Circle Compactifications

Giancarlo Oancia[a]

[a] *University of Bologna*

E-mail: giancarlo.oancia@studio.unibo.it

Abstract: This document discusses the compactification of strings on circles, and how type IIA and IIB are related by T-duality. This is *not* an original work, and its purpose is to prepare the string theory exam of UNIBO. We follow the conventions of the university lectures by professor Ling Lin [1], while the development mostly follows [2]. Other references used are [3–8].

Keywords: Strings, Compactification, T-duality

Contents

Kaluza-Klein Compactification in Field Theory

Consider a field theory in D dimensions propagating in a spacetime $\mathcal{M}_D = \mathcal{M}_d \times S^1$, where $D = d+1$, with *mostly plus metric* $\eta_{\hat{\mu}\hat{\nu}} = \mathrm{diag}(-,+,\dots,+)$. Let's call $x^{\hat{\mu}} = (x^\mu, y)$ the coordinates in $\mathcal{M}_D$, where $\hat{\mu} = 0,\dots,D-1$, $\mu = 0,\dots,d-1$. On the internal space S^1, we identify $y \simeq y + 2\pi R$.

The *compactification scale* is $M_c = \frac{1}{R}$, and we're interested in the limit

$$E \ll \frac{1}{R}, \tag{1.0.1}$$

to be able to ignore the Kaluza-Klein massive modes, as will be explained in the following sections.

1.1 Kaluza-Klein Reduction of a Scalar Field on S^1.

Consider a real scalar field $\phi(x^{\hat{\mu}})$ propagating with action

$$S_D = \frac{\Lambda^{D-4}}{2} \int_{\mathcal{M}_d \times S^1} \mathrm{d}^D x \, \partial_{\hat{\mu}} \phi(x^{\hat{\mu}}) \, \partial^{\hat{\mu}} \phi(x^{\hat{\mu}}), \tag{1.1.1}$$

where Λ is some scale introduced for dimensional reasons. The equations of motion are

$$\partial_{\hat{\mu}} \partial^{\hat{\mu}} \phi(x^{\hat{\mu}}) = 0. \tag{1.1.2}$$

Exploiting the periodicity in the internal space, we can decompose the field in Fourier modes as follows,

$$\phi(x^{\hat{\mu}}) = \sum_{s\in\mathbb{Z}} e^{isy/R} \phi_s(x^\mu), \tag{1.1.3}$$

with $\phi_0(x^\mu)$ real and $\phi_s(x^\mu)$ complex such that $\phi_s^* = \phi_{-s}$. The orthogonality condition is

$$\int_0^{2\pi R} \mathrm{d}y \, e^{-iny/R} e^{+imy/R} = 2\pi R \delta_{n,m}. \tag{1.1.4}$$

Substituting the expansion (1.1.3) into the equations of motion (1.1.2), we get

$$\partial_{\hat{\mu}} \partial^{\hat{\mu}} \phi(x^{\hat{\mu}}) = (\partial_\mu \partial^\mu + \partial_y \partial^y) \left(\sum_{s\in\mathbb{Z}} e^{isy/R} \phi_s(x^\mu) \right) = \sum_{s\in\mathbb{Z}} \left(\partial_\mu \partial^\mu - \frac{s^2}{R^2} \right) \phi_s(x^\mu) e^{isy/R}.$$

It must vanish to satisfy (1.1.2), and using the completeness of Fourier basis, we get

$$\left(\partial_\mu \partial^\mu - \frac{s^2}{R^2} \right) \phi_s(x^\mu) = 0, \quad \forall s \in \mathbb{Z}, \tag{1.1.5}$$

which is the Klein-Gordon equation for a scalar field in d dimensions with mass

$$m_s^2 = \frac{s^2}{R^2} = s^2 M_c^2. \tag{1.1.6}$$

Then, from a d-dimensional point of view, we observe one massless scalar and an infinite tower of massive Kaluza-Klein modes, which can be neglected for energies $E \ll 1/R$. Starting from a 5d field with mass M, and applying the very same procedure, we'd have found

$$m_s^2 = M^2 + \frac{s^2}{R^2}, \tag{1.1.7}$$

which is the same result as the compactification of a string in the zero-winding sector, as we'll see.

1.2 Kaluza-Klein Reduction of Einstein Theory on S^1.

Consider now a D-dimensional Einstein theory, with metric $G_{\hat{\mu}\hat{\nu}}(x^{\hat{\mu}})$. Exploiting the periodicity in the internal space, we may expand the metric in Fourier modes as

$$G_{\hat{\mu}\hat{\nu}}(x^{\hat{\mu}}) = G_{\hat{\mu}\hat{\nu}}(x^\mu, y) = G^{(0)}_{\hat{\mu}\hat{\nu}}(x^\mu) + \sum_{s\neq 0} G^{(s)}_{\hat{\mu}\hat{\nu}}(x^\mu) e^{isy/R}. \tag{1.2.1}$$

However, we're interested in the d-dimensional theory, so it's more convenient to first decompose $SO(1, D-1)$'s representations into representations of $SO(1, d-1)$. After the decomposition, we can exploit the periodicity of the internal space and Fourier expand as usual. In particular, recalling that we defined $x^{D-1} = x^d = y$, the graviton decomposes as

$$G_{\hat{\mu}\hat{\nu}}(x^\mu, y) \to G_{\mu\nu}(x^\mu, y) = G^{(0)}_{\mu\nu}(x^\mu) + \sum_{s\neq 0} G^{(s)}_{\mu\nu}(x^\mu) e^{isy/R}, \tag{1.2.2}$$

$$\to G_{\mu d}(x^\mu, y) = G^{(0)}_{\mu d}(x^\mu) + \sum_{s\neq 0} G^{(s)}_{\mu d}(x^\mu) e^{isy/R}, \tag{1.2.3}$$

$$\to G_{dd}(x^\mu, y) = G^{(0)}_{dd}(x^\mu) + \sum_{s\neq 0} G^{(s)}_{dd}(x^\mu) e^{isy/R}. \tag{1.2.4}$$

It's straightforward to observe that the massless modes are a d-dimensional graviton, a d-dimensional $U(1)$ gauge boson and a d-dimensional scalar. To be more precise, if we call the zero-modes fields $g_{\mu\nu}(x^\mu)$, $A_\mu(x^\mu)$ and $\phi(x^\mu)$, then, they're related to the zero-mode five-dimensional metric by

$$G^{(0)}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{2\alpha_d} g_{\mu\nu} + e^{-2(d-2)\alpha_d\phi} A_\mu A_\nu & e^{-2(d-2)\alpha_d\phi} A_\mu \\ e^{-2(d-2)\alpha_d\phi} A_\nu & e^{-2(d-2)\alpha_d\phi} \end{pmatrix}. \tag{1.2.5}$$

We defined α_d in such a way that the action in dimension $D-1=d$ is canonically normalized, that is,

$$\alpha_d = \sqrt{\frac{1}{2(d-1)(d-2)}}. \tag{1.2.6}$$

From a zero-mode Kaluza-Klein point of view, this leads to the metric

$$\mathrm{d}s^2 = G^{(0)}_{\hat{\mu}\hat{\nu}}\,\mathrm{d}x^{\hat{\mu}}\,\mathrm{d}x^{\hat{\nu}} + \cdots = e^{2\alpha_d\phi} g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + e^{-2(d-2)\alpha_d\phi}(\mathrm{d}y + A_\mu\,\mathrm{d}x^\mu)^2 + \ldots . \qquad (1.2.7)$$

In particular, the above theory allows for d-dimensional reparametrizations $x'^\mu(x^\nu)$ and for reparametrizations of the internal space coordinates $y' = y + \lambda(x^\mu)$. To make the theory invariant under the latter transformations, the field A_μ must transform under the $U(1)$ transformation $A'_\mu = A_\mu - \partial_\mu\lambda$, which is straightforward if we look at (1.2.7).

Therefore, we observe that the gauge transformations of the vector boson follow from coordinate reparametrization in the internal dimension. At the end, gauge invariance after dimensional reduction follows from diffeomorphism invariance in higher dimensions. Basically, under the compactification ansatz $\mathcal{M}_D \to \mathcal{M}_d \times S^1$, the group of diffeomorphisms decomposes as

$$Gl(D,\mathbb{R}) \to Gl(d,\mathbb{R}) \times U(1). \qquad (1.2.8)$$

This was the original motivation of Kaluza-Klein, to unify gravity with gauge theories. However, this way we can't obtain charged chiral fermions, so this attempt failed, but the idea remained up to these days, as we shall see.

Another interesting feature of Kaluza-Klein compactifications is the arising of *moduli*, which are massless scalar fields in d-dimensions with no potential. In this example, this is represented by the field ϕ, called *radion*. It sets the volume of the internal space, since

$$\mathrm{Vol}(S^1) = \int_0^{2\pi R} \mathrm{d}y\,\sqrt{G^{(0)}_{yy}} = e^{-(d-2)\alpha_d\phi}\cdot 2\pi R. \qquad (1.2.9)$$

To observe that it isn't constraint by a potential, let's fix the ideas and consider the case $D = 5$, $d = 4$. The theory is represented by the action

$$S_5 = \frac{M_5^3}{2}\int \mathrm{d}^5x\sqrt{-G}R_5, \qquad (1.2.10)$$

where $G = \det(G_{\hat{\mu}\hat{\nu}})$, and R_5 is the Ricci scalar. Substituting the above metric for the zero-modes, after working out the Ricci scalar, we get a 4 dimensional action which reads

$$S_4 = M_5^3\pi R\int \mathrm{d}^4x\sqrt{-g}\left(R_4 - \frac{1}{6}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4e^\phi}F_{\mu\nu}F^{\mu\nu} + \text{KK tower}\ldots\right), \qquad (1.2.11)$$

where the constant is compatible with the $4d$ gravity action

$$S_4 = \frac{M_p^2}{16\pi}\int \mathrm{d}^4x\sqrt{-g}R_4, \qquad (1.2.12)$$

if

$$M_p^2 = 16\pi^2 M_5^3 R, \qquad (1.2.13)$$

and the gauge coupling is $g^2 = e^\phi$. The above relation (1.2.13) suggests us that there's the possibility to have a low fundamental gravity scale M_5, if we take R to be large enough. However, this approach would fail if we couple the theory with the Standard Model. Indeed, for large R, the Kaluza-Klein masses, proportional to $M_c = 1/R$, would be too light, and should've been already observed. This leads to the idea of a *brane-world scenario*.

Brane-World scenario. Even if in the first years people focused on type I string theories since they contain gauge theories, there had been a revival of type II theories after the discovery of branes. In particular, the idea is that *closed strings*, which contains the graviton, propagates on the bulk of the theory, which is 10-dimensional for the superstring. Further, open strings are attached to branes, which are submanifolds of the bulk on which Standard Model fields live in, considering that stack of branes could lead to non-abelian gauge theories.

One simple example is considering the Standard Model fields localized on a 3-brane, which exists for a type IIB theory, so that the action splits into a 4 dimensional piece for the 3-brane worldvolume, and a bulk piece in the larger 10-dimensional spacetime

$$S_{\text{brane-world}} = S_{\text{brane}} + S_{\text{bulk}} = \int \mathrm{d}^4x\, \mathcal{L}_{\text{brane}} + \int d^{10}x\, \mathcal{L}_{\text{bulk}}. \tag{1.2.14}$$

Then, gravity is compactified as discussed earlier, while gauge and matter fields are insensitive of the compactification procedure, and then they don't have massive Kaluza-Klein modes. This allows for *large extra dimensions*, whose size is in principle detectable only through the effect on gravity. Looking again at (1.2.13), we can even have $M_5 \simeq$ TeV, so that there would be no hierarchy between the EW and the gravitational scale. However, this is not a fully satisfactory resolution, since we've moved the problem into a hierarchy problem between the size $1/R$ of the extra dimension, and the fundamental scale $1/M_5$.

Closed Bosonic String

We use mostly plus metric $\eta = \text{diag}(-,+,\dots,+)$, and focus on the *closed string*. We take the *critical string*, with background $\mathcal{M}_{26}$. The worldsheet coordinates are $\xi^a = (\tau, \sigma)$, $a = 1, 2$, where $\sigma \in (0, l)$. The string is described by the worldsheet fields X^μ, with $\mu = 0, \dots, 25$.

2.1 Mode Expansion.

Calling the $2d$ metric $\gamma_{ab}(\xi^a)$, the Polyakov action reads

$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\xi\sqrt{-\gamma}\,\gamma^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}, \tag{2.1.1}$$

where the string tension is

$$T = \frac{1}{2\pi\alpha'}. \tag{2.1.2}$$

Exploiting the reparametrisation invariance and the Weyl symmetry, we can go to *conformal gauge*, in which $\gamma_{ab} = \eta_{ab}$, such that Polyakov action becomes

$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\xi\, \partial_a X^\mu \partial^a X_\nu. \tag{2.1.3}$$

Defining *worldsheet lightcone coordinates* $\xi^\pm \equiv \tau \pm \sigma$, the equations of motion read $\partial_+\partial_- X^\mu = 0$, which is solved by the ansatz

$$X^\mu(\xi^a) = X^\mu_L(\xi^+) + X^\mu_R(\xi^-). \tag{2.1.4}$$

For a closed string, we must further impose the boundary condition

$$X^\mu(\tau,\sigma) = X^\mu(\tau, \sigma + l), \tag{2.1.5}$$

which is compatible with (2.1.4) for the following mode decomposition

$$X^\mu_L(\xi^+) = \frac{x^\mu}{2} + \sqrt{2\alpha'}\frac{\pi}{l}\tilde{\alpha}^\mu_0\xi^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}^\mu_n}{n}e^{-\frac{2\pi i}{l}n\xi^+}, \tag{2.1.6a}$$

$$X^\mu_R(\xi^-) = \frac{x^\mu}{2} + \sqrt{2\alpha'}\frac{\pi}{l}\alpha^\mu_0\xi^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-\frac{2\pi i}{l}n\xi^-}. \tag{2.1.6b}$$

Summing them, we find

$$X^\mu(\tau,\sigma) = x^\mu + \sqrt{2\alpha'}\frac{\pi}{l}(\tilde{\alpha}_0^\mu + \alpha_0^\mu)\tau + \sqrt{2\alpha'}\frac{\pi}{l}(\tilde{\alpha}_0^\mu - \alpha_0^\mu)\sigma + \text{oscillators}, \tag{2.1.7}$$

which, under $\sigma \to \sigma + l$, transforms as $\delta X^\mu = \sqrt{2\alpha'}\frac{\pi}{l}(\tilde{\alpha}_0^\mu - \alpha_0^\mu)l \stackrel{!}{=} 0$. So, to satisfy the boundary condition (2.1.5), it must be $\alpha_0^\mu = \tilde{\alpha}_0^\mu$. Moreover, using Noether's theorem for the global Poincaré symmetry, the conserved charge under translations $\delta X^\mu = \varepsilon^\mu$ is

$$\frac{1}{\sqrt{2\alpha'}}(\tilde{\alpha}_0^\mu + \alpha_0^\mu) \equiv p^\mu, \tag{2.1.8}$$

so

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu. \tag{2.1.9}$$

Proof. Under a transformation

$$\phi_k(x) \to \phi_k(x) + \varepsilon\Delta\phi_k(x) + O(\varepsilon^2), \tag{2.1.10}$$

the Lagrangian transforms as $\mathcal{L} \to \mathcal{L} + \Delta\mathcal{L}$, with

$$\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_k}(\varepsilon\Delta\phi_k) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)}\partial_\mu(\varepsilon\Delta\phi_k). \tag{2.1.11}$$

After integration by parts and using the equations of motion, we easily see that the conserved current, such that $\partial_\mu J^\mu = 0$ is

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_k)}\Delta\phi_k - k^\mu \tag{2.1.12}$$

Considering Polyakov action in conformal gauge,

$$S_P = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\,\xi\partial_a X\cdot\partial^a X, \tag{2.1.13}$$

we can compute

$$\delta X^\mu = \varepsilon^\mu, \quad \frac{\partial\mathcal{L}}{\partial(\partial^a X_\mu)} = -\frac{1}{2\pi\alpha'}\partial_a X^\mu, \quad q_a = -\frac{1}{2\pi\alpha'}\partial_a X^\mu\varepsilon_\mu.$$

So, the conserved current is

$$q_a^\mu = -\frac{1}{2\pi\alpha'}\partial_a X^\mu, \tag{2.1.14}$$

and the conserved charge is, up to a sign

$$p^\mu \equiv -\int_0^l \mathrm{d}\sigma(q^\mu)_\tau = \frac{1}{2\pi\alpha'}\partial_\tau X^\mu. \tag{2.1.15}$$

Using, then, the mode decomposition (2.1.6), giving that the spacial integral of the oscillators' exponentials vanish, we get

$$p^\mu = \frac{1}{2\pi\alpha'}\int_0^l \mathrm{d}\sigma\,\sqrt{2\alpha'}\frac{\pi}{l}(\tilde{\alpha}_0^\mu + \alpha_0^\mu) + \cdots = \frac{1}{\sqrt{2\alpha'}}(\tilde{\alpha}_0^\mu + \alpha_0^\mu). \tag{2.1.16}$$

2.2 Lightcone Gauge.

In lightcone quantization, we impose the Virasoro constraints $T_{ab} = 0$ on the classical theory. In worldsheet lightcone coordinates, they ensure that the reparametrizations

$$\xi^+ \to \tilde{\xi}^+(\xi^+), \quad \xi^- \to \tilde{\xi}^-(\xi^-), \tag{2.2.1}$$

which induce a conformal transformation, is a gauge symmetry of the classical theory. It's convenient to define *spacetime lightcone coordinates*

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1), \quad X^i, i = 2, \ldots, 25, \tag{2.2.2}$$

so that, the above freedom (2.2.1) can be used to fix

$$X^+(\tau,\sigma) = x^+ + \frac{2\pi\alpha'}{l}p^+\tau, \tag{2.2.3}$$

which is called *lightcone gauge*. Then, the mode expansion (2.1.6), with (2.1.9), would be, in lightcone gauge,

$$X_L^+(\xi^+) = \frac{x^+}{2} + \frac{\pi\alpha'}{l}p^+\xi^+, \tag{2.2.4a}$$

$$X_R^+(\xi^-) = \frac{x^+}{2} + \frac{\pi\alpha'}{l}p^+\xi^-, \tag{2.2.4b}$$

$$X_L^-(\xi^+) = \frac{1}{2}x^- + \frac{\pi\alpha'}{l}p^-\xi^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^-}{n}e^{-\frac{2\pi i}{l}n\xi^+}, \tag{2.2.4c}$$

$$X_R^-(\xi^-) = \frac{1}{2}x^- + \frac{\pi\alpha'}{l}p^-\xi^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^-}{n}e^{-\frac{2\pi i}{l}n\xi^-}, \tag{2.2.4d}$$

where, in particular, the constraints $T_{ab} = 0$ with the choice (2.2.3), fix all the α^- and $\tilde{\alpha}^-$ modes in terms of the α^i and $\tilde{\alpha}^i$, via the relations[1]

$$\tilde{\alpha}_n^- = \frac{1}{\sqrt{2\alpha'}p^+}\sum_{m\in\mathbb{Z}}\tilde{\alpha}_{n-m}^i\tilde{\alpha}_m^i, \quad \alpha_n^- = \frac{1}{\sqrt{2\alpha'}p^+}\sum_{m\in\mathbb{Z}}\alpha_{n-m}^i\alpha_m^i. \tag{2.2.5}$$

In this whole document, unless otherwise stated, repeated i-indices are summed over, with $i = 2, \ldots 25$.

[1] Details in Ling Lin's notes [1]. We're not interested in those here.

Since the oscillators are fixed, the only remaining degree of freedom for X^- is the centre of mass position,

$$q^-(\tau) \equiv \frac{1}{l}\int_0^l \mathrm{d}\sigma X^-(\tau,\sigma), \tag{2.2.6}$$

as we'll also see writing down the action and the hamiltonian.

The Polyakov action (2.1.3) reads

$$S = \frac{1}{4\pi\alpha'}\int \mathrm{d}\tau\,\mathrm{d}\sigma\left((\partial_\tau X^i)^2 - (\partial_\sigma X^i)^2\right) - \int \mathrm{d}\tau p^+\partial_\tau q^-, \tag{2.2.7}$$

where q^- is defined by (2.2.6)

Proof.

$$\begin{aligned}
S &= -\frac{1}{4\pi\alpha'}\int \mathrm{d}\tau\int_0^l \mathrm{d}\sigma(\partial_a X)\cdot(\partial^a X) \\
&= -\frac{1}{4\pi\alpha'}\int \mathrm{d}\tau\int_0^l \mathrm{d}\sigma\left(-2\partial_a X^+\partial^a X^- + \partial_a X^i\partial^a X^i\right) \\
&= \frac{1}{4\pi\alpha'}\int \mathrm{d}\tau\int_0^l \mathrm{d}\sigma\left((\partial_\tau X^i)^2 - (\partial_\sigma X^i)^2\right) - \frac{1}{2\pi\alpha'}\int \mathrm{d}\tau\int_0^l \mathrm{d}\sigma\partial_\tau X^+\partial_\tau X^-.
\end{aligned}$$

Now, using that $\partial_\tau X^+ = \frac{2\pi\alpha'}{l}p^+$, bringing it outside the spacial integral and further exchanging $\int_0^l \mathrm{d}\sigma$ with ∂_τ for X^-, we can rewrite the second part of the last expression as

$$-\frac{1}{l}\int \mathrm{d}\tau\partial_\tau p^+\left(\int_0^l \mathrm{d}\sigma X^-\right) = -\int \mathrm{d}\tau p^+\partial_\tau q^-,$$

where we used the definition (2.2.6). Putting all together, we obtain (2.2.7).

From the action (2.2.7), it's easy to compute the conjugate momenta[2] to q^- and X^i

$$q^i: \quad p_- \equiv \frac{\partial L}{\partial(\partial_\tau q^-)} = -p^+, \tag{2.2.8a}$$

$$X^i: \quad \Pi_i \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^i)} = \frac{1}{2\pi\alpha'}\partial_\tau X^i. \tag{2.2.8b}$$

The Hamiltonian reads

$$\begin{aligned}
H &= p_-\partial_\tau q^- + \int_0^l \mathrm{d}\sigma\Pi_i\partial_\tau X^i - L \\
&= \frac{1}{4\pi\alpha'}\int_0^l \mathrm{d}\sigma\left[(\partial_\tau X^i)^2 + (\partial_\sigma X^i)^2\right] \\
&= \frac{1}{2}\int_0^l \mathrm{d}\sigma\left[2\pi\alpha'\Pi_i\Pi_i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i\partial_\sigma X^i\right],
\end{aligned} \tag{2.2.9}$$

[2] Pay attention that the first piece of (2.2.7) contains the Lagrangian density, integrated over $\mathrm{d}^2\xi$, while the second piece is a standard Lagrangian, integrated over $\mathrm{d}t$. So, to find the Hamiltonian, we have to integrate over $\mathrm{d}\sigma$ the conjugate momentum Π_i to X^i.

where we used (2.2.8) and, looking at (2.2.7), the Lagrangian[3]

$$L = \frac{1}{4\pi\alpha'}\int_0^l \mathrm{d}\sigma\,\left[(\partial_\tau X^i)^2 - (\partial_\sigma X^i)^2\right] - p^+\partial_\tau q^- . \tag{2.2.10}$$

2.3 Lightcone Quantization.

Observing that the conjugate variables (X^i, Π_i) and $(q^-, p_- = -p^+)$ satisfy canonical Poisson bracket relations, we can promote them to operators acting on a Hilbert space, with commutation relations

$$\left[\alpha^i_m, \alpha^j_n\right] = \left[\tilde{\alpha}^i_m, \tilde{\alpha}^j_n\right] = m\delta_{n+m}\delta^{ij} \tag{2.3.1}$$

$$\left[x^i, p^i\right] = i\delta^{ij}, \quad \left[p^+, q^-\right] = i. \tag{2.3.2}$$

In lightcone gauge, we completely removed the oscillators α^+_n and $\tilde{\alpha}^+_n$, while the α^-_n $(\tilde{\alpha}^-_n)$ are functions of α^i_n $(\tilde{\alpha}^i_n)$ via eq. (2.2.5). Since $\left[\alpha^i_{n-m}, \alpha^i_m\right] \neq 0$ for $n = 0$, there's a normal ordering ambiguity in the definition of α^-_0 $(\tilde{\alpha}^-_0)$. We can relate α^-_n to the normal ordered expression by

$$\begin{aligned} \alpha^-_n &= \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}\left(:\sum_{m\in\mathbb{Z}} \alpha^i_{n-m}\alpha^i_m: -2a\delta_{n,0}\right) \\ \tilde{\alpha}^-_n &= \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}\left(:\sum_{m\in\mathbb{Z}} \tilde{\alpha}^i_{n-m}\tilde{\alpha}^i_m: -2\tilde{a}\delta_{n,0}\right) \end{aligned} \tag{2.3.3}$$

where a and $\tilde{a}$ are the normal ordering constants. The factor 2 appears to be consistent with the usual definition of a, that is, to be consistent with

$$\frac{1}{2}\sum_{n\neq 0}\alpha^i_{-n}\alpha^i_n = \sum_{n>0}\alpha^i_{-n}\alpha^i_n - a, \quad \frac{1}{2}\sum_{n\neq 0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n = \sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n - a \tag{2.3.4}$$

which, after regularization, gives $a = 1$ and $\tilde{a} = 1$. Further, gravitational anomaly arguments leads to $a = \tilde{a}$, so that, at the end

$$a = \tilde{a} = 1. \tag{2.3.5}$$

The oscillators α^i_m and $\tilde{\alpha}^i_m$ are creation (for $m < 0$) and annihilation (for $m > 0$) operators and $((x^i, p_i), (q^-, p_-))$ form 25 Heisenberg pairs. Therefore, the vacuum is $|0, k\rangle$, where k has 25 components, being eigenvalues of (p_-, p^i). Then, physical states are created by applying creation operators $\alpha^i_{m<0}$ and $\tilde{\alpha}^i_{m<0}$ in all possible ways.

Moreover, after defining the *transverse number operators*

$$\tilde{N}_\perp \equiv \sum_{i=2}^{25}\sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n, \quad N_\perp \equiv \sum_{i=2}^{25}\sum_{n>0}\alpha^i_{-n}\alpha^i_n, \tag{2.3.6}$$

[3]Here we mean the standard Lagrangian, and *not* the Lagrangian density.

a careful rewriting of (2.3.3) leads to the *level matching condition*

$$\tilde{N}_\perp = N_\perp, \tag{2.3.7}$$

and to the *mass-shell condition*

$$M^2 = \frac{2}{\alpha'}(\tilde{N}_\perp + N_\perp - 2). \tag{2.3.8}$$

Proof. Let's consider (2.3.3) for $n = 0$, and use (2.1.9), i.e., $\alpha_0^- = \tilde{\alpha}_0^- = \sqrt{\frac{\alpha'}{2}}p^-$ and $\alpha_0^i = \tilde{\alpha}_0^i = \sqrt{\frac{\alpha'}{2}}p^i$. We get

$$\begin{aligned}
\alpha' p^- p^+ &= \frac{\alpha'}{2}p^i p^i + :\sum_{n\neq 0}\alpha_{-n}^i\alpha_n^i: -2 = \frac{\alpha'}{2}p^i p^i + 2\sum_{n>0}\alpha_{-n}^i\alpha_n^i - 2,\\
\alpha' p^- p^+ &= \frac{\alpha'}{2}p^i p^i + :\sum_{n\neq 0}\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i: -2 = \frac{\alpha'}{2}p^i p^i + 2\sum_{n>0}\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i - 2,
\end{aligned} \tag{2.3.9}$$

where we used (2.3.5) and the definition of normal ordering. Therefore, by means of (2.3.6), we get $\tilde{N}_\perp = N_\perp$. Further, dividing by α' and summing the above equations yields

$$2p^+p^- = p^i p^i + \frac{2}{\alpha'}(N_\perp + \tilde{N}_\perp) - \frac{4}{\alpha'}, \tag{2.3.10}$$

which can be used to compute

$$M^2 = -p_\mu p^\mu = 2p^+p^- - p^i p^i = \frac{2}{\alpha'}(\tilde{N}_\perp + N_\perp - 2), \tag{2.3.11}$$

as claimed.

2.4 Compactification on S^1.

Let's now compactify the *closed string* on $\mathcal{M}_{26}$ on $\mathcal{M}_{25} \times S^1$. We'll refer to lightcone quantization, with the only difference being the name of the indices. In particular, we'll denote with $\hat{\mu}$ the indices of $\mathcal{M}_{26}$, with $\hat{\mu} = 0, \dots, 25$, and with μ the indices of $\mathcal{M}_{25}$, with $\mu = 0, \dots, 24$. This means that we can write $X^{\hat{\mu}} = (X^\mu, X^{25})$. Similarly, the transversal indices will be called $\hat{i}$, with $\hat{i} = 2, \dots 25$, while i will denote the transversal indices in the non-compact space $\mathcal{M}_{25}$, with $i = 2, \dots 24$. Therefore, in spacetime lightcone coordinates, we have $X^{\hat{\mu}} \to (X^\pm, X^{\hat{i}}) = (X^\pm, X^i, X^{25})$.

The dynamics of the string oscillations is *locally* identical for the theories on $\mathcal{M}_{26}$ and $\mathcal{M}_{25}\times S^1$. Indeed, the difference is in a global effect, i.e., the identification $X^{25} \simeq X^{25} + 2\pi R$. Therefore, the local dynamics is still described by the action (2.2.7), that is,

$$S = \frac{1}{4\pi\alpha'}\int \mathrm{d}\tau\,\mathrm{d}\sigma \left((\partial_\tau X^{\hat{i}})^2 - (\partial_\sigma X^{\hat{i}})^2\right) - \int \mathrm{d}\tau p^+ \partial_\tau q^-, \tag{2.4.1}$$

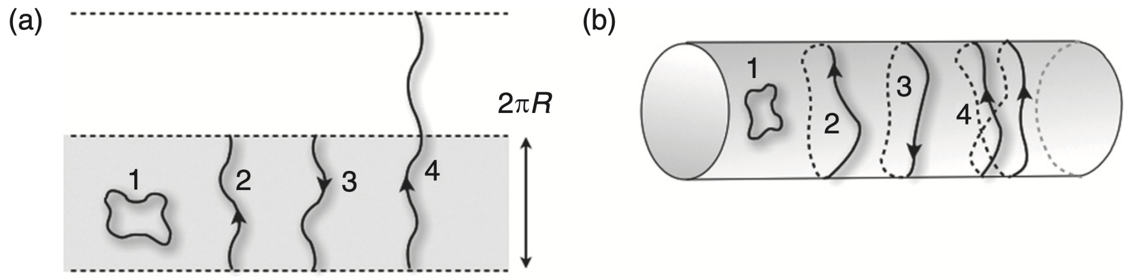

Figure 2.1. Closed string states in S^1 compactifications. The closed string 1 is closed already in flat $26d$ space, and has no winding. Closed strings 2, 3 and 4 are closed only after periodic identification defining the circle. States 2 and 3 have opposite winding numbers $\omega = \pm 1$, and state 4 has a multiple winding $\omega = 2$.

with the difference being the boundary conditions of the worldsheet fields. Then, in light-cone quantization, the relevant degrees of freedom are still $X^{\hat{i}}(\tau, \sigma)$, $\hat{i} = 2, \ldots, 25$, with Hamiltonian (2.2.9), i.e.,

$$H = \frac{1}{2} \int_0^l \mathrm{d}\sigma \left[2\pi\alpha' \Pi_{\hat{i}} \Pi_{\hat{i}} + \frac{1}{2\pi\alpha'} \partial_\sigma X^{\hat{i}} \partial_\sigma X^{\hat{i}} \right], \tag{2.4.2}$$

To express it in terms of the oscillator modes, we first need to specify the boundary conditions. In particular, writing $X^{\hat{i}} = (X^i, X^{25})$, we have the usual periodic boundary condition for the X^i with $i = 2, \ldots, 24$, while X^{25} can have a more general boundary condition due to the periodicity $X^{25} \simeq X^{25} + 2\pi R$ in the internal space. Specifically,

$$X^i(\tau, \sigma + l) = X^i(\tau, \sigma), \quad i = 2, \ldots, 24 \tag{2.4.3a}$$

$$X^{25}(\tau, \sigma + l) = X^{25}(\tau, \sigma) + 2\pi R\omega, \quad \omega \in \mathbb{Z}. \tag{2.4.3b}$$

For $\omega = 0$, it describes the usual boundary condition, while for $\omega \neq 0$ it describes a closed string winding around S^1 ω-times, as showed in figure 2.1.

Since the X^i with $i = 2, \ldots, 24$ has the same boundary conditions as before, their mode expansion is unchanged, i.e, is given by (2.1.6), with $\mu = i$ and $\alpha_0^i = \tilde{\alpha}_0^i = \sqrt{\frac{\alpha'}{2}} p^i$, that is,

$$X_L^i(\xi^+) = \frac{x^i}{2} + \frac{\alpha'\pi}{l} p^i \xi^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\tilde{\alpha}_n^i}{n} e^{-\frac{2\pi i}{l} n \xi^+}, \tag{2.4.4a}$$

$$X_R^i(\xi^-) = \frac{x^i}{2} + \frac{\alpha'\pi}{l} p^i \xi^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^i}{n} e^{-\frac{2\pi i}{l} n \xi^-}. \tag{2.4.4b}$$

Conversely, the new boundary condition for X^{25} is compatible with a different expansion. In particular, recalling the generic expansion (2.1.7), we find

$$X^{25}(\tau, \sigma) = x^{25} + \sqrt{2\alpha'} \frac{\pi}{l} (\tilde{\alpha}_0^{25} + \alpha_0^{25}) \tau + \sqrt{2\alpha'} \frac{\pi}{l} (\tilde{\alpha}_0^{25} - \alpha_0^{25}) \sigma + \text{oscillators}, \tag{2.4.5}$$

so that, after $\sigma \to \sigma + l$, in order to have $\delta X^{25} = 2\pi R\omega$, we must impose

$$\delta X^{25} = \sqrt{2\alpha'}\frac{\pi}{l}(\tilde{\alpha}_0^{25} - \alpha_0^{25})l \overset{!}{=} 2\pi R\omega \iff \tilde{\alpha}_0^{25} - \alpha_0^{25} = \sqrt{\frac{2}{\alpha'}}R\omega, \quad \omega \in \mathbb{Z}. \tag{2.4.6}$$

Moreover, since we identified $X^{25} \sim X^{25} + 2\pi R\omega$, in order for the wavefunction $e^{ip_{25}X^{25}}$ to be single-valued the momentum must be quantized, i.e,

$$p_{25} = \frac{s}{R}, \quad s \in \mathbb{Z}. \tag{2.4.7}$$

Then, recalling the expression for the Noether's charge associated to spacetime translations, eq. (2.1.8), we get

$$p^{25} = \frac{1}{\sqrt{2\alpha'}}(\tilde{\alpha}_0^{25} + \alpha_0^{25}) \overset{!}{=} \frac{s}{R} \iff \tilde{\alpha}_0^{25} + \alpha_0^{25} = \sqrt{2\alpha'}\frac{s}{R}, \quad s \in \mathbb{Z}. \tag{2.4.8}$$

Solving (2.4.6) and (2.4.8) for $\tilde{\alpha}_0^{25}$ and α_0^{25}, we get

$$\tilde{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(\frac{s}{R} + \frac{\omega R}{\alpha'}\right), \quad \alpha_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(\frac{s}{R} - \frac{\omega R}{\alpha'}\right), \quad \omega, s \in \mathbb{Z}. \tag{2.4.9}$$

Now, we can perform the same computation which led to (2.3.7) and (2.3.8), to find that the *level-matching* condition reads

$$\tilde{N}_\perp - N_\perp + s\omega = 0, \quad s, \omega \in \mathbb{Z}, \tag{2.4.10}$$

where the transverse number operators (2.3.6) are defined as before, i.e.,

$$\tilde{N}_\perp = \sum_{n>0} \tilde{\alpha}_{-n}^{\hat{i}}\tilde{\alpha}_n^{\hat{i}}, \quad N_\perp = \sum_{n>0} \alpha_{-n}^{\hat{i}}\alpha_n^{\hat{i}}. \tag{2.4.11}$$

Further, *mass-shell* condition in *the* 25d *space* $\mathcal{M}_{25}$ reads[4]

$$M^2 = -p_\mu p^\mu = \frac{s^2}{R^2} + \frac{\omega^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N_\perp + \tilde{N}_\perp - 2). \tag{2.4.12}$$

Proof. Once again, the starting point is the relation (2.3.3) for $n = 0$, which, using $\tilde{\alpha}_0^- = \alpha_0^- = \sqrt{\frac{\alpha'}{2}}p^-$ and (2.3.5), reads

$$p^+p^- = \frac{1}{\alpha'}\left(\alpha_0^{\hat{i}}\alpha_0^{\hat{i}} + :\sum_{n\neq 0}\alpha_{-n}^{\hat{i}}\alpha_n^{\hat{i}}: -2\right) = \frac{1}{\alpha'}\left(\alpha_0^i\alpha_0^i + \alpha_0^{25}\alpha_0^{25} + 2\sum_{n>0}\alpha_{-n}^{\hat{i}}\alpha_n^{\hat{i}} - 2\right)$$
$$p^+p^- = \frac{1}{\tilde{\alpha}'}\left(\tilde{\alpha}_0^{\hat{i}}\tilde{\alpha}_0^{\hat{i}} + :\sum_{n\neq 0}\tilde{\alpha}_{-n}^{\hat{i}}\tilde{\alpha}_n^{\hat{i}}: -2\right) = \frac{1}{\tilde{\alpha}'}\left(\tilde{\alpha}_0^i\tilde{\alpha}_0^i + \tilde{\alpha}_0^{25}\tilde{\alpha}_0^{25} + 2\sum_{n>0}\tilde{\alpha}_{-n}^{\hat{i}}\tilde{\alpha}_n^{\hat{i}} - 2\right).$$

Using, now (2.4.11) and

$$\alpha_0^i = \tilde{\alpha}_0^i = \sqrt{\frac{\alpha'}{2}}p^i, \quad \tilde{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(\frac{s}{R} + \frac{\omega R}{\alpha'}\right), \quad \alpha_0^{25} = \sqrt{\frac{\alpha'}{2}}\left(\frac{s}{R} - \frac{\omega R}{\alpha'}\right),$$

[4] Pay attention to the fact that we're only focusing on $\mathcal{M}_{25}$. Indeed, we compute $-p_\mu p^\mu$, with $\mu = 0, \ldots, 24$, *not* $-p_{\hat{\mu}}p^{\hat{\mu}}$, with $\hat{\mu} = 0, \ldots, 25$.

we get

$$p^+p^- = \frac{1}{2}p^ip^i + \frac{1}{2}\left(\frac{s}{R} - \frac{\omega R}{\alpha'}\right)^2 + \frac{2}{\alpha'}N_\perp - \frac{2}{\alpha'},$$

$$p^+p^- = \frac{1}{2}p^ip^i + \frac{1}{2}\left(\frac{s}{R} + \frac{\omega R}{\alpha'}\right)^2 + \frac{2}{\alpha'}\tilde{N}_\perp - \frac{2}{\alpha'}.$$

Then, by comparison, the new level-matching condition is

$$\frac{1}{2}\left(\frac{s}{R} - \frac{\omega R}{\alpha'}\right)^2 + \frac{2}{\alpha'}N_\perp = \frac{1}{2}\left(\frac{s}{R} + \frac{\omega R}{\alpha'}\right)^2 + \frac{2}{\alpha'}\tilde{N}_\perp \iff \tilde{N}_\perp - N_\perp + s\omega = 0,$$

in agreement with (2.4.10). Finally, to find the mass-shell condition, we sum the above relations to obtain

$$2p^+p^- = p^ip^i + \frac{1}{2}\left(\frac{s}{R} + \frac{\omega R}{\alpha'}\right)^2 + \frac{1}{2}\left(\frac{s}{R} - \frac{\omega R}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N_\perp + \tilde{N}_\perp - 2),$$

so that

$$M^2 = -p_\mu p^\mu = 2p^+p^- - p^ip^i = \frac{s^2}{R^2} + \frac{\omega^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N_\perp + \tilde{N}_\perp - 2), \tag{2.4.13}$$

in agreement with (2.4.12).

Notice that in old covariant quantization, those results would have been related to Virasoro constraints, with N and $\tilde{N}$ the usual operators. In particular, recall that the first constraint for a physical state is $(L_0 - a)\,|\phi\rangle$, $(\tilde{L}_0 - \tilde{a})\,|\phi\rangle = 0$, while the level matching condition reads $(L_0 - \tilde{L}_0)\,|\phi\rangle = 0$, with $L_0 = \frac{\alpha_0^2}{2} + N$ and $\tilde{L}_0 = \frac{\tilde{\alpha}_0^2}{2} + \tilde{N}$.

From the mass formula (2.4.12) we observe that for $s = \omega = 0$ we recover the previous formula (2.3.8). This is compatible with the *decompactification limit* $R \to \infty$, where the *winding modes* become infinitely massive and decouple, while the *Kaluza-Klein* modes can have continuous momentum values.

For $\omega = 0$, $s \neq 0$, we have the usual Kaluza-Klein tower of massive excitations (1.1.7), with

$$m_s^2 = M^2 + \frac{s^2}{R^2}, \quad M^2 = \frac{2}{\alpha'}(N_\perp + \tilde{N}_\perp - 2). \tag{2.4.14}$$

However, there's a new feature with respect to the field theory compactification, that is, the *winding sector* $\omega \neq 0$, $s = 0$. Here, the winding states have mass $\omega^2 R^2/\alpha'^2$, which reflects the physical intuition that winding costs energy due to the string tension. This is a pure string effect. Indeed, taking the limit $R \to 0$, the Kaluza-Klein modes decouples, and for a field theory compactification we'd recover the d dimensional theory. However, here we have the winding states too, which become light in the limit, and contribute to the low-energy spectrum.

For future convenience, let's re-derive everything in terms of left and right movers, independently. Define

$$\tilde{\alpha}_0^{25} \equiv \sqrt{\frac{\alpha'}{2}}p_L, \quad \alpha_0^{25} \equiv \sqrt{\frac{\alpha'}{2}}p_R, \quad p_L \equiv \left(\frac{s}{R} + \frac{\omega R}{\alpha'}\right), \quad p_R \equiv \left(\frac{s}{R} - \frac{\omega R}{\alpha'}\right), \tag{2.4.15}$$

so that we can divide the expansion of X^{25} into a left and a right moving part, i.e,

$$\begin{aligned} X^{25}(\tau,\sigma) &= X_L^{25}(\xi^+) + X_R^{25}(\xi^-) && \text{(2.4.16a)} \\ X_L^{25}(\xi^+) &= \frac{x^{25}}{2} + \frac{\alpha'\pi}{l} p_L \xi^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\tilde{\alpha}_n^{25}}{n} e^{-\frac{2\pi i}{l} n\xi^+} && \text{(2.4.16b)} \\ X_R^{25}(\xi^-) &= \frac{x^{25}}{2} + \frac{\alpha'\pi}{l} p_R \xi^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^{25}}{n} e^{-\frac{2\pi i}{l} n\xi^-}. && \text{(2.4.16c)} \end{aligned}$$

Without delving into the details, the Hamiltonian of the system splits into a left and a right part, which are completely independent. So, we can carry out the quantization of the left and right moving coordinates independently, with mass-shell conditions on $\mathcal{M}_{25}$

$$M_L^2 = \frac{{p_L}^2}{2} + \frac{2}{\alpha'}(\tilde{N}_\perp - 1), \quad M_R^2 = \frac{{p_R}^2}{2} + \frac{2}{\alpha'}(N_\perp - 1), \tag{2.4.17}$$

and only at the end combine the two sectors using the level-matching condition

$$M_L^2 = M_R^2, \quad M^2 = M_L^2 + M_R^2. \tag{2.4.18}$$

This implies that the $2d$ field theory of purely left moving and purely right moving fields make sense independently.

Let's finally study the spectrum. The vacuum for $\mathcal{M}_{26}$ was $\left|0,\tilde{0};k^{\hat{\mu}}\right\rangle$, with $k^{\hat{\mu}} \in \mathcal{M}_{26}$, $\hat{\mu} = 0,\ldots 25$. In our case, it is $\left|0,\tilde{0};k^\mu,s,\omega\right\rangle$, with $k^\mu \in \mathcal{M}_{25}$, $\mu = 0,\ldots,24$, and $s,\omega \in \mathbb{Z}$. In particular

$$\begin{aligned} \hat{p}^\mu \left|0,\tilde{0};k^\mu,s,\omega\right\rangle &= k^\mu \left|0,\tilde{0};k^\mu,s,\omega\right\rangle, \\ p_{L/R} \left|0,\tilde{0};k^\mu,s,\omega\right\rangle &= \left(\frac{s}{R} \pm \frac{\omega R}{\alpha'}\right) \left|0,\tilde{0};k^\mu,s,\omega\right\rangle. \end{aligned} \tag{2.4.19}$$

Then, the spectrum is built by acting in all possible ways with $\alpha^{\hat{i}}_{n<0}$ and $\alpha^{\hat{i}}_{n<0}$. In particular, the massless spectrum corresponds to $s = 0, \omega = 0, \tilde{N} = N = 1$. The massless states are, then,

$$\alpha^{\hat{i}}_{-1}\tilde{\alpha}^{\hat{j}}_{-1} \left|0,\tilde{0};k^\mu,s=0,\omega=0\right\rangle. \tag{2.4.20}$$

To simplify the notation, let's denote the vacuum simply by $|s,\omega\rangle$. Before compactification, the symmetry group of $\mathcal{M}_{26}$ was $SO(1,25)$, and the little group for massless representations $SO(24)$[5]. The states (2.4.20) transform under the representation $\mathbf{24}\otimes\mathbf{24}$ of $SO(24)$. This is reducible into $\mathbf{24}\otimes\mathbf{24} = (\mathbf{2}) \oplus [\mathbf{2}] \oplus \mathbf{1}$. These correspond to $G_{(\hat{\mu}\hat{\nu})}(X^{\hat{\mu}})$, $B_{[\hat{\mu}\hat{\nu}]}(X^{\hat{\mu}})$, and $\phi(X^{\hat{\mu}})$, respectively. We use the indices $\hat{\mu}$ instead of $\hat{i}$ because of the presence of gauge symmetries, which allow preserving manifest gauge invariance of the fields.

After compactification, the index $\hat{i}$ can be either i or 25, so there are various possibilities. First, we can have

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1}\left|0,0\right\rangle, \tag{2.4.21}$$

which correspond to the graviton, Kalb-Ramond and scalar fields in the non-compact $\mathcal{M}_{25}$ space. The symmetry group is $SO(1,24)$, the massless little group is $SO(23)$[6] and the

[5]Indeed, $\hat{i} = 2,\ldots,25$ can take 24 different values.
[6]Indeed, $i = 2,\ldots,24$ can take 23 different values.

above state transforms under the representation $\mathbf{23} \otimes \mathbf{23} \simeq (\mathbf{2}) \oplus [\mathbf{2}] \oplus \mathbf{1}$ of $SO(23)$, which correspond to the *graviton* $G_{(\mu\nu)}(X^\mu)$, the *Kalb-Ramond* $B_{[\mu\nu]}(X^\mu)$ and the *dilaton* $\phi(X^\mu)$, respectively. Recall that we're looking at massless Kaluza-Klein modes here, so we can safely conclude that those fields depend only on the non-compact coordinates X^μ.

Further, we can have the states

$$\alpha^i_{-1}\tilde{\alpha}^{25}_{-1}\,|0,0\rangle\,, \quad \alpha^{25}_{-1}\tilde{\alpha}^i_{-1}\,|0,0\rangle\,, \tag{2.4.22}$$

which represents $25d$ gauge bosons. In particular, the symmetric combination $(\alpha^i_{-1}\tilde{\alpha}^{25}_{-1} + \alpha^{25}_{-1}\tilde{\alpha}^i_{-1})\,|0,0\rangle$ is called *graviphoton* and corresponds to the decomposition of the $26d$ metric, that is, to $G_{\mu 25}(X^\mu)$. It is the same gauge vector we found after compactification of the Einstein theory. On the other hand, the antisymmetric combination $(\alpha^i_{-1}\tilde{\alpha}^{25}_{-1} - \alpha^{25}_{-1}\tilde{\alpha}^i_{-1})\,|0,0\rangle$ is called *Kalb-Ramond photon* and corresponds to the decomposition of the $26d$ Kalb-Ramond, that is, to $B_{\mu 25}(X^\mu)$.

Finally, we get a scalar

$$\alpha^{25}_{-1}\tilde{\alpha}^{25}_{-1}\,|0\rangle\,, \tag{2.4.23}$$

which corresponds to the scalar component $G_{25\,25}(X^\mu)$ after compactification. It is related to the radius of the extra-dimension, as discussed before, around eq. (1.2.9).

To sum up, everything is coherent with the expected decomposition of the fields under $\mathcal{M}_{26} \to \mathcal{M}_{25} \times S^1$,

$$\begin{aligned} G_{\hat{\mu}\hat{\nu}}(X^{\hat{\mu}}) &\to \{G_{\mu\nu}(X^\mu), G_{\mu 25}(X^\mu), G_{25\,25}(X^i)\}, \\ B_{\hat{\mu}\hat{\nu}}(X^{\hat{\mu}}) &\to \{B_{\mu\nu}(X^\mu), B_{\mu 25}(X^\mu)\}, \\ \phi(X^{\hat{\mu}}) &\to \phi(X^\mu), \end{aligned} \tag{2.4.24}$$

where we only focused on the massless Kaluza-Klein modes.

Without delving into the technical details, let's just mention a few words about a pure string effect, which is *enhancement*.

2.5 Enhancement.

Looking at the vector bosons

$$(\alpha^i_{-1}\tilde{\alpha}^{25}_{-1} + \alpha^{25}_{-1}\tilde{\alpha}^i_{-1})\,|0,0\rangle\,, \quad (\alpha^i_{-1}\tilde{\alpha}^{25}_{-1} - \alpha^{25}_{-1}\tilde{\alpha}^i_{-1})\,|0,0\rangle\,, \tag{2.5.1}$$

their $U(1)$ gauge symmetry is a consequence of the compactification. This was previously discussed for the metric in Hilbert-Einstein theory and can be generalized for a two-form.

One can verify that for the particular value $R = \sqrt{\alpha'}$, there are 4 additional gauge bosons, which can *enhance* the $U(1) \times U(1)$ gauge symmetry, up to $SU(2) \times SU(2)$, together with 8 additional scalar fields coupled to them. This can be checked at the level of interactions, and it's beyond our purposes. Let's just observe that (2.4.12) and (2.4.10) can be simultaneously satisfied with $M^2 = 0$ for the following values, which lead to the above-mentioned additional fields,

$$s=\omega=\pm 1, N_\perp=0, \tilde{N}_\perp=1 \implies \tilde{\alpha}^i_{-1}\left|\pm 1,\pm 1\right\rangle, \quad \tilde{\alpha}^{25}_{-1}\left|\pm 1,\pm 1\right\rangle \tag{2.5.2a}$$

$$s=-\omega=\pm 1, N_\perp=1, \tilde{N}_\perp=0 \implies \alpha^i_{-1}\left|\pm 1,\mp 1\right\rangle, \quad \alpha^{25}_{-1}\left|\pm 1,\mp 1\right\rangle \tag{2.5.2b}$$

$$s=\pm 2, \omega=0, N_\perp=\tilde{N}_\perp=0 \implies \left|\pm 2,0\right\rangle \tag{2.5.2c}$$

$$s=0, \omega=\pm 2, N_\perp=\tilde{N}_\perp=0 \implies \left|0,\pm 2\right\rangle. \tag{2.5.2d}$$

One can show that the four additional massless vector, together with (2.4.22) are in the adjoint representation of $SU(2)\times SU(2)$, while the eight scalars, together with (2.4.23), form the $(3,3)$ representation of the same group.

We could've started from the $26d$ effective field theory of the bosonic string on $\mathcal{M}_{26}$ and then compactify it on $\mathcal{M}_{25}\times S^1$. However, this would've been a good approximation only in the *large volume approximation*, i.e.,

$$E \ll \frac{1}{R} \ll M_s, \quad \frac{\alpha'}{R^2} \ll 1. \tag{2.5.3}$$

Indeed, the latter would've missed the feature of winding states, since for large R they decouple, as can be seen from the mass-shell (2.4.12). Then, we can write down another effective field theory which is a good approximation for $R \simeq \sqrt{\alpha'}$ by considering the massless spectrum we found for $R=\sqrt{\alpha'}$. This theory includes gravity, non-abelian $SU(2)\times SU(2)$ interactions and 9 scalars, coupled to the gauge bosons. However, we should pay attention in doing so. Indeed, as R varies away from $\sqrt{\alpha'}$, some gauge bosons and scalars get masses, leading to a sort of *stringy Higgs mechanism*. In particular, we have the breaking of the gauge symmetry $SU(2)\times SU(2) \to U(1)\times U(1)$, triggered by the radius of the extra dimension, given by the V.E.V of the modulus (2.4.23).

Without focusing on those details, let's study another interesting, purely stringy, effect, which is T-duality.

2.6 T-duality.

From the discussion above, it could seem that the space of physically inequivalent S^1 compactifications is parametrized by $R/\alpha' \in (0,\infty)$. However, inspection of the mass formula (2.4.12) reveals that the spectrum is invariant under the so-called *T-duality transformation*

$$R \to R' = \frac{\alpha'}{R}, \quad (s,\omega)\to(s',\omega')=(\omega,s), \tag{2.6.1}$$

which has fixed point $R=\sqrt{\alpha'}$. Then, the spectrum is completely characterized by the values $R\geq\sqrt{\alpha'}$, or equivalently, by $0<R\leq\sqrt{\alpha'}$.

A consequence of T-duality is that the $R\to 0$ limit corresponds to a decompactification limit of the T-dual theory, in which $R'\to\infty$, so the infinite tower of winding states becoming light in the $R\to 0$ limit are interpreted as an infinite tower of Kaluza-Klein modes becoming light in the $R'\to\infty$ limit.

One could wonder if T-duality is an accidental property of the spectrum, rather than a symmetry of the full theory. It's possible to show that the T-dual theories are described by the very same worldsheet theory, so they're physically equivalent. To see this, recall that we can describe the left movers and the right movers independently, through the coordinates $X_L^{\hat{i}}(\xi^+)$ and $X_R^{\hat{i}}(\xi^-)$. Out of them, there are two different, but at the end physically equivalent, ways of constructing spacetime coordinates $X^{\hat{i}}(\tau, \sigma)$ out of them.

Indeed, looking at the T-duality transformation (2.6.1), it acts on the momenta $p_{L/R}$ as

$$\begin{aligned} p_L &= \frac{n}{R} + \frac{\omega R}{\alpha'} \to \frac{\omega R}{\alpha'} + \frac{n}{R} = p_L \quad \implies \tilde{\alpha}_0^{25} \to \tilde{\alpha}_0^{25}, \\ p_R &= \frac{n}{R} - \frac{\omega R}{\alpha'} \to \frac{\omega R}{\alpha'} - \frac{n}{R} = -p_R \implies \alpha_0^{25} \to -\alpha_0^{25}. \end{aligned} \tag{2.6.2}$$

This suggests us to define a new set of coordinate fields

$$\begin{aligned} Y^i &= X^i, \quad i = 2 \ldots, 24, \\ Y^{25} &= X_L^{25}(\xi^+) - X_R^{25}(\xi^-), \end{aligned} \tag{2.6.3}$$

so that the mode expansion of Y^{25} differs from the one of X^{25}, (2.4.16), by the transformation (2.6.2). Then, looking again at (2.4.16) and using (2.4.15) and (2.6.1), after a transformation $\sigma \to \sigma + l$, we get

$$\begin{aligned} Y^{25}(\tau, \sigma) \to Y^{25}(\tau, \sigma + l) = X_L^{25}(\xi^+ + l) - X_R^{25}(\xi^- - l) &= Y^{25}(\tau, \sigma) + \alpha'\pi(p_L + p_R) \\ &= Y^{25}(\tau, \sigma) + \frac{2\pi\alpha' s}{R} = Y^{25}(\tau, \sigma) + 2\pi R' s, \end{aligned} \tag{2.6.4}$$

So that we can treat Y^{25} exactly as before, which means that X^{25} and Y^{25} describe string compactifications on radii R and $R' = \alpha'/R$, respectively.

One can show that X and Y have the same energy-momentum tensor and the same operator product expansions, so that the conformal field theories of X and Y are identical. This proves that T-duality is a symmetry of the bosonic string theory and not just of its spectrum.

This construction suggests that somehow spacetime is a secondary concept in string theory, and that it's derived from more fundamental concepts like the worldsheet theory. As argued by Uranga [2], what this tells us about the nature of spacetime is still unclear.

Type II String

We use mostly plus metric $\eta = \text{diag}(-,+,\dots,+)$ and focus on closed strings. We consider the critical string, with background $\mathcal{M}_{10}$. The worldsheet coordinates are ξ^a, $a = 1,2$, where $\sigma \in (0,l)$.

3.1 Ramond-Neveu-Schwarz Superstring.

The supersymmetric extension of Polyakov action (2.1.1) is reached by enlarging the field content of the theory. In particular, the idea is to supersymmetrise and couple the 10 bosonic fields on the worldsheet, $X^\mu(\xi^a)$, $\mu = 0,\dots,9$, to two-dimensional supergravity. As a result, we obtain additional worldsheet spinors, which are the superpartners of the X^μ, and will be denoted by $\psi^\mu_\alpha(\xi^a)$, where α are the spinorial indices. They are *Majorana-Weyl spinors.*

In particular, in dimension $d = 2$ the Clifford algebra reads

$$\{\gamma^a, \gamma^b\}_{\alpha\beta} = 2\eta^{ab}\mathbb{1}_{\alpha\beta}, \tag{3.1.1}$$

with $a, b = 0,1$ are the worldsheet indices, while $\alpha, \beta = 1,2$ the spinorial indices. Indeed, in $d = 2$, the spinor representation turns out to be two-dimensional as well. A basis for the γ-matrices is

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.1.2}$$

Further, the Majorana condition is equivalent to the requirement that the spinors are real i.e.,

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} \psi_+^* \\ \psi_-^* \end{pmatrix} = \psi^*, \tag{3.1.3}$$

while the chirality distinguishes between the two inequivalent Weyl representations. In particular, defining the chirality operator $\gamma = \gamma^0\gamma^1$, we have

$$\gamma \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix} = \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix}, \quad \gamma \begin{pmatrix} 0 \\ \psi_- \end{pmatrix} = -\begin{pmatrix} 0 \\ \psi_- \end{pmatrix}. \tag{3.1.4}$$

The two condition are compatible for $d = 2 \mod 8$, dimensions in which Majorana-Weyl spinors exist.

Skipping the details, the classical RNS action adds a Majorana-Weyl spinor for each scalar field. After gauge-fixing to $\gamma_{ab} = \eta_{ab}$, and considering a flat spacetime metric $g_{\mu\nu} = \eta_{\mu\nu}$, it reads

$$S = -\frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\xi \left(\frac{1}{\alpha'}\partial_a X^\mu \partial^a X_\mu + i\bar{\psi}^\mu_A \gamma^a_{AB}\partial_a \psi_\mu\right), \tag{3.1.5}$$

where the spinor conjugate is defined by

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 = \psi^T \gamma^0 = (-\psi_-, \psi_+). \tag{3.1.6}$$

It's equations of motion are

$$\partial_a \partial^a X^\mu = 0, \quad \gamma^a \partial_a \psi^\mu = 0. \tag{3.1.7}$$

Taking *worldsheet lightcone coordinates*, $\xi^\pm = \tau \pm \sigma$, the action (3.1.5) reads

$$S = \frac{1}{\pi}\int \mathrm{d}^2\xi \left(\frac{1}{\alpha'}\partial_+ X \cdot \partial_- X + \frac{i}{2}(\psi_+ \cdot \partial_- \psi_+ + \psi_- \cdot \partial_+ \psi_-)\right), \tag{3.1.8}$$

while the equations of motion (3.1.7) become

$$\partial_+ \partial_- X^\mu = 0, \quad \partial_- \psi^\mu_+ = \partial_+ \psi^\mu_- = 0. \tag{3.1.9}$$

This means that in *spacetime lightcone coordinates* we have

$$X^\mu(\xi^\pm) = X^\mu_L(\xi^+) + X^\mu_R(\xi^-), \quad \psi^\mu_+(\xi^\pm) = \psi^\mu_+(\xi^+), \quad \psi^\mu_-(\xi^\pm) = \psi^\mu_-(\xi^-). \tag{3.1.10}$$

The residual symmetries after gauge fixing the superconformal symmetry have conserved currents

$$\begin{aligned} T_{\pm\pm} &= -\frac{1}{\alpha'}\partial_\pm X \cdot \partial_\pm X - \frac{i}{2}(\psi^\mu)_\pm \partial_\pm (\psi_\mu)_\pm, \\ J_\pm &= -\sqrt{\frac{1}{2\alpha'}}(\psi^\mu)_\pm \partial_\pm X_\mu. \end{aligned} \tag{3.1.11}$$

Then, *gauge-fixing* is achieved by imposing the *superconformal Virasoro constraints*

$$T_{\pm\pm} =, \quad J_\pm = 0, \tag{3.1.12}$$

on the equations of motion.

Let's turn to the mode expansion. Because of (3.1.10), for the bosonic sector the analysis is the same as before. However, while finding the equations of motion (3.1.7) from (3.1.5), other than the condition $\delta\psi^\mu(\tau_0) = \delta\psi^\mu(\tau_1) = 0$, which is what we impose in a variational principle, we must be sure that the following boundary term vanishes,

$$\delta S = \frac{1}{2\pi}\int_{\tau_0}^{\tau_1} \mathrm{d}\tau \left(\psi_+ \cdot \delta\psi_+ - \psi_- \cdot \delta\psi_-\right)\Big|_{\sigma=0}^{\sigma=l} \stackrel{!}{=} 0. \tag{3.1.13}$$

For the closed string, in which we have periodicity $\sigma \sim \sigma + l$, the above condition is satisfied for

$$\begin{aligned} \psi^\mu_+(\tau,\sigma) &= \pm\psi^\mu_+(\tau,\sigma+l), \\ \psi^\mu_-(\tau,\sigma) &= \pm\psi^\mu_-(\tau,\sigma+l), \end{aligned} \tag{3.1.14}$$

with the same conditions on $\delta\psi_\pm$. Indeed, anti-periodic boundary conditions for $\psi_\pm$ are possible since observables are built out of fermion bilinears. In particular, periodic boundary conditions are referred as *Ramond* (R) boundary conditions, while anti-periodic ones are called *Neveu-Schwartz* (NS). Therefore, fermions on the worldsheet satisfy

$$\psi(\tau, \sigma + l) = e^{2\pi i\phi}\psi(\tau, \sigma), \quad \phi = \begin{cases} 0, & \text{for R-sector} \\ \frac{1}{2}, & \text{for NS-sector} \end{cases} \tag{3.1.15}$$

where more general phases are not allowed since ψ must be real.

The conditions for the two spinor components ψ_+ and ψ_- can be chosen independently, but Lorentz invariance requires that in a given sector, fermions fields ψ^μ have the same boundary condition for all μ. This leads to a total of four possibilities: (R,R), (NS,NS), (NS,R) and (R,NS). One can see that, after quantization, modular invariance requires these different boundary conditions to coexist within the same theory. Roughly speaking, as we have to sum over different topologies to get a consistent string theory, we need to sum over different topological sectors, i.e., boundary conditions, as well. We won't focus on such details and take them for granted.

The mode expansion for the bosonic coordinates is the same as in section 2.1, while for the fermions we get

$$\begin{aligned} \psi_+^\mu(\xi^+) &= \sqrt{\frac{2\pi}{l}} \sum_{r\in\mathbb{Z}+\phi} \tilde{b}_r^\mu e^{-\frac{2\pi i}{l} r\xi^+}, \\ \psi_-^\mu(\xi^-) &= \sqrt{\frac{2\pi}{l}} \sum_{r\in\mathbb{Z}+\phi} b_r^\mu e^{-\frac{2\pi i}{l} r\xi^-}, \end{aligned} \tag{3.1.16}$$

where

$$\phi = \begin{cases} 0, & \text{for R-sector} \\ \frac{1}{2}, & \text{for NS-sector.} \end{cases} \tag{3.1.17}$$

Here, ϕ can be chosen independently for the left and right movers, and the reality of the Majorana-Weyl spinors translates into

$$(b_r^\mu)^* = b_{-r}^\mu, \quad (\tilde{b}_r^\mu)^* = \tilde{b}_{-r}^\mu. \tag{3.1.18}$$

3.2 Quantum Superstring.

We quantize as usual, by finding the canonical conjugate variables, compute their equal-time Poisson brackets and promoting the latter to commutators and anti-commutators on a Hilbert space. Then, we consider the mode expansions and work out the commutators and anti-commutators of the mode operators. The result is

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}, \tag{3.2.1a}$$

$$[\alpha_m^\mu, \tilde{\alpha}_n^\nu] = 0, \tag{3.2.1b}$$

$$\{b_m^\mu, b_n^\nu\} = \{\tilde{b}_m^\mu, \tilde{b}_n^\nu\} = \delta_{m+n}\eta^{\mu\nu} \tag{3.2.1c}$$

$$\{b_m^\mu, \tilde{b}_n^\nu\} = 0 \tag{3.2.1d}$$

$$[\alpha_m^\mu, b_n^\nu] = 0. \tag{3.2.1e}$$

Further, the reality condition of the fermions, $\psi_\pm^* = \psi_\pm$, translates into

$$(b_n^\mu)^\dagger = b_{-n}^\mu, \quad (\tilde{b}_n^\mu)^\dagger = \tilde{b}_{-n}^\mu \tag{3.2.2}$$

Without delving into the details, let's consider lightcone quantization, where we define the *spacetime lightcone coordinates*

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1), \quad \psi^\pm = \frac{1}{\sqrt{2}}(\psi^0 \pm \psi^1), \tag{3.2.3}$$

and the remaining fields are $X_L^i(\xi^+)$, $X_R^i(\xi^-)$, $\psi_+^i(\xi^+)$ and $\psi_-^i(\xi^-)$, with $i = 2, \dots 9$ in spacetime dimension $D = 10$. As in the bosonic case, we can quantize independently the left and the right moving sectors and glue them together at the end. Other than the level-matching condition, there will be additional constraints to be imposed here.

One finds the usual normal ordering constants, which are a priori different for (R) and (NS) sectors. After renormalization, and in the critical setting, we find

$$a_R = \tilde{a}_R = 0, \quad a_{NS} = \tilde{a}_{NS} = \frac{1}{2}. \tag{3.2.4}$$

In particular, a is considered to be the sum of the zero point energies of the bosons and of the fermions. The fact that $a_R = 0$ suggests that in the (R) sector, supersymmetry is globally preserved by the boundary conditions of bosons and fermions. Conversely, in the (NS) sectors, the local $2d$ supersymmetry is broken by the different boundary conditions.

Similarly to (2.3.6), the transverse number operators read

$$\begin{aligned} \tilde{N}_\perp &= \sum_{n>0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + \sum_{k \geq 0+\phi} k \tilde{b}_{-k}^i \tilde{b}_k^i, \\ N_\perp &= \sum_{n>0} \alpha_{-n}^i \alpha_n^i + \sum_{k \geq 0+\phi} k b_{-k}^i b_k^i, \end{aligned} \tag{3.2.5}$$

with ϕ given by (3.1.17), different in (R) and (NS) sectors.

For the mass-shell condition, we have a formula similar to (2.4.17)[1], with two independent contributions, M_L^2 for the left moving sector and M_R^2 for the right moving one. Since the normal ordering constants enter the mass-shell[2], we should distinguish between the two sectors. In both cases, the *level-matching condition* reads

$$M_L^2 = M_R^2. \tag{3.2.6}$$

From now on, we'll focus only on the right moving part, the equations for the left moving being the same but with tilde operators. We'll separately study the vacuum and the spectrum of the (NS) and (R) sectors, gluing them together to find the full spectrum of the closed string.

At the end, we will focus on the massless spectrum of the theory, which is made of representations of the little group $SO(8)$ of $SO(1,9)$. Let's denote by $\mathbf{8}_{\boldsymbol{v}}$ the vector

[1]Here we aren't considering compactifications yet, so $s = \omega = 0$, and $p_{L/R} = 0$.

[2]Via the normal-ordering of L_0 in old covariant quantization, or of the hamiltonian in lightcone quantization.

representation, by $\mathbf{8_s}$ the positive-chirality spinor representation and by $\mathbf{8_c}$ the negative-chirality co-spinor representation. Recall that, looking at spinors of $SO(1, D-1)$, we can have Majorana-Weyl spinors, with a number of real degrees of freedom equal to $2^{[D/2]-1}$, for $D = 2 \mod 8$. Moreover, looking at spinors of $SO(D)$, we can have real Weyl spinors for $D = 0 \mod 8$. Therefore, we denote the representations of $SO(8)$ by the number of its real degrees of freedom.

A detailed discussion can be found in the appendix of [4], from which we take only the summary tables.

$SO(1, d-1)$				
$d \mod 8$	Majorana	Weyl	Majorana-Weyl	min rep
2	Yes	Self	Yes	1
3	Yes	-	-	2
4	Yes	Complex	-	4
5	-	-	-	8
6	-	Self	-	8
7	-	-	-	16
8	Yes	Complex	-	16
$8+1=9$	Yes	-	-	16
$8+2=10$	Yes	Self	Yes	16
$8+3=11$	Yes	-	-	32
$8+4=12$	Yes	Complex	-	64

(3.2.7)

$SO(d)$			
$d \mod 8$	Real	Weyl	Real and Weyl
0	Yes	Self	Yes
1	Yes	-	-
2	Yes	Complex	-
3	Pseudo	-	-
4	Pseudo	Self	-
5	Pseudo	-	-
6	Yes	Complex	-
7	Yes	-	-

(3.2.8)

Let's now analyse the (NS) and (R) sectors, for the right movers. The formulas for the left ones are the same, substituting operators with the tilde version.

3.2.1 NS Sector.

Here, $\phi = 1/2$ and $a_{NS} = 1/2$. The *mass-shell condition* reads

$$\frac{\alpha' M_R^2}{2} = N_\perp - \frac{1}{2}, \tag{3.2.9}$$

with number operator given by (3.2.5).

The spectrum is built by defining a ground state $|0;k\rangle_{NS}$ with spacetime momentum k_i, and annihilated by all positive mode operators, i.e.,

$$\begin{aligned} b^i_{k+1/2} |0;k\rangle_{NS} &= 0, \quad \forall k \geq 0, \\ \alpha^i_n |0;k\rangle_{NS} &= 0, \quad \forall n > 0. \end{aligned} \tag{3.2.10}$$

Then, we apply negative mode operators in all possible ways. Note that there's no zero mode b^i_0. Taking for simplicity zero spacetime momentum, the lightest right moving states are

State	$\alpha' M_R^2/2$	$SO(8)$
$\|0\rangle_{NS}$	$-1/2$	$\mathbf{1}$
$b^i_{-1/2}\|0\rangle_{NS}$	0	$\mathbf{8_v}$

(3.2.11)

3.2.2 R Sector.

Here, $\phi = 0$ and $a_R = 0$. The mass-shell condition reads

$$\frac{\alpha' M_R^2}{2} = N_\perp, \tag{3.2.12}$$

with number operator given by (3.2.5).

The ground state is now $|0\rangle_R$, with $M_R^2 = 0$[3]. However, we must pay attention to the existence of zero-modes b^i_0. Indeed, since $\left[M_R^2, b^i_0\right] = 0$, the application of b^i_0 on the ground state does not change its mass. Hence, it is *degenerate*, and we must find how fermionic mode operators act on them. Indeed, we can require that all positive mode operators annihilate it, but we can't consistently require that all the fermionic zero-mode operators annihilate it, since it wouldn't be consistent with the anticommutators

$$\{b^i_0, b^j_0\} = \delta^{ij}. \tag{3.2.13}$$

In particular, the relation (3.2.13) defines a Cliffod algebra, and defining the action of b^i_0 on the degenerate ground states is equivalent to find a representation for it. To construct such representation, let's recall how to build generic representations for the Clifford algebra.

Let's focus on $SO(1,9)$. The Clifford algebra is defined by

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}, \quad \mu = 0, \dots, 9, \tag{3.2.14}$$

where we're considering an even dimension $d = 10$, $k = 4$. We group the Γ^μ into two

[3]Beware the notation! The R in the mass stands for right component, *not* Ramond.

sets of 5 anticommuting raising and lowering operators

$$\Gamma_0^{\pm} = \frac{1}{2}(\pm\Gamma^0 + \Gamma^1), \tag{3.2.15a}$$

$$\Gamma_a^{\pm} = \frac{1}{2}(\Gamma^{2a} \pm i\Gamma^{2a+1}), \quad a = 1, \ldots, 4. \tag{3.2.15b}$$

They satisfy

$$\{\Gamma_a^+, \Gamma_b^-\} = \delta_{ab}, \tag{3.2.16a}$$

$$\{\Gamma_a^+, \Gamma_b^+\} = \{\Gamma_a^-, \Gamma_b^-\} = 0, \tag{3.2.16b}$$

with $a = 0, \ldots, 4$. In particular $(\Gamma_a^+)^2 = (\Gamma_a^-)^2 = 0$. So, we can find the lowest weight state by acting repeatedly with the Γ_a^- until we reach a spinor annihilated by all of them, i.e.,

$$\Gamma_a^- |\zeta\rangle = 0, \quad \forall a. \tag{3.2.17}$$

Then, by starting from $|\zeta\rangle$, we obtain a $2^5 = 32$-dimensional representation by acting with Γ_a^+, at most once, in all possible ways. We label those states by $|s_0, s_1, s_2, s_3, s_4\rangle$, with $s_a = \pm 1/2$:

$$|s_0, s_1, s_2, s_3, s_4\rangle \equiv (\Gamma_4^+)^{s_4+1/2}(\Gamma_3^+)^{s_3+1/2}(\Gamma_2^+)^{s_2+1/2}(\Gamma_1^+)^{s_1+1/2}(\Gamma_0^+)^{s_0+1/2} |\zeta\rangle,$$

where, in particular,

$$|\zeta\rangle = \left| -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\rangle. \tag{3.2.18}$$

One can verify the Lorentz generators

$$\Sigma^{\mu\nu} = -\frac{i}{4}[\Gamma^\mu, \Gamma^\nu] \tag{3.2.19}$$

indeed satisfy the $SO(1,9)$ algebra, that is,

$$i[\Sigma^{\mu\nu}, \Sigma^{\sigma\rho}] = \eta^{\nu\sigma}\Sigma^{\mu\rho} + \eta^{\mu\rho}\Sigma^{\nu\sigma} - \eta^{\nu\rho}\Sigma^{\mu\sigma} - \eta^{\mu\sigma}\Sigma^{\nu\rho}. \tag{3.2.20}$$

In particular, the generators $\Sigma^{2a,2a+1}$ commute and can be simultaneously diagonalized. In terms of the raising and lowering operators,

$$S_a \equiv i^{\delta_a}\Sigma^{2a,2a+1} = \Gamma_a^+\Gamma_a^- - \frac{1}{2}, \tag{3.2.21}$$

so that

$$S_a |s_0, s_1, s_2, s_3, s_4\rangle = s_a |s_0, s_1, s_2, s_3, s_4\rangle. \tag{3.2.22}$$

The half-integer values show that this is indeed a spinor representation. The spinors form a $2^5 = 32$-dimensional Dirac representation of the Lorentz algebra $SO(1,9)$.

The Dirac representation is reducible as a representation of the Lorentz algebra. Indeed, because $\Sigma^{\mu\nu}$ is quadratic in the Γ matrices, the $|s_0, s_1, s_2, s_3, s_4\rangle$ with even or odd numbers of $+\frac{1}{2}$ do not mix. In particular, we can define the chirality matrix

$$\Gamma = \Gamma^0\Gamma^1 \dots \Gamma^9, \tag{3.2.23}$$

which satisfies

$$(\Gamma)^2 = 1, \quad \{\Gamma, \Gamma^\mu\} = 0, \quad [\Gamma, \Sigma^{\mu\nu}] = 0. \tag{3.2.24}$$

The eigenvalues of Γ are ± 1 and one can easily show

$$\Gamma = 2^5 S_0 S_1 S_2 S_3 S_4. \tag{3.2.25}$$

Then, as a matrix acting on $|s_0, s_1, s_2, s_3, s_4\rangle$, Γ is diagonal, with matrix element taking the value $+1$ when s_a include an even number of $-1/2$ and -1 for an odd number of $-1/2$. Its eigenvalue is called chirality, and the two $2^4 = 16$ states with definite chirality form two inequivalent Weyl representations of the Lorentz algebra. Therefore, for $d = 10$, we obtained

$$\mathbf{32}_{\text{Dirac}} = \mathbf{16} \oplus \mathbf{16'}. \tag{3.2.26}$$

A priori, the dimensionalities we mentioned above should be regarded as complex. However, in dimensions $d = 2 \mod 8$, we can define Majorana-Weyl spinors. Taking, then, real Majorana spinors from the beginning leads to real degrees of freedom. In particular

$$\mathbf{32}_{\text{Majorana}} = \mathbf{16} \oplus \mathbf{16'}, \tag{3.2.27}$$

where the dimensions are now real.

Turning back to the string, we notice that (3.2.13) satisfies (3.2.14) for $\Gamma^i = \sqrt{2} b_0^i$, and that there are no $0, 1$ gamma matrices, since we're in lightcone gauge. Basically, we'll find representations of $SO(8)$, which is the little group for massless representations of $SO(1, 9)$.

From a group theoretic point of view the possibility is twofold. On the one hand, we could've started from the old covariant quantization, with anticommutators (3.2.1c), for $m = n = 0$,

$$\{b_0^\mu, b_0^\nu\} = \eta^{\mu\nu}, \tag{3.2.28}$$

defining a Clifford algebra for $SO(1, 9)$, up to a factor $\sqrt{2}$. Then, we'd have found (3.2.27), namely

$$\mathbf{32}_{\text{Majorana}} = \mathbf{16} \oplus \mathbf{16'}, \tag{3.2.29}$$

where the degrees of freedom are real and the Majorana spinor decomposes into two set of Majorana-Weyl spinors with opposite chirality. Further, since we're interested in the little group $SO(8)$, to study the massless spectrum of the superstring, we should've considered the branching

$$\begin{aligned} SO(1,9) &\to SO(1,1) \times SO(8) \\ \mathbf{16} &\to (+, \mathbf{8_s}) \oplus (-, \mathbf{8_c}) \\ \mathbf{16'} &\to (-, \mathbf{8_s}) \oplus (+, \mathbf{8_c}), \end{aligned} \tag{3.2.30}$$

where $\pm$ denotes two different irreducible representations of $SO(1,1)$. Then we should've applied the physicality condition $G_0 |\phi\rangle = 0$, to find that the physical state is

$$|0\rangle_R = (+, \mathbf{8_s}) \oplus (+, \mathbf{8_c}). \tag{3.2.31}$$

On the other hand, and this is what we'll actually do, we can get advantage of the lightcone gauge ad directly find representations of $SO(8)$, focusing on the Clifford algebra (3.2.13). As discussed at the beginning of this section, for $SO(8)$ the reality and Weyl conditions are compatible. Indeed, we start from real gamma matrices, due to (3.1.18), and we'll find that the $2^4 = 16$-dimensional real representation will split into two inequivalent 8-dimensional real representations of definite chirality.

Let's then focus again on $\Gamma^i = \sqrt{2} b_0^i$. We haven't the matrices $\Gamma_0^\pm$ of eq. (3.2.15a), but only (3.2.15b), that is

$$B_a^\pm = \frac{1}{\sqrt{2}} \left(b_0^{2a} \pm i b_0^{2a+1} \right), \quad a = 1, \dots, 4. \tag{3.2.32}$$

Then, the lowest weight state is

$$B_a^- |0\rangle_R = 0, \quad \forall a = 1, \dots, 4, \tag{3.2.33}$$

which is an eigenstate of the spin operator such that

$$|0\rangle_R = \left| -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\rangle, \tag{3.2.34}$$

with

$$S_a |s_1, s_2, s_3, s_4\rangle = s_a |s_1, s_2, s_3, s_4\rangle. \tag{3.2.35}$$

Then, we obtain a $2^4 = 16$-dimensional real representation by application of the B_a^+ operators

state	eigenstate of $2S_a$	number of states
$\|0\rangle_R$	$\|-,-,-,-\rangle$	$\binom{4}{0} = 1$
$B_{a_1}^+ \|0\rangle_R$	$\|+,-,-,-\rangle, \|-,+,-,-\rangle,$ $\|-,-,+,-\rangle, \|-,-,-,+\rangle$	$\binom{4}{1} = 4$
$B_{a_1}^+ B_{a_2}^+ \|0\rangle_R$	$\|+,+,-,-\rangle, \|+,-,+,-\rangle, \|+,-,-,+\rangle,$ $\|-,+,+,-\rangle, \|-,+,-,+\rangle, \|-,-,+,+\rangle$	$\binom{4}{2} = 6$
$B_{a_1}^+ B_{a_2}^+ B_{a_3}^+ \|0\rangle_R$	$\|-,+,+,+\rangle, \|+,-,+,+\rangle,$ $\|+,+,-,+\rangle, \|+,+,+,-\rangle$	$\binom{4}{3} = 4$
$B_1^+ B_2^+ B_3^+ B_4^+ \|0\rangle_R$	$\|+,+,+,+\rangle$	$\binom{4}{4} = 1$

The states $(|0\rangle_R, B_{a_1}^+ B_{a_2}^+ |0\rangle_R, B_1^+ B_2^+ B_3^+ B_4^+ |0\rangle_R)$ are characterized by an even number of creation operators, so they have positive chirality and gather to form the spinor representation $\mathbf{8_s}$ of $SO(8)$. Those with negative chirality, namely $(B_{a_1}^+ |0\rangle_R, B_{a_1}^+ B_{a_2}^+ B_{a_3}^+ |0\rangle_R)$, form the co-spinor representation $\mathbf{8_c}$ of $SO(8)$. Recall that the dimensions are real.

In conclusion, the massless modes of (R) sector are[4]

state	$\alpha' M_R^2/2$	$SO(8)$
$\|0\rangle_R\,, B^+_{a_1}B^+_{a_2}\|0\rangle_R\,,$ $B^+_1 B^+_2 B^+_3 B^+_4\|0\rangle_R$	0	$\mathbf{8_s}$
$B^+_{a_1}\|0\rangle_R\,, B^+_{a_1}B^+_{a_2}B^+_{a_3}\|0\rangle_R$	0	$\mathbf{8_c}$

(3.2.36)

For the left movers the analysis is completely the same, with tilde operators. We need to understand how to glue those sectors together, consistently.

3.2.3 Closed String Spectrum.

To obtain the closed string spectrum, we must glue together the left and the right moving sectors, constraint by the level-matching condition (3.2.6). Since left and right movers are themselves divided into (R) and (NS) sectors, we have, a priori, 16 combinations. Basically, we have to look at (3.2.11) and (3.2.36). Since **1** is the only one to have half-integer value for the mass-squared, it can be tensored only with itself to be consistent with (3.2.6). Further, a priori all the massless representations can be tensored, pairwise.

To have a superstring which have spacetime supersymmetry, it's convenient to define *G-parity*, and use it in the context of the *GSO-projection*. We're not interested in those details, so we only say that G-parity essentially counts the evenness and oddness of fermionic excitations. Gathering the information of (3.2.11) and (3.2.36) into one table, and citing just the result for G-parity, we get the following table, for the right movers.

sector	G-parity	state	little group rep.	$\alpha' M_R^2/2$	statistics
NS	$-$	$\|0\rangle_{NS}$	$SO(9): \mathbf{1}$	$-1/2$	boson
NS	$+$	$b^i_{-1/2}\|0\rangle_{NS}$	$SO(8): \mathbf{8_v}$	0	boson
R	$+$	$\|0\rangle_R\,, B^+_{a_1}B^+_{a_2}\|0\rangle_R\,,$ $B^+_1 B^+_2 B^+_3 B^+_4\|0\rangle_R$	$SO(8): \mathbf{8_s}$	0	fermion
R	$-$	$B^+_{a_1}\|0\rangle_R\,,$ $B^+_{a_1}B^+_{a_2}B^+_{a_3}\|0\rangle_R$	$SO(8): \mathbf{8_c}$	0	fermion

For the left movers the situation is completely analogous. We call $\mathrm{NS}_\pm$ and $\mathrm{R}_\pm$ the sectors with G-parity $\pm$. Then, the 10 possibilities to tensor those sectors and glue together left

[4] Again, the R in the mass stands for right movers, *not* Ramond.

and right movers are:

sector	state rep.	$\alpha' M^2$	statistics	$SO(8)$ (indices)	$SO(8)$ (dim.)
$(\mathrm{NS}_-,\mathrm{NS}_-)$	$\mathbf{1}\otimes\mathbf{1}$	-2	boson	/	/
$(\mathrm{NS}_+,\mathrm{NS}_+)$	$\mathbf{8_v}\otimes\mathbf{8_v}$	0	boson	$[\mathbf{0}]\oplus[\mathbf{2}]\oplus(\mathbf{2})$	$\mathbf{1}\oplus\mathbf{28_v}\oplus\mathbf{35_v}$
$(\mathrm{R}_+,\mathrm{R}_+)$	$\mathbf{8_s}\otimes\mathbf{8_s}$	0	boson	$[\mathbf{0}]\oplus[\mathbf{2}]\oplus[\mathbf{4}]_+$	$\mathbf{1_s}\oplus\mathbf{28_s}\oplus\mathbf{35_s}$
$(\mathrm{R}_-,\mathrm{R}_-)$	$\mathbf{8_c}\otimes\mathbf{8_c}$	0	boson	$[\mathbf{0}]\oplus[\mathbf{2}]\oplus[\mathbf{4}]_-$	$\mathbf{1_c}\oplus\mathbf{28_c}\oplus\mathbf{35_c}$
$(\mathrm{R}_-,\mathrm{R}_+)$	$\mathbf{8_c}\otimes\mathbf{8_s}$	0	boson	$[\mathbf{1}]\oplus[\mathbf{3}]$	$\mathbf{8_v}\oplus\mathbf{56_v}$
$(\mathrm{R}_+,\mathrm{R}_-)$	$\mathbf{8_s}\otimes\mathbf{8_c}$	0	boson	$[\mathbf{1}]\oplus[\mathbf{3}]$	$\mathbf{8_v}\oplus\mathbf{56_v}$
$(\mathrm{R}_+,\mathrm{NS}_+)$	$\mathbf{8_s}\otimes\mathbf{8_v}$	0	fermion	/	$\mathbf{8_c}\oplus\mathbf{56_s}$
$(\mathrm{R}_-,\mathrm{NS}_+)$	$\mathbf{8_c}\otimes\mathbf{8_v}$	0	fermion	/	$\mathbf{8_s}\oplus\mathbf{56_c}$
$(\mathrm{NS}_+,\mathrm{R}_+)$	$\mathbf{8_v}\otimes\mathbf{8_s}$	0	fermion	/	$\mathbf{8_c}\oplus\mathbf{56_s}$
$(\mathrm{NS}_+,\mathrm{R}_-)$	$\mathbf{8_v}\otimes\mathbf{8_c}$	0	fermion	/	$\mathbf{8_s}\oplus\mathbf{56_c}$

In the last two columns, we've decomposed the tensor product representations into irreducible representations of $SO(8)$. In particular, in the second to last column, for the bosons, $(\boldsymbol{n})$ denotes a symmetric tensor with n indices, while $[\boldsymbol{n}]$ a completely antisymmetric tensor with n indices. Moreover, in the last column, we've counted the *real* degrees of freedom, *on-shell*. Then,

- $\mathbf{1}$, $\mathbf{28_v}$ and $\mathbf{35_v}$ are the usual dilaton Φ, Kalb-Ramond $B_{[\mu\nu]}$ and graviton $G_{(\mu\nu)}$ in 10 dimensions;
- $\mathbf{1_s}$ and $\mathbf{1_c}$ represents zero-forms, C_0 and $\tilde{C}_0$;
- $\mathbf{28_s}$ and $\mathbf{28_c}$ are the degrees of freedom of two-forms C_2 and $\tilde{C}_2$;
- $\mathbf{35_s}$ and $\mathbf{35_c}$ are the degrees of freedom of four-forms C_4^+ and C_4^-, whose field strenghts are dual and self-dual, respectively;
- $\mathbf{8_v}$ and $\mathbf{56}_v$ are the degrees of freedom of a vector and an antisymmetric three-tensor, respectively;
- $\mathbf{8_s}$ and $\mathbf{8_c}$ are the on-shell degrees of freedom of two dilatini of spin 1/2, one of each handedness;
- $\mathbf{56}_s$ and $\mathbf{56}_c$ are the on-shell degrees of freedom of two gravitini of spin 3/2, one of each handedness.

In particular, for the vector representations $\mathbf{8_v}\otimes\mathbf{8_v}$, we used the familiar decomposition

$$\mathbf{8_v}\otimes\mathbf{8_v} = [\mathbf{0}]\oplus(\mathbf{2})\otimes[\mathbf{2}]. \tag{3.2.37}$$

For the tensor product of two spinor representations, we just cite the group theoretical result, with no proof. For $SO(d)$ in even dimension, with $d = 2l$, the product representations can be decomposed as

$$\mathbf{2^{l-1}} \otimes \mathbf{2^{l-1}} = \begin{cases} [\mathbf{0}] \oplus [\mathbf{2}] \oplus \cdots \oplus [\boldsymbol{l}]_+, & l \text{ even}, \\ [\mathbf{1}] \oplus [\mathbf{3}] \oplus \cdots \oplus [\boldsymbol{l}]_+, & l \text{ odd}, \end{cases} \tag{3.2.38a}$$

$$\mathbf{2^{l-1\prime}} \otimes \mathbf{2^{l-1\prime}} = \begin{cases} [\mathbf{0}] \oplus [\mathbf{2}] \oplus \cdots \oplus [\boldsymbol{l}]_-, & l \text{ even}, \\ [\mathbf{1}] \oplus [\mathbf{3}] \oplus \cdots \oplus [\boldsymbol{l}]_-, & l \text{ odd}, \end{cases} \tag{3.2.38b}$$

$$\mathbf{2^{l-1}} \otimes \mathbf{2^{l-1\prime}} = \begin{cases} [\mathbf{1}] \oplus [\mathbf{3}] \oplus \cdots \oplus [\boldsymbol{l-1}], & l \text{ even}, \\ [\mathbf{0}] \oplus [\mathbf{2}] \oplus \cdots \oplus [\boldsymbol{l-1}], & l \text{ odd}, \end{cases} \tag{3.2.38c}$$

where $[\boldsymbol{n}]_\pm$ denote states which are identified with spacetime fields which are (anti-)self-dual under Hodge star operator.

Applied to our case, in which $d = 8$ and $l = 4$, we get

$$\mathbf{8_s} \otimes \mathbf{8_s} = [\mathbf{0}] \oplus [\mathbf{2}] \oplus [\mathbf{4}]_+, \tag{3.2.39a}$$
$$\mathbf{8_c} \otimes \mathbf{8_c} = [\mathbf{0}] \oplus [\mathbf{2}] \oplus [\mathbf{4}]_-, \tag{3.2.39b}$$
$$\mathbf{8_s} \otimes \mathbf{8_c} = [\mathbf{1}] \oplus [\mathbf{3}]. \tag{3.2.39c}$$
$$\tag{3.2.39d}$$

Finally, for the tensor product of a spinor and vector-bilinears, we get

$$\mathbf{8_v} \otimes \mathbf{8_s} = \mathbf{8_c} \oplus \mathbf{56_s}, \tag{3.2.40a}$$
$$\mathbf{8_v} \otimes \mathbf{8_c} = \mathbf{8_s} \oplus \mathbf{56_c}. \tag{3.2.40b}$$

3.2.4 Type II Superstring Theories.

In order to obtain consistent spacetime theories, one should combine the above described sectors in a way which is consistent with CFT on the worldsheet and supersymmetry of spacetime. The mathematical tool to achieve this is the *GSO projection.* Without proving it, we just quote the result, in particular focusing on *type II* theories, which are characterized by $\mathcal{N} = 2$ susy on spacetime. They're characterized by the sectors

$$\begin{aligned} \text{IIA}: &\quad (\text{NS}_+, \text{NS}_+),\ (\text{R}_+, \text{R}_-),\ (\text{NS}_+, \text{R}_-),\ (\text{R}_+, \text{NS}_+), \\ \text{IIB}: &\quad (\text{NS}_+, \text{NS}_+),\ (\text{R}_+, \text{R}_+),\ (\text{NS}_+, \text{R}_+),\ (\text{R}_+, \text{NS}_+), \end{aligned} \tag{3.2.41}$$

where we could exchange $\text{R}_\pm \to \text{R}_\mp$ in type IIA, and $\text{R}_+ \to \text{R}_-$ in type IIB, obtaining equivalent theories in spacetime. The field content from the massless spectrum is the following:

Type IIA			Type IIB		
sector	fields	$SO(8)$	sector	fields	$SO(8)$
$(\mathrm{NS}_+, \mathrm{NS}_+)$	Φ, $B_{[\mu\nu]}$, $G_{(\mu\nu)}$	$\boldsymbol{8_v} \otimes \boldsymbol{8_v}$	$(\mathrm{NS}_+, \mathrm{NS}_+)$	Φ, $B_{[\mu\nu]}$, $G_{(\mu\nu)}$	$\boldsymbol{8_v} \otimes \boldsymbol{8_v}$
$(\mathrm{R}_+, \mathrm{R}_-)$	C_1, C_3	$\boldsymbol{8_s} \otimes \boldsymbol{8_c}$	$(\mathrm{R}_+, \mathrm{R}_+)$	C_0, C_2, C_4^+	$\boldsymbol{8_s} \otimes \boldsymbol{8_s}$
$(\mathrm{NS}_+, \mathrm{R}_-)$	$\tilde{\lambda}_a$, $\tilde{\psi}_a^\mu$	$\boldsymbol{8_v} \otimes \boldsymbol{8_c}$	$(\mathrm{NS}_+, \mathrm{R}_+)$	$\lambda_a^{(1)}$, $\psi_a^{(1)\mu}$	$\boldsymbol{8_v} \otimes \boldsymbol{8_s}$
$(\mathrm{R}_+, \mathrm{NS}_+)$	λ_a, ψ_a^μ	$\boldsymbol{8_s} \otimes \boldsymbol{8_v}$	$(\mathrm{R}_+, \mathrm{NS}_+)$	$\lambda_a^{(2)}$, $\psi_a^{(2)\mu}$	$\boldsymbol{8_s} \otimes \boldsymbol{8_v}$

3.3 Compactification of Type II Superstrings on S^1.

We're now able to generalize the T-duality discussion of section 2.6 to type IIA/IIB superstrings. We focus on the *closed string* and compactify $\mathcal{M}_{10}$ on $\mathcal{M}_9 \times S^1$. Differently than above, we change the notation for the indices. In particular, we'll denote with $\hat{\mu}$ the indices of $\mathcal{M}_{10}$, with $\hat{\mu} = 0, \ldots, 9$, and with μ the indices of $\mathcal{M}_9$, with $\mu = 0, \ldots, 8$. This means that we can write $X^{\hat{\mu}} = (X^\mu, X^9)$. Similarly, the transversal indices will be called $\hat{i}$ in the following, with $\hat{i} = 2, \ldots 9$, while i will denote the transversal indices in the non-compact space $\mathcal{M}_9$, with $i = 2, \ldots 8$. Therefore, in spacetime lightcone coordinates, we have $X^{\hat{\mu}} \to (X^\pm, X^{\hat{i}}) = (X^\pm, X^i, X^9)$. The same conventions will be used for the fermions on the worldsheet.

An important detail is that the 2d fermion sector on the worldsheet is completely unchanged by the compactification. Indeed, we suppose that the only effect of the compactification is on the bosonic fields, which describe how the string in embedded in spacetime, while the fermionic fields are added to guarantee worldsheet supersymmetry. Therefore, the fermions will have the usual (R) or (NS) boundary conditions, with no change due to the compactification. For what concerns the bosons, the analysis is parallel to the bosonic string.

Due to the identification $X^9 \simeq X^9 + 2\pi R$, the possible boundary conditions for the bosonic fields are

$$\begin{aligned} X^i(\tau, \sigma + l) &= X^i(\tau, \sigma), \quad i = 2, \ldots, 8, \\ X^9(\tau, \sigma + l) &= X^9(\tau, \sigma) + 2\pi R\omega, \quad \omega \in \mathbb{Z}. \end{aligned} \tag{3.3.1}$$

Further, the momentum along X^9 is quantized, i.e.,

$$p_9 = \frac{s}{R}, \quad s \in \mathbb{Z}. \tag{3.3.2}$$

Regarding the mode decompositions, for $X^i_{L/R}$ is given by (2.4.4), with $i = 2, \ldots, 8$, while for $X^9(\tau, \sigma) = X^9_L(\xi^+) + X^9_R(\xi^-)$, it reads

$$X^9_L(\xi^+) = \frac{x^9}{2} + \frac{\alpha' \pi}{l} p_L \xi^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\tilde{\alpha}^9_n}{n} e^{-\frac{2\pi i}{l} n \xi^+} \tag{3.3.3a}$$

$$X^9_R(\xi^-) = \frac{x^9}{2} + \frac{\alpha' \pi}{l} p_R \xi^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha^9_n}{n} e^{-\frac{2\pi i}{l} n \xi^-}, \tag{3.3.3b}$$

with

$$p_L \equiv \left(\frac{s}{R}+\frac{\omega R}{\alpha'}\right), \quad p_R \equiv \left(\frac{s}{R}-\frac{\omega R}{\alpha'}\right). \tag{3.3.4}$$

From (2.4.17), (3.2.9) and (3.2.12), since the fermions don't enter this analysis, we easily see that the *mass-shell condition* on $\mathcal{M}_9$ reads

$$M_L^2 = \frac{p_L^2}{2}+\frac{2}{\alpha'}\left(\tilde{N}_\perp - \tilde{a}_\phi\right), \quad M_R^2 = \frac{p_R^2}{2}+\frac{2}{\alpha'}\left(N_\perp - a_\phi\right). \tag{3.3.5}$$

The number operators are given by (3.2.5), i.e.,

$$\begin{aligned} \tilde{N}_\perp &= \sum_{n>0} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + \sum_{k\geq 0+\phi} k\tilde{b}^i_{-k}\tilde{b}^i_k, \\ N_\perp &= \sum_{n>0} \alpha^i_{-n}\alpha^i_n + \sum_{k\geq 0+\phi} k b^i_{-k} b^i_k, \end{aligned} \tag{3.3.6}$$

and a_ϕ and $\tilde{a}_\phi$ are the ordering constants for the (R) or (NS) sector. In particular, for $\phi = 0$, we have $a_0 = \tilde{a}_0 = a_R = 0$, while for $\phi = 1/2$, we get $a_{1/2} = \tilde{a}_{1/2} = a_{NS} = 1/2$.

For a generic R, the only massless states are in the sector $(s = 0, \omega = 0)$. These states correspond to zero modes of the Kaluza-Klein reduction of the effective field theory of 10d massless modes. Indeed, the internal momentum is zero and there's no winding. Therefore, performing the Kaluza-Klein reduction to 9d, keeping just the zero-modes, is equivalent to decoupling representations with respect $SO(8)$, the little group for massless representations of $SO(1,9)$ in $\mathcal{M}_{10}$, into representations of $SO(7)$, which is the little group for massless representations of $SO(1,8)$ in $\mathcal{M}_9$.

In particular, looking at (N) and (NS) sectors for left or right movers, separately, we have

sector	$SO(8)$ rep.	$SO(7)$ rep.
NS_+	$\boldsymbol{8_v}$	$\mathbf{7}\oplus\mathbf{1}$
R_+	$\boldsymbol{8_s}$	$\mathbf{8}$
R_-	$\boldsymbol{8_c}$	$\mathbf{8}$

(3.3.7)

where $\mathbf{1}$ is the scalar representation of $SO(7)$, $\mathbf{7}$ is the vector one and $\mathbf{8}$ the spinor representation. Notice that, looking at $SO(d)$, for $d = 7 \mod 8$ we can impose the reality condition, but *not* the Weyl one, since there's no chirality in odd dimensions. Therefore, there's a real, non-chiral, spinor representation of $SO(7)$ of dimension 8 and both $\boldsymbol{8_s}$ and $\boldsymbol{8_c}$ are decomposed into this $\mathbf{8}$.

Then, we have to glue together the left and right movers. To do so, we can decompose the left and right movers with respect to $SO(7)$, separately, and then tensor them[5].

Type IIA					
sector	$SO(8)$	10d fields	$SO(7)$	$SO(7)$ irrep	9d fields
(NS$_+$,NS$_+$)	$\mathbf{8_v} \otimes \mathbf{8_v}$	$\Phi, B_{[\hat{\mu}\hat{\nu}]}, G_{(\hat{\mu}\hat{\nu})}$	$\mathbf{7} \otimes \mathbf{7}$	$\mathbf{1} \oplus \mathbf{21} \oplus \mathbf{27}$	$\phi, B_{[\mu\nu]}, G_{(\mu\nu)}$
			$(\mathbf{7} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{7})$	$\mathbf{7} \oplus \mathbf{7}$	$G_{\mu 9}, B_{\mu 9}$
			$\mathbf{1} \otimes \mathbf{1}$	$\mathbf{1}$	G_{99}
(R$_+$,R$_-$)	$\mathbf{8_s} \otimes \mathbf{8_c}$	$C_{\hat{\mu}}, C_{[\hat{\mu}\hat{\nu}\hat{\rho}]}$	$\mathbf{8} \otimes \mathbf{8}$	$\mathbf{1} \oplus \mathbf{7} \oplus \mathbf{21} \oplus \mathbf{35}$	$A_9, A_\mu, C_{9\mu\nu}, C_{\mu\nu\rho}$
(NS$_+$,R$_-$)	$\mathbf{8_v} \otimes \mathbf{8_c}$	$\tilde{\lambda}_a, \tilde{\psi}^{\hat{\mu}}_a$	$\mathbf{7} \otimes \mathbf{8}$	$\mathbf{8} \oplus \mathbf{48}$	$\tilde{\psi}^{\mu}_a, \tilde{\psi}^{9}_a$
			$\mathbf{1} \otimes \mathbf{8}$	$\mathbf{8}$	$\tilde{\lambda}_a$
(R$_+$,NS$_+$)	$\mathbf{8_s} \otimes \mathbf{8_v}$	$\lambda_a, \psi^{\hat{\mu}}_a$	$\mathbf{8} \otimes \mathbf{7}$	$\mathbf{8} \oplus \mathbf{48}$	ψ^{μ}_a, ψ^{9}_a
			$\mathbf{8} \otimes \mathbf{1}$	$\mathbf{8}$	λ_a
Type IIB					
sector	$SO(8)$	10d fields	$SO(7)$	$SO(7)$ irrep	9d fields
(NS$_+$,NS$_+$)	$\mathbf{8_v} \otimes \mathbf{8_v}$	$\Phi, B_{[\hat{\mu}\hat{\nu}]}, G_{(\hat{\mu}\hat{\nu})}$	$\mathbf{7} \otimes \mathbf{7}$	$\mathbf{1} \oplus \mathbf{21} \oplus \mathbf{27}$	$\phi, B_{[\mu\nu]}, G_{(\mu\nu)}$
			$(\mathbf{7} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{7})$	$\mathbf{7} \oplus \mathbf{7}$	$G_{\mu 9}, B_{\mu 9}$
			$\mathbf{1} \otimes \mathbf{1}$	$\mathbf{1}$	G_{99}
(R$_+$,R$_+$)	$\mathbf{8_s} \otimes \mathbf{8_s}$	$C_0, C_{[\hat{\mu}\hat{\nu}]}, C^{+}_{[\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}]}$	$\mathbf{8} \otimes \mathbf{8}$	$\mathbf{1} \oplus \mathbf{7} \oplus \mathbf{21} \oplus \mathbf{35}$	$a, C_{\mu 9}, C_{\mu\nu}, C_{\mu\nu\rho 9}$
(NS$_+$,R$_+$)	$\mathbf{8_v} \otimes \mathbf{8_s}$	$\lambda^{(1)}_a, \psi^{(1)\hat{\mu}}_a$	$\mathbf{7} \otimes \mathbf{8}$	$\mathbf{8} \oplus \mathbf{48}$	$\psi^{(1)9}_a, \psi^{(1)\mu}_a$
			$\mathbf{1} \otimes \mathbf{8}$	$\mathbf{8}$	$\lambda^{(1)}_a$
(R$_+$,NS$_+$)	$\mathbf{8_s} \otimes \mathbf{8_v}$	$\lambda^{(2)}_a, \psi^{(2)\hat{\mu}}_a$	$\mathbf{8} \otimes \mathbf{7}$	$\mathbf{8} \oplus \mathbf{48}$	$\psi^{(2)9}_a, \psi^{(2)\mu}_a$
			$\mathbf{8} \otimes \mathbf{1}$	$\mathbf{8}$	$\lambda^{(2)}_a$

Notice that, in decomposing in irreducible representations of $SO(7)$, each representation is labelled by its real dimension, which is equal to the *on-shell* degrees of freedom of the corresponding field. For the bosons it's enough to count the number of independent lightcone indices, taking into account the symmetry/antisymmetry property. For fermions, one can show that the handedness of the dilatini is opposite to that of the gravitini. Together, a gravitino and a dilatino form a reducible vector-spinor ψ^μ_a of $SO(1,9)$, where μ are the spacetime indices while a the spinorial ones. Its traceless part, $\Gamma^\mu \psi_\mu = 0$ it the gravitino, while the trace is the dilatino. Lightcone quantization can be carefully carried over, obtaining, for the gravitino, a state like ψ^i_a, but with a constraint deriving from $\Gamma^\mu \psi_\mu = 0$. Without the derivation, we just remark that this constraint will impose 8 conditions for both $SO(8)$ and $SO(7)$, leading to the correct number of degrees of freedom on-shell.

As usual, similarly to (1.2.9), the scalar which arise from the metric, i.e., G_{99}, sets the volume of extra dimension, or rather, the compactification radius in this case. There's, however, another scalar, arising from the R-R sector, which is A_9. Then, it would be interesting

[5]An equivalent procedure would be to tensor the representations of $SO(8)$ and then decompose them with respect to $SO(7)$. Both methods give the same result.

to describe the compactification for an arbitrary background of this field. Unfortunately, it's not known how to couple R-R fields to the 2d worldsheet theory.

Recall that NS-NS fields are the same as the bosonic string. For them, we know how to couple the background to the 2d theory. This is provided by the *non-linear σ-model* action, which reads

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\xi\sqrt{-\det\gamma}\left[\left(\gamma^{ab}G_{\mu\nu}(X)+i\varepsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu\partial_b X^\nu+\alpha'\mathcal{R}\phi(X)\right],$$

and it's usually exactly solvable. There's no analogue for R-R fields, to couple them directly to the 2d worldsheet theory. Indeed, they must be coupled to the worldvolume of D-branes, through *Cern-Simons* actions, raising many complications.

The interesting thing to notice is that the 9d massless spectrum for type IIA and IIB is the same. In particular, chirality of type IIB is lost after compactification, since there's no notion of chirality in odd dimensions. It's easy to see from the 9d spectrum that the theory has $\mathcal{N}=2$ susy, since there are two gravitini. The spectrum, indeed, corresponds to 9d supergravity with 32 supercharges, which is a unique theory.

In general, *toroidal compactifications* don't break any supersymmetry, and the *number of supercharges is conserved*. Indeed, compactification on T^n preserves the spinorial degrees of freedom, since the representation of $SO(1,9)$ simply reorganizes under $SO(1,9-n)$, without loosing degrees of freedom. Then, compactification of type IIA and IIB on T^6 to obtain a 4d spacetime theory, would lead to $\mathcal{N}=8$ 4d supergravity, since the 32 supercharges organize into 8 independent Majorana spinors in 4d. This is *not* a chiral theory, and so it's useless for phenomenology.

Let's see what happens for non-zero winding and internal momentum. After imposing the *level-matching condition*

$$M_L^2 = M_R^2, \tag{3.3.8}$$

with $M^2_{L/R}$ given by (3.3.5), the mass-shell condition on $\mathcal{M}_9$ reads

$$M^2 = \frac{s^2}{R^2}+\frac{\omega^2R^2}{(\alpha')^2}+\frac{2}{\alpha'}(\tilde{N}_\perp+N_\perp-a_\phi-\tilde{a}_\phi), \tag{3.3.9}$$

with number operators given by (3.3.6) and $a_R=0, a_{NS}=1/2$.

Then, as $R\to\infty$, the winding states decouple from the light spectrum and the internal momenta assume continuous values, which corresponds to the *decompactification limit*. Conversely, for $R\to 0$, the Kaluza-Klein modes decouples and the winding states contributes to the light spectrum. As in the bosonic case, it's natural to think of this as the decompactification limit of the *dual theory*, to be studied.

3.4 T-duality for Type II Superstrings.

Recall from the bosonic case that a *T-duality transformation* (2.6.1) reads

$$R\to R'=\frac{\alpha'}{R},\quad (s,\omega)\to(s',\omega')=(\omega,s). \tag{3.4.1}$$

By defining (2.6.3), we've seen that T-duality is a symmetry of the full string theory, which acts as a parity transformation on the coordinates, in particular as

$$X^9_L \to X^9_L, \quad X^9_R \to -X^9_R, \tag{3.4.2}$$

which maps

$$X^9(\tau,\sigma) = X^9_L(\xi^+) + X^9_R(\xi^-) \to X'^9(\tau,\sigma) = X^9_L(\xi^+) - X^9_R(\xi^-), \tag{3.4.3}$$

where $X'^9(\tau,\sigma)$ describes a compactification on a circle of radius $R' = \alpha'/R$. However, this transformation acts non-trivially on the worldsheet fermions. Indeed, worldsheet supersymmetry imposes that ψ^9 transforms accordingly, namely

$$\psi^9_L \to \psi^9_L, \quad \psi^9_R \to -\psi^9_R, \tag{3.4.4}$$

giving

$$\psi^9(\tau,\sigma) = \psi^9_+(\xi^+) + \psi^9_-(\xi^-) \to \psi'^9(\tau,\sigma) = \psi^9_+(\xi^+) - \psi^9_-(\xi^-). \tag{3.4.5}$$

Then, T-duality acts as a spacetime parity on the right movers.

To see the implications, let's first recall the mode expansion of the fermions, given by (3.1.16). Therefore, the T-duality (3.4.4) acts on the modes as

$$\tilde{b}^9_r \to \tilde{b}'^9_r = \tilde{b}^9_r, \quad b^9_r \to b'^9_r = -b^9_r. \tag{3.4.6}$$

Recall, then, the construction of the Ramond vacuum in section (3.2.2), where we defined creation/annihilation operators for a fermionic harmonic oscillator in (3.2.32). Since it's only b^9_r which changes, we focus on $B^\pm_4$, which is the only one containing it. Under T-duality, it transforms as

$$B^\pm_4 = \frac{1}{\sqrt{2}}\left(b^8_0 \pm i b^9_0\right) \to \frac{1}{\sqrt{2}}\left(b^8_0 \mp i b^9_0\right) = B^\mp_4. \tag{3.4.7}$$

Analysing (3.2.21), (3.2.16) and (3.2.25), we conclude that T-duality flips the chirality for the right moving spinors, and therefore transforms the various superstring sectors as

$$\begin{aligned} (\mathrm{R}_+, \mathrm{R}_\pm) &\to (\mathrm{R}_+, \mathrm{R}_\mp), \\ (\mathrm{NS}_+, \mathrm{R}_\pm) &\to (\mathrm{NS}_+, \mathrm{R}_\mp), \end{aligned} \tag{3.4.8}$$

Indeed, as an example, let's focus on the R_+ sector, with degenerate vacuum

$$|0\rangle_R\,, \quad B^+_{a_1} B^+_{a_1} |0\rangle_R\,, \quad B^+_1 B^+_2 B^+_3 B^+_4 |0\rangle_R\,. \tag{3.4.9}$$

Here, $|0\rangle_R$ is defined by

$$B^-_a |0\rangle_R = 0, \quad \forall a = 1, \ldots, 4. \tag{3.4.10}$$

In the T-dual theory, we'd define the state $|0\rangle'_R$ by

$$B'^-_a |0\rangle'_R = 0, \quad \forall a = 1, \ldots, 4. \tag{3.4.11}$$

In terms of the original operators, we have $B^-_a |0\rangle'_R = 0$ for $a = 1, 2, 3$, and $B^+_4 |0\rangle'_R = 0$, so

$$|0\rangle'_R = B^+_4 |0\rangle_R\,. \tag{3.4.12}$$

This exchanges the G-parity of $|0\rangle'_R$ with respect to $|0\rangle_R$, so that the GSO projection of the T-dual theory is the opposite. In particular, the degenerate vacuum in the T-dual theory will be

$$A_a^+ |0\rangle_R \,, \quad A_{a_1}^+ A_{a_2}^+ A_{a_3}^+ |0\rangle_R \,, \tag{3.4.13}$$

which is the same as R_-, as previously mentioned.

The overall effect of T-duality is exchanging type IIA and IIB, by means of (3.2.41). Summarizing,

$$\text{Type IIB on } S^1 \text{ with radius } R \;\cong\; \text{Type IIA on } S'^1 \text{ with radius } R' = \frac{\alpha'}{R}.$$

Then, the $9d$ spectrum of the two theories is the same, up to T-duality transformation.

3.5 Postilla on Energies and Limits.

- The compactification scale is $M_c = \frac{1}{R}$;
- The string scale is $M_s = \frac{1}{l_s} = \frac{1}{2\pi\sqrt{\alpha'}}$;
- In field theory compactification, we work at energies

$$E \ll M_c \ll M_s \iff \quad L \gg R \gg \sqrt{\alpha'}, \tag{3.5.1}$$

  which allows us to neglect stringy effect and Kaluza-Klein heavy modes;
- In string theory, we can either work in $R \gg \sqrt{\alpha'}$ or $R \ll \sqrt{\alpha'}$, in the T-dual frame;
- To forget about both Kaluza-Klein and winding states, we may work at energies

$$E \ll \frac{1}{R} \ll \frac{R}{\alpha'}, \tag{3.5.2}$$

  which is compatible with the large volume scenario (2.5.3).

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