Two-Dimensional Minimum Sensor Array: A New Perspective to Array Design

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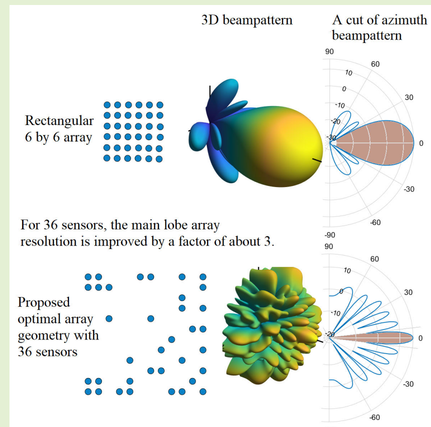
***Abstract*—In this article, we design/propose a new class of planar arrays sparsely located on a two-dimensional (2-D) lattice to achieve the highest degrees of freedom (DOF) for a given number of sensors. We formulate an optimization problem to search for a hole-free 2-D sparse array with the fewest possible sensors. We convert this problem into a nonlinear binary problem by changing the variables, then into a binary linear programming problem, and solve it efficiently by employing branch and bound programming. Our results show that this optimal array outperforms alternative state-of-the-art 2-D array geometries in terms of a target resolution and direction of arrival (DOA) estimation accuracy given the number of sensors. Moreover, the proposed array outperforms other geometries in terms of the resolution probability of close targets and in the presence of mutual coupling.**



***Index Terms*—Branch and bound optimization, degree of freedom (DOF), linear integer programming, sparse two-dimensional (2-D) planar array.**

I. Introduction

Two-dimensional planar arrays (2DPAs) are widely used in a variety of applications allowing localization of sources in two dimensions such as in radar, communications, sound systems, and sonar applications [1], [2], [3], [4], [5], [6]. These arrays enable the joint estimation of the azimuth and the elevation angles of sources [7]. The sparse arrays have long been investigated to enhance the spatial resolution of the uniform linear arrays in source localization and separation without increasing the number of sensors. Using a sparse array, the average distance amongst sensors is considerably increased which leads to a significant reduction in undesired mutual coupling effects. In contrast to conventional configurations such as uniform rectangular arrays (URAs) [8], [9] and uniform circular arrays (UCAs) [10] and hexagonal arrays [11], optimizing sparse planar arrays (SPAs) can achieve features such as increasing the degree of freedom (DOFs), enhancing the spatial resolution in separating closer sources, improving localization accuracy, and reducing the mutual coupling effects. The idea of linear sparse arrays has been investigated in many papers, e.g., minimum redundancy array (MRA) [12] and Golomb array [13]. There is a complex relationship between the array geometry and the ultimate performance criterion [1]. This is why nonconvex numerical optimization such as search algorithms is used to optimize sensor positions. Usually, there is no closed-form solution for optimal sensor positioning using most criterions [1]. As alternative, several suboptimal arrays with simple element positioning are suggested, e.g., nested arrays [14], super nested arrays [15], and coprime arrays [16] with its generations [17], [18]. These arrays are proposed based on the difference co-array (DCA) concept.

The SPAs have evolved along with the sparse linear arrays, e.g., Billboard arrays [19], open box arrays [20], hourglass arrays [21], 2-D nested arrays [22], and 2-D coprime arrays [23]. These arrays do not achieve the maximum for a given number of sensors. The open box array achieves the highest DOF among them. Unlike linear sparse arrays, conventional direction of arrival (DOA) estimators (e.g., MUSIC [24] and ESPRIT [25]) cannot be directly applied to these SPAs as they cause ambiguity in DOA estimation. However, these algorithms can be applied to the uniform part of the DCA to estimate the DOAs without ambiguity. In this case, by optimizing the sensor configuration, we can considerably enlarge the dimensions of the uniform part of the DCA in order

Manuscript received 21 February 2023; revised 22 April 2023; accepted 1 May 2023. Date of publication 11 May 2023; date of current version 29 June 2023. The associate editor coordinating the review of this article and approving it for publication was Dr. Qammer H. Abbasi. *(Corresponding author: Mohammad Ebrahimi.)*

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Digital Object Identifier 10.1109/JSEN.2023.3273401

to enable the discovery of most targets and improve the array's resolution, a topic that has received little attention in the literature. The existing arrays do not always achieve the maximum of the attainable continuous component of the DCA.

In this article, we formulate the design of a 2-D hole-free array as a novel integer minimization problem, this is equivalent to maximizing the DOF to achieve the best target resolution among all 2-D arrays with the same number of sensors. Our simulation results reveal that the proposed array outperforms the existing arrays using the same number of sensors and not only archives the best DOA estimation accuracy but also allows the discovery of most targets and provides the highest probability of target resolution. This proposed problem is a binary nonlinear problem which we converted into an equivalent linear problem by changing the variables. We solve this problem by branch and bound method to find the global optimal geometry.

A. Organization

In Section II, we review the background and the signal model. In Section III, we present the sparse linear array design as a linear optimization problem. In Section IV, our performance studies and simulation results show that the proposed arrays outperform the existing state-of-the-art arrays with respect to the number of sensors for a given DOF, the ability to resoling more targets and the accuracy of the DOA estimation. We finally conclude in Section V.

B. Notations

The lower-case and upper-case bold letters denote vectors and matrices, respectively. We denote $N \times N$ identity matrix by $\mathbf{I}_N$. We denote the conjugate transpose and transpose of a matrix by $(.)^*$ and $(.)^T$, respectively. The Kronecker and the Kharti–Rao products are denoted by $\otimes$ and $\odot$ [26]. We use fonts such as $\mathbb{S}$ for sets of pairs of integer numbers, e.g., $\mathbb{Z}$ and $\mathbb{N}_0$ are the set of integers and nonnegative integers. We denote the concatenations of columns of (.) by vec(.). We denote $|.|$ as the cardinality of a set and $*$ as the discrete convolution operator for two sequences. We use $\lceil . \rceil$ and $\lfloor . \rfloor$ for the integer ceiling and floor functions, respectively.

II. 2-D Spare Array on a 2-D Rectangular Lattice

We consider the 2-D lattice $\mathbb{Z}^2 = \{(n_x, n_y) | n_x, n_y \in \mathbb{Z}\}$ as the set of possible normalized locations of N sensors for a 2-D planar array. We aim to find a subset $\mathbb{S} \subseteq \mathbb{Z}^2$ with $|\mathbb{S}| = N$. Consider D sources impinging on sensors located at entries of $\mathbb{S}$, we can write the received signal as [21]

$$\mathbf{x}_s = \sum_{i=1}^{D} A_i \mathbf{v}_s\left(\bar{\theta}_i, \bar{\phi}_i\right) + \mathbf{n}_s \quad \forall s = (n_x, n_y) \in \mathbb{S} \tag{1}$$

where $\bar{\theta}_i = (d/\lambda)\sin\theta_i \cos\phi_i$, $\bar{\phi}_i = (d/\lambda)\sin\theta_i \sin\phi_i$, and A_i are the normalized elevation, azimuth, and amplitude of ith target, respectively. Moreover, we have $\mathbf{v}_s(\bar{\theta}_i, \bar{\phi}_i) = e^{j2\pi(\bar{\theta}_i n_x + \bar{\Phi}_i n_y)}$. We assume that the signals are uncorrelated, i.e., $\mathbb{E}[A_i A_j^*] = \sigma_i^2 \delta_{i,j}$ and there is no correlation between signal and white noise, i.e., $\mathbb{E}[\mathbf{n}_s \mathbf{n}_s^H] = \sigma^2 \mathbf{I}$ and $\mathbb{E}[A_i \mathbf{n}_s^H] = \mathbf{0}$. This allows us to write the covariance matrix of the received signal as

$$\mathbf{R}_s \doteq \mathbb{E}\left[\mathbf{x}_s \mathbf{x}_s^H\right] = \sum_{i=1}^{D} \sigma_i^2 \mathbf{v}_s\left(\bar{\theta}_i, \bar{\phi}_i\right) \mathbf{v}_s^H\left(\bar{\theta}_i, \bar{\phi}_i\right) + \sigma^2 \mathbf{I}. \tag{2}$$

By vectorizing (2), we obtain the DCA of the array as [14]

$$\mathbf{x}_d = \mathrm{vec}(\mathbf{R}_s) = \sum_{i=1}^{D} \sigma_i^2 \mathbf{v}_d\left(\bar{\theta}_i, \bar{\phi}_i\right) + \sigma^2 \mathbf{i} \quad \forall d \in \mathbb{D} \tag{3}$$

where $\mathbf{i} = \mathrm{vec}(\mathbf{I}_N)$ and $d \in \mathbb{D}$ are the difference of two sensor locations in $\mathbb{S}$ and the DCA is defined by

$$\mathbb{D} \doteq \{\mathbf{n}_1 - \mathbf{n}_2 \mid \mathbf{n}_1, \mathbf{n}_2 \in \mathbb{S}\}. \tag{4}$$

We denote the URA of $\mathbb{D}$ by $\mathbb{U}$ [21] and the hole-set in the DCA by $\mathbb{H} = \mathbb{V} - \mathbb{D}$, where

$$\begin{aligned} \mathbb{V} &\doteq \big\{(v_x, v_y) | \min(\mathbb{D}_x) \le v_x \le \max(\mathbb{D}_x), \\ &\qquad \min(\mathbb{D}_y) \le v_y \le \max(\mathbb{D}_y)\big\} \\ \mathbb{D}_x &\doteq \{x \mid (x, y) \in \mathbb{D}, y \in \mathbb{Z}\}, \quad \mathbb{D}_y \doteq \{y \mid (x, y) \in \mathbb{D}, x \in \mathbb{Z}\}. \end{aligned} \tag{5}$$

Definition 1 (URA): We define $\mathbb{U} = \{(p, q) \in \mathbb{D} : p \le m, q \le n\}$ as *the central URA segment* of $\mathbb{D}$ where m and n are the largest integers for which we have

$$\{0, \pm 1, \pm 2, \ldots, \pm m\} \subseteq \mathbb{D}_x, \ \{0, \pm 1, \pm 2, \ldots, \pm n\} \subseteq \mathbb{D}_y. \tag{6}$$

Similar to 1-D arrays, the number of detectable targets (the DOF) is directly related to the $|\mathbb{U}|$ [14], [23], [24], [25], [26], [27].

Definition 2 (Array Sequence): For a 2-D sparse array with N sensors located in $\mathbb{S}$, we define a 2-D sequence for $\mathbb{S}$ as

$$a_{i,j}^{\mathbb{S}} = \begin{cases} 1, & (i, j) \in \mathbb{S} \\ 0, & (i, j) \notin \mathbb{S}. \end{cases} \tag{7}$$

Definition 3 (Weight): For a 2-D array $\mathbb{S}$, the weight is [21]

$$w_{m,n}^{\mathbb{S}} = \begin{cases} |\mathbb{F}(m, n)|, & \forall (m, n) \in \mathbb{D} \\ 0, & \forall (m, n) \in \mathbb{H} \end{cases} \tag{8}$$

where $\mathbb{F}(m, n) = \{(m, n) \in \mathbb{S}^2 : m \in \mathbb{D}_x, n \in \mathbb{D}_y\}$. We have $|\mathbb{S}| = \sum_{m=0}^{M} \sum_{n=0}^{N} a_{m,n}^{\mathbb{S}} = w_{0,0}^{\mathbb{S}}$ and $a_{m,n}^{\mathbb{D}} = \min(w_{m,n}^{\mathbb{S}}, 1)$.

Example 1: Fig. 1 illustrates an SPA with ten sensors located at $\mathbb{S}$ its URA and the weight function.

Corollary 1: From (3), we can simply deduce that for any hole-free array we have $w_{m,n}^{\mathbb{S}} > 0$ for all $(m, n) \in \mathbb{V}$.

Theorem 1: The weight function of any 2-D planner array $\mathbb{S}$ with the array sequence $a(i, j)$ is given by $w_{m,n}^{\mathbb{S}} = a_{i,j}^{\mathbb{S}} \otimes a_{i,j}^{\mathbb{S}}$, where $\otimes$ is the 2-D discrete convolution defined by $a_{n_1,n_2} \otimes b_{n_1,n_2} = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} b_{k_1,k_2} a_{n_1-k_1,n_2-k_2}$.

Proof: Without loss of generality suppose that all n_x and n_y are non-negative and one of the sensors are at $(0, 0)$. In this case for $0 \le k, k - m \le M, 0 \le l,$ and $l - n \le N$, we have

$$a_{k,l}^{\mathbb{S}} a_{k-m,l-n}^{\mathbb{S}} = \begin{cases} 1, & m \in \mathbb{D}_x, \ n \in \mathbb{D}_y \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

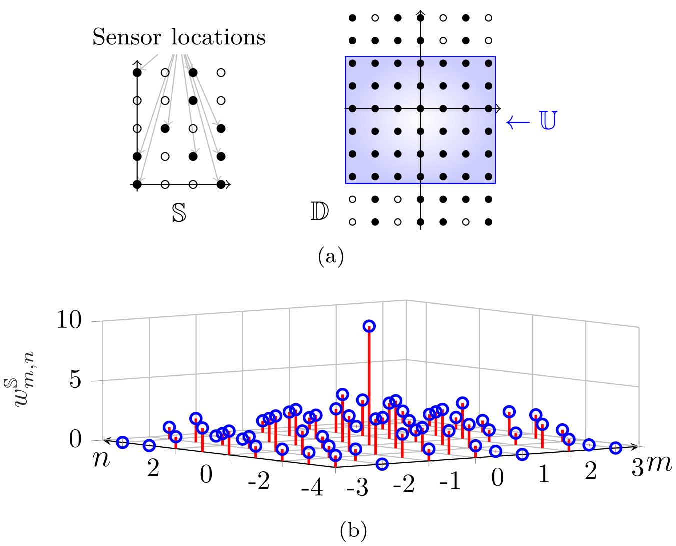


Fig. 1. Illustration of a 2-D SPA with ten sensors at $S = \{(0,0),(0,1),(0,4),(1,2),(2,1),(2,3),(2,4),(3,0),(3,1),(3,2)\}$, its DCA D, and its URA U. (a) Array $\mathbb{S}$, its DCA $\mathbb{D}$, and its URA $\mathbb{U}$. (b) Weight function $w^{\mathbb{S}}_{m,n}$ and $w^{\mathbb{S}}_{\mathbf{0},\mathbf{0}} = |\mathbb{S}|$.

Thus, Definition 3 obviously yields to

$$w^{\mathbb{S}}_{m,n} = \begin{cases} \sum_{k=0}^{M+m}\sum_{l=0}^{N+n} a^{\mathbb{S}}_{k,l}a^{\mathbb{S}}_{k-m,l-n} & -M \le m \le 0,\ -N \le n \le 0, \\ \sum_{k=0}^{M+m}\sum_{l=n}^{N} a^{\mathbb{S}}_{k,l}a^{\mathbb{S}}_{k-m,l-n} & -M \le m \le 0,\ 0 \le n \le N, \\ \sum_{k=m}^{M}\sum_{l=0}^{N+n} a^{\mathbb{S}}_{k,l}a^{\mathbb{S}}_{k-m,l-n} & 0 \le m \le M,\ -N \le n \le 0, \\ \sum_{k=m}^{M}\sum_{l=n}^{N} a^{\mathbb{S}}_{k,l}a^{\mathbb{S}}_{k-m,l-n} & 0 \le m \le M,\ 0 \le n \le N \end{cases}$$
$$= \sum_{0\le k,k-m\le M}\ \sum_{0\le l,l-n\le N} a^{\mathbb{S}}_{k,l}a^{\mathbb{S}}_{k-m,l-n} = a^{\mathbb{S}}_{i,j} \otimes a^{\mathbb{S}}_{i,j}. \quad \blacksquare \tag{10}$$

Corollary 2: We conclude that $w^{\mathbb{S}}_{m,n} = w^{\mathbb{S}}_{-m,-n}$.

III. Proposed 2-D Array

We optimize a 2DSPA to have a minimal number of sensor arrays (MSAs) for a predetermined DOF in this section, i.e., our 2DMSA is filled and has no holes in its DCA. In other words, we obtain 2DMSA by solving the following problem:

$$\mathbb{S}_{2\text{DMSA}} = \arg\min |\mathbb{S}| \quad \text{s.t. } \mathbb{D} = \mathbb{U} \quad \max(\mathbb{S}_x) = M, \quad \max(\mathbb{S}_y) = N \tag{11}$$

which simply finds an arrangement of sensors on a rectangular lattice with $M+1$ columns and $N+1$ rows. This problem is solved under constraints that the resulting filled DCA has $2M+1$ columns and $2N+1$ rows with a URA of size $|\mathbb{U}| = (2M+1)(2N+1)$. Remark 1 identifies the maximum number of targets that a 2-D array can potentially detect.

Remark 1: The number of resolvable targets using a 2-D rectangular array with $K \times L$ sensors is bounded by DOF $= \lfloor K/2 \rfloor \lceil L/2 \rceil$ [28]. Thus, the number of detectable targets by a 2-D hole-free array using DCA is bounded by

$$\text{DOF} = \lfloor (2M+1)/2 \rfloor \lceil (2N+1)/2 \rceil = M(N+1). \tag{12}$$

Therefore, among all hole-free arrays with the same number of sensors our proposed method searches for the array which provides the maximum aperture size. From $|\mathbb{S}| = \sum_{m=0}^{M}\sum_{n=0}^{N} a^{\mathbb{S}}_{m,n}$ and by assuming that the DCA has no hole, i.e., $\mathbb{D} = \mathbb{U}$) and from Corollary 1, we obtain $w^{\mathbb{S}}_{m,n} > 0$ for all $(m,n) \in \mathbb{U}$. Therefore, we rewrite (11) as

$$a^{\mathbb{S}}_{m,n} = \arg\min_{a^{\mathbb{S}}_{m,n}} \sum_{m=0}^{M}\sum_{n=0}^{N} a^{\mathbb{S}}_{m,n} \quad \text{s.t. } w^{\mathbb{S}}_{m,n} > 0. \tag{13}$$

From Corollary (2) for $w^{\mathbb{S}}_{m,n} > 0$, we shall enforce

$$\sum_{k=0}^{M+m}\sum_{l=0}^{N+n} a^{\mathbb{S}}_{k,l}a^{\mathbb{S}}_{k-m,l-n} > 0, \quad -M \le m \le 0, \quad -N \le n \le 0$$
$$\sum_{k=0}^{M+m}\sum_{l=n}^{N} a^{\mathbb{S}}_{k,l}a^{\mathbb{S}}_{k-m,l-n} > 0, \quad -M \le m \le 0, \ 0 \le n \le N.$$

Thus for (13), we shall solve

$$a^{\mathbb{S}}_{m,n} = \arg\min_{a_{m,n}} \sum_{m=0}^{M}\sum_{n=0}^{N} a^{\mathbb{S}}_{m,n}$$
$$\sum_{k=0}^{M+m}\sum_{l=0}^{N+n} a^{\mathbb{S}}_{k,l}a^{\mathbb{S}}_{k-m,l-n} > 0, \quad -M \le m \le 0$$
$$-N \le n \le 0$$
$$\sum_{k=0}^{M+m}\sum_{l=n}^{N} a^{\mathbb{S}}_{k,l}a^{\mathbb{S}}_{k-m,l-n} > 0, \quad -M \le m \le 0$$
$$0 \le n \le N. \tag{14}$$

We rewrite (14) by denoting $z_{k,l,k-m,l-n} \doteq a^{\mathbb{S}}_{k,l}a^{\mathbb{S}}_{k-m,l-n}$ as

$$a^{\mathbb{S}}_{m,n} = \arg\min_{a_{m,n}} \sum_{m=0}^{M}\sum_{n=0}^{N} a^{\mathbb{S}}_{m,n}$$
$$\text{s.t. } \sum_{k=0}^{M+m}\sum_{l=0}^{N+n} z_{k,l,k-m,l-n} > 0, \quad -M \le m \le 0$$
$$-N \le n \le 0$$
$$\sum_{k=0}^{M+m}\sum_{l=n}^{N} z_{k,l,k-m,l-n} > 0, \quad -M \le m \le 0,$$
$$0 \le n \le N \tag{15}$$

where $C_1 : z_{k,l,k-m,l-n} = a^{\mathbb{S}}_{k,l}a^{\mathbb{S}}_{k-m,l-n}$ can be rewritten as

$$z_{k,l,k-m,l-n} \le \min\{a^{\mathbb{S}}_{k,l}, a^{\mathbb{S}}_{k-m,l-n}\}$$
$$z_{k,l,k-m,l-n} \ge a^{\mathbb{S}}_{k-m,l-n} + a^{\mathbb{S}}_{k,l} - 1. \tag{16}$$

The first inequality ensure $z_{k,l,k-m,l-n} = 0$ if either $a^{\mathbb{S}}_{k,l}=0$ or $a^{\mathbb{S}}_{k-m,l-n}=0$. The last one ensures $z_{k,l,k-m,l-n} = 1$ if $a^{\mathbb{S}}_{k,l}$ and $a^{\mathbb{S}}_{k-m,l-n}$ are both 1. Thus, (13) simplifies to

$$a^{\mathbb{S}}_{m,n} = \arg\min_{a_{m,n}} \sum_{m=0}^{M}\sum_{n=0}^{N} a^{\mathbb{S}}_{m,n}$$

TABLE I
NUMBER OF REQUIRED SENSORS IN SOME POPULAR 2-D ARRAYS AND OUR PROPOSED 2DMSA TO ACHIEVE GIVEN DOF

		Array Geometry			
Size	DOF	Billboard	Open Box	Hourglass	Proposed
4×5	15	11	11	11	11
6×5	25	14	15	15	13
6×6	30	15	16	16	15
7×6	36	17	18	18	16
6×8	40	18	18	18	17
13×5	60	21	29	N.A.	20
8×9	63	23	23	23	22
9×10	80	26	26	26	24
11×11	110	30	31	31	29

$$\begin{aligned}
\text{s.t. } & \sum_{k=0}^{M+m}\sum_{l=0}^{N+n} z_{k,l,k-m,l-n} > 0, \quad -M \le m \le 0,\\
& -N \le n \le 0\\
& \sum_{k=0}^{M+m}\sum_{l=n}^{N} z_{k,l,k-m,l-n} > 0, \quad -M \le m \le 0,\\
& 0 \le n \le N\\
& z_{k,l,k-m,l-n} \le a^{\mathbb{S}}_{k,l}, \quad z_{k,l,k-m,l-n} \le a^{\mathbb{S}}_{k-m,l-n}\\
& z_{k,l,k-m,l-n} \ge a^{\mathbb{S}}_{k-m,l-n} + a^{\mathbb{S}}_{k,l} - 1. \qquad (17)
\end{aligned}$$

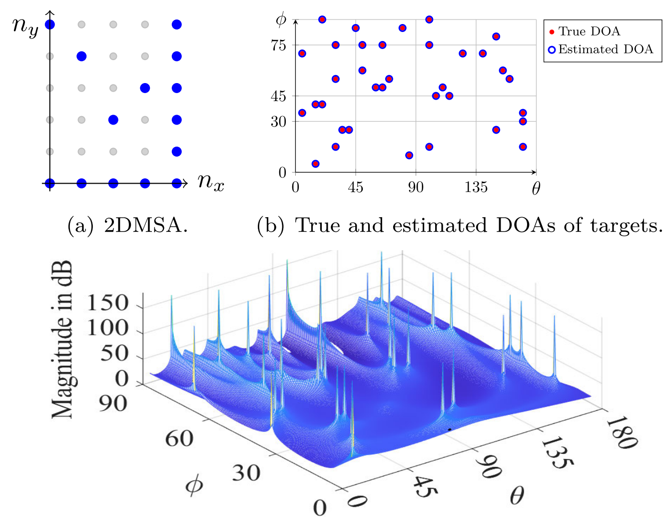
The problem (17) has no closed-form solution, thus we solve it by using branch-and-bound (B&B) algorithm [29] through the CPLEX solver in GAMS software with (M, N) as input and $a^{\mathbb{S}}_{m,n}$ or $\mathbb{S}$ as output. The B&B is a fundamental algorithm that finds the exact optimal solutions of the integers programs. Another low-complexity approach to find a suboptimal solution for this BLP problem is to convert it into a convex quadratic programming (CQP) [30] problem which could be solved in polynomial time complexity.

IV. SIMULATION RESULTS

Here, we first compare our proposed 2DMSA with the state-of-the-art and popular arrays. Table I compares our proposed array with several other arrays in terms of the number of required sensors to achieve some given DOF. In Table I, the array dimensions are indicated $m \times n$, e.g., 6×5 means that the array has six rows and five columns. We observe that as expected the proposed 2DMSA requires less sensors to achieve the same DOF compared with other geometries. This means that the proposed 2DMSA provides an increased DOF for the same number of sensors among all array configurations.

A. Spatial Spectrum

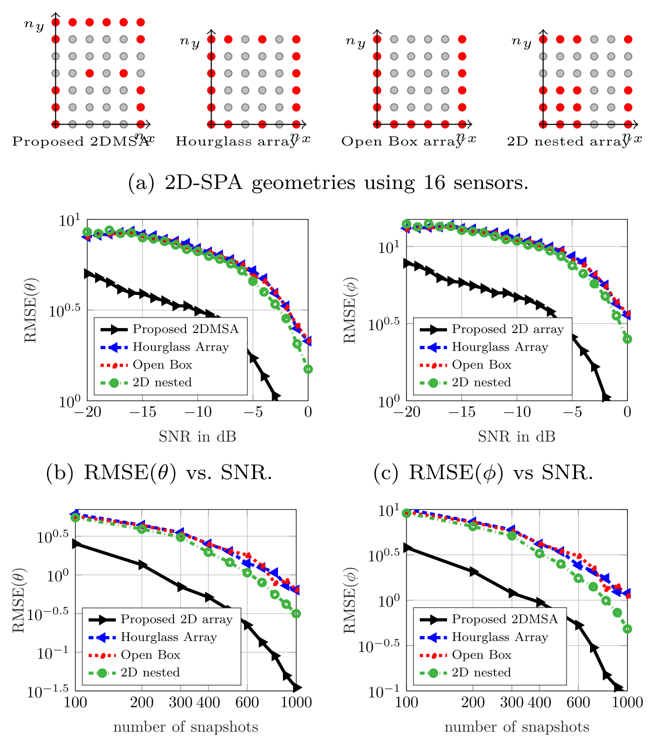
In this section, we use the results of the first simulation and compare the performance of the MUSIC spectrum in extracting the maximum number of targets using the proposed 2DMSA arrangement with other arrays. We set $M = 6$ and $N = 5$ in (17) and find the optimal planar MSA as illustrated in Fig. 2. Using this 2DMSA which has 14 sensors arranged in a 6×5 planar array, the MUSIC spectrum allowed us to detect all targets up to a maximum 36 targets which are randomly distributed between $\theta = [5^\circ : 90^\circ]$ and $\phi = [5^\circ : 180^\circ]$ (no other existing 7×6 array arrangement allows to simultaneously detect the such number of targets).



(a) 2DMSA. (b) True and estimated DOAs of targets.

(c) 2D MUSIC spectrum by using the proposed 2DMSA.

Fig. 2. (a) Geometry of 6×5 proposed 2DMSA with 14 sensors and DOF $= M(N + 1) = 36$. (b) True and estimated DOAs of 36 randomly located targets. (c) Obtained 2-D MUSIC spectrum.



(a) 2D-SPA geometries using 16 sensors.

(b) RMSE(θ) vs. SNR. (c) RMSE(ϕ) vs SNR.

(d) RMSE(θ) vs. # of snapshots. (e) RMSE(ϕ) vs. # of snapshots.

Fig. 3. Averaged localization accuracy comparison of 2-D SPAs for ten targets. (a) Array geometries using 16 sensors. (b) and (c) RMSE(θ) and RMSE(ϕ) versus SNR. (d) and (e) RMSE(θ) and RMSE(ϕ) versus the number of snapshots.

B. RMSE Versus SNR and Snapshots

We now compare the 2DMSA with three other state-of-the-art 2-D arrays shown in Fig. 3(a) in terms of their ability to accurately estimate the elevation and azimuth of targets. These arrays have 16 sensors. We first asses the RMSE versus the signal to noise ratio (SNR) for $P = 500$ snapshots in θ and ϕ. We define the SNR as the ratio of the total power of

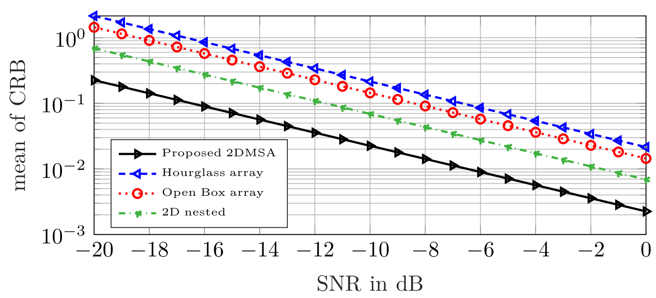


Fig. 4. Mean of the CRB values as a function of SNR in dB.

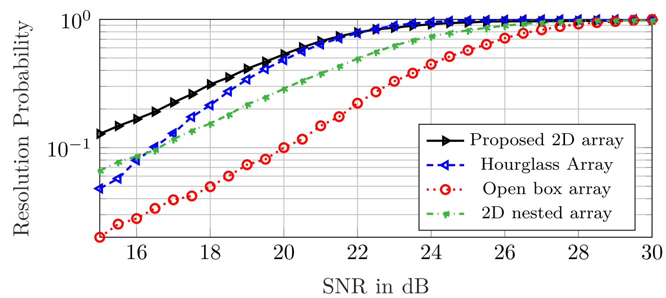


Fig. 5. Resolution probability of two close targets versus SNR.

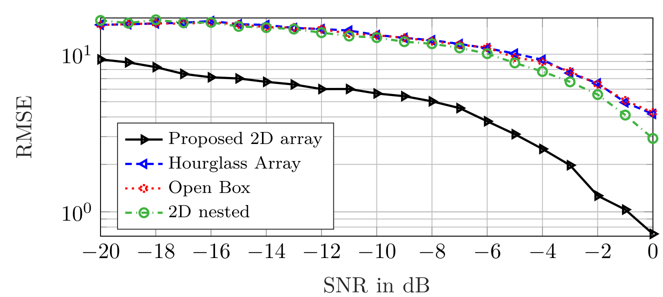


Fig. 6. RMSE versus SNR in the presence of MC between sensors.

signals of all targets to the noise power. We calculate RMSE by averaging over the squared of the estimation error of ϕ (and θ) for $K = 10$ targets and over $N_o = 500$ independent Monte Carlo runs for each SNR as follows:

$$\text{RMSE}(\theta) = \sqrt{\frac{1}{N_o K} \sum\nolimits_{i=1}^{N_o} \sum\nolimits_{k=1}^{K} |\theta_k - \hat{\theta}_{k,i}|^2} \tag{18}$$

where θ_k and $\hat{\theta}_{k,i}$ are the true and estimated angles, respectively. We calculate RMSE(ϕ) in the same manner. Fig. 3(b) and (c) shows the RMSE for estimating θ and ϕ, respectively, by using these arrays and 500 snapshots. These figures reveal that the 2DMSA outperforms these other geometrise and provides at least an SNR improvement of about 3 and 2 dB, respectively, in estimating θ and ϕ. Fig. 3(d) and (e) shows RMSE(θ) and RMSE(ϕ) versus the number of snapshots for 500 independent Monte Carlo runs and at SNR = 0 dB. As expected, the estimation accuracy improves as the number of snapshots is increased. However, the proposed 2DMSA requires considerably less number of snapshots compared with the state-of-the-art to achieve the same accuracy. Interestingly, these gains are roughly equal to the ratios of the apertures of the employed arrays.

C. Crammer–Rao Band

Fig. 4 shows the Cramér–Rao bound (CRB) (see [31]) in estimating (θ, ϕ) for these planar arrays versus the SNR. The proposed 2DMSA outperforms other geometries and gives about 4.5-dB SNR improvement compared to their best.

D. Array Resolution

We here compare the resolution of the 2DMSA with other arrays for a given number of sensors. The aperture of the 2DMSA is maximum for a given number of sensors due to its maximal DOF. Hence, the 2DMSA shall provide the highest resolution compared to all other configurations on a 2-D lattice. To assess this resolution, we simulate two targets with equal powers, located at $(\phi_1, \theta_1) = (25°, 60°)$ and $(\phi_2, \theta_2) = (27°, 62°)$, i.e., they are 2° away from each other in both angles. Thus we say these targets are resolvable if their estimated errors are less than 1°, i.e., we shall have $|\hat{\theta}_i - \theta_i| < 1°$ and $|\hat{\phi}_i - \phi_i| < 1°$ for $i \in \{1, 2\}$. The resolution probability of these targets is plotted in Fig. 5 versus SNR using 500 Monte Carlo runs and 512 signal snapshots. We observe that by using 16 sensors, the proposed array provides the highest target resolution probability among these state-of-the-art geometries.

E. Robustness Against Mutual Coupling

The MC of sensors impacts various performances of an antenna array. Here, we only investigate its impact on localization accuracy and compare the 2DMSA with other geometries. Obviously, closer sensors impose greater undesired MC. Thus, we intuitively anticipate that the mutual cupeling is lower for a geometry that has less the number of sensor pairs with the minimum distance. Without a claim, we compare the accuracy of the arrays in Fig. 3(a) by using the model presented in [21] and [32] which takes into account the MCs as follows:

$$\mathbf{x}_s = \sum\nolimits_{i=1}^{D} A_i \mathbf{C} \mathbf{v}_s \left(\bar{\theta}_i, \bar{\phi}_i\right) + \mathbf{n}_s \quad \forall s \in \mathbb{S} \tag{19}$$

where the entries of $\mathbf{C}$, i.e., $\mathbf{C}_{\mathbf{n}_1,\mathbf{n}_2} = c(\|\mathbf{n}_1 - \mathbf{n}_2\|_2)$ are zero for $\|\mathbf{n}_1 - \mathbf{n}_2\|_2 \geq B$ [21] and $c(\|\mathbf{n}_1 - \mathbf{n}_2\|_2)$ are decreasing in $\|\mathbf{n}_1 - \mathbf{n}_2\|_2$. We set $B = 1$, $c(0) = 1$, $c(1) = 0.2$ and $c(\ell) = c(1)e^{-j(\ell-1)2\pi/8}/\ell$ in our simulation. We generate four targets located at $(\theta, \phi) = (30°, 90°)$, $(50°, 80°)$, $(60°, 110°)$, and $(40°, 100°)$ with 500 snapshots and calculated the absolute RMSE over 500 independent Monte Carlo runs as

$$\text{RMSE} = \sqrt{\frac{1}{N_0 K} \sum\nolimits_{i=1}^{N_0} \sum\nolimits_{k=1}^{K} |\theta_k - \hat{\theta}_{k,i}|^2 + |\phi_k - \hat{\phi}_{k,i}|^2}.$$

Fig. 6 shows the RMSE of DOA obtained from these arrays and reveals that the 2DMSA is superior geometry in the presence of MC. The proposed array contains the least number of pairs with unit distance among the arrays in Fig. 3(a). This justifies our intuitive expectation of this superiority against MC.

V. Conclusion

In this article, we developed a framework to optimize a hole-free 2-D array geometry with a minimum number of sensors. We converted this problem into a nonlinear integer problem and then transformed it into binary linear programming. This class of problems can be easily solved by using the branch and bound method. Our numerical results show that the proposed optimal array which has the least number of sensors among all arrays with a given DOF outperforms the state-of-the-art 2-D array in terms of DOA estimation accuracy by using a given number of sensors. Moreover, the proposed optimal 2DMSA provides the highest resolution probability of close targets and outperforms other geometries in the presence of mutual coupling in our example.

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