Project Financial Econometrics

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The objective of this project is pricing a Call option using Monté Carlo method and a GARCH model with the assumption of no risk premium. We also study the relation between price and moneyness of the option and investigate its imlplied volatility.

1 GARCH option pricing model and implied volatility

1.1 M-GARCH(1,1) model and and risk neutralisation

Suppose that the asset price is $(S_t)_t$ and the returns $(\epsilon_t)_t$, $\epsilon_t = \frac{S_t - S_{t-1}}{S_{t-1}}$ are given by a GARCH model.

1. For pricing a Call option, in this project, we consider the M-GARCH model. A M-GARCH(1,1) model can be written as follow

$$\begin{cases} \epsilon_t = \sigma_t Z_t + r + \lambda \sigma_t \\ \sigma_t^2 = \omega + \alpha \sigma_{t-1}^2 Z_{t-1}^2 + \beta \sigma_{t-1}^2 \end{cases}$$

   where Z_t are i.i.d $\mathcal{N}(0,1)$, r is the interest rate and λ is the risk premium parameter.

2. Let $\boldsymbol{Q}$ be a risk-neutral measure equivalent to the historical measure $\boldsymbol{P}$. Under $\boldsymbol{Q}$, the dynamics of the returns are given by

$$\begin{cases} \epsilon_t = \sigma_t Z_t^{'} + r \\ \sigma_t^2 = \omega + \alpha \sigma_{t-1}^2 (Z_{t-1}^{'} - \lambda)^2 + \beta \sigma_{t-1}^2 \end{cases}$$

   where $Z_t^{'}$ are i.i.d $\mathcal{N}(0,1)$.

   Then $Var(\epsilon_t) = \dfrac{\omega}{1 - \alpha(1+\lambda^2) - \beta}$. The condition for stationary is $\alpha(1+\lambda^2) + \beta < 1$.

Throughout this project, we will assume that $\lambda = 0$.

1.2 Call Option pricing

1. The payoff of an European Call option with strike price K at maturity T is

$$f(S_T) = (S_T - K)_+.$$

   The price of this option at time $t = 0$ can be calculated by

$$P = \frac{1}{(1+r)^T} \mathbb{E}_{\boldsymbol{Q}}[f(S_T)].$$

2. Assume the normal distribution of log returns and a constant volatility, according to Black-Scholes formula, the value of the Call option at time t is

$$P_{BSCall}(S_t, K, t, T, r, \sigma) = S_t N(d_1) - K e^{-r(T-t)} N(d_2)$$

where $d_1 = \dfrac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$

$d_2 = d_1 - \sigma\sqrt{T-t}$

$N(.)$ is the cumulative distribution function of $\mathcal{N}(0,1)$.

1.3 Implied Volatility

Implied volatility is the value σ^* such that

$$P_{market}(S_t, K, t, T) = P_{BSCall}(S_t, K, t, T, r, \sigma^*).$$

where P_{market} is the market price of the option.
With σ being the variable, for solving this equation we will use the Newton-Raphson method:

$$\sigma^{(i+1)} = \sigma^{(i)} - \frac{P_{BSCall}(\sigma^{(i)}) - P_{market}}{\mathcal{V}(\sigma^{(i)})}$$

where $\mathcal{V}(\sigma^{(i)}) = \dfrac{\partial P_{BSCall}(\sigma^{(i)})}{\partial \sigma^{(i)}}$.
We also try using Secant method where $\mathcal{V}(\sigma^{(i)})$ is approximately $\dfrac{P_{BSCall}(\sigma^{(i)}) - P_{BSCall}(\sigma^{(i-1)})}{\sigma^{(i)} - \sigma^{(i-1)}}$

2 Simulation and results

2.1 Monte Carlo simulation

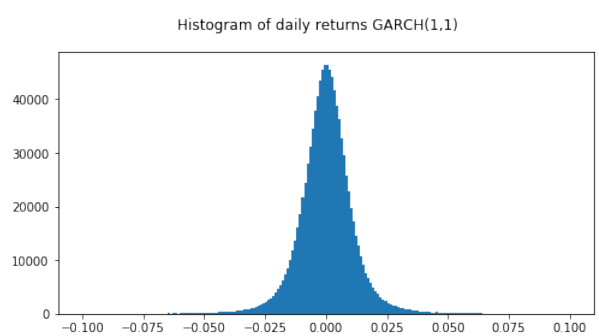
Our steps for pricing the option using Monte Carlo method are as follow:

1. Simulate GARCH(1,1) process with the dynamic under the risk neutral probability and $\lambda = 0$ in case of discrete time. We set the initial volatility equals to the invariant volatility.
2. For each simulation, we compute the corresponding option payoffs
3. Average discounted payoffs to get the price of the option.

2.2 GARCH returns and asset prices

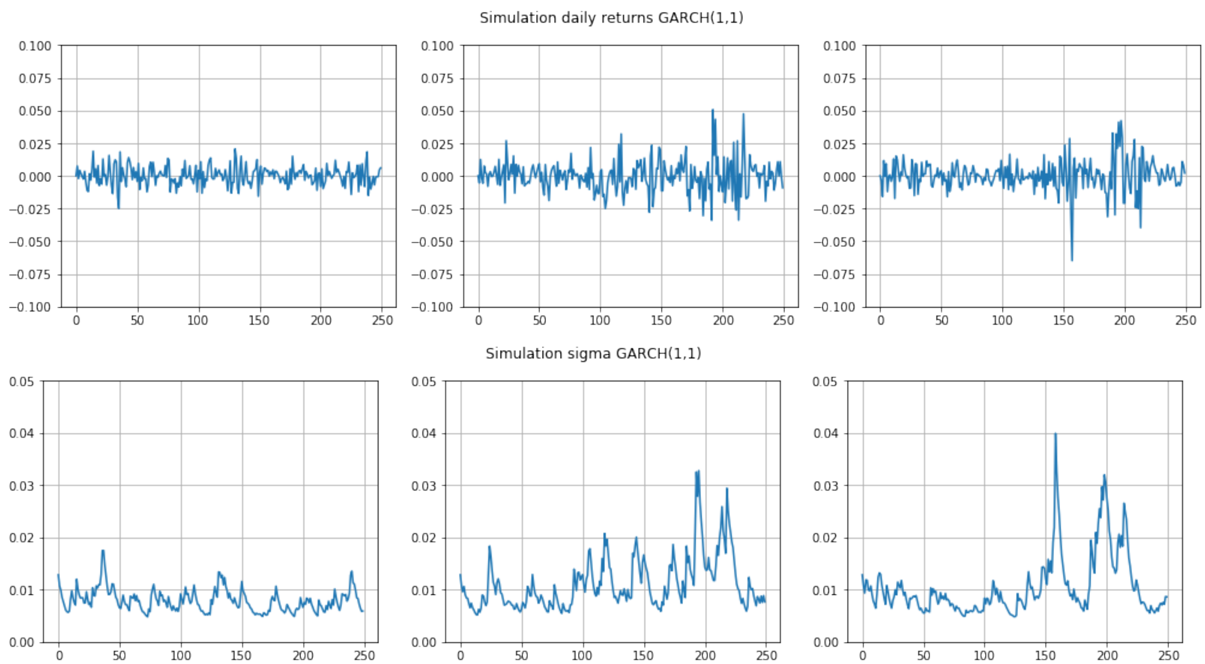
1. Using values of the parameters for stock's price on a daily scale: T =250 and r = 0.0002 we simulate processes of daily returns. We will check whether GARCH(1,1) with parameter $\omega = 6.6.10^{-6}$, $\alpha = 0.3$, $\beta = 0.66$ can capture some main characteristics of the returns of financial assets.

   From 1 000 000 simulations, we have the histogram of simulated daily returns.

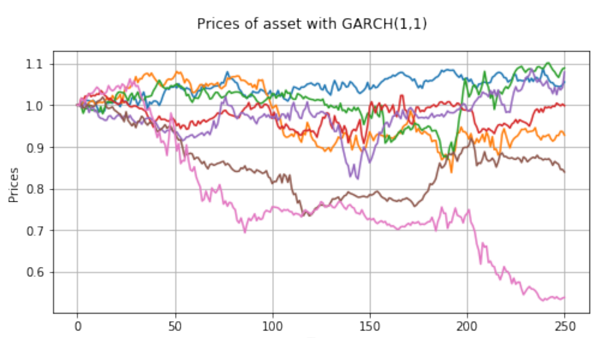


With the chosen parameters, the simulated sample has Mean = 0.0002, Skewness = 0.5775 and Kurtosis = 79.3568. High Kurtosis indicator shows us that the distribution of GARCH(1,1) process is heavy-tail, which is indeed a feature of time series of financial returns. It means that investors will occasionally experience more extreme returns (either positive or negative) than the normal distribution of returns.

Below are the graphs of some simulated processes of daily returns and the estimate of σ_t. We can see that large (small) changes tend to be followed by large (small) changes (not necessary the same sign). This reflect the feature "persistence of volatility" of returns.



2. From the above returns, we then obtain this graph, which shows the daily prices of assets



2.3 Call pricing

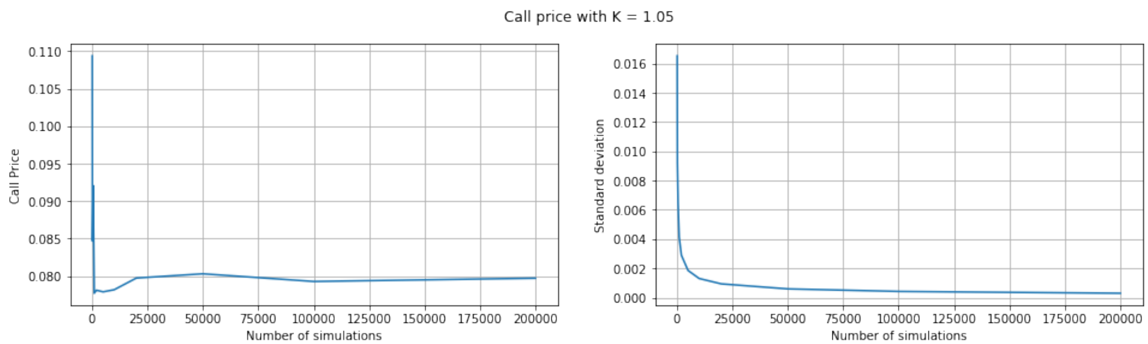
We have:

$$P = \frac{1}{(1+r)^T} \mathbb{E}_{\mathbb{Q}}[f(S_T)]$$

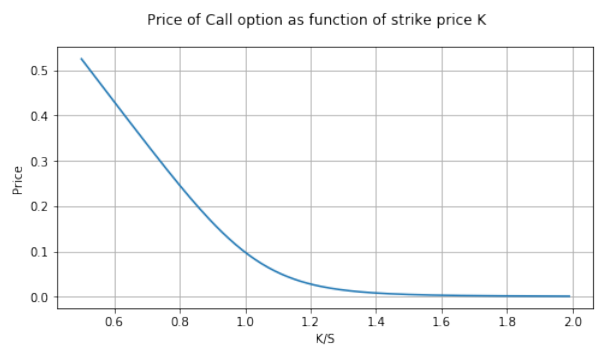
$$Var(P) = \frac{1}{\sqrt{N}(1+r)^{2T}} Var_{\mathbb{Q}}[f(S_T)]$$

With different number of simulations, we obtain the prices of Call option and calculate the standard deviation of these estimates. Given K = 1.05, we obtain

Number of simulations	200	500	1000	2000	5000	10000	20000	50000	100000	200000
Call price	0.0846	0.0920	0.0776	0.0780	0.0778	0.0781	0.0797	0.0802	0.0792	0.0796
Standard deviation	0.0092	0.0064	0.0041	0.0028	0.0018	0.0013	0.0009	0.0005	0.0004	0.0003



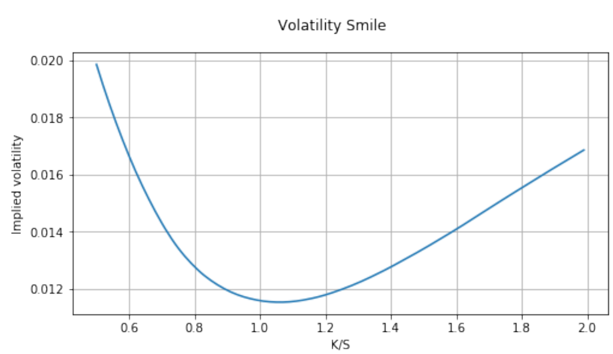
As number of simulations increases, the price of Call option with strike price $K = 1.05$ converges to around 0.0796.



Above is the graph of price of Call option with different strike price. As we see the price of Call option is much higher when the option is deeply in the money at time $t = 0$, and it decrease as the strike price goes up. When the option is deeply out the money at time $t = 0$, the value of the option is nearly 0.

2.4 Volatility smile

We use different values of strike price K, and estimate the implied volatility using Secant method. And we obtain the result:



We can see that implied volatility rises when the option is further out of the money or in the money, compared to at the money.

3 Conclusion

In this project, we make a program to compute the price of a Call option with no risk premium using a GARCH Model and Monte Carlo method. For the calibration of the model, we used typical values of the parameters for financial data on a daily scale instead of estimated ones. Then we study how the price of the Call option depends on the value of the ratio S/K and compute the values of the implied volatility using Newton-Raphson and Secant method with different value of S/K.

References

[1] Peter J. Brockwell Richard A. Davis. *Introduction to Time Series and Forecasting*. Springer Texts in Statistics. Springer, 2016.

[2] Jin Chuan Duan. *GARCH and stochastic volatility option pricing*. URL: `http://www.math.ntu.edu.tw/~hchen/jointseminar/garchopt.pdf`.