Georgia Tech | School of Electrical and Computer Engineering
College of Engineering

GRITS — Georgia Robotics and InTelligent Systems Laboratory

Control of Mobile Robots: Glue Lectures

Instructor:

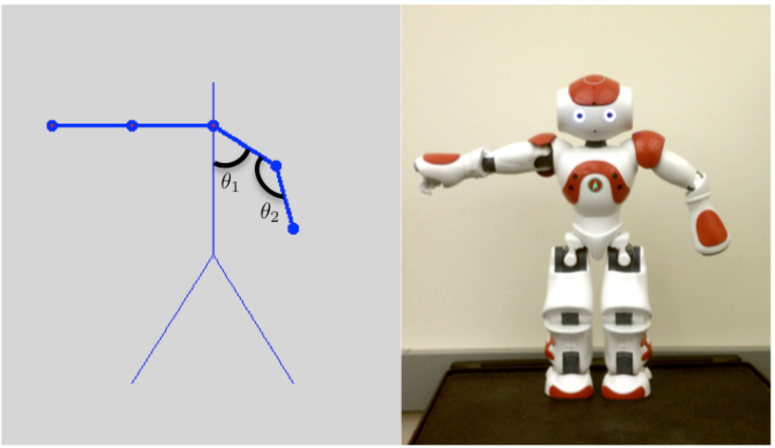
Smriti Chopra
ECE Ph.D. Candidate
Georgia Tech

Glue Lecture 4: Controllability and Observability

Pay attention, this lecture will help you all with Quiz 4!

System Stability

$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



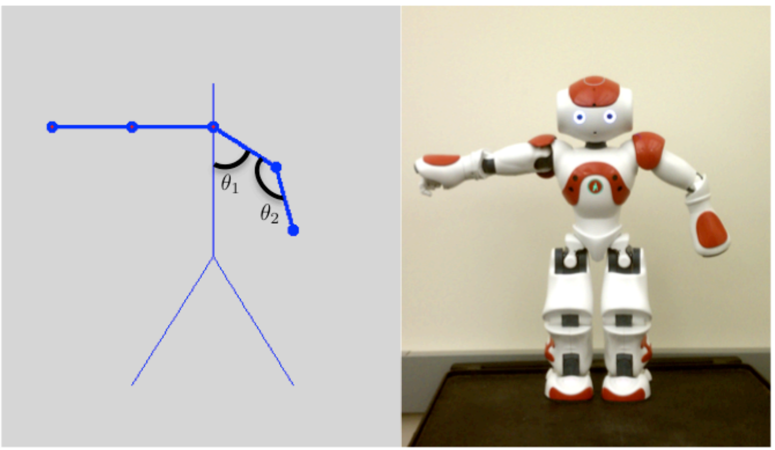
$$\dot{x} = Ax$$

In Matlab: `>> eig(A)`

All eigen values of A are zero! Unstable !!!

Introduce Control…

$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Say we can control $\ddot{\theta}_1$

$$\dot{x} = Ax + Bu$$

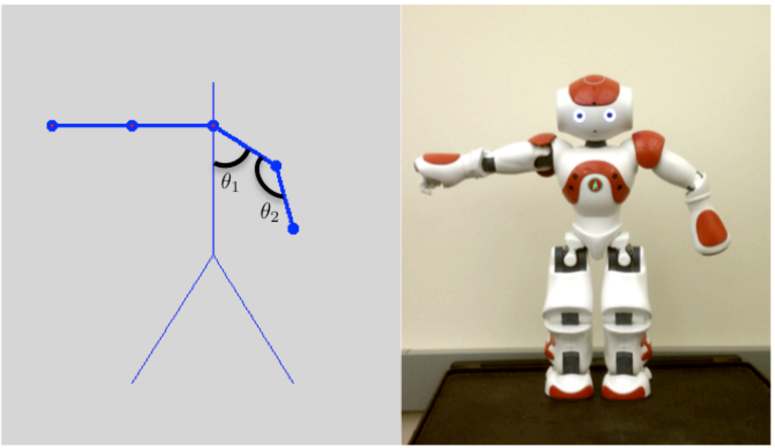
Controllable ??

$$\Gamma = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$rank(\Gamma) = 2 \neq 4$$

Introduce Control…

$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

n = number of states (x)

Say we can control $\ddot{\theta}_1$

$$\dot{x} = Ax + Bu$$

Uncontrollable !!!

$$\Gamma = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad rank(\Gamma) = 2 \neq 4$$

In Matlab…

```
EDU>> A=[0 1 0 0; 0 0 0 0; 0 0 0 1; 0 0 0 0]

A =

     0     1     0     0
     0     0     0     0
     0     0     0     1
     0     0     0     0

EDU>> B=[0; 1; 0; 0]

B =

     0
     1
     0
     0

EDU>> ctrb(A,B)

ans =

     0     1     0     0
     1     0     0     0
     0     0     0     0
     0     0     0     0

EDU>> rank(ctrb(A,B))

ans =

     2
```

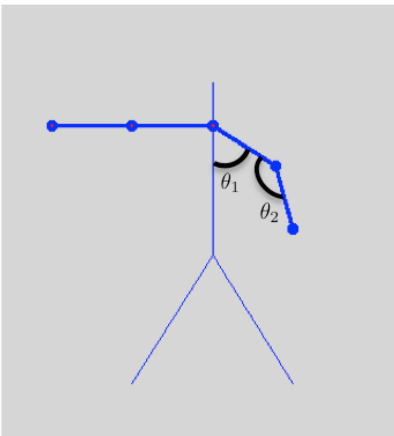
$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

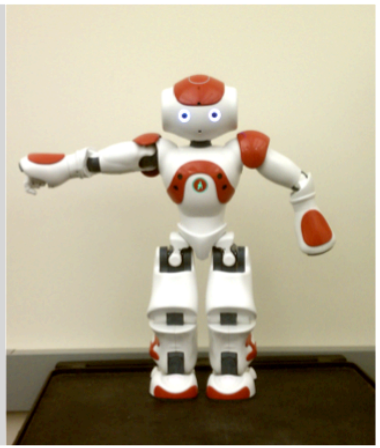
$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Uncontrollable !!!

Need "more" control…

$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



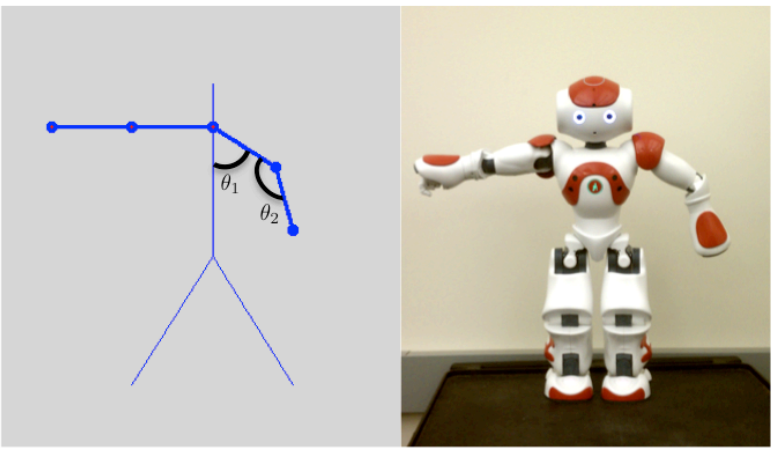


$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$

Need "more" control…

$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Say we can control $\ddot{\theta}_1$ and $\ddot{\theta}_2$

$$\dot{x} = Ax + Bu$$ Controllable ??

$$\Gamma = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

In Matlab again ☺…

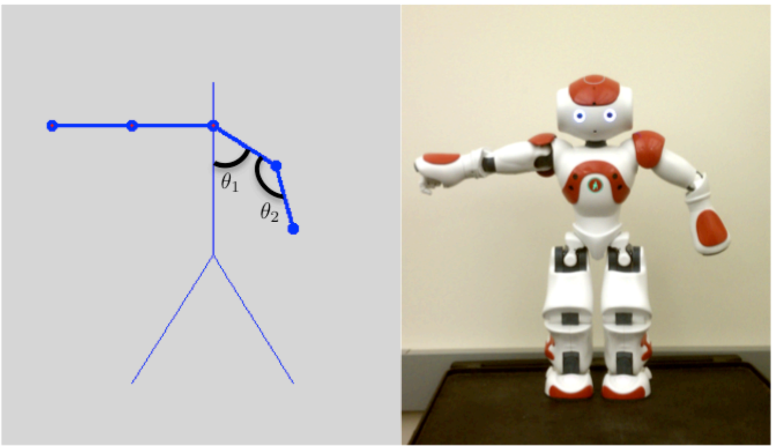
```
EDU>> A=[0 1 0 0; 0 0 0 0; 0 0 0 1; 0 0 0 0]

A =

     0     1     0     0
     0     0     0     0
     0     0     0     1
     0     0     0     0

EDU>> B=[0 0; 1 0; 0 0; 0 1]

B =

     0     0
     1     0
     0     0
     0     1

EDU>> ctrb(A,B)

ans =

     0     0     1     0     0     0     0     0
     1     0     0     0     0     0     0     0
     0     0     0     1     0     0     0     0
     0     1     0     0     0     0     0     0

EDU>> rank(ctrb(A,B))

ans =

     4
```

$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Controllable !!!

State feedback…



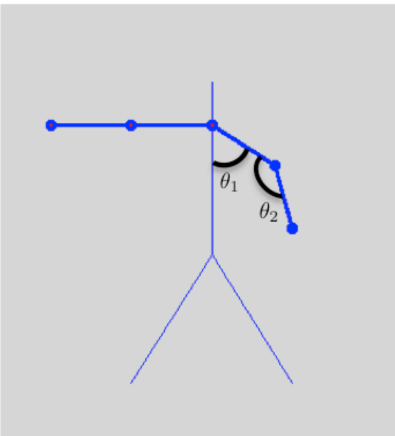
$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

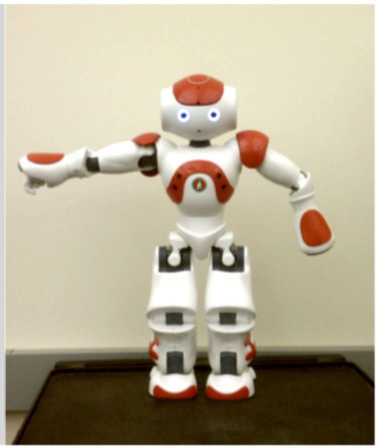
$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$ Controllable

$$u = -Kx$$

State feedback… on a simpler system



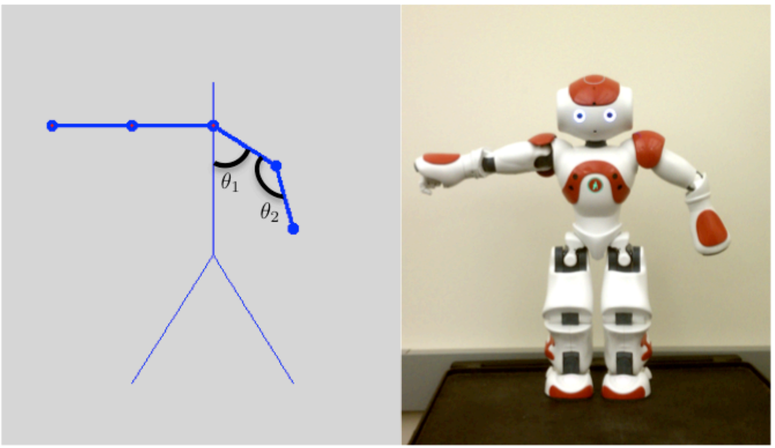


$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\dot{x} = Ax$$

Unstable !!!

State feedback… on a simpler system



$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

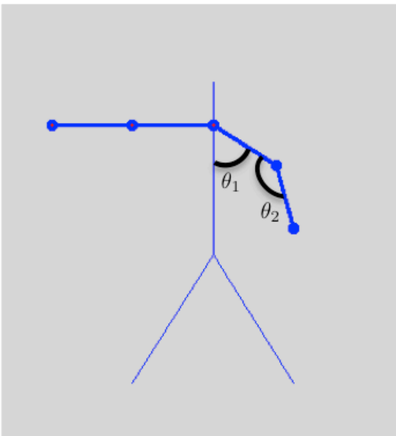
$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

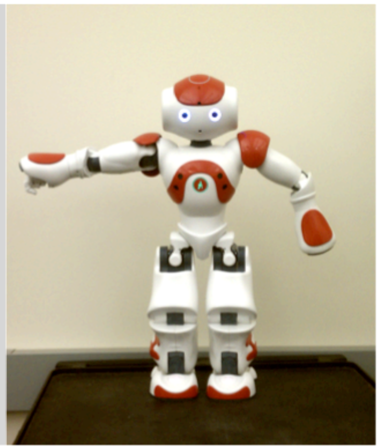
Say we can control $\ddot{\theta}_1$

$$\dot{x} = Ax + Bu$$

Controllable ???

State feedback… on a simpler system





$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$ Controllable !!!

$$u = -Kx$$

State feedback… on a simpler system

$$\dot{x} = Ax + Bu$$

Controllable

$$u = -Kx$$

$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\dot{x} = (A - BK)x$$

$$\dot{x} = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) x$$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}}_{A'} x$$

State feedback... on a simpler system

$$\dot{x} = \boxed{\begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}} x \Rightarrow A'$$

$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Characteristic polynomial:

$$det(A' - \lambda I)$$

$$\begin{matrix} -\lambda & 1 \\ -k_1 & -k_2 - \lambda \end{matrix} = \lambda^2 + k_2\lambda + k_1$$

$$\lambda^2 + k_2\lambda + k_1$$

Pick your favorite 2 eigen values (LHP) :

$$\lambda_1 = -1$$

$$\lambda_2 = -2$$

$$(\lambda + 1)(\lambda + 2)$$

$$\lambda^2 + 3\lambda + 2$$

$$\prod_{n=1}^{n=\infty} (\lambda - \lambda_n)$$

$$k_1 = 2$$

$$k_2 = 3$$

State feedback… on a simpler system

$$\dot{x} = \boxed{\begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}} x \Rightarrow A'$$

$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Characteristic polynomial:

$$det(A' - \lambda I)$$

$$\lambda^2 + k_2\lambda + k_1$$

Pick your favorite 2 eigen values (LHP) :

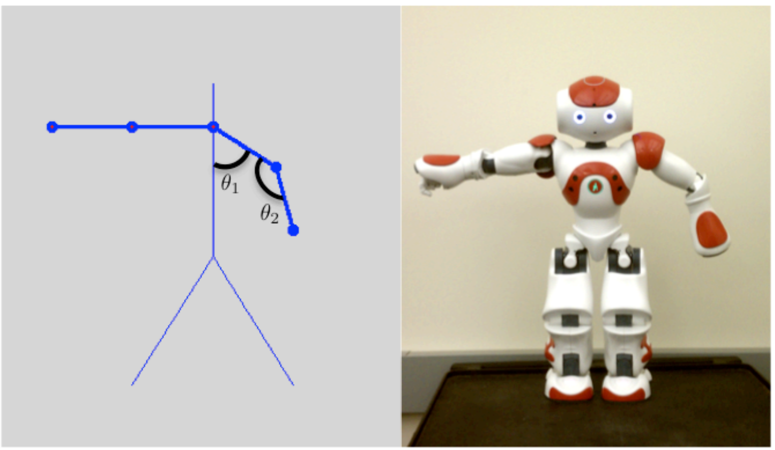
$$\lambda_1 = -1$$

$$\lambda_2 = -2$$

$$\lambda^2 + 3\lambda + 2$$

$$\boxed{\begin{aligned} k_2 &= 3 \\ k_1 &= 2 \end{aligned}}$$

State feedback… on a simpler system



$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

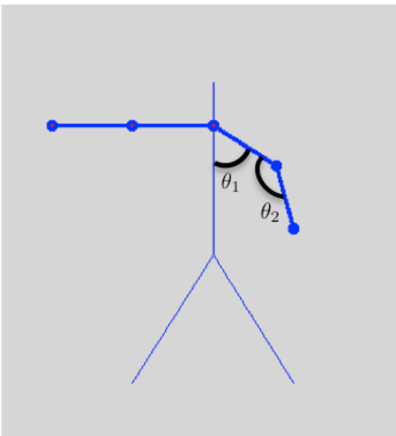
$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

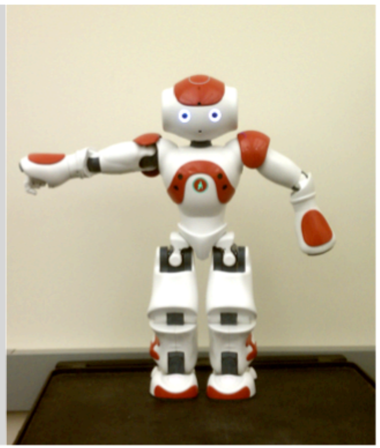
$$\dot{x} = Ax + Bu$$ Stable !!!

$$u = -Kx$$

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

But we don't know our state…





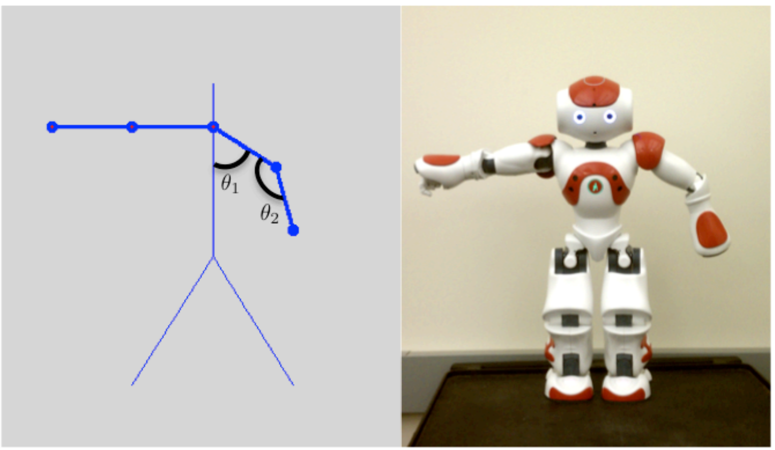
$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$

$$u = -Kx$$

Need to estimate our state…



$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$ ↙ Physics

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$ ↙ Sensors

↳ actuators

Say we can "see" θ_1

$$\dot{x} = Ax + Bu \qquad y = Cx$$

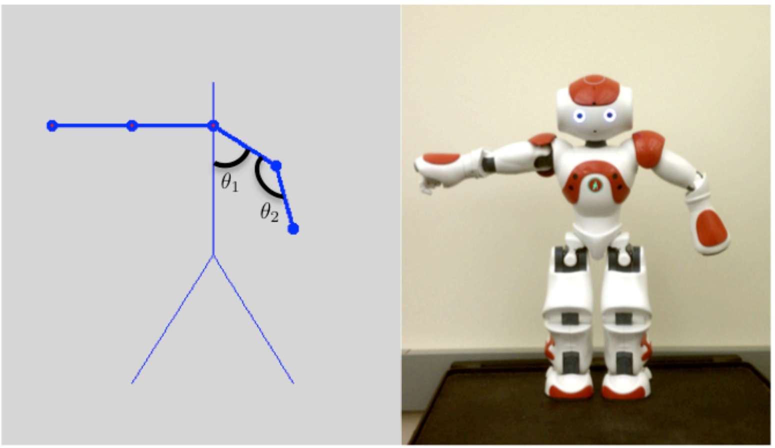
Observable ???

↳ $\mathrm{rank}(\Omega) = n$

$$\Omega = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}$$

$$\boxed{u = -K\hat{x}}$$

Need to estimate our state…



$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

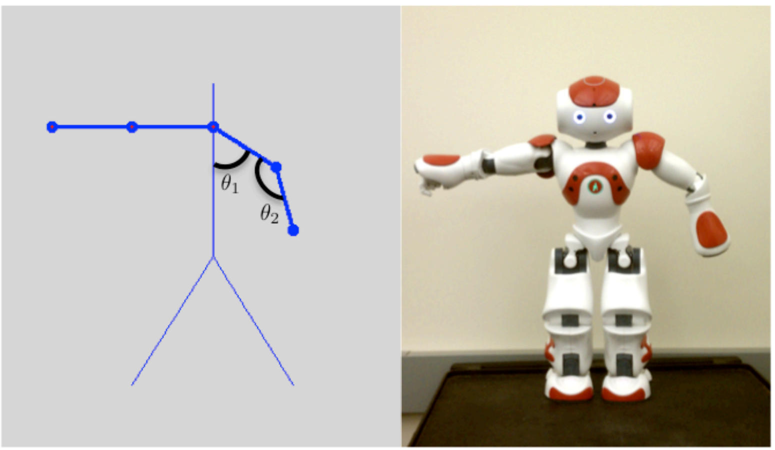
$$\dot{x} = Ax + Bu \qquad y = Cx$$

Observable ???

$$\Omega = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad rank(\Omega) = 2$$

equals the state (x)

Need to estimate our state…



$$x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$\dot{x} = A - BK$

$$\dot{x} = Ax + Bu \qquad y = Cx$$

Observable !!!

output

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

↳ expected

$$\dot{e} = \dot{x} - \dot{\hat{x}} = (A - LC)e$$

↳ We pick L same way we pick K

All together… execution

$$\dot{x} = Ax + Bu \qquad y = Cx \qquad x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Controllable and Observable

$$K \qquad L \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

a) Wake up $t = t_0, \quad x = x_0, \quad \hat{x} = \hat{x}_0$

b) Start Loop (dt increments)

c) Read output $y = Cx$

d) Compute control $u = -K\hat{x}$

e) Send control u

f) Update $\hat{x}$ using dynamics $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$

g) Repeat

$$+ \quad \hat{x}_{k+1} = \hat{x}_k + dt\,\dot{\hat{x}}$$

Check the forums, and good luck with Quiz 4!