Lecture 19

Partial autocorrelation

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ACF of an MA(q) process (Recap)

- Consider the ACF of an MA(q) process, $X_t = \theta(B)W_t$, where

$$\theta(B) = 1 + \theta_1 B + \ldots + \theta_q B^q.$$

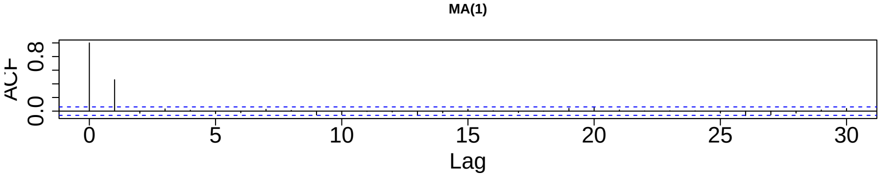
- If $1 \le h \le q$,

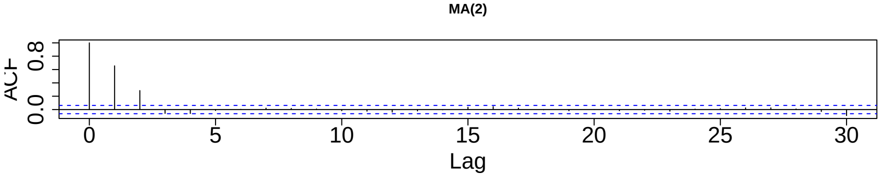
$$\rho(h) = \frac{\sum_{j=0}^{q-h} \theta_j \theta_{j+h}}{1 + \theta_1^2 + \ldots + \theta_q^2}.$$

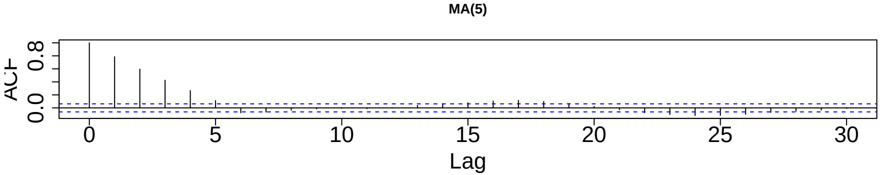
- If $h > q$, $\rho(h) = 0$.

- Hence, if we look at the ACF plot and the ACF becomes insignificant after lag q, there is an indication that the process might be an MA process and the last lag with a significant value indicates the order.

ACF of MA process







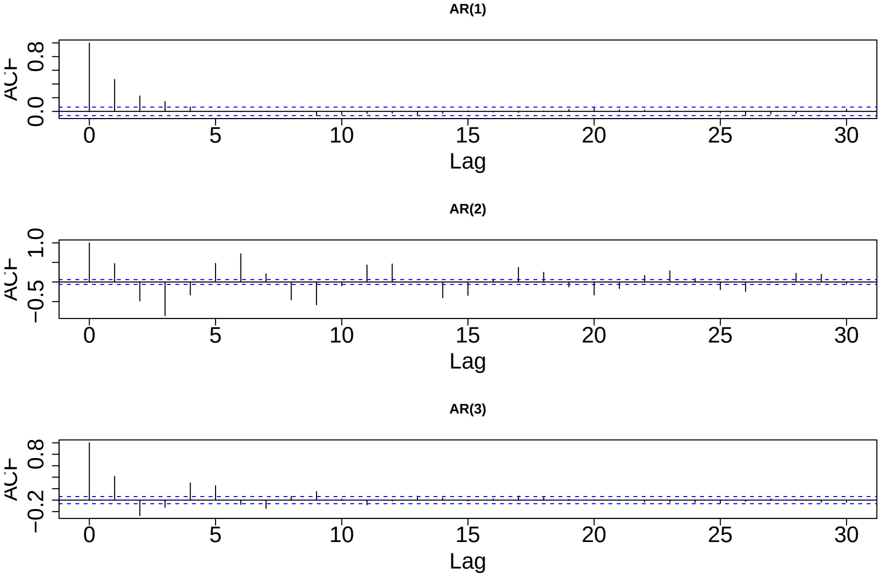
ACF of an AR(p) process (Recap)

- Suppose $X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + W_t$ is a causal AR(p) process.
- The general solution for ACF $\rho(h)$ is

$$\rho(h) = z_1^{-h} P_1(h) + z_2^{-h} P_2(h) + \ldots + z_r^{-h} P_r(h), \quad h = p, p+1, \ldots$$

  where $P_j(h)$, for $j = 1, 2, \ldots, r$, is a polynomial in h, of degree $m_j - 1$.
- Depending on the behaviors of the polynomials, the actual values for small h are different but when $h \uparrow \infty$, the z_i^{-h} terms are dominating and they decay to zero as $|z_i| > 1$ for all i follows from causality.
- Overall, ACF decay to zero as $h \uparrow \infty$ but does not cutoff after a lag.

ACF of AR process



ACF of an ARMA(p, q) process (Recap)

- A causal ARMA(p, q) model $\{X_t; t = 0, \pm 1, \pm 2, \ldots\}$ can be written as a one-sided linear process $X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j} = \psi(B) W_t$.

- Here, we have $\gamma(h) - \sum_{j=1}^{p} \phi_j \gamma(h-j) = 0, \quad h \geq \max\{p, q+1\}$, with initial conditions

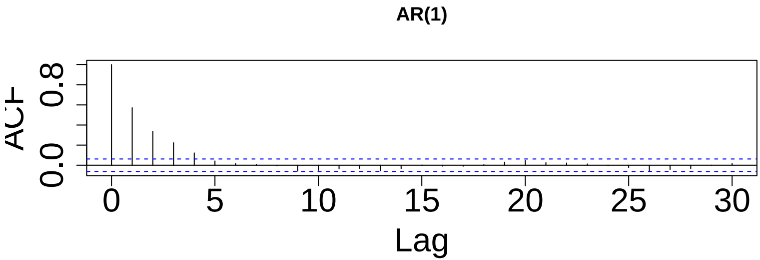
$$\gamma(h) - \sum_{j=1}^{p} \phi_j \gamma(h-j) - \sigma_w^2 \sum_{j=h}^{q} \theta_j \psi_{j-h} = 0, \quad 0 \leq h < \max\{p, q+1\}.$$

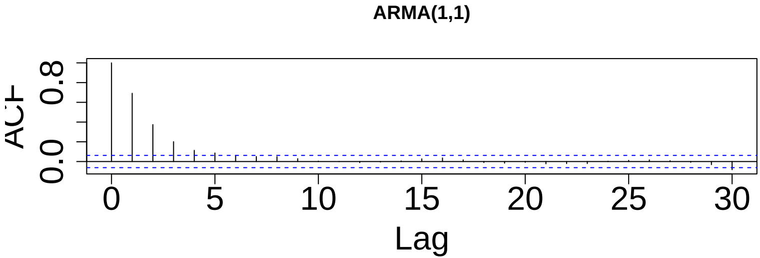
- Finally by dividing $\gamma(h)$ by $\gamma(0)$, we obtain $\rho(h)$.

Example: ACF of an ARMA$(1, 1)$ process (Recap)

- Consider the model: $X_t = \phi X_{t-1} + \theta W_{t-1} + W_t$, where $|\phi| < 1$.
- We have $\gamma(h) - \phi\gamma(h-1) = 0,\ h \geq 2$, which implies $\gamma(h) = \gamma(1)\phi^{h-1}, h \geq 2$.
- Initial conditions can be solved as: $\gamma(0) = \phi\gamma(1) + \sigma_W^2[1 + \theta\phi + \theta^2]$ and $\gamma(1) = \phi\gamma(0) + \sigma_W^2\theta$.
- The final solution is $\rho(h) = \dfrac{(1+\theta\phi)(\phi+\theta)}{1+2\theta\phi+\theta^2}\phi^{h-1},\quad h \geq 1$.
- Thus, the dominating terms in $\rho(h)$ for AR(1) and ARMA(1,1) are the exponentially decaying terms ϕ^h.
- As a result, it is not possible to distinguish between them based on ACF.

ACFs of AR and ARMA processes





Correlation versus partial correlation

- If X, Y, and Z are random variables, then the partial correlation between X and Y given Z is obtained by regressing X on Z to obtain $\hat{X}$, regressing Y on Z to obtain $\hat{Y}$, and then calculating

$$\rho_{XY|Z} = \mathrm{corr}\{X - \hat{X}, Y - \hat{Y}\}.$$

- If the variables are multivariate normal, then this definition coincides with $\rho_{XY|Z} = \mathrm{corr}(X, Y|Z)$.
- For an MA(1) model, X_{t+2} and X_t are uncorrelated. However, for an AR(1) model, they are correlated.
- Here, X_{t+2} and X_t are correlated through X_{t+1}. Thus, our goal is to break the dependence by conditioning on X_{t+1}.
- Show that $\mathrm{Corr}\{X_{t+2} - \phi X_{t+1}, X_t - \phi X_{t+1}\} = 0$.

PACF

- The PACF of a stationary process, X_t, denoted $\phi_{h,h}$, for $h = 1, 2, \ldots$, is

$$\phi_{1,1} = \mathrm{corr}(X_{t+1}, X_t) = \rho(1)$$

  and

$$\phi_{h,h} = \mathrm{corr}(X_{t+h} - \hat{X}_{t+h}, X_t - \hat{X}_t), \quad h \geq 2.$$

- $\hat{X}_{t+h}, \quad h \geq 2$, denote the regression of X_{t+h} on $\{X_{t+h-1}, X_{t+h-2}, \ldots, X_{t+1}\}$, and we write

$$\hat{X}_{t+h} = \beta_1 X_{t+h-1} + \beta_2 X_{t+h-2} + \ldots + \beta_{h-1} X_{t+1}.$$

- Let $\hat{X}_t$ denote the regression of X_t on $\{X_{t+1}, X_{t+2}, \ldots, X_{t+h-1}\}$, then

$$\hat{X}_t = \beta_1 X_{t+1} + \beta_2 X_{t+2} + \ldots + \beta_{h-1} X_{t+h-1}.$$

- The coefficients, $\beta_1, \ldots, \beta_{h-1}$ are the same due to stationarity.

The PACF of AR(1)

- Consider the AR(1) process $X_t = \phi X_{t-1} + W_t$, with $|\phi| < 1$.
- By definition, $\phi_{1,1} = \rho(1) = \phi$.
- To calculate $\phi_{2,2}$, consider the regression of X_{t+2} on X_{t+1}, say, $\hat{X}_{t+2} = \beta X_{t+1}$.
- We choose β to minimize

$$E[(X_{t+2} - \hat{X}_{t+2})^2] = E[(X_{t+2} - \beta X_{t+1})^2] = \gamma(0) - 2\beta\gamma(1) + \beta^2\gamma(0).$$

- The solution is $\beta = \phi$. Similarly, $E[(X_t - \beta X_{t+1})^2]$ is minimum when $\beta = \phi$.
- Hence,

$$\phi_{2,2} = \mathrm{Corr}(X_{t+2} - \beta X_{t+1}, X_t - \beta X_{t+1}) = \mathrm{Corr}(W_{t+2}, X_t - \beta X_{t+1}) = 0$$

by causality. Thus, $\phi_{2,2} = 0$.

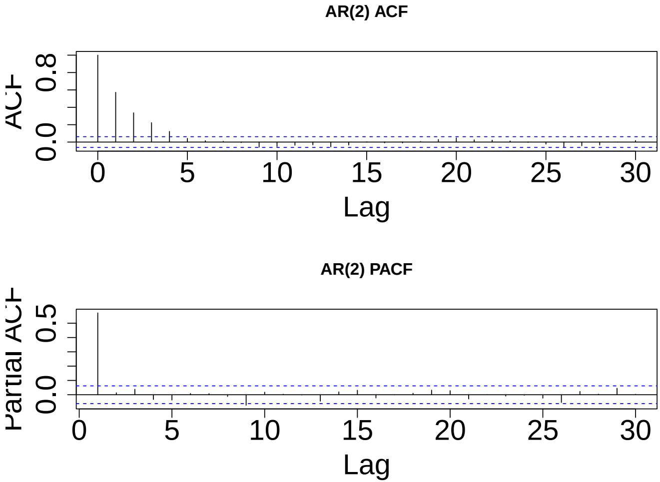
The PACF of AR(p)

- Consider the causal AR(p) process $X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + W_t$.
- Consider calculating $\phi_{h,h} = \mathrm{Corr}(X_{t+h} - \hat{X}_{t+h}, X_t - \hat{X}_t)$ for $h > p$.
- We can show $\hat{X}_{t+h} = \sum_{i=1}^{p} \phi_i X_{t+h-i}$.
- Thus,

$$\phi_{h,h} = \mathrm{Corr}(X_{t+h} - \hat{X}_{t+h}, X_t - \hat{X}_t) = \mathrm{Corr}(W_{t+h}, X_t - \hat{X}_t) = 0$$

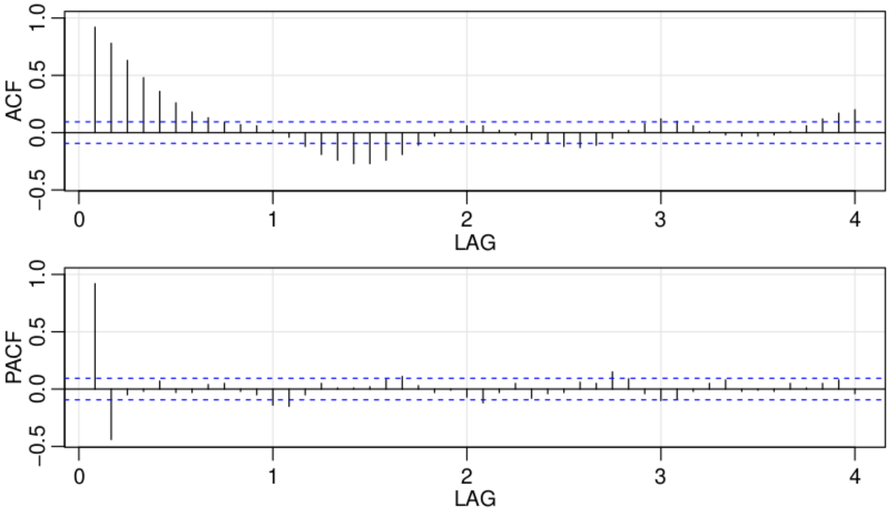
  follows from causality.

ACF and PACF of an AR(2) model



The ACF and PACF of an AR(2) model with $\phi_1 = 1.5$ and $\phi_2 = -0.75$.

ACF and PACF of the Recruitment data



Behavior of ACF and PACF for ARMA models

	AR(p)	MA(q)	ARMA(p, q)
ACF	Tails off	Cuts off after lag q	Tails off
PACF	Cuts off after lag p	Tails off	Tails off

Thank you!