MTH442 Assignment 3

Jiyanshu Dhaka 220481 jiyanshud22@iitk.ac.in

Q1:

Claim: If $\gamma(0) > 0 \;\; \gamma(h) \to 0$ as $h \to \infty$, then Γ_T is positive defnite.

Proof by contradiction:

1. assume Γ_T is singular. from class notes condition for that is like if Γ_T wass singulur for some T

$$\implies \exists u \neq 0 \quad (\text{non zero vector}) such that \quad \Gamma_T u = 0. \tag{i}$$

2. condition: from given cond$^{\text{n}}$ in question $\gamma(0) > 0$:

$$\Gamma_1 = \big(\gamma(0)\big)$$

as $\gamma(0) > 0$: $\implies$ Γ_1 is nonsingular.

3. existence of non singular matrices: so from point 2 , $\exists s \geq 1$ where Γ_s is nonsingulur. take sequence:

$$\Gamma_1, \Gamma_2, \ldots.$$

and let Γ_{s+1} is first singular matirx in this sequenc.

4. linear Combination: let $z_1, z_2, \ldots$ be vector representing the components in space where cov. matrixes Γ are defined. vectors form a sequence, now as Γ_{s+1} is singular (from point 3) so we can write z_{s+1} as linear combination:

$$z_{s+1} = c' z, \quad c = (c_1, c_2, \ldots, c_s)',$$

here c is the vector of coefficients $c_1, c_2, \ldots, c_s$ and $z = (z_1, z_2, \ldots, z_s)'$.

5. using stationarity condition from notes: it says that covariance between any two points depends only on time difference so using condition and point 4

$$z_{s+h+1} = c' z_h \quad \forall h \geq 1. \tag{ii}$$

here $\mathrm{z}_h = (z_h, \ldots, z_{r+h-1})'$

so, for $T \geq s+1$,linear combination condition holds, and we can write z_T as:

$$z_T = c_T' z \;\; . \tag{iii}$$

here $\quad c_T = (c_{T1}, c_{T2}, \ldots, c_{Ts})' \;\; and \;\; z = (z_1, z_2, \ldots, z_s)'$.

6. Variance of z_T: The variance $\gamma(0)$ can be expressed as:

$$\gamma(0) = \text{var}(z_T) = c_T' \Gamma_s c_T = c_T' P E P' c_T,$$

c_T is a column vector of order s

$$PP' = I_s \quad (\text{identity matrix of order } s)$$

and

$$E = \begin{pmatrix} e_1 & 0 & \cdots & 0 \\ 0 & e_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_s \end{pmatrix} \quad (\text{diagonal matrix of order } s \text{ with positive eigenvalues})$$

with $0 < e_1 \leq e_2 \leq \cdots \leq e_s$ as eigenvalues of Γ_s.

7. bound on coefficients: using eigen value condition from pt. 6 and $\gamma(0) = c_T' P E P' c_T$:
as $0 < e_1 \leq e_2 \leq \cdots \leq e_s$. e_1 is smallest, so: $c_T' P E P' c_T \geq e_1 c_T' P P' c_T$
$\implies \gamma(0) = c_T' P E P' c_T \geq e_1 c_T' P P' c_T = e_1 c_T' I c_T = e_1 c_T' c_T = e_1 \sum_{i=1}^{s} c_{Ti}^2$.
$\implies \gamma(0) \geq e_1 \sum_{i=1}^{s} (c_{Ti})^2$, $\implies$ coefficients c_{Ti} are bounded in T $\forall i = 1, .., s$.

8. Contradiction:
as $\gamma(0) = \text{cov}(z_T, z_T) = \text{cov}(z_T, c_T' z)$, $\implies$ $\text{cov}(z_T, c_T' z) \leq \sum_{i=1}^{s} |c_{Ti}||\gamma(T-i)|$.
so: $\implies$ $0 < \gamma(0) \leq \sum_{i=1}^{s} |c_{Ti}||\gamma(T-i)|$.

using this inequality and from point 7 coeff. c_{Ti} are bounded: as all terms $\gamma(T-i) \to 0$ as $h \to \infty$, $\implies$ $\gamma(h) \to 0$ as $h \to \infty$. This contradicts our assumption $\gamma(0) > 0$.

so, Γ_T is not singulur, $\implies$ Γ_T (positive definite).