Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

Time Series Analysis (MTH442)

End-semester examination, Date: Nov 19, 2024, Tuesday

Time: 8:00–9:30 AM (+30 minutes for DAP candidates) Max point: 25

Carrying mobile phones during the exam, even in silent mode, is not allowed. It should be kept in your bag in silent mode or switched off and the bag should be kept in front of the invigilator only. A maximum of only one bio-break will be given during the exam for each student and the duration of leaving the room to coming back cannot exceed 5 minutes overall. Anyone asked to change seat during the exam should do that immediately without any further argument.

All answers should be to the point. Any unnecessary extra sentences will be penalized. If any rough work is necessary, you can use the end of the answer booklet. Calculation mistakes will be penalized. No extra time will be provided. Answers should be written in a neat and clean handwriting. No point will be given for unreadble answers.

1. The autocovariance function of a stationary process can be written in terms of its spectral density as $\gamma(h) = \int_{-1/2}^{1/2} \exp(\iota\ 2\pi\omega h) f(\omega) d\omega,\ \ h = 0, \pm 1, \pm 2, \ldots$. If the spectral density is $f(\omega) = \sigma^2$ for $\omega \in [-1/2, 1/2]$, calculate $\gamma(h)$ for $h = 0, 1$. (1+1 = 2 points)

2. The spectral density can be written as $f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) \exp(-\iota\ 2\pi\omega h),\ \ -1/2 \leq \omega \leq 1/2$ using the inverse spectral transformation. Calculate the spectral density of an AR(1) model $X_t = \phi X_{t-1} + W_t$, where W_t are white noise with marginal variance σ^2. You can use any direct result for the spectral densities of linear processes. Do NOT use any direct formula for spectral densities of ARMA(p, q) models. (2 points)

3. Suppose the values of a scaled periodogram are obtained as $P(0.1) = 2, P(0.3) = 8, P(0.45) = 6$. What are the values of $P(0.7)$ and $P(0.9)$? State any result that supports your claim. (0.5+0.5 =1 point)

4. Consider an ARCH(1) model with $\alpha_0 = 0.5$ and $\alpha_1 = 0.5$. Calculate the marginal mean, marginal variance, and the autocovariance at lag 1. You should derive the results. No point if you just state some known result. You can use the fact that if a process R_t follows an ARCH(1) model, it satisfies the properties of a martingale difference. (1+1+1=3 points)

5. Calculate $\mathrm{Var}[\mathrm{Var}(R_t|R_{t-1})|R_{t-1}]$ for an ARCH(1) model. (1 point)

6. If the ACF of an ARMA time series satisfies the property that $\rho(h) \sim h^{2d-1}$, where d is the order of fractional differencing, under what condition and why we call that a long-memory process? Write very neatly on the first page of your answer sheet only. No partial credit will be given. (1 point)

7. What is the purpose of a unit root test and what test statistic do we use here? Write very neatly on the first page of your answer sheet only. Proper scaling of the test statistic should be used as otherwise the asymptotic distribution of the test statistic would be degenerate. No partial credit will be given. (1+1 =2 points)

Consider a univariate state-space model where the observations are noisy, i.e., $Y_t = X_t + V_t$, and the signal (state) is an AR(1) process, $X_t = \phi X_{t-1} + W_t$, where $V_t \overset{IID}{\sim} \mathcal{N}(0, \sigma_V^2)$, $W_t \overset{IID}{\sim} \mathcal{N}(0, \sigma_W^2)$, and $X_0 \sim \mathcal{N}(0, (1-\phi^2)^{-1}\sigma_W^2)$, with X_0, $\{W_t\}$, and $\{V_t\}$ are independent. We use the notations $X_t^s = E[X_t|Y_1, \ldots, Y_s]$ and $P_t^s = \text{Var}(X_t|Y_1, \ldots, Y_s)$, for $t, s = 0, \ldots, T$. We only observe $Y_1, \ldots, Y_T$. Our goal also lies in estimating the parameters ϕ, σ_W^2, σ_V^2, and we want to implement an E-M algorithm. As a byproduct, we also obtain Kalman smoother. You may need to use numerical optimization within E-M algorithm. The following questions 8–13 are based on this setup.

8. Here, given the state variables X_t's, Y_t's are independent. Are Y_t's unconditionally independent as well? Justify, the derivation of any result that supports your claim needs to be shown. A summary of Yes or No and a one line summary of any result that supports your result should be added to the first page of the answer booklet. (1.5 points)

9. Write down the expressions for X_t^{t-1} and P_t^{t-1} in terms of X_{t-1}^{t-1} and P_{t-1}^{t-1}, respectively, and the model parameters. (0.5+1=1.5 points)

10. Write down a method-of-moments based starting values of ϕ, σ_W^2, σ_V^2. (1+1+1=3 points)

11. Write down the expression of the 2-times negative logarithm of the complete likelihood, i.e., $-2\log L_{\mathcal{X},\mathcal{Y}}(\boldsymbol{\Theta})$. You can skip the constant terms appearing as sums within the objective function as they are unnecessary. (1 point)

12. Calculate the objective function of the E-step $Q(\boldsymbol{\Theta}|\boldsymbol{\Theta}^{(j-1)}) = E[-2\log L_{\mathcal{X},\mathcal{Y}}(\boldsymbol{\Theta}|\mathcal{Y}_{1:T}, \boldsymbol{\Theta}^{(j-1)})]$. Write down the final expression on the first page. (3 points)

13. Write down how you would obtain $\phi^{(j)}$, $\sigma_W^{2(j)}$, $\sigma_V^{2(j)}$ in the M-step from $Q(\boldsymbol{\Theta}|\boldsymbol{\Theta}^{(j-1)})$. Write down the final expressions on the first page. (3 points)