{aligned}$$

- Thus, $f_X(\omega) = \sigma_W^2|\theta(\exp[-\iota\, 2\pi\omega])|^2$.

Example: Autoregression $X_t - X_{t-1} + 0.9X_{t-2} = W_t$

- Here

$$\begin{aligned}
\gamma_W(h) &= \mathrm{Cov}(W_{t+h}, W_t) \\
&= \mathrm{Cov}(X_{t+h} - X_{t+h-1} + 0.9X_{t+h-2}, X_t - X_{t-1} + 0.9X_{t-2}) \\
&= 2.81\gamma_X(h) - 1.9[\gamma_X(h+1) + \gamma_X(h-1)] + 0.9[\gamma_X(h+2) + \gamma_X(h-2)].
\end{aligned}$$

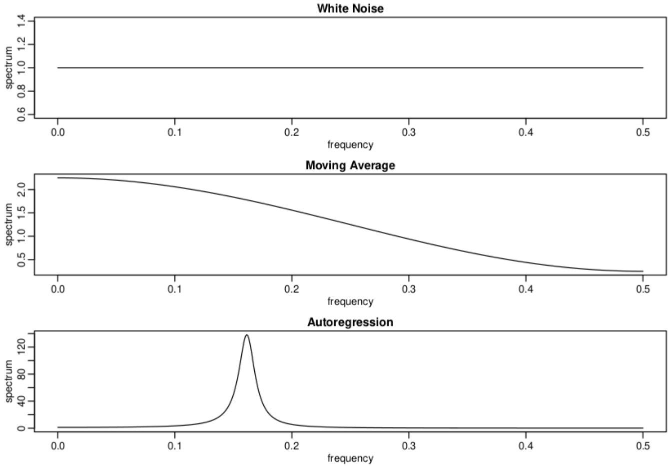
- By uniqueness of spectral density,

$$g_W(\omega) = [2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega)] f_X(\omega).$$

- Hence,

$$f_X(\omega) = \frac{\sigma_W^2}{2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega)}.$$

Different spectral densities



MA(1) case is $X_t = W_t + 0.5W_{t-1}, \quad \sigma_W^2 = 1$

AR(2) case is $X_t - X_{t-1} + 0.9X_{t-2} = W_t, \quad \sigma_W^2 = 1$

Periodic process (contd.)

- Recall the stationary process $X_t = U_1 \cos(2\pi\omega_0 t) + U_2 \sin(2\pi\omega_0 t)$
- We can write this as $X_t = \frac{1}{2}(U_1 + \iota U_2)\exp[-\iota 2\pi\omega_0 t] + \frac{1}{2}(U_1 - \iota U_2)\exp[\iota 2\pi\omega_0 t]$.
- Suppose we call $Z = \frac{1}{2}[U_1 + \iota U_2]$ and $\bar{Z} = \frac{1}{2}[U_1 - \iota U_2]$.
- Clearly, $E(Z) = E(\bar{Z}) = 0$ because $E(U_1) = E(U_2) = 0$.
- Besides, $\mathrm{Var}(Z) = E[|Z|^2] = E[Z\bar{Z}] = \frac{1}{4}(E[U_1^2] + E[U_2^2]) = \frac{\sigma^2}{2}$.
- Finally, $\mathrm{Cov}(Z, \bar{Z}) = \frac{1}{4}E([U_1 + \iota U_2][U_1 - \iota U_2]) = \frac{1}{4}(E[U_1^2] - E[U_2^2]) = 0$.
- Hence, $X_t = Z\exp[-\iota 2\pi\omega_0 t] + \bar{Z}\exp[\iota 2\pi\omega_0 t] = \int_{-1/2}^{1/2} \exp[\iota 2\pi\omega t] dZ(\omega)$.
- Here $Z(\omega)$ is a complex-valued random process that makes uncorrelated jumps at $-\omega_0$ and ω_0 with mean-zero and variance $\sigma^2/2$.

Spectral Representation of a Stationary Process

- If X_t is a mean-zero stationary process, with spectral distribution $F(\omega)$, then there exists a complex-valued stochastic process $Z(\omega)$, on the interval $\omega \in [-1/2, 1/2]$, having stationary uncorrelated non-overlapping increments, such that X_t can be written as the stochastic integral

$$X_t = \int_{-1/2}^{1/2} \exp[\iota 2\pi\omega t] dZ(\omega),$$

where, for $-1/2 \leq \omega_1 \leq \omega_2 \leq 1/2$,

$$\mathrm{Var}\{Z(\omega_2) - Z(\omega_1)\} = F(\omega_2) - F(\omega_1).$$

Thank you!