Indian Institute of Technology Kanpur

Department of Mathematics and Statistics

Time Series Analysis (MTH442)

Assignment 5, Due date: November 14, 2024, Thursday

Answers should be provided neatly. In case the handwriting is unreadable, the instructor and the teaching assistants hold the right to give a zero score. In case of cheating, all students involved will get zero, irrespective of who copied from whom. Write on a paper using some dark ink and scan using a good scanner to obtain a clearly readable PDF file. For the coding related question, use R markdown and generate the PDF output with echo = TRUE mode (so that the codes are also visible along with the outputs). Finally, join the two PDFs and submit the single PDF file only. The final file should be named as RollNo_Lastname_Firstname.pdf. If the file is not submitted in this nomenclature format, marks will be zero. No request should be made in that case.

1. Consider a system process given by $X_t = \phi X_{t-2} + W_t,\ t = 1, \ldots, T$ where $X_0 \sim \mathrm{N}(0, \sigma_0^2)$, $X_{-1} \sim \mathrm{N}(0, \sigma_1^2)$, and W_t is Gaussian white noise with variance σ_W^2. The system process is observed with noise, say, $Y_t = X_t + V_t$, where V_t is Gaussian white noise with variance σ_V^2. Further, suppose $X_0, X_{-1}, \{W_t\}$, and $\{V_t\}$ are independent.
(a) Write the state and observation equations in the form of a state space model.
(b) Find the values of σ_0^2 and σ_1^2 that make the observations, Y_t, stationary (as a function of ϕ). (1+1=2 points)

2. Consider the state-space model $Y_t = X_t + V_t$ and $X_t = \phi X_{t-1} + W_t$, where $V_t \overset{IID}{\sim} \mathrm{N}(0, \sigma_V^2)$, $W_t \overset{IID}{\sim} \mathrm{N}(0, \sigma_W^2)$, and $X_0 \sim \mathrm{N}(0, \sigma_W^2(1-\phi^2)^{-1})$; $X_0, \{W_t\}$, and $\{V_t\}$ are independent and $t = 1, 2, \ldots$. Let $X_t^{t-1} = E(X_t | Y_{t-1}, \ldots, Y_1)$ and let $P_t^{t-1} = E[(X_t - X_t^{t-1})^2]$. The innovation sequence or residuals are $\varepsilon_t = Y_t - Y_t^{t-1}$, where $Y_t^{t-1} = E(Y_t | Y_{t-1}, \ldots, Y_1)$. Find $\mathrm{Cov}(\varepsilon_s, \varepsilon_t)$ in terms of X_t^{t-1} and P_t^{t-1} for (i) $s \neq t$ and (ii) $s = t$. (1+1=2 points)

3. Consider the univariate state-space model given by state conditions $X_0 = W_0$, $X_t = X_{t-1} + W_t$ and observations $Y_t = X_t + V_t,\ t = 1, 2, \ldots$, where W_t and V_t are independent, Gaussian, white noise processes with $\mathrm{Var}(W_t) = \sigma_W^2$ and $\mathrm{Var}(V_t) = \sigma_V^2$. (a) Show that Y_t follows an IMA(1,1) model, that is, ∇Y_t follows an MA(1) model. (b) Fit the model specified in part (a) to the logarithm of the glacial varve series, available as `varve` from the `astsa` package, and summarize the findings. (1+2 = 3 points)

4. We discussed the maximum likelihood estimation for state-space models, where it is possible to obtain a recursion for the gradient vector, $-\partial \log L_Y(\mathbf{\Theta})/\partial \mathbf{\Theta}$. Assume the model is given by $X_t = \Phi X_{t-1} + W_t$ and $Y_t = A_t X_t + V_t$ and A_t is a known design matrix that does not depend on $\mathbf{\Theta}$. We use the notation $\partial_i g = \partial g(\Theta)/\partial \Theta_i$. For the gradient vector,
$$\partial \log L_Y(\mathbf{\Theta})/\partial \mathbf{\Theta}_i = \sum_{t=1}^{T} \left\{ \epsilon_t' \Sigma_t^{-1} \frac{\partial \epsilon_t}{\partial \Theta_i} - -0.5 \epsilon_t' \Sigma_t^{-1} \frac{\partial \Sigma_t}{\partial \Theta_i} \Sigma_t^{-1} \epsilon_t + 0.5 \mathrm{tr} \left(\Sigma_t^{-1} \frac{\partial \Sigma_t}{\partial \Theta_i} \right) \right\}$$
show the following
(a) $\partial_i \epsilon_t = -A_t \partial_i X_t^{t-1}$
(b) $\partial_i X_t^{t-1} = \partial_i \Phi X_{t-1}^{t-2} + \Phi \partial_i X_{t-1}^{t-2} + \partial_i K_{t-1} \epsilon_{t-1} + K_{t-1} \partial_i \epsilon_{t-1}$
(c) $\partial_i \Sigma_t = A_t \partial_i P_t^{t-1} A_t' + \partial_i R$
(d) $\partial_i K_t = [\partial_i \Phi P_t^{t-1} A_t' + \Phi \partial_i P_t^{t-1} A_t' - K_t \partial_i \Sigma_t] \Sigma_t^{-1}$
(e) $\partial_i P_t^{t-1} = \partial_i \Phi P_{t-1}^{t-2} \Phi' + \Phi \partial_i P_{t-1}^{t-2} \Phi' + \Phi P_{t-1}^{t-2} \partial_i \Phi' + \partial_i Q - \partial_i K_{t-1} \Sigma_t K_{t-1}' - K_{t-1} \partial_i \Sigma_t K_{t-1}' - K_{t-1} \Sigma_t \partial_i K_{t-1}'$,
using the fact that $P_t^{t-1} = \Phi P_{t-1}^{t-2} \Phi' + Q - K_{t-1} \Sigma_t K_{t-1}'$. (0.5+0.5+0.5+0.5+1= 3 points)