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Control of Mobile Robots

Module 5

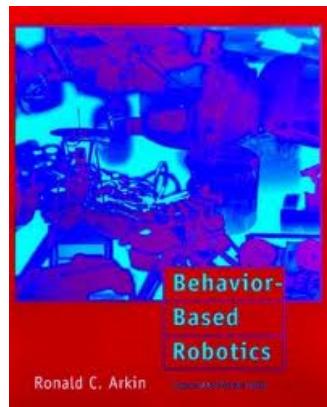
Hybrid Systems

*How make mobile robots move in effective, safe,
predictable, and collaborative ways using modern
control theory?*

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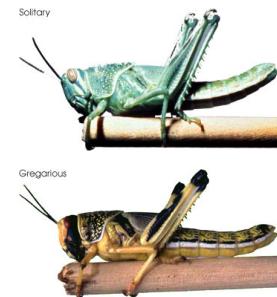
Lecture 5.1 – Switches Everywhere

- So far, the models stay the same over time
 - NOT ALWAYS TRUE!
- We have a designed one-size-fits-all controllers
 - “NEVER” TRUE IN ROBOTICS!



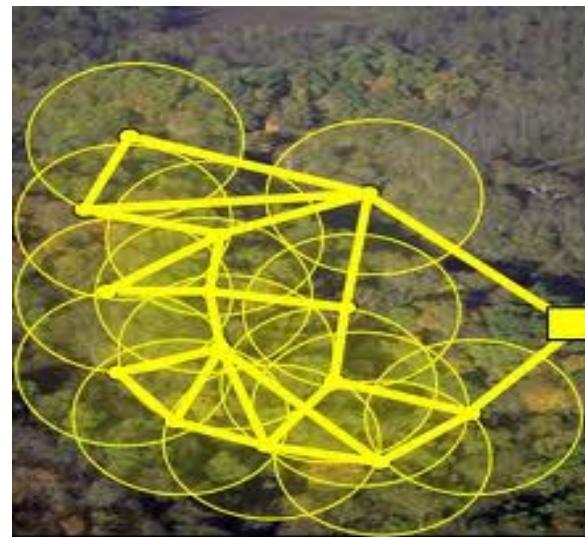
- Recall the behavior-based control paradigm
- We need to be able to deal with these types of phenomena

Switches by Necessity



Switches by Design

✓ for the better
not because
we have
to



Switches by Design



Issues

- Models?
- Stability and Performance? ($t \not\rightarrow \infty$)
- Compositionality?
 - ↳ t is now stable
- Traps?
 - ↳ Put all controllers together

Lecture 5.2 – Hybrid Automata

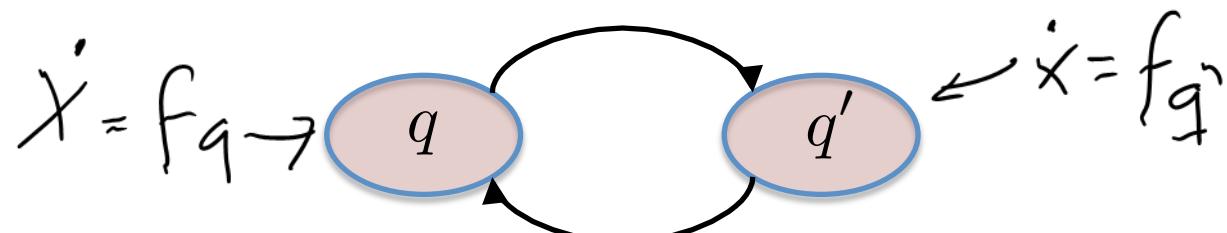
- We need to be able to describe systems that contain both the continuous dynamics and the discrete switch logic
- ***Hybrid Automata*** = Finite state machines (discrete logic) on steroids (continuous dynamics)

↳ multiple
controllers

Modes, Transitions, Guards, and Resets

- Let, as before, the (continuous) state of the system be x
- As we will be switching between different *modes of operation*, let's add an additional discrete state q which mode I am in
- Dynamics:

$$\dot{x} = f_q(x, u)$$
- The *transitions* between different discrete modes can be encoded in a state machine:



how it affects the state

Modes, Transitions, Guards, and Resets

- The conditions under which a transition occurs are called guard *conditions*, i.e., a transition occurs from q to q' if

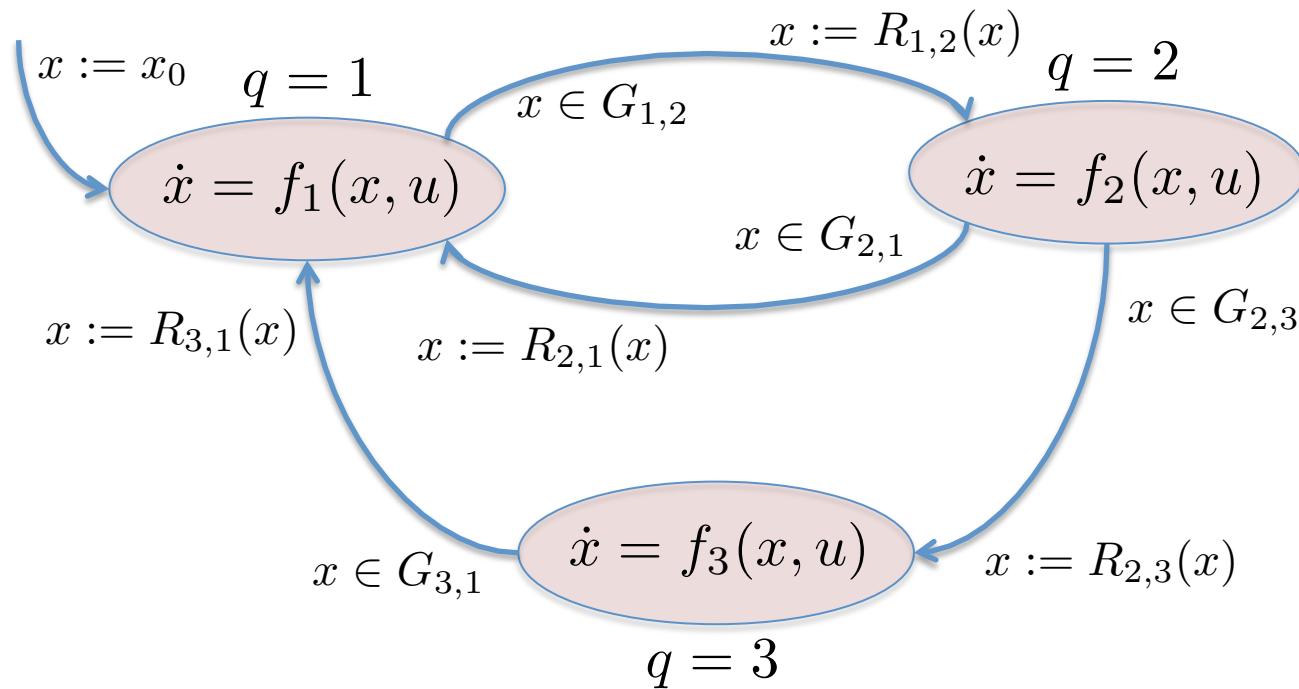
$x \in G_{q,q'}$ Guard (when is time
to jump)

- As a final component, we would like to allow for abrupt changes in the continuous state as the transitions occur, which we will call *resets*:

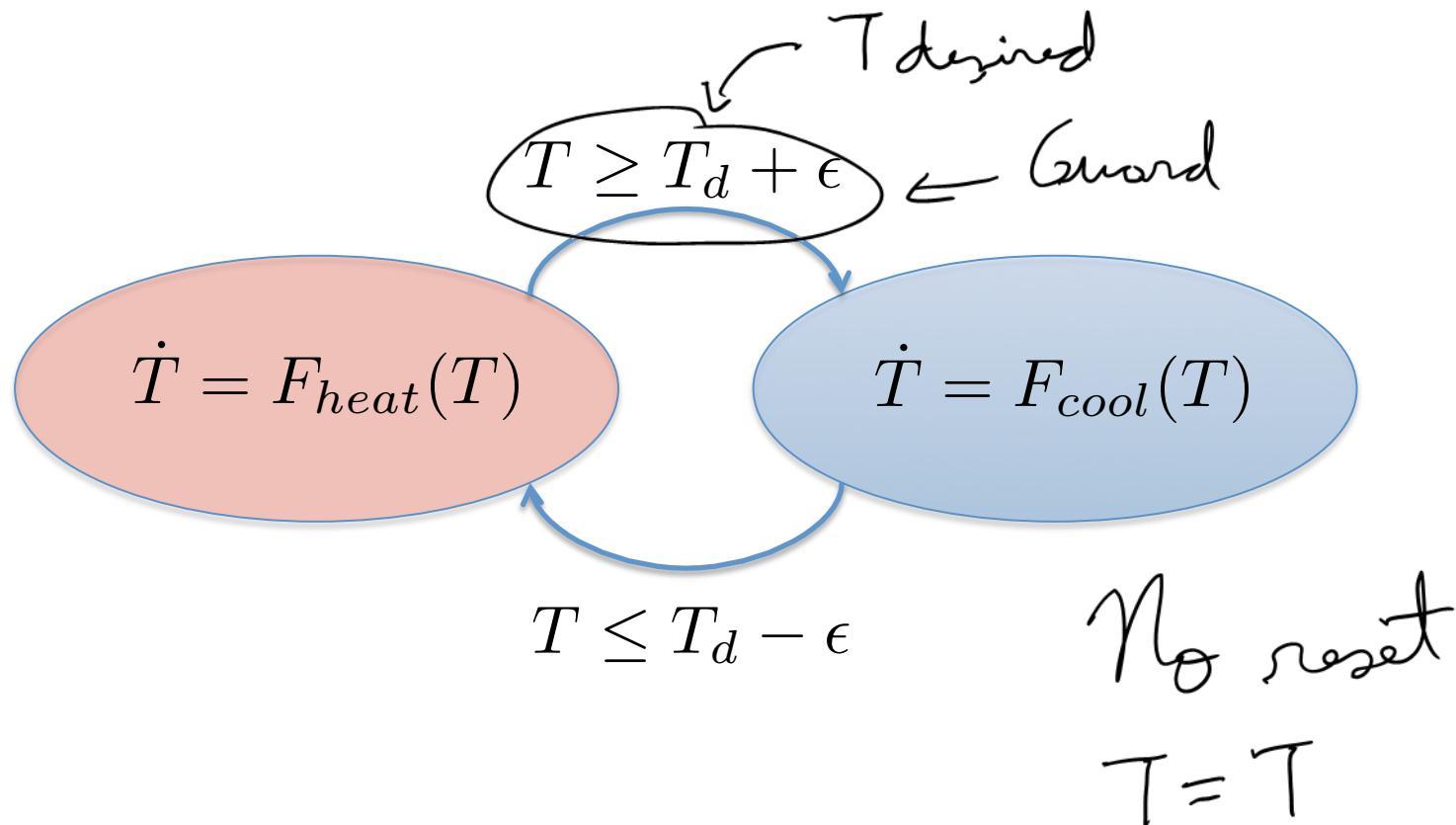
$x := R_{q,q'}(x)$

The Hybrid Automata Model

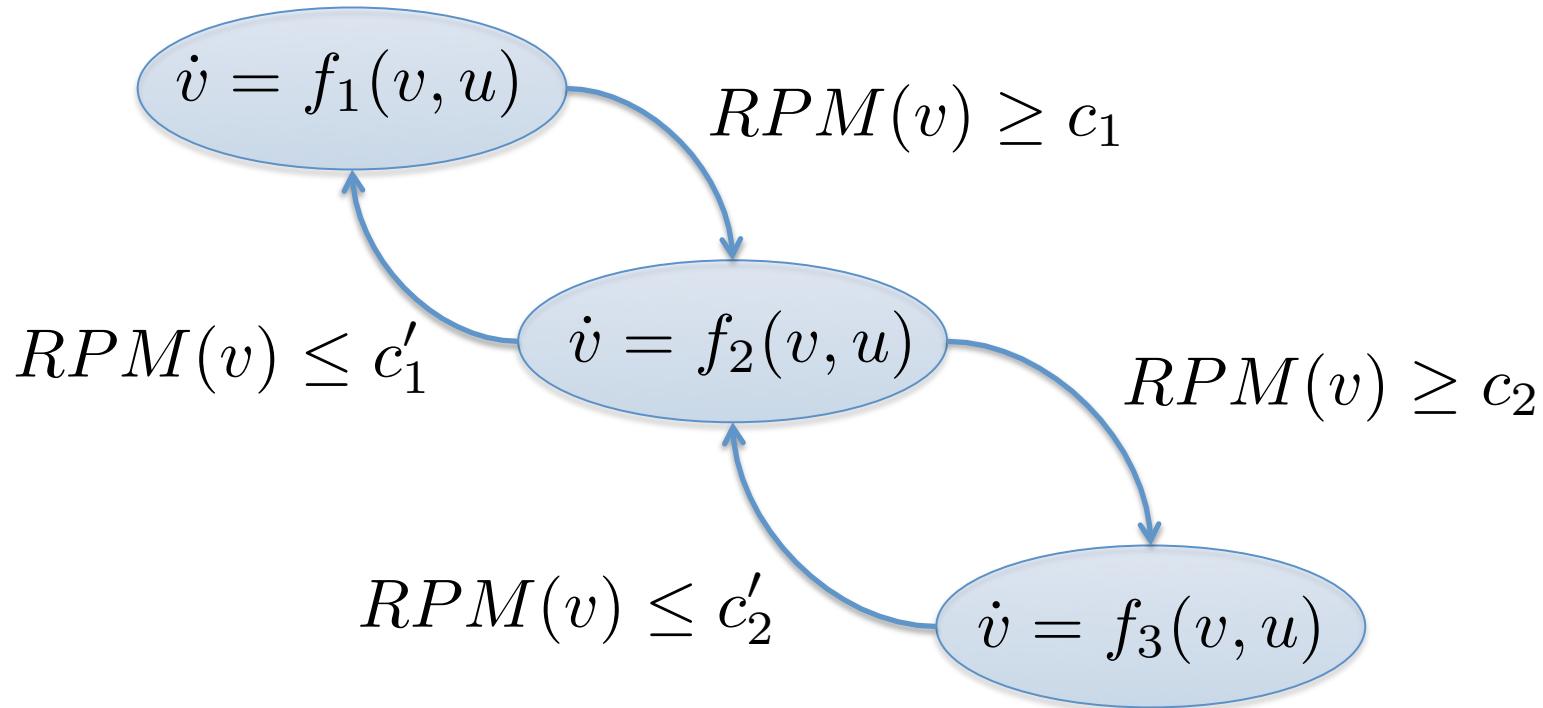
- Putting all of this together yields a very rich model known as a hybrid automata (HA) model:



HA Example 1 - Thermostat

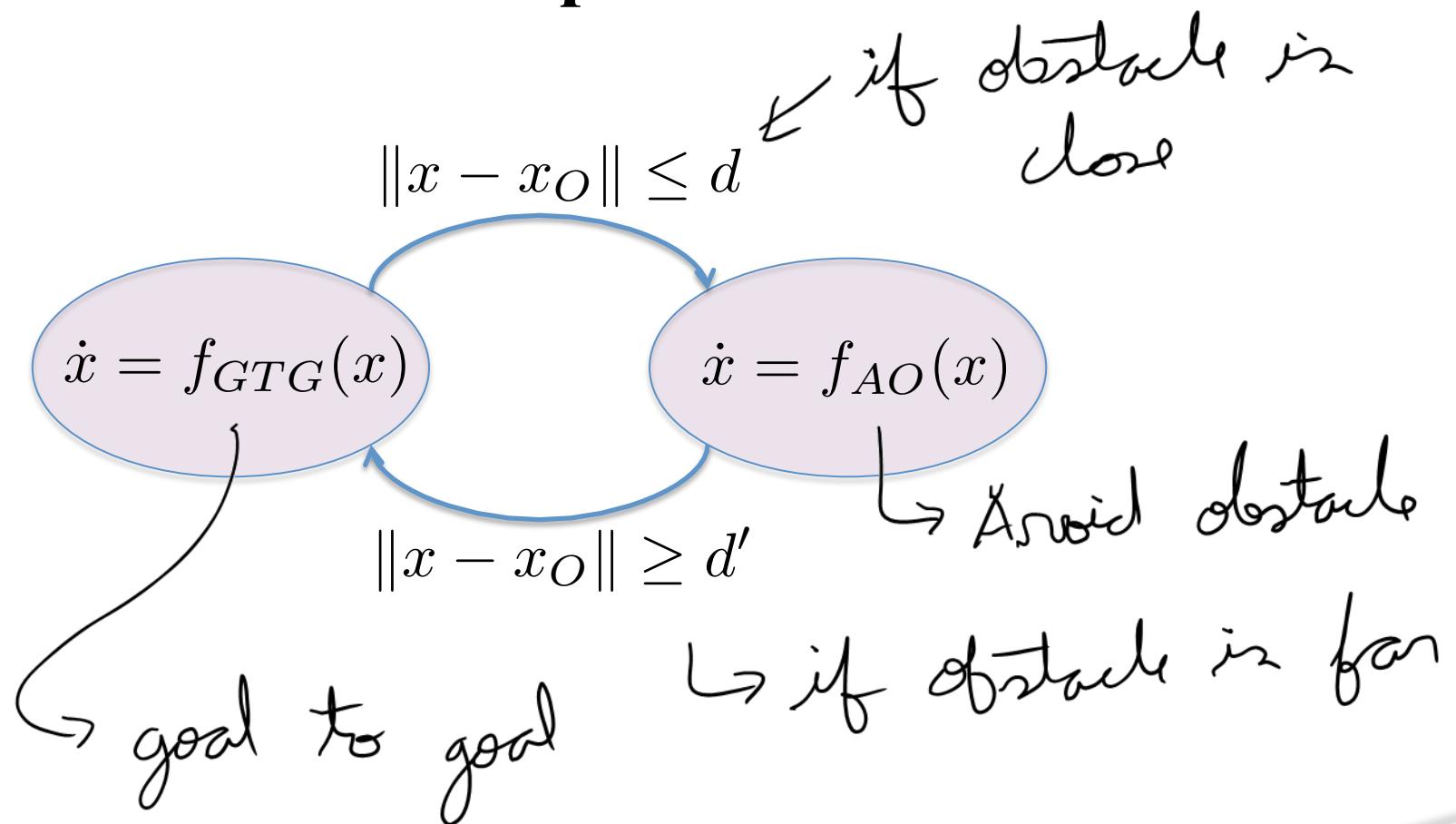


HA Example 2 – Gear Shift



c_1 has to be different to c'_1 so
we 2 guards are not true at the same time

HA Example 3 – Behaviors



Lecture 5.3 – A Counter Example

- What can possibly go wrong when you start switching between different controllers?



A Simple 2-Mode System

- Two modes:

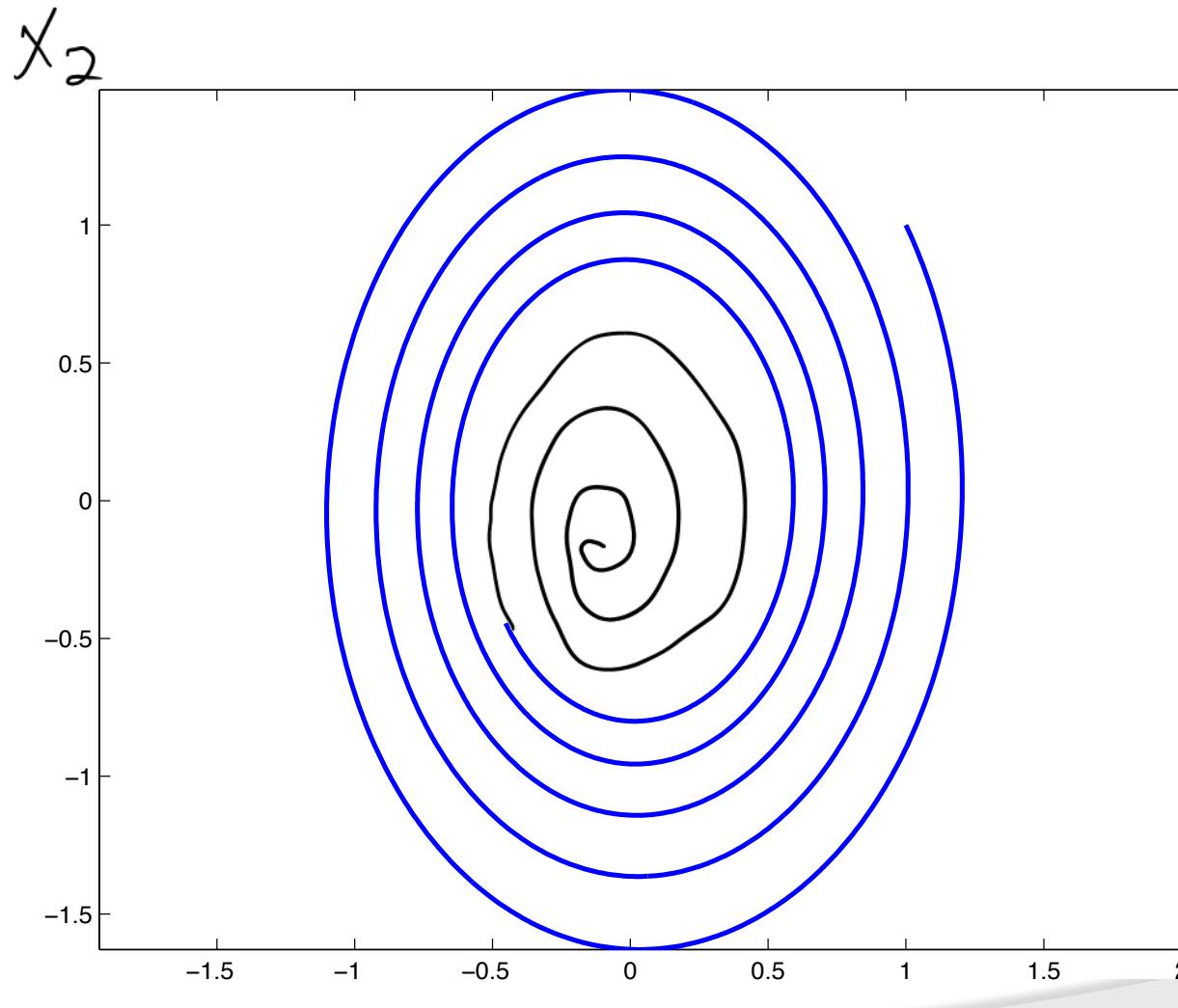
$$\dot{x} = A_1 x = \begin{bmatrix} -\epsilon & 1 \\ -2 & -\epsilon \end{bmatrix} x \quad \text{eig}(A_i) = -\epsilon \pm 1.41j$$

normal oscillation

$$\dot{x} = A_2 x = \begin{bmatrix} -\epsilon & 2 \\ -1 & -\epsilon \end{bmatrix} x$$

- Both modes are asymptotically stable!

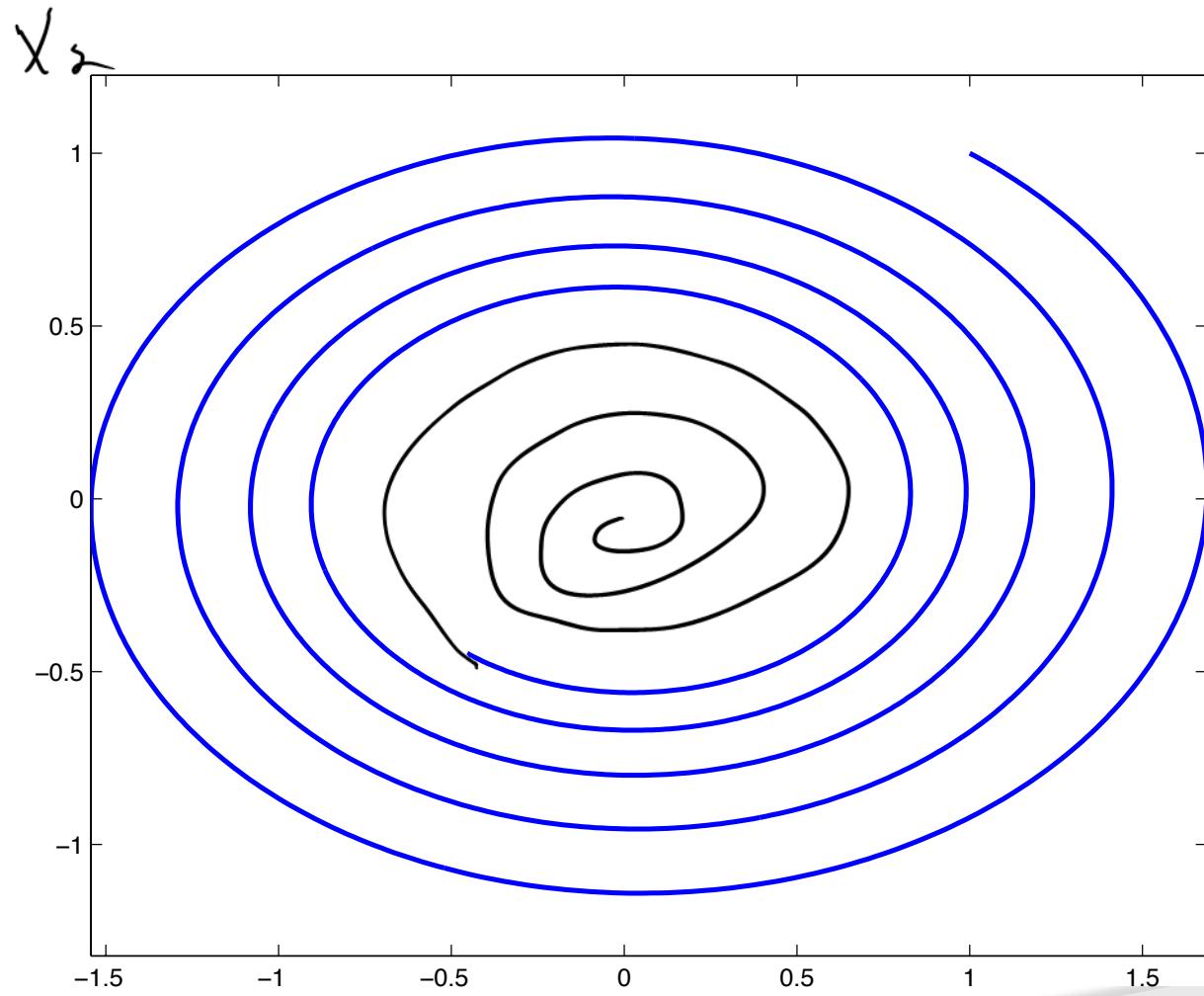
Mode 1



$$\dot{x} = A_1 x$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

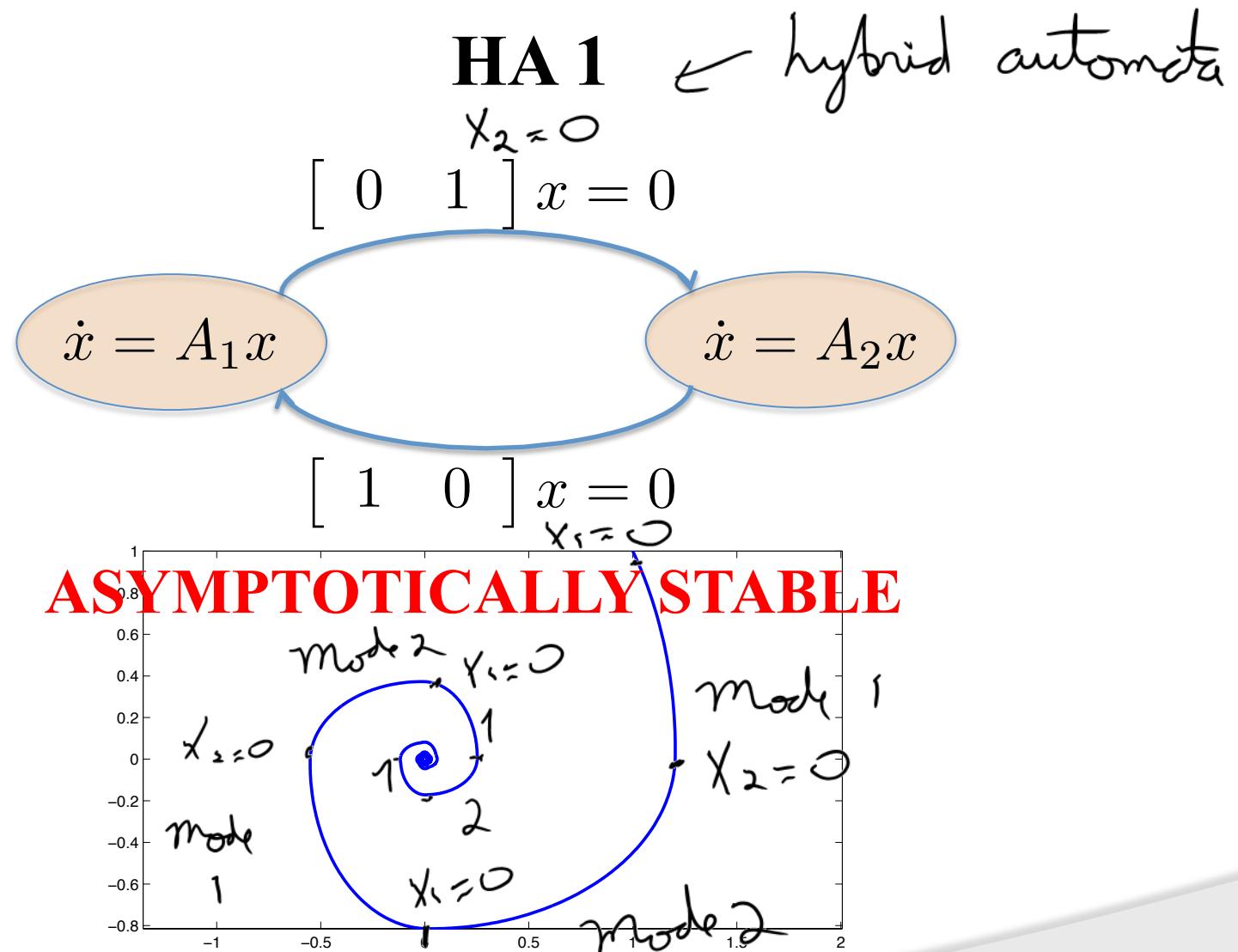
Mode 2

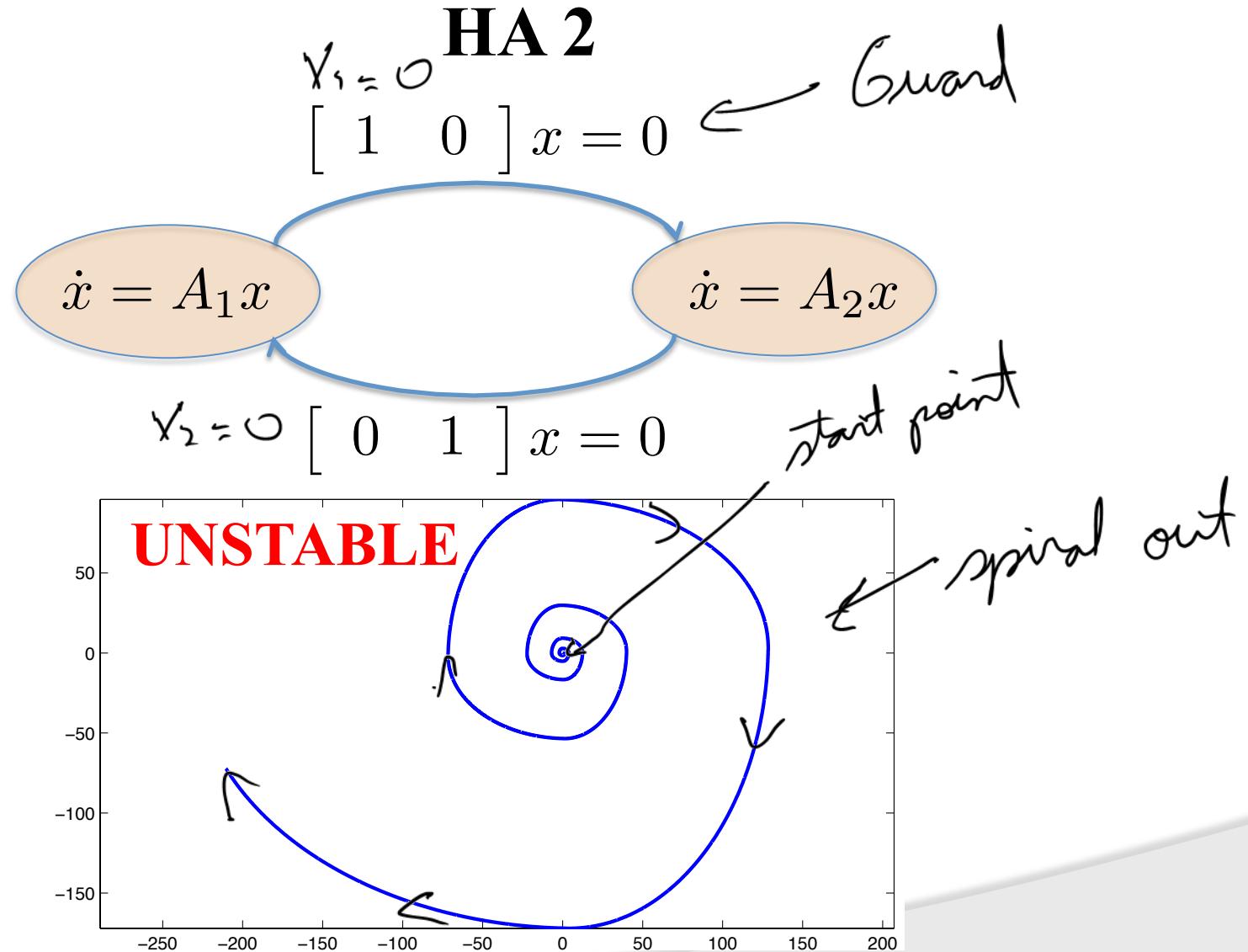


$$\dot{x} = A_2 x$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

x_1





Punchlines

- By combining stable modes, the resulting hybrid system may be unstable!
- By combining unstable modes, the resulting hybrid system may be stable!
- Design stable modes but be aware that this is a risk one may face!

↳ it will be good for design to check this

Lecture 5.4 – Danger, Beware!

- Stable subsystems do not guarantee a stable hybrid system



Switched Systems

- We saw last time that it was possible to destabilize stable subsystems by an unfortunate series of switches
- Ignoring resets, we can write the hybrid system as a *switched system*:

$$\dot{x} = f_{\sigma}(x, u)$$

switch signal

- The *switch signal* dictates which discrete mode the system is in

$$\sigma(t) \in \Sigma = \underbrace{\{1, \dots, p\}}_{\text{modes}}$$

Different Kinds of Stability

- Given a switched system $\dot{x} = f_\sigma(x)$

– **universal**, asymptotic stability:

$$x \rightarrow 0, \forall \sigma$$

it will always be stable. No matter how we switch

– **existential**, asymptotic stability:

$$\exists \sigma \text{ s.t. } x \rightarrow 0 \quad \text{There is a signal that will make it stable}$$

- If the switch signal is generated by an underlying hybrid automaton:

– **hybrid**, asymptotic stability:

$$x \rightarrow 0$$

Some Results

If all the individual modes are A.S., then

Existentially A.S.

- *Why?* Simply pick a mode and never switch!

Not always universally A.S.

- *Why?* See the counter example
- *What to do?* (common Lyapunov function)

read

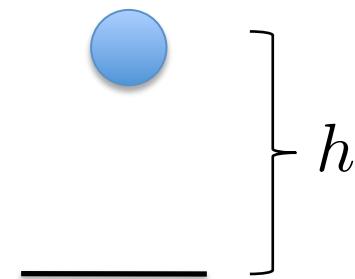
Practically Speaking

↙ all modes
should be AS

- Design stabilizing controllers for the subsystems
- Design the switching logic in the HA
- (Find common Lyapunov function) → really hard
- Be aware of the potential danger here and test, test, test!

Lecture 5.5 – The Bouncing Ball

- Let's model a ball bouncing on a surface:



- Equations of motion in-between bounces:

$$\ddot{h} = -g \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -g \end{bmatrix}, \quad y = [1 \ 0] x$$

- Bounces:

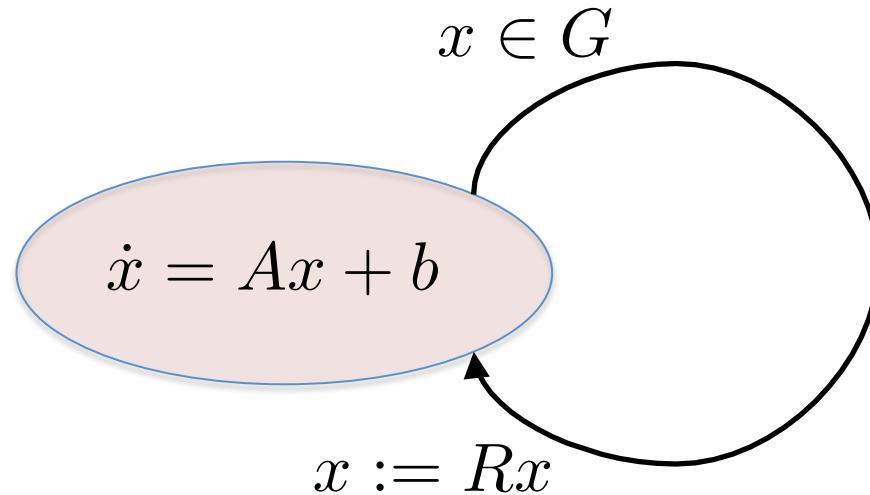
$$\dot{h} := -\gamma \dot{h} \Rightarrow x := \begin{bmatrix} 1 & 0 \\ 0 & -\gamma \end{bmatrix} x$$

reset condition

factor that accounts for energy lost

5.5.1

The Ball HA



Guards?

~~$$h \leq 0 \Rightarrow \begin{bmatrix} 1 & 0 \end{bmatrix} x \leq 0$$~~

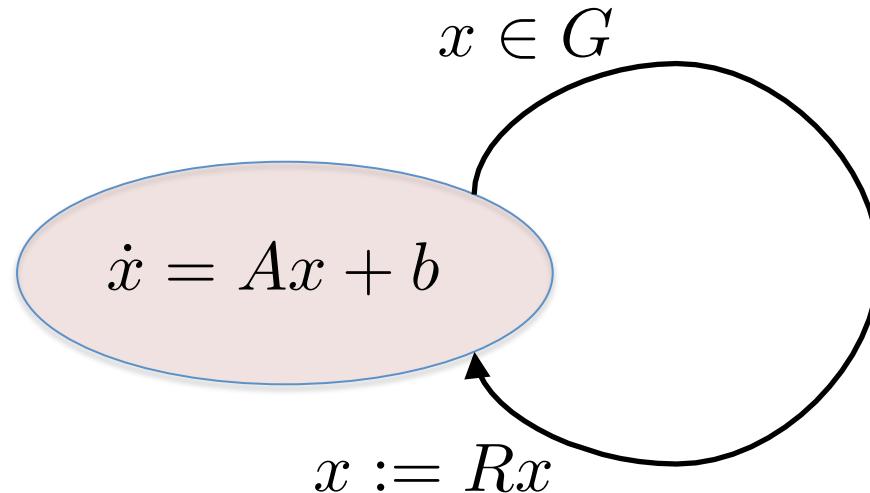
$$h \leq 0 \text{ and } \dot{h} \leq 0 \Rightarrow x \leq 0$$

goes on infinity loop

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & -\gamma \end{bmatrix}$$

The Ball HA



- Is this any good?
- To answer that we need to figure out how the system actually behaves!

Solving for the Output

$$\ddot{h} = -g \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -g \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$y = C\Phi(t, t_0)x(t_0) + C \int_{t_0}^t \Phi(t, \tau)Bu(\tau)d\tau$$

$$\Phi(t, \tau) = e^{A(t-\tau)}$$

$$A^2 = A^3 = \dots = 0$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{At^k}{k!} = I + At + 0 = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Solving for the Output

$$\ddot{h} = -g \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -g \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$y = C\Phi(t, t_0)x(t_0) + C \int_{t_0}^t \Phi(t, \tau)Bu(\tau)d\tau$$

$$\begin{aligned} y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & t - t_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h_0 \\ \dot{h}_0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 1 & 0 \end{bmatrix} \int_{t_0}^t \begin{bmatrix} 1 & t - \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -g \end{bmatrix} d\tau \end{aligned}$$

$$= h_0 + \dot{h}_0(t - t_0) - \frac{g}{2}(t - t_0)^2$$

Time In-Between Bounces?

$$y = h_0 + \dot{h}_0(t - t_0) - \frac{g}{2}(t - t_0)^2$$

$$h_0 = 0, \quad t_0 = 0$$

$$y(T) = 0 = \dot{h}_0 T - \frac{g}{2} T^2 = T(\dot{h}_0 - g/2T)$$

$$T = 0, \quad T = \frac{2\dot{h}_0}{g}$$

Accumulated Bounce Times

velocity at the beginning

$$\dot{h}_{0,0} = v$$

velocity after first bounce

$$\dot{h}_{0,1} = \gamma v$$

\vdots

$$\dot{h}_{0,k} = \gamma^k v$$

$$T = \frac{2\dot{h}_0}{g} \quad (\text{time in-between bounces})$$

time of first bounce

$$T_1 = \frac{2v}{g} \quad \text{time of second bounce}$$

$$T_2 = \frac{2v}{g} + \gamma \frac{2v}{g}$$

$$T_N = \frac{2v}{g} \sum_{k=0}^{N-1} \gamma^k \rightarrow \frac{2v}{g} \frac{1}{1-\gamma}$$

$$\gamma < 1$$

So What?

$$T_{\infty} = \frac{2v}{g} \frac{1}{1 - \gamma} < \infty$$

- The ball bounces an infinite number of times in finite time.
- More than just a mathematical curiosity
 - Simulations crash!
 - Hybrid system is undefined beyond this time!
- This is known as the **Zeno Phenomenon** – next lecture

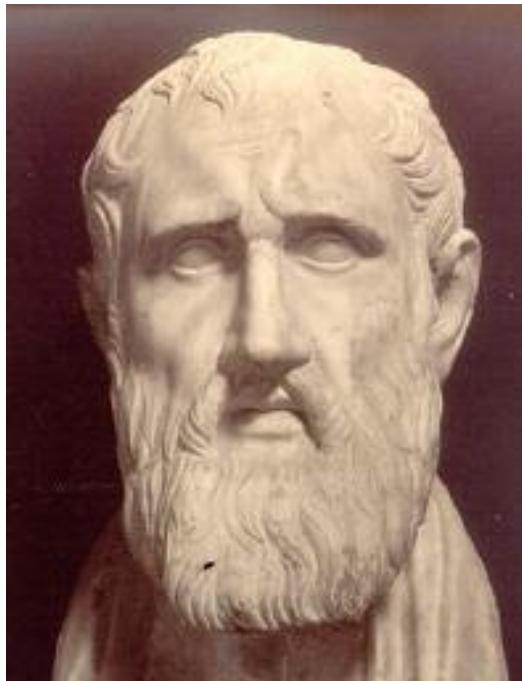
Lecture 5.6 – The Zeno Phenomenon

- Problem with the bouncing ball: Infinitely many switches in finite time
- This is bad:
 - Simulations crash
 - Model is not accurate
 - System behavior fundamentally ill-defined beyond “Zeno point”

*in real life, the ball
stop bouncing*

*↳ when the $t \rightarrow \infty$ series ends into
finite time*

The Zeno Phenomenon



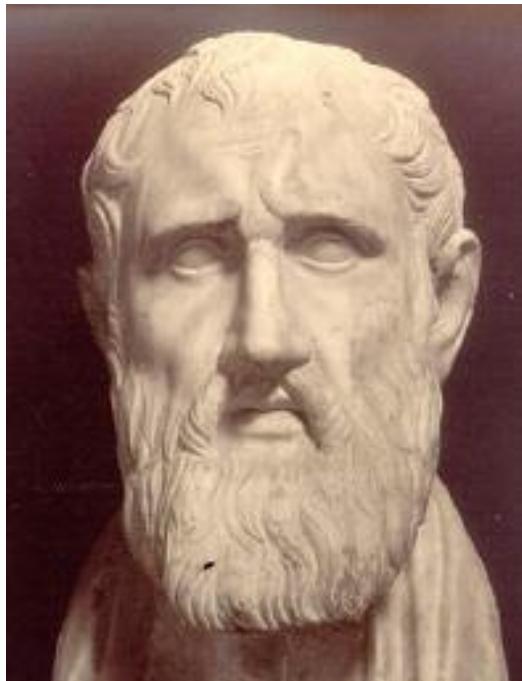
The hare



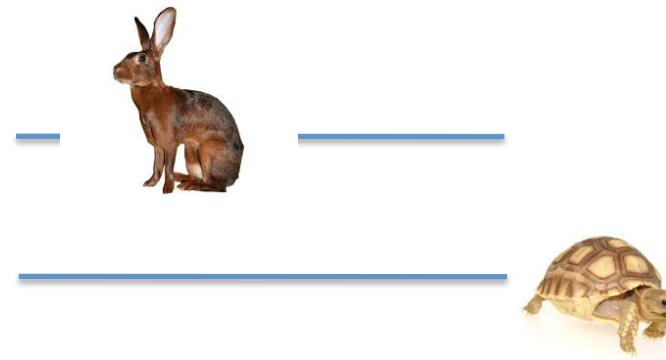
The tortoise

Zeno of Elea: 490 BC-430 BC

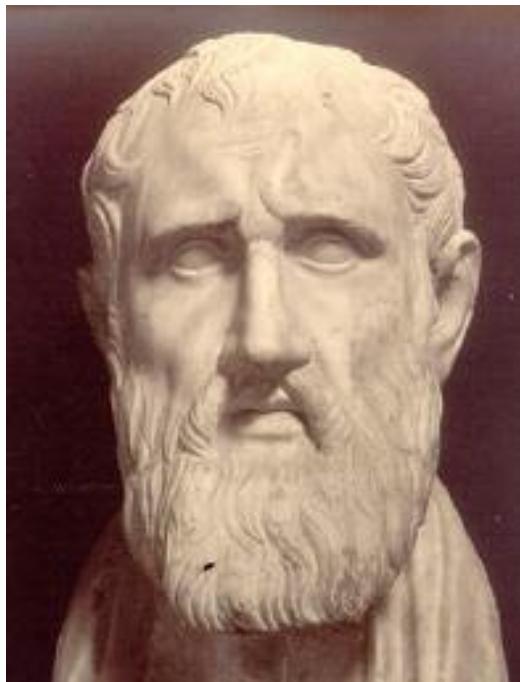
The Zeno Phenomenon



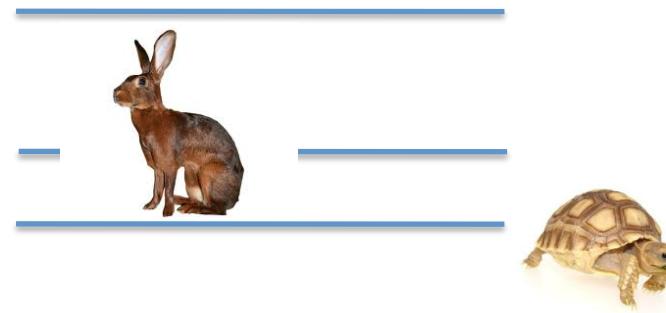
Zeno of Elea: 490 BC-430 BC



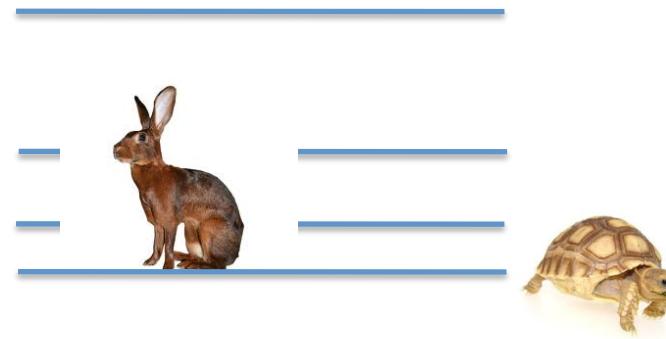
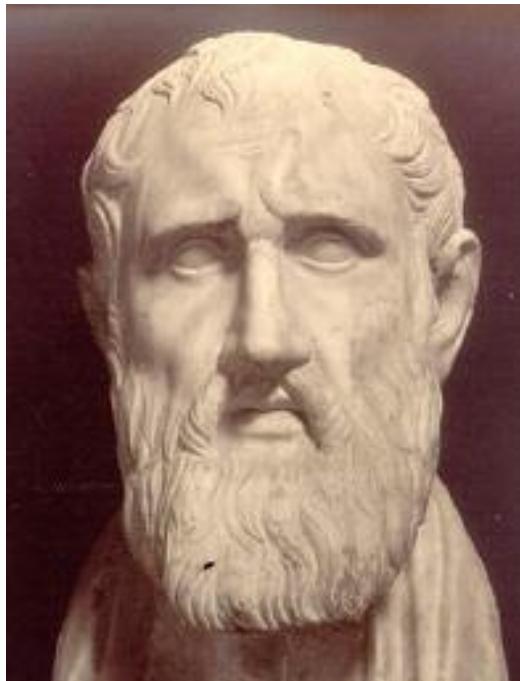
The Zeno Phenomenon



Zeno of Elea: 490 BC-430 BC



The Zeno Phenomenon



Zeno of Elea: 490 BC-430 BC

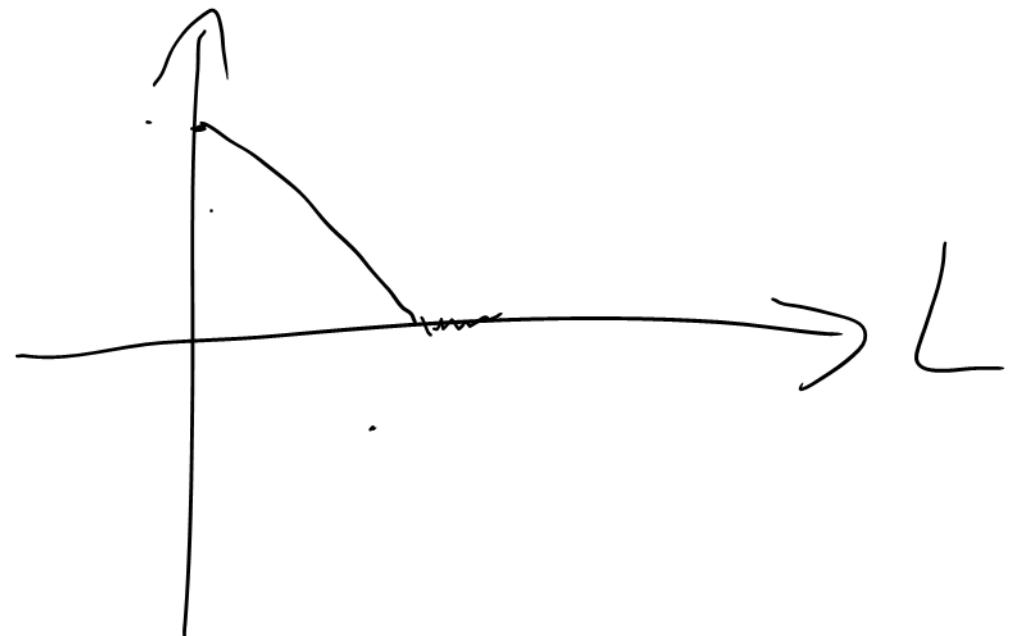
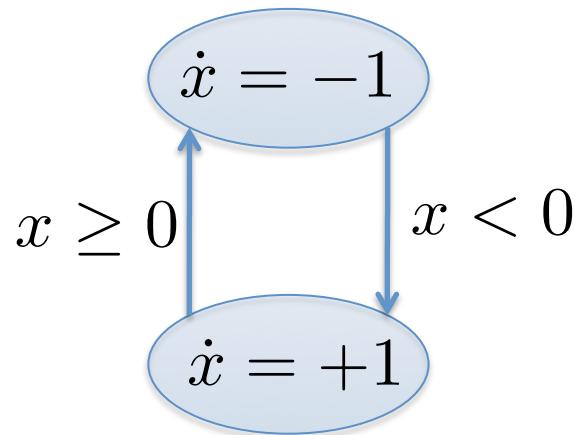
The Paradox: The hare never catches up with the tortoise

The Solution: Convergent series

The Problem: Infinitely many switches in finite time

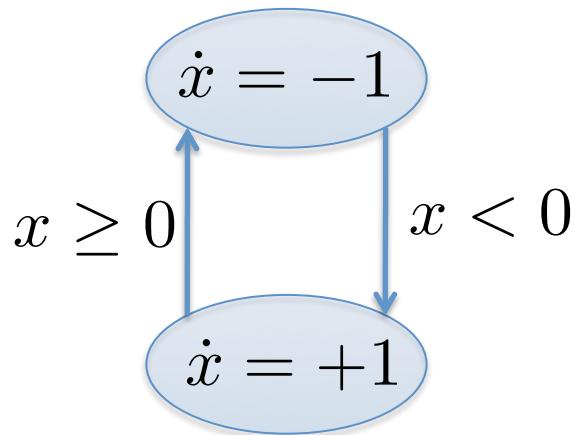
Example

$$\dot{x} = \begin{cases} -1 & x \geq 0 \\ +1 & x < 0 \end{cases}$$



Super-Zeno?

$$\dot{x} = \begin{cases} -1 & x \geq 0 \\ +1 & x < 0 \end{cases}$$



- This system switches infinitely many times in a single time-instant
- Zeno: Infinitely many switches in finite time
 - Type 1: In a single time-instant (*example on left*)
 - Type 2: Not Type 1 (bouncing ball)

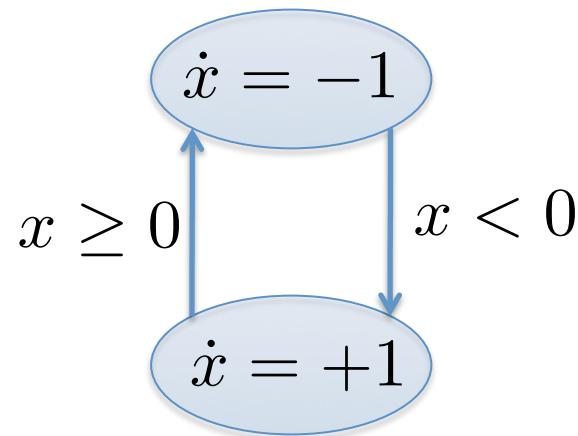
Good News and Bad News

- Zeno is a problem
- Type 1 is not only detectable, but one can deal with it in a rather straightforward manner
- Type 2 is overall hard to handle!

Test, test, test

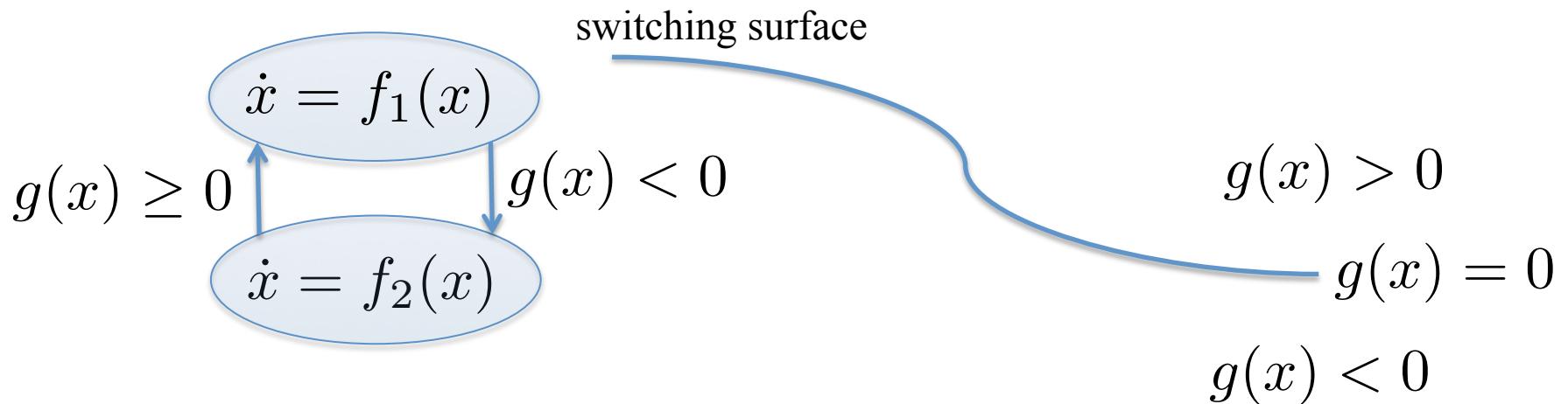
most common

Lecture 5.7 – Sliding Mode Control

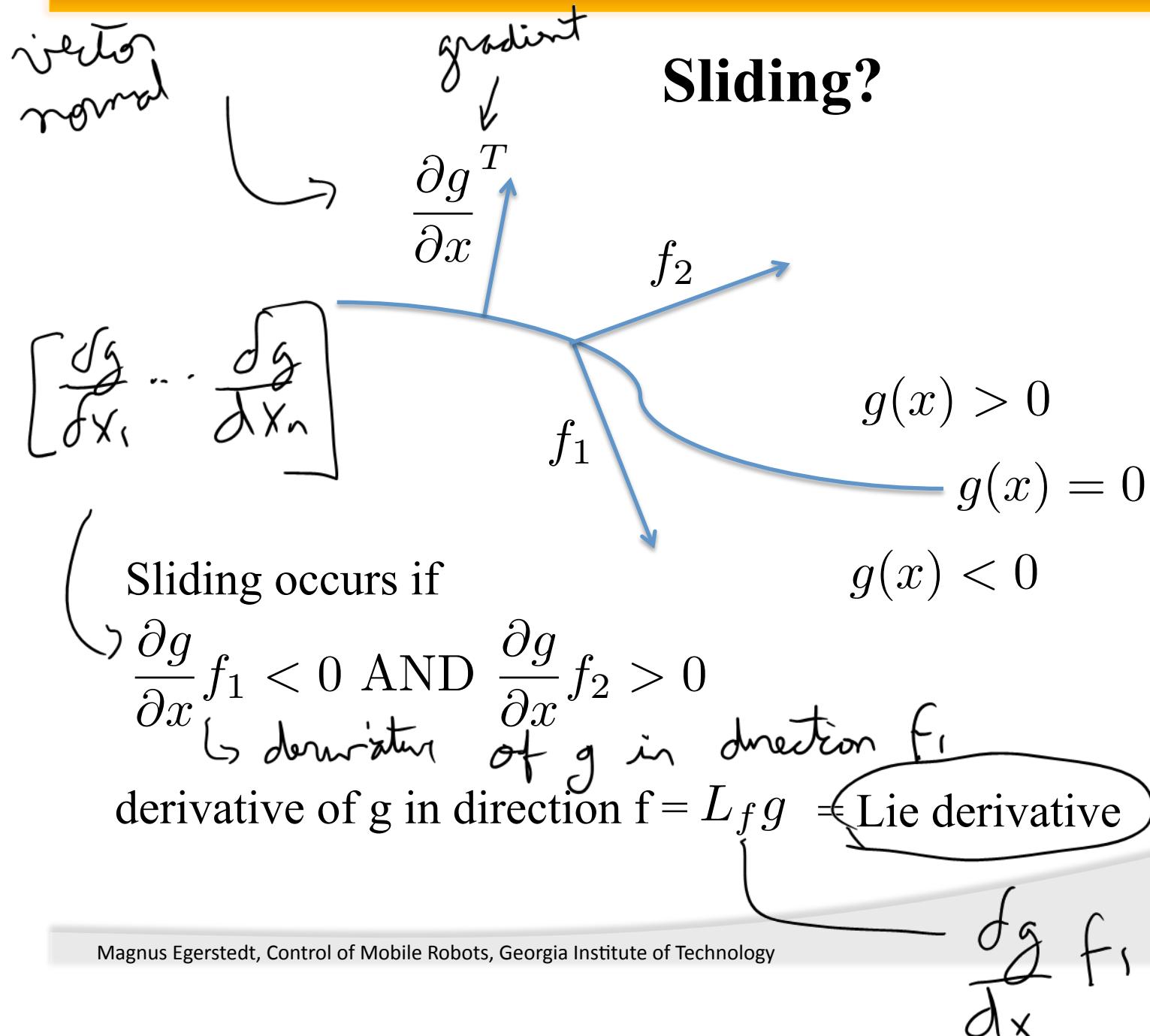


- It is clear what should happen!
- How do we make that mathematically sound?
- Sliding Mode Control

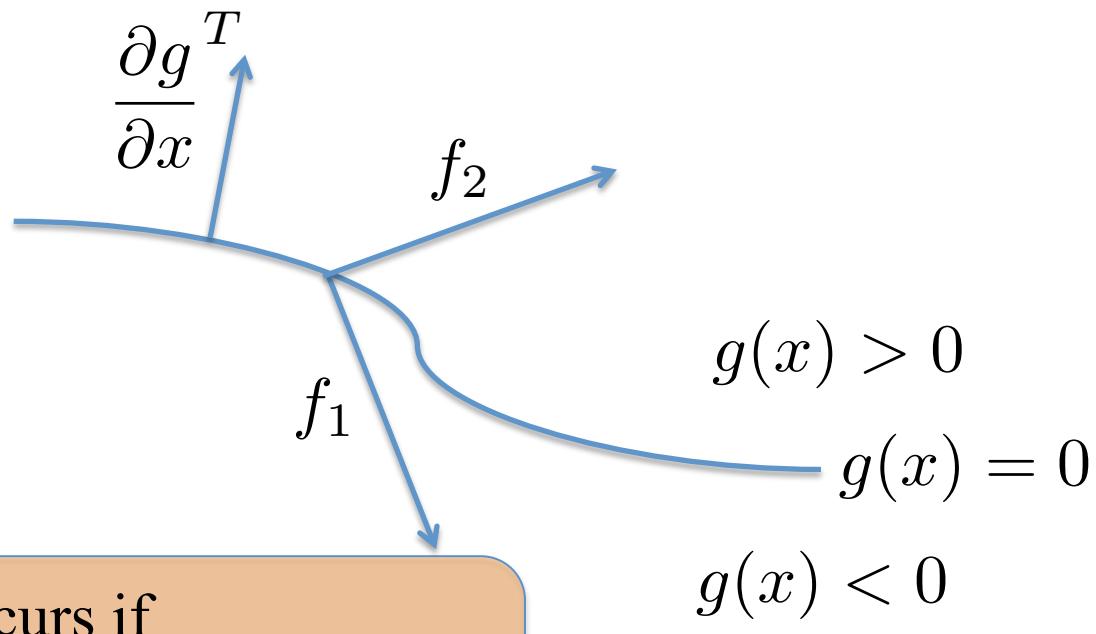
Switching Surfaces



- Both vector fields point “inwards” = bad!
- We should keep sliding along the switching surface
- Sliding Mode Control



Sliding?



derivative of g in direction $f = L_f g$ = Lie derivative

A Test For Type 1 Zeno

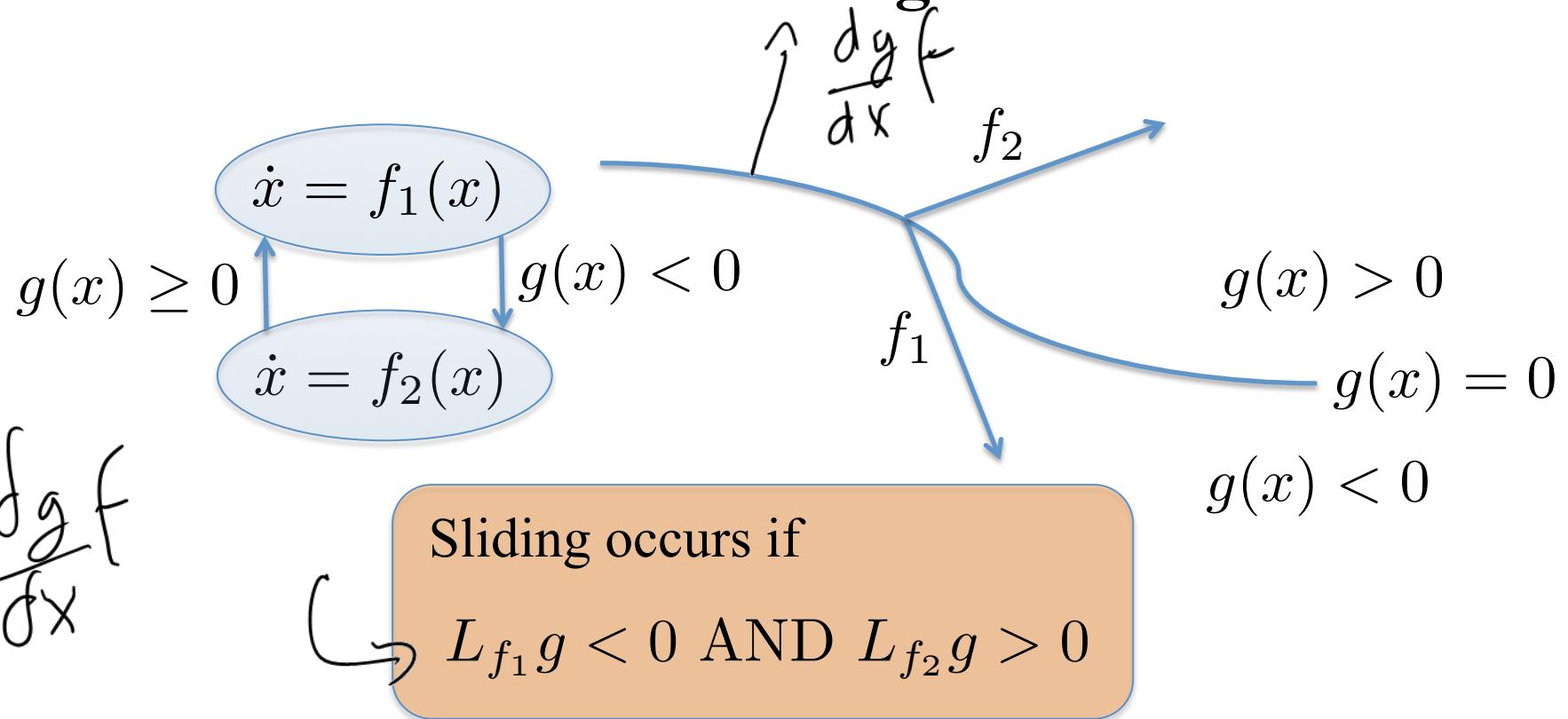
Sliding occurs if

$$L_{f_1}g < 0 \text{ AND } L_{f_2}g > 0$$

at $g(x) = 0$

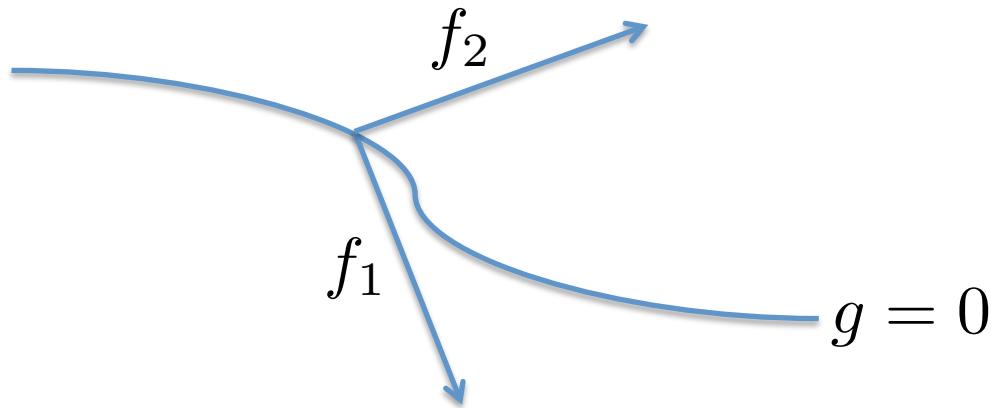
- Next time: But what happens beyond the Zeno point?

Lecture 5.8 – Regularizations



- How do we move beyond the Zeno point?

The Sliding “Mode”



$$\frac{dg}{dt} = 0$$

*when
sliding*

$$\begin{aligned} \frac{dg}{dt} &= \frac{\partial g}{\partial x} \dot{x} = \frac{\partial g}{\partial x} (\sigma_1 f_1 + \sigma_2 f_2) = \sigma_1 L_{f_1} g + \sigma_2 L_{f_2} g \\ &= 0 \Rightarrow \sigma_2 = -\sigma_1 \frac{L_{f_1} g}{L_{f_2} g} \end{aligned}$$

$$\sigma_1, \sigma_2 \geq 0, \quad \sigma_1 + \sigma_2 = 1$$

Back to the Example

$$\dot{x} = \begin{cases} -1 & x \geq 0 \\ +1 & x < 0 \end{cases} \quad g(x) = x = 0$$

Places we have zero

$$L_{f_1}g = \frac{\partial g}{\partial x} f_1 = 1(-1) = -1$$

$$L_{f_1}g = \frac{\partial g}{\partial x} f_2 = 1 \cdot 1 = 1$$

$$\sigma_2 = -\sigma_1 \frac{L_{f_1}g}{L_{f_2}g} = -\sigma_1 \frac{-1}{1} = \sigma_1$$

$$\sigma_1 = \sigma_2 = 0.5 \xrightarrow{\text{Since}} \sigma_1 + \sigma_2 = 1$$

The Induced Mode

$$\dot{x} = 0.5f_1 + 0.5f_2 = 0$$

$$\dot{x} = \sigma_1 f_1 + \sigma_2 f_2$$

↳ slide mode

The Induced Mode

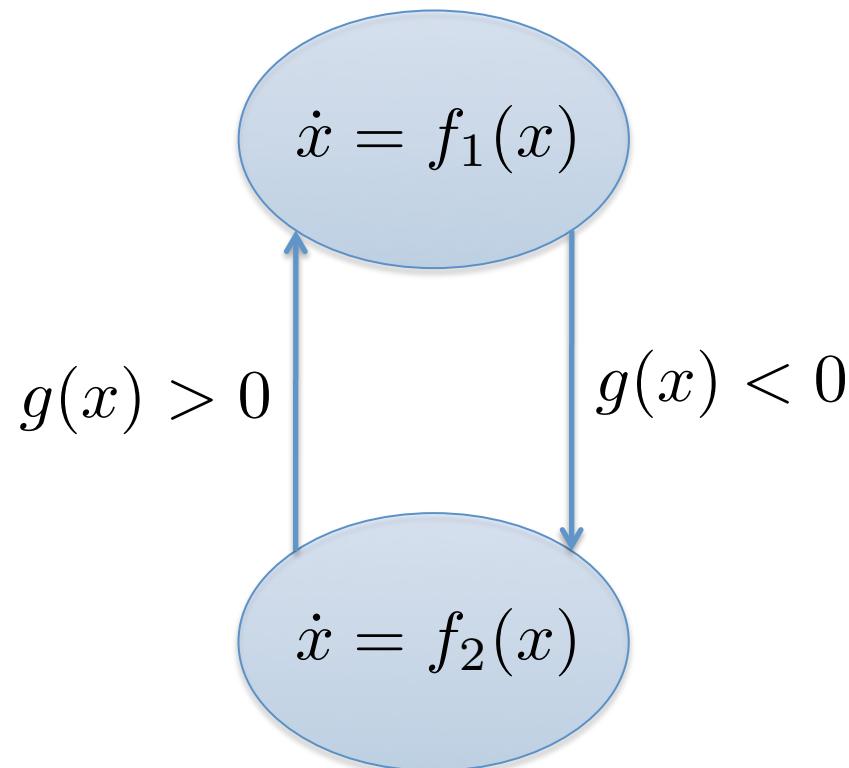
$$\sigma_2 = -\sigma_1 \frac{L_{f_1}g}{L_{f_2}g} \quad \sigma_1 + \sigma_2 = \sigma_1 \left(1 - \frac{L_{f_1}g}{L_{f_2}g}\right) = 1$$

$$\sigma_1 = \frac{1}{1 - \frac{L_{f_1}g}{L_{f_2}g}} = \frac{L_{f_2}g}{L_{f_2}g - L_{f_1}g}$$

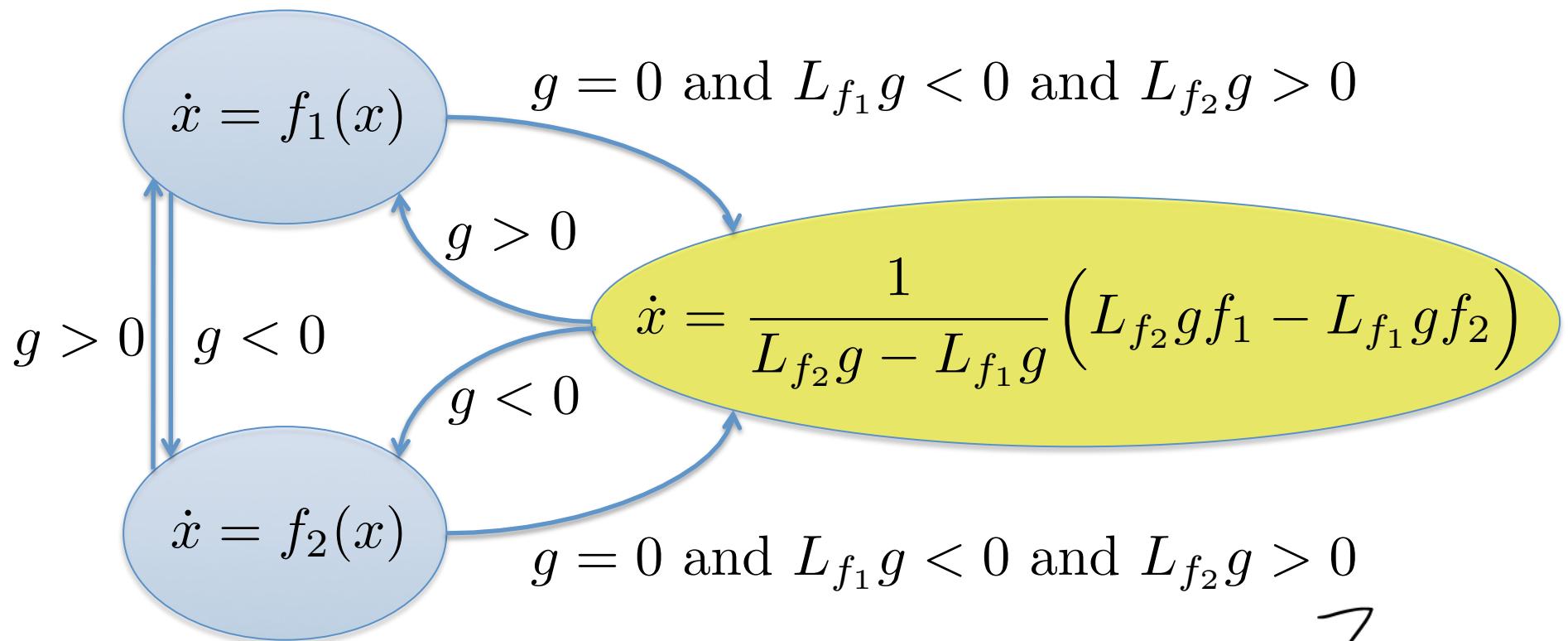
$$\sigma_2 = -\frac{L_{f_1}g}{L_{f_2}g - L_{f_1}g}$$

$$\dot{x} = \frac{1}{L_{f_2}g - L_{f_1}g} \left(L_{f_2}g f_1 - L_{f_1}g f_2 \right)$$

Regularizations of Type 1 Zeno HA



Regularizations of Type 1 Zeno HA



Hybrid system $\xrightarrow{\text{Zeno proof}}$ 5.8.5

Hybrid Systems: In Summary

- We have $\dot{x} = \Delta x + \beta u$ $y = Cx$
 - Models
 - Stability Awareness – Cc and CJ
 - Zeno \rightarrow Type 1, Type 2
 - Regularizations \rightarrow Solving Zeno
- Next Module: Back to **ROBOTICS!**