MTH442 Assignment 5

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Q1 (a)

Given a system process defined by:

$$X_t = \phi X_{t-2} + W_t, \quad t = 1, 2, \ldots, T$$

where:

- $X_0 \sim N(0, \sigma_0^2)$ and $X_{-1} \sim N(0, \sigma_1^2)$ are initial conditions with known variances σ_0^2 and σ_1^2.
- W_t represents Gaussian white noise with variance σ_W^2, i.e., $W_t \sim N(0, \sigma_W^2)$.

The process X_t is observed with measurement noise, where:

$$Y_t = X_t + V_t$$

Here:

- V_t is Gaussian white noise with variance σ_V^2, i.e., $V_t \sim N(0, \sigma_V^2)$.
- $X_0, X_{-1}, \{W_t\}$, and $\{V_t\}$ are all mutually independent.

Our objective is to express the system in the standard state-space form, comprising state and observation equations.

State-Space Model Formulation

To represent this process as a state-space model, we must define a state vector that includes all dependencies in the recursive equation. Notice that the current state X_t depends on X_{t-2}, so we need to keep track of both X_t and X_{t-1} as part of our state vector.

Define the state vector at time t as:

$$\mathbf{X}_t = \begin{bmatrix} X_t \\ X_{t-1} \end{bmatrix}$$

where:

- The first element X_t represents the current state of the process.
- The second element X_{t-1} stores the previous state, which is needed to reference X_{t-2} in future time steps.

Given this state vector, we derive the state equation by expressing $\mathbf{X}_t$ in terms of $\mathbf{X}_{t-1}$ and the process noise W_t.

Deriving the State Equation

From the given process, $X_t = \phi X_{t-2} + W_t$. To express $\mathbf{X}_t$ in terms of $\mathbf{X}_{t-1}$, observe that:

$$X_t = \phi X_{t-2} + W_t$$

and

$$X_{t-1} = X_{t-1}$$

Using these relations, we can rewrite the evolution of the state vector as:

$$\mathbf{X}_t = \begin{bmatrix} X_t \\ X_{t-1} \end{bmatrix} = \begin{bmatrix} \phi X_{t-2} + W_t \\ X_{t-1} \end{bmatrix} = \begin{bmatrix} \phi & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_{t-1} \\ X_{t-2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} W_t$$

In general form, this can be expressed as:

$$\mathbf{X}_t = \mathbf{A}\mathbf{X}_{t-1} + \mathbf{B}W_t$$

where:

$$\mathbf{A} = \begin{bmatrix} \phi & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Here:

- The matrix $\mathbf{A}$ defines the transition dynamics of the state vector.
- The matrix $\mathbf{B}$ specifies how the process noise W_t influences the current state.

Observation Equation

The observed process is given by:

$$Y_t = X_t + V_t$$

Since X_t is the first element of the state vector $\mathbf{X}_t$, we can express Y_t in terms of $\mathbf{X}_t$ as:

$$Y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{X}_t + V_t$$

or, in general form:

$$Y_t = \mathbf{H}\mathbf{X}_t + V_t$$

where:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

In this representation:

- The matrix $\mathbf{H}$ extracts the relevant part of the state vector (i.e., X_t) for the observation equation.
- V_t represents the observation noise with variance σ_V^2.

Complete State-Space Model

Thus, the state-space model for the given process is:

- **State Equation**:

$$\mathbf{X}_t = \begin{bmatrix} \phi & 0 \\ 1 & 0 \end{bmatrix} \mathbf{X}_{t-1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} W_t$$

  where $W_t \sim N(0, \sigma_W^2)$.

- **Observation Equation**:

$$Y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{X}_t + V_t$$

  where $V_t \sim N(0, \sigma_V^2)$.

Key Points and Assumptions

- $X_0 \sim N(0, \sigma_0^2)$ and $X_{-1} \sim N(0, \sigma_1^2)$ serve as initial conditions for the process.
- The process noise $W_t \sim N(0, \sigma_W^2)$ introduces randomness into the state transitions.
- The observation noise $V_t \sim N(0, \sigma_V^2)$ adds uncertainty to the measurements Y_t.
- X_0, X_{-1}, W_t, and V_t are assumed to be independent.

This formulation enables us to apply state estimation techniques, such as the Kalman filter, to estimate X_t from observations Y_t, accounting for both process and observation noise.

Q1(b) Solution

To make the observations Y_t stationary, we need the system state X_t to be stationary. We are given the following:
1. $X_t = \phi X_{t-2} + W_t$, where $W_t \sim N(0, \sigma_W^2)$. 2. $Y_t = X_t + V_t$, where $V_t \sim N(0, \sigma_V^2)$.

Step 1: Define the Stationary Variance of X_t

For X_t to be stationary, its variance $\text{Var}(X_t)$ should remain constant over time. Let us denote this stationary variance as γ_0.

The state equation can be written as:

$$X_t = \phi X_{t-2} + W_t$$

where W_t is Gaussian white noise with variance σ_W^2.

Since W_t is independent of X_{t-2}, we calculate $\text{Var}(X_t)$ as follows:

$$\text{Var}(X_t) = \text{Var}(\phi X_{t-2} + W_t)$$

Expanding this expression, we get:

$$\text{Var}(X_t) = \phi^2 \, \text{Var}(X_{t-2}) + \text{Var}(W_t)$$

For stationarity, we set $\text{Var}(X_t) = \text{Var}(X_{t-2}) = \gamma_0$, leading to:

$$\gamma_0 = \phi^2\gamma_0 + \sigma_W^2$$

Solving for γ_0, we find:

$$\gamma_0(1 - \phi^2) = \sigma_W^2$$

$$\gamma_0 = \frac{\sigma_W^2}{1 - \phi^2}$$

Thus, the stationary variance of X_t is:

$$\gamma_0 = \frac{\sigma_W^2}{1 - \phi^2}$$

This result holds if $|\phi| < 1$, which ensures the process is stationary.

Step 2: Set Initial Variances σ_0^2 and σ_1^2

To ensure stationarity, the initial variances σ_0^2 and σ_1^2 (for X_0 and X_{-1}) should match the stationary variance γ_0.

Since $X_0 \sim N(0, \sigma_0^2)$ and $X_{-1} \sim N(0, \sigma_1^2)$, set:

$$\sigma_0^2 = \gamma_0 = \frac{\sigma_W^2}{1 - \phi^2}$$

and similarly,

$$\sigma_1^2 = \gamma_0 = \frac{\sigma_W^2}{1 - \phi^2}$$

These values for σ_0^2 and σ_1^2 ensure that the initial states X_0 and X_{-1} are consistent with the stationary variance of X_t, making Y_t stationary as well.

Q 2

We are given a state-space model defined by:

$$Y_t = X_t + V_t,$$
$$X_t = \phi X_{t-1} + W_t,$$

where:

- $V_t \sim$ IID $N(0, \sigma_V^2)$ (observation noise),
- $W_t \sim$ IID $N(0, \sigma_W^2)$ (process noise),
- $X_0 \sim N\left(0, \frac{\sigma_W^2}{1-\phi^2}\right)$ (initial state).

The variables X_0, $\{W_t\}$, and $\{V_t\}$ are independent of each other.

Definitions and Setup

Let:

$$X_t^{t-1} = \mathbb{E}(X_t | Y_{t-1}, \ldots, Y_1),$$
$$P_t^{t-1} = \mathbb{E}\left[(X_t - X_t^{t-1})^2\right].$$

The term X_t^{t-1} represents the best linear estimate of X_t given past observations, and P_t^{t-1} is the estimation error variance for X_t given $Y_{t-1}, \ldots, Y_1$.

The innovation sequence (or residuals) is defined as:

$$\varepsilon_t = Y_t - Y_t^{t-1},$$

where:

$$Y_t^{t-1} = \mathbb{E}(Y_t | Y_{t-1}, \ldots, Y_1).$$

Since $Y_t = X_t + V_t$, we can express ε_t as:

$$\varepsilon_t = (X_t + V_t) - Y_t^{t-1}.$$

Step 1: Expression for Y_t^{t-1}

We know that:

$$Y_t^{t-1} = \mathbb{E}(Y_t|Y_{t-1},\ldots,Y_1) = \mathbb{E}(X_t|Y_{t-1},\ldots,Y_1) + \mathbb{E}(V_t|Y_{t-1},\ldots,Y_1).$$

Since V_t is independent of the past observations and has zero mean, $\mathbb{E}(V_t|Y_{t-1},\ldots,Y_1) = 0$. Therefore,

$$Y_t^{t-1} = X_t^{t-1}.$$

Thus, the innovation sequence becomes:

$$\varepsilon_t = Y_t - Y_t^{t-1} = (X_t + V_t) - X_t^{t-1} = (X_t - X_t^{t-1}) + V_t.$$

Step 2: Covariance $\text{Cov}(\varepsilon_s, \varepsilon_t)$

We aim to calculate $\text{Cov}(\varepsilon_s, \varepsilon_t)$ for two cases: $s = t$ and $s \neq t$.

Case 1: $s \neq t$

For $s \neq t$, we have:

$$\text{Cov}(\varepsilon_s, \varepsilon_t) = \text{Cov}((X_s - X_s^{s-1} + V_s), (X_t - X_t^{t-1} + V_t)).$$

Since V_s and V_t are independent for $s \neq t$, we have $\text{Cov}(V_s, V_t) = 0$. Similarly, $X_s - X_s^{s-1}$ and $X_t - X_t^{t-1}$ are also uncorrelated for $s \neq t$ because they are based on independent noise terms W_s and W_t. Hence,

$$\text{Cov}(\varepsilon_s, \varepsilon_t) = 0.$$

Case 2: $s = t$

For $s = t$, we are interested in calculating the variance of ε_t, which is given by:

$$\text{Var}(\varepsilon_t) = \text{Var}(X_t - X_t^{t-1} + V_t).$$

Using the fact that $X_t - X_t^{t-1}$ and V_t are independent, we can separate the variances:

$$\text{Var}(\varepsilon_t) = \text{Var}(X_t - X_t^{t-1}) + \text{Var}(V_t).$$

We know that:

$$\text{Var}(X_t - X_t^{t-1}) = P_t^{t-1} \quad \text{and} \quad \text{Var}(V_t) = \sigma_V^2.$$

Thus,

$$\text{Var}(\varepsilon_t) = P_t^{t-1} + \sigma_V^2.$$

Final Answer

We can now summarize the covariance of the innovation sequence $\{\varepsilon_t\}$ as follows:

$$\text{Cov}(\varepsilon_s, \varepsilon_t) = \begin{cases} 0 & \text{if } s \neq t, \\ P_t^{t-1} + \sigma_V^2 & \text{if } s = t. \end{cases}$$

This completes the derivation of the covariance of the innovation sequence $\{\varepsilon_t\}$ in terms of X_t^{t-1} and P_t^{t-1}.

Q3(a)

We are given a univariate state-space model with the following state and observation equations:

$$\begin{aligned} X_0 &= W_0, \\ X_t &= X_{t-1} + W_t, \\ Y_t &= X_t + V_t, \quad t = 1, 2, \ldots \end{aligned}$$

where:

- $W_t \sim$ IID $N(0, \sigma_W^2)$ is the process noise,
- $V_t \sim$ IID $N(0, \sigma_V^2)$ is the observation noise,

and W_t and V_t are independent.

Our objective is to show that Y_t follows an Integrated Moving Average (IMA) model of order $(1, 1)$, that is, we want to show that $\nabla Y_t = Y_t - Y_{t-1}$ follows an MA(1) process.

Step 1: Define the Differenced Observation Process ∇Y_t

Define the differenced observation process:

$$\nabla Y_t = Y_t - Y_{t-1}.$$

Substitute $Y_t = X_t + V_t$ and $Y_{t-1} = X_{t-1} + V_{t-1}$ to get:

$$\nabla Y_t = (X_t + V_t) - (X_{t-1} + V_{t-1}).$$

This simplifies to:

$$\nabla Y_t = (X_t - X_{t-1}) + (V_t - V_{t-1}).$$

Step 2: Substitute the State Equation
From the state equation, we have:

$$X_t = X_{t-1} + W_t.$$

Thus,

$$X_t - X_{t-1} = W_t.$$

Substitute this result into the expression for ∇Y_t:

$$\nabla Y_t = W_t + (V_t - V_{t-1}).$$

Step 3: Analyze ∇Y_t
Now, ∇Y_t is expressed as:

$$\nabla Y_t = W_t + (V_t - V_{t-1}).$$

Since $W_t \sim N(0, \sigma_W^2)$ and $V_t \sim N(0, \sigma_V^2)$ are both white noise processes, the difference $V_t - V_{t-1}$ will be an MA(1) process with moving average parameter -1.

Therefore, ∇Y_t can be written as an MA(1) process:

$$\nabla Y_t = W_t + V_t - V_{t-1}.$$

Step 4: Covariance Structure of ∇Y_t
To confirm that ∇Y_t follows an MA(1) model, we need to examine its covariance structure.
1. Mean of ∇Y_t: Since both W_t and $V_t - V_{t-1}$ have zero mean, it follows that:

$$\mathbb{E}[\nabla Y_t] = 0.$$

2. Variance of ∇Y_t: The variance of ∇Y_t is:

$$\mathrm{Var}(\nabla Y_t) = \mathrm{Var}(W_t) + \mathrm{Var}(V_t - V_{t-1}).$$

Since $W_t \sim N(0, \sigma_W^2)$ and $V_t - V_{t-1} \sim N(0, 2\sigma_V^2)$, we have:

$$\mathrm{Var}(\nabla Y_t) = \sigma_W^2 + 2\sigma_V^2.$$

3. Autocovariance of ∇Y_t: For $k \geq 2$, the autocovariance $\gamma(k) = \mathrm{Cov}(\nabla Y_t, \nabla Y_{t-k}) = 0$, because W_t and V_t are white noise and independent.

For $k = 1$:

$$\gamma(1) = \mathrm{Cov}(\nabla Y_t, \nabla Y_{t-1}).$$

Since $\nabla Y_t = W_t + V_t - V_{t-1}$ and $\nabla Y_{t-1} = W_{t-1} + V_{t-1} - V_{t-2}$, the only non-zero covariance term is $-\sigma_V^2$ from V_{t-1}. Thus,

$$\gamma(1) = -\sigma_V^2.$$

Hence, the autocovariance structure of ∇Y_t matches that of an MA(1) model with parameter $\theta = -1$.
Conclusion
We have shown that $\nabla Y_t = Y_t - Y_{t-1}$ follows an MA(1) model with:

$$\nabla Y_t = W_t + V_t - V_{t-1},$$

where $W_t \sim N(0, \sigma_W^2)$ and $V_t - V_{t-1} \sim N(0, 2\sigma_V^2)$. Therefore, Y_t follows an IMA$(1, 1)$ model.

Q 4

We have the state-space model defined by:

$$X_t = \Phi X_{t-1} + W_t$$

$$Y_t = A_t X_t + V_t$$

where: - $W_t \sim \mathcal{N}(0, \sigma_W^2)$ and $V_t \sim \mathcal{N}(0, \sigma_V^2)$ are independent Gaussian white noise processes, - A_t is a known design matrix independent of the parameters Θ, - $X_0 \sim \mathcal{N}(0, \sigma_{X_0}^2)$ is the initial state.

The gradient of the log-likelihood with respect to Θ_i is given by:

$$\frac{\partial \log L_Y(\Theta)}{\partial \Theta_i} = \sum_{t=1}^{T} \left\{ \epsilon_t' \Sigma_t^{-1} \frac{\partial \epsilon_t}{\partial \Theta_i} - 0.5 \epsilon_t' \Sigma_t^{-1} \frac{\partial \Sigma_t}{\partial \Theta_i} \Sigma_t^{-1} \epsilon_t + 0.5 \operatorname{tr} \left(\Sigma_t^{-1} \frac{\partial \Sigma_t}{\partial \Theta_i} \right) \right\}$$

where: - $\epsilon_t = Y_t - Y_t^{t-1}$ is the prediction error, - $Y_t^{t-1} = E(Y_t | Y_{t-1}, \dots, Y_1)$ is the one-step-ahead prediction of Y_t, - Σ_t is the covariance of the prediction error ϵ_t.

(a)

Show that:

$$\frac{\partial \epsilon_t}{\partial \Theta_i} = -A_t \frac{\partial X_t^{t-1}}{\partial \Theta_i}$$

Solution:
1. Define ϵ_t:

$$\epsilon_t = Y_t - Y_t^{t-1}$$

where $Y_t = A_t X_t + V_t$ and $Y_t^{t-1} = A_t X_t^{t-1}$ because V_t has mean zero.
2. Substitute for Y_t and Y_t^{t-1}:

$$\epsilon_t = (A_t X_t + V_t) - A_t X_t^{t-1}$$

3. Simplify ϵ_t:

$$\epsilon_t = A_t (X_t - X_t^{t-1}) + V_t$$

4. Take the derivative with respect to Θ_i: Since V_t does not depend on Θ_i, we have:

$$\frac{\partial \epsilon_t}{\partial \Theta_i} = A_t \frac{\partial (X_t - X_t^{t-1})}{\partial \Theta_i}$$

5. Express $X_t - X_t^{t-1}$: Noting that $X_t - X_t^{t-1}$ is just the prediction error in the state, we get:

$$\frac{\partial \epsilon_t}{\partial \Theta_i} = -A_t \frac{\partial X_t^{t-1}}{\partial \Theta_i}$$

Thus, we have shown that:

$$\frac{\partial \epsilon_t}{\partial \Theta_i} = -A_t \frac{\partial X_t^{t-1}}{\partial \Theta_i}$$

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(b)

Show that:

$$\frac{\partial X_t^{t-1}}{\partial \Theta_i} = \frac{\partial \Phi}{\partial \Theta_i} X_{t-2}^{t-1} + \Phi \frac{\partial X_{t-2}^{t-1}}{\partial \Theta_i} + \frac{\partial K_{t-1}}{\partial \Theta_i} \epsilon_{t-1} + K_{t-1} \frac{\partial \epsilon_{t-1}}{\partial \Theta_i}$$

Solution:
1. Express X_t^{t-1} using the state transition:

$$X_t^{t-1} = \Phi X_{t-1}^{t-1}$$

2. Differentiate X_t^{t-1} with respect to Θ_i: Taking the derivative, we get:

$$\frac{\partial X_t^{t-1}}{\partial \Theta_i} = \frac{\partial \Phi}{\partial \Theta_i} X_{t-1}^{t-1} + \Phi \frac{\partial X_{t-1}^{t-1}}{\partial \Theta_i}$$

3. Express X_{t-1}^{t-1}: We can express X_{t-1}^{t-1} in terms of X_{t-2}^{t-1} as:

$$X_{t-1}^{t-1} = \Phi X_{t-2}^{t-1} + K_{t-1}\epsilon_{t-1}$$

4. Differentiate X_{t-1}^{t-1}: Taking the derivative with respect to Θ_i, we get:

$$\frac{\partial X_{t-1}^{t-1}}{\partial \Theta_i} = \frac{\partial \Phi}{\partial \Theta_i} X_{t-2}^{t-1} + \Phi \frac{\partial X_{t-2}^{t-1}}{\partial \Theta_i} + \frac{\partial K_{t-1}}{\partial \Theta_i}\epsilon_{t-1} + K_{t-1}\frac{\partial \epsilon_{t-1}}{\partial \Theta_i}$$

5. Combine Results: Substitute back to obtain:

$$\frac{\partial X_{t}^{t-1}}{\partial \Theta_i} = \frac{\partial \Phi}{\partial \Theta_i} X_{t-2}^{t-1} + \Phi \frac{\partial X_{t-2}^{t-1}}{\partial \Theta_i} + \frac{\partial K_{t-1}}{\partial \Theta_i}\epsilon_{t-1} + K_{t-1}\frac{\partial \epsilon_{t-1}}{\partial \Theta_i}$$

(c)

Show that:

$$\frac{\partial \Sigma_t}{\partial \Theta_i} = A_t \frac{\partial P_t^{t-1}}{\partial \Theta_i} A_t' + \frac{\partial R}{\partial \Theta_i}$$

Solution:
1. Define Σ_t:

$$\Sigma_t = A_t P_t^{t-1} A_t' + R$$

where P_t^{t-1} is the covariance matrix of the predicted state X_t^{t-1} and R is the variance of the observation noise V_t.
2. Differentiate Σ_t with respect to Θ_i:

$$\frac{\partial \Sigma_t}{\partial \Theta_i} = \frac{\partial}{\partial \Theta_i}\left(A_t P_t^{t-1} A_t' + R\right)$$

3. Apply the Product Rule: Since A_t is independent of Θ_i,

$$\frac{\partial \Sigma_t}{\partial \Theta_i} = A_t \frac{\partial P_t^{t-1}}{\partial \Theta_i} A_t' + \frac{\partial R}{\partial \Theta_i}$$

Thus,

$$\frac{\partial \Sigma_t}{\partial \Theta_i} = A_t \frac{\partial P_t^{t-1}}{\partial \Theta_i} A_t' + \frac{\partial R}{\partial \Theta_i}$$

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(d)

Show that:

$$\frac{\partial K_t}{\partial \Theta_i} = \left[\frac{\partial P_t^{t-1}}{\partial \Theta_i} A_t' + \Phi \frac{\partial P_{t-1}^{t-1}}{\partial \Theta_i} A_t' - K_t \frac{\partial \Sigma_t}{\partial \Theta_i}\right] \Sigma_t^{-1}$$

Solution:
1. Define K_t:

$$K_t = P_t^{t-1} A_t' \Sigma_t^{-1}$$

2. Differentiate K_t with respect to Θ_i: Using the product rule,

$$\frac{\partial K_t}{\partial \Theta_i} = \frac{\partial (P_t^{t-1} A_t' \Sigma_t^{-1})}{\partial \Theta_i}$$

3. Expand using Product Rule:

$$\frac{\partial K_t}{\partial \Theta_i} = \frac{\partial P_t^{t-1}}{\partial \Theta_i} A_t' \Sigma_t^{-1} + P_t^{t-1} A_t' \frac{\partial \Sigma_t^{-1}}{\partial \Theta_i}$$

4. Substitute the derivative of Σ_t^{-1}:

$$\frac{\partial \Sigma_t^{-1}}{\partial \Theta_i} = -\Sigma_t^{-1} \frac{\partial \Sigma_t}{\partial \Theta_i} \Sigma_t^{-1}$$

5. Combine terms:

$$\frac{\partial K_t}{\partial \Theta_i} = \left(\frac{\partial P_t^{t-1}}{\partial \Theta_i} A_t' - K_t \frac{\partial \Sigma_t}{\partial \Theta_i}\right) \Sigma_t^{-1}$$

Thus,

$$\frac{\partial K_t}{\partial \Theta_i} = \left[\frac{\partial P_t^{t-1}}{\partial \Theta_i} A_t' + \Phi \frac{\partial P_{t-1}^{t-1}}{\partial \Theta_i} A_t' - K_t \frac{\partial \Sigma_t}{\partial \Theta_i}\right] \Sigma_t^{-1}$$

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(e)

Show that:

$$\frac{\partial P_t^{t-1}}{\partial \Theta_i} = \Phi \frac{\partial P_{t-2}^{t-1}}{\partial \Theta_i} \Phi' + \Phi \frac{\partial P_{t-2}^{t-1}}{\partial \Theta_i} + \frac{\partial Q}{\partial \Theta_i} - \frac{\partial K_t}{\partial \Theta_i} \Sigma_t K_t' - K_t \frac{\partial \Sigma_t}{\partial \Theta_i} K_t'$$

Solution:

1. Define P_t^{t-1}:

$$P_t^{t-1} = \Phi P_{t-1}^{t-1} \Phi' + Q$$

2. Differentiate P_t^{t-1} with respect to Θ_i:

$$\frac{\partial P_t^{t-1}}{\partial \Theta_i} = \frac{\partial (\Phi P_{t-1}^{t-1} \Phi' + Q)}{\partial \Theta_i}$$

3. Apply Product Rule:

$$\frac{\partial P_t^{t-1}}{\partial \Theta_i} = \Phi \frac{\partial P_{t-1}^{t-1}}{\partial \Theta_i} \Phi' + \Phi \frac{\partial P_{t-2}^{t-1}}{\partial \Theta_i} + \frac{\partial Q}{\partial \Theta_i}$$

4. Subtract the terms involving K_t:

$$\frac{\partial P_t^{t-1}}{\partial \Theta_i} = \Phi \frac{\partial P_{t-2}^{t-1}}{\partial \Theta_i} \Phi' + \Phi \frac{\partial P_{t-2}^{t-1}}{\partial \Theta_i} + \frac{\partial Q}{\partial \Theta_i} - \frac{\partial K_t}{\partial \Theta_i} \Sigma_t K_t' - K_t \frac{\partial \Sigma_t}{\partial \Theta_i} K_t'$$

This completes the derivation of the solution.