MTH 517: Time Series Analysis
Problem Set # 1

[1] Consider the following decomposition of the time series $\{Y_t\}$;

$$Y_t = m_t + \varepsilon_t,$$

where, $\{\varepsilon_t\}$ is a sequence of i.i.d. $\left(0,\sigma^2\right)$ sequence of random variables. Compute the mean and variance of the processes $\nabla^2 Y_t$ and $\nabla_2 Y_t$ for each of the following cases (a) $m_t = a + bt$; (b) $m_t = a + bt + ct^2$.

[2] Suppose that the time series $\{Y_t\}$ has the decomposition;

$$Y_t = m_t + s_t + \varepsilon_t,$$

where, $\{\varepsilon_t\}$ is a sequence of i.i.d. $N\left(0,\sigma^2\right)$ process, the trend component m_t is $m_t = a + bt$ and s_t is the seasonal component with period 4.

(a) Prove or disprove the following statements:

(i) ∇_4 applied on Y_t eliminates trend from Y_t.

(ii) $\nabla_4 Y_t$ does not have a seasonal component.

(iii) $\nabla_4\, Y_t$ dampens the noise present in Y_t.

(b) Find the distributions of **(i)** ∇Y_t, **(ii)** $\nabla_4 Y_t$ and **(iii)** $\nabla\nabla_4 Y_t$.

[3] Consider equally weighted moving-average filter with a window length of $(2q+1)$.

(i) If $m_t = c_0 + c_1 t$, show that $\sum_{j=-q}^{q} a_j m_{t-j} = m_t$. [The result shows that a linear trend is passed undistorted through the MA filter]

(ii) If $Z_t, t = 0, \pm 1, \ldots$, are independent random variables with mean 0 and variance σ^2, show that the moving averages $A_t = \sum_{j=-q}^{q} a_j Z_{t-j}$ is a time series process with $E\left(A_t\right) = 0$ and $V(A_t) = \sigma^2 / (2q+1)$.

[4] Consider the time series decomposition $Y_t = m_t + s_t + \varepsilon_t$; where, m_t is a polynomial trend of order 2; S_t a seasonal component of periodicity 3 and $\{\varepsilon_t\}$ is a sequence of i.i.d. $\left(0,\sigma^2\right)$ process. Apply an equally weighted two-sided 3-point moving average filter with coefficients $\left[a_{-1}, a_0, a_1\right] = \frac{1}{3}[1,1,1]$. Show that the filter eliminates the seasonal component and passes the trend with a distortion.

[5] The Spencer 15-point linear filter $\left\{a_j\right\}$ is given by

$$\left[a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7\right] = \frac{1}{320}\left[74, 67, 46, 21, 3, -5, -6, -3\right]$$
$$a_i = 0, \qquad |i| > 7$$
$$a_i = a_{-i}, \qquad |i| \le 7.$$

Show that the above filter passes an arbitrary trend component $m_t = a + bt$ without distortion.

[6] Show that a moving average filter with coefficients

$$[a_{-2}, a_{-1}, a_0, a_1, a_2] = \frac{1}{9}[-1, 4, 3, 4, -1]$$

passes a degree one polynomial trend without distortion and eliminates seasonal components with period 3.

[7] Suppose that $m_t = a + bt + ct^2, t = 0, \pm 1, \ldots$

(a) Show that $m_t = \sum_{j=-2}^{2} a_j\, m_{t+j} = \sum_{j=-3}^{3} b_j\, m_{t+j}; t = 0, \pm 1, \pm 2, \ldots$.. where,

$$[a_{-2}, a_{-1}, a_0, a_1, a_2] = \frac{1}{35}[-3, 12, 17, 12, -3]$$

and $[b_{-3}, b_{-2}, b_{-1}, b_0, b_1, b_2, b_3] = \frac{1}{21}[-2, 3, 6, 7, 6, 3, -2]$.

(b) Suppose that $Y_t = m_t + \varepsilon_t$, where $\{\varepsilon_t\}$ is an independent sequence of normal random variables, each with mean 0 and variance σ^2. Define $U_t = \sum_{j=-2}^{2} a_j\, Y_{t+j}$ and $V_t = \sum_{j=-3}^{3} b_j\, Y_{t+j}$.

(i) Find the covariances (U_t, U_{t+1}), (V_t, V_{t+1}), (U_t, V_t) and (U_{t+1}, V_t).

(ii) Which of the two filtered series $\{U_t\}$and $\{V_t\}$ would you expect to produce a smoother series?

[8] Consider a time series given by $Y_t = (a_1 + a_2 t)S_t + X_t$, where S_t is a seasonal component of periodicity 4 and X_t is a stochastic component with $E(X_t) = 0$; $\text{cov}(X_t, X_s) = 0$, if $t \neq s$ and $= \sigma^2$ if $t = s$. Apply appropriate order of differencing to eliminate time trend and seasonality from the given series $\{Y_t\}$.

[9] Let $\{X_t\}$ be a time series given by $X_t = \alpha + \beta t + \gamma t^2 + S_t + Y_t$; where, α, β, γ are constants, S_t is a seasonal component with period 12 and $\{Y_t\}$ is a sequence of uncorrelated random variables with mean μ and variance θ. Apply appropriate lag difference operator (s) to reduce $\{X_t\}$ to a process which does not have trend and seasonal component.

[10] $\{Y_t\}$ is a time series such that $Y_t = m_t + S_t + \epsilon_t$; $t=1,\ldots,n$. $m_t=\alpha+\beta t$ is the linear time trend component, S_t is the seasonal component having period 3 and $\{\epsilon_t\}$ is a sequence of i.i.d. $N(0,1)$ random variables. $\{a_k\}$ is a 3-point linear filter such that $(a_{-1}, a_0, a_1) = \frac{1}{3}(-1,5,-1)$ and $a_k = 0$ for all $|k|>1$. Let $Q_t = \sum_{k=-1}^{1} a_k\, Y_{t+k}$, $2 \leq t \leq n-1$ denote the output series obtained when $\{Y_t\}$ passes through the filter $\{a_k\}$. Prove or disprove the following statements:

(a) The filter $\{a_k\}$ eliminates seasonality component in $\{Y_t\}$, i.e. $\{Q_t\}$ does not have any seasonal component.

(b) The filter $\{a_k\}$ passes the trend line m_t, present in Y_t, without any distortion.

(c) $V(Q_t) > V(Y_t)\ \forall t$.

(d) $\{\Delta_6 Q_t\}$ is free from trend.