(1) $\{Y_t\}$ is covariance stationary $(0, \sigma^2)$ & ACVF $\gamma_Y(h)$

$X_t = (\alpha + \beta t) s_t + Y_t$; s_t – seasonal with period 4

$\nabla_4 X_t = X_t - X_{t-4}$

$= \left((\cancel{\alpha} + \cancel{\beta t}) s_t + Y_t\right) - \left((\cancel{\alpha} + \beta(\cancel{t} - 4)) s_{t-4} + Y_{t-4}\right)$

i.e. $\nabla_4 X_t = Y_t - Y_{t-4} + 4\beta s_{t-4}$

$\nabla_4^2 X_t = Y_t - 2Y_{t-4} + Y_{t-8} = g_t$

$E g_t = 0 \quad \forall t$

$\mathrm{Cov}(g_t, g_{t+h}) = \mathrm{Cov}(Y_t - 2Y_{t-4} + Y_{t-8},$

$Y_{t+h} - 2Y_{t+h-4} + Y_{t+h-8})$

$\gamma_g(h) = \gamma_Y(h) - 2\gamma_Y(h-4) + \gamma_Y(h-8)$

$- 2\gamma_Y(h+4) + 4\gamma_Y(h) - 2\gamma_Y(h-4)$

$+ \gamma_Y(h+8) - 2\gamma_Y(h+4) + \gamma_Y(h)$

f^n of h only, indep of t

$\Rightarrow g_t = \nabla_4^2 X_t$ is covariance stationary

$\gamma_g(\cdot)$ is ACVF of $\{g_t\}$ in terms of ACVF $\gamma_Y(\cdot)$.

(2) $X_t = A + Bt$

$A \& B$ r.v.s. $\ni E\begin{pmatrix} A \\ B \end{pmatrix} = \underset{\sim}{0}$; $\text{cov}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ & \sigma_{22} \end{pmatrix}$

$E X_t = 0 \quad \forall t \Rightarrow \{X_t\}$ is mean stationary

$\text{cov}(X_t, X_{t+h}) = \text{cov}(A + Bt, A + B(t+h))$

$= \sigma_{11} + (t+h)\sigma_{12} + t\sigma_{12} + t(t+h)\sigma_{22}$

f^n of t

$\Rightarrow \{X_t\}$ is not covariance stationary

(3) $X_t = \mu + X_{t-1} + \epsilon_t \qquad \epsilon_t$ i.i.d. $N(0, \sigma^2)$

$X_1 = \mu + \epsilon_1$ (process initialized at 0)

$X_2 = \mu + X_1 + \epsilon_2 = 2\mu + \epsilon_1 + \epsilon_2$

$\vdots$

$X_t = t\mu + \sum_{i=1}^{t} \epsilon_i$

(a) $E X_t = t\mu \Rightarrow \{X_t\}$ is not mean stationary

$\Rightarrow \{X_t\}$ can't be cov stationary

(b) If $\mu = 0$, $E X_t = 0 \quad \forall t$

$\Rightarrow \{X_t\}$ is mean stationary

$X_t = \sum_{i=1}^{t} \epsilon_i \qquad V X_t = \underline{\underline{t}}\,\sigma^2$

f^n of t

$\{X_t\}$ is not covariance stationary

(4) $X_t = \epsilon_t \cos w_0 t + \epsilon_{t-1} \sin w_0 t + \epsilon_{t-2}$

ϵ_t i.i.d. $N(0, \sigma^2)$; w_0 is fixed const

(a) $\underline{h=0}$ $\quad Cov(X_t, X_t) = V(X_t)$

$$= \cos^2 w_0 t \, \sigma^2 + \sin^2 w_0 t \, \sigma^2 + \sigma^2$$

$$= 2\sigma^2$$

$h = 1$, $\quad Cov(X_{t+1}, X_t) = Cov\Big[\epsilon_{t+1} \cos w_0(t+1) + \epsilon_t \sin w_0(t+1) + \epsilon_{t-1} ,$

$\epsilon_t \cos w_0 t + \epsilon_{t-1} \sin w_0 t + \epsilon_{t-2}\Big)$

$$= \sigma^2 \cos w_0 t \sin w_0(t+1) + \sigma^2 \sin w_0 t$$

$h = -1$

$Cov(X_{t-1}, X_t) = Cov\Big(\epsilon_{t-1} \cos w_0(t-1) + \epsilon_{t-2} \sin w_0(t-1) + \epsilon_{t-3} ,$

$\epsilon_t \cos w_0 t + \epsilon_{t-1} \sin w_0 t + \epsilon_{t-2}\Big)$

$$= \sigma^2 \cos w_0(t-1) \sin w_0 t + \sigma^2 \sin w_0 (t-1)$$

Sly for $h = \pm 2$

(b) $E X_t = 0 \quad \forall t \Rightarrow \{X_t\}$ is mean stationary

(c) As observed in (a) $Cov(X_{t+h}, X_t)$ is not of f^n of h only, it depends on t

$\Rightarrow \{X_t\}$ is not Covariance stationary.

(d) $X_t = \epsilon_t \cos \omega_0 t + \epsilon_{t-1} \sin \omega_0 t + \epsilon_{t-2}$

$\forall\, t_1, \ldots, t_n$ and $(X_{t_1}, \ldots, X_{t_n})$

Consider the random vector

$$\underset{\sim}{X}_t = \begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix} \quad \text{and any } \underset{\sim}{\alpha} \in \mathbb{R}^n \ (\underset{\sim}{\alpha} \neq \underset{\sim}{0})$$

$$\begin{aligned}
\underset{\sim}{\alpha}' \underset{\sim}{X}_t &= \alpha_1 X_{t_1} + \cdots + \alpha_n (X_{t_n}) \\
&= \alpha_1 (\epsilon_{t_1} \cos w_0 t_1 + \epsilon_{t_1-1} \sin w_0 t_1 + \epsilon_{t_1-2}) \\
&\quad + \cdots \\
&\quad + \alpha_n (\epsilon_{t_n} \cos \omega_0 t_n + \epsilon_{t_n-1} \sin w_0 t_n + \epsilon_{t_n-2})
\end{aligned}$$

= linear combination of indep. normal random variables — $(*)$

$\Rightarrow \underset{\sim}{\alpha}' \underset{\sim}{X}_t \sim N_1 \quad \forall\, \underset{\sim}{\alpha} \in \mathbb{R}^n\ (\underset{\sim}{\alpha} \neq \underset{\sim}{0})$ — $(*^2)$

Note: # of indep normals in $(*)$ depends on the spacing between $t_1, \ldots, t_n$

$(*^2) \Rightarrow \underset{\sim}{X}_t \sim N_n$

$\Rightarrow \{X_t\}$ is a Gaussian process.

(e) $\{X_t\}$ is not covariance stationary

$\Rightarrow$ $\{X_t\}$ cannot be strict stationary

(5) $$X_t = \alpha + \beta t + S_t + Y_t$$

S_t: seasonal comp with period 6

$$Y_t = \epsilon_t - \epsilon_{t-1} \; ; \; \epsilon_t \text{ i.i.d. } N(0,\sigma^2)$$

$$E X_t = \alpha + \beta t + S_t \quad (E Y_t = 0 \; \forall t)$$

$$\begin{aligned} \text{Cov}(X_t, X_{t+h}) &= E\big(X_t - (\alpha+\beta t + S_t)\big)\big(X_{t+h} - (\alpha + \beta(t+h) + S_{t+h})\big) \\ &= E(Y_t Y_{t+h}) \\ &= E(\epsilon_t - \epsilon_{t-1})(\epsilon_{t+h} - \epsilon_{t+h-1}) \end{aligned}$$

$$h=2 \rightarrow \text{Cov}(X_t, X_{t+2}) = E(\epsilon_t - \epsilon_{t-1})(\epsilon_{t+2} - \epsilon_{t+1}) = 0$$

Sly $\forall \; |h| \geq 2$ there are no common terms

$\Rightarrow$ $\text{Cov}(X_t, X_{t+h}) = 0 \quad \forall |h| \geq 2$

(6) $$E X_t = \alpha + \beta t + S_t$$

$\Rightarrow$ $\{X_t\}$ is not even mean stationary

$\Rightarrow$ $\{X_t\}$ is not Cov stationary

Note: Note that covariance structure will be a f^n of h.

(c) Consider any $t_1, \ldots, t_n$ and corresp $X_{t_1}, \ldots, X_{t_n}$

Let $\underset{\sim}{X}_t = \begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix}$ & take $\underset{\sim}{\alpha} \in \mathbb{R}^n$ ($\underset{\sim}{\alpha} \neq \underset{\sim}{0}$)

$$\underset{\sim}{\alpha}' \underset{\sim}{X}_t = \alpha_1 (\alpha + \beta t_1 + S_{t_1} + \epsilon_{t_1} - \epsilon_{t_1 - 1})$$
$$+ \alpha_2 (\alpha + \beta t_2 + S_{t_2} + \epsilon_{t_2} - \epsilon_{t_2 - 1})$$
$$\vdots$$
$$+ \alpha_n (\alpha + \beta t_n + S_{t_n} + \epsilon_{t_n} - \epsilon_{t_n - 1})$$

= A deterministic const (f^n of $\alpha, \beta, S_{t_1}, \ldots S_{t_n}$). + a linear combination of indep. univariate normal random variables

$\sim N_1 \quad \forall\, \underset{\sim}{\alpha} \in \mathbb{R}^n$ ($\underset{\sim}{\alpha} \neq \underset{\sim}{0}$

$\Rightarrow \{X_t\}$ is a Gaussian process.

(d) $\nabla X_t = X_t - X_{t-1}$

$$= (\alpha + \beta t + S_t + Y_t) - (\alpha + \beta(t-1) + S_{t-1} + Y_{t-1})$$
$$= \beta + S_t - S_{t-1} + Y_t - Y_{t-1}$$

$E\, \nabla X_t = \beta + S_t - S_{t-1} \leftarrow f^n$ of t

$\Rightarrow \{X_t\}$ is not covariance stationary

(e) $\nabla_6 X_t = X_t - X_{t-6}$

$$= (\cancel{\alpha} + \cancel{\beta t} + \cancel{s_t} + Y_t) - (\cancel{\alpha} + \beta(\cancel{t}-6) + \cancel{s_{t-6}} + Y_{t-6})$$

$$\nabla_6 X_t = 6\beta + Y_t - Y_{t-6}$$

$$E\, \nabla_6 X_t = 6\beta$$

<u>h=2</u>

$$Cov(\nabla_6 X_t, \nabla_6 X_{t+2}) = Cov(6\beta + Y_t - Y_{t-6},\ 6\beta + Y_{t+2} - Y_{t-4})$$

$$= Cov(Y_t - Y_{t-6},\ Y_{t+2} - Y_{t-4})$$

$$= Cov[\epsilon_t - \epsilon_{t-1} - (\epsilon_{t-6} - \epsilon_{t-7}),$$

$$\epsilon_{t+2} - \epsilon_{t+1} - (\epsilon_{t-4} - \epsilon_{t-5})] = 0$$

Sly $h = -2$ $\quad Cov = 0$

Sly $h = \pm 3$, will not have any common term

<u>h=4</u>

$$Cov(\nabla_6 X_t, \nabla_6 X_{t+4})$$

$$= Cov(Y_t - Y_{t-6},\ Y_{t+4} - Y_{t-2})$$

$$= Cov(\epsilon_t - \epsilon_{t-1} - (\epsilon_{t-6} - \epsilon_{t-7}),$$

$$\epsilon_{t+4} - \epsilon_{t+3} - (\epsilon_{t-2} - \epsilon_{t-3})) = 0$$

Expected !!

(f) $Z_t = \nabla_6 X_t = 6\beta + Y_t - Y_{t-6}$

$Y_t = \epsilon_t - \epsilon_{t-1}$

Note the $\{Y_t\}$ is $\ni$ $E Y_t = 0$ &

$$\gamma_Y(h) = Cov(Y_t, Y_{t+h}) = \begin{cases} 2\sigma^2, & h = 0 \\ -\sigma^2, & h = \pm 1 \\ 0, & o/w \end{cases}$$

$\Rightarrow \{Y_t\}$ is cov stat (in fact MA(1))

$E Z_t = 6\beta \ \forall t$

$$Cov(Z_t, Z_{t+h}) = Cov(Y_t - Y_{t-6}, Y_{t+h} - Y_{t+h-6})$$
$$= \gamma_Y(h) - \gamma_Y(h-6) - \gamma_Y(h+6) + \gamma_Y(h)$$

f^n of h only

$\Rightarrow \{Z_t\}$ is Covariance stationary

(g) Use the similar logic as used in (e) to conclude that $\{Z_t\}$ is Gaussian

Further, $\{Z_t\}$ is covariance stationary

$\Rightarrow \{Z_t\}$ $(Z_t = \nabla_6 X_t)$ is strict stationary.

(6) $\{X_t\}$ – covariance stationary with mean μ_X & ACVF $\gamma_X(\cdot)$

$Z_t = X_t - X_{t-1}$

$E\, Z_t = 0 \quad \forall t$

$Cov(Z_t, Z_{t+h}) = Cov(X_t - X_{t-1}, X_{t+h} - X_{t+h-1})$

$= \gamma_X(h) - \gamma_X(h-1) - \gamma_X(h+1) + \gamma_X(h)$

$\Rightarrow \{X_t\}$ is cov stationary

(7) Z_t is cov stationary given by

$Z_t = X_t + Y_t$

Counter example

$$Z_t = \underbrace{A \cos \omega t}_{X_t} + \underbrace{B \sin \omega t}_{Y_t}$$

A & B are uncorrelated r.v.s $\ni$

$E(A) = E(B) = 0$

$V(A) = V(B) = 1$

Z_t is covariance stationary with

$E\, Z_t = 0 \;\forall t$

$Cov(Z_t, Z_{t+h}) = \cos \omega h \quad \forall h$

$X_t = A \cos \omega t$ is $\ni$ $V(X_t) = \cos^2 \omega t$

$\Rightarrow \{X_t\}$ is not covariance stationary ($\{Y_t\}$ is also not cov stat)

(8) ϵ_t i.i.d. $N(0,1)$

(a) $X_t = a + b\epsilon_t + c\epsilon_{t-2}$

$E X_t = a \quad \forall t$

$$\begin{aligned} Cov(X_t, X_{t+h}) &= Cov(b\epsilon_t + c\epsilon_{t-2}, b\epsilon_{t+h} + c\epsilon_{t+h-2}) \\ &= b^2 I_0(h) + bc\, I_2(h) + bc\, I_{-2}(h) + c^2 I_0(h) \\ &= \begin{cases} b^2+c^2, & h=0 \\ bc, & h=\pm 2 \\ 0, & o/w \end{cases} \leftarrow f^n \text{ of } h \text{ only} \end{aligned}$$

$\Rightarrow \{X_t\}$ is Covariance stationary

(b) Similar to an earlier problem

(c) $X_t = \epsilon_t \epsilon_{t-1}$

$E X_t = 0 \quad \forall t$

$$\begin{aligned} Cov(X_t, X_{t+h}) &= E(X_t X_{t+h}) \\ &= E(\epsilon_t \epsilon_{t-1} \epsilon_{t+h} \epsilon_{t+h-1}) \\ &= \begin{cases} 1, & h=0 \\ 0, & o/w \end{cases} \leftarrow f^n \text{ of } h \text{ only} \end{aligned}$$

$\Rightarrow \{X_t\}$ is Covariance stationary

(9) $X_t = A\cos\omega_0 t + B\sin\omega_0 t + Z_t$

(a) A, B, ω_0 are fixed consts.

$\{Z_t\}$ seq of indep $(0, \sigma^2)$ r.v.s.

$$EX_t = A\cos\omega_0 t + B\sin\omega_0 t \leftarrow f^n \text{ of } t$$

$\Rightarrow \{X_t\}$ is not covariance stationary

(b) A & B are indep $(0, \sigma_0^2)$ r.v.s.

$\{Z_t\}$ seq of indep $(0, \sigma^2)$ r.v.s.

$\{Z_t\}$ & A, B are indep.

$\omega_0 \in (0, \pi)$ is fixed

$$EX_t = 0 \;\forall t$$

$$X_t = Y_t + Z_t$$

$$Y_t = A\cos\omega_0 t + B\sin\omega_0 t$$

$\{Y_t\}$ & $\{Z_t\}$ are indep.

$$\begin{aligned}Cov(X_t, X_{t+h}) &= Cov(Y_t + Z_t, Y_{t+h} + Z_{t+h}) \\ &= Cov(Y_t, Y_{t+h}) + \cancel{Cov(Y_t, Z_{t+h})}^{0} \\ &\quad + \cancel{Cov(Z_t, Y_{t+h})}^{0} + Cov(Z_t, Z_{t+h}) \quad -(*)\end{aligned}$$

Realise that $\{Y_t\}$ is covariance stationary with

$$EY_t = 0 \;\forall t \quad \gamma_Y(h) = (\cos\omega_0 h)\,\sigma_0^2$$

$\{Z_t\}$ is also cov stat with $EZ_t = 0 \;\forall t$

$$\gamma_Z(h) = \begin{cases}\sigma^2; & h = 0 \\ 0; & o/w\end{cases}$$

We have due to indep (refer to *)

$$\gamma_X(h) = \underbrace{\gamma_Y(h) + \gamma_Z(h)}_{f^n \text{ of } h \text{ only}}$$

(this would have been true even with uncorrelatedness of $\{Y_t\}$ & $\{Z_t\}$)

$\Rightarrow \{X_t\}$ is covariance stationary

(10)

$$\gamma(h) = (-1)^{|h|} + \cos\left(\frac{\pi}{4} h\right).$$

Let $X_t = (-1)^t A$; A is a r.v. with mean 0 and var 1

alternating time series ($X_1 = -A,\ X_2 = A,\ X_3 = -A, \ldots$

$E X_t = 0$

$\mathrm{cov}(X_t, X_{t+h}) = E\left((-1)^t A\ (-1)^{t+h} A\right) = (-1)^h$

$= (-1)^{-h}$

$$\gamma_X(h) = (-1)^{|h|}$$

$\cos\left(\frac{\pi}{4}h\right)$ is ACVF of $Y_t = A_1 \cos(\pi/4\, t) + B_1 \sin(\pi/4\, t)$

A_1 & B_1 are uncorrelated $(0,1)$

$\gamma_Y(h) = \cos(\pi/4\, h)$

Suppose A is indep of (A_1, B_1) (or even u. c).

then $Z_t = X_t + Y_t$ will be covariance stationary with ACVF $\gamma_Z(h) = \gamma_X(h) + \gamma_Y(h)$

i.e. $\gamma_Z(h) = (-1)^{|h|} + \cos \pi/4\, h$

(12)

$$X_t = Y_t \left(\cos(\omega_0 t + \theta)\right) + Z_t$$

$\{Y_t\}$ & $\{Z_t\}$ are indep covariance stationary

ACVFs: $\gamma_Y(h)$ $\gamma_Z(h)$; means: μ_X & μ_Z, say

ω_0 is a fixed constant

$\theta \sim U(-\pi, \pi)$ and is indep of $\{Y_t\}$ & $\{Z_t\}$

Let $P_t = Y_t \cos(\omega_0 t + \theta)$

$$E P_t = E Y_t \, E\left(\cos(\omega_0 t + \theta)\right)$$

$$= \mu_Y \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\omega_0 t + \theta)\, d\theta = 0 \ \forall t$$

$$\left[\int_{-\pi}^{\pi} (\cos \omega_0 t \cos\theta - \sin \omega_0 t \sin\theta)\, d\theta = 0 \right]$$

$$E P_t P_{t+h} = E(Y_t Y_{t+h}) \, E\left(\cos(\omega_0 t + \theta) \cos(\omega_0 (t+h) + \theta)\right)$$

$$= \gamma_Y(h) \underset{+\mu_Y^2}{} \left(\frac{1}{2} E\left(\cos \omega_0 h + \cos(2\omega_0 t + 2\theta + \omega_0 h)\right)\right)$$

$$= \gamma_Y(h) \underset{+\mu_Y^2}{} \left(\frac{1}{2} \cos \omega_0 h + \frac{1}{2} E\left(\cos(2\omega_0 t + 2\theta + \omega_0 h)\right)\right)$$

$$\left[E\left(\cos(\omega_0(2t+h) + 2\theta)\right) \right.$$

$$= \cos \omega_0 (2t+h) \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos 2\theta - \sin\left(\omega_0(2t+h)\right) \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin 2\theta \, d\theta$$

$$\left. = 0 \right]$$

$$\Rightarrow \gamma_P(h) = \frac{1}{2} \left(\gamma_Y(h) + \mu_Y^2\right) \cos \omega_0 h$$

$\{P_t\}$ is covariance stationary

$\{P_t\}$ & $\{Z_t\}$ are indep covariance stationary

hence

$X_t = P_t + Z_t$ is Covariance stationary

with $E X_t = E P_t + E Z_t = \mu_Z$

$$\gamma_X(h) = \gamma_P(h) + \gamma_Z(h)$$

(13) $Y_t = I_t X_t + (1-I_t) X_{t-1}$

$E Y_t = E(I_t) E X_t + E(1-I_t) E X_{t-1}$

$= 0 \quad (E X_t = 0 \ \forall t)$

X_t is covar stat with mean 0 & ACVF $\gamma_X(h)$

I_t is $\left.\begin{matrix} 1 & \text{w.p } 1-p \\ 0 & \text{w.p. } p \end{matrix}\right\}$ I_t is i.i.d. seq indep of $\{X_t\}$.

$V Y_t = E Y_t^2 = E\left(I_t^2 X_t^2 + (1-I_t)^2 X_{t-1}^2 + 2 I_t (1-I_t) X_t X_{t-1}\right)$

$\gamma_Y(0) = (1-p)\gamma_X(0) + p\gamma_X(0) + 2E\underbrace{(\cancel{I_t} - I_t^2)}_{0} E(X_t X_{t-1})$

i.e $\gamma_Y(0) = (1-p)\gamma_X(0) + p\gamma_X(0)$

$h \neq 0$

$\mathrm{cov}(Y_t, Y_{t+h}) = E\, Y_t Y_{t+h}$

i.e. $\operatorname{Cov}(Y_t, Y_{t+h})$

$$= E\Big(\big(I_t X_t + (1-I_t) X_{t-1}\big)\big(I_{t+h} X_{t+h} + (1-I_{t+h} X_{t+h-1})\big)\Big)$$

$$= E(I_t I_{t+h}) E(X_t X_{t+h}) + E\big(I_t(1-I_{t+h})\big) E(X_t X_{t+h-1}) + E(1-I_t) I_{t+h}\; E(X_{t-1} X_{t+h}) + E(1-I_t)(1-I_{t+h}) E(X_{t-1} X_{t+h-1})$$

$$\gamma_Y(h) = (1-p)^2 \gamma_X(h) + p(1-p)\gamma_X(h-1) + p(1-p)\gamma_X(h+1) + p^2 \gamma_X(h)$$ $\leftarrow f^n$ of h only

$\Rightarrow \{Y_t\}$ is Covariance stationary

(14) $X_t = e^Y t^2 + \epsilon_t$

$\epsilon_t \sim WN(0,\sigma^2)$; $Y \sim U(0,1)$

u.c.

Y & $\{\epsilon_t\}$ are indep

$Z_t = \nabla^2 X_t$

$$\nabla X_t = (e^Y t^2 + \epsilon_t) - (e^Y (t-1)^2 + \epsilon_{t-1})$$

$$\nabla X_t = (\cancel{e^Y t^2} + \epsilon_t) - (e^Y (\cancel{t^2} + 1 - 2t) + \epsilon_{t-1})$$

$$= 2te^Y - e^Y + \epsilon_t - \epsilon_{t-1}$$

$$\nabla^2 X_t = (2\cancel{t}e^Y - e^Y) - (2(\cancel{t}-1)e^Y - e^Y) + \epsilon_t - 2\epsilon_{t-1} + \epsilon_{t-2}$$

$$= 2e^Y + \underbrace{\epsilon_t - 2\epsilon_{t-1} + \epsilon_{t-2}}_{\varrho_t \text{ — MA(2)}}$$

Let $Z_t = \nabla^2 X_t$

i.e. $Z_t = 2e^Y + \varrho_t$; $EZ_t = 2E(e^Y) \; \forall t$ — (i)

$$Cov(Z_t, Z_{t+h}) = Cov(2e^Y + \rho_t, 2e^Y + \rho_{t+h})$$
$$= 4V(e^Y) + Cov(\rho_t, \rho_{t+h}) \quad \text{— (ii)}$$

$$Cov(\rho_t, \rho_{t+h}) = \begin{cases} 6\sigma^2, & h=0 \\ -4\sigma^2, & h=\pm1 \\ \sigma^2, & h=\pm 2 \\ 0, & o/w \end{cases} \quad \text{— (iii)}$$

$$E(e^Y) = \int_0^1 e^y\,dy = (e^1 - 1)$$

$$E(e^Y) = (e-1)$$

$$E(e^{2Y}) = \int_0^1 e^{2y}\,dy = \frac{1}{2}(e^2 - 1)$$

$$V(e^Y) = E(e^{2Y}) - (E(e^Y))^2$$
$$= \frac{1}{2}(e^2-1) - (e-1)^2 = g(e) \quad \text{— (iv)}$$

(iii) & (iv) $\Rightarrow Cov(Z_t, Z_{t+h}) \rightarrow f^n$ of h only, indep of t

$\Rightarrow \{Z_t\}$ is covariance stationary

(15) $\{X_t\}$ - Gaussian process $\ni$

$$E X_t = 0 \;\forall t$$

$$Cov(X_t, X_{t+s}) = e^{-|t-s|} \;\forall t, s.$$

$$Y_t = e^{X_t}$$

(i) $Z_t = \nabla X_t = X_t - X_{t-1}$

$$E Z_t = 0$$

$$Cov(Z_t, Z_{t+h}) = Cov(X_t - X_{t-1}, X_{t+h} - X_{t+h-1})$$

$$= Cov(X_t, X_{t+h}) - Cov(X_t, X_{t+h-1})$$
$$- Cov(X_{t-1}, X_{t+h})$$
$$+ Cov(X_{t-1}, X_{t+h-1})$$
$$= e^{-|t-h|} - e^{-|t-h+1|} - e^{-|(t-1)-(h+1)|}$$
$$+ e^{-|t-1-h|} \quad - *$$

$t=0,\ h=1;\ (*) = e^{-1} - e^{0} - e^{-3} + e^{-2}$

$t=1;\ h=1;\ (*) = e^{0} - e^{-1} - e^{-2} + e^{-1}$

$\Rightarrow \nabla X_t$ is not covariance stationary

$\Rightarrow \nabla X_t$ is not strict stationary

(ii) $Y_t = e^{X_t} \not\sim N_1$

$\Rightarrow \{Y_t\}$ is not a Gaussian process.

(iii) $E\,Y_t = E\,(e^{X_t})$

$$= M_{X_t}(1) = \exp\left(\frac{e^{-|t|}}{2}\right) \leftarrow f^n \text{ of } t$$

m.g.f. of X_t at 1

$X_t \sim N(0, e^{-|t|})$

$\Rightarrow \{Y_t\}$ is not covariance stationary

(16) $X_t = \epsilon_t + \epsilon_{t-1}$

$\{\epsilon_t\}$ seq of i.i.d. $N(0, \sigma^2)$

$\{Y_t\}$ is EWMA derived from $\{X_t\}$

$Y_1 = X_1$

$\forall\, t \geq 2\,;\quad Y_t = \alpha X_t + (1-\alpha)\, Y_{t-1} \quad ;\ \alpha = \frac{3}{4}$

(a) $$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} X_1 \\ \alpha X_2 + (1-\alpha) X_1 \\ \alpha X_3 + (1-\alpha)\left(\alpha X_2 + (1-\alpha) X_1\right) \end{pmatrix}$$

$$\underset{\sim}{Y} = \begin{pmatrix} 1 & 0 & 0 \\ 1-\alpha & \alpha & 0 \\ (1-\alpha)^2 & \alpha(1-\alpha) & \alpha \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}.$$

$$\underset{\sim}{Y} = A(\alpha)\, \underset{\sim}{X}$$

Realise that $\{X_t\}$ is a Gaussian process

$\Rightarrow \underset{\sim}{X} \sim N_3(\underset{\sim}{0}, \Sigma)$

$$\Sigma = \begin{pmatrix} 2\sigma^2 & \sigma^2 & 0 \\ & 2\sigma^2 & \sigma^2 \\ & & 2\sigma^2 \end{pmatrix}.$$

$\Rightarrow \underset{\sim}{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N_3(\underset{\sim}{0}, A(\alpha)\Sigma A'(\alpha)).$

(b) $\underset{\sim}{Y}_n = \begin{pmatrix} Y_{t_1} \\ \vdots \\ Y_{t_n} \end{pmatrix}.$

$\forall \underset{\sim}{\alpha} \in \mathbb{R}^n (\underline{\alpha} \neq \underline{0})$

$\underline{\alpha}' \underset{\sim}{Y}_n$ is linear comb of indep normal

$\Rightarrow \underline{\alpha}' \underset{\sim}{Y}_n \sim N_1 \quad \forall \underset{\sim}{\alpha} \in \mathbb{R}^n (\underline{\alpha} \neq \underline{0})$

$\Rightarrow \underset{\sim}{Y}_n \sim N_n$

$\Rightarrow \{Y_t\}$ is Gaussian

(c) $V(Y_1) = 2\sigma^2 \neq V(Y_2) = V(\alpha X_2 + (1-\alpha)X_1)$

$= (\alpha^2 + (1-\alpha)^2)2\sigma^2 + 2\,\mathrm{Cov}(\alpha X_2, (1-\alpha)X_1)$

$\Rightarrow \{Y_t\}$ is not even covariance stationary

$\Rightarrow \{Y_t\}$ is not strict stationary

—x—