MTH442 Assignment 2 Solutions

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Q1:

MA(1) model:

$$X_t = W_t + \theta W_{t-1}$$

variance:

$$\text{Var}(X_t) = \text{Var}(W_t + \theta W_{t-1}) = \text{Var}(W_t) + \theta^2 \text{Var}(W_{t-1}) = \sigma^2 + \theta^2\sigma^2 = \sigma^2(1+\theta^2)$$

covariance:

$$\text{Cov}(X_t, X_{t-1}) = \text{Cov}(W_t + \theta W_{t-1}, W_{t-1}) = \theta\sigma^2$$

(ACF) at lag 1 is:

$$\rho_1 = \frac{\text{Cov}(X_t, X_{t-1})}{\text{Var}(X_t)} = \frac{\theta\sigma^2}{\sigma^2(1+\theta^2)} = \frac{\theta}{1+\theta^2}$$

finding max. value of $|\rho_1|$:

$$f(\theta) = \frac{\theta}{1+\theta^2}$$

$$f'(\theta) = \frac{1-\theta^2}{(1+\theta^2)^2}$$

Set $f'(\theta) = 0$ to find the critical points:

$$1 - \theta^2 = 0 \Rightarrow \theta^2 = 1 \Rightarrow \theta = \pm 1$$

$f(\theta)$ at $\theta = 1$ and $\theta = -1$:

$$f(1) = \frac{1}{1+1^2} = \frac{1}{2}, \quad f(-1) = \frac{-1}{1+1^2} = -\frac{1}{2}$$

as $\rho_1 = \frac{\theta}{1+\theta^2} \Rightarrow |\rho_1| = \left|\frac{\theta}{1+\theta^2}\right| \leq \frac{1}{2}$
so ACF is bounded above by 0.5.

Q2:

(a) X_t

given $X_0 = W_0$. so i can recursively put values, $X_t = \phi X_{t-1} + W_t = \phi(\phi X_{t-2} + W_{t-1}) + W_t = \ldots = \sum_{j=0}^{t} \phi^j W_{t-j}$.

(b) E(X_t)

$$E(X_t) = E\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right) = \sum_{j=0}^{t} \phi^j E(W_{t-j}) = 0$$

because $E(W_t) = 0$ for all t.

(c) Variance of X_t

$$\text{Var}(X_t) = \text{Var}\left(\sum_{j=0}^{t} \phi^j W_{t-j}\right) = \sum_{j=0}^{t} \phi^{2j}\,\text{Var}(W_{t-j}) = \sigma_W^2 \sum_{j=0}^{t} \phi^{2j} = \sigma_W^2 \frac{1-\phi^{2(t+1)}}{1-\phi^2} \text{for } |\phi| < 1.$$

As $t \to \infty$:

$$\text{Var}(X_t) \to \frac{\sigma_W^2}{1-\phi^2}.$$

(d) Covariance Calculation

$$\mathrm{Cov}(X_{t+h}, X_t) = E(X_{t+h}X_t) - E(X_{t+h})E(X_t) = E\left(\left(\sum_{j=0}^{t+h} \phi^j W_{t+h-j}\right)\left(\sum_{k=0}^{t} \phi^k W_{t-k}\right)\right)$$

W_t are independent, so terms like W_t and W_{t+h}have 0 covariance for all non 0 integer h.

$$\mathrm{Cov}(X_{t+h}, X_t) = \sigma_W^2 \sum_{j=0}^{t} \phi^{j+h}\phi^j = \sigma_W^2 \phi^h \sum_{j=0}^{t} \phi^{2j} = \sigma_W^2 \phi^h \frac{1-\phi^{2(t+1)}}{1-\phi^2}$$

(e) Stationarity of X_t

from classnotes defn, process $\{X_t\}$ is weakly stationary if $E(X_t)$ and $\mathrm{Var}(X_t)$ are constant for all t, and $\mathrm{Cov}(X_{t+h}, X_t)$ depends only on lag h.

from part(b), $E(X_t) = 0$. But from part c, as variance depends on t so it is not stationary process.

(f) Asymptotic Stationarity

Conditions are

$$\lim_{t\to\infty} \mathbb{E}[X_t] = \mu$$

$$\lim_{t\to\infty} \mathrm{Var}(X_t) = \sigma^2 < \infty$$

$$\lim_{t\to\infty} \mathrm{Cov}(X_{t+h}, X_t) = \gamma(h)$$

so from part(b), $E(X_t) = 0$, and from part c and d $t \to \infty$, $\mathrm{var}(X_t)$ approaches $\frac{\sigma_W^2}{1-\phi^2}$, which is constant. covariance approaches $\mathrm{Cov}(X_{t+h}, X_t) = \phi^h \frac{\sigma_W^2}{1-\phi^2}$ also depends only on lag h. So the process $\{X_t\}$ is asymptotically stationary as $t \to \infty$ when $|\phi| < 1$.

(g)

So for simulating from stationary Gaussian AR(1) process, we assume large value of t such that $\phi^t \approx 0$ and then generate $X_t, X_{t+1}, \ldots, X_{t+n}$ as

$$X_{t+k} = \sum_{j=0}^{t} \phi^j W_{t+k-j} \quad \text{for } k = 0, 1, 2, \ldots, n$$

h)

Now consider $X_0 = \frac{W_0}{1-\phi^2}$.

$$X_t = \phi X_{t-1} + W_t$$

using result

$$X_t = \sum_{j=0}^{t-1} \phi^j X_{t-j} + \frac{\phi^t W_0}{\sqrt{1-\phi^2}}$$

$$E(X_t) = E\left(\sum_{j=0}^{t-1} \phi^j X_{t-j} + \frac{\phi^t W_0}{\sqrt{1-\phi^2}}\right) = 0 \quad \text{(independent of } t)$$

$$\mathrm{Var}(X_t) = \mathrm{Var}\left(\sum_{j=0}^{t-1} \phi^j W_{t-j} + \frac{\phi^t W_0}{\sqrt{1-\phi^2}}\right)$$

$$= \sum_{j=0}^{t-1} \phi^{2j}\mathrm{Var}(W_{t-j}) + \mathrm{Var}\left(\frac{\phi^t W_0}{\sqrt{1-\phi^2}}\right)$$

$$= \sum_{j=0}^{t-1} \phi^{2j}\sigma^2 + \frac{\sigma^2\phi^{2t}}{1-\phi^2}$$

$$= \sigma^2 \left(1 + \phi^2 + \ldots + \phi^{2(t-1)}\right) + \frac{\sigma^2 \phi^{2t}}{1-\phi^2}$$
$$= \sigma^2 \left[\frac{1-\phi^{2t}}{1-\phi^2} + \frac{\sigma^2 \phi^{2t}}{1-\phi^2}\right]$$
$$= \frac{\sigma^2}{1-\phi^2} \text{(independent of } t)$$

$$\operatorname{Cov}(X_{t+h}, X_t) = \operatorname{Cov}\left(\sum_{j=0}^{t+h-1} \phi^j W_{t+h-j} + \frac{\phi^{t+h} W_0}{\sqrt{1-\phi^2}}, \sum_{j=0}^{t-1} \phi^j W_{t-j} + \frac{\phi^t W_0}{\sqrt{1-\phi^2}}\right)$$
$$= \operatorname{Cov}\left(\sum_{j=0}^{t+h-1} \phi^j W_{t+h-j}, \sum_{j=0}^{t-1} \phi^j W_{t-j}\right) + \operatorname{Cov}\left(\frac{\phi^{t+h} W_0}{\sqrt{1-\phi^2}}, \frac{\phi^t W_0}{\sqrt{1-\phi^2}}\right)$$
$$= \sum_{j=0}^{t-1} \phi^{j+h} \sigma^2 + \frac{\sigma^2 \phi^{2t}}{1-\phi^2}$$
$$= \sigma^2 \phi^h \left(1 + \phi^2 + \ldots + \phi^{2(t-1)}\right) + \frac{\sigma^2 \phi^{2t}}{1-\phi^2}$$
$$= \sigma^2 \phi^h \left[\frac{1-\phi^{2t}}{1-\phi^2}\right] + \frac{\sigma^2 \phi^{2t}}{1-\phi^2}$$
$$= \frac{\sigma^2 \phi^h}{1-\phi^2} \text{ (depends only on } h]$$

so process is stationary.

Q3:

Let AR(2) process:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t$$

X_t is stationary and W_t is WN(0, σ^2).

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} = W_t$$
$$(1 - \phi_1 B - \phi_2 B^2) X_t = W_t$$

$$\Phi(z) = 1 - \phi_1 z - \phi_2 z^2$$

Condition of causal on roots:$|z_1| > 1$,$|z_2| > 1$.

$$1 - \phi_1 z - \phi_2 z^2 = 0$$

root by quad. formula

$$z = \frac{-\phi_1 \pm \sqrt{\phi_1^2 - 4\phi_2}}{2\phi_2}$$

let z1, z2 are root

if $\phi_1^2 - 4\phi_2 > 0$ (real distinct root)

$$|z_1| < 1 \Rightarrow z_1 < 1$$
$$\Rightarrow \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} < 1$$
$$\phi_1 + \sqrt{\phi_1^2 + 4\phi_2} < 2$$

$$\phi_1^2 + 4\phi_2 < (2 - \phi_1)^2$$
$$\phi_1^2 + 4\phi_2 < 4 - 4\phi_1 + \phi_1^2$$
$$4\phi_2 < 4 - 4\phi_1$$

$$\phi_2 < 1 - \phi_1$$

And for z2:

$$|z_2| < 1 \Rightarrow z_2 > -1$$

$$\phi_1 - \sqrt{\phi_1^2 + 4\phi_2} < 1$$

$$\phi_1 - \sqrt{\phi_1^2 + 4\phi_2} > -1$$

$$\phi_1 + 2\sqrt{\phi_1^2 + 4\phi_2} > \phi_1^2 + 4\phi_2$$

$$4\phi_2 - 4\phi_1 < 1$$

$$\phi_2 - \phi_1 < 1$$

and

$$|z_1||z_2| < 1$$

$$\left| \frac{(\phi_1 + \sqrt{\phi_1^2 + 4\phi_2})(\phi_1 - \sqrt{\phi_1^2 + 4\phi_2})}{4} \right| < 1$$

$$|\phi_2| < 1$$

now if roots are complex:

$$z = re^{i\theta}$$

$$|z| = r < 1$$

$$\sqrt{\phi_1 - \sqrt{\phi_1^2 - 4\phi_2}} < 1$$

$$\sqrt{\phi_1^2 - 4\phi_2} < 1$$

$$\sqrt{\phi_2} < 1$$

$$-\phi_2 < 1$$

So causal condn. are:

$$|z_1| < 1, \quad |z_2| < 1, \quad \phi_1 < 1, \quad \phi_2 > 0.$$

Q3:

given

$$(1 - \phi_1 B - \phi_2 B^2)X_t = W_t$$

characteristic eqn.

$$1 - \phi_1 z - \phi_2 z^2 = 0$$

$$z^2 + \frac{\phi_1}{\phi_2}z + \frac{1}{\phi_2} = 0$$

root by quad. formula

$$z = \frac{-\phi_1 \pm \sqrt{\phi_1^2 - 4\phi_2}}{2\phi_2}$$

causality condition for root is $|z| > 1$.
D of quad.eqn. is $\phi_1^2 - 4\phi_2$.let z1, z2 are root

if $\phi_1^2 - 4\phi_2 > 0$ (real and distinct root)

$$z_1 = \frac{-\phi_1 + \sqrt{\phi_1^2 - 4\phi_2}}{2\phi_2}, \quad z_2 = \frac{-\phi_1 - \sqrt{\phi_1^2 - 4\phi_2}}{2\phi_2}$$

as $|z_1| > 1$ and $|z_2| > 1 \Rightarrow |z_1 z_2| > 1$:

$$|z_1 z_2| = \frac{1}{|\phi_2|}, \quad \left|\frac{1}{\phi_2}\right| > 1 \Rightarrow 1 > |\phi_2|$$

$$z_1 + z_2 = \frac{-\phi_1}{\phi_2}$$

as both roots outside unit circle

$$\phi_1 + \phi_2 < 1, \quad \phi_2 - \phi_1 < 1$$

if $\phi_1^2 - 4\phi_2 = 0$ (Equal roots)

$$z_1 = z_2 = \frac{-\phi_1}{2\phi_2}$$

Condition $|z_1| > 1$:

$$\left|\frac{-\phi_1}{2\phi_2}\right| > 1 \Rightarrow \phi_2 < \frac{1}{4}$$

if $\phi_1^2 - 4\phi_2 < 0$ (Complex roots) let root are $z = re^{i\theta}$ and z bar

$$|z| = \sqrt{\frac{1}{\phi_2}}$$

$$|z| > 1 \Rightarrow \sqrt{\frac{1}{\phi_2}} > 1 \Rightarrow \frac{1}{|\phi_2|} > 1 \Rightarrow |\phi_2| < 1$$

so conditions are: $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$ and $|\phi_2| < 1$

Q5:

given model

$$X_t = 0.80X_{t-1} - 0.15X_{t-2} + W_t - 0.30W_{t-1}$$

simplifying

$$X_t - 0.80X_{t-1} + 0.15X_{t-2} = W_t - 0.30W_{t-1}$$

$$(1 - 0.80B + 0.15B^2)X_t = (1 - 0.30B)W_t$$

$$(1 - 0.3B)(1 - 0.5B)X_t = (1 - 0.30B)W_t$$

AR polynomial $\phi(z)$:

$$\phi(z) = (1 - 0.3z)(1 - 0.5z)$$

and MA polynomial $\theta(z)$:

$$\theta(z) = (1 - 0.3z)$$

has common root. So there is parameter redundancy.

So the model is:

$$(1 - 0.5B)X_t = W_t$$

$$X_t = 0.5X_{t-1} + W_t$$

is AR(1) model.

Also,i can see root of the AR polynomial $\phi(z)$:

$$\phi(z) = 1 - 0.5z = 0$$

$$z = 2 \quad \text{and} \quad |z| > 1$$

So process is causal.

Invertibility Check: reduced AR(1) model:

$$X_t = 0.50X_{t-1} + W_t$$

W_t in terms of past X_t values, (MA) representation:

$$W_t = X_t - 0.50X_{t-1}$$

$$W_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$

For AR(1) model:

$$\pi_0 = 1, \quad \pi_1 = -0.5, \quad \text{and} \quad \pi_j = 0 \text{ for } j \geq 2$$

So, W_t is:

$$W_t = X_t - 0.5X_{t-1}$$

So it is invertible also.

Q 4:

models:

1. $X_t = 0.25X_{t-2} + W_t$
2. $X_t = -0.9X_{t-2} + W_t$

Root Calculation

Model 1:

$$X_t = 0.25X_{t-2} + W_t$$

Characteristic polynomial is

$$1 - 0.25z^2 = 0$$
$$0.25z^2 = 1$$
$$z^2 = 4$$
$$z = \pm 2$$

Roots: $z_1 = 2$, $z_2 = -2$ are outside unit circle.

Model 2:

$$X_t = -0.9X_{t-2} + W_t$$

Characteristic polynomial is:

$$1 + 0.9z^2 = 0$$

$$0.9z^2 = -1$$
$$z^2 = -\frac{1}{0.9}$$
$$z = \pm i\frac{\sqrt{10}}{3}$$

roots are complex: $z_1 = i\frac{\sqrt{10}}{3}$, $z_2 = -i\frac{\sqrt{10}}{3}$

$|z| = \frac{\sqrt{10}}{3}$ for both roots, so outside unit circle.

ACF

Model 1

$$\rho(h) = c_1 r_1^h + c_2 r_2^h, \quad r_1 = 2, r_2 = -2$$
$$\rho(0) = c_1 + c_2$$
$$\rho(1) = 2c_1 - 2c_2$$

$$\rho(h) = c_1(2)^h + c_2(-2)^h$$

Model 2

$$\rho(h) = Ae^{-\alpha h}\cos(\omega h)$$
$$\alpha = \sin^{-1}\left(\frac{\sqrt{10}}{3}\right), \quad \omega = \cos^{-1}(0.9)$$
$$\rho(0) = A$$

$$\rho(h) = A\cos\left(h\cos^{-1}(0.9)\right)e^{-h\sin^{-1}\left(\frac{\sqrt{10}}{3}\right)}$$