Lecture 12

Autoregressive Models Part 1

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Motivation

- The classical regression model only allows the dependent variable to be influenced by current values of the independent variables.
- In the time series case, allowing the response to be influenced by the past values of the independent variables and possibly by its own past values is desirable.
- If the present can be plausibly modeled in terms of only the past values of the independent inputs, forecasting will be possible.
- Autoregressive models are based on the idea that X_t can be explained as a function of p past values, $X_{t-1}, X_{t-2}, \dots, X_{t-p}$ for some p.
- As an example, we have seen the model $X_t = X_{t-1} - 0.9X_{t-2} + W_t$ where W_t is white Gaussian noise with $\sigma_W^2 = 1$.
- Here the forecast can be done as $\hat{X}_{t+1} = X_t - 0.9X_{t-1}$

Definition

- An autoregressive model of order p, abbreviated $AR(p)$, is of the form

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + W_t.$$

- Here X_t is stationary, $W_t \sim WN(0, \sigma_W^2)$, and $\phi_1, \phi_2, \ldots, \phi_p$ are constants with $\phi_p \neq 0$.
- Here we assumed the mean of X_t to be zero. Otherwise, we need to replace all X_s terms by $X_s - \mu$, i.e.,

$$X_t - \mu = \phi_1(X_{t-1} - \mu) + \phi_2(X_{t-2} - \mu) + \ldots + \phi_p(X_{t-p} - \mu) + W_t.$$

- Equivalently, for $\alpha = (1 - \phi_1 - \ldots - \phi_p)\mu$, we can write

$$X_t = \alpha + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + W_t.$$

Autoregressive operator

- We can rewrite the model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + W_t$$

  by

$$X_t = \phi_1 B X_t + \phi_2 B^2 X_t + \ldots + \phi_p B^p X_t + W_t,$$

  where B is the backshift operator, and hence,

$$\phi(B) X_t = (1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p) X_t = W_t.$$

- Here $\phi(B)$ is called the autoregressive operator, where

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p.$$

- For a AR(1) process $X_t = \phi X_{t-1} + W_t$, we have $\phi(B) = 1 - \phi B$

Autoregressive process as a linear process

- We can represent a AR(1) process $X_t = \phi X_{t-1} + W_t$ as a linear process

$$X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}.$$

- We can write iteratively as

$$\begin{aligned} X_t &= \phi X_{t-1} + W_t \\ &= \phi(\phi X_{t-2} + W_{t-1}) + W_t = \phi^2 X_{t-2} + (\phi W_{t-1} + W_t) \\ &\dots \\ &= \phi^k X_{t-k} + \sum_{j=0}^{k-1} \phi^j W_{t-j} \end{aligned}$$

- Here $E[(\phi^k X_{t-k})^2] = \phi^{2k}\mathrm{Var}(X_{t-k}) \to 0$ with $k \uparrow \infty$ as $\mathrm{Var}(X_{t-k})$ is assumed to be finite and $|\phi| < 1$.
- The condition of the linear process is satisfied: $\sum_{j=0}^{\infty} |\phi|^j = (1 - |\phi|^2)^{-1} < \infty$.

Mean and covariance function

- We can write iteratively as $X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}$.
- $E(X_t) = E\left(\sum_{j=0}^{\infty} \phi^j W_{t-j}\right) = \sum_{j=0}^{\infty} \phi^j E(W_{t-j}) = 0$.
- $$\begin{aligned}
\gamma(h) &= \mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}\left(\sum_{j=0}^{\infty} \phi^j W_{t+h-j}, \sum_{j=0}^{\infty} \phi^j W_{t-j}\right) \\
&= \mathrm{Cov}\left(\sum_{j'=-h}^{\infty} \phi^{j'+h} W_{t-j'}, \sum_{j=0}^{\infty} \phi^j W_{t-j}\right) \quad \text{(replacing } j' = j - h\text{)} \\
&= \mathrm{Cov}\left(\sum_{j'=0}^{\infty} \phi^{j'+h} W_{t-j'}, \sum_{j=0}^{\infty} \phi^j W_{t-j}\right) \quad (W_{t+1}, W_{t+2}, \ldots \text{indep. of } W_t, W_{t-1}, \ldots) \\
&= \phi^h \sum_{j=0}^{\infty} \phi^{2j} \mathrm{Var}(W_t) = \phi^h \sigma_W^2 (1 - \phi^2)^{-1}
\end{aligned}$$

Thank you!