Week 1: Review material [1]

Reading material from the books

- *Burgess-Moore, Chapter 1*
- *Weiberg, Chapter 2*

1 The irreducible representations of the Poincare group.

Relativistic systems are those invariant under the Poincare group. If we add quantum mechanics to the problem, we need to consider how the Poincare group acts on the Hilbert space of states of the system.

In quantum mechanics continuous symmetries have associated to them operators that generate infinitesimal transformations. These operators arise from the quantization of the associated Noether charge of a classical system.

The Poincare group has ten generators: 6 of them associated to the Lorentz group of transformations, and four more associated to translations in space and time.

The generators are usually called $M_{\mu\nu}$ or $J_{\mu\nu}$ and P_μ. These are four-vector indices. Moreover, one can check that these satisfy the following Lie-algebra relations:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\eta_{\mu\rho}M_{\nu\sigma} + i\eta_{\mu\sigma}M_{\rho\nu} - (\mu \leftrightarrow \nu) \quad (1)$$

$$[M_{\mu\nu}, P_\sigma] = i\eta_{\mu\sigma}P_\nu - i\eta_{\nu\sigma}P_\mu \quad (2)$$

$$[P_\mu, P_\nu] = 0 \quad (3)$$

In the conventions above, all M and P are Hermitian, hence the factors of i. We also have the convention that $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and it is used to raise and lower indices. By convention, the operator P^0 is called the energy operator and it should be positive relative to a stable vacuum.

One can see that P transforms as a four vector under the Lorentz group.

The energy operator is what defines evolution in a quantum system. The operators that commute with P^0 are of major importance to classify states.

From the Lorentz algebra, these are the momentum operators $P^i \sim \vec{P}$ and angular momentum operators $J^i \sim \frac{1}{2}\epsilon^{ijk}J_{kj}$.

Usually it is convenient to find a maximally commuting set of operators. Thus, we will diagonalize all of the P^μ operators, to have eigenvalue p^μ. Thus, our states are classified by momentum as follows

$$P^\mu|p,\sigma\rangle = p^\mu|p,\sigma\rangle \tag{4}$$

We are using the convention that σ is the additional degeneracy of a state, while we use the convention that the lower case letter is an eigenvalue.

If we perform a Poincare transformation, it can always be exponentiated from M, P, and moreover we can always write it in the form

$$U \sim \exp(i/2\theta^{\mu\nu}M_{\mu\nu})\exp(ia^\mu P_\mu) \tag{5}$$

where the translation is taken first. It is also common to denote the the rotations and boosts by a matrix Λ^μ_ν, that acts on points in spacetime by sending $x^\nu \to (x^\mu)' = \Lambda^\mu{}_\nu x^\nu$.

The effect on the operator space of states is to send

$$P^\mu \to (P^\nu)' = (\Lambda^\nu{}_\mu)^{-1}P^\mu \tag{6}$$

This is, the transformation should be

$$U(\Lambda)P^\mu U(\Lambda^{-1}) = (\Lambda^\nu{}_\mu)^{-1}P^\mu \tag{7}$$

and because it is a symmetry of the quantum system it should be a unitary transformation.

The equation implies that it sends $p^\mu \to (p^\nu)' = \Lambda^\nu_\mu p^\mu$ for the eigenvalues. It can also act on the σ labels.

The difference between Λ and Λ^{-1} is due to the difference between Active and Passive transformations. The transformations are active on states, but they should be passive on operators (or viceversa).

Given a four momentum p^μ, it is easy to show that $p^2 = p^\mu p_\mu = -m^2$ is invariant. One can also check that the operator $P^\mu P_\mu$ commutes with all of the Lorentz generators. ***** Check this as an exercise *****

This property makes it into a Cassimir operator of the Lorentz algebra. It is called the mass squared operator of a representation. States can be interpreted as particles only if $m^2 \geq 0$ and $p^0 \geq 0$. A state with $P = 0$ is called a vacuum.

The reason for this constraint is that if the vacuum is translation invariant (a property under Lorentz transformations) and if it the lowest energy state (as required for stability), then all other excitations should have positive energy. In particular, if one has a particle with $m^2 < 0$, one can show that there is a Lorentz frame where the vacuum would not be the Lowest energy state, and hence it would be unstable.

Given this information, we find that $p^0 > 0$ and we have two options, $m^2 > 0$ and $m^2 = 0$.

In the first case, $m^2 > 0$, we can go to a Lorentz frame where $\vec{p}$ vanishes. This special Lorentz frame is the center of mass frame. Any element of the Hilbert space of states with fixed momentum can therefore be transformed to elements of the Hilbert space in the center of mass frame.

Since the Lorentz transformations also involve rotations, we see that we still have to consider the spectrum of states under rotations. Since in the center of mass frame $J^{1,2,3}$ commute with P^0, and the $\vec{p}$ vanishes, the rotations rotate between different σ labels, but keep the condition $\vec{p} = 0$. We find that the standard theory of angular momentum holds.

This group of rotations in the center of mass frame is called the little group of the representation, and this method of building representations is called the method of induced representations. Given a representation of the group of rotations ($SO(3)$), that is classified by a spin j, and given the value of $m^2 > 0$ we can build a complete representation of the Lorentz group. It will have $2j+1$ components (usually classified by j_z), and $J^2 = j(j+1)$ is the square of the angular momentum operator.

This is called the spin of the representation. It is a quantum number of a particle. The state is degenerate.

Notice that we can pick the components of J that are transverse to the center of mass frame by considering the combinations

$$W^\mu \sim \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} p_\sigma \tag{8}$$

It is obvious that $W^\mu p_\mu = 0$, and in the center of mass, only $W^{1,2,3}$ are non-zero.

When we look more closely and consider the operator

$$W^\mu \sim \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma \tag{9}$$

we see that this is free of ordering ambiguities: the P and M in the definition commute with each other because of the ϵ symbol. ***** Check this as an exercise *****

Because of this, it is obvious that $W \cdot P = 0$ as an operator equation.

Also, in the center of mass frame, we find that $W^i \sim mJ^i$ and that therefore $W^2 = W^\mu W_\mu \sim m^2 J^2 = m^2 j(j+1)$.

We can show further that $[W^2, P\mu] = 0$ and that $[W^2, M_{\mu\nu}] = 0$. ***** Check this as an exercise *****

Thus, the operator combination W^2 is also an invariant of the Poincare group. This is the one that carries the information on spin of a state.

For single particle states they are classified uniquely by mass and spin (here spin is in the same sense as in non-relativisiitic mechanics). There can be additional degeneracies in mass and these would be due to symmetries.

Also, because boosts along the z direction commute with rotations along z, we see that if start a system at rest and boost it, we keep the angular momentum along the z axis fixed.

Thus, it is convenient to introduce the projection of spin along the axis of momentum for a moving particle as a convenience reference on how to get the state from zero momentum.

This property is called helicity and it depends on the frame of reference.

For massless particles, $P^2 = 0$, we can only go to a frame where $P \sim (p, p, 0, 0)$. The little group in this case is called $ISO(2)$ (the standard group of Euclidean geometry) (see Weinberg for more details).

This has one compact generator of rotations, and two 'translations'. If single particle states are to have a discrete set of states at fixed momentum, only the rotations are allowed to have non-trivial eigenvalues on the Hilbert space.

This rotation is the spin along the axis of momentum and it is still a valid object, so the notion of helicity remains. In fact, helicity is the only quantum number that remains and it can not change when we do Lorentz transformations and it is the same for all states in the representation of the Lorentz group.

This is the complete set of representations of the Lorentz group that are used in particle physics.

There is one extra ingredient that is required. This is how we choose the normalization of the states $|p, \sigma\rangle$. We choose them so that

$$\langle q, \sigma' | p, \sigma\rangle = f(E_p)(2\pi)^3 \delta^3(\vec{p} - \vec{q})\delta_{\sigma,\sigma'} \tag{10}$$

We should remember that in non-relativistic situations, we choose $f(E_p) = 1$. This is a normalization of the plane wave function so that the density of particles is fixed and the same for all velocities.

in a relativistic theory, it is convenient to normalize it so that the density of particles in the rest frame is constant. This means that in a moving frame there will be a Lorentz contraction factor of $\gamma^{-1} = \sqrt{1-v^2}$, so the density gets multiplied by γ. Now $E = m\gamma$, so it is better to choose $f(E) = 2E$. This differs from the naive density normalization by a factor of m, so it is more appropriate to talk about this as having normalized energy density per unit volume (this also works well for massless particles which is why it is preferred over factors of γ).

This is also the inverse of the normalization of the relativistically invariant integral over the on-shell momenta, given by

$$\int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2+m^2))\theta(p^0)G(p) \sim \int \frac{d^3p}{(2\pi)^3 2E}G(p) \tag{11}$$

In $G(p)$ over three momenta, the energy is evaluated on-shell: $p^0 = +\sqrt{\vec{p}^2+m^2}$.

Notation: integrals over p will always be written with factors of 2π to the appropriate number of momenta in the denominator, and δ functions in momentum variables will always carry factors of 2π in the numerator (one factor of 2π per δ-function).

So this describes the one-particle states in a relativistic quantum field theory.

2 Fock space: raising and lowering operators

Having described the single particle states, we can now consider multiparticle states. These are described by taking the Hilbert space to be given by

$$\mathcal{H} = \oplus_{i=0}^{\infty}\mathcal{H}_i \tag{12}$$

and where the Hilbert space of states $\mathcal{H}_0 = |0\rangle$ is called the vacuum and has a single basis element. It is normalized to one $\langle 0|0\rangle = 1$.

The space $\mathcal{H}_\infty$ is the single particle set of states (it looks as above with various masses and particle species. It can include composite particles (a hydrogen atom).

In perturbation theory we expect that interactions are weak between particles, so an initial condition with some fixed number of particles stays approximately that way for a long time until we get an interaction and particles are produced/destroyed. Thus in the free field limit we can describe a collection of many particles with the single particle states as building blocks.

The states with N particles are schematically a product $\otimes_{i=1}^{N}\mathcal{H}_1$ where identical particles can be exchanged and the answer does not change.

The particles are classified into bosons and fermions according to spin (the spin-statistics theorem). Thus a two particle state can be thought of as being described by

$$|p_1,\sigma_1;p_2,\sigma_2\rangle \simeq \pm|p_2,\sigma_2;p_1,\sigma_1\rangle \tag{13}$$

where the minus sign applies only if the two particles are fermions. In the convention above, the σ might includes the identity of the particle. We would normalize these with the product of the norms of the one-particle states, with statistics taken into account:

$$\begin{aligned}\langle q_1,\tau_1;q_2,\tau_2|p_1,\sigma_1;p_2,\sigma_2\rangle &= (2E_1)(2\pi)^3\delta^3(p_1-q_1)\delta_{\sigma_1,\tau_1} && (14)\\ &\quad\times(2E_2)(2\pi)^3\delta^3(p_2-q_2)\delta_{\sigma_2,\tau_2} && (15)\\ &\quad\pm[(q_1,\tau_1)\leftrightarrow(q_2,\tau_2)] && (16)\end{aligned}$$

It is convenient to introduce raising and lowering operators for particles: operators that create single particle states. They are defined by

$$a^\dagger(p,\sigma)|0\rangle = |p,\sigma\rangle \tag{17}$$

And more schematically

$$a^\dagger(p,\sigma)|\alpha\rangle = |p,\sigma;\alpha\rangle \sim |p,\sigma\rangle\otimes|\alpha\rangle \tag{18}$$

for $|\alpha\rangle\in\mathcal{H}_{>0}$.

The lowering operators $a(p,\sigma)$ are the adjoint of $a^\dagger(\sigma)$. Notice that

$$\langle 0|a^\dagger|\alpha\rangle = 0, \tag{19}$$

for all $|\alpha\rangle$, so that if we take the adjoint of this equation we find that

$$a|0\rangle = 0 \tag{20}$$

is necessarily zero (it is orthogonal to every other state).

Exercise: Show that if $a^\dagger(q,\sigma)$ is fermionic, then $(a^\dagger)^2(q,\sigma)|\alpha\rangle = 0$ (Calculate the norm of the state).

It is easy to show that the following commutation relations are satisfied

$$a(p,\sigma)a^\dagger(q,\sigma') \mp a^\dagger(q,\sigma')a(p,\sigma) = 2E_p(2\pi)^3\delta^3(p-q) \tag{21}$$

The minus sign is if any of $a, a^\dagger$ are bosons and the plus sign is for fermions only. ***** Check this as an exercise *****

The Hilbert space $\mathcal{H}$ is called a Fock space.

It can be proved by induction on the number of particles of a state that any operator acting on $\mathcal{H}$ can be written as a Taylor series in the raising and lowering operators with all lowering operators to the right of the raising operators.

In the free theory, we have that momentum is additive

$$P^\mu|p_1,\sigma_1;\ldots;p_N,\sigma_N\rangle = \sum_{i=1}^{M} p_i|p_1,\sigma_1;\ldots;p_N,\sigma_N\rangle \tag{22}$$

and that the number operator $\hat{N}$ has the representation

$$\hat{N}|p_1,\sigma_1;\ldots;p_N,\sigma_N\rangle = N|p_1,\sigma_1;\ldots;p_N,\sigma_N\rangle \tag{23}$$

Thus it is easy to show that

$$[N, a^\dagger] = a^\dagger \tag{24}$$

and that similarly $[N,a] = a$. This defines the operator N.

The number operator is given by the following polynomial in the raising and lowering operators

$$N = \textstyle\sum_\sigma \int \frac{d^3p}{(2\pi)^3 2E_p} a^\dagger(p,\sigma)a(p,\sigma) \tag{25}$$

$$= \textstyle\sum_{\sigma,\sigma'} \int \frac{d^3p}{(2\pi)^3 2E_p} \int \frac{d^3q}{(2\pi)^3 2E_q} a^\dagger(p,\sigma)a(q,\sigma')(2\pi)^3(2E_q)\delta^3(p-q)\delta_{\sigma,\sigma'} \tag{26}$$

***** Check this as an exercise *****

Also check that

$$P^\mu = \int \frac{d^3p}{(2\pi)^3 E_p} p^\mu a^\dagger(p,\sigma)a(p,\sigma) \tag{27}$$

and that the corresponding operators transform as in equation 7

Notice that in the definitions above we have very pedantically included all the δ functions and integrals with the correct Lorentz factors and factors of $(2\pi)^3$.

The Lorentz transformations are encoded into $U(\Lambda)|p,\sigma\rangle \sim |\Lambda p, \Lambda\sigma\rangle$, because we chose an invariant normalization of the density in the rest frame, and so

$$U(\Lambda)a^\dagger(p,\sigma)U^{-1}(\Lambda) = a^\dagger(\Lambda p, \Lambda\sigma) \tag{28}$$

You should check that this makes P^μ into a four vector in the Lorentz sense.