Correspondence

Almost Sure Identifiability of Constant Modulus Multidimensional Harmonic Retrieval

Xiangqian Liu and Nicholas D. Sidiropoulos

***Abstract*—In a recent paper by Jiang *et al.* in this Transactions, it has been shown that up to $\lfloor K/2 \rfloor \lceil L/2 \rceil$ two-dimensional (2-D) exponentials are almost surely identifiable from a $K \times L$ mixture, assuming regular sampling at or above Nyquist in both dimensions. This holds for damped or undamped exponentials. As a complement, in this correspondence, we show that up to $\lceil K/2 \rceil \lceil L/2 \rceil$ *undamped* exponentials can be uniquely recovered almost surely. Multidimensional conjugate folding is used to achieve this improvement. The main result is then generalized to $N > 2$ dimensions. The gain is interesting from a theoretical standpoint but also for small 2-D sensor arrays or higher dimensions and odd sample sizes.**

***Index Terms*—Array signal processing, frequency estimation, harmonic analysis, multidimensional signal processing, spectral analysis.**

I. Introduction

Constant modulus two-dimensional (2-D) and more generally multidimensional harmonic retrieval (HR) has a wide range of applications, e.g., in sensor array processing [9], wireless communication [4], and radar [2], [5]. An important issue is to determine the maximum number of harmonics that can be resolved for a given sample size. In the one-dimensional (1-D) case, the answer can be traced to Carathéodory [1] (see also [8]). A generalization of Carathéodory's uniqueness to the N-D case has been given in [6]. For the 2-D case, the result of [6] indicates that $(K+L-1)/2$ harmonics are deterministically identifiable, where K is the number of (equispaced) samples along one dimension, and L likewise for the other dimension. In follow-up work, [3] showed that up to $\lfloor K/2 \rfloor \lceil L/2 \rceil$ exponentials are uniquely resolvable almost surely (a.s.). This is the most relaxed identifiability result regarding 2-D HR to date. It holds for damped or undamped exponentials. In this paper, we derive an improved stochastic identifiability result for undamped multidimensional exponentials. In particular, we show that up to $\lceil K/2 \rceil \lceil L/2 \rceil$ undamped 2-D exponentials can be uniquely recovered almost surely. The main result is then generalized to $N > 2$ dimensions.

Throughout the paper, uppercase (or lowercase) boldface letters will be used for matrices (column vectors). Superscript ^{H} will denote Hermitian, * conjugate, ^{T} transpose, and † matrix pseudo-inverse. We will use $\odot$ for Khatri-Rao (column-wise Kronecker) product, $\lceil \cdot \rceil$ for integer ceiling, and $\lfloor \cdot \rfloor$ for integer floor.

Manuscript received December 11, 2001; revised June 1, 2002. This work was supported by NSF/Wireless under Grant CCR-0096164, subcontract participation in the ARL Communications and Networks CTA, and DARPA/ATO under Contract MDA 972-01-0056. The associate editor coordinating the review of this paper and approving it for publication was Dr. Olivier Cappe.

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Publisher Item Identifier 10.1109/TSP.2002.801933.

II. Problem Formulation

In general terms, the 2-D harmonic retrieval problem can be stated as follows: Given a mixture of F 2-D exponentials

$$x_{k,l} = \sum_{f=1}^{F} c_f a_f^{k-1} b_f^{l-1} \tag{1}$$

for $k = 1, \ldots, K$ and $l = 1, \ldots, L$, where $a_f, b_f, c_f \in \mathbb{C}$, find the parameter triples (a_f, b_f, c_f) for $f = 1, \ldots, F$. In the case of constant modulus exponentials, i.e., $a_f = e^{j\omega_f}$ and $b_f = e^{j\nu_f}$, (1) yields

$$x_{k,l} = \sum_{f=1}^{F} c_f e^{j\omega_f (k-1)} e^{j\nu_f (l-1)} \tag{2}$$

where $\omega_f, \nu_f \in \Pi$, and the set $\Pi := (-\pi, \pi]$. Define $\mathbf{X} \in \mathbb{C}^{K \times L}$ with $\mathbf{X}(k,l) = x_{k,l}$, $\mathbf{A} \in \mathbb{C}^{K \times F}$ with $\mathbf{A}(k,f) = e^{j\omega_f(k-1)}$, $\mathbf{B} \in \mathbb{C}^{L \times F}$ with $\mathbf{B}(l,f) = e^{j\omega_f(l-1)}$, and a diagonal matrix $\mathbf{C} \in \mathbb{C}^{F \times F}$ with $\mathbf{C}(f,f) = c_f$. Then, the 2-D harmonic mixture in (2) can be written in matrix form as

$$\mathbf{X} = \mathbf{A}\mathbf{C}\mathbf{B}^T. \tag{3}$$

A. Preliminaries

The following result will be used in our derivation.

Theorem 1 (a.s. Full Rank of Khatri-Rao Product of Vandermonde Matrices [3]): For a pair of Vandermonde matrices $\mathbf{A} \in \mathbb{C}^{K \times F}$ and $\mathbf{B} \in \mathbb{C}^{L \times F}$, with generators on the unit circle

$$\text{rank}(\mathbf{A} \odot \mathbf{B}) = \min(KL, F), \quad P_{\mathcal{L}}(\mathbb{U}^{2F})\text{-a.s.} \tag{4}$$

where $\mathbb{U}$ is the unit circle, and $P_{\mathcal{L}}(\mathbb{U}^{2F})$ is the distribution used to draw the $2F$ generators for $\mathbf{A}$ and $\mathbf{B}$, which is assumed continuous with respect to the Lebesgue measure in $\mathbb{U}^{2F}$.

A recently obtained stochastic identifiability result regarding 2-D harmonic retrieval [3] is reproduced next.

Theorem 2 (a.s. Identifiability of 2-D Harmonic Retrieval [3]): Given a sum of F 2-D exponentials as in (1) with $K \geq 4$ and $L \geq 4$, if

$$F \leq \left\lfloor \frac{K}{2} \right\rfloor \left\lceil \frac{L}{2} \right\rceil \tag{5}$$

and the distribution used to draw the $2F$ complex exponential parameters (a_f, b_f), $f = 1, \ldots, F$, which is denoted by $P_{\mathcal{L}}(\mathbb{C}^{2F})$, is continuous with respect to the Lebesgue measure in $\mathbb{C}^{2F}$, then the parameter triples (a_f, b_f, c_f), $f = 1, \ldots, F$ are $P_{\mathcal{L}}(\mathbb{C}^{2F})$-a.s. unique. Note that switching K and L in (5) yields an alternative sufficient condition.

III. Identifiability of Constant Modulus 2-D HR

Theorem 3: Given a sum of F 2-D undamped exponentials

$$x_{k,l} = \sum_{f=1}^{F} c_f e^{j\omega_f (k-1)} e^{j\nu_f (l-1)} \tag{6}$$

for $k = 1, \ldots, K \geq 3$ and $l = 1, \ldots, L \geq 3$, if

$$F \leq \left\lceil \frac{K}{2} \right\rceil \left\lceil \frac{L}{2} \right\rceil \tag{7}$$

and the distribution used to draw the $2F$ frequencies (ω_f, ν_f), $f = 1, \ldots, F$, which is denoted by $P_{\mathcal{L}}(\Pi^{2F})$, is continuous with respect to the Lebesgue measure in Π^{2F}, then the parameter triples (ω_f, ν_f, c_f), $f = 1, \ldots, F$ are $P_{\mathcal{L}}(\Pi^{2F})$-a.s. unique.

Proof: See the Appendix.

Compared with Theorem 2, the improvement of Theorem 3 is nontrivial. For example, when $K = L = 5$, the maximum number of identifiable 2-D harmonics given by Theorem 3 is 9 [cf. (7)], whereas Theorem 2 indicates only 6. In addition, our identifiability condition allows as few as three measurements along both dimensions, in which case, it yields four identifiable 2-D harmonics, which is a case with which [3] cannot deal. This is important when one is constrained in the number of measurements that can be taken along certain dimensions, usually due to hardware and/or cost limitations, e.g., in spatial sampling for direction-of-arrival estimation using a uniform rectangular array.

IV. Identifiability of Constant Modulus N-D HR

The result in Theorem 3 can be generalized to the N-D case, as stated in the following.

Theorem 4: Given a sum of F N-D undamped exponentials

$$x_{i_1,\ldots,i_N} = \sum_{f=1}^{F} c_f \prod_{n=1}^{N} e^{j\omega_{f,n}(i_n - 1)} \tag{8}$$

for $i_n = 1, \ldots, I_n \geq 3$, $n = 1, \ldots, N$, if

$$F \leq \prod_{n=1}^{N} \left\lceil \frac{I_n}{2} \right\rceil \tag{9}$$

and the distribution used to draw the NF frequencies $(\omega_{f,1}, \ldots, \omega_{f,N})$, for $f = 1, \ldots, F$, which is denoted by $P_{\mathcal{L}}(\Pi^{NF})$, is continuous with respect to Lebesgue measure in Π^{NF}, then the parameter $(N+1)$-tuples $(\omega_{f,1}, \ldots, \omega_{f,N}, c_f)$, $f = 1, \ldots, F$ are $P_{\mathcal{L}}(\Pi^{NF})$-a.s. unique.

The idea of the associated proof is similar to the 2-D case. First, one can extend the given N-way array to a $2N$-way array and then nest the $2N$-way to a matrix $\tilde{\mathbf{X}} = \mathbf{GCH}^T$ by recursive dimensionality embedding[1] in such a way that $\mathbf{G}$ and $\mathbf{H}$ are Khatri–Rao products of N matrices. The same procedure can be carried out for the conjugate of the N-way data to form a matrix $\tilde{\mathbf{Y}}$. Finally, the N-D harmonic retrieval problem can be reduced to an eigenvalue decomposition problem that is similar to (19), and consequently, (9) is obtained.

V. Conclusion

Using multidimensional conjugate folding, it has been shown that up to $\lceil K/2 \rceil \lceil L/2 \rceil$ *undamped* exponentials can be uniquely recovered a.s. from a 2-D harmonic mixture. The result has also been generalized to $N > 2$ dimensions. The gain is interesting from a theoretical standpoint but is also nontrivial for small 2-D sensor arrays or higher dimensions and odd sample sizes.

[1]A similar detailed mathematical argument can be found in [3], where the given N-way array is first extended to a $(2N+1)$-way array and then nested to a three-way array by recursive dimensionality embedding.

Appendix
Proof of Theorem 3

A. Proof

Given $\mathbf{X}$ [cf. (6)], define a four-way array $\hat{\mathbf{X}}$ with typical element

$$\begin{aligned} \hat{x}_{k_1,k_2,l_1,l_2} &:= x_{k_1+k_2-1,l_1+l_2-1} \\ &= \sum_{f=1}^{F} c_f e^{j\omega_f(k_1-1)} e^{j\omega_f(k_2-1)} e^{j\nu_f(l_1-1)} e^{j\nu_f(l_2-1)} \end{aligned} \tag{10}$$

where $k_i = 1, \ldots, K_i \geq 2$, $l_i = 1, \ldots, L_i \geq 2$, for $i = 1, 2$, and

$$K_1 + K_2 = K + 1, \quad L_1 + L_2 = L + 1. \tag{11}$$

Since $\min(K, L) \geq 3$ has been assumed, such extension to a four-way array is always feasible. For $i = 1, 2$, define matrices

$$\begin{aligned} \mathbf{A}_i &:= (e^{j\omega_f(k_i - 1)}) \in \mathbb{C}^{K_i \times F} \\ \mathbf{B}_i &:= (e^{j\nu_f(l_i - 1)}) \in \mathbb{C}^{L_i \times F}. \end{aligned}$$

Then, nest the four-way array $\hat{\mathbf{X}}$ into a matrix $\tilde{\mathbf{X}} \in \mathbb{C}^{K_1L_1 \times K_2L_2}$ by collapsing two pairs of dimensions such that

$$\begin{aligned} \tilde{x}_{p,q} &:= \hat{x}_{\lceil p/L_1 \rceil, \lceil q/L_2 \rceil, p - (\lceil p/L_1 \rceil - 1)L_1, q - (\lceil q/L_2 \rceil - 1)L_2} \\ &= \sum_{f=1}^{F} c_f g_{p,f} h_{q,f} \end{aligned} \tag{12}$$

where

$$\begin{aligned} g_{p,f} &:= e^{j\omega_f(\lceil p/L_1 \rceil - 1)} e^{j\nu_f(p - (\lceil p/L_1 \rceil - 1)L_1 - 1)} \\ h_{q,f} &:= e^{j\omega_f(\lceil q/L_2 \rceil - 1)} e^{j\nu_f(q - (\lceil q/L_2 \rceil - 1)L_2 - 1)} \end{aligned}$$

for $p = 1, \ldots, K_1L_1$, and $q = 1, \ldots, K_2L_2$. Define

$$\begin{aligned} \mathbf{G} &:= (g_{p,f}) \in \mathbb{C}^{K_1L_1 \times F} \\ \mathbf{H} &:= (h_{q,f}) \in \mathbb{C}^{K_2L_2 \times F}. \end{aligned}$$

It can be verified that

$$\mathbf{G} = \mathbf{A}_1 \odot \mathbf{B}_1, \quad \mathbf{H} = \mathbf{A}_2 \odot \mathbf{B}_2. \tag{13}$$

Hence, (12) can be written in compact matrix form as

$$\tilde{\mathbf{X}} = \mathbf{GCH}^T. \tag{14}$$

This is important because under the premise of Theorem 1, $\mathbf{G}$ and $\mathbf{H}$ are guaranteed to be full column rank a.s. if $K_1L_1 \geq F$ and $K_2L_2 \geq F$, and consequently, $\tilde{\mathbf{X}}$ is of rank F.

Next, taking the conjugate of $x_{k,l}$ in (6), we obtain

$$x^*_{k,l} = \sum_{f=1}^{F} \tilde{c}_f e^{j\omega_f(K-k)} e^{j\nu_f(L-l)}$$

where $\tilde{c}_f := c^*_f e^{-j\omega_f(K-1) - j\nu_f(L-1)}$. Define

$$\begin{aligned} y_{k,l} &:= x^*_{K-k+1, L-l+1} \\ &= \sum_{f=1}^{F} \tilde{c}_f e^{j\omega_f(k-1)} e^{j\nu_f(l-1)} \end{aligned} \tag{15}$$

for $k = 1, \ldots, K, l = 1, \ldots, L$ and, correspondingly, the matrix $\mathbf{Y} := (y_{k,l}) \in \mathbb{C}^{K \times L}$. This means that by conjugation and folding[2] of the lower-left quadrant, we obtain another harmonic mixture with the same harmonics as $\mathbf{X}$ but different proportions. This is key. Following the same procedure as in the construction of $\tilde{\mathbf{X}}$ from $\mathbf{X}$, we can construct a matrix $\tilde{\mathbf{Y}} \in \mathbb{C}^{K_1 L_1 \times K_2 L_2}$ from $\mathbf{Y}$ such that

$$\tilde{y}_{p,q} = \sum_{f=1}^{F} \tilde{c}_f g_{p,f} h_{q,f} \tag{16}$$

i.e.,

$$\tilde{\mathbf{Y}} = \mathbf{G}\tilde{\mathbf{C}}\mathbf{H}^T \tag{17}$$

where $\tilde{\mathbf{C}} = \text{diag}(\tilde{c}_1, \ldots, \tilde{c}_F)$.

Invoking Theorem 1, if $K_1 L_1 \geq F$ and $K_2 L_2 \geq F$, then both $\mathbf{G}$ and $\mathbf{H}$ in (13) are a.s. full column rank. Hence, $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ are of rank F, and the singular value decomposition of the stacked data yields

$$\begin{bmatrix} \tilde{\mathbf{X}} \\ \tilde{\mathbf{Y}} \end{bmatrix} = \begin{bmatrix} \mathbf{GC} \\ \mathbf{G}\tilde{\mathbf{C}} \end{bmatrix} \mathbf{H}^T = \mathbf{U}_{2K_1 L_1 \times F} \mathbf{\Sigma}_{F \times F} \mathbf{V}^H_{K_2 L_2 \times F}$$

where $\mathbf{U}$ has F columns, which, together, span the column space of $\left[\tilde{\mathbf{X}}^T \ \tilde{\mathbf{Y}}^T\right]^T$. Since the same space is spanned by the columns of $\left[(\mathbf{GC})^T \ (\mathbf{G}\tilde{\mathbf{C}})^T\right]^T$, there exists an $F \times F$ nonsingular matrix $\mathbf{T}$ such that

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{GC} \\ \mathbf{G}\tilde{\mathbf{C}} \end{bmatrix} \mathbf{T}. \tag{18}$$

It then follows that

$$\mathbf{U}_1^\dagger \mathbf{U}_2 = \mathbf{T}^{-1}\mathbf{C}^{-1}\tilde{\mathbf{C}}\mathbf{T} \tag{19}$$

which is an eigenvalue decomposition problem. $\mathbf{T}^{-1}$ contains the eigenvectors of $\mathbf{U}_1^\dagger \mathbf{U}_2$ (scaled to unit norm). Other parameters are given by

$$\mathbf{GC} = \mathbf{U}_1 \mathbf{T}^{-1} \tag{20}$$
$$\mathbf{H}^T = (\mathbf{GC})^\dagger \tilde{\mathbf{X}}. \tag{21}$$

Notice that the first row of the product $\mathbf{GC}$ is the diagonal of $\mathbf{C}$, i.e., $[c_1, \ldots, c_F]$. Now, the (ω_f, ν_f) can be readily recovered from $\mathbf{G}$ and/or $\mathbf{H}$, for example, the second and $(L_2 + 1)$th rows of $\mathbf{H}$ are $[e^{j\nu_1}, \ldots, e^{j\nu_F}]$ and $[e^{j\omega_1}, \ldots, e^{j\omega_F}]$, respectively. Note that no pairing issue exists, i.e., (ω_f, ν_f, c_f) are paired up automatically.

Hence, we have shown that the parameter triples (ω_f, ν_f, c_f), $f = 1, \ldots, F$ can be uniquely recovered a.s., provided there exist positive integers K_1, K_2, L_1, L_2 such that

$$K_1 L_1 \geq F, \quad K_2 L_2 \geq F \tag{22}$$

subject to (11). If the integers are chosen such that

$$\begin{cases} \text{if } K \text{ is odd}, & \text{pick } K_1 = K_2 = \frac{K+1}{2} \\ \text{if } K \text{ is even}, & \text{pick } K_1 = \frac{K}{2}, K_2 = \frac{K+2}{2} \end{cases} \tag{23}$$

and similarly for L_1 and L_2, then (11) is satisfied. Once we pick four integers following the rules in (23), (7) assures that inequality (22) holds for those particular integers. This completes the proof. □

[2]Similar to the well-known "forward-backward averaging" trick used in the context of 1-D harmonic retrieval (see, e.g., [7, pp. 165–167]). In the context of multidimensional harmonic retrieval, this trick has been used by Harrdt [4] in a similar fashion.

REFERENCES

[1] C. Carathéodory and L. Fejér, "Uber den zusammenghang der extemen von harmonischen funktionen mit ihren koeffizienten und uber den Picard-Landauschen Satz," *Rendiconti del Circolo Matematico di Palermo*, vol. 32, pp. 218–239, 1911.
[2] Y. Hua, "High resolution imaging of continuously moving object using stepped frequency radar," *Signal Process.*, vol. 35, no. 1, pp. 33–40, Jan. 1994.
[3] T. Jiang, N. D. Sidiropoulos, and J. M. F. ten Berge, "Almost sure identifiability of multidimensional harmonic retrieval," *IEEE Trans. Signal Processing*, vol. 49, pp. 1849–1859, Sept. 2001.
[4] M. Haardt and J. A. Nossek, "Simultaneous Schur decomposition of several nonsymmetric matrices to achieve automatic pairing in multidimensional harmonic retrieval problems," *IEEE Trans. Signal Processing*, vol. 46, pp. 161–169, Jan. 1998.
[5] J. Li, P. Stoica, and D. Zheng, "An efficient algoritm for two-dimensional frequency estimation," *Multidimen. Syst. Signal Process.*, vol. 7, no. 2, pp. 151–178, Apr. 1996.
[6] N. D. Sidiropoulos, "Generalizing Carathéodory's uniqueness of harmonic parameterization to N dimensions," *IEEE Trans. Inform. Theory*, vol. 47, pp. 1687–1690, May 2001.
[7] P. Stoica and R. L. Moses, *Introduction to Spectral Analysis*. Upper Saddle River, NJ: Prentice-Hall, 1997.
[8] P. Stoica and T. Söderstöm, "Parameter identifiability problem in signal processing," *Proc. Inst. El;ect. Eng., Radar, Sonar, Navig.*, vol. 141, no. 3, pp. 133–136, June 1994.
[9] M. D. Zoltowski, M. Haardt, and C. P. Mathews, "Closed-form 2-D angle estimation with rectangular arrays in element space or beamspace via unitary ESPRIT," *IEEE Trans. Signal Processing*, vol. 44, pp. 316–328, Feb. 1996.

Limitations on SNR Estimator Accuracy

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***Abstract*—We consider the samples of a pure tone in additive white Gaussian noise (AWGN) for which we wish to determine the signal-to-noise ratio (SNR) defined here to be $\alpha = (A^2/2\sigma^2)$, where A is the tone amplitude, and σ^2 is the noise variance. A and σ^2 are assumed to be deterministic but unknown *a priori*. If the variance of an unbiased estimator of α is $\sigma^2_{\hat{\alpha}}$, we show that at high SNR, the normalized standard deviation satisfies the Cramér-Rao lower bound (CRLB) according to $\sigma_{\hat{\alpha}}/\alpha \geq \sqrt{2/N}$, where N is the number of independent observables used to obtain the SNR estimate $\hat{\alpha}$.**

***Index Terms*—Cramér-Rao bound (CRB), sinusoid signal-to-noise ratio (SNR) estimator.**

I. INTRODUCTION

The need to estimate the parameters of a pure tone in additive white Gaussian noise (AWGN) arises in many places. The present work is motivated by the problem of estimating the velocity of a fluid by processing laser velocimetry data (LVD) [1]. In this instance, we assume the instantaneous frequency of the data (which gives fluid velocity information) is constant over N samples. In this case, we might consider applying the frequency estimators in, for example, either [2], [3, Sec.

Manuscript received May 11, 1999; revised May 29, 2002. The associate editor coordinating the review of this paper and approving it for publication was Editor-In-Chief Arye Nehorai.

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Publisher Item Identifier 10.1109/TSP.2002.801936.